The Local Fundamental Class

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Abstract

We compute the local fundamental class of the extension $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ when p is an odd prime. This requires making a number of standard group cohomology constructions fully explicit in the process.

Contents

1 Set-Up

Fix an extension local fields L/K. Then let K_m be the largest unramified subextension, which we will give degree m; let $\overline{\sigma}_K \in \operatorname{Gal}(L/K)$ denote the Frobenius automorphism, which lets us set

$$K_{\pi,\nu} := L^{\langle \overline{\sigma}_K \rangle}.$$

In particular, $K_{\pi,\nu}/K$ is totally ramified because, for example, the residue fields of $K_{\pi,\nu}$ and K have the same order.

Example 1. For $K = \mathbb{Q}_p$, we can take $K_m = \mathbb{Q}_p\left(\zeta_{p^m-1}\right)$ and $K_{\pi,\nu} = \mathbb{Q}_p\left(\zeta_{p^\nu}\right)$.

For some fixed ν and m, we let $L := K_{\pi,\nu}K_m$. This gives us the following tower of fields.

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash] & \\
K_{\pi,\nu} \arrow[dr, dash] & & K_m \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

To help us a little later, we will assume that the extension L/K is neither totally ramified nor unramified.

Remark 2. Assuming that L/K is neither totally ramified nor unramified is not actually very big of a problem because we can apply inflation to $u_{L/K}$ to read off the fundamental class for the totally ramified and unramified parts.

We provide some quick commentary on these extensions.

- The extension K_m/K is unramified of degree $f := m$; note we are assuming $1 < f < n$. Its Galois group is thus generated by the Frobenius element defined by $\overline{\sigma}_K$.
- The extension $K_{\pi,\nu}/K$ is totally ramified of degree $[K_{\pi,\nu} : K]$. Because we are assuming this Galois group is abelian, we may write
$$\operatorname{Gal}(K_{\pi,\nu}/K) \simeq \Gamma_1 \times \cdots \times \Gamma_t$$
where $\Gamma_i = \langle \tau_i \rangle \subseteq \operatorname{Gal}(K_{\pi,\nu}/K)$ is a cyclic group of order n_i.
- Because $K_{\pi,\nu}/K$ is totally ramified and K_m/K is unramified, we have that the fields $K_{\pi,\nu}$ and K_m are linearly disjoint over K. As such, $L = K_{\pi,\nu}K_m$ has
$$\begin{aligned}
\operatorname{Gal}(L/K_{\pi,\nu}) &\simeq \operatorname{Gal}(K_m/K) = \langle \overline{\sigma}_K \rangle \\
\operatorname{Gal}(L/K_m) &\simeq \operatorname{Gal}(K_{\pi,\nu}/K) = \Gamma_1 \times \cdots \times \Gamma_t \\
\operatorname{Gal}(L/K) &\simeq \operatorname{Gal}(K_m/K) \times \operatorname{Gal}(K_{\pi,\nu}/K) = \langle \overline{\sigma}_K \rangle \times \Gamma_1 \times \cdots \times \Gamma_t.
\end{aligned}$$
In light of these isomorphisms, we will upgrade $\overline{\sigma}_K$ to the automorphism of L/K which restricts properly on K_m/K and fixing $K_{\pi,\nu}$; we do analogously for the τ_i. We also acknowledge that our degree is
$$n := [L : K] = [K_m : K] \cdot [K_{\pi,\nu} : K] = f \cdot [K_{\pi,\nu} : K].$$

For brevity, we will also set $L_i := L^{\langle \tau_i \rangle}$ for each i, which makes the fields under L look like the following.

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash] & & & \\
K_{\pi,\nu} \arrow[ddr, dash] & & L_1 \arrow[d, dash] & \cdots & L_t \arrow[dll, dash] \\
 & & K_m \arrow[dl, dash] & & \\
 & K & & &
\end{tikzcd}$$

In particular, $\operatorname{Gal}(L/L_i)$ is cyclic for each i.

Now, the main idea in the computation is to use an unramified extension $M := K_n$ of the same degree as L/K. This modifies our diagram of fields as follows.

$$\begin{tikzcd}
& ML \arrow[dl, dash, "\text{unr}"'] \arrow[dr, dash, "\text{ram}"] & \\
L \arrow[d, dash, "\text{unr}"'] \arrow[dr, dash, "\text{ram}"] & & M \arrow[dl, dash, "\text{unr}"] \\
K_{\pi,\nu} \arrow[dr, dash, "\text{ram}"'] & K_m \arrow[d, dash, "\text{unr}"] & \\
& K &
\end{tikzcd}$$

We have labeled the unramified extensions by "unr" and the totally ramified extensions by "ram."

As before, we provide some comments on the field extensions.

- The extension M/K is unramified of degree n. As before, its Galois group is cyclic, generated by the Frobenius element σ_K. Observe that σ_K restricted to K_m is $\overline{\sigma}_K$, explaining our notation. In particular, σ_K has order n, but $\overline{\sigma}_K$ has order $f < n$.
- As before, note that $K_{\pi,\nu}$ and M are linearly disjoint over K because $K_{\pi,\nu}/K$ is totally ramified while M/K is unramified. As such, we may say that

  $$\begin{aligned}
  \operatorname{Gal}(ML/M) &\simeq \operatorname{Gal}(K_{\pi,\nu}/K) = \Gamma_1 \times \cdots \times \Gamma_t \\
  \operatorname{Gal}(ML/K_{\pi,\nu}) &\simeq \operatorname{Gal}(M/K) = \langle \sigma_K \rangle \\
  \operatorname{Gal}(ML/K) &\simeq \operatorname{Gal}(M/K) \times \operatorname{Gal}(K_{\pi,\nu}/K) = \langle \sigma_K \rangle \times \Gamma_1 \times \cdots \times \Gamma_t.
  \end{aligned}$$

  Again, we will upgrade σ_K and the τ_i to their corresponding automorphisms on any subfield of ML.
- We take a moment to compute

  $$\operatorname{Gal}(ML/L) \simeq \left\{ \sigma_K^a \tau \in \operatorname{Gal}(ML/K) : \sigma_K^a \tau|_L = \operatorname{id}_L \right\}.$$

  Because L is $K_{\pi,\nu}K_m$, it suffices to fix each of these fields individually. Well, to fix $K_{\pi,\nu}$, we need τ to vanish, so we might as well force $\tau = 1$. But to fix K_m, we need $\sigma_K^a|_{K_m} = \overline{\sigma}_K^a$ to be the identity, so we are actually requiring that $f \mid a$ here. As such,

  $$\operatorname{Gal}(ML/L) = \langle \sigma_K^f \rangle.$$

These comments complete the Galois-theoretic portion of the analysis.

2 Idea

We will begin by briefly describe the outline for the computation. For a finite extension of local fields L/K, let $u_{L/K} \in H^2(L/K)$ denote the fundamental class.

Now, take variables as in our set-up in section 1. The main idea is to translate what we know about the unramified extension M/K over to the general extension L/K. In particular, we are able to compute the fundamental class $u_{M/K} \in H^2(M/K)$, so we observe that

$$\operatorname{Inf}_{M/K}^{ML/K} u_{M/K} = [ML : M] u_{M/K} = n \cdot u_{ML/K} = [ML : L] u_{ML/L} = \operatorname{Inf}_{L/K}^{ML/K} u_{L/K}.$$

As such, we will be able to compute $u_{L/K}$ as long as we are able to invert the inflation map $\operatorname{Inf}\colon H^2(L/K) \to H^2(ML/K)$. This is not actually very easy to do in general,[1] but we are in luck because this inflation map here comes from the Inflation–Restriction exact sequence

$$0 \to H^2(L/K) \stackrel{\text{Inf}}{\to} H^2(ML/K) \stackrel{\text{Res}}{\to} H^2(ML/L).$$

[1] The difficulty comes from the fact that a generic cocycle might be off from an inflated cocycle by some truly hideous coboundary.

The argument for the Inflation–Restriction exact sequence is an explicit computation on cocycles (involving some dimension shifting), but it can be tracked backwards to give the desired cocycle.

3 Computation

In this section we record the details of the computation.

3.1 Group Cohomology

Throughout this section, G will be a group (usually finite) and $H \subseteq G$ will be a subgroup (usually normal). We denote $\mathbb{Z}[G]$ by the group ring and $I_G \subseteq \mathbb{Z}[G]$ by the augmentation ideal, defined as the kernel of the map $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ which sends $g \mapsto 1$ for all $g \in G$.

We begin by recalling the statement of the Inflation–Restriction exact sequence.

Theorem 3 (Inflation–Restriction)**.** Let G be a finite group with normal subgroup $H \subseteq G$. Given a G-module A, suppose that the $H^i(H, A) = 0$ for $1 \le i < q$ for some index $q \ge 1$. Then the sequence

$$0 \to H^q\left(G/H, A^H\right) \overset{\text{Inf}}{\to} H^q(G, A) \overset{\text{Res}}{\to} H^q(H, A)$$

is exact.

Sketch. The proof is by induction on q, via dimension shifting. For $q = 1$, we can just directly check this on 1-cocycles. The main point is the exactness at $H^q(G, A)$: if $c \in Z^1(G, A)$ has $\operatorname{Res}(c) \in B^1(H, A)$, then find $a \in A$ with

$$\operatorname{Res}(c)(a) := h \cdot a - a.$$

As such, we define $f_a \in B^1(G, A)$ by $f_a(g) := g \cdot a - a$, which implies that $c - f_a$ vanishes on H. It is then possible to stare at the 1-cocycle condition

$$(c - f_a)(gg') = (c - f_a)(g) + g \cdot (c - f_a)(g')$$

to check that $c - f_a$ only depends on the cosets of H (e.g., by taking $g' \in H$) and that $\operatorname{im}(c - f_a) \subseteq A^H$ (e.g., by taking $g \in H$).

For $q > 1$, we use dimension shifting via the following lemma.

Lemma 4 (Dimension shifting)**.** Let G be a group with subgroup $H \subseteq G$. Given a G-module A, all indices $q \ge 1$ have

$$\delta\colon H^q(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^{q+1}(H, A).$$

Sketch. Recall that we have the short exact sequence of $\mathbb{Z}[H]$-modules

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

In fact, this short exact sequence splits over $\mathbb{Z}$, so it will still be short exact after applying $\operatorname{Hom}_{\mathbb{Z}}(-, A)$, which gives the short exact sequence

$$0 \to A \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to \operatorname{Hom}_{\mathbb{Z}}(I_G, A) \to 0$$

of $\mathbb{Z}[H]$-modules. The result now follows from the long exact sequence of cohomology upon noting that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ is coinduced and hence acyclic for cohomology. ■

Using the above lemma, we have the following the commutative diagram with vertical arrows which are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & H^q\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right) \arrow[r] \arrow[d, "\delta"] & H^q(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[r] \arrow[d, "\delta"] & H^q(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[d, "\delta"] \\
0 \arrow[r] & H^{q+1}\left(G/H, A^H\right) \arrow[r] & H^{q+1}(G, A) \arrow[r] & H^{q+1}(H, A)
\end{tikzcd}$$

The top row is exact by the inductive hypothesis, so the bottom row is therefore also exact. ■

Our goal is to make the above proof explicit in the case of $q = 2$, which is the only reason we sketched the above proofs at all. We begin by making the dimension shifting explicit.

Lemma 5. Let G be a group with subgroup $H \subseteq G$, and let $\{g_\alpha\}_{\alpha\in\lambda}$ be coset representatives for $H\backslash G$. Now, given a G-module A, the maps

$$\begin{aligned}
\delta_H\colon Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) &\to Z^2(H, A) \\
c &\mapsto \left[(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)\right] \\
\left[h \mapsto \left((h'g_\bullet - 1) \mapsto h' \cdot u((h')^{-1}, h)\right)\right] &\leftarrow u
\end{aligned}$$

are group homomorphisms which descend to the isomorphism $\overline{\delta}\colon H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^2(H, A)$ of Lemma 4. The map δ above is surjective, and the reverse map is a section; when $H = G$, these are isomorphisms.

Proof. We begin by noting that our short exact sequence can be written more explicitly as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(I_G, A) & \longrightarrow & 0 \\
 & & a & \longmapsto & (z \mapsto \varepsilon(z)a) & & & & \\
 & & & & f & \longmapsto & f|_{I_G} & &
\end{array}$$

We now track through the induced boundary morphism $\delta\colon H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \to H^2(H, Q)$.

- We begin with $c \in Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$, which means that we have $c(h)\colon I_G \to A$ for each $h, h' \in H$, and we satisfy
$$c(hh') = c(h) + h \cdot c(h').$$
Tracking through the action of H on $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)$, this means that
$$c(hh')(g-1) = c(h)(g-1) + h \cdot c(h')(h^{-1}g - h^{-1})$$
for any $g \in G$.
- To pull c back to $C^1(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$, we need to lift $c(h)\colon I_G \to A$ to a $\widetilde{c}(h)\colon \mathbb{Z}[G] \to A$. Recalling that we only need to preserve group structure, we simply precompose $c(h)$ with the map $\mathbb{Z}[G] \to I_G$ given by $z \mapsto z - \varepsilon(z)$. That is, we define
$$\widetilde{c}(h)(z) := c(h)(z - \varepsilon(z)).$$
- We now push $\widetilde{c}$ through $d\colon C^1(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \to Z^2(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$. This gives
$$(d\widetilde{c})(h, h') = g\widetilde{c}(h') - \widetilde{c}(hh') + \widetilde{c}(h)$$

for any $h, h' \in H$. Concretely, plugging in some $z \in \mathbb{Z}[G]$ makes this look like

$$\begin{aligned}(d\widetilde{c})(h,h')(z) &= (h\widetilde{c}(h'))(z) - \widetilde{c}(hh')(z) + \widetilde{c}(h)(z) \\ &= h \cdot c(h')\left(h^{-1}z - \varepsilon(h^{-1}z)\right) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) \\ &= h \cdot c(h')\left(h^{-1}z - \varepsilon(z)\right) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)).\end{aligned}$$

Now, from the 1-cocycle condition on c, we recall

$$-c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) = -h \cdot (c(h')(h^{-1}z - \varepsilon(z)h^{-1})),$$

so

$$\begin{aligned}(d\widetilde{c})(h,h')(z) &= h \cdot c(h')\left(\varepsilon(z)h^{-1} - \varepsilon(z)\right) \\ &= \varepsilon(z) \cdot \left(h \cdot c(h')\left(h^{-1} - 1\right)\right).\end{aligned}$$

In particular, we see that $d\widetilde{c} \in Z^2(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ pulls back to $(h,h') \mapsto h \cdot c(h')\left(h^{-1} - 1\right)$ in $Z^2(H, A)$. It is not too difficult to check that we have in fact defined a 2-cocycle, but we will not do so because it is not necessary for the proof.

Now, we do know that δ_H is a homomorphism abstractly on elements of our cohomology classes by the Snake lemma, but it is also not too hard to see that

$$\delta_H \colon Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \to Z^2(H, A)$$

is in fact a homomorphism of groups directly from the construction. In short,

$$\delta_H(c + c')(h,h') = h' \cdot c(h)\left(h^{-1} - 1\right) + h' \cdot c'(h)\left(h^{-1} - 1\right) = (\delta_H(c) + \delta_H(c'))(h,h')$$

for any $h, h' \in H$.

It remains to prove the last sentence. We run the following checks; given $u \in Z^2(H, A)$, define $c_u \in C^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$ by

$$c_u(h)(h'g_\bullet - 1) = h' \cdot u\left((h')^{-1}, h\right).$$

Note that this is enough data to define $c_u(h) \colon I_G \to A$ because I_G is a free $\mathbb{Z}$-module generated by $\{g - 1 : g \in G\}$.

- We verify that c_u is a 1-cocycle. This is a matter of force. Pick up $h, h' \in H$ and $g_\bullet h'' \in G$ and write

$$\begin{aligned}&(hc_u(h'))(h''g_\bullet - 1) + c_u(hh')(h''g_\bullet - 1) + c_u(h)(h''g_\bullet - 1) \\ &= h \cdot c_u(h')\left(h^{-1}h''g_\bullet - h^{-1}\right) + c_u(hh')(h''g_\bullet - 1) + c_u(h)(h''g_\bullet - 1) \\ &= h \cdot \left(h^{-1}h''u\left((h'')^{-1}h, h'\right) - h^{-1}u(h,h')\right) + h''u\left((h'')^{-1}, hh'\right) + h''u\left((h'')^{-1}, h\right) \\ &= h''u\left((h'')^{-1}h, h'\right) - u(h,h') + h''u\left((h'')^{-1}, hh'\right) + h''u\left((h'')^{-1}, h\right).\end{aligned}$$

  This is just the 2-cocycle condition for u upon dividing out by h'', so we are done.

- For $u \in Z^2(H, A)$, we verify that $\delta_H(c_u) = u$. Indeed, given $h, h' \in H$, we check

$$\begin{aligned}\delta_H(c_u)(h,h') &= h \cdot c_u(h')\left(h^{-1} - 1\right) \\ &= h \cdot h^{-1} \cdot u(h,h') \\ &= u(h,h').\end{aligned}$$

So far we have verified that δ has section $u \mapsto c_u$ and hence must be surjective. Lastly, we take $H = G$ and show that $c_{\delta c} = c$ to finish. Indeed, for $g, g' \in G = H$, we write

$$\begin{aligned}c_{\delta_H c}(g)(g' - 1) &= g' \cdot (\delta_H c)\left((g')^{-1}, g\right) \\ &= g'(g')^{-1} \cdot c(g)(g' - 1) \\ &= c(g)(g' - 1),\end{aligned}$$

which is what we wanted. ■

We also have used dimension shifting to show that $H^1\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right) \to H^2\left(G/H, A^H\right)$ is an isomorphism, but this requires a little more trickery. To begin, we discuss how to lift from $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ to $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$.

Lemma 6. Let G be a group with subgroup $H \subseteq G$. Fix a G-module A with $H^1(H, A) = 0$. Then, for any $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$, the function $h \mapsto h\psi\left(h^{-1} - 1\right)$ is a cocycle in $Z^1(H, A) = B^1(H, A)$, so we can define a function $\eta_\bullet \colon \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H \to A$ such that

$$\psi(h-1) = h \cdot \eta_\varphi - \eta_\varphi$$

for all $h \in H$. In fact, given $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$, we can construct $\widetilde{\varphi} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$ by

$$\widetilde{\varphi}(z) := \varphi(z - \varepsilon(z)) + \varepsilon(z)\eta_\varphi$$

so that $\widetilde{\varphi}|_{I_G} = \varphi$.

Proof. We will just run the checks directly.

- We start by checking $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ give 1-cocycles $c(h) := \varphi(h-1)$ in $Z^1(A, H)$. To begin, we note that $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ simply means that any $z - \varepsilon(z) \in I_G$ has
$$\psi(z - \varepsilon(z)) = (h\psi)(z - \varepsilon(z)) = h\psi\left(h^{-1}z - h^{-1}\varepsilon(z)\right)$$
for all $h \in H$. In particular, replacing h with h^{-1} tells us that
$$h\psi(z - \varepsilon(z)) = \psi(hz - h\varepsilon(z)).$$
Now, we can just compute
$$\begin{aligned}(dc)(h, h') &= hc(h') - c(hh') + c(h) \\ &= hc\left(h' - 1\right) - c\left(hh' - 1\right) + c\left(h - 1\right) \\ &= c\left(hh' - h\right) - c\left(hh' - 1\right) + c\left(h - 1\right),\end{aligned}$$
where in the last equality we used the fact that $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$. Now, $(dc)(h, h')$ manifestly vanishes, so we are done.
- Note that $\widetilde{\varphi} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ because it is a linear combination of (compositions of) homomorphisms.
- Note that any $z \in I_G$ has $\varepsilon(z) = 0$, so
$$\widetilde{\varphi}(z) = \varphi(z - 0) + 0 \cdot \eta_\varphi = \varphi(z),$$
so $\widetilde{\varphi}|_{I_G} = \varphi$.
- It remains to check that $\widetilde{\varphi}$ is fixed by H. This requires a little more effort. Recall that $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ means that any $z - \varepsilon(z) \in I_G$ has
$$h\varphi(z - \varepsilon(z)) = \varphi\left(hz - h\varepsilon(z)\right)$$
for any $h \in H$. Now, we just compute
$$\begin{aligned}(h\widetilde{\varphi})(z) &= h\widetilde{\varphi}\left(h^{-1}z\right) \\ &= h\left(\varphi\left(h^{-1}z - \varepsilon(h^{-1}z)\right) + \varepsilon(h^{-1}z)\eta_\varphi\right) \\ &= \varphi\left(z - h\varepsilon(z)\right) + \varepsilon(z) \cdot h\eta_\varphi \\ &= \varphi\left(z - h\varepsilon(z)\right) + \varepsilon(z)\varphi(h-1) + \varepsilon(z)\eta_\varphi \\ &= \varphi(z - \varepsilon(z)) + \varepsilon(z)\eta_\varphi \\ &= \widetilde{\varphi}(z).\end{aligned}$$

The above checks complete the proof. ■

Remark 7. For motivation, the $\widetilde{\varphi}$ was constructed by tracking through the following diagram.

$$\begin{tikzcd} \dfrac{C^0(H,A)}{B^0(H,A)} \arrow[r] \arrow[d] & \dfrac{C^0(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))}{B^0(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))} \arrow[r] \arrow[d] & \dfrac{C^0(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A))}{B^0(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A))} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z^1(H,A)=B^1(H,A) \arrow[r] & Z^1(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)) \arrow[r] & Z^1(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A)) \end{tikzcd}$$

In short, take $\varphi \in Z^0(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A)) = \operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H$, pull it back to $z \mapsto \varphi(z-\varepsilon(z))$. Pushing this down to $Z^1(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))$ and pulling back to $Z^1(H,A)$ takes us to the 1-cocycle $h \mapsto h\varphi\left(h^{-1}-1\right)$. Here we use the $H^1(H,A)=0$ condition above and adjust our lift $z \mapsto \varphi(z-\varepsilon(z))$ accordingly.

And now we can now make our dimension shifting explicit.

Lemma 8. Work in the context of Lemma 6 and assume that $H \subseteq G$ is normal. We track through the isomorphism
$$\delta\colon H^1\left(G/H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H\right) \simeq H^2\left(G/H,A^H\right)$$
given by the exact sequence
$$0 \to A^H \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)^H \to \operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H \to 0.$$

Proof. We begin with some $c \in H^1\left(G/H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H\right)$. To track through the δ, we define
$$\widetilde{c}(gH) \coloneqq c(gH)(z-\varepsilon(z)) + \eta_{c(gH)}\varepsilon(z)$$
to be the lift given in Lemma 6. Now, we are given that $dc=0$, which here means that any $z \in \mathbb{Z}[G]$ and $gH, g'H \in G/H$ will have
$$\begin{aligned} 0 &= (dc)(gH,g'H)(z-\varepsilon(z)) \\ 0 &= (gH\cdot c(g'H) - c(gg'H) + c(gH))(z-\varepsilon(z)) \\ 0 &= g\cdot c(g'H)\left(g^{-1}z - g^{-1}\varepsilon(z)\right) - c(gg'H)(z-\varepsilon(z)) + c(gH)(z-\varepsilon(z)) \\ g\cdot c(g'H)\left(g^{-1}-1\right)\varepsilon(z) &= g\cdot c(g'H)\left(g^{-1}z-\varepsilon(z)\right) - c(gg'H)(z-\varepsilon(z)) + c(gH)(z-\varepsilon(z)) \\ g\cdot c(g'H)\left(g^{-1}-1\right)\varepsilon(z) &= g\cdot c(g'H)\left(g^{-1}z-\varepsilon(g^{-1}z)\right) - c(gg'H)(z-\varepsilon(z)) + c(gH)(z-\varepsilon(z)). \end{aligned}$$
We now directly compute that
$$\begin{aligned} (d\widetilde{c})(gH,g'H)(z) &= (gH\cdot c(g'H) - c(gg'H) + c(gH))(z) \\ &= g\cdot c(g'H)\left(g^{-1}z-\varepsilon(g^{-1}z)\right) + g\eta_{c(g'H)}\varepsilon(z) \\ &\quad - c(gg'H)(z-\varepsilon(z)) - \eta_{c(gg'H)}\varepsilon(z) \\ &\quad + c(gH)(z-\varepsilon(z)) + \eta_{c(gH)}\varepsilon(z) \\ &= \left(g\cdot c(g'H)\left(g^{-1}-1\right) + g\cdot\eta_{c(g'H)} - \eta_{c(gg'H)} + \eta_{c(gH)}\right)\varepsilon(z) \end{aligned}$$
As such, we have pulled ourselves back to the 2-cocycle given by
$$\boxed{u(gH,g'H) \coloneqq g\cdot c(g'H)\left(g^{-1}-1\right) + g\cdot\eta_{c(g'H)} - \eta_{c(gg'H)} + \eta_{c(gH)}}.$$
We quickly note that this is in fact independent of our choice of representative $g \in gH$: changing representative of g to gh for $h \in H$ will only affect the terms
$$h\cdot c(g'H)\left(h^{-1}g^{-1}-1\right) + h\eta_{c(g'H)} = c(g'H)\left(g^{-1}-h\right) + c(g'H)(h-1) + \eta_{c(g'H)} = c(g'H)\left(g^{-1}-1\right) + \eta_{c(g'H)},$$
so we are indeed safe. This completes the proof. ■

We now make Theorem 3 explicit in the case of $q=2$.

Lemma 9. Let G be a group with normal subgroup $H \subseteq G$. Fix a G-module A with $H^1(H,A)=0$, and define the function $\eta_\bullet\colon \operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H \to A$ of Lemma 6. Given $c \in Z^2(G,A)$ such that $\operatorname{Res}^G_H c \in B^2(H,A)$; in particular, suppose we have $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G,A)$ such that all $h \in H$ have

$$\operatorname{Res}^G_H(\delta^{-1}c)(h) = (db)(h) = h \cdot b - h,$$

where δ^{-1} is the inverse isomorphism of Lemma 5. Then we find $u \in Z^2\left(G/H, A^H\right)$ such that

$$[\operatorname{Inf} u] = [c]$$

in $H^2(G,A)$.

Proof. The main point is that boundary morphisms δ commute with Res and Inf. By construction, we have that $\left(\operatorname{Res}^G_H \delta^{-1}c\right) - db = 0$ in $Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G,A))$. Pulling back to $Z^1(G,\operatorname{Hom}_{\mathbb{Z}}(I_G,A))$, we note that

$$c' \coloneqq \left(\delta^{-1}c - db\right) \in Z^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G,A))$$

vanishes on H by hypothesis. Because $\delta^{-1}c - db$ is a 1-cocycle, we are able to write

$$c'(gg') = c'(g) + gc'(g').$$

Letting g' vary over H, we see that $\delta^{-1}c - db$ is well-defined on G/H. On the other hand, for any $h \in H$ and $g \in G$, we note that $g^{-1}hg \in H$, so

$$c'(g) = c'\left(g \cdot g^{-1}hg\right) = c'\left(hg\right) = c'\left(h\right) + hc(g),$$

implying that $c'(g) \in \operatorname{Hom}_{\mathbb{Z}}(I_G,A)^H$.

We are now ready to apply Lemma 8, which we use on c', thus defining $u \coloneqq \delta(c')$. Explicitly, we have

$$\boxed{u(gH,g'H) = g \cdot c'(g'H)\left(g^{-1}-1\right) + g \cdot \eta_{c'(g'H)} - \eta_{c'(gg'H)} + \eta_{c'(gH)}}.$$

This is explicit enough for our purposes. Observe that $[\operatorname{Inf} u] = [c]$ because $[\operatorname{Inf} c'] = [\delta^{-1}c]$, and δ commutes with Inf. ■

3.2 Number Theory

Throughout, we will let $u_{L/K}$ denote a representative of the fundamental class in $H^2(L/K)$ rather than the actual cohomology class, mostly out of laziness.

We now return to the set-up in section 1 and track through Lemma 9 in our case. For reference, the following is the diagram that we will be chasing around; here $G \coloneqq \operatorname{Gal}(ML/K)$ and $H \coloneqq \operatorname{Gal}(ML/L)$.

$$\begin{tikzcd}
 & & H^2(\operatorname{Gal}(M/K), M^\times) \arrow[d, "\operatorname{Inf}"] & \\
0 \arrow[r] & H^2(\operatorname{Gal}(L/K), L^\times) \arrow[r, "\operatorname{Inf}"] & H^2(G, ML^\times) \arrow[r, "\operatorname{Res}"] & H^2(\operatorname{Gal}(ML/L), ML^\times) \\
0 \arrow[r] & H^1(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)^H) \arrow[r, "\operatorname{Inf}"] \arrow[u, "\delta"] & H^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \arrow[r, "\operatorname{Res}"] \arrow[u, "\delta"] & H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \arrow[u, "\delta"]
\end{tikzcd}$$

To begin, we know that we can write

$$u_{M/K}\left(\sigma_K^i, \sigma_K^j\right) = \pi^{\left\lfloor \frac{i+j}{n} \right\rfloor} = \begin{cases} 1 & i+j<n, \\ \pi & i+j \geq n, \end{cases}$$

where π is a uniformizer of K. Inflating this down to $H^2(G, ML^\times)$ gives

$$(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\tau, \sigma_K^{b_1}\tau'\right) = \pi^{\left\lfloor \frac{a_1+b_1}{n}\right\rfloor}.$$

Now, we use Lemma 4 to move down to $H^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times))$ as

$$\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\tau\right)\left(\sigma_K^{b_1}\tau' - 1\right) = \sigma_K^{b_1}\tau' \cdot (\operatorname{Inf} u_{M/K})\left(\sigma_K^{[-b_1]}(\tau')^{-1}, \sigma_K^{a_1}\tau\right) = p^{\left\lfloor \frac{a_1+[-b_1]}{n}\right\rfloor},$$

where $[k]$ denote the integer $0 \le [k] < n$ such that $k \equiv [k] \pmod{n}$.

Now, we need to show that the restriction to $H = \langle \sigma_k^f \rangle$ is a coboundary. That is, we need to find $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ such that

$$\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{fa_1}\right) = \frac{\sigma_K^{fa_1} \cdot b}{b}.$$

Because I_G is freely generated by elements of the form $g - 1$ for $g \in G$, it suffices to plug in some arbitrary $\sigma_K^{b_1}\tau' - 1$, which we see requires

$$\begin{aligned}
\pi^{\left\lfloor \frac{fa_1+[-b_1]}{n}\right\rfloor} &= \frac{\left(\sigma_K^{fa_1} \cdot b\right)\left(\sigma_K^{b_1}\tau' - 1\right)}{b\left(\sigma_K^{b_1}\tau' - 1\right)} \\
&= \frac{\sigma_K^{fa_1} b\left(\sigma_K^{b_1 - fa_1}\tau' - 1\right)}{\sigma_K^{fa_1} b\left(\sigma_K^{-fa_1} - 1\right) b\left(\sigma_K^{b_1}\tau' - 1\right)}.
\end{aligned}$$

We can see that b should not depend on τ', so we define $\hat{b}\left(\sigma_K^a\right) = b\left(\sigma_K^a\tau' - 1\right)$; the above is then equivalent to

$$\begin{aligned}
\pi^{\left\lfloor \frac{fa_1+[-b_1]}{n}\right\rfloor} &= \frac{\sigma_K^{fa_1}\hat{b}\left(\sigma_K^{b_1 - fa_1}\right)}{\sigma_K^{fa_1}\hat{b}\left(\sigma_K^{-fa_1}\right)\hat{b}\left(\sigma_K^{b_1}\right)} \\
\pi^{\left\lfloor \frac{fa_1+b_1}{n}\right\rfloor} &= \frac{\hat{b}\left(\sigma_K^{-b_1 - fa_1}\right)}{\hat{b}\left(\sigma_K^{-fa_1}\right)\sigma_K^{-fa_1}\hat{b}\left(\sigma_K^{-b_1}\right)},
\end{aligned}$$

where we have negated b_1 in the last step. At this point, the right-hand side will look a lot more natural if we set $\tau := \sigma_K^{-1}$, which turns this into

$$\frac{\hat{b}\left(\tau_K^{fa_1}\right)\tau_K^{fa_1}\hat{b}\left(\tau_K^{b_1}\right)}{\hat{b}\left(\tau_K^{b_1 fa_1}\right)} = (1/\pi)^{\left\lfloor \frac{fa_1+b_1}{n}\right\rfloor}$$

after taking reciprocals. Thus, we see that $\hat{b}$ should be counting carries of τs. With this in mind, we let ϖ be a uniformizer of $K_{\pi,\nu}$ and note that $\varpi \in L$ be a uniformizer because $L/K_{\pi,\nu}$ is an unramified extension. It follows that

$$\varpi^{[ML:L]} \in \mathrm{N}_{ML/L}\left(ML^\times\right).$$

Further, $\varpi^{[ML:L]}$ has the same absolute value as π because $K_{\pi,\nu}/K$ is a totally ramified extension of degree $[K_{\pi,\nu} : K] = [ML : M] = [ML : L]$. Thus, π is a norm in $\mathrm{N}_{ML/L}(ML^\times)$ because ML/L is unramified and so $\mathcal{O}_L^\times \subseteq \mathrm{N}_{ML/L}(ML^\times)$. Thus, we find $\gamma \in ML^\times$ such that

$$\mathrm{N}_{ML/L}(\gamma) = \pi.$$

The point of doing all of this is so that we can codify our carrying by writing

$$\hat{b}\left(\tau_K^a\right) := \prod_{i=0}^{\lfloor a/f\rfloor - 1} \tau^{if}(\gamma)^{-1}.$$

Tracking out $\hat{b}$ backwards to b, our desired $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ is given by

$$\boxed{b\left(\sigma_K^{a_1}\tau - 1\right) = \prod_{i=0}^{\lfloor [-a_1]/f\rfloor - 1} \sigma_K^{-if}(\gamma)^{-1}}.$$

We take a moment to write out $c := \delta^{-1}(\operatorname{Inf} u_{M/K})/db$, which looks like

$$\begin{aligned}
c\left(\sigma_K^{a_1}\tau\right)\left(\sigma_K^{b_1}\tau' - 1\right) &= \frac{\delta^{-1}(\operatorname{Inf} u_{M/K})}{db}\left(\sigma_K^{a_1}\tau\right)\left(\sigma_K^{b_1}\tau' - 1\right) \\
&= \frac{\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\tau\right)\left(\sigma_K^{b_1}\tau' - 1\right)}{\left(\sigma_K^{a_1}\tau b\right)\left(\sigma_K^{b_1}\tau' - 1\right)/b\left(\sigma_K^{b_1}\tau' - 1\right)} \\
&= \frac{\pi^{\lfloor (a_1 + [-b_1])/n\rfloor}}{\sigma_K^{a_1}\tau b\left(\sigma_K^{b_1 - a_1}\tau'\tau^{-1} - \sigma_K^{-a_1}\tau^{-1}\right)/b\left(\sigma_K^{b_1}\tau' - 1\right)} \\
&= \pi^{\lfloor (a_1 + [-b_1])/n\rfloor}\cdot \hat{b}\left(\sigma_K^{b_1}\right)\cdot \sigma_K^{a_1}\tau\left(\frac{\hat{b}\left(\sigma_K^{-a_1}\right)}{\hat{b}\left(\sigma_K^{b_1 - a_1}\right)}\right).
\end{aligned}$$

Before proceeding, we discuss a few special cases.

- Taking $\sigma_K^{a_1}\tau = \tau_i$ for some τ_i, we get

$$\begin{aligned}
c\left(\tau_i\right)\left(\sigma_K^{b_1}\tau' - 1\right) &= \pi^{\lfloor (0 + [-b_1])/n\rfloor}\cdot \hat{b}\left(\sigma_K^{b_1}\right)\cdot \tau_i\left(\frac{1}{\hat{b}\left(\sigma_K^{b_1}\right)}\right) \\
&= \hat{b}\left(\sigma_K^{b_1}\right)/\tau_i\hat{b}\left(\sigma_K^{b_1}\right).
\end{aligned}$$

  In particular, $c\left(\sigma_x\right)\left(\sigma_K^{-1} - 1\right) = 1$, provided that $f > 1$. Additionally, $c(\tau_i)\left(\tau' - 1\right) = 1$.

  Our general theory says that $h \mapsto c(\sigma_x)(h-1)$ is a 1-cocycle in $Z^1(H, ML^\times)$ (though we could also check this directly), so Hilbert's Theorem 90 promises us a magical element $\eta_i \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1}\eta_i}{\eta_i} = \frac{\hat{b}\left(\sigma_K^{fb_1}\right)}{\tau_i\hat{b}\left(\sigma_K^{fb_1}\right)}$$

  for all $\sigma_K^{fb_1} \in H$. This condition will be a little clearer if we write everything in terms of $\tau_K := \sigma_K^{-1}$, which transforms this into

$$\frac{\tau_K^{fb_1}\eta_i}{\eta_i} = \frac{\hat{b}\left(\tau_K^{-fb_1}\right)}{\tau_i\hat{b}\left(\tau_K^{-fb_1}\right)} = \prod_{i=0}^{b_1 - 1}\frac{\tau_K^{if}\left(\gamma^{-1}\right)}{\tau_i\tau_K^{if}\left(\gamma^{-1}\right)} = \prod_{i=0}^{b_1 - 1}\frac{\tau_i\tau_K^{if}(\gamma)}{\tau_K^{if}(\gamma)}.$$

  Because we are dealing with a cyclic group H, it is not too hard to see that it suffices merely for $b_1 = 1$ to hold, so our magical element $\eta_{c(\sigma_x)}$ merely requires

$$\boxed{\frac{\sigma_K^{-f}\left(\eta_i\right)}{\eta_i} = \frac{\tau_i(\gamma)}{\gamma}}$$

  after inverting τ_K back to σ_K.

- Taking $\sigma_K^{a_1}\tau = \sigma_K$, we get
$$c(\sigma_K)\left(\sigma_K^{b_1}\tau' - 1\right) = \pi^{\lfloor(1+[-b_1])/n\rfloor}\cdot\hat{b}\left(\sigma_K^{b_1}\right)\cdot\sigma_K\left(\frac{\hat{b}\left(\sigma_K^{-1}\right)}{\hat{b}\left(\sigma_K^{b_1-1}\right)}\right).$$
In particular, $\sigma_K^{b_1}\tau' = \tau_i^{-1}$ will give $c(\sigma_K)\left(\tau_i^{-1}-1\right) = 1$. We will also want $c(\sigma_K)\left(\sigma_K^{-b_1}-1\right)$ for $0 \le b_1 < f$. Using the fact that $f < n$ and $f > 1$, it is not too hard to see that everything will cancel down to 1 except in the case where $b_1 = f-1$, where we get
$$c(\sigma_K)\left(\sigma_K^{-(f-1)}-1\right) = \sigma_K\left(\frac{1}{\hat{b}\left(\sigma_K^{-f}\right)}\right) = \sigma_K(\gamma).$$
Continuing as before, our general theory says that $h \mapsto c(\sigma_K)(h-1)$ is a 1-cocycle in $Z^1(H, ML^\times)$, though again we could just check this directly. It follows that Hilbert's Theorem 90 promises us a magical element $\eta_K \in ML^\times$ such that
$$\frac{\sigma_K^{fb_1}\eta_K}{\eta_K} = p^{\lfloor(1+[-fb_1])/n\rfloor}\cdot\hat{b}\left(\sigma_K^{fb_1}\right)\cdot\sigma_K\left(\frac{\hat{b}\left(\sigma_K^{-1}\right)}{\hat{b}\left(\sigma_K^{fb_1-1}\right)}\right)$$
for all $\sigma_K^{fb_1}\in H$. Using $f>1$, this collapses down to
$$\frac{\sigma_K^{fb_1}\eta_K}{\eta_K} = \frac{\hat{b}\left(\sigma_K^{fb_1}\right)}{\sigma_K\hat{b}\left(\sigma_K^{fb_1-1}\right)}.$$
As before, this condition will be a little clearer if we set $\tau_K \coloneqq \sigma_K^{-1}$, which turns the condition into
$$\frac{\tau_K^{fb_1}\eta_K}{\eta_K} = \frac{\hat{b}\left(\tau_K^{fb_1}\right)}{\sigma_K\hat{b}\left(\tau_K^{fb_1+1}\right)} = \prod_{i=0}^{b_1-1}\frac{\tau_K^{if}(\gamma^{-1})}{\sigma_K\tau_K^{if}(\gamma^{-1})} = \prod_{i=0}^{b_1-1}\frac{\sigma_K\tau_K^{if}(\gamma)}{\tau^{if}(\gamma)}.$$
(Notably, $\hat{b}\left(\tau^{fb_1}\right) = \hat{b}\left(\tau^{fb_1+1}\right)$ because $f>1$.) Again, because H is cyclic generated by τ^f, an induction shows that it suffices to check this condition for $b_1 = 1$, which means that our magical element $\eta_K \in ML^\times$ is constructed so that
$$\boxed{\frac{\sigma_K^{-f}(\eta_K)}{\eta_K} = \frac{\sigma_K(\gamma)}{\gamma}}$$
where we have again inverted back from τ_K to σ_K.
- We will not actually need a more concrete description of this, but we remark that we can run the same story for any $g\in G$ through to get an element $\eta_g \in ML^\times$ such that
$$\frac{\sigma_K^{fb_1}\eta_g}{\eta_g} = \frac{1}{c(g)(\sigma_K^{fb_1}-1)}$$
for any $\sigma_K^{fb_1}\in H$. As usual, this follows from our general theory.

We are now ready to describe the local fundamental class. Piecing what we have so far, we know from Lemma 9 that we can write
$$u_{L/K}(g,g') \coloneqq gc(g')\left(g^{-1}-1\right)\cdot\frac{g\eta_{g'}\cdot\eta_g}{\eta_{gg'}}.$$
Here are the values that we care about for our specific computation; for consistency, we set $\tau_0 \coloneqq \sigma_K$ and $n_0 \coloneqq f$ to be the order of τ_0.

- We write

$$\begin{aligned} u_{L/K}(\sigma_K, \tau_i) &= \sigma_K c(\tau_i)\left(\sigma_K^{-1} - 1\right) \cdot \frac{\sigma_K \eta_i \cdot \eta_K}{\eta_{\sigma_K \tau_i}} \\ &= \frac{\sigma_K \eta_i \cdot \eta_K}{\eta_{\sigma_K \sigma_x}}. \end{aligned}$$

- We write

$$\begin{aligned} u_{L/K}(\tau_i, \sigma_K) &= \tau_i c(\sigma_K)\left(\tau_i^{-1} - 1\right) \cdot \frac{\tau_i \eta_K \cdot \eta_i}{\eta_{\sigma_x \sigma_K}} \\ &= \frac{\tau_i \eta_K \cdot \eta_i}{\eta_{\sigma_x \sigma_K}}. \end{aligned}$$

- In particular, we know that we can set β_{i0} to

$$\begin{aligned} \beta_{i0} &\coloneqq \frac{u_{L/K}(\tau_i, \sigma_K)}{u_{L/K}(\sigma_K, \tau_i)} \\ &= \frac{\tau_i \eta_K \cdot \eta_i / \eta_{\sigma_x \sigma_K}}{\sigma_K \eta_i \cdot \eta_K / \eta_{\sigma_K \sigma_x}} \\ &\boxed{\beta_{i0} = \frac{\eta_i}{\sigma_K(\eta_i)} \cdot \frac{\tau_i(\eta_K)}{\eta_K}}. \end{aligned}$$

- We write

$$\begin{aligned} u_{L/K}(\tau_i, \tau_j) &= \tau_i c(\tau_j)\left(\tau_j^{-1} - 1\right) \cdot \frac{\tau_i \eta_j \cdot \eta_i}{\eta_{\tau_i \tau_j}} \\ &= \frac{\tau_i \eta_j \cdot \eta_i}{\eta_{\tau_i \tau_j}}. \end{aligned}$$

- Thus, for $i > j > 0$, we can set β_{ij} to

$$\begin{aligned} \beta_{ij} &\coloneqq \frac{u_{L/K}(\tau_i, \tau_j)}{u_{L/K}(\tau_j, \tau_i)} \\ &= \frac{\tau_i \eta_j \cdot \eta_i / \eta_{\tau_i \tau_j}}{\tau_j \eta_i \cdot \eta_j / \eta_{\tau_i \tau_j}} \\ &\boxed{\beta_{ij} = \frac{\eta_i}{\tau_j \eta_i} \cdot \frac{\tau_i \eta_j}{\eta_j}}. \end{aligned}$$

- We will go ahead and compute α_0 and the α_i, for completeness. For α_0, our element is given by

$$\begin{aligned} \alpha_0 &\coloneqq \prod_{i=0}^{f-1} u_{L/K}\left(\sigma_K^i, \sigma_K\right) \\ &= \prod_{i=0}^{f-1} \left(\sigma_K^i c\left(\sigma_K, \sigma_K^{-i} - 1\right) \cdot \frac{\sigma_K^i \eta_K \cdot \eta_{\sigma_K^i}}{\eta_{\sigma_K^{i+1}}}\right). \end{aligned}$$

  Recall from our general theory that η_g only depends on the coset of g in G/H, so we see that the product of the quotients $\eta_{\sigma_K^i}/\eta_{\sigma_K^{i+1}}$ will cancel out. As for the c term, we know from our computation that this is 1 until $i = f-1$, which gives $\sigma_K(\gamma)$. As such, we collapse down to

$$\boxed{\alpha_0 = \sigma_K^f(\gamma) \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\eta_K\right)}.$$

- For α_i with $i > 0$, our element is given by

$$\begin{aligned}\alpha_i &:= \prod_{p=0}^{n_i-1} u_{L/K}\left(\tau_i^p, \tau_i\right) \\ &= \prod_{p=0}^{n_i-1} \tau_i^p c(\tau_i)\left(\tau_i^{-p} - 1\right) \cdot \frac{\tau_i^p \eta_i \cdot \eta_{\tau_i^p}}{\eta_{\tau_i^{p+1}}}.\end{aligned}$$

Recalling that τ_i has order n_i, our quotient term $\eta_{\tau_i^i}/\eta_{\tau_i^{i+1}}$ will again cancel out. Additionally, the cocycle c always spits out 1 on these inputs, so we are left with

$$\boxed{\alpha_i = \prod_{p=0}^{n_i-1} \tau_i^p\left(\eta_i\right)}.$$

We summarize the results above in the following theorem.

Theorem 10. Fix everything as in the set-up. Then there exists some $\gamma \in ML^\times$ such that $\mathrm{N}_{ML/L}(\gamma) = \pi$ and elements in $\eta_K, \eta_i \in ML^\times$ (for $1 \le i \le t$) such that

$$\frac{\sigma_K^{-f}\left(\eta_K\right)}{\eta_K} = \frac{\sigma_K(\gamma)}{\gamma} \qquad \text{and} \qquad \frac{\sigma_K^{-f}\left(\eta_i\right)}{\eta_i} = \frac{\tau_i(\gamma)}{\gamma}.$$

Then the tuple

$$((\alpha_0, \alpha_i), (\beta_{ij})) := \left(\sigma_K^f(\gamma) \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\eta_K\right), \quad \prod_{p=0}^{n_i-1} \tau_i^p\left(\eta_i\right), \quad \frac{\eta_i}{\tau_j \eta_i} \cdot \frac{\tau_i \eta_j}{\eta_j}\right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$.

We remark that we can replace γ with $\sigma_K^f(\gamma)$ (which still has norm p) while keeping all other variables the same; this gives us the following slightly prettier presentation. Note that we have multiplied the equations for $\eta_\bullet$ by σ_K^f on both sides.

Corollary 11. Fix everything as in the set-up. Then there exists some $\gamma \in ML^\times$ such that $\mathrm{N}_{ML/L}(\gamma) = \pi$ and elements in $\eta_K, \eta_i \in ML^\times$ (for $1 \le i \le t$) such that

$$\frac{\eta_K}{\sigma_K^f\left(\eta_K\right)} = \frac{\sigma_K(\gamma)}{\gamma} \qquad \text{and} \qquad \frac{\eta_i}{\sigma_K^f\left(\eta_i\right)} = \frac{\tau_i(\gamma)}{\gamma}.$$

Then the tuple

$$((\alpha_0, \alpha_i), (\beta_{ij})) := \left(\gamma \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\eta_K\right), \quad \prod_{p=0}^{n_i-1} \tau_i^p\left(\eta_i\right), \quad \frac{\eta_i}{\tau_j \eta_i} \cdot \frac{\tau_i \eta_j}{\eta_j}\right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$.

For brevity later on, we will give a name to these conditions.

Definition 12. Fix an extension L/K. The $\{\sigma_i\}_{i=1}^m$-tuples constructed in Corollary 11 will be called *fundamental tuples.*

We will show shortly that fundamental tuples actually give the entire equivalence class of $\{\sigma_i\}_{i=1}$-tuples associated to the fundamental class.

Remark 13. This result is essentially a stronger version of Dwork's theorem (1958), recorded in Serre's *Local Fields*, chapter XIII, Theorem 2. Namely, Dwork and Serre are interested in computing the reciprocity map, which roughly means we only want access to the αs, but above we are interested in computing the full fundamental class.

3.3 Checks

In this section we run some checks and discuss some consequences of Theorem 10, in the form of Corollary 11. For these results, we recall that we set $L := \mathbb{Q}_p(\zeta_N)$ and $L_1 := \mathbb{Q}_p(\zeta_{p^\nu})$ and $L_2 := \mathbb{Q}_p(\zeta_m)$ so that $\overline{\sigma}_K = \sigma_K|_{L_1}$ generates $\operatorname{Gal}(L/L_1)$ and σ_x generates $\operatorname{Gal}(L/L_2)$.

In the discussion which follows, we will make repeated use of the fact that (using notation of Corollary 11)

$$\sigma_K^f(\eta_K) = \frac{\gamma}{\sigma_K(\gamma)} \cdot \eta_K \qquad \text{and} \qquad \sigma_K^f(\eta_i) = \frac{\gamma}{\tau_i(\gamma)} \cdot \eta_i.$$

And here are our checks; we start by showing that our elements are in the right field.

Lemma 14. Fix a tuple $(\alpha_0, \alpha_i), (\beta_{ij})$ as in Corollary 11. Then the following are true.

(a) $\alpha_0 \in K_{\pi,\nu}^\times$.

(b) $\alpha_i \in L_i^\times$ for each $i \geq 1$.

(c) $\beta_{ij} \in L^\times$ for each $i > j$.

Proof. We run the checks one at a time.

(a) It suffices to show that α_0 is fixed by $\operatorname{Gal}(M/K_{\pi,\nu}) = \langle \sigma_K \rangle$. As such, we simply compute

$$\begin{aligned}
\sigma_K(\alpha_0) &= \sigma_K\left(\gamma \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_K)\right) \\
&= \sigma_K(\gamma) \cdot \prod_{i=0}^{f-1} \sigma_K^{i+1}(\eta_K) \\
&= \sigma_K(\gamma) \cdot \sigma_K^f(\eta_K) \prod_{i=1}^{f-1} \sigma_K^{i+1}(\eta_K) \\
&= \gamma \cdot \eta_K \prod_{i=1}^{f-1} \sigma_K^{i+1}(\eta_K) \\
&= \gamma \cdot \prod_{i=0}^{f-1} \sigma_K^{i+1}(\eta_K) \\
&= \alpha_0.
\end{aligned}$$

(b) It suffices to show that α_i is fixed by $\mathrm{Gal}(M/L_i) = \langle \sigma_K^f, \tau_i \rangle$. On one hand,

$$\begin{aligned}
\sigma_K^f(\alpha_i) &= \sigma_K^f\left(\prod_{p=0}^{n_i-1} \tau_i^p(\eta_i)\right) \\
&= \prod_{p=0}^{n_i-1} \tau_i^p\left(\sigma_K^f \eta_i\right) \\
&= \left(\prod_{p=0}^{n_i-1} \tau_i^p\left(\frac{\gamma}{\tau_i(\gamma)}\right)\right) \cdot \left(\prod_{p=0}^{n_i-1} \tau_i^p(\eta_i)\right) \\
&= \left(\prod_{p=0}^{n_i-1} \frac{\tau_i^p(\gamma)}{\tau_i^{p+1}(\gamma)}\right) \cdot \alpha_i \\
&= \alpha_i,
\end{aligned}$$

where the product telescopes because τ_i has order n_i.
On the other hand,

$$\begin{aligned}
\tau_i(\alpha_i) &= \tau_i\left(\prod_{p=0}^{n_i-1} \tau_i^p(\eta_i)\right) \\
&= \prod_{p=0}^{n_i-1} \tau_i^{p+1}(\eta_i) \\
&= \prod_{p=0}^{n_i-1} \tau_i^p(\eta_i),
\end{aligned}$$

where we have again used the fact that τ_i has order n_i. This last product is α_i, so we are done.

(c) It suffices to show that β_{ij} is fixed by $\mathrm{Gal}(M/L) = \langle \sigma_K^f \rangle$. Applying force, we see

$$\begin{aligned}
\sigma_K^f(\beta_{ij}) &= \sigma_K^f\left(\frac{\eta_i}{\tau_j\eta_i} \cdot \frac{\tau_i\eta_j}{\eta_j}\right) \\
&= \frac{\sigma_K^f\eta_i}{\tau_j\sigma_K^f\eta_i} \cdot \frac{\tau_i\sigma_K^f\eta_j}{\sigma_K^f\eta_j} \\
&= \frac{\eta_i \cdot \gamma/\tau_i\gamma}{\tau_j\eta_i \cdot \tau_j\gamma/\tau_i\tau_j\gamma} \cdot \frac{\eta_j \cdot \tau_i\gamma/\tau_i\tau_j\gamma}{\eta_j \cdot \gamma/\tau_j\gamma} \\
&= \frac{\eta_i}{\tau_j\eta_i} \cdot \frac{\eta_j}{\eta_j} \\
&= \beta_{ij}.
\end{aligned}$$

The above checks complete the proof. ■

Next we show the relations between the αs and βs.

Lemma 15. Fix a tuple $(\alpha_0, \alpha_i), (\beta_{ij})$ as in Corollary 11. Then the following are true.

(a) $\mathrm{N}_{L/L_i}(\beta_{ij}) = \alpha_i/\tau_j\alpha_i$ for $i > j \geq 0$.

(b) $\mathrm{N}_{L/L_0}(\beta_{i0}^{-1}) = \alpha_0/\tau_i\alpha_0$.

(c) $\mathrm{N}_{L/L_j}(\beta_{ij}^{-1}) = \alpha_j/\tau_i\alpha_j$ for $i > j > 0$.

Proof. We go one at a time.

(a) Note $\operatorname{Gal}(L/L_i) = \langle\tau_i\rangle$, so we compute

$$\begin{aligned}
\mathrm{N}_{L/L_i}(\beta_{ij}) &= \prod_{p=0}^{n_i-1} \tau_i^p(\beta_{ij}) \\
&= \prod_{p=0}^{n_i-1} \tau_i^p\left(\frac{\eta_i}{\tau_j\eta_i}\cdot\frac{\tau_i\eta_j}{\eta_j}\right) \\
&= \prod_{p=0}^{n_i-1} \frac{\tau_i^p\eta_i}{\tau_j\tau_i^p\eta_i}\cdot\prod_{p=0}^{n_i-1}\frac{\tau_i^{p+1}\eta_j}{\tau_i^p\eta_j} \\
&= \left(\prod_{p=0}^{n_i-1}\tau_i^p\eta_i\Big/\tau_j\prod_{p=0}^{n_i-1}\tau_i^p\eta_i\right)\cdot\frac{\tau_i^{n_i}\eta_j}{\eta_j},
\end{aligned}$$

which collapses into $\alpha_i/\tau_j\alpha_i$, as needed.

(b) Note $\operatorname{Gal}(L/L_0) = \langle\overline{\sigma}_K\rangle$. In particular, $\overline{\sigma}_K$ has order f, so we can just compute out

$$\begin{aligned}
\mathrm{N}_{L/L_0}(\beta_{i0}) &= \prod_{p=0}^{f-1} \sigma_K^p(\beta_{i0}) \\
&= \prod_{p=0}^{f-1} \sigma_K^p\left(\frac{\eta_i}{\sigma_K\eta_i}\cdot\frac{\tau_i\eta_K}{\eta_K}\right) \\
&= \prod_{p=0}^{f-1}\frac{\sigma_K^p\eta_i}{\sigma_K^{p+1}\eta_i}\cdot\prod_{p=0}^{f-1}\frac{\tau_i\sigma_K^p\eta_K}{\sigma_K^p\eta_K} \\
&= \frac{\eta_i}{\sigma_K^f\eta_i}\cdot\prod_{p=0}^{f-1}\tau_i\sigma_K^p\eta_K\Big/\prod_{p=0}^{f-1}\sigma_K^p\eta_K \\
&= \tau_i\left(\gamma\prod_{p=0}^{f-1}\sigma_K^p\eta_K\right)\Big/\left(\gamma\prod_{p=0}^{f-1}\sigma_K^p\eta_K\right),
\end{aligned}$$

which is what we wanted after taking reciprocals.

(c) This time around, we have $\operatorname{Gal}(L/L_j) = \langle\tau_j\rangle$. As such, we proceed similarly to (a), writing

$$\begin{aligned}
\mathrm{N}_{L/L_j}(\beta_{ij}) &= \prod_{p=0}^{n_j-1} \tau_j^p(\beta_{ij}) \\
&= \prod_{p=0}^{n_j-1} \tau_j^p\left(\frac{\eta_i}{\tau_j\eta_i}\cdot\frac{\tau_i\eta_j}{\eta_j}\right) \\
&= \prod_{p=0}^{n_j-1}\frac{\tau_j^p\eta_i}{\tau_j^{p+1}\eta_i}\cdot\prod_{p=0}^{n_j-1}\frac{\tau_i\tau_j^p\eta_j}{\tau_j^p\eta_j} \\
&= \frac{\eta_i}{\tau_j^{n_j}\eta_i}\cdot\left(\tau_i\prod_{p=0}^{n_j-1}\tau_j^p\eta_j\Big/\prod_{p=0}^{n_j-1}\tau_j^p\eta_j\right),
\end{aligned}$$

which again collapses into $\tau_i\alpha_j/\alpha_j$. Taking reciprocals finishes.

The above checks complete the proof. ■

Lastly, here are the relations between the βs.

Lemma 16. Fix a tuple $(\alpha_0, \alpha_i), (\beta_{ij})$ as in Corollary 11. Then, for $i > j > k$, we have

$$\frac{\tau_j \beta_{ik}}{\beta_{ik}} = \frac{\tau_k \beta_{ij}}{\beta_{ij}} \cdot \frac{\tau_i \beta_{jk}}{\beta_{jk}}.$$

Proof. As usual, we apply force. Note

$$\begin{aligned}
\frac{\tau_k \beta_{ij}}{\beta_{ij}} \cdot \frac{\tau_i \beta_{jk}}{\beta_{jk}} &= \frac{\dfrac{\tau_k \eta_i}{\tau_k \tau_j \eta_i} \cdot \dfrac{\tau_k \tau_i \eta_j}{\tau_k \eta_j}}{\dfrac{\eta_i}{\tau_j \eta_i} \cdot \dfrac{\tau_i \eta_j}{\eta_j}} \cdot \frac{\dfrac{\tau_i \eta_j}{\tau_i \tau_k \eta_j} \cdot \dfrac{\tau_i \tau_j \eta_k}{\tau_i \eta_k}}{\dfrac{\eta_j}{\tau_k \eta_j} \cdot \dfrac{\tau_j \eta_k}{\eta_k}} \\
&= \frac{\tau_k \eta_i}{\tau_k \tau_j \eta_i} \cdot \frac{\tau_k \tau_i \eta_j}{\tau_k \eta_j} \cdot \frac{\tau_j \eta_i}{\eta_i} \cdot \frac{\eta_j}{\tau_i \eta_j} \cdot \frac{\tau_i \eta_j}{\tau_i \tau_k \eta_j} \cdot \frac{\tau_i \tau_j \eta_k}{\tau_i \eta_k} \cdot \frac{\tau_k \eta_j}{\eta_j} \cdot \frac{\eta_k}{\tau_j \eta_k} \\
&= \frac{\tau_k \eta_i}{\tau_k \tau_j \eta_i} \cdot \frac{1}{1} \cdot \frac{\tau_j \eta_i}{\eta_i} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{\tau_i \tau_j \eta_k}{\tau_i \eta_k} \cdot \frac{1}{1} \cdot \frac{\eta_k}{\tau_j \eta_k} \\
&= \frac{\tau_j \eta_i}{\tau_k \tau_j \eta_i} \cdot \frac{\tau_i \tau_j \eta_k}{\tau_j \eta_k} \cdot \frac{\tau_k \eta_i}{\eta_i} \cdot \frac{\eta_k}{\tau_i \eta_k},
\end{aligned}$$

which is what we wanted. ■

3.4 Consequences

Warning 17. The following section does not use the notation of the rest of the article.

With some checks out of the way, here are some actual consequences. To begin, we state Hilbert's Theorem 90.

Lemma 18. Suppose that L/K is a (finite) cyclic extension of fields such that $\Gamma := \operatorname{Gal}(L/K)$ is generated by $\sigma \in \Gamma$. Given some $\alpha \in L^\times$ such that $\operatorname{N}(\alpha) = 1$, there exists $\beta_0 \in L^\times$ such that $\alpha = \beta_0/\sigma\beta_0$. In fact, this β_0 is unique "up to a multiple in $K^\times$" in the sense that

$$\left\{\beta \in L^\times : \alpha = \beta/\sigma\beta\right\} = \left\{x\beta_0 : x \in K^\times\right\}.$$

Proof. That such a β_0 exists follows directly from Hilbert's Theorem 90. For the last sentence, of course any $\beta := x\beta_0 \in L^\times$ with $x \in K^\times$ will have

$$\frac{\beta}{\sigma\beta} = \frac{\beta_0}{\sigma\beta_0} = \alpha.$$

In the other direction, if $\beta \in L^\times$ has $\beta/\sigma\beta = \alpha$, then

$$\sigma(\beta/\beta_0) = (\sigma\beta)/(\sigma\beta_0) = \beta/\beta_0,$$

so $\beta/\beta_0 \in K^\times$ and $\beta = (\beta/\beta_0) \cdot \beta_0$. ■

And here are some quick consequences of this.

Corollary 19. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $\operatorname{N}_{ML/L}(\alpha) = p$. Choosing some $\sigma \in \{\sigma_K, \sigma_x\}$, the elements η_σ satisfying

$$\frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{\sigma(\alpha)}{\alpha}$$

are unique up to a multiple in $L^\times$, in the sense of Lemma 18.

Proof. Note that $\operatorname{Gal}(ML/L) = \langle \sigma_K^f \rangle$ is cyclic generated by σ_K^f and $\mathrm{N}_{ML/L}(\sigma\alpha/\alpha) = p/p = 1$, so we may simply apply Lemma 18 directly to get the result. ■

We might be worried that our choice α is affecting the set of $\eta_{c(\sigma_K)}$ or $\eta_{c(\sigma_x)}$, but in fact they are not, more or less.

Corollary 20. Fix everything as in the set-up, and choose $\sigma \in \{\sigma_K, \sigma_x\}$. Given $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$, define

$$S_\alpha := \left\{ \eta_\sigma \in ML^\times : \frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{\sigma(\alpha)}{\alpha} \right\}.$$

Then the set S_α is "unique up to a multiple in $ML^\times$" in the sense that two $\alpha, \alpha' \in ML^\times$ with $\mathrm{N}_{ML/L}(\alpha) = \mathrm{N}_{ML/L}(\alpha') = p$ have some $\chi \in ML^\times$ such that

$$S_\alpha = \chi \cdot S_{\alpha'} := \{\chi \cdot \eta_\sigma : \eta_\sigma \in S_{\alpha'}\}.$$

Proof. Suppose $\alpha, \alpha' \in ML^\times$ satisfy $\mathrm{N}_{ML/L}(\alpha) = \mathrm{N}_{ML/L}(\alpha') = p$. The key point is that

$$\mathrm{N}_{ML/L}(\alpha/\alpha') = p/p = 1,$$

so Lemma 18 promises us some $\gamma \in ML^\times$ such that $\alpha/\alpha' = \gamma/\sigma_K^f(\gamma)$. As such, we see that

$$\frac{\sigma(\alpha)}{\alpha} = \frac{\sigma(\alpha/\alpha')}{\alpha/\alpha'} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'}.$$

As such, we set $\chi := (\sigma\gamma/\gamma)$.

To finish, we check that $S_\alpha \subseteq \chi \cdot S_{\alpha'}$, and the other inclusion is similar. Well, if $\eta_\sigma \in S_{\alpha'}$, then

$$\frac{\chi\eta_\sigma}{\sigma_K^f(\chi\eta_\sigma)} = \frac{\chi}{\sigma_K^f(\chi)} \cdot \frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{\sigma(\alpha)}{\alpha},$$

so $\chi\eta_\sigma \in S_\alpha$. This finishes. ■

We now return to describing triples.

Corollary 21. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$. Then, for any triple $(\alpha_1', \alpha_2', \beta')$ corresponding to the fundamental class, there exist elements $\eta'_{c(\sigma_K)}, \eta'_{c(\sigma_x)} \in ML^\times$ with

$$\frac{\eta'_{c(\sigma_K)}}{\sigma_K^f\left(\eta'_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{\eta'_{c(\sigma_x)}}{\sigma_K^f\left(\eta'_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha}$$

such that

$$(\alpha_1', \alpha_2', \beta') = \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\eta'_{c(\sigma_K)}\right),\ \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\eta'_{c(\sigma_x)}\right),\ \frac{\sigma_K\left(\eta'_{c(\sigma_x)}\right)}{\eta'_{c(\sigma_x)}} \cdot \frac{\eta'_{c(\sigma_K)}}{\sigma_x\left(\eta'_{c(\sigma_K)}\right)} \right).$$

In other words, all triples corresponding to the fundamental class come from the recipe described in Corollary 11.

Proof. By Corollary 11, we can certainly find some elements $\eta_{c(\sigma_K)}, \eta_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\eta_{c(\sigma_K)}}{\sigma_K^f\left(\eta_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{\eta_{c(\sigma_x)}}{\sigma_K^f\left(\eta_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha},$$

for which

$$(\alpha_1,\alpha_2,\beta) := \left(\alpha\cdot\prod_{i=0}^{f-1}\sigma_K^i\left(\eta_{c(\sigma_K)}\right),\quad \prod_{i=0}^{\varphi(p^\nu)-1}\sigma_x^i\left(\eta_{c(\sigma_x)}\right),\quad \frac{\sigma_K\left(\eta_{c(\sigma_x)}\right)}{\eta_{c(\sigma_x)}}\cdot\frac{\eta_{c(\sigma_K)}}{\sigma_x\left(\eta_{c(\sigma_K)}\right)}\right)$$

corresponds to the fundamental class $u_{L/K}\in H^2(\mathrm{Gal}(L/K),L^\times)$. In particular, $(\alpha_1,\alpha_2,\beta)$ and $(\alpha_1',\alpha_2',\beta')$ both correspond to the same cohomology class and hence in the same equivalence class of triples, so we know that there exist $m_1,m_2\in L^\times$ such that

$$\alpha_1'=\alpha_1\cdot\mathrm{N}_{L/L_1}(m_1),\qquad \alpha_2'=\alpha_2\cdot\mathrm{N}_{L/L_2}(m_2),\qquad \beta'=\beta\cdot\frac{\sigma_K(m_2)}{m_2}\cdot\frac{m_1}{\sigma_x(m_1)}.$$

As such, we set $\eta'_{c(\sigma_K)}:=\eta_{c(\sigma_K)}\cdot m_1$ and $\eta'_{c(\sigma_x)}:=\eta_{c(\sigma_x)}\cdot m_2$, and these can be checked to work. For example, $\eta'_{c(\sigma_K)}$ satisfies

$$\frac{\eta'_{c(\sigma_K)}}{\sigma_K^f\left(\eta'_{c(\sigma_K)}\right)}=\frac{\sigma_K(\alpha)}{\alpha}\qquad\text{and}\qquad\frac{\eta'_{c(\sigma_x)}}{\sigma_K^f\left(\eta'_{c(\sigma_x)}\right)}=\frac{\sigma_x(\alpha)}{\alpha}$$

by Lemma 18. The rest of the checks are similar. ■

Corollary 22. Fix everything as in the set-up, and let $\pi_1\in L_1^\times$ be a uniformizer. If the triple $(\alpha_1,\alpha_2,\beta)$ is a triple corresponding to the fundamental class, then

$$\alpha_1\equiv\pi_1\pmod{\mathrm{N}_{L/L_1}(L^\times)}.$$

Proof by triples. Note that L/L_1 is an unramified extension, so all elements of absolute value 1 are norms, so there is in fact a class of elements containing all uniformizers in $L_1^\times/\mathrm{N}_{L/L_1}(L^\times)$. Further, because α_1 is also only defined up to a multiple in $\mathrm{N}_{L/L_1}(L^\times)$, to show that the classes in $L^\times/\mathrm{N}_{L/L_1}(L^\times)$ coincide, it thus suffices to exhibit a single triple $(\alpha_1,\alpha_2,\beta)$ such that $\alpha_1\in L_1^\times$ is a uniformizer.

This is a matter of force. To begin, we can use Corollary 11 to find some α with $\mathrm{N}_{ML/L}(\alpha)=p$ and $\eta_{c(\sigma_K)},\eta_{c(\sigma_x)}\in ML^\times$ giving the triple $(\alpha_1,\alpha_2,\beta)$ as described. The idea is to force $\eta_{c(\sigma_K)}$ to have valuation zero.

Let v_{ML} be the fixed valuation of ML extending the standard valuation $v_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and let v_L be its restriction to L. Because ML/L is an unramified, the image of v_{ML} and v_L in $\mathbb{Q}$ is the same. In particular, we can find some $m_1\in L_1^\times$ such that

$$v_{ML}\left(\eta_{c(\sigma_K)}\right)=v_L(m_1).$$

Thus, we replace $\eta_{c(\sigma_K)}$ with $\eta_{c(\sigma_K)}/m_1$, and we still satisfy the conditions of Corollary 11 by Lemma 18 while getting $v_{ML}\left(\eta_{c(\sigma_K)}\right)=0$. Now, the corresponding α_1 looks like

$$\alpha_1=\alpha\cdot\prod_{i=0}^{f-1}\sigma_K^i\left(\eta_{c(\sigma_K)}\right).$$

In particular, defining $v_{L_1}:=v_L|_{L_1}$, it follows

$$v_{L_1}(\alpha_1)=v_{ML}(\alpha_1)=v_{ML}(\alpha),$$

However, $\mathrm{N}_{ML/L}(\alpha)=p$ by construction, so we see that

$$[ML:L]v_{ML}(\alpha)=v_{ML}(p)=v_{\mathbb{Q}_p}(p)=1.$$

Explicitly, we see that

$$[ML:L]=[\mathbb{Q}(\zeta_{N'}):\mathbb{Q}(\zeta_m)]=\frac{[\mathbb{Q}(\zeta_{N'}):\mathbb{Q}_p]}{[\mathbb{Q}_p(\zeta_m):\mathbb{Q}_p]}=\frac{n}{f}=\varphi\left(p^\nu\right).$$

However, L_1/K has ramification degree $\varphi(p^\nu)$ (from the maximal totally ramified subextension $\mathbb{Q}_p(\zeta_{p^\nu})$), so its uniformizers are the elements of valuation $1/\varphi(p^\nu)$. Thus, we have computed that α_1 has the correct valuation and hence is a uniformizer. ∎

Proof by the Artin map. We take a moment to say that there is an alternate derivation of Corollary 22 using the Artin map: one can show that, if $u \in Z^2(L/K)$ is a representative of the fundamental class of an abelian extension L/K, then

$$\begin{aligned} \operatorname{Gal}(L/K) &\to K^\times / \operatorname{N}(L^\times) \\ \sigma &\mapsto \prod_{g \in \operatorname{Gal}(L/K)} u(g, \sigma) \end{aligned}$$

is the inverse Artin map. In particular, from our explicit formula for α_1, we see

$$\alpha_1 = \prod_{g \in \operatorname{Gal}(L/L_1)} u(g, \overline{\sigma}_K) = \theta_{L/L_1}^{-1}(\overline{\sigma}_K).$$

However, $\overline{\sigma}_K$ is the Frobenius automorphism of L/L_1 because the extension L_1/K is totally ramified, implying that the residue field of L_1 is the same as $K = \mathbb{Q}_p$. Thus, $\theta_{L/L_1}^{-1}(\overline{\sigma}_K)$ is the class containing the uniformizers of $L_1^\times$. ∎

We close with a sanity check.

Corollary 23. Fix everything as in the set-up, and let T_α denote the set of triples $(\alpha_1, \alpha_2, \beta)$ generated by some element $\alpha \in ML^\times$ with $\operatorname{N}_{ML/L}(\alpha) = p$ via Corollary 11. Then T_α is independent of α.

Proof. The main idea is to use (the proof of) Corollary 20. Fix $\alpha, \alpha' \in ML^\times$ with $\operatorname{N}_{ML/L}(\alpha) = \operatorname{N}_{ML/L}(\alpha') = p$, and we need to show that $T_\alpha = T_{\alpha'}$. By symmetry, it will be enough for $T_\alpha \subseteq T_{\alpha'}$.

Following the proof of Corollary 20, note that $\operatorname{N}_{ML/L}(\alpha/\alpha') = 1$, so we are promised $\gamma \in ML^\times$ such that $\alpha/\alpha' = \gamma/\sigma_K^f(\gamma)$. Then we showed that any $\sigma \in \{\sigma_K, \sigma_x\}$ can set

$$\chi_\sigma \coloneqq \frac{\sigma(\gamma)}{\gamma}$$

to give $S_{\alpha,\sigma} x = \cdot S_{\alpha',\sigma}$, where $S_{\alpha,\sigma}$ is the set of possible η_σ defined in Corollary 20.

We now proceed directly with the proof. Suppose that we have some triple $(\alpha_1, \alpha_2, \beta) \in T_\alpha$, which we know that we can write down as

$$(\alpha_1, \alpha_2, \beta) = \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i \left(\eta_{\sigma_K}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i \left(\eta_{\sigma_x}\right), \quad \frac{\sigma_K\left(\eta_{\sigma_x}\right)}{\eta_{\sigma_x}} \cdot \frac{\eta_{\sigma_K}}{\sigma_x\left(\eta_{\sigma_K}\right)} \right)$$

for some $\eta_{\sigma_K} \in S_{\alpha,\sigma_K}$ and $\eta_{\sigma_x} \in S_{\alpha,\sigma_x}$. We need to show that $(\alpha_1, \alpha_2, \beta) \in T_{\alpha'}$. Well, by Corollary 20, we can set

$$I'_\sigma \coloneqq \eta_\sigma / \chi_\sigma \in S_{\alpha',\sigma}$$

for $\sigma \in \{\sigma_K, \sigma_x\}$. We now compute

$$\begin{aligned}
\alpha_1 &= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K}) \\
&= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\chi_\sigma) \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K'}) \\
&= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\frac{\sigma_K\gamma}{\gamma}\right) \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K'}) \\
&= \alpha \cdot \frac{\sigma_K^f(\gamma)}{\gamma} \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K'}) \\
&= \alpha' \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K'}),
\end{aligned}$$

where the last equality holds by definition of γ. Similarly, we see

$$\begin{aligned}
\alpha_2 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{\sigma_x}) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\chi_{\sigma_x}) \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\frac{\sigma_x(\gamma)}{\gamma}\right) \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}),
\end{aligned}$$

where the product telescopes in the last equality because σ_x has order $\varphi\left(p^\nu\right)$. Lastly, we set

$$\begin{aligned}
\beta &= \frac{\sigma_K\left(\eta_{\sigma_x}\right)}{\eta_{\sigma_x}} \cdot \frac{\eta_{\sigma_K}}{\sigma_x\left(\eta_{\sigma_K}\right)} \\
&= \frac{\sigma_K\left(\chi_{\sigma_x}\right)}{\chi_{\sigma_x}} \cdot \frac{\chi_{\sigma_K}}{\sigma_x\left(\chi_{\sigma_K}\right)} \cdot \frac{\sigma_K\left(\eta'_{\sigma_x}\right)}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x\left(\eta'_{\sigma_K}\right)} \\
&= \frac{\sigma_K\sigma_x\gamma/\sigma_K\gamma}{\sigma_x\gamma/\gamma} \cdot \frac{\sigma_K\gamma/\gamma}{\sigma_x\sigma_K\gamma/\sigma_x\gamma} \cdot \frac{\sigma_K\left(\eta'_{\sigma_x}\right)}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x\left(\eta'_{\sigma_K}\right)} \\
&= \frac{\sigma_K\left(\eta'_{\sigma_x}\right)}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x\left(\eta'_{\sigma_K}\right)}.
\end{aligned}$$

Thus,

$$(\alpha_1, \alpha_2, \beta) = \left(\alpha' \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(\eta'_{\sigma_K}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\eta'_{\sigma_x}\right), \quad \frac{\sigma_K\left(\eta'_{\sigma_x}\right)}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x\left(\eta'_{\sigma_K}\right)}\right) \in T_{\alpha'},$$

which finishes. ■

4 An Explicit Example

In this section, we work through Corollary 11 very explicitly in a basic case. Let p be an odd prime because the following discussion has no content in the case of $p = 2$. Set $K := \mathbb{Q}_p$ and $K_m := \mathbb{Q}_p(\zeta_m)$ with $f := [\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p]$.

The main simplification we will make which allows explicit computation is that we will set $K_{\pi,\nu} := \mathbb{Q}_p(\zeta_p)$. Continuing with the set-up, we see $L = \mathbb{Q}_p(\zeta_p, \zeta_m)$ with $n := (p-1)\cdot f$; as such, set $N' := p^n - 1$ so that $M = \mathbb{Q}_p(\zeta_{N'})$. Here is the diagram of our fields.

$$\begin{tikzcd}
 & & \mathbb{Q}_p(\zeta_p, \zeta_{N'}) \arrow[dl, dash] \arrow[dr, dash] & \\
 & \mathbb{Q}_p(\zeta_p, \zeta_m) \arrow[dl, dash] \arrow[dr, dash] & & \mathbb{Q}_p(\zeta_{N'}) \arrow[dl, dash] \\
\mathbb{Q}_p(\zeta_p) \arrow[dr, dash] & & \mathbb{Q}_p(\zeta_m) \arrow[dl, dash] & \\
 & \mathbb{Q}_p & &
\end{tikzcd}$$

Now, the reason we set $K_{\pi,\nu} = \mathbb{Q}_p(\zeta_p)$ is that we can show that

$$\gamma := (-p)^{1/(p-1)} \in \mathbb{Q}_p(\zeta_p).$$

Indeed, we sneakily set $\pi = -p$ to be our uniformizer of $\mathbb{Q}_p$ so that $\mathrm{N}_{ML/L}(\gamma) = \gamma^{p-1} = -p$. Because it will be helpful for us shortly, we will actually give a construction of $(-p)^{1/(p-1)}$.

Lemma 24. Let p be a prime. Then we can find some $\gamma := (-p)^{1/(p-1)}$ in $\mathbb{Q}_p(\zeta_p)$. In fact, we can take $\gamma \equiv c\pi \pmod{\pi^2}$ for any $c \in \mathbb{F}_p^\times$.

Sketch. We follow Professor Andrew Sutherland's Lemma 20.5. Set $\pi := \zeta_p - 1$ to be a uniformizer of $\mathbb{Q}_p(\zeta_p)$. Now, the minimal polynomial of ζ_p is

$$f(T) := \frac{(T+1)^p - 1}{T},$$

which is p-Eisenstein. To properly apply Hensel's lemma to solve $T^{p-1} + p$, we see that any solution should be divisible by π, so we divide out by this first. Note $v(\pi) = 1/(p-1)$, so $u := -\pi^{p-1}/p \in \mathcal{O}_{\mathbb{Q}_p(\zeta_p)}^\times$. In fact, we can see from the polynomial f that

$$\pi^{p-1} + p \equiv 0 \pmod{p\pi},$$

so $u \equiv -1 \pmod{\pi}$. As such, we now note that $g(T) := T^{p-1} - u$ has

$$g(c) \equiv 0 \pmod{\pi} \qquad \text{and} \qquad g'(c) = (p-1)c \not\equiv 0 \pmod{\pi},$$

for any $c \in \mathbb{F}_p^\times$, so we can lift c to a root $\beta_c \in \mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$. From here, we see $(\pi/\beta_c)^{p-1} = \pi^{p-1}/u = -p$, so π/β_c is our desired root. For the last statement, we see

$$\pi/\beta_c \equiv c^{-1}\pi \pmod{\pi^2},$$

so as $c \in \mathbb{F}_p^\times$ varies, we do indeed get all equivalence classes. ■

In light of Lemma 24, we will just take γ to have $\gamma^{p-1} = -p$ with $\gamma \equiv c\pi \pmod{\pi^2}$. This satisfies $\mathrm{N}_{ML/L}(\gamma) = -p$ as discussed above.

We start with the unramified side because it is easier. Namely, $\gamma \in \mathbb{Q}_p(\zeta_p)$ is fixed by the Frobenius automorphism σ_K, so we may set $\eta_K := 1$ to have

$$\frac{\eta_K}{\sigma_K^f(\eta_K)} = 1 = \frac{\sigma_K(\gamma)}{\gamma}.$$

The corresponding α_0 is thus

$$\boxed{\alpha_0 = \gamma}.$$

We now deal with ramification. Observe $\operatorname{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic, but we must choose a generator nonetheless. Let $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ be a generator, and let $\sigma_x \colon \zeta_p \mapsto \zeta_p^x$ be the corresponding automorphism; namely, $\tau_1 := \sigma_x$. (Notably, this is not the automorphism generated by the Artin map; we will return to this point later.) Here is the corresponding computation.

Lemma 25. Fix everything as above. Then $\zeta_{p-1} := \sigma_x(\gamma)/\gamma$ is a primitive $p-1$st root of unity and in particular lies in $\mathbb{Q}_p$. In fact, $\zeta_{p-1} \equiv x \pmod p$.

Note that we are defining ζ_{p-1} above, which is okay: in the worst case, we might have to adjust the definitions of $\zeta_{N'}$ and ζ_m to correspond with this particular ζ_{p-1}, but otherwise ζ_{p-1} may be any fixed primitive $p-1$st root of unity.

Proof. To see that ζ_{p-1} is a $p-1$st root of unity, we note that $\sigma_x(\gamma) = \zeta_{p-1}\cdot\gamma$, so an induction shows that

$$\sigma_x^k(\gamma) = \zeta_{p-1}^k \cdot \gamma.$$

Setting $k = p-1$ shows that $\zeta_{p-1}^{p-1} = 1$, so ζ_{p-1} is a $p-1$st root of unity. To show that ζ_{p-1} is primitive, we know that $\zeta_{p-1}^k = 1$ above would imply that $\sigma_x^k(\gamma) = \gamma$, but $\mathbb{Q}_p(\gamma) = \mathbb{Q}_p(\zeta_p)$ (we already know $\mathbb{Q}_p(\gamma) \subseteq \mathbb{Q}_p(\zeta_p)$, but both of these extensions have degree $p-1$), so in fact $\sigma_x^k = \mathrm{id}$. So $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ being a generator requires $p-1 \mid k$. So indeed, the least positive integer k with $\zeta_p^k = 1$ is $k = p-1$.

We now quickly note that $\mathbb{Q}_p$ contains all $p-1$st roots of unity by Hensel's lemma because the polynomial $T^{p-1} - 1 \in \mathbb{F}_p[T]$ fully splits into $p-1$ distinct factors; in particular, $\zeta_{p-1} \in \mathbb{Q}_p$. In fact, Hensel's lemma tells us that the $p-1$st roots of unity of $\mathbb{Q}_p$ fully represent $(\mathbb{Z}/p\mathbb{Z})^\times$, so there is a chance for $\zeta_{p-1} \equiv x \pmod p$.

Well, it remains to show $\zeta_{p-1} \equiv x \pmod p$. Let $\pi := \zeta_p - 1$ be a uniformizer of $\mathbb{Q}_p(\zeta_p)$. Because $\zeta_{p-1}, x \in \mathbb{Q}_p$, it is enough for $v_{\mathbb{Q}}(\zeta_{p-1} - x) > 0$; as such, we will show that

$$\zeta_{p-1} \stackrel{?}{\equiv} x \pmod \pi.$$

To see this, recall $\gamma \equiv c\pi \pmod{\pi^2}$, so find $c_1 \in \mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$ with $\gamma = c\pi + c_1\pi^2$. Now, observe $\sigma_x(\pi) = \zeta_p^x - 1$ is another uniformizer and in particular divisible by p, so we may write

$$\sigma_x(\gamma) = \left(\zeta_p^x - 1\right) + c_2\pi^2$$

for some $c_2 \in \mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$. Thus, we note

$$\frac{\zeta_p^x - 1}{\zeta_p - 1} \equiv 1 + \zeta_p + \cdots + \zeta_p^{x-1} \equiv \underbrace{1 + \cdots + 1}_{x} \equiv x \pmod \pi.$$

It follows that

$$\sigma_x(\gamma) \equiv \zeta_p^x - 1 \equiv x(\zeta_p - 1) \equiv x\gamma \pmod{\pi^2},$$

which is enough. ■

We are almost able to compute $\eta_x := \eta_1$. To do this, we pick up a quick lemma.

Lemma 26. Let p and f be integers. Then

$$\frac{p^{f(p-1)} - 1}{(p-1)\left(p^f - 1\right)} \in \mathbb{Z}.$$

Proof. Observe

$$\frac{p^{f(p-1)}-1}{p^f-1}=\sum_{k=0}^{p-1}p^{fk}\equiv\sum_{k=0}^{p-1}1=p-1\equiv 0 \pmod{p-1}.$$

This finishes. ■

In light of the above lemma, we define

$$z:=-\frac{p^{f(p-1)}-1}{(p-1)\left(p^f-1\right)}.$$

Note the sign here; it is very important! It follows that $\eta_x:=\zeta_{N'}^z$ will have

$$\begin{aligned}\frac{\eta_x}{\sigma_K^f(\eta_x)}&=\frac{\zeta_{N'}^z}{\zeta_{N'}^{zp^f}}\\&=\zeta_{N'}^{-z\left(p^f-1\right)}\\&=\zeta_{N'}^{N'/(p-1)}\\&=\zeta_{p-1},\end{aligned}$$

which is indeed $\sigma_x(\gamma)/\gamma$. Notably, we have $\eta_x\in\mathbb{Q}_p(\zeta_{N'})$, which is fixed by σ_x. Thus, the corresponding α_1 is thus

$$\begin{aligned}\alpha_1&=\prod_{i=0}^{p-1}\sigma_x^i(\eta_i)\\&=\eta_i^{p-1}\\&=\zeta_{N'}^{z(p-1)}\\&=\zeta_{N'}^{-N'/\left(p^f-1\right)}\\&\boxed{\alpha_1=\zeta_{p^f-1}^{-1}}.\end{aligned}$$

Lastly, we compute our β_{10} as

$$\begin{aligned}\beta_{10}&=\frac{\eta_K}{\sigma_x\eta_K}\cdot\frac{\sigma_K\eta_x}{\eta_x}\\&=\zeta_{N'}^{z(p-1)}\\&\boxed{\beta_{10}=\zeta_{p^f-1}^{-1}}.\end{aligned}$$

In total, we get the following nice result.

Theorem 27. Let p be an odd prime, and fix $K:=\mathbb{Q}_p$ and $L:=\mathbb{Q}_p(\zeta_p,\zeta_m)$, where $p\nmid m$. Further, set $L_0:=\mathbb{Q}_p(\zeta_p)$ and $L_1:=\mathbb{Q}_p(\zeta_m)$ so that $L=L_0L_1$ and $L_0\cap L_1=K$. Now, pick up the following data.

- Suppose the order of p modulo m is f.
- Let $\sigma_x\colon\zeta_p\mapsto\zeta_p^x$ be a generator of $\operatorname{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$.
- Find $\gamma\in\mathbb{Q}_p(\zeta_p)$ such that $\gamma^{p-1}+p=0$ and $\sigma_x(\gamma)/\gamma=\zeta_{p-1}$. (Equivalently, set $\zeta_{p-1}:=\sigma_x(\gamma)/\gamma$.)

Then the fundamental class $u_{L/K}\in H^2(\operatorname{Gal}(L/K),L^\times)$ is represented by the triple

$$(\alpha_0,\alpha_1,\beta_{10})=\left(\gamma,\zeta_{p^f-1}^{-1},\zeta_{p^f-1}^{-1}\right).$$

Remark 28. We verify Artin reciprocity for $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$. Let $c \in Z^2(\operatorname{Gal}(L/K), L^\times)$ represent the fundamental class. The explicit formula for α_1 tells us that

$$\alpha_1 = \prod_{i=0}^{p-1} c\left(\sigma_x^i, \sigma_x\right) = [\sigma_x] \cup \operatorname{Res} u_{L/\mathbb{Q}_p} = [\sigma_x] \cup u_{L/\mathbb{Q}_p(\zeta_m)} = \theta^{-1}_{L/\mathbb{Q}_p(\zeta_m)}(\sigma_x).$$

Taking norms down to $K^\times$, we see on one hand that

$$\mathrm{N}_{\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p}(\alpha_1) = \prod_{i=0}^{f-1} \zeta_{p^f-1}^{-p^i} = \zeta_{p^f-1}^{-\left(1+p+\cdots+p^{f-1}\right)} = \zeta_{p^f-1}^{-\left(p^f-1\right)/(p-1)} = \zeta_{p-1}^{-1} \equiv x^{-1} \pmod p.$$

On the other hand,

$$\mathrm{N}_{\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p}\, \theta^{-1}_{L/\mathbb{Q}_p(\zeta_m)}(\sigma_x) = \theta^{-1}_{L/\mathbb{Q}_p}(\sigma_x) = \theta^{-1}_{\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p}(\sigma_x).$$

So $\theta^{-1}_{\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p}$ sends $\sigma_x \colon \zeta_p \mapsto \zeta_p^x$ to $x^{-1} \pmod p$, as predicted by Lubin–Tate theory.

5 Towers

In this section, we will use the notions but not the exact notation as in the set-up. Instead, we will build a "tower set-up" below. Our goal is to be able to force some compatibility among the data in the tuples of Corollary 11 in towers. This is particularly simple in the case where we fix some unramified extension and allow our ramification to ascend in a tower.

As such, fix a base field K and unramified extension K_m, and we also fix a tower of totally ramified extensions

$$K \coloneqq K_{\pi,0} \subseteq K_{\pi,1} \subseteq K_{\pi,2} \subseteq \cdots.$$

For example, we might choose Lubin–Tate extensions for this purpose. For brevity, we set

$$K_\pi \coloneqq \bigcup_{i \geq 0} K_{\pi,i}$$

to be the (very large) composite totally ramified extension. Now, for each $i \geq 0$, we define $L_i \coloneqq K_m K_{\pi,i}$ for each and M_i to be the unramified extension of degree $[L_i : K]$ over K; notably, $[K_m : K] \mid [L_i : K]$, so $K_m \subseteq M_i$ for each $i \geq 0$. Here is our diagram.

$$\begin{tikzcd}
 & & \ddots & & \iddots & & \\
 & \ddots & & M_1L_2 \arrow[ul, dash] \arrow[ur, dash] & & \iddots & \\
\ddots & & L_2 \arrow[ul, dash] \arrow[ur, dash] & & M_1L_1 \arrow[ul, dash] \arrow[ur, dash] & & \iddots \\
 & K_{\pi,2} \arrow[ul, dash] \arrow[ur, dash] & & L_1 \arrow[ul, dash] \arrow[ur, dash] & & M_1 \arrow[ul, dash] \arrow[ur, dash] & \\
 & & K_{\pi,1} \arrow[ul, dash] \arrow[ur, dash] & & K_m \arrow[ul, dash] \arrow[ur, dash] & & \\
 & & & K \arrow[ul, dash] \arrow[ur, dash] & & &
\end{tikzcd}$$

Arrows going up and to the left are unramified; arrows going up and to the right are (totally) ramified. Now, we are interested in constructing a "compatible" system of tuples representing fundamental classes for the ascending chain of extensions L_1/K, L_2/K, L_3/K, etc.

For coherence reasons, we will also place a few assumptions on our Galois groups. Namely, we will assume that

$$\mathrm{Gal}(K_\pi/K) = \bigoplus_{i=1}^{m} \overline{\langle \tau_i \rangle}$$

is a direct sum of finitely many procyclic groups. For example, if we are using Lubin–Tate extensions, and we are in characteristic 0, then this is automatic. Additionally, we will assume that our quotients are

$$\mathrm{Gal}(K_{\pi,i}/K) = \bigoplus_{i=1}^{m} \langle \tau_i|_{K_i} \rangle$$

for each $i \geq 0$. This requirement, though strong, is essentially the only way we could hope for compatibility among our tuples—namely, it tells us that each L_i/K has Galois group generated by the same elements (up to restriction) and hence have more or less the same requirements to yield a fundamental tuple. As an example, this requirement is satisfied when $K = \mathbb{Q}_p$ and $K_i = \mathbb{Q}_p(\zeta_{p^i})$; in fact, $m \in \{1, 2\}$ in this case.

The main focus of the construction is to construct compatible γ elements, but the notion of compatibility will in fact extend. As such, we will codify this into the following definition.

Definition 29. Fix everything as above. Then a sequence $\{x_i\}_{i=0}^{\infty}$ of elements $x_i \in M_i L_i$ is *compatible in towers* if and only if

$$\mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}(x_{i+1}) = x_i.$$

This definition is written down sequentially so that verifying its existence is easy.

Lemma 30. Fix a uniformizer $\pi_K \in K$. There is a sequence $\{\gamma_i\}_{i=0}^{\infty}$ of elements compatible in towers such that $\gamma_0 \in M_0 L_0 = K_m$ is $\gamma_0 = \pi_K$.

Proof. This comes down to a norm argument and an induction. Extend a valuation $v_K : K \to \mathbb{Z}$ to all fields above. Suppose we have constructed γ_i such that γ_i is a uniformizer of $M_i L_i$. We claim that we can construct γ_{i+1} to also be a uniformizer of $M_{i+1}L_{i+1}$ and with

$$\mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}(\gamma_{i+1}) = \gamma_i.$$

This claim will finish the proof inductively.

Now, observe that the extension $M_i L_{i+1}/M_i L_i$ is a totally ramified extension, so if we let ϖ denote a uniformizer of $M_i L_{i+1}$, we have

$$v\left(\varpi^{[M_iL_{i+1}:M_iL_i]}\right) = v(\gamma_i). \tag{5.1}$$

Continuing, $M_{i+1}L_{i+1}/M_iL_{i+1}$ is an unramified extension, so in fact ϖ continues to be a uniformizer up in $M_{i+1}L_{i+1}$. As such, we see that it suffices to construct $u \in M_{i+1}L_{i+1}$ such that

$$\mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}(u) = \frac{\gamma_i}{\mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}(\varpi)}.$$

But the right-hand side is a unit because it has valuation 0 from (5.1), so we can construct a unit u for the left-hand side as well because the norm map surjects from units to units in unramified extensions. In total, $\gamma_{i+1} := u\varpi$ is the element we are looking for. ■

However, the definition of compatibility does not actually tell us that each of these γ_i will behave the way that we need them to as required by Corollary 11. The compatibility is also a little unnatural because it only moves one step at a time. To fix both of these issues, we have the following.

Lemma 31. Suppose that the sequence $\{x_i\}_{i=0}^{\infty}$ is compatible in towers. Then for any nonnegative integers $p \geq q$, we have

$$\mathrm{N}_{M_pL_p/M_qL_p}(x_p) = x_q.$$

Proof. This will require us to actually describe the Galois groups involved. Set $\sigma_K \in \operatorname{Gal}(K^{\mathrm{unr}}/K)$ to the Frobenius automorphism on K, but extend σ_K to all K^{ab} by acting trivially on totally ramified extensions. Additionally, for brevity we set

$$f := [K_m : K] \qquad \text{and} \qquad e_i := [K_{\pi,i} : K]$$

for each $i \geq 0$. Now, the extension M_pL_p/M_qL_p is unramified and hence has Galois group generated by its Frobenius element. The Frobenius element of M_qL_p is equal to the Frobenius element of M_q because the extension M_jL_i/M_j is totally ramified, and because M_q/K is unramified, we may compute the Frobenius element of M_q as

$$\sigma_K^{[M_q:K]},$$

where $[M_q : K] = [L_q : K] = [L_q : K_{\pi,q}] \cdot [K_{\pi,q} : K] = [K_m : K] \cdot [K_{\pi,q} : K] = fe_q$. As for the order of $\operatorname{Gal}(M_pL_p/M_qL_p)$, we first compute, for any $i \geq 0$,

$$[M_iL_i : L_i] = \frac{[M_iL_i : K]}{[L_i : K]} = \frac{[M_iL_i : M_i] \cdot [M_i : K]}{[L_i : K]} = [M_iL_i : M_i] = [K_{\pi,i} : K] = e_i,$$

so the degree we want is e_p/e_q. Thus,

$$\mathrm{N}_{M_pL_p/M_qL_p}(x_p) = \prod_{i=0}^{e_p/e_q-1} \sigma_K^{fe_qi}(x_p).$$

Now, we show that this equals x_q by induction on p. When $p = q$, there is nothing to say. Then, supposing we have the equality at p, we write

$$\begin{aligned}
\mathrm{N}_{M_{p+1}L_{p+1}/M_qL_{p+1}}(x_{p+1}) &= \prod_{i=0}^{e_{p+1}/e_q-1} \sigma_K^{fe_qi}(x_{p+1}) \\
&= \prod_{b=0}^{e_p/e_q-1} \prod_{a=0}^{e_{p+1}/e_p-1} \sigma_K^{fe_q(a(e_p/e_q)+b)}(x_{p+1}) \\
&= \prod_{b=0}^{e_p/e_q-1} \sigma_K^{fe_qb}\left(\prod_{a=0}^{e_{p+1}/e_p-1} \sigma_K^{fe_pa}(x_{p+1})\right) \\
&= \prod_{b=0}^{e_p/e_q-1} \sigma_K^{fe_qb}\left(\prod_{a=0}^{e_{p+1}/e_p-1} \sigma_K^{fe_pa}(x_{p+1})\right).
\end{aligned}$$

Doing the same Galois theory, we see $\operatorname{Gal}(M_{p+1}L_{p+1}/M_pL_{p+1})$ is cyclic generated by $\sigma_K^{fe_p}$ of order e_{p+1}/e_p, so the inner term is $\mathrm{N}_{M_{p+1}L_{p+1}/M_pL_{p+1}}(x_{p+1})$, which we know to be x_p. Now, $x_p \in M_pL_p$, so in fact the entire product collapses to

$$\mathrm{N}_{M_{p+1}L_{p+1}/M_qL_{p+1}}(x_{p+1}) = \mathrm{N}_{M_pL_p/M_qL_p}(x_p) = x_q,$$

which is what we wanted. This completes the proof. ■

In particular, our sequence $\{\gamma_i\}_{i=0}^\infty$ compatible in towers with $\gamma_0 = 0$ will have

$$\mathrm{N}_{M_iL_i/L_i}(\gamma_i) = \mathrm{N}_{M_iL_i/M_0L_i}(\gamma_i) = \gamma_0 = \pi_K$$

for each $i \geq 0$, so these $\gamma_i \in M_iL_i$ do in fact satisfy the needed requirement of Corollary 11.

Thus, we have described how to construct our γ terms in the tower, from which the rest of the fundamental tuple follows. However, we do remark that it is possible to choose the η terms to be compatible in towers as well.

Lemma 32. Fix everything as above. Further, fix some $\sigma \in \operatorname{Gal}\left(\bigcup_{i\geq 0} L_i/K\right)$. Then there exists a sequence $\{\eta_i\}_{i=0}^\infty$ compatible in towers such that

$$\frac{\eta_i}{\sigma_K^f(\eta_i)} = \frac{\sigma(\gamma_i)}{\gamma_i} \tag{5.2}$$

for each $i \geq 0$.

Proof. Well, to begin we have $\gamma_0 = \pi_K$, which is fixed by σ, so the right-hand side is 1, meaning that we might as well take $\eta_0 = 1$. We now claim that, given η_i satisfying (5.2) which is a unit, we can construct η_{i+1} with

$$\mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}(\eta_{i+1}) = \eta_i$$

also satisfying (5.2) (for $i+1$) which is a unit. For brevity, set $\mathrm{N} := \mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}$. To begin, we note that η_i is a unit in M_iL_{i+1} as well, so because $M_{i+1}L_{i+1}/M_iL_{i+1}$ is unramified, we may simply guess any $\eta \in M_{i+1}L_{i+1}$ such that

$$\mathrm{N}(\eta) = \eta_i.$$

We now need to correct for (5.2). Well, we start by noting we're pretty close because

$$\begin{aligned} \mathrm{N}\left(\frac{\eta}{\sigma_K^f(\eta)}\Big/\frac{\sigma(\gamma_{i+1})}{\gamma_{i+1}}\right) &= \frac{\mathrm{N}\,\eta}{\sigma_K^f(\mathrm{N}\,\eta)}\Big/\frac{\sigma(\mathrm{N}\,\gamma_{i+1})}{\mathrm{N}\,\gamma_{i+1}} \\ &= \frac{\eta_i}{\sigma_K^f(\eta_i)}\Big/\frac{\sigma(\gamma_i)}{\gamma_i} \\ &= 1. \end{aligned}$$

Now, $M_{i+1}L_{i+1}/M_iL_{i+1}$ is unramified and hence cyclic, and we know that its Galois group is generated by $\sigma_K^{fe_i}$ as computed earlier, so Hilbert's theorem 90 allows us to find some $u \in M_{i+1}L_{i+1}$ such that

$$\frac{\eta}{\sigma_K^f(\eta)}\Big/\frac{\sigma(\gamma_{i+1})}{\gamma_{i+1}} = \frac{u}{\sigma_K^{fe_i}u}.$$

Quickly, note that we may multiply u by any element in M_iL_{i+1} without adjusting the equality. Thus, taking ϖ to be a uniformizer of M_iL_{i+1}, we note that we can divide out u by some number of ϖs to force u to be a unit because the extension $M_{i+1}L_{i+1}/M_iL_{i+1}$ is unramified, making ϖ also a uniformizer of $M_{i+1}L_{i+1}$. This is all to say that we may assume that u is a unit.

Now, we note that

$$\frac{u}{\sigma_K^{fe_i}u} = \prod_{k=0}^{e_i-1}\frac{\sigma_K^{fk}u}{\sigma_K^{f(k+1)}u} = \underbrace{\left(\prod_{k=0}^{e_i-1}\sigma_K^{fk}u\right)}_{v:=}/\sigma_K^f\left(\prod_{k=0}^{e_i-1}\sigma_K^{fk}u\right) = \frac{v}{\sigma_K^f v}.$$

Because u is a unit, v is as well. In total, we see that

$$\frac{\eta}{\sigma_K^f(\eta)}\Big/\frac{\sigma(\gamma_{i+1})}{\gamma_{i+1}} = \frac{u}{\sigma_K^{fe_i}u} = \frac{v}{\sigma_K^f v}$$

now implies that

$$\frac{\eta/v}{\sigma_K^f(\eta/v)} = \frac{\sigma(\gamma_{i+1})}{\gamma_{i+1}}.$$

Thus, we set $\eta_{i+1} := \eta/v$, which we know to be a unit because both η and v are. This completes the inductive step and hence the proof. ∎

As such, we define $\{\eta_{\sigma,i}\}_{i=0}^{\infty}$ for each $\sigma \in \mathrm{Gal}\left(\bigcup_{i\geq 0} L_i/K\right)$ as constructed above, and we know these to be compatible in towers.

To finish our discussion, we note that because the expressions for the α_i and β_{ij} are multiplicative and because norms commute with automorphisms in abelian extensions, choosing the γs and ηs to be compatible in towers will imply that the entire fundamental tuples will be (pointwise) compatible in towers.

As an example, we write this compatibility out for α_0; the rest of the terms are similar. We define

$$\alpha_{0,i} \coloneqq \gamma_i \cdot \prod_{k=0}^{f-1} \sigma_K^k(\eta_{\sigma_K,i})$$

in accordance with Corollary 11. To check that this is compatible in towers, we set $\mathrm{N} \coloneqq \mathrm{N}_{M_{i+1}L_{i+1}/M_iL_{i+1}}$ for some index i and compute

$$\begin{aligned}
\mathrm{N}(\alpha_{0,i+1}) &= \mathrm{N}\left(\gamma_{i+1} \cdot \prod_{k=0}^{f-1} \sigma_K^k(\eta_{\sigma_K,i+1})\right) \\
&= \mathrm{N}\,\gamma_{i+1} \cdot \prod_{k=0}^{f-1} \sigma_K^k(\mathrm{N}\,\eta_{\sigma_K,i+1}) \\
&= \gamma_i \cdot \prod_{k=0}^{f-1} \sigma_K^k(\eta_{\sigma_K,i}) \\
&= \alpha_{0,i},
\end{aligned}$$

which is what we wanted.

6 Global Gerbs

In this section we provide a concrete description of the Kottwitz gerb $\mathcal{E}_2$ associated to the global extension $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$ when p is a prime.

6.1 Set-Up

We quickly recall the construction of $\mathcal{E}_2$. Given a global field K, let V_K denote the set of places of K. We follow [Kot14] and [Tat66].

Fix an extension of global fields L/K with Galois group $G \coloneqq \mathrm{Gal}(L/K)$. For later use, we will also let $G_v \subseteq G$ denote the decomposition group of a place $v \in V_L$. Now, we have the two short exact sequences. To begin, we note that the augmentation map $\mathbb{Z}[V_K] \twoheadrightarrow \mathbb{Z}$ induces the short exact sequence

$$0 \to \mathbb{Z}[V_L]_0 \to \mathbb{Z}[V_L] \to \mathbb{Z} \to 0 \tag{X}$$

where $\mathbb{Z}[V_L]$ is the kernel of $\mathbb{Z}[V_L] \twoheadrightarrow \mathbb{Z}$. We also have the short exact sequence

$$1 \to L^\times \to \mathbb{A}_L^\times \to \mathbb{A}_L^\times/L^\times \to 1 \tag{A}$$

where the inclusion $L^\times \hookrightarrow \mathbb{A}_L^\times$ is diagonal.

Let $\mathbb{D}_2 \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_L], -)$ denote the protorus with character group $\mathbb{Z}[V_L]$. Then $\mathcal{E}_2(L/K)$ is the Galois gerb associated to a particular class $\alpha_2 \in H^2\left(G, \mathbb{D}(\mathbb{A}_L)\right)$. To construct this class, we need the following lemma.

Lemma 33 ([Tat66], p. 714)**.** Let L/K be an extension of global fields with Galois group G, and let V_L and V_K denote the set of places of L and K respectively. Given a place $v \in V_L$, let $G_v \subseteq G$ denote its decomposition group. Then, for any $i \in \mathbb{Z}$,

$$\widehat{H}^i(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_L], M)) \simeq \prod_{u \in V_K} \widehat{H}^i(G_{v(u)}, M),$$

where the product is over places $u \in V_K$ taking a fixed place $v(u) \in V_L$ above u.

Proof. We give the proof for later use. This is essentially a matter of separating our places and then applying Shapiro's lemma. For each $u \in V_K$, let $V_u \subseteq V_L$ denote the set of places in L above u. Then we see

$$\mathbb{Z}[V_L] \simeq \bigoplus_{u \in V_K} \mathbb{Z}[V_u]$$

as G-modules because the G-orbit of a place $v \in V_L$ lying over a place $u \in V_K$ is exactly V_u. Thus, we have the isomorphisms

$$\begin{aligned}
\widehat{H}^i(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_L], M)) &\simeq \widehat{H}^i\left(G, \operatorname{Hom}_{\mathbb{Z}}\left(\bigoplus_{u \in V_L} \mathbb{Z}[V_u], M\right)\right) \\
&\simeq \widehat{H}^i\left(G, \prod_{u \in V_K} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_u], M)\right) \\
&\simeq \prod_{u \in V_K} \widehat{H}^i\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_u], M)\right).
\end{aligned}$$

It remains to show that

$$\widehat{H}^i\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_u], M)\right) \overset{?}{\simeq} \widehat{H}^i(G_{v(u)}, M).$$

Well, for each place $u \in V_K$, find a place $v(u) \in V_L$ above it. As discussed above, V_u is a transitive G-set, and the stabilizer of $v(u)$ is $G_{v(u)}$. Thus, $V_u \simeq G_{v(u)} \backslash G$ as G-sets (note the distinction between left and right G-sets is somewhat irrelevant because $gG_v = G_v g$ for each $g \in G_v$), so $\mathbb{Z}[V_u] \simeq \mathbb{Z}[G_{v(u)} \backslash G]$ as G-modules. Thus, we may write

$$\begin{aligned}
\widehat{H}^i\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_u], M)\right) &\simeq \widehat{H}^i\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_{v(u)} \backslash G], M)\right) \\
&\simeq \widehat{H}^i\left(G, \operatorname{Mor}_{\mathrm{Set}}(G_{v(u)} \backslash G, M)\right) \\
&\simeq \widehat{H}^i\big(G, \operatorname{CoInd}_{G_{v(u)}}^G(M)\big),
\end{aligned}$$

where the last isomorphism is because $\operatorname{Mor}_{\mathrm{Set}}(G_{v(u)} \backslash G, M) \simeq \operatorname{CoInd}_H^G(M)$ by taking $f \colon G_{v(u)} \backslash G \to M$ to the function $g \mapsto gf\left(G_v g^{-1}\right)$. Now, this last cohomology group is isomorphic to $\widehat{H}^i(G_{v(u)}, M)$ by Shapiro's lemma, thus finishing. ■

Thus, to specify $\alpha_2 \in \widehat{H}^2(G, \mathbb{D}_2(\mathbb{A}_L))$, it is enough to specify a set of classes

$$\alpha_2(u) \in \widehat{H}^2\left(G_{v(u)}, \mathbb{A}_L^\times\right)$$

for each $u \in V_K$. To do so, we note that $G_{v(u)} = \operatorname{Gal}(L_{v(u)}/K_u)$, so we use the natural embedding $i_v \colon L_v \hookrightarrow \mathbb{A}_L^\times$ (for $u \in V_L$) to set

$$\alpha_2(u) \coloneqq i_{v(u)}\big(\alpha(L_{v(u)}/K_u)\big),$$

where $\alpha(L_{v(u)}/K_u) \in \widehat{H}^2\left(G_{v(u)}, L_{v(u)}^\times\right)$ is the local fundamental class.

6.2 An Explicit Cocycle

We continue in the context of subsection 6.1, in the case of $K := \mathbb{Q}$ and $L := \mathbb{Q}(\zeta_{p^m})$ when p is odd; for brevity, set $\zeta := \zeta_{p^m}$. The goal of the computation is to fully reverse Lemma 33 to be able to write down a 2-cocycle in $Z^2(G, \mathbb{D}_2(\mathbb{A}_L))$ representing α_2, which will then specify a gerb in the correct equivalence class of $\mathcal{E}_2$.

As such, we start from the bottom and work our way upwards. To begin, we need to write down $G := \operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ in some concrete way, so we pick a generator $x \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ (recall that p is odd) so that $\sigma\colon \zeta \mapsto \zeta^x$ is a generator of G of order $n := \varphi(p^m) = (p-1)p^{m-1}$. To be able to properly localize, for each prime $q \neq p$, we define $k_q \geq 0$ to have

$$x^{k_q} \equiv q \pmod{p}$$

so that $\sigma^{k_q}\colon \zeta \mapsto \zeta^q$. We also set $d_q := \gcd(k_q, n)$ so that $\langle \sigma^{k_q} \rangle = \langle \sigma^{d_q} \rangle$ with order n/d_q.

6.2.1 Defining Local Fundamental Classes

To work up Lemma 33, we must find explicit 2-cocycles to represent the various $\alpha(L_{v(u)}/K_u)$s. It will be helpful to be able to change between generators freely, so we pick up the following lemma.

Lemma 34. Let $G = \langle \sigma \rangle$ be a finite cyclic group of order n. Further, suppose $k \in \mathbb{Z}$ has $\gcd(k, n) = 1$. Then define $\chi, \chi_d \in Z^2(G, \mathbb{Z})$ by

$$\chi\left(\sigma^i, \sigma^j\right) := \left\lfloor \frac{i+j}{n} \right\rfloor \qquad \text{and} \qquad \chi_k\left(\sigma^{ki}, \sigma^{kj}\right) := \left\lfloor \frac{i+j}{n} \right\rfloor,$$

where $0 \leq i, j < n$. Then $[\chi] = k[\chi_k]$ in $H^2(G, \mathbb{Z})$.

Citation?

Proof. It is well-known that

$$(- \cup [\chi_k])\colon \widehat{H}^0(G, \mathbb{Z}) \to \widehat{H}^2(G, \mathbb{Z}) \tag{6.1}$$

is an isomorphism. Now, for $m \in \mathbb{Z}$, we see that $[m] \cup [\chi_k] = [m\chi_k]$, so we see that we can actually invert the above isomorphism explicitly because

$$\sum_{g \in G} (m\chi_k)\left(g, \sigma^k\right) = m \sum_{\ell=0}^{n-1} \chi_k\left(\sigma^{\ell k}, \sigma^k\right) = m,$$

so $[c] \mapsto \left[\sum_{g \in G} c(g, \sigma^k)\right]$ describes the inverse of (6.1). As such, we pick up χ and compute

$$\sum_{g \in G} \chi\left(g, \sigma^k\right) = \sum_{\ell=0}^{n-1} \chi\left(\sigma^\ell, \sigma^k\right) = k.$$

Thus, $[k] \cup [\chi_k] = [\chi]$, which is what we wanted. ∎

With that out of the way, here are our local fundamental classes.

- Now, for a finite place $u := q \neq p$, we note that q is unramified, so $v(q) \in V_L$ has decomposition group $G_{v(q)}$ cyclic generated by the Frobenius automorphism $\sigma^{k_q}\colon \zeta \mapsto \zeta^{k_q}$. As such, the local fundamental class here is represented by

  $$\left(\sigma^{k_q i}, \sigma^{k_q j}\right) \mapsto q^{\lfloor (i+j)/n \rfloor}.$$

  In particular, if we set $\chi_{k_q} \in Z^2(G, \mathbb{Z})$ by $\chi_{k_q}\colon \left(\sigma^{k_q i}, \sigma^{k_q j}\right) \mapsto \left\lfloor \frac{i+j}{n} \right\rfloor$, we see that $\alpha(L_{v(u)}/K_u) = [q] \cup [\chi_{k_q}]$, where $[q] \in \widehat{H}^0(G, L_{v(u)}^\times)$.

  It will be beneficial, psychologically speaking, to change generators from σ^{k_q} to σ^{d_q}. As such, we set $\chi_{d_q} \in Z^2(G, \mathbb{Z})$ by $\chi_{d_q}\colon \left(\sigma^{d_q i}, \sigma^{d_q j}\right) \mapsto \left\lfloor \frac{i+j}{n} \right\rfloor$. Then Lemma 34 tells us that

  $$[\chi_{d_q}] = (k_q/d_q)[\chi_{k_q}].$$

Thus, we find $y_q \in \mathbb{Z}$ with $y_q \cdot k_q/d_q \equiv 1 \pmod{n_q}$ so that we can represent $\alpha(L_{v(u)}/K_u)$ by

$$(q \cup y_q\chi_{d_q})\colon\ \left(\sigma^{d_q i}, \sigma^{d_q j}\right) \mapsto q^{y_q\lfloor (i+j)/n\rfloor}.$$

For brevity, let this 2-cocycle be $c_q \in Z^2\big(G_{v(u)}, L_{v(u)}^\times\big)$.

- For the finite place $u := q = p$, we note that $L_{v(p)}/K_p$ is totally ramified. Using Lubin–Tate theory and the fact that the local fundamental class is uniquely determined by the local Artin reciprocity map for cyclic extensions, we can just directly compute that

$$\left(\sigma^i, \sigma^j\right) \mapsto x^{\lfloor (i+j)/n\rfloor}$$

represents $\alpha\left(L_{v(p)}/K_p\right)$. Let this 2-cocycle be $c_p \in Z^2\big(G, L_{v(p)}^\times\big)$.

- Lastly, for the infinite place $u := \infty$, set $v(\infty)$ to be a complex place L. Then $G_v := \mathrm{Gal}(L_{v(u)}/K_u) = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic generated by $\sigma^{n/2}$ of order 2. As such, the 2-cocycle

$$\left(\sigma^{in/2}, \sigma^{jn/2}\right) \mapsto (-1)^{\lfloor (i+j)/2\rfloor}$$

represents $\alpha(L_{v(u)}/K_u)$. Let this 2-cocycle be $c_\infty \in Z^2\big(G_{v(u)}, L_{v(\infty)}^\times\big)$.

In order to talk about our 2-cocycles in a unified way, we define

$$\omega_u := \begin{cases} q^{y_q} & u = q \neq p \text{ is finite,} \\ x & u = p, \\ -1 & u = \infty, \end{cases} \qquad \text{and} \qquad d_u := \begin{cases} d_q & u = q \neq p \text{ is finite,} \\ 1 & u = p, \\ n/2 & u = \infty, \end{cases}$$

and $n_u := n/d_u$ for each $u \in V_K$. Thus, we see that $c_u \in Z^2\big(G_{v(u)}, L_{v(u)}^\times\big)$ is defined by

$$c_u\left(\sigma^{d_u i}, \sigma^{d_u j}\right) = \omega_u^{\lfloor (i+j)/n_u\rfloor}$$

for any place $u \in V_K$. Thus, we see that $\alpha_2(u)$ is now represented by

$$(i_{v(u)}c_u)\left(\sigma^{d_u i}, \sigma^{d_u j}\right) = (i_{v(u)}\omega_u)^{\lfloor (i+j)/n_u\rfloor},$$

where $i_{v(u)}\colon L_{v(u)} \hookrightarrow \mathbb{A}_L$ is the canonical embedding. We quickly observe that our construction of c_u has the remarkable properties that $d_u \mid n$ and $\omega_u \in K$ for each place $u \in V_K$.

6.2.2 Inverting Shapiro's Lemma

The next step in reversing Lemma 33 is to invert the Shapiro's lemma isomorphism

$$\widehat{H}^2\left(G_{v(u)}, \mathbb{A}_L^\times\right) \simeq \widehat{H}^2\big(G, \mathrm{CoInd}_{G_{v(u)}}^G(\mathbb{A}_L^\times)\big)$$

for each place $u \in V_K$. For the purposes of this section, we will fix the place $u \in V_K$ and set $v := v(u) \in V_L$ and $H := G_v = G_{v(u)}$ for brevity. It is known that (e.g., see [Kal18]) this inverse morphism can be constructed as the composite

$$\widehat{H}^2\left(H, \mathbb{A}_L^\times\right) \overset{\iota}{\to} \widehat{H}^2\big(H, \mathrm{CoInd}_H^G \mathbb{A}_L^\times\big) \overset{\mathrm{cor}}{\to} \widehat{H}^2\big(G, \mathrm{CoInd}_H^G \mathbb{A}_L^\times\big),$$

where $\iota\colon \mathbb{A}_L^\times \to \mathrm{CoInd}_H^G \mathbb{A}_L^\times$ takes a to $\iota(a)\colon g \mapsto \big(g1_{g\in H}\big)a$.

Thus, we have two maps to track on the level of our 2-cocycles. To track ι, we start by computing

$$(\iota i_v c_u)\left(\sigma^{d_u i}, \sigma^{d_u j}\right) : \sigma^c \mapsto (\sigma^c \iota_v \omega_u)^{1_{d_u\mid c}\lfloor (i+j)/n_u\rfloor}.$$

Because $\omega_u \in K$, it is fixed by σ^c, so in fact this is just

$$(\iota i_v c_u)\left(\sigma^{d_u i}, \sigma^{d_u j}\right) : \sigma^c \mapsto \omega_u^{1_{d_u\mid c}\lfloor (i+j)/n_u\rfloor}.$$

Next we must track through cor. This is more difficult; we follow [NSW08]. We start by noting that we can write the inhomogeneous 2-cocycle as the homogeneous 2-cocycle

$$\left(1, \sigma^{d_u i}, \sigma^{d_u(i+j)}\right) : \sigma^c \mapsto \iota_v \omega_u^{1_{d_u \mid c} \lfloor (i+j)/n_u \rfloor}.$$

Denote this homogeneous 2-cocycle by $\iota i_v \widetilde{c}_u$. To set up our evaluation of cor, we note that we can represent any coset in $H\backslash G$ by some $\sigma^c H$ with $0 \le c < d_u$, and this c is unique; here we are using the fact that $d_u \mid n$. So for any coset $Hg \in H\backslash G$, we let $\overline{Hg}$ denote the correct σ^c. With this notation, we may compute

$$\begin{aligned}
\left(\operatorname{cor} \iota i_v \widetilde{c}_u\right)\left(\sigma^i, \sigma^j\right) &= \sum_{Hg \in H\backslash G} (\overline{H}g)^{-1} \cdot \left(\iota i_v \widetilde{c}_u\right)\left(1, \overline{Hg}\sigma^i \overline{Hg\sigma^i}^{-1}, \overline{Hg}\sigma^{i+j}\overline{Hg\sigma^{i+j}}^{-1}\right) \\
&= \sum_{\ell=0}^{d_q-1} \sigma^{-\ell} \cdot \left(\iota i_v \widetilde{c}_u\right)\left(1, \sigma^{i+\ell-[i+\ell]_{d_q}}, \sigma^{i+j+\ell-[i+j+\ell]_{d_q}}\right).
\end{aligned}$$

Now, the G-action on $\operatorname{CoInd}_H^G \mathbb{A}_L^\times$ takes $f\colon G \to \mathbb{A}_L^\times$ to $(gf)\colon x \mapsto f(xg)$. So when we plug in $\sigma^c \in G$, we get

$$\left(\operatorname{cor} \iota i_v \widetilde{c}_u\right)\left(\sigma^i, \sigma^j\right)\left(\sigma^c\right) = \prod_{\ell=0}^{d_q-1} \left(\iota i_v \widetilde{c}_u\right)\left(1, \sigma^{i+\ell-[i+\ell]_{d_q}}, \sigma^{i+j+\ell-[i+j+\ell]_{d_q}}\right)\left(\sigma^{c-\ell}\right).$$

The only opportunity for a factor in the product to not output 1 is when d_q divides $c-\ell$, so we in fact only care about the term $\ell = [c]_{d_q}$, leaving us with

$$\left(\operatorname{cor} \iota i_v \widetilde{c}_u\right)\left(\sigma^i, \sigma^j\right)\left(\sigma^c\right) = \left(\iota i_v \widetilde{c}_u\right)\left(1, \sigma^{i+[c]_{d_q}-[i+c]_{d_q}}, \sigma^{i+j+[c]_{d_q}-[i+j+c]_{d_q}}\right)\left(\sigma^{c-[c]_{d_q}}\right).$$

Now, transitioning back to an inhomogeneous 2-cocycle, we have

$$\left(\operatorname{cor} \iota i_v \widetilde{c}_u\right)\left(\sigma^i, \sigma^j\right)\left(\sigma^c\right) = \left(\iota i_v c_u\right)\left(\sigma^{i+[c]_{d_q}-[i+c]_{d_q}}, \sigma^{j-[i+j+c]_{d_q}+[i+c]_{d_q}}\right)\left(\sigma^{c-[c]_{d_q}}\right).$$

We can simplify this some, but not much. Observe $i+[c]_{d_q}-[i+c]_{d_q} = d_q\left\lfloor \frac{i+[c]_{d_q}}{d_q}\right\rfloor$ and $i+j+[c]_{d_q}-[i+j+c]_{d_q} = d_q\left\lfloor \frac{i+j+[c]_{d_q}}{d_q}\right\rfloor$, so we have

$$\left(\operatorname{cor} \iota i_v \widetilde{c}_u\right)\left(\sigma^i, \sigma^j\right)\left(\sigma^c\right) = \iota_v \omega_u^{\left\lfloor \frac{\left[\left\lfloor \frac{i+[c]_{d_q}}{d_q}\right\rfloor\right]_n + \left[\left\lfloor \frac{i+j+[c]_{d_q}}{d_q}\right\rfloor - \left\lfloor \frac{i+[c]_{d_q}}{d_q}\right\rfloor\right]_n}{n}\right\rfloor}$$

as our 2-cocycle in $Z^2\left(G, \operatorname{CoInd}_H^G \mathbb{A}_L^\times\right)$.

6.2.3 Finishing Up

We are ready to finish tracking upwards through Lemma 33. Our next step is to go from $Z^2\left(G, \operatorname{CoInd}_{G_v}^G \mathbb{A}_L^\times\right)$ up to $Z^2\left(G, \operatorname{Mor}_{Set}(G_{v(u)}\backslash G, \mathbb{A}_L^\times)\right)$, for which we note that $f \in \operatorname{CoInd}_{G_v}^G \mathbb{A}_L^\times$ should be sent to $G_{v(u)}g \mapsto gf\left(g^{-1}\right)$. (This is well-defined because $f(hg) = hf(g)$ for $h \in G_{v(u)}$ here.) This gives the 2-cocycle

$$\left(\sigma^i, \sigma^j\right) \mapsto G_{v(u)}\sigma^c \mapsto \iota_v \omega_u^{\left\lfloor \frac{\left[\left\lfloor \frac{i+[-c]_{d_q}}{d_q}\right\rfloor\right]_n + \left[\left\lfloor \frac{i+j+[-c]_{d_q}}{d_q}\right\rfloor - \left\lfloor \frac{i+[-c]_{d_q}}{d_q}\right\rfloor\right]_n}{n}\right\rfloor},$$

where we are now assuming $0 \le c < d_q$ without loss of generality. The above immediately extends to a 2-cocycle in $Z^2\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_{v(u)}\backslash G], \mathbb{A}_L^\times)\right)$, which then turns into the 2-cocycle

$$\left(\sigma^i, \sigma^j\right) \mapsto \sigma^c v(u) \mapsto \iota_v \omega_u^{\left\lfloor \frac{\left[\left\lfloor \frac{i+c}{d_q}\right\rfloor\right]_n + \left[\left\lfloor \frac{i+j+c}{d_q}\right\rfloor - \left\lfloor \frac{i+c}{d_q}\right\rfloor\right]_n}{n}\right\rfloor}$$

in $Z^2\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_u], \mathbb{A}_L^\times)\right)$. Lastly, we note that letting u vary in the above expression immediately pushes the 2-cocycle to $Z^2\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[V_L], \mathbb{A}_L^\times)\right)$, which is exactly the representative of α_2 we have been looking for.

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