Encoding Cohomology and Classifying Extensions

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Abstract

We use group cohomology to provide some general theory to classify all group extensions of a G-module A in the case of an abelian group G. The main idea is to use a group presentation of G provide a group presentation of the extension using specially chosen elements of A. It turns out that this "encoding" of the extension into elements of A enjoys a number of homological niceties.

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1 Generalized Periodic Cohomology

The goal of this section is to separate out what we can, a priori, expect from "encoding" modules from what is a special property of the specific encoding module we study in the rest of the paper. As such, we should begin by motivating encoding modules.

Throughout this section, let G be a finite group. When $G = \langle \sigma \rangle$ is a cyclic group of order n, it is an amazing feature that there is some $\chi \in \widehat{H}^2(G, \mathbb{Z})$ granting isomorphisms

$$(\chi \cup -) \colon \widehat{H}^0(G, M) \to \widehat{H}^2(G, M) \tag{1.1}$$

for any G-module M. In fact, it is not too hard to write down χ as being represented by the "carrying" 2-cocycle

$$\left(\sigma^i, \sigma^j\right) \mapsto \left\lfloor \frac{i+j}{n} \right\rfloor,$$

so (1.1) is telling us that we can represent each cohomology class of $\widehat{H}^2(G, M)$ by a 2-cocycle of the form

$$\left(\sigma^i, \sigma^j\right) \mapsto \left\lfloor \frac{i+j}{n} \right\rfloor \alpha$$

for some $\alpha \in M^G$. This "classification" of 2-cocycles in $\widehat{H}^2(G, M)$ is incredibly useful and makes cyclic groups very easy to work with computationally.

From one perspective, this classification of 2-cocycles for cyclic groups says that we can retrieve all 2-cocycles by keeping track of the single element $\alpha \in M^G = H^0(G, M)$, modulo some equivalence relation coming from Tate cohomology. The algebraic way to choose a single element of M^G is by elements in

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, M^G\right) = \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

As such, one can phrase (1.1) as providing a natural isomorphism

$$\widehat{H}^0(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)) \Rightarrow \widehat{H}^2(G, -).$$

Here, the choice of G-module $\mathbb{Z}$ in some sense "encodes" 2-cocycles from $\widehat{H}^2(G, M)$ into a single morphism from $\mathbb{Z}$ to M, modulo some equivalence relations.

More generally, permit G to be non-cyclic, and suppose we have a G-module $\mathbb{Z}[G]^m/I$ for some $m \geq 0$ and G-submodule I with isomorphisms

$$\widehat{H}^0\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m/I, M)\right) \to \widehat{H}^2(G, M),$$

for any G-module M. In this case, we see we are still encoding 2-cocycles into morphisms, where these morphisms look like

$$\widehat{H}^0\left(G, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m/I, M)\right) = \frac{\operatorname{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[G]^m/I, M\right)}{N_G \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m/I, M)}.$$

To see the encoding here, view the numerator as choosing out an m-tuple of elements of M in the same way that we would choose morphisms $\operatorname{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[G]^m, M\right)$, but we can't just choose any m elements because they must saitsfy some relations dictated by I. Then the denominator provides an equivalence relation of the m-tuples which determines if two tuples live in the same "class."

The above discussion is intended to motivate the following definition.

Definition 1. Let G be a finite group and $r \in \mathbb{Z}$ be an index. Then a G-module X is an *r-encoding G-module* if and only if there is a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^i(G, \mathrm{Hom}_\mathbb{Z}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -)$$

for some $i \in \mathbb{Z}$.

We will abbreviate the G from "r-encoding G-module" whenever confusion is unlikely to arise.

It will turn out that the index $i \in \mathbb{Z}$ is more or less irrelevant. Indeed, we will be able to show the following equivalences.

Theorem 2. Let G be a finite group. Given a G-module X and index $r \in \mathbb{Z}$, the following are equivalent.

(a) X is an r-encoding module.

(b) If $r \geq 0$, then X is cohomologically equivalent to $I_G^{\otimes r}$; if $r < 0$, then X is cohomologically equivalent to $\mathrm{Hom}_\mathbb{Z}(I_G^{\otimes r}, \mathbb{Z})$.

(c) There is $x \in \widehat{H}^r(G, X)$ granting a natural isomorphism

$$(x \cup -) \colon \widehat{H}^i(G, \mathrm{Hom}_\mathbb{Z}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -)$$

for any $i \in \mathbb{Z}$.

(d) There are $x \in \widehat{H}^r(G, X)$ and $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_\mathbb{Z}(X, \mathbb{Z}))$ such that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_\mathbb{Z}(X, X)).$$

(e) We can find $x \in \widehat{H}^r(G, X)$ yielding natural isomorphisms

$$(- \cup x) \colon \widehat{H}^i(G, -) \Rightarrow \widehat{H}^{i+r}(G, - \otimes_\mathbb{Z} X)$$

for all $i \in \mathbb{Z}$.

If X is also $\mathbb{Z}$-free, these are equivalent to

(f) $\widehat{H}^r(G, X) \cong \mathbb{Z}/\#G\mathbb{Z}$ and $\widehat{H}^0(G, \mathrm{Hom}_\mathbb{Z}(X, X))$ is cyclic.

Proof. The equivalence of (a) and (b) follow from combining Proposition 31 with Example 32 and Example 63. The equivalence of (c) follows from Corollary 35. Continuing, the equivalence of (d) follows from Proposition 44. The equivalence of (e) follows from Theorem 50 and the discussion in Remark 51. Lastly, the equivalence of (f) follows from Proposition 55. ∎

When we may take $X = \mathbb{Z}$ (e.g., when G is cyclic), we are essentially studying groups with periodic cohomology, so many results in this section will mimic these results. However, periodic cohomology requires somewhat stringent conditions on the group itself, and allowing this "free parameter" X will permit general groups at the cost of a perhaps more complex X. For example, when $r \geq 0$, we can take $X = I_G^{\otimes r}$ for any finite group G, though this G-module is quite rough to handle.

In general, it can be an interesting question what specified abelian groups X can be turned into encoding modules or dually what the encoding modules for a given group G look like. Many of the results in this section are motivated by a desire to provide partial answers or intuition towards answers to these questions.

1.1 Shiftable Functors

The main point of this section is to set up some theory around what we call shiftable functors, whose main application will be in the proofs of Corollary 14 and Corollary 15.

Definition 3. Let G be a finite group. Then a functor $F\colon \mathrm{Mod}_G \to \mathrm{Mod}_G$ is a *shiftable functor* if and only if F is both additive and sends induced modules to induced modules.

The main point to shiftable functors F is that the dimension-shifting short exact sequences

$$\begin{aligned} 0 &\to I_G \otimes_{\mathbb{Z}} A \to \quad \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \quad \to \quad A \quad \to 0 \\ 0 &\to \quad A \quad \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to \mathrm{Hom}_{\mathbb{Z}}(I_G, A) \to 0 \end{aligned}$$

will remain exact upon applying F (because F is additive, and these short exact sequences are $\mathbb{Z}$-split), and the middle term will remain induced.

Here are our key examples of shiftable functors.

Lemma 4. Let G be a finite group and X a G-module. Then $\mathrm{Hom}_{\mathbb{Z}}(-, X)$ is a (contravariant) shiftable functor.

Proof. We already know that $\mathrm{Hom}_{\mathbb{Z}}(-, X)$ is additive, so the main check is that we send induced modules to induced modules. Well, without loss of generality, let $M \coloneqq \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ be our induced module. Then the tensor–hom adjunction gives

$$\mathrm{Hom}_{\mathbb{Z}}(M, X) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes_{\mathbb{Z}} A, X) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(A, X)),$$

which is also a G-module isomorphism. This finishes. ■

Lemma 5. Let G be a finite group and X a G-module. Then $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ is a shiftable functor.

Proof. It is known that $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ is an additive functor, so we just need to check that it sends induced modules to induced modules. Well, pick up some induced module $M \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ for some G-module A. Then we see

$$\mathrm{Hom}_{\mathbb{Z}}(X, M) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \simeq \mathrm{Hom}_{\mathbb{Z}}(X \otimes_{\mathbb{Z}} \mathbb{Z}[G], A),$$

which is induced by noting $X \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is induced and using Lemma 4. ■

Lemma 6. Let G be a finite group and X a G-module. Then $X \otimes_{\mathbb{Z}} -$ is a shiftable functor.

Proof. Again, $X \otimes_{\mathbb{Z}} -$ is additive, so we just need to check that it sends induced modules to induced modules. Well, suppose $M \coloneqq \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ is an induced module. Then we note the isomorphisms

$$X \otimes_{\mathbb{Z}} M = X \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} (X \otimes_{\mathbb{Z}} A)$$

are all also isomorphisms of G-modules. Because $\mathbb{Z}[G] \otimes_{\mathbb{Z}} (X \otimes_{\mathbb{Z}} A)$ is induced, we are done. ■

Of course, we can create some crazier examples of shiftable functors by melding them together.

Lemma 7. Let G be a finite group. If F and F' are shiftable functors, then $F \circ F'$ is a shiftable functor.

Proof. This follows directly from the definition. ■

Example 8. The functor

$$A \mapsto \mathrm{Hom}_{\mathbb{Z}}\left(A \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(I_G, I_G \otimes_{\mathbb{Z}} A), I_G\right)$$

is a shiftable functor.

1.2 Shifting by Cup Products

A key property of shiftable functors is how we will be able to relate them to each other via cup products. With this in mind, we have the following definition.

Definition 9. Let G be a finite group. Then we define a *shifting pair* (F, F', X, η) to be a pair of shiftable functors F and F' equipped with a natural transformation

$$\eta_\bullet \colon X \otimes_{\mathbb{Z}} F \Rightarrow F'.$$

The following will be our key example.

Example 10. Given G-modules X and X', there is a canonical composition map

$$\begin{array}{ccccc} \eta_\bullet \colon \mathrm{Hom}_{\mathbb{Z}}(X', X) & \otimes_{\mathbb{Z}} & \mathrm{Hom}_{\mathbb{Z}}(X, -) & \Rightarrow & \mathrm{Hom}_{\mathbb{Z}}(X', -) \\ \varphi & \otimes & f & \mapsto & f \circ \varphi \end{array}$$

so $(\mathrm{Hom}_{\mathbb{Z}}(X, -), \mathrm{Hom}_{\mathbb{Z}}(X', -), \mathrm{Hom}_{\mathbb{Z}}(X', X), \eta_\bullet)$ is a shifting pair.

In particular, cup products assemble into natural transformations.

Lemma 11. Let G be a finite group, and let (F, F', X, η) be a shifting pair. Then, given indices $r, s \in \mathbb{Z}$ and $c \in \widehat{H}^r(G, X)$, the cup-product maps

$$(c \cup -) \colon \widehat{H}^s(G, F-) \Rightarrow \widehat{H}^{r+s}(G, F'-)$$

make a natural transformation of cohomology functors.

Proof. Given a G-module A, we note that our cup-product map is defined by

$$\widehat{H}^s(G, FA) \xrightarrow{c \cup -} \widehat{H}^{r+s}(G, X \otimes_{\mathbb{Z}} FA) \xrightarrow{\eta_A} \widehat{H}^{r+s}(G, F'A).$$

So, to check naturality, we pick up a G-module homomorphism $\varphi \colon A \to B$ and draw the following diagram.

$$\begin{array}{ccccc} \widehat{H}^s(G, FA) & \xrightarrow{c \cup -} & \widehat{H}^{r+s}(G, X \otimes_{\mathbb{Z}} FA) & \xrightarrow{\eta_A} & \widehat{H}^{r+s}(G, F'A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \widehat{H}^s(G, FB) & \xrightarrow{c \cup -} & \widehat{H}^{r+s}(G, X \otimes_{\mathbb{Z}} FB) & \xrightarrow{\eta_B} & \widehat{H}^{r+s}(G, F'B) \end{array}$$

The left square commutes by functoriality of cup products (see [Neu13], Proposition I.5.3), and the right square commutes by the naturality of η and functoriality of $\widehat{H}^{r+s}(G, -)$. ∎

It will turn out occasionally that we have multiple evaluation maps flying around, so we pick up the following lemma for reassurance.

Lemma 12. Let G be a finite group, and let A, B, C be G-modules equipped with maps

$$\begin{aligned}\varphi_{AB} &\colon A \otimes_{\mathbb{Z}} B \to X \\ \varphi_{XC} &\colon X \otimes_{\mathbb{Z}} C \to Z \\ \varphi_{BC} &\colon B \otimes_{\mathbb{Z}} C \to Y \\ \varphi_{AY} &\colon A \otimes_{\mathbb{Z}} Y \to Z\end{aligned}$$

making the diagram

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}} B \otimes_{\mathbb{Z}} C & \xrightarrow{\varphi_{AB}} & X \otimes_{\mathbb{Z}} C \\ \downarrow{\scriptstyle \varphi_{BC}} & & \downarrow{\scriptstyle \varphi_{XC}} \\ A \otimes_{\mathbb{Z}} Y & \xrightarrow{\varphi_{AY}} & Z \end{array}$$

commute. Then, for any $a \in \widehat{H}^r(G, A)$ and $b \in \widehat{H}^s(G, B)$ and $c \in \widehat{H}^t(G, C)$, we have

$$(a \cup b) \cup c = a \cup (b \cup c) \in \widehat{H}^{r+s+t}(G, Z).$$

Proof. The point is to track $b \in \widehat{H}^s(G, B)$ through the following very large commutative diagram.

$$\begin{array}{ccccc} \widehat{H}^s(G, B) & \xrightarrow{a\cup -} & \widehat{H}^{r+s}(G, A \otimes_{\mathbb{Z}} B) & \xrightarrow{\varphi_{AB}} & \widehat{H}^{r+s}(G, X) \\ \downarrow{\scriptstyle c\cup -} & (1) & \downarrow{\scriptstyle c\cup -} & (2) & \downarrow{\scriptstyle c\cup -} \\ \widehat{H}^{s+t}(G, B \otimes_{\mathbb{Z}} C) & \xrightarrow{a\cup -} & \widehat{H}^{r+s+t}(G, A \otimes_{\mathbb{Z}} B \otimes_{\mathbb{Z}} C) & \xrightarrow{\varphi_{AB}} & \widehat{H}^{r+s+t}(G, X \otimes_{\mathbb{Z}} C) \\ \downarrow{\scriptstyle \varphi_{BC}} & (3) & \downarrow{\scriptstyle \varphi_{BC}} & (4) & \downarrow{\scriptstyle \varphi_{XC}} \\ \widehat{H}^{s+t}(G, Y) & \xrightarrow{a\cup -} & \widehat{H}^{r+s+t}(G, A \otimes_{\mathbb{Z}} Y) & \xrightarrow{\varphi_{AY}} & \widehat{H}^{r+s+t}(G, Z) \end{array}$$

Namely, $(a \cup b) \cup c$ corresponds to following the top and then right legs of the diagram, and $a \cup (b \cup c)$ corresponds to following the left and then bottom legs of the diagram.

Thus, it suffices to show that the entire diagram square commutes. Well, (1) commutes by associativity of the cup product, (2) and (3) commute by naturality of the cup product, and (4) commutes by the hypothesized square and functoriality of $\widehat{H}^{r+s+t}(G, -)$. This finishes. ■

Let's start with a key result on shiftable functors, which gives a taste for why our hypotheses are so specially chosen.

Proposition 13. Let G be a finite group, and let (F, F', X, η) be a shifting pair. If we have indices $r, s \in \mathbb{Z}$ and $c \in H^r(G, X)$ such that the cup-product map

$$(c \cup -) \colon \widehat{H}^s(G, F-) \Rightarrow \widehat{H}^{r+s}(G, F'-)$$

is a natural isomorphism, then the cup-product map

$$(c \cup -) \colon \widehat{H}^j(G, F-) \Rightarrow \widehat{H}^{r+j}(G, F'-)$$

is a natural isomorphism for all indices $j \in \mathbb{Z}$.

Proof. This proof is by dimension-shifting on j. Note that it suffices by Lemma 11 to only worry about the component morphisms being isomorphisms.

To shift downwards, we suppose that the cup-product map is always an isomorphism for j, and we show that it is always an isomorphism $j - 1$. Namely, fix a G-module A, and we are interested in showing that the cup-product map

$$(c \cup -) \colon \widehat{H}^{j-1}(G, FA) \to \widehat{H}^{r+j-1}(G, F'A)$$

is an isomorphism. To do so, we note the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \tag{1.2}$$

which splits over $\mathbb{Z}$ and thus gives us the short exact sequences

$$0 \longrightarrow F(I_G \otimes_{\mathbb{Z}} A) \longrightarrow F(\mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \longrightarrow FA \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes_{\mathbb{Z}} F(I_G \otimes_{\mathbb{Z}} A) & \longrightarrow & X \otimes_{\mathbb{Z}} F(\mathbb{Z}[G] \otimes_{\mathbb{Z}} A) & \longrightarrow & X \otimes_{\mathbb{Z}} FA & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_{I_G}} & & \downarrow{\scriptstyle \eta_{\mathbb{Z}[G]}} & & \downarrow{\scriptstyle \eta_A} & & \\ 0 & \longrightarrow & F'(I_G \otimes_{\mathbb{Z}} A) & \longrightarrow & F'(\mathbb{Z}[G] \otimes_{\mathbb{Z}} A) & \longrightarrow & F'A & \longrightarrow & 0 \end{array}$$

where the bottom two rows commute by definition of η and thus give a morphism of short exact sequences. These short exact sequences give us boundary morphisms

$$\begin{aligned} \delta\colon &\quad \widehat{H}^{r+j-1}(G, F'A) &\to &\quad \widehat{H}^{r+j}(G, F'(I_G \otimes_{\mathbb{Z}} A)) \\ \delta_h\colon &\quad \widehat{H}^{j-1}(G, FA) &\to &\quad \widehat{H}^{j}(G, F(I_G \otimes_{\mathbb{Z}} A)) \\ \delta_t\colon &\, \widehat{H}^{r+j-1}(G, X \otimes_{\mathbb{Z}} FA) &\to &\, \widehat{H}^{r+j}(G, X \otimes_{\mathbb{Z}} F(I_G \otimes_{\mathbb{Z}} A)). \end{aligned}$$

Notably, all these δ morphisms because their short exact sequences have induced middle terms: all of F and $X \otimes_{\mathbb{Z}} F$ and F' are shiftable functors.

Now, the key to this dimension-shifting is claiming that the diagram

$$\begin{array}{ccc} \widehat{H}^{j-1}(G, FA) & \xrightarrow{c\cup -} & \widehat{H}^{r+j-1}(G, F'A) \\ \downarrow{\scriptstyle \delta_h} & & \downarrow{\scriptstyle (-1)^r\delta} \\ \widehat{H}^{j}(G, F(I_G \otimes_{\mathbb{Z}} A)) & \xrightarrow{c\cup -} & \widehat{H}^{r+j}(G, F'(I_G \otimes_{\mathbb{Z}} A)) \end{array}$$

commutes. Indeed, this will be enough because the bottom row is an isomorphism by the inductive hypothesis, and the left and morphisms are isomorphisms as discussed above, which makes the top row into an isomorphism. Well, to see that the diagram commutes, we expand the diagram as follows.

$$\begin{array}{ccccc} \widehat{H}^{j-1}(G, FA) & \xrightarrow{c\cup -} & \widehat{H}^{r+j-1}(G, X \otimes_{\mathbb{Z}} FA) & \xrightarrow{\eta_A} & \widehat{H}^{r+j-1}(G, F'A) \\ \downarrow{\scriptstyle \delta_h} & & \downarrow{\scriptstyle (-1)^r\delta_t} & & \downarrow{\scriptstyle (-1)^r\delta} \\ \widehat{H}^{j}(G, F(I_G \otimes_{\mathbb{Z}} A)) & \xrightarrow{c\cup -} & \widehat{H}^{r+j}(G, X \otimes_{\mathbb{Z}} F(I_G \otimes_{\mathbb{Z}} A)) & \xrightarrow{\eta_{I_G}} & \widehat{H}^{r+j}(G, F'(I_G \otimes_{\mathbb{Z}} A)) \end{array}$$

The left square commutes because cup products commute with boundary morphisms; the right square commutes by functoriality of boundary morphisms.

Shifting upwards is similar. Suppose that the cup-product in question is always an isomorphism for j, and we show that it is always an isomorphism for $j+1$. Namely, fix a G-module A, and we are interested in showing that the cup-product map

$$(c \cup -)\colon \widehat{H}^{j+1}(G, FA) \to \widehat{H}^{r+j+1}(G, F'A)$$

is an isomorphism. As before, we use (1.2) to induce the short exact sequences

$$0 \longrightarrow FA \longrightarrow F(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \longrightarrow F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes_{\mathbb{Z}} FA & \longrightarrow & X \otimes_{\mathbb{Z}} F(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) & \longrightarrow & X \otimes_{\mathbb{Z}} F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_{\mathbb{Z}[G]}} & & \downarrow{\scriptstyle \eta_{I_G}} & & \\ 0 & \longrightarrow & F'A & \longrightarrow & F'(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) & \longrightarrow & F'(\mathrm{Hom}_{\mathbb{Z}}(I_G, A)) & \longrightarrow & 0 \end{array}$$

where again the bottom rows commute by definition of η. As before, we have the boundary morphisms

$$\begin{aligned}\delta\colon &\quad \widehat{H}^{r+j}(G, F'(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) &\to &\quad \widehat{H}^{r+j+1}(G, F'A)\\ \delta_h\colon &\quad \widehat{H}^{j}(G, F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) &\to &\quad \widehat{H}^{j+1}(G, FA)\\ \delta_t\colon &\ \widehat{H}^{r+j}(G, X\otimes_{\mathbb{Z}} F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) &\to &\ \widehat{H}^{r+j+1}(G, X\otimes_{\mathbb{Z}} FA).\end{aligned}$$

We again note that all δ are isomorphisms because the middle terms of our short exact sequences are induced: all of F and $X\otimes_{\mathbb{Z}} F$ and F' are shiftable functors.

Once more, the key to the dimension-shifting will be the claim that the diagram

$$\begin{array}{ccc}\widehat{H}^{j}(G, F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & \xrightarrow{c\cup -} & \widehat{H}^{r+j}(G, F'(\mathrm{Hom}_{\mathbb{Z}}(I_G, A)))\\ \downarrow{\scriptstyle \delta_h} & & \downarrow{\scriptstyle (-1)^r\delta}\\ \widehat{H}^{j+1}(G, FA) & \xrightarrow{c\cup -} & \widehat{H}^{r+j+1}(G, F'A)\end{array}$$

commutes. This will be enough because the top arrow is an isomorphism by the inductive hypothesis, and the left and right arrows are isomorphisms as discussed above, thus making the bottom arrow also an isomorphism. Now, to see that the diagram commutes, we expand out our cup products as follows.

$$\begin{array}{ccccc}\widehat{H}^{j}(G, F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & \xrightarrow{c\cup -} & \widehat{H}^{r+j}(G, X\otimes_{\mathbb{Z}} F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & \xrightarrow{\eta_{I_G}} & \widehat{H}^{r+j}(G, F'(\mathrm{Hom}_{\mathbb{Z}}(I_G, A)))\\ \downarrow{\scriptstyle \delta_h} & & \downarrow{\scriptstyle (-1)^r\delta_t} & & \downarrow{\scriptstyle (-1)^r\delta}\\ \widehat{H}^{j+1}(G, FA) & \xrightarrow{c\cup -} & \widehat{H}^{r+j+1}(G, X\otimes_{\mathbb{Z}} FA) & \xrightarrow{\eta_A} & \widehat{H}^{r+j+1}(G, F'A)\end{array}$$

The left square commutes because cup products commute with boundary morphisms, and the right square commutes by functoriality of boundary morphisms. This finishes. ■

Here are some applications.

Corollary 14. Let G be a finite group. There exists $c\in \widehat{H}^1(G, I_G)$ such that, for any G-module X,

$$(c\cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} -))$$

is a natural isomorphism for any $i\in\mathbb{Z}$.

Proof. Here, we are using the shifting pair $(\mathrm{Hom}_{\mathbb{Z}}(X, -), \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} -), I_G, \eta)$, where

$$\eta_A\colon I_G\otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)\to \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} A)$$

is the canonical map sending $z\otimes f$ to $x\mapsto z\otimes f(x)$.

Now, in light of Proposition 13, we merely have to find $c\in\widehat{H}^1(G, I_G)$ and show that we have a natural isomorphism at $i=0$. Because we already have a natural transformation by Lemma 11, we are only worried about making the component morphisms

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\to \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} A))$$

isomorphisms for all G-modules A. Well, we note that we have the $\mathbb{Z}$-split short exact sequence

$$0\to \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} A)\to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G]\otimes_{\mathbb{Z}} A)\to \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} A)\to 0$$

which will induce a δ morphism between the correct modules. In fact, because $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ is a shiftable functor, the middle term here is induced, so the δ morphism

$$\delta\colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\to \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G\otimes_{\mathbb{Z}} A))$$

is an isomorphism.

To finish, we claim that this δ morphism arises as a cup product. We simply show this by hand by tracking through the δ morphism. Given $[f] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ where $f\colon X \to A$ is a G-module homomorphism, we can pull this back to the 0-chain $\widetilde{f}\colon X \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ defined by

$$\widetilde{f}\colon x \mapsto 1 \otimes f(x).$$

Applying the differential, we get the 1-cocycle $d\widetilde{f} \in B^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A))$ defined by

$$\begin{aligned}(d\widetilde{f})(g)(x) &= (g\widetilde{f})(x) - \widetilde{f}(x) \\ &= g \cdot \widetilde{f}\left(g^{-1}x\right) - \widetilde{f}(x) \\ &= g\left(1 \otimes f(g^{-1}x)\right) - (1 \otimes f(x)) \\ &= (g-1) \otimes f(x),\end{aligned}$$

which we know must be a 1-cocycle representing $\delta([f]) \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A))$.

Thus, we see that we should set $c \in \widehat{H}^1(G, I_G)$ to be represented by $g \mapsto (g-1)$. This will work as long as $g \mapsto (g-1)$ is actually 1-cocycle in $\widehat{H}^1(G, I_G)$. Well, take $X = A = \mathbb{Z}$ and $f = \mathrm{id}_{\mathbb{Z}}$ in the above argument so that $\delta(f)$ is exactly $g \mapsto (x \mapsto (g-1) \otimes x)$, which is $g \mapsto (g-1)$ after applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, I_G) \simeq I_G$. ■

Corollary 15. Let G be a finite group. There exists $c \in \widehat{H}^1(G, I_G)$ such that, for any G-module X,

$$(c \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, -))) \Rightarrow \widehat{H}^{i+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, -))$$

is a natural isomorphism for any $i \in \mathbb{Z}$.

Proof. Similar to before, we are using the shifting pair $(\mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, -)), \mathrm{Hom}_{\mathbb{Z}}(X, -), I_G, \eta)$, where

$$\eta_A\colon I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \Rightarrow \mathrm{Hom}_{\mathbb{Z}}(X, -)$$

is the canonical map sending $z \otimes f$ to $x \mapsto f(x)(z)$.

Using Proposition 13 and Lemma 11 again, it will suffice to find $c \in \widehat{H}^1(G, I_G)$ such that we have isomorphisms

$$(c \cup -)\colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \to \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$$

for all G-modules A. This time around we use the $\mathbb{Z}$-split short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, A) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \to 0$$

which will induce a boundary morphism

$$\delta\colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \to \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)).$$

In fact, δ is an isomorphism because our middle term $\mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ is induced.

We now show that δ is a cup product by hand. Pick up some $[f] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)))$ where $f\colon X \to \mathrm{Hom}_{\mathbb{Z}}(I_G, A)$ is a G-module homomorphism. This pulls back to the 0-cochain

$$\widetilde{f}\colon x \mapsto \left(z \mapsto f(x)(z - \varepsilon(z))\right).$$

Applying the differential, we compute

$$\begin{aligned}(d\widetilde{f})(g)(x)(z) &= (g\widetilde{f} - \widetilde{f})(x)(z)\\ &= (g\widetilde{f})(x)(z) - \widetilde{f}(x)(z)\\ &= \left(g \cdot \widetilde{f}\left(g^{-1}x\right)\right)(z) - \widetilde{f}(x)(z)\\ &= g \cdot \widetilde{f}\left(g^{-1}x\right)\left(g^{-1}z\right) - \widetilde{f}(x)(z)\\ &= g \cdot f\left(g^{-1}x\right)\left(g^{-1}z - \varepsilon(z)\right) - f(x)(z - \varepsilon(z))\\ &= g \cdot \left(g^{-1}f(x)\right)\left(g^{-1}z - \varepsilon(z)\right) - f(x)(z - \varepsilon(z))\\ &= f(x)\left(z - g\varepsilon(z)\right) - f(x)(z - \varepsilon(z))\\ &= \varepsilon(z)f(x)\left(1 - g\right).\end{aligned}$$

Thus, this pulls back to the 1-cocycle $g \mapsto (x \mapsto f(x)(1-g))$ in $\widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$.

In particular, we see that we should take c represented by $g \mapsto (1-g)$, which will work as soon as we know that $g \mapsto (1-g)$ is a 1-cocycle. Well, this is the negation of the 1-cocycle $g \mapsto (g-1)$ found in Corollary 14. We close by remarking that we can actually take c represented by $g \mapsto (g-1)$ because negating c does not change the fact that the cup product gives an isomorphism. ■

Remark 16. Essentially the same proofs for Corollary 14 and Corollary 15 will work when $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ is replaced by $X \otimes_{\mathbb{Z}} -$, or any composite of these. There isn't an analogue for arbitrary shiftable functors because, for example, there is no way obvious way to construct η in general. Regardless, we will not need to work in these levels of generality.

The point of Corollary 14 and Corollary 15 is that have a somewhat general version of dimension-shifting granted by cup products. In fact, we see that we can use the same $c \in \widehat{H}^1(G, I_G)$ represented by $g \mapsto (g-1)$ for both shifting isomorphisms.

1.3 Shifting Natural Transformations

Observe that a natural transformation $F \Rightarrow F'$ of shiftable functors will induce natural transformations in cohomology

$$\widehat{H}^i(G, F-) \Rightarrow \widehat{H}^i(G, F'-)$$

It will turn out that, when $F = \mathrm{Hom}_{\mathbb{Z}}(X, -)$ and $F' = \mathrm{Hom}_{\mathbb{Z}}(X', -)$, we will be able to force all natural transformations in cohomology will come from natural transformations $F \Rightarrow F'$.

To begin, we show this result for $i = 0$.

Lemma 17. Let G be a finite group, and let X and X' be G-modules. Suppose that, for given index $r \in \mathbb{Z}$, there is a natural transformation

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

Then there exists $[x] \in \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ such that $\Phi_\bullet = ([x] \cup -)$, where the cup product is induced by the shifting pair of Example 10.

Proof. This is essentially the Yoneda lemma. As such, set $[x] \coloneqq \Phi_X([\mathrm{id}_X])$. The point is to fix some G-module A and $[\overline{f}] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ in order to track through the commutativity of the following diagram.

$$\begin{array}{ccc} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) & \xrightarrow{\Phi_X} & \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', X)) \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{f}} \\ \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) & \xrightarrow{\Phi_A} & \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \end{array} \tag{1.3}$$

Because we will need to deal with the cup products with negative indices, we will use the standard resolution of [AW10]. For example, we interpret $f \in [\overline{f}] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ as a constant function $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(X, A))$ outputting $\overline{f} \in \mathrm{Hom}_{\mathbb{Z}[G]}(X, A)$, which means that $f(z)$ is the same G-module homomorphism for each $z \in \mathbb{Z}[G]$.

As such, we can track the left arrow of (1.3) as

$$\begin{aligned} \overline{f} \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) &\to \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ [z \mapsto \mathrm{id}_X] &\mapsto [z \mapsto f(z) \circ \mathrm{id}_X] = [\overline{f}]. \end{aligned}$$

So, along the bottom of (1.3), we are evaluating $\Phi_A([\overline{f}])$.

Along the top of (1.3), we immediately send $[z \mapsto \mathrm{id}_X]$ to $\Phi_X([z \mapsto \mathrm{id}_X]) = [x]$, so to finish the proof, we need to show that

$$\overline{f}([x]) \stackrel{?}{=} [x] \cup [\overline{f}],$$

which will be enough by the commutativity of (1.3). We have two similar cases to appropriately deal with the cup product.

- Suppose that $r \geq 0$ so that we can interpret x as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[G^{r+1}], X\right)$, using the standard resolution. As such, we compute
$$(x \cup f)(g_0, \ldots, g_p) = x(g_0, \ldots, g_p) \otimes f(g_r),$$
where our output is in $\mathrm{Hom}_{\mathbb{Z}}(X', X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)$. Applying evaluation, the cup product is outputting
$$(g_0, \ldots, g_r) \mapsto \overline{f} \circ x(g_0, \ldots, g_r)$$
as our element of $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{r+1}], \mathrm{Hom}_{\mathbb{Z}}(X', A))$. Indeed, this morphism represents $\overline{f}([x])$.
- Analogously, suppose that $r < 0$ so that we interpret x as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^r, \mathbb{Z}), X\right)$. To decrease headaches, we let $g^* \colon \mathbb{Z}[G] \to \mathbb{Z}$ denote the G-module homomorphism sending $g \mapsto 1$ and other group elements to 0. Then r-tuples $(g_1^*, \ldots, g_r^*)$ form a $\mathbb{Z}$-basis of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^r, \mathbb{Z})$, so it's enough to specify
$$(x \cup f)(g_1^*, \ldots, g_r^*) = x(g_1^*, \ldots, g_r^*) \otimes f(g_r),$$
where the output is in $\mathrm{Hom}_{\mathbb{Z}}(X', X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)$. Applying evaluation, the cup product is outputting
$$(g_1^*, \ldots, g_r^*) \mapsto \overline{f} \circ x(g_1^*, \ldots, g_r^*)$$
as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^r, \mathbb{Z}), \mathrm{Hom}_{\mathbb{Z}}(X', A)\right)$. Indeed, this represents $\overline{f}([x])$.

The above cases finish tracking through (1.3) and hence finish the proof. ■

The case of $r = 0$ will be particularly interesting to us, so we note that we have the following more concrete description.

Lemma 18. Let G be a finite group, and let X and X' be G-modules. Then, given a G-module morphism $\varphi \colon X' \to X$, the maps $(- \circ \varphi)$ and $([\varphi] \cup -)$ on

$$\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', -))$$

assemble into the same natural transformation for any $i \in \mathbb{Z}$.

Proof. This follows from unpacking the definitions.

We already know that $([\varphi] \cup -)$ is a natural transformation by Lemma 11, so it suffices to show that the two maps agree on components. (Namely, naturality of $(- \circ \varphi)$ will immediately follow.) To see this, we fix a G-module A to evaluate the morphism

$$\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)),$$

for which we use the standard resolution of [AW10]. For this, we represent $[\varphi]$ by the morphism $\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(X, A))$ which constantly outputs φ.

Now, pick up $[x] \in \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$. We have two cases.

- If $i \geq 0$, we can interpret x as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[G]^{i+1}, \mathrm{Hom}_{\mathbb{Z}}(X, A)\right)$. Then our cup product is
$$(\widetilde{\varphi} \cup x)(g_0, \ldots, g_i) = \widetilde{\varphi}(x_0) \otimes x(g_0, \ldots, g_i),$$
which evaluates to
$$(g_0, \ldots, g_i) \mapsto x(g_0, \ldots, g_i) \circ \varphi,$$
which represents the desired class $(- \circ \varphi)([x])$.
- If $i < 0$, we can interpret x as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{i}, \mathbb{Z}), \mathrm{Hom}_{\mathbb{Z}}(X, A)\right)$. Our cup product is
$$(\widetilde{\varphi} \cup x)(g_1^*, \ldots, g_i^*) = \widetilde{\varphi}(g_1) \otimes x(g_1^*, \ldots, g_i^*),$$
which evaluates to
$$(g_1^*, \ldots, g_i^*) \mapsto x(g_1^*, \ldots, g_i^*) \circ \varphi,$$
which represents the desired class $(- \circ \varphi)([x])$.

The above cases finish the proof. ■

We now get the main result by dimension-shifting.

Proposition 19. Let G be a finite group, and let X and X' be G-modules. Then, given indices $r, s \in \mathbb{Z}$, any natural transformation
$$\Phi_\bullet^{(s)} \colon \widehat{H}^s(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{r+s}(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)),$$
is $\Phi_\bullet^{(s)} = (x \cup -)$ for some $x \in \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$.

Proof. This argument is by dimension-shifting the s upwards and downwards. Namely, we show the conclusion of the statement by induction on s; for $s = 0$, this is Lemma 17. We will show how to induct upwards to $s \geq 0$ in detail, and inducting downwards is similar. For brevity, we set $F := \mathrm{Hom}_{\mathbb{Z}}(X, -)$ and $F' := \mathrm{Hom}_{\mathbb{Z}}(X', -)$.

To induct upwards, suppose the statement is true for $s = i$, and we show $s = i + 1$, so fix a natural transformation
$$\Phi_\bullet^{(i+1)} \colon \widehat{H}^{i+1}(G, F-) \Rightarrow \widehat{H}^{p+i+1}(G, F'-),$$
which we would like to know arises as $(x \cup -)$ for some $x \in \widehat{H}^p(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$. The main idea is to use $\Phi_\bullet^{(i+1)}$ in order to construct $\Phi_\bullet^{(i)}$. Well, using Corollary 14, we have some $c \in \widehat{H}^1(G, I_G)$ given by $g \mapsto (g - 1)$ yielding the following isomorphisms for any G-module A.
$$\begin{aligned} (c \cup -)_d &\colon \widehat{H}^i(G, FA) \to \widehat{H}^{i+1}(G, F(I_G \otimes_{\mathbb{Z}} A)) \\ (c \cup -)'_d &\colon \widehat{H}^{r+i}(G, F'A) \to \widehat{H}^{r+i+1}(G, F'(I_G \otimes_{\mathbb{Z}} A)) \end{aligned}$$
As such, we have the diagram
$$\begin{array}{ccc} \widehat{H}^i(G, FA) & \xrightarrow{(c\cup-)_d} & \widehat{H}^{i+1}(G, F(I_G \otimes_{\mathbb{Z}} A)) \\ \downarrow & & \downarrow \Phi^{(i+1)}_{I_G \otimes_{\mathbb{Z}} A} \\ \widehat{H}^{r+i}(G, F'A) & \xrightarrow{(c\cup-)'_d} & \widehat{H}^{r+i+1}(G, F'(I_G \otimes_{\mathbb{Z}} A)) \end{array}$$
where the horizontal arrows are isomorphisms. Thus, we induce a morphism
$$\Phi_A^{(i)} := ((c \cup -)'_d)^{-1} \circ \Phi^{(i+1)}_{I_G \otimes_{\mathbb{Z}} A} \circ (c \cup -)_d.$$
Note that $\Phi_\bullet^{(i)}$ is the composition of natural transformations (the cup product is a natural transformation by construction) and therefore is a natural transformation.

Thus, the inductive hypothesis now tells us that $\Phi_\bullet^{(i)} = (x \cup -)$ for some $x \in \widehat{H}^p(G, \mathrm{Hom}_\mathbb{Z}(X', X))$. We now need to turn this around on $\Phi_\bullet^{(i+1)}$, which essentially means we need to shift back in the other direction. As such, we use Corollary 15 to give the following isomorphisms for any G-module A.

$$\begin{aligned}(c \cup -)_u &\colon \widehat{H}^i(G, F(\mathrm{Hom}_\mathbb{Z}(I_G, A))) \to \widehat{H}^{i+1}(G, FA) \\ (c \cup -)'_u &\colon \widehat{H}^{r+i}(G, F(\mathrm{Hom}_\mathbb{Z}(I_G, A))) \to \widehat{H}^{r+i+1}(G, FA)\end{aligned}$$

Now, to deal with $\Phi_\bullet^{(i+1)}$, we claim that associativity and commutativity of cup products implies $\big((-1)^i x \cup -\big)$ can be used to make the right arrow in the diagram

$$\begin{array}{ccc} \widehat{H}^i(G, F(\mathrm{Hom}_\mathbb{Z}(I_G, A))) & \xrightarrow{(c\cup-)_u} & \widehat{H}^{i+1}(G, FA) \\ \downarrow{\scriptstyle x\cup-} & & \vdots \\ \widehat{H}^{r+i}(G, F'(\mathrm{Hom}_\mathbb{Z}(I_G, A))) & \xrightarrow{(c\cup-)'_u} & \widehat{H}^{r+i+1}(G, F'A) \end{array} \tag{1.4}$$

commute. Indeed, applying Lemma 12 to the square

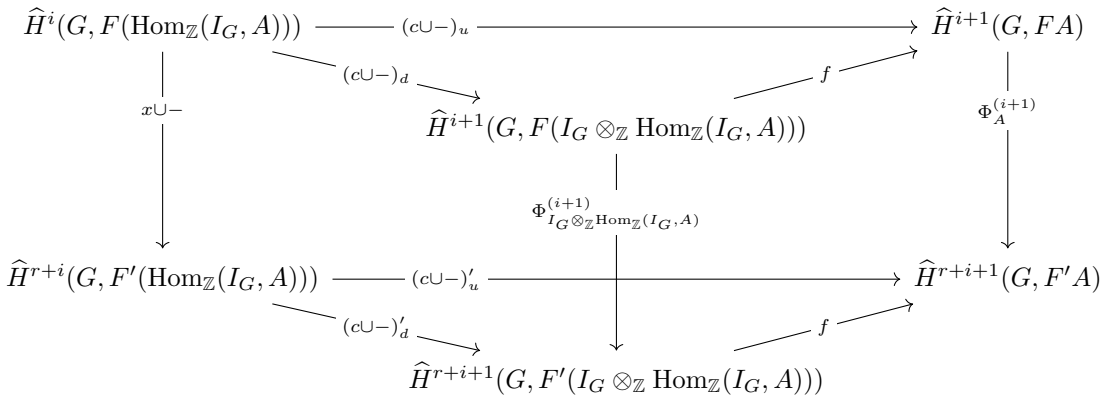
$$\begin{array}{ccc} I_G \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(X, \mathrm{Hom}_\mathbb{Z}(I_G, A)) \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(X', X) & \longrightarrow & \mathrm{Hom}_\mathbb{Z}(X, A) \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(X', X) \\ \downarrow & & \downarrow \\ I_G \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(X', \mathrm{Hom}_\mathbb{Z}(I_G, A)) & \longrightarrow & \mathrm{Hom}_\mathbb{Z}(X', A) \end{array}$$

shows that any $a \in \widehat{H}^i(G, F(\mathrm{Hom}_\mathbb{Z}(I_G, A)))$ has

$$c \cup (x \cup a) = (-1)^{ir} c \cup (a \cup x) = (-1)^{ir} (c \cup a) \cup x = (-1)^{ir+i(r+1)} x \cup (c \cup a) = (-1)^i x \cup (c \cup a),$$

which is what we wanted.

Now, this right arrow of (1.4) is unique because the horizontal arrows are isomorphisms, so we will be done if we can show that we can place $\Phi_A^{(i+1)}$ in the right arrow to also make the diagram commute as well. For this, we draw the following very large diagram.



Here, the f maps are induced by the evaluation map

$$f \colon I_G \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(I_G, A) \to A.$$

We want the outer rectangle to commute, for which it suffices to show that each parallelogram and the small top and bottom triangles to commute.

- The left parallelogram commutes by definition of $\Phi_A^{(i)}$.
- The right parallelogram commutes by naturality of $\Phi_\bullet^{(i+1)}$.

- Showing that the bottom triangle commutes will be analogous to showing that the top triangle commutes, so we will only show the top. Unwinding Corollary 14 and Corollary 15, we see that this triangle is actually induced by the following diagram.

$$\begin{array}{ccccc} \widehat{H}^i(G, F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & \xrightarrow{c\cup -} & \widehat{H}^{i+1}(G, I_G \otimes_{\mathbb{Z}} F(\mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & \xrightarrow{\eta_u} & \widehat{H}^{i+1}(G, FA) \\ & & \downarrow{\scriptstyle \eta_d} & \nearrow_{f} & \\ & & \widehat{H}^{i+1}(G, F(I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) & & \end{array}$$

  Here, $\eta_u \colon I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \to \mathrm{Hom}_{\mathbb{Z}}(X, A)$ behaves as

$$\eta_u \colon z \otimes f \mapsto \big(x \mapsto f(x)(z)\big),$$

  and $\eta_d \colon I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \to \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(I_G, A))$ behaves as

$$\eta_d \colon z \otimes f \mapsto \big(x \mapsto z \otimes f(x)\big).$$

  Now, to check our commutativity, it suffices to show that the triangle

$$\begin{array}{ccc} I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) & \xrightarrow{\eta_u} & \mathrm{Hom}_{\mathbb{Z}}(X, A) \\ \downarrow{\scriptstyle \eta_d} & \nearrow_{f} & \\ \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) & & \end{array}$$

  commutes. Well, we can simply track through the diagram as follows.

$$\begin{array}{ccc} z \otimes f & \longmapsto & \big(x \mapsto f(x)(z)\big) \\ \downarrow & \nearrow & \\ \big(x \mapsto z \otimes f(x)\big) & & \end{array}$$

The above commutativity checks finish the induction upwards.

We will not give detail for the induction downwards from $i-1$ to i, except to say that we reverse the applications of Corollary 14 and Corollary 15. The rest of the approach essentially goes through verbatim, constructing $\Phi_\bullet^{(i)}$ from a given $\Phi_\bullet^{(i-1)}$, applying the inducting hypothesis to $\Phi_\bullet^{(i)}$, and then finishing by shifting back to $\Phi_\bullet^{(i-1)}$. ■

Remark 20. Essentially the same proof can show that, for any pair of shiftable functors $F, F' \colon \mathrm{Mod}_G \to \mathrm{Mod}_G$, a natural transformation (respectively, isomorphism)

$$\Phi_\bullet^{(i)} \colon \widehat{H}^i(G, F-) \Rightarrow \widehat{H}^i(G, F'-),$$

at $i = r$ induces natural transformations (respectively, isomorphisms) at all $i \in \mathbb{Z}$. Instead of using Corollary 14 and Corollary 15, we must instead dimension-shifting using the usual short exact sequences.

Corollary 21. Let G be a finite group, and let X and X' be G-modules. Then, given indices $s \in \mathbb{Z}$, any natural transformation

$$\Phi_\bullet^{(s)} \colon \widehat{H}^q(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^s(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)),$$

is $\Phi_\bullet^{(s)} = (- \circ \varphi)$ for some G-module morphism $\varphi \colon X' \to X$.

Proof. Proposition 19 tells us that the natural transformation takes the form $([\varphi] \cup -)$ for some G-module morphism $\varphi \colon X' \to X$. Then $([\varphi] \cup -)$ is simply $(- \circ \varphi)$ by Lemma 18. ■

1.4 Cohomological Equivalence

It might be the case that "many" different shiftable functors give the same cohomology groups. Because we are mostly interested in the case of $\mathrm{Hom}_{\mathbb{Z}}(X, -)$, we now have the tools to talk fairly concretely about what this means. We have the following definition.

Definition 22. Let G be a finite group. We say that two G-modules X, X' are *cohomologically equivalent* if and only if there exist morphisms $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that

$$[\varphi \circ \varphi'] = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \qquad \text{and} \qquad [\varphi' \circ \varphi] = [\mathrm{id}_{X'}] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X')).$$

Example 23. All induced modules X are cohomologically equivalent to 0. To see this, we set $\varphi\colon 0 \to X$ and $\varphi'\colon X \to 0$ equal to the zero maps (which are our only options). Then note that $\mathrm{Hom}_{\mathbb{Z}}(X, X)$ is induced by Lemma 5 and $\mathrm{Hom}_{\mathbb{Z}}(0, 0) = 0$, so

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) = \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X')) = 0,$$

making the checks on φ and φ' both trivial.

More concretely, X and X' are cohomologically equivalent if and only if we have two G-module morphisms $\varphi\colon X' \to X$ and $\varphi'\colon X \to X'$ and two $\mathbb{Z}$-module morphisms $f\colon X \to X$ and $f'\colon X' \to X'$ such that

$$\varphi \circ \varphi' = \mathrm{id}_X + N_G f \qquad \text{and} \qquad \varphi' \circ \varphi = \mathrm{id}_{X'} + N_G f'.$$

As a quick sanity check that this is a reasonable notion of equivalence of modules, we have the following.

Lemma 24. Let G be a finite group. If the G-modules X and X' are equivalent and Y and Y' are equivalent, then $X \oplus Y$ is equivalent to $X' \oplus Y'$.

Proof. We are promised the morphisms

- $\varphi\colon X' \to X$ and $\varphi'\colon X \to X'$ (as morphisms of G-modules),
- $f\colon X \to X$ and $f'\colon X' \to X'$ (as morphisms of $\mathbb{Z}$-modules),
- $\psi\colon Y' \to Y$ and $\psi'\colon Y \to Y'$ (as morphisms of G-modules),
- $g\colon Y \to Y$ and $g'\colon Y' \to Y'$ (as morphisms of $\mathbb{Z}$-modules),

which are required to satisfy

$$\begin{aligned} \varphi \circ \varphi' &= \mathrm{id}_X + N_G f \qquad \text{and} \qquad \varphi' \circ \varphi = \mathrm{id}_{X'} + N_G f', \\ \psi \circ \psi' &= \mathrm{id}_Y + N_G g \qquad \text{and} \qquad \psi' \circ \psi = \mathrm{id}_{Y'} + N_G g'. \end{aligned}$$

Summing everywhere, we get the G-module homomorphisms $\varphi \oplus \psi\colon X \oplus Y \to X' \oplus Y'$ and $\varphi' \oplus \psi'\colon X' \oplus Y' \to X \oplus Y$ satisfying

$$\begin{aligned} (\varphi \oplus \psi) \circ (\varphi' \oplus \psi') &= (\varphi \circ \varphi') \oplus (\psi \circ \psi') \\ &= (\mathrm{id}_X + N_G f) \oplus (\mathrm{id}_Y + N_G g) \\ &= \mathrm{id}_X \oplus \mathrm{id}_Y + N_G(f \oplus g). \end{aligned}$$

The other check is analogous, switching primed and unprimed variables. ∎

We now show that this notion of equivalence correctly translates to shiftable functors.

Proposition 25. Let G be a finite group, and let X and X' be G-modules. Then X and X' are cohomologically equivalent if and only if there is a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

Proof. In the forward direction, suppose X and X' are cohomologically equivalent so that we have $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that

$$[\varphi] \cup [\varphi'] = [\varphi \circ \varphi'] = [\mathrm{id}_X] \qquad \text{and} \qquad [\varphi'] \cup [\varphi] = [\varphi' \circ \varphi] = [\mathrm{id}_{X'}],$$

where we are using the canonical evaluation maps for the cup products. Now, we note that, for any G-module A, we have inverse morphisms

$$\begin{array}{ccc} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) & \simeq & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ [f] & \mapsto & [f \circ \varphi] \\ [f' \circ \varphi'] & \mapsfrom & [f']. \end{array} \tag{1.5}$$

Indeed, these are mutually inverse because

$$[f \circ \varphi \circ \varphi'] = [f] \cup [\varphi \circ \varphi'] = [f] \circ [\mathrm{id}_X] = [f]$$

and similar on the other side. To finish, we note that the isomorphisms (1.5) assemble into a natural isomorphism by Lemma 11 and Lemma 18.

We now show the backwards direction. Suppose we have a natural isomorphism $\Phi_\bullet$. Then applying Lemma 17 in both directions, we get $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that the morphisms

$$\begin{array}{ccc} \Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) & \simeq & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \\ [f] & \mapsto & [f \circ \varphi] \\ [f' \circ \varphi'] & \mapsfrom & [f'] \end{array}$$

are mutually inverse. In particular, we see that

$$[\mathrm{id}_X] = [\mathrm{id}_X \circ \varphi \circ \varphi'] = [\varphi \circ \varphi'],$$

so $[\varphi \circ \varphi'] = [\mathrm{id}_X]$. Swapping primed and unprimed variables, we see $[\varphi' \circ \varphi] = [\mathrm{id}_{X'}]$ as well. ∎

Remark 26. The above result makes it fairly clear that cohomological equivalence actually makes an equivalence relation. In particular, we can invert and compose natural isomorphisms, which gives symmetry and transitivity of cohomological equivalence respectively.

Corollary 27. Let G be a finite group, and let X be a G-module. Then X is cohomologically equivalent to 0 if and only if $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) = 0$.

Proof. On one hand, if X is cohomologically equivalent to 0, then Proposition 25 promises a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(0, -)) = 0,$$

so it follows $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) = 0$ by plugging in X.

On the other hand, if $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) = 0$, then the G-module homomorphism $\mathrm{id}_X \colon X \to X$ must be equivalent to 0 in this group. So let $\varphi \colon 0 \to X$ and $\varphi' \colon X \to 0$ be the canonical zero morphisms so that

$$[\varphi \circ \varphi'] = [0] = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$$

and

$$[\varphi' \circ \varphi] = [0] = [\mathrm{id}_0] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(0, 0)),$$

which finishes. ∎

Example 28. It is not in general true that two G-modules X and X' are cohomologically equivalent if and only if $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \cong \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X'))$. Indeed, let $G = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ act on $X = \mathbb{Z}$ trivially and on $X' = \mathbb{Z}i$ by $\sigma \colon i \mapsto -i$. Then

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}i, \mathbb{Z}i)$$

as G-modules (!), but these modules are not cohomologically equivalent because

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z} \ncong 0 \cong \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}i)).$$

Namely, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}i) = 0$ as G-modules.

This alternate definition also provides us with a way to multiply.

Corollary 29. Let G be a finite group. If X and X' are cohomologically equivalent and Y and Y' are cohomologically equivalent, then $X \otimes_{\mathbb{Z}} X'$ is cohomologically equivalent to $Y \otimes_{\mathbb{Z}} Y'$.

Proof. We are granted natural isomorphisms as follows.

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -))$$
$$\Psi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y', -))$$

Now, repeatedly using the hom–tensor adjunction, we can chain together natural isomorphisms

$$\begin{aligned}
\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X \otimes_{\mathbb{Z}} Y, -)) &\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(Y, -))) \\
&\overset{\Phi \mathrm{Hom}(Y,-)}{\simeq} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', \mathrm{Hom}_{\mathbb{Z}}(Y, -))) \\
&\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X' \otimes_{\mathbb{Z}} Y, -)) \\
&\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y \otimes_{\mathbb{Z}} X', -)) \\
&\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y, \mathrm{Hom}_{\mathbb{Z}}(X', -))) \\
&\overset{\Psi \mathrm{Hom}_{\mathbb{Z}}(X',-)}{\simeq} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y', \mathrm{Hom}_{\mathbb{Z}}(X', -))) \\
&\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(Y' \otimes_{\mathbb{Z}} X', -)) \\
&\simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X' \otimes_{\mathbb{Z}} Y', -)),
\end{aligned}$$

which is what we wanted. ∎

One might hope that we can get more information by using indices away from 0, but in fact we cannot.

Proposition 30. Let G be a finite group, and let X and X' be G-modules. Then the following are equivalent.

(a) X and X' are cohomologically equivalent.

(b) For some $r \in \mathbb{Z}$, there is a natural isomorphism

$$\Phi_\bullet^{(r)} \colon \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

(c) There is a G-module homomorphism $\varphi \colon X' \to X$ such that the induced maps

$$(- \circ \varphi) \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', -))$$

are natural isomorphisms for all $i \in \mathbb{Z}$.

Proof. Note that (a) implies (b) by taking $r = 0$ and applying Proposition 25. Also, (c) implies (a) by taking $i = 0$ and again applying Proposition 25. Lastly, to show (b) implies (c), we note that Proposition 19 promises us $\varphi\colon X' \to X$ such that

$$\Phi_\bullet^{(r)} = (- \circ \varphi).$$

We would like to use Proposition 13. Let our shifting pair be $(\mathrm{Hom}_\mathbb{Z}(X,-), \mathrm{Hom}_\mathbb{Z}(X',-), \mathrm{Hom}_\mathbb{Z}(X',X), \eta)$, where $\eta_\bullet$ is the canonical pre-composition map

$$\eta_\bullet\colon \mathrm{Hom}_\mathbb{Z}(X',X) \otimes_\mathbb{Z} \mathrm{Hom}_\mathbb{Z}(X,-) \to \mathrm{Hom}_\mathbb{Z}(X',-).$$

Then we take $r = r$ and $s = 0$ and $c = [\varphi]$ as above so that the cup-product natural transformation

$$([\varphi] \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_\mathbb{Z}(X,-)) \Rightarrow \widehat{H}^i(G, \mathrm{Hom}_\mathbb{Z}(X',-))$$

is simply induced by $(- \circ \varphi)$ for any $i \in \mathbb{Z}$ by Lemma 18. So we are given that this is a natural isomorphism at $i = r$, so Proposition 13 gives us this isomorphism at all $i \in \mathbb{Z}$, which proves (c). ■

1.5 Encoding Modules

Lastly, we arrive at the application we care about: encoding cohomology. Cohomological equivalence is exactly what we need to talk about uniqueness.

Proposition 31. Let G be a finite group, and let $r, s \in \mathbb{Z}$ be indices. Then, if nonempty, the set of G-module X with a natural isomorphism

$$\Phi_\bullet\colon \widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X,-)) \Rightarrow \widehat{H}^{r+s}(G,-)$$

make up exactly one cohomological equivalence class.

Proof. Fix some G-module X with such a natural isomorphism

$$\Psi_\bullet\colon \widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X,-)) \Rightarrow \widehat{H}^{r+s}(G,-).$$

We would like to show that a G-module X' has a natural isomorphism $\Phi_\bullet$ between the analogous functors if and only if X and X' are cohomologically equivalent.

If X and X' are cohomologically equivalent, then we can compose the promised natural isomorphism of Proposition 30 (c) with $\Psi_\bullet$, giving a natural isomorphism

$$\widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X',-)) \Rightarrow \widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X,-)) \overset{\Psi_\bullet}{\Rightarrow} \widehat{H}^{r+s}(G,-).$$

In the other direction, if we have a natural isomorphism

$$\Phi_\bullet\colon \widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X',-)) \Rightarrow \widehat{H}^{r+s}(G,-),$$

then we can compose with $\Psi_\bullet^{-1}$ to build a natural isomorphism

$$\widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X',-)) \overset{\Phi_\bullet}{\Rightarrow} \widehat{H}^{r+s}(G,-) \overset{\Psi_\bullet^{-1}}{\Rightarrow} \widehat{H}^r(G, \mathrm{Hom}_\mathbb{Z}(X,-)),$$

from which it follows that X and X' are cohomologically equivalent by Proposition 25 (b). ■

Example 32. Take $s \geq 0$. Dimension-shifting iteratively with the short exact sequence

$$0 \to I_G \otimes_\mathbb{Z} A \to \mathbb{Z}[G] \otimes_\mathbb{Z} A \to A \to 0$$

shows that

$$\widehat{H}^{r+s}(G, A) \simeq \widehat{H}^r\left(G, \mathrm{Hom}_\mathbb{Z}(I_G^{\otimes s}, A)\right),$$

and in fact these isomorphisms are natural by the functoriality of boundary morphisms. So the equivalence class of Proposition 31 is represented by $I_G^{\otimes s}$.

Remark 33. We will show in Example 63 that all of these equivalence classes are nonempty.

Example 34. Not all r-encoding modules are $\mathbb{Z}$-torsion-free. For example, if M is an r-encoding module, and A is induced, then $M \oplus A$ is cohomologically equivalent to M, so $M \oplus A$ is an r-encoding module. However, not all induced modules A are $\mathbb{Z}$-torsion-free.

In fact, akin to the classification of natural transformations from Proposition 19, we can show that these encoding maps must be cup products.

Corollary 35. Let G be a finite group, and let $r \in \mathbb{Z}$ be an index. Suppose we have a G-module X and index $i \in \mathbb{Z}$ with a natural transformation

$$\Phi_\bullet \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -).$$

Then there exists $[x] \in \widehat{H}^r(G, X)$ such that $\Phi_\bullet$ is the cup-product map $([x] \cup -)$.

Proof. The point is to set $X' = \mathbb{Z}$ in Proposition 19. For technical reasons, we note that we have a natural isomorphism

$$([1] \cup -) \colon \widehat{H}^{i+r}(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)) \Rightarrow \widehat{H}^{i+r}(G, -)$$

by checking at index 0 and then using Proposition 13. Thus, $\Phi_\bullet$ will induce a natural transformation

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \overset{\Phi_\bullet}{\Rightarrow} \widehat{H}^{i+r}(G, -) \overset{([1]\cup -)^{-1}}{\Rightarrow} \widehat{H}^{i+r}(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)).$$

By Proposition 19, we are promised $[x] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, X))$ such that this composite is $([x] \cup -)$. It follows that $\Phi_\bullet$ is

$$[1] \cup \big([x] \cup -\big) \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -).$$

Associating, this natural isomorphism is the same as $\big(([1] \cup [x]) \cup -\big)$; indeed, fixing a G-module A to plug in, we get the result by noting the commutativity of

$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} A \\ \downarrow & & \downarrow \\ X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} z \otimes y \otimes f & \longmapsto & z \otimes f(y) \\ \downarrow & & \downarrow \\ zy \otimes f & \longmapsto & f(zy) \end{array}$$

and passing through Lemma 12. ∎

Example 36. For $r \geq 0$, we can continue Example 32 to note that standard dimension-shifting arguments give natural isomorphisms

$$\widehat{H}^0\left(G, \mathrm{Hom}_{\mathbb{Z}}(I_G^{\otimes r}, -)\right) \Rightarrow \widehat{H}^r(G, -),$$

so Corollary 35 implies that these isomorphisms are cup products with an element of $\widehat{H}^r(G, I_G^{\otimes r})$. For example, when $r = 0$, we have $[1] \in \widehat{H}^0(G, \mathbb{Z})$; and when $r = 1$, we have $g \mapsto (1 - g)$ in $\widehat{H}^1(G, I_G)$. Observe that we could also see this by inductively dimension-shifting with Corollary 15.

Because cup products are better-behaved than just general natural transformations, we get the following nice statement.

Corollary 37. Let G be a finite group, and let $r \in \mathbb{Z}$ an index. Then an r-encoding module X has $x \in \widehat{H}^r(G, X)$ such that

$$(x \cup -) \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -)$$

is a natural isomorphism for all $i \in \mathbb{Z}$.

Proof. By definition of X, we know that there is some $i \in \mathbb{Z}$ such that we have a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -).$$

Then Corollary 35 tells us that this natural isomorphism arises as $(x \cup -)$ for some $x \in \widehat{H}^r(G, X)$.

To finish, we extend $(x \cup -)$ being a natural isomorphism from a single i to all $i \in \mathbb{Z}$ by using Proposition 13. Indeed, take $F = \mathrm{Hom}_{\mathbb{Z}}(X, -)$ and $F' = \mathrm{id}$ and $X = X$ and $\eta \colon X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, -) \Rightarrow \mathrm{id}$ to be the canonical evaluation maps. This finishes. ■

Remark 38. Taking $X = \mathbb{Z}$ above, we are asserting that, if G is a group such that all G-modules admit period-r cohomology which is natural in some sense at a single index i, then this periodicity extends to all indices and arises from a cup product with an element of $\widehat{H}^r(G, \mathbb{Z})$.

Observe that the naturality in the isomorphisms is important: letting $G \coloneqq \mathbb{Z}/p\mathbb{Z}$ act on $A \coloneqq \mathbb{Z}/p\mathbb{Z}$ trivially,

$$\widehat{H}^{-1}(G, A) = \frac{\mathbb{Z}/p\mathbb{Z}}{0} \simeq \widehat{H}^0(G, A),$$

but this does not extend to all G-modules. For example,

$$\widehat{H}^{-1}(G, \mathbb{Z}) = 0 \ncong \frac{\mathbb{Z}}{p\mathbb{Z}} = \widehat{H}^0(G, \mathbb{Z}).$$

The element defined in Corollary 37 is so special that we will give it a name.

Definition 39. Let G be a finite group and X an r-encoding module. An element $x \in \widehat{H}^r(G, X)$ as constructed in Corollary 37 is the *encoding element*.

It will turn out that encoding elements are unique, though they will be almost unique.

1.6 Encoding Is Unique

Fix an r-encoding module X. As a brief intermission, we will show that there is essentially one way to do the encoding

$$\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(G, -).$$

Namely, we know from Corollary 35, that this natural isomorphism must come from a cup-product with an element $x \in \widehat{H}^r(G, X)$, so we might wonder how unique this element x is. The answer to this, roughly speaking, will be that $\widehat{H}^r(G, X)$ is cyclic of order $\#G$ generated by x. In the process, we will be able to compute all the cohomology groups $\widehat{H}^i(G, X)$.

Anyway, the main idea will be to use the following duality result.

Proposition 40 ([CE56, Corollary XII.6.5])**.** Let G be a finite group and A be any G-module. Then the cup-product pairing induces an isomorphism

$$\widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G, A), \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})\right)$$

for all $i \in \mathbb{Z}$. Indeed, this is a duality upon embedding $\widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})$ into $\mathbb{Q}/\mathbb{Z}$.

And here is our computation.

Corollary 41. Let G be a finite group and X an r-encoding module. Picking up an encoding element $x \in \widehat{H}^r(G,X)$, $\widehat{H}^r(G,X)$ is cyclic of order $\#G$ generated by x.

Proof. For brevity, set $n \coloneqq \#G$. By Corollary 37, we have the isomorphism

$$x \cup -\colon \widehat{H}^{-p-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \to \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) = \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

In particular, $\widehat{H}^{-p-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z})) \simeq \mathbb{Z}/n\mathbb{Z}$, generated by some element $x^\vee$ such that $x \cup x^\vee = [1/n]$.

Now, we apply Proposition 40 to say that the cup-product pairing induces an isomorphism

$$\tfrac{1}{n}\mathbb{Z}/n\mathbb{Z} \simeq \widehat{H}^{-p-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^p(G,X), \widehat{H}^{-1}(G,\mathbb{Q}/\mathbb{Z})\right) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^p(G,X), \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}\right).$$

Because $\widehat{H}^p(G,X)$ is n-torsion, homomorphisms $\widehat{H}^2(G,X) \to \mathbb{Q}/\mathbb{Z}$ must have image in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so in fact the rightmost group is the dual of $\widehat{H}^p(G,X)$. Because an abelian group is isomorphic to its dual, we see that $\widehat{H}^p(G,X)$ is in fact cyclic of order n.

It remains to show that x is a generator; for this, we show that x has order at least n, which will be enough because $H^2(G,X)$ is cyclic of order n. Well, if we have $k \in \mathbb{Z}$ such that $kx = 0$, then

$$[k/n] = k\big(x \cup x^\vee\big) = kx \cup x^\vee = [0] \cup x^\vee = [0]$$

in $\widehat{H}^{-1}(G,\mathbb{Q}/\mathbb{Z}) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so $n \mid k$. This finishes. ∎

Remark 42. Conversely, if $x \in \widehat{H}^r(G,X)$ is any generator, then

$$(x \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X,-)) \Rightarrow \widehat{H}^{i+r}(G,-)$$

is a natural isomorphism. Indeed, certainly some generator $x_0 \in \widehat{H}^p(G,X)$ conjured from Corollary 37 suffices, but then $x = kx_0$ for some $k \in (\mathbb{Z}/\#G\mathbb{Z})^\times$, so we have the equality

$$(x \cup -) = \big((kx_0) \cup -\big) = k(x_0 \cup -)$$

of natural transformations. But multiplication by k is a natural isomorphism $\widehat{H}^\bullet(G,-) \Rightarrow \widehat{H}^\bullet(G,-)$ because these cohomology groups are $\#G$-torsion, so we conclude $(x \cup -) = k(x_0 \cup -)$ is a natural isomorphism.

And here is our uniqueness result.

Corollary 43. Let G be a finite group, and let X be a finitely generated r-encoding module. Then, given $i \in \mathbb{Z}$ and two natural isomorphisms

$$\Phi_\bullet, \Phi'_\bullet\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X,-)) \Rightarrow \widehat{H}^{i+r}(G,-),$$

there exists a unique $k \in (\mathbb{Z}/\#G\mathbb{Z})^\times$ such that $\Phi'_\bullet = k\Phi_\bullet$.

Proof. Note that we are allowed to interpret $k \pmod n$ because these cohomology groups are $\#G$-torsion, so $\#G \cdot \Phi_\bullet = 0$.

Anyway, by Corollary 35, we know that there are $x, x' \in \widehat{H}^r(G,X)$ such that

$$\Phi_\bullet = (x \cup -) \qquad \text{and} \qquad \Phi'_\bullet = (x' \cup -).$$

However, by Corollary 41, we see that $\widehat{H}^r(G,X)$ is cyclic generated by x of order $\#G$, so we can write $x' = kx$ for a unique $k \in \mathbb{Z}/\#G\mathbb{Z}$.

It remains to show that $\Phi'_\bullet = k\Phi_\bullet$. Well, for any G-module A and $c \in \widehat{H}^i(G, \operatorname{Hom}_\mathbb{Z}(X,A))$, we write

$$\Phi'_A(c) = x' \cup c = kx \cup c = k(x \cup c) = k\Phi_A(c).$$

It follows that $\Phi'_\bullet = k\Phi_\bullet$. ∎

1.7 The Dual Element

In the theory of periodic cohomology (e.g., see [CE56, Section XII.11]), it is helpful to phrase the theory in terms of having some elements $x \in \widehat{H}^r(G,\mathbb{Z})$ and $y \in \widehat{H}^{-r}(G,\mathbb{Z})$ such that

$$x \cup y = [1] \in \widehat{H}^0(G,\mathbb{Z}).$$

In contrast, given an r-encoding module X, we cannot hope to have $x \in \widehat{H}^r(G,X)$ and $y \in \widehat{H}^{-r}(G,X)$ with $x \cup y \in \widehat{H}^0(G,\mathbb{Z})$ because there is no obvious map $X \otimes_\mathbb{Z} X \to \mathbb{Z}$. To remedy this, we observe that this is a canonical map

$$X \otimes_\mathbb{Z} \operatorname{Hom}_\mathbb{Z}(X,\mathbb{Z}) \to \mathbb{Z}.$$

This idea gives the following result.

Proposition 44. Let G be a finite group, and let X be a G-module and $r \in \mathbb{Z}$ be an index. The following are equivalent.

(a) X is an r-encoding module.

(b) There are $x \in \widehat{H}^r(G,X)$ and $x^\vee \in \widehat{H}^{-r}(G, \operatorname{Hom}_\mathbb{Z}(X,\mathbb{Z}))$ such that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G,\mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \operatorname{Hom}_\mathbb{Z}(X,X)).$$

Proof. For brevity, set $n := \#G$.

We start by showing (a) implies (b). By Corollary 37, we can find an encoding element $x \in \widehat{H}^r(G,X)$ yielding the isomorphism

$$(x \cup -) \colon \widehat{H}^{-r}(G, \operatorname{Hom}_\mathbb{Z}(X,\mathbb{Z})) \to \widehat{H}^0(G,\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

As such, we can find a unique $x^\vee \in \widehat{H}^{-r}(G, \operatorname{Hom}_\mathbb{Z}(X,\mathbb{Z}))$ such that $x \cup x^\vee = [1]$. It remains to show that $x^\vee \cup x = [\mathrm{id}_X]$.

Note that $(x \cup -)$ and $(x^\vee \cup -)$ induce morphisms

$$\begin{aligned} (x \cup -) \colon \widehat{H}^0(G, \operatorname{Hom}_\mathbb{Z}(X,X)) &\to \widehat{H}^r(G,X) \\ (x^\vee \cup -) \colon \widehat{H}^r(G,X) &\to \widehat{H}^0(G, \operatorname{Hom}_\mathbb{Z}(X,X)) \end{aligned}$$

We claim that these are inverse. Because $(x \cup -)$ is already an isomorphism, it suffices to show that we have an inverse on one side. Well, $\widehat{H}^r(G,X)$ is cyclic generated by x by Corollary 41, so it suffices to note that any $kx \in \widehat{H}^r(G,X)$ has

$$\big((x \cup -) \circ (x^\vee \cup -)\big)(kx) = x \cup (x^\vee \cup kx) \overset{*}{=} (x \cup x^\vee) \cup kx = [1] \cup kx = kx.$$

Notably, $\overset{*}{=}$ has used Lemma 12, noting that the square

$$\begin{array}{ccc} X \otimes_\mathbb{Z} \operatorname{Hom}_\mathbb{Z}(X,\mathbb{Z}) \otimes_\mathbb{Z} X & \longrightarrow & X \otimes_\mathbb{Z} \operatorname{Hom}_\mathbb{Z}(X,X) \\ \downarrow & & \downarrow \\ \mathbb{Z} \otimes_\mathbb{Z} X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} x_1 \otimes f \otimes x_2 & \longmapsto & x_1 \otimes (y \mapsto f(y)x_2) \\ \downarrow & & \downarrow \\ f(x_1) \otimes x_2 & \longmapsto & f(x_1)x_2 \end{array}$$

commutes. Anyway, we now see that we have inverse morphisms, so

$$x \cup [\mathrm{id}_X] = \mathrm{id}_X(x) = x$$

implies that $x^\vee \cup x = [\mathrm{id}_X]$, finishing.

We now show (b) implies (a). Let $i \in \mathbb{Z}$ be any index. The main point is that

$$\begin{aligned} (x \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) &\Rightarrow \quad \widehat{H}^{i+r}(G, -) \\ (x^\vee \cup -)\colon \quad \widehat{H}^{i+r}(G, -) \quad &\Rightarrow \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \end{aligned}$$

ought to be inverse natural transformations. More formally, we want to show $(x \cup -)$ is a natural isomorphism, for which we note naturality follows from Lemma 11.

Thus, given a G-module A, it remains to show that its component morphisms

$$(x \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{i+r}(G, A).$$

In fact, we claim that the corresponding map

$$(x^\vee \cup -)\colon \widehat{H}^{i+r}(G, A) \to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$$

is the inverse morphism. We have two checks.

- In one direction, we note that any $a \in \widehat{H}^{i+r}(G, A)$ has

$$\big((x \cup -) \circ (x^\vee \cup -)\big)(a) = x \cup (x^\vee \cup a) \overset{*}{=} (x \cup x^\vee) \cup a = [1] \cup a = a$$

  where $\overset{*}{=}$ holds by using Lemma 12 on the following commuting square.

$$\begin{array}{ccc} X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A & \longrightarrow & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \\ \downarrow & & \downarrow \\ \mathbb{Z} \otimes_{\mathbb{Z}} A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} y \otimes f \otimes b & \longmapsto & y \otimes (y_0 \mapsto f(y_0)b) \\ \downarrow & & \downarrow \\ f(y) \otimes b & \longmapsto & f(y)b \end{array}$$

- In the other direction, we note that $a^\vee \in \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ will have

$$\big((x^\vee \cup -) \circ (x \cup -)\big)(a^\vee) = x^\vee \cup (x \cup a) \overset{*}{=} (x^\vee \cup x) \cup a = [\mathrm{id}_X] \cup a = \mathrm{id}_X(a) = a,$$

  where $\overset{*}{=}$ holds by using Lemma 12 on the following commuting square.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) & \to & \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(X, X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, A) \end{array} \qquad \begin{array}{ccc} f \otimes y \otimes g & \longrightarrow & f \otimes g(y) \\ \downarrow & & \downarrow \\ (y_0 \mapsto f(y_0)y) \otimes g & \to & \big(y_0 \mapsto f(y_0)g(y)\big) \end{array}$$

This finishes the proof. ∎

We quickly note that the proof of Proposition 44 actually managed to conjure the inverse natural transformation to $(x \cup -)$.

Corollary 45. Let G be a finite group, and let X be an r-encoding module. Constructing $x \in \widehat{H}^r(G, X)$ and $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ from Proposition 44, the natural transformations

$$\begin{aligned} (x \cup -)\colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) &\to \quad \widehat{H}^{i+r}(G, -) \\ (x^\vee \cup -)\colon \quad \widehat{H}^{i+r}(G, -) \quad &\to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \end{aligned}$$

are inverse for each $i \in \mathbb{Z}$.

Proof. In the proof, we showed that, given a G-module A, the morphisms

$$(x^\vee \cup -)\colon \widehat{H}^{i+r}(G, A) \to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$$

provide the inverses for $(x \cup -)$. This is what we wanted. ■

As such, we will give the element $x^\vee$ a name.

Definition 46. Let G be a finite group and X an r-encoding module. Element $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ as constructed in Proposition 44 are *dual elements*.

Here is another amusing corollary we get from this.

Corollary 47. Let G be a finite group, and let X be an r-encoding module with encoding element $x \in \widehat{H}^r(G, X)$. Then, for any subgroup $H \subseteq G$, X is an r-encoding H-module so that any index $i \in \mathbb{Z}$ has the natural isomorphism

$$(\mathrm{Res}\, x \cup -)\colon \widehat{H}^i(H, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{i+r}(H, -).$$

Proof. The point is that restriction commutes with cup products, so we may use Proposition 44. Indeed, we are given $x \in \widehat{H}^r(G, X)$ and $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ such that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)).$$

Applying restriction to H everywhere, we see

$$\begin{aligned} \mathrm{Res}\, x \cup \mathrm{Res}\, x^\vee &= \mathrm{Res}(x \cup x^\vee) \\ &= \mathrm{Res}([1]) \\ &= [1] \in \widehat{H}^0(H, \mathbb{Z}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Res}\, x^\vee \cup \mathrm{Res}\, x &= \mathrm{Res}(x^\vee \cup x) \\ &= \mathrm{Res}([\mathrm{id}_X]) \\ &= [\mathrm{id}_X] \in \widehat{H}^0(H, \mathrm{Hom}_{\mathbb{Z}}(X, X)), \end{aligned}$$

which is enough by Proposition 44 to show that X is an r-encoding H-module. The remarks from Corollary 45 explain why the needed isomorphism is given by $(\mathrm{Res}\, x \cup -)$. ■

Remark 48. Essentially the same proof should hold for inflation.

Example 49. It is not true that, if X is an r-encoding G_p-module for all Sylow p-subgroups $G_p \subseteq G$, then X is an r-encoding G-module. Indeed, take $X = \mathbb{Z}$ and $G = S_3$: all Sylow p-subgroups of S_3 are cyclic, so $\mathbb{Z}$ is a 2-encoding module for all these subgroups. However, S_3 is not cyclic, so

$$\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \simeq \widehat{H}^{-2}(G, \mathbb{Z}) \simeq S_3/[S_3, S_3] \ncong \mathbb{Z}/6\mathbb{Z} = \widehat{H}^0(G, \mathbb{Z}).$$

1.8 Encoding by Tensoring

It turns out that we can also encode "on the other side," in the following sense.

Theorem 50. Let G be a finite group, and let X be an r-encoding module with encoding element $x \in \widehat{H}^r(G, X)$. Then the cup products

$$(- \cup x) \colon \widehat{H}^i(G, -) \Rightarrow \widehat{H}^{i+r}(G, - \otimes_{\mathbb{Z}} X)$$

assemble into a natural isomorphism for any $i \in \mathbb{Z}$.

Proof. That we have a natural transformation follows from the naturality of cup products. Thus, it suffices to pick up a G-module A and show that the component morphisms

$$(- \cup x) \colon \widehat{H}^i(G, A) \to \widehat{H}^{i+r}(G, A \otimes_{\mathbb{Z}} X)$$

is an isomorphism. For this, we pick up a dual element $x^\vee \in \widehat{H}^{i+r}(G, \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ so that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, X)).$$

As such, we claim that the morphisms

$$\begin{aligned} (- \cup x) &\colon \widehat{H}^i(G, A) \to \widehat{H}^{i+r}(G, A \otimes_{\mathbb{Z}} X) \\ (- \cup x^\vee) &\colon \widehat{H}^{i+r}(G, A \otimes_{\mathbb{Z}} X) \to \widehat{H}^i(G, A) \end{aligned}$$

are inverse, which will finish; here $(- \cup x^\vee)$ is using the following evaluation map.

$$\begin{array}{ccccc} (A \otimes_{\mathbb{Z}} X) & \otimes_{\mathbb{Z}} & \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) & \to & A \\ b & \otimes & y & \mapsto & f(y)b \end{array}$$

We have now checks for our morphisms to be inverse.

- On one hand, pick up $a \in \widehat{H}^i(G, A)$. Then we can use Lemma 12 on the commuting square

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) & \longrightarrow & A \otimes_{\mathbb{Z}} \mathbb{Z} \\ \| & & \downarrow \\ A \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} b \otimes y \otimes f & \longmapsto & b \otimes f(y) \\ \| & & \downarrow \\ b \otimes y \otimes f & \longmapsto & f(y)b \end{array}$$

  to evaluate

$$\big((- \cup x^\vee) \circ (- \cup x)\big)(a) = (a \cup x) \cup x^\vee = a \cup (x \cup x^\vee) = a \cup [1] = a.$$

- On the other hand, pick up $a^\vee \in \widehat{H}^{i+r}(G, A \otimes_{\mathbb{Z}} X)$. Then we can use Lemma 12 on the commuting square

$$\begin{array}{ccc} (A \otimes_{\mathbb{Z}} X) \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X & \longrightarrow & A \otimes_{\mathbb{Z}} X \\ \downarrow & & \| \\ (A \otimes_{\mathbb{Z}} X) \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, X) & \longrightarrow & A \otimes_{\mathbb{Z}} X \end{array} \qquad \begin{array}{ccc} b \otimes y \otimes f \otimes y' & \longmapsto & f(y)b \otimes y' \\ \downarrow & & \| \\ b \otimes y \otimes (y_0 \mapsto f(y_0)y') & \longmapsto & b \otimes f(y)y' \end{array}$$

  to evaluate

$$\big((- \cup x) \circ (- \cup x^\vee)\big)(a) = (a \cup x^\vee) \cup x = a^\vee \cup (x^\vee \cup x) = a^\vee \cup [\mathrm{id}_X] = a^\vee.$$

The above checks complete the proof. ∎

Remark 51. One could rebuild the theory we have in the previous sections, in particular showing that all natural isomorphisms of the form

$$\widehat{H}^i(G, - \otimes_{\mathbb{Z}} A) \Rightarrow \widehat{H}^{i+r}(G, - \otimes_{\mathbb{Z}} B)$$

are cup products from an element of $\widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(A, B))$. Thus, for some G-module X, natural isomorphisms

$$\widehat{H}^i(G, -) \Rightarrow \widehat{H}^{i+r}(G, - \otimes_{\mathbb{Z}} X)$$

promise $x \in \widehat{H}^r(G, X)$ (from using $A = \mathbb{Z}$ and $B = X$ with $i = 0$) and $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ (from using $A = X$ and $B = \mathbb{Z}$ with $i = -r$) which we can evaluate to have

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)).$$

Namely, X is an r-encoding module!

Corollary 52. Let G be a finite group, and let X be an r-encoding module with encoding element $x \in \widehat{H}^r(G, X)$. Then the cup products

$$(- \cup x) \colon \widehat{H}^i(G, \mathbb{Z}) \to \widehat{H}^{i+r}(G, X)$$

are isomorphisms for all $i \in \mathbb{Z}$.

Proof. Plug in $\mathbb{Z}$ into Theorem 50. ∎

1.9 Torsion-Free Encoding

In the theory of periodic cohomology, one can show that it is enough to check the single cohomology group

$$\widehat{H}^r(G, \mathbb{Z}) \cong \mathbb{Z}/\#G\mathbb{Z}$$

for some index $r \in \mathbb{Z}$ to get r-periodic cohomology. We might hope that something similar is true for our r-encoding modules. To this end, we pick up the following "integral" duality statement.

Proposition 53. Let G be a finite group, and let X be a $\mathbb{Z}$-free G-module. Then the cup-product pairing induces an isomorphism

$$\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G, X), \widehat{H}^0(G, \mathbb{Z})\right)$$

for all $i \in \mathbb{Z}$. Indeed, this is a duality upon identifying $\widehat{H}^0(G, \mathbb{Z})$ with $\frac{1}{\#G}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$.

Proof. This proof is analogous to [CE56, Theorem XII.6.6]. The key to the proof is the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{1.6}$$

The main point is that X being $\mathbb{Z}$-free implies that X is projective (as an abelian group), so we can apply $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ to get out the short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \to 0. \tag{1.7}$$

Now, note that the multiplication-by-n endomorphism on $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})$ is an isomorphism (namely, $\mathbb{Q}$ is a divisible abelian group), so the same will be true of $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}))$ for any $i \in \mathbb{Z}$. However, these cohomology groups must be $\#G$-torsion, so in fact $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})) = 0$ for all $i \in \mathbb{Z}$.

Similarly, we note that we can hit (1.7) with the functor $- \otimes_{\mathbb{Z}} X$ to get another short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} X \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X \to 0. \tag{1.8}$$

Notably, this is exact because X is $\mathbb{Z}$-free and hence flat as a $\mathbb{Z}$-module. Now, $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} X$ is still a divisible abelian group, so again $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})) = 0$ for all $i \in \mathbb{Z}$.

The rest of the proof is tracking boundary morphisms around. Fix some $i \in \mathbb{Z}$. Note (1.6) and (1.7) and (1.8) induce boundary isomorphisms

$$\begin{aligned}
\delta &: \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \to \widehat{H}^0(G, \mathbb{Z}) \\
\delta_h &: \widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \\
\delta_t &: \widehat{H}^{-1}(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X) \to \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X).
\end{aligned}$$

We also note that we have a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} X & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_{\mathbb{Z}}} & & \downarrow{\scriptstyle \eta_{\mathbb{Q}}} & & \downarrow{\scriptstyle \eta_{\mathbb{Q}/\mathbb{Z}}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where the $\eta_\bullet$ are evaluation maps. Now, Proposition 40 tells us that

$$\begin{array}{ccc}
\widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G, X), \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})\right) \\
a & \mapsto & (b \mapsto \eta_{\mathbb{Q}/\mathbb{Z}}(a \cup b))
\end{array}$$

is an isomorphism. Composing this with various other isomorphisms, we can build the isomorphism

$$\begin{array}{ccccccc}
\widehat{H}^i(G, X^\vee) & \to & \widehat{H}^{i-1}(G, X^*) & \to & \mathrm{Hom}\left(\widehat{H}^{-i}(G, X), \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})\right) & \to & \mathrm{Hom}\left(\widehat{H}^{-i}(G, X), \widehat{H}^0(G, \mathbb{Q}/\mathbb{Z})\right) \\
a & \mapsto & \delta_h^{-1} a & \mapsto & \left(b \mapsto \eta_{\mathbb{Q}/\mathbb{Z}}(\delta_h^{-1} a \cup b)\right) & \mapsto & \left(b \mapsto \delta\eta_{\mathbb{Q}/\mathbb{Z}}(\delta_h^{-1} a \cup b)\right)
\end{array}$$

where $X^\vee \coloneqq \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $X^* \coloneqq \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$, for brevity. This gives an isomorphism between the desired objects, but to prove the result we need to show that the above map is $a \mapsto (b \mapsto \eta_{\mathbb{Z}}(a \cup b))$. Well, given $a \in \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ and $b \in \widehat{H}^{-i}(G, X)$, properties of the boundary morphisms tells us

$$\begin{aligned}
\delta\eta_{\mathbb{Q}/\mathbb{Z}}\left(\delta_h^{-1} a \cup b\right) &= \eta_{\mathbb{Z}}\delta_t\left(\delta_h^{-1} a \cup b\right) \\
&= \eta_{\mathbb{Z}}\left(\delta_h \delta_h^{-1} a \cup b\right) \\
&= \eta_{\mathbb{Z}}(a \cup b),
\end{aligned}$$

which is what we wanted. ∎

Remark 54. The hypothesis that X be $\mathbb{Z}$-free is necessary: the statement is false for $X = \mathbb{Z}/\#G\mathbb{Z}$ and $i = 0$, for example.

And here is our result.

Proposition 55. Let G be a finite group, and let X be a $\mathbb{Z}$-free G-module. The following are equivalent.

(a) X is an r-encoding module.

(b) $\widehat{H}^r(G, X) \cong \mathbb{Z}/\#G\mathbb{Z}$ and $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \cong \mathbb{Z}/\#G\mathbb{Z}$.

(c) $\widehat{H}^r(G, X) \cong \mathbb{Z}/\#G\mathbb{Z}$ and $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ is cyclic.

Proof. For brevity, set $n \coloneqq \#G$. That (a) implies (b) is not hard: Corollary 41 tells us that $\widehat{H}^r(G, X) \cong \mathbb{Z}/n\mathbb{Z}$, and then being an r-encoding module promises an isomorphism

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \simeq \widehat{H}^r(G, X) \cong \mathbb{Z}/n\mathbb{Z}.$$

Continuing, we see that (b) implies (c) easily. Thus, the interesting direction is showing that (c) implies (a).

For this, we use Proposition 53 and Proposition 44. We are given $x \in \widehat{H}^r(G, X)$ of order n, so we note that there is a morphism

$$\widehat{H}^r(G, X) \simeq \mathbb{Z}/n\mathbb{Z} = \widehat{H}^0(G, \mathbb{Z})$$

sending x to $[1]$. Thus, Proposition 53 grants $x^\vee \in \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ such that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}).$$

It remains to check that $x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$. This is more difficult.

For this, we will show that

$$(x \cup -) \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \to \widehat{H}^r(G, X) \tag{1.9}$$

is injective while showing that $x^\vee \cup x$ and id_X have the same image under $(x \cup -)$.

Indeed, on one hand, let A be a G-module (which we will set to be X shortly), and we claim that the composite

$$\widehat{H}^r(G, A) \overset{x^\vee \cup -}{\to} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \overset{x \cup -}{\to} \widehat{H}^r(G, A)$$

is the identity. Then the commutativity of the diagram

$$\begin{array}{ccc} X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A & \longrightarrow & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \\ \downarrow & & \downarrow \\ \mathbb{Z} \otimes_{\mathbb{Z}} A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} x_0 \otimes f \otimes a_0 & \longmapsto & x_0 \otimes \big(y \mapsto f(y)a_0\big) \\ \downarrow & & \downarrow \\ f(x_0) \otimes a_0 & \longmapsto & f(x_0)a_0 \end{array}$$

allows us to compute, for any $a \in \widehat{H}^p(G, A)$,

$$x \cup x^\vee \cup a = [1] \cup a = a, \tag{1.10}$$

as desired.

Now, taking $A = X$, we note

$$x \cup [\mathrm{id}_X] = x \in \widehat{H}^r(G, X). \tag{1.11}$$

So we will be done once we show that (1.9)is injective. Well, note $[\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ has order n: if $k[\mathrm{id}_X] = 0$, then $0 = k(x \cup [\mathrm{id}_X]) = kx$, so $n \mid k$. Because $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ is cyclic (by hypothesis!) and n-torsion, we conclude that in fact $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ is cyclic of order n generated by $[\mathrm{id}_X]$. Thus, we note that there is a unique isomorphism

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \cong \mathbb{Z}/n\mathbb{Z} \cong \widehat{H}^r(G, X)$$

sending $[\mathrm{id}_X]$ to 1 to x, so this isomorphism must be $(x \cup -)$ by (1.11); in particular, $(x \cup -)$ is injective.

Finishing up, comparing the injectivity of $(x \cup -)$ with (1.10) and (1.11) forces us to conclude that $x^\vee \cup x = [\mathrm{id}_X]$. This finishes. ∎

Example 56. The $\mathbb{Z}$-free condition is necessary. As in Remark 38, let $G \coloneqq \mathbb{Z}/p\mathbb{Z}$ act on $X \coloneqq \mathbb{Z}/p\mathbb{Z}$ trivially. Then

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \simeq \widehat{H}^0(G, X) \cong \widehat{H}^{-1}(G, X) = \mathbb{Z}/p\mathbb{Z}.$$

However, X is not a (-1)-encoding module because

$$\widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \simeq \widehat{H}^1(G, 0) = 0 \ncong \mathbb{Z}/p\mathbb{Z} = \widehat{H}^0(G, \mathbb{Z}).$$

Remark 57. As we will discuss later in Remark 65, the requirement that X be $\mathbb{Z}$-free is not too serious.

Example 58. To see that $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X))$ being cyclic is necessary, we use the example from Example 28. Let $G = \langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ act on $X := \mathbb{Z}[i] = \mathbb{Z} \oplus \mathbb{Z}i$ by conjugation. Then

$$\widehat{H}^0(G, X) \simeq \widehat{H}^0(G, \mathbb{Z}) \oplus \widehat{H}^0(G, \mathbb{Z}i) \simeq \mathbb{Z}/2\mathbb{Z},$$

but

$$\begin{aligned}\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})) \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}i)) \\ \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}i, \mathbb{Z})) \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}i, \mathbb{Z}i))\end{aligned}$$

comes out to $\mathbb{Z}/2\mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z}/2\mathbb{Z}$. Thus,

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \ncong \widehat{H}^0(G, X),$$

so X is not a 0-encoding module even though X is $\mathbb{Z}$-free and $\widehat{H}^0(G, X) \cong \mathbb{Z}/\#G\mathbb{Z}$.

In some sense, the issue with the above example is that we could decompose our G-module into $A \oplus B$ when in fact there is no reason to talk about these sorts of G-modules as encoding modules.

Corollary 59. Let G be a finite p-group. If $A \oplus B$ is a finitely generated $\mathbb{Z}$-free r-encoding module, then one of A or B is an r-encoding module and the other is cohomologically equivalent to 0.

Proof. This follows quickly from the check in Proposition 55. On one hand,

$$\begin{aligned}\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A \oplus B, A \oplus B)) \simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, A)) \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, B)) \\ \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(B, A)) \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(B, B))\end{aligned}$$

tells us that both $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, A))$ and $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(B, B))$ are both cyclic because $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A \oplus B, A \oplus B))$ is.

On the other hand, we note

$$\widehat{H}^r(G, A) \oplus \widehat{H}^r(G, B) \simeq \widehat{H}^r(G, A \oplus B) \cong \mathbb{Z}/\#G\mathbb{Z},$$

so we are forced to have $\widehat{H}^r(G, A) \cong \mathbb{Z}/\#G\mathbb{Z}$ or $\widehat{H}^r(G, B) \cong \mathbb{Z}/\#G\mathbb{Z}$ because G is a finite p-group.

Thus, one of A or B is an r-encoding module; without loss of generality, say that A is. It remains to show that B is cohomologically equivalent to 0. Well, we have

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, A)) \cong \widehat{H}^r(G, A) = \mathbb{Z}/\#G\mathbb{Z}$$

because A is an r-encoding module, so the embedding

$$\underbrace{\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, A))}_{\mathbb{Z}/\#G\mathbb{Z}} \oplus \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(B, B)) \subseteq \underbrace{\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A \oplus B, A \oplus B))}_{\mathbb{Z}/\#G\mathbb{Z}}$$

forces $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(B, B)) = 0$. As such, Corollary 27 finishes. ∎

Remark 60. It is conceivable that Corollary 59 is true without requiring $A \oplus B$ to be $\mathbb{Z}$-free nor G to be a p-group.

1.10 New Encoding Modules From Old

The goal of this section is to build encoding modules up from smaller ones.

Proposition 61. Let G be a finite group. Given an r-encoding module A and an s-encoding module B, the G-module $A \otimes_{\mathbb{Z}} B$ is an $(r+s)$-encoding module.

Proof. By Corollary 37, we have natural isomorphisms

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A, -)) \simeq \widehat{H}^r(G, -) \qquad \text{and} \qquad \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(B, -)) \simeq \widehat{H}^{r+s}(G, -).$$

Whiskering, we have a natural isomorphism

$$\begin{aligned}\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, -)) &\simeq \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}}(B, -))) \\ &\simeq \widehat{H}^r(G, \mathrm{Hom}_{\mathbb{Z}}(B, -)) \\ &\simeq \widehat{H}^{r+s}(G, -),\end{aligned}$$

which is what we wanted. ∎

Proposition 62. Let G be a finite group, and let X be an r-encoding module. Then $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ is a $(-r)$-encoding module.

Proof. We use Proposition 44. For brevity, we set $X^\vee \coloneqq \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $X^{\vee\vee} \coloneqq \mathrm{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$. Observe that there is a (canonical) map $\varphi\colon X \to X^{\vee\vee}$ by

$$\varphi\colon f \mapsto f \circ \varphi.$$

By Proposition 44, we may find $x \in \widehat{H}^r(G, X)$ and $x^\vee \in \widehat{H}^{-r}(G, X^\vee)$ such that

$$x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}) \qquad \text{and} \qquad x^\vee \cup x = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)).$$

As such, we set $y \coloneqq x^\vee$ and $y^\vee \coloneqq (-1)^r \varphi(x)$. The commutative diagram

$$\begin{array}{ccc} X \otimes_{\mathbb{Z}} X^\vee & \longrightarrow & \mathbb{Z} \\ {\scriptstyle \varphi \otimes \mathrm{id}}\downarrow & & \| \\ X^{\vee\vee} \otimes_{\mathbb{Z}} X^\vee & \longrightarrow & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} x \otimes f & \longmapsto & f(x) \\ \downarrow & & \| \\ (g \mapsto g(x)) \otimes f & \longmapsto & f(x) \end{array}$$

tells us that we may evaluate

$$\varphi(x) \cup x^\vee = x \cup x^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z}),$$

so $y \cup y^\vee = [1] \in \widehat{H}^0(G, \mathbb{Z})$ after being careful with signs.

On the other hand, we set $A = B = X$ for clarity and define $\psi\colon \mathrm{Hom}_{\mathbb{Z}}(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(B^*, A^*)$ by

$$\psi(f)\colon g \mapsto (g \circ f),$$

yielding the commutative diagram

$$\begin{array}{ccc} A^\vee \otimes_{\mathbb{Z}} B & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(A, B) \\ {\scriptstyle \mathrm{id} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ A^\vee \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(B^\vee, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(B^\vee, A^\vee) \end{array} \qquad \begin{array}{ccc} f \otimes b & \longmapsto & \big(a \mapsto f(a)b\big) \\ \downarrow & & \downarrow \\ f \otimes \big(g \mapsto g(b)\big) & \longmapsto & \big(g \mapsto g(b)f\big) \end{array}$$

which tells us that we may evaluate

$$x^\vee \cup \varphi(x) = \psi(x^\vee \cup x) = \psi([\mathrm{id}_X]) = [\mathrm{id}_{X^\vee}] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X^\vee, X^\vee)),$$

so $y^\vee \cup y = [\mathrm{id}_{X^\vee}]$ after being careful with signs. This completes the proof. ∎

Example 63. Example 32 established that $I_G^{\otimes r}$ is an r-encoding module for $r \geq 0$. As such, $\mathrm{Hom}_{\mathbb{Z}}\left(I_G^{\otimes r}, \mathbb{Z}\right)$ is a $(-r)$-encoding module for $-r \leq 0$. Thus, we have established existence for r-encoding modules for all $r \in \mathbb{Z}$.

Corollary 64. Let G be a finite group, and let X be a finitely generated r-encoding module. Letting X_t denote the $\mathbb{Z}$-torsion subgroup of X, we have that X_t is a G-submodule of X, and X/X_t is an r-encoding module.

Proof. To see that X_t is a G-submodule, we note that any $x \in X_t$ has some $k \in \mathbb{Z}$ such that $kx = 0$, so any $g \in G$ will have

$$k \cdot gx = g(kx) = g \cdot 0 = 0.$$

Thus, $X_t \subseteq X$ is preserved by G.

It remains to show that $X_f \coloneqq X/X_t$ is an r-encoding module. To begin, we claim that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X_f, \mathbb{Z}), \mathbb{Z}) \tag{1.12}$$

as G-modules; by Proposition 62, this will imply that $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X_f, \mathbb{Z}), \mathbb{Z})$ is an r-encoding module. To see this, we note that the short exact sequence

$$0 \to X_t \to X \to X_f \to 0$$

becomes the left exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X_f, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(X_t, \mathbb{Z}).$$

However, $\mathrm{Hom}_{\mathbb{Z}}(X_t, \mathbb{Z}) = 0$ because X_t is $\mathbb{Z}$-torsion, so the above left exact sequence witnesses the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(X_f, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ again yields (1.12).

To finish, we note that

$$\varphi \colon X_f \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X_f, \mathbb{Z}), \mathbb{Z})$$

by $x \mapsto (f \mapsto f(x))$ is a G-module morphism and isomorphism of abelian groups because X_f is torsion-free and finitely generated. Thus, φ is an isomorphism of G-modules, implying that X_f is an r-encoding module. ∎

Remark 65. Even though Example 34 asserts that not all p-encoding modules X are $\mathbb{Z}$-torsion-free, Corollary 64 explains that we can canonically obtain a $\mathbb{Z}$-torsion-free p-encoding module from X in the form of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}), \mathbb{Z}) \cong X/X_t$.

Proposition 66. Let G be a finite group, and let

$$0 \to X' \to M \to X \to 0$$

be a $\mathbb{Z}$-split short exact sequence such that M is an induced G-module. Then X is an r-encoding module if and only if X' is an $(r+1)$-encoding module.

Proof. Given a G-module A, we recall that $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ is a shiftable functor by Lemma 4, so $\mathrm{Hom}_{\mathbb{Z}}(M, A)$ is induced. Now, because the short exact sequence is $\mathbb{Z}$-split, we have the short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, A) \to \mathrm{Hom}_{\mathbb{Z}}(M, A) \to \mathrm{Hom}_{\mathbb{Z}}(X', A) \to 0$$

which gives the isomorphism

$$\delta_A \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \to \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$$

because $\mathrm{Hom}_{\mathbb{Z}}(M, A)$ is induced. In fact, the δ_A make a natural isomorphism $\delta_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)) \Rightarrow \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, -))$: given a G-module morphism $f \colon A \to B$, the morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M, A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X', A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X, B) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M, B) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X', B) & \longrightarrow & 0 \end{array}$$

induces the desired commuting square, as follows.

$$\begin{array}{ccc} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) & \xrightarrow{\delta_A} & \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', B)) & \xrightarrow{\delta_B} & \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, B)) \end{array}$$

We now proceed with the proof. In one direction, if X is an r-encoding module, then Corollary 37 promises us a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{r+1}(G, -),$$

so the composite

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)) \overset{\delta_\bullet}{\Rightarrow} \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \overset{\Phi_\bullet}{\Rightarrow} \widehat{H}^{r+1}(G, -)$$

shows that X' is a $(r+1)$-encoding module. The other direction is analogous, concatenating with $\delta_\bullet^{-1}$. ■

Example 67. Fix a finite group G generated by $S \coloneqq \langle \sigma_1, \dots, \sigma_n \rangle$, and let $M \coloneqq \mathbb{Z}[G]^{\#S}$ have basis $\{e_i\}_{i=1}^m$. Then there is a projection $\pi \colon \mathbb{Z}[G]^{\#G} \twoheadrightarrow I_G$ by sending $e_i \mapsto (\sigma_i - 1)$, giving the short exact sequence

$$0 \to \ker \pi \to \mathbb{Z}[G]^{\#S} \to I_G \to 0.$$

This short exact sequence is $\mathbb{Z}$-split because I_G is $\mathbb{Z}$-free. Because $\mathbb{Z}[G]^{\#S} \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}^{\#S}$ is induced and I_G is a 1-encoding module, we conclude that $\ker \pi$ is a 2-encoding module by Proposition 66.

By this point, we have a wide array of ways of making p-encoding modules, so we call it quits here.

1.11 A Perfect Pairing

We close this section with a hint of Artin reciprocity. The main goal of this subsection is to prove the following result.

Theorem 68. Let G be a finite group, and let X and A be G-modules and $r \in \mathbb{Z}$ be an index. Then, if there exists an element $c \in H^r(G, X)$ such that the cup-product maps

$$\begin{aligned} (c \cup -) \colon \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) &\to \widehat{H}^0(G, \mathbb{Z}) \\ (c \cup -) \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) &\to \widehat{H}^r(G, A) \end{aligned}$$

are isomorphisms, then the cup-product pairing induces an isomorphism

$$\widehat{H}^r(G, A) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})), \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\right).$$

Proof. Applying Lemma 12 to the commutative square

$$\begin{array}{ccc} X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A & \longrightarrow & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \\ \downarrow & & \downarrow \\ \mathbb{Z} \otimes_{\mathbb{Z}} A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} x \otimes f \otimes a & \longmapsto & x \otimes (y \mapsto f(y)a) \\ \downarrow & & \downarrow \\ f(x) \otimes a & \longmapsto & f(x)a \end{array}$$

we are able to conclude that any $u \in H^p(G, A)$ makes the diagram

$$\begin{array}{ccc} \widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) & \xrightarrow{-\cup u} & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ {\scriptstyle c\cup -}\downarrow & & \downarrow{\scriptstyle c\cup -} \\ \widehat{H}^0(G, \mathbb{Z}) & \xrightarrow[-\cup u]{} & \widehat{H}^r(G, A) \end{array}$$

commute. Now, by hypothesis, the left and right arrows are isomorphisms, so the commutativity means that showing

$$\begin{array}{ccc} \widehat{H}^r(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})), \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\right) \\ u & \mapsto & (a \mapsto (a \cup u)) \end{array}$$

is an isomorphism is the same as showing that

$$\begin{array}{ccc} \widehat{H}^r(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^0(G, \mathbb{Z}), \widehat{H}^r(G, A)\right) \\ u & \mapsto & (k \mapsto (k \cup u)) \end{array}$$

is an isomorphism.

Setting $n \coloneqq \#G$, we see $\widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and the cup product we are looking at sends $k \in \mathbb{Z}/n\mathbb{Z}$ and $u \in \widehat{H}^2(G, A)$ to $k \cup u = ku$ by how the isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} A \simeq A$ behaves. Thus, we are showing that

$$\begin{array}{ccc} \widehat{H}^p(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, \widehat{H}^r(G, A)\right) \\ u & \mapsto & (k \mapsto ku) \end{array}$$

is an isomorphism.

However, $\widehat{H}^r(G, A)$ is n-torsion, so in fact maps $\mathbb{Z} \to \widehat{H}^p(G, A)$ automatically have $n\mathbb{Z}$ in their kernel and hence reduce to maps $\mathbb{Z}/n\mathbb{Z} \to \widehat{H}^r(G, A)$. Conversely, any map $\mathbb{Z}/n\mathbb{Z} \to \widehat{H}^p(G, A)$ can be extended by $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ to a map $\mathbb{Z} \to \widehat{H}^r(G, A)$, so we have a natural isomorphism

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, \widehat{H}^r(G, A)\right) & \simeq & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \widehat{H}^r(G, A)\right) \\ f & \mapsto & (k \mapsto f([k])) \\ ([k] \mapsto f(k)) & \mapsfrom & f. \end{array}$$

In particular, it suffices to show that

$$\begin{array}{ccc} \widehat{H}^r(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \widehat{H}^r(G, A)\right) \\ u & \mapsto & (k \mapsto ku) \end{array}$$

is an isomorphism. But this is a standard fact about the functor $\mathrm{Hom}_{\mathbb{Z}}$, so we are done. ∎

We now synthesize this with the theory we have been building.

Corollary 69. Let G be a finite group, and let X be an r-encoding module. Then, given a G-module A, the cup-product pairing induces an isomorphism

$$\widehat{H}^r(G, A) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})), \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\right).$$

Proof. We apply Theorem 68 to our case; we take c to be the x of Corollary 35. The cup-product maps in question are isomorphisms by Corollary 37. Thus, Theorem 68 kicks in, completing the proof. ∎

Remark 70. The other side of the pairing

$$\widehat{H}^{-r}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{r}(G, A), \widehat{H}^{0}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\right)$$

need not be an isomorphism; for example, take $A = 0$.

Remark 71. When X is a $\mathbb{Z}$-free 2-encoding module, we can think about $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ as a torus T. For example, if L/K is an extension of local fields, and the torus T splits over L, then the above statement says that the Artin reciprocity map

$$\widehat{H}^{-2}(L/K, X_*(T)) \to \widehat{H}^{0}(L/K, TL)$$

uniquely determines $u_{L/K} \in \widehat{H}^2(L/K, L^\times)$. It is conceivable that a sufficiently concrete description of this reciprocity map might then be able to describe $u_{L/K}$.

2 General Group Extensions

Having established some background of what we expect from our encoding modules, we will spend the next few sections building a particularly nice example of a 2-encoding module.

Throughout this section, G will be a finite group and A will be a G-module; we will write the group operation of A and the group action of G on A multiplicatively. To sketch the idea here, begin with an extension

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1.$$

We know that we can abstractly represent $\mathcal{E}$ as the set $A \times G$ with some group law dictated by a 2-cocycle in $H^2(G, A)$, so we expect that $\mathcal{E}$ can be presented by A and a choice of lifts from G, with some specially chosen relations.

Here are some basic observations realizing this idea. We start by lifting a single element of G.

Lemma 72. Let A be a G-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

denote a group extension. Further, fix some $\sigma \in G$ of order n_σ, and find $F \in \mathcal{E}$ such that $\sigma := \pi(F)$. Then

$$\alpha := F^{n_\sigma}$$

has $\alpha \in A^{\langle \sigma \rangle}$.

Proof. A priori, we only know that $\alpha \in \mathcal{E}$, so we compute

$$\pi(\alpha) = \pi\left(F^{n_\sigma}\right) = \sigma^{n_\sigma} = 1,$$

so $\alpha \in \ker \pi = A$. Thus, we may say that

$$\sigma(\alpha) = F \alpha F^{-1} = F^{n_\sigma} = \alpha,$$

so $\alpha \in A^{\langle \sigma \rangle}$, as desired. ∎

We can make the above proof more explicit by specifying the group law of $\mathcal{E}$.

Lemma 73. Let A be a G-module. Picking up some 2-cocycle $c \in Z^2(G, A)$, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be the corresponding extension. Fixing $\sigma \in G$ of order n_σ, let $F \coloneqq (m, \sigma) \in \mathcal{E}$ be a lift. Then

$$\alpha \coloneqq F^{n_\sigma} = N_\sigma(m) \prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right),$$

where $N_\sigma \coloneqq \sum_{i=0}^{n_\sigma - 1} \sigma^i$.

Proof. This is a direct computation. By induction, we have that

$$F^k = \left(\prod_{i=0}^{k-1} \sigma^i(m) c\left(\sigma^i, \sigma\right), \sigma^k \right)$$

for $k \in \mathbb{N}$. Indeed, there is nothing to say for $k = 0$, and the inductive step merely expands out $F^k \cdot F$.

It follows that

$$\alpha = F^{n_\sigma} = \left(\prod_{i=0}^{n_\sigma - 1} \sigma^i(m) \cdot \prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right), 1 \right),$$

which is what we wanted. ∎

Having this explicit formula lets us say how α changes as we vary the lift.

Proposition 74. Let A be a G-module. Fixing a cohomology class $u \in H^2(G, A)$, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension whose isomorphism class corresponds to u. Further, fix some $\sigma \in G$ of order n_σ, and let $A_\sigma \coloneqq A^{\langle \sigma \rangle}$ be the fixed submodule. Then the set

$$S_{\mathcal{E},\sigma} \coloneqq \{F^{n_\sigma} : \pi(F) = \sigma\}$$

is an equivalence class in $A_\sigma / N_\sigma(A)$, independent of the choice of $\mathcal{E}$. Again, $N_\sigma \coloneqq \sum_{i=1}^{n_\sigma - 1} \sigma^i$.

Proof. Note that $S_{\mathcal{E},\sigma} \subseteq A_\sigma$ already from Lemma 72.

The point is to use Lemma 73. Note the extension $\mathcal{E}$ corresponds to the equivalence class $u \in H^2(G, A)$, so let $c \in Z^2(G, A)$ be a representative. Letting $\mathcal{E}_c$ be the extension constructed from c, we are promised an isomorphism $\varphi \colon \mathcal{E} \simeq \mathcal{E}_c$ making the following diagram commute.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & \mathcal{E} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \\
1 & \longrightarrow & A & \longrightarrow & \mathcal{E}_c & \xrightarrow{\pi_c} & G & \longrightarrow & 1
\end{array}$$

We start by claiming that $S_{\mathcal{E},\sigma} = S_{\mathcal{E}_c,\sigma}$, which will show that $S_{\mathcal{E},\sigma}$ is independent of the choice of representative $\mathcal{E}$. To show $S_{\mathcal{E},\sigma} \subseteq S_{\mathcal{E}_c,\sigma}$, note that $\alpha \in S_{\mathcal{E},\sigma}$ has $F \in \mathcal{E}$ with $\pi(F) = \sigma$ and $\alpha = F^{n_\sigma}$. Pushing this through φ, we see $\varphi(F) \in \mathcal{E}_c$ has

$$\pi_c(\varphi(F)) = \varphi(\pi(F)) = \sigma \qquad \text{and} \qquad \varphi(F)^{n_\sigma} = \varphi(F^{n_\sigma}) = \alpha,$$

so $\alpha \in S_{\mathcal{E}_c,\sigma}$ follows. An analogous argument with φ^{-1} shows the other needed inclusion.

It thus suffices to show that $S_{\mathcal{E}_c,\sigma}$ is an equivalence class in $A_\sigma / N_\sigma(A)$. However, this is exactly what Lemma 73 says as we let the possible lifts $F = (m, \sigma) \in \mathcal{E}_c$ of σ vary over $m \in A$. ∎

The fact that we are taking elements of G to equivalence classes in $A_\sigma^\times / N_\sigma(A)$ is reminiscent of the (inverse) Artin reciprocity map, and indeed that is exactly what is going on.

Corollary 75. Work in the context of Proposition 74. Then

$$S_\sigma \coloneqq S_{\mathcal{E},\sigma} = [\sigma] \cup [c],$$

where $\cup \colon \widehat{H}^{-2}(G,A) \times \widehat{H}^2(G,A) \to \widehat{H}^0(G,A)$ is the cup product in Tate cohomology.

Proof. Using notation as in the proof of Proposition 74, we recall that $S_\sigma = S_{\mathcal{E}_c,\sigma}$, so it suffices to prove the result for $\mathcal{E}_c$. Well, by Lemma 73, S_σ is represented by

$$\prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right).$$

However, this product is exactly the cup product $[\sigma] \cup [c]$. ∎

Corollary 76. Let L/K be a finite Galois extension of local fields with Galois group $G \coloneqq \mathrm{Gal}(L/K)$. Further, let

$$1 \to L^\times \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be an L/K-gerb bound by $\mathbb{G}_m$ whose isomorphism class corresponds to the fundamental class $u_{L/K} \in H^2(G, L^\times)$. Further, fix some $\sigma \in G$ of order n_σ, and let $L_\sigma \coloneqq L^{\langle\sigma\rangle}$ be the fixed field. Then

$$\theta_{L/L_\sigma}^{-1}(\sigma) = \left\{F^{n_\sigma} : \pi(F) = \sigma\right\}.$$

Proof. Recalling θ_{L/L_σ}^{-1} is a cup product map, note that $\theta_{L/L_\sigma}^{-1}(\sigma)$ is given by $[\sigma] \cup u_{L/K}$. So we are done by Corollary 75. ∎

The above results are all interested in lifting single elements of G and studying how they behave on their own. In the discussion that follows, we will need to study how the lifts interact with each other, but for now, we will justify why lifts are adequate to study as follows.

Proposition 77. Let A be a G-module. Further, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Given elements $\{\sigma_i\}_{i=1}^m$ which generate G, then $\mathcal{E}$ is generated by A and a set of lifts $\{F_i\}_{i=1}^m$ with $\pi(F_i) = \sigma_i$ for each i.

Proof. Fix some element $e \in \mathcal{E}$, which we need to exhibit as a product of elements in A and F_is. Well, because the σ_i generate G, we know that $\pi(e) \in G$ can be written as

$$\pi(e) = \prod_{i=1}^m \sigma_i^{a_i}$$

for some sequence of integers $\{a_i\}_{i=1}^m$. It follows that

$$\pi\left(\frac{e}{\prod_{i=1}^m F_i^{a_i}}\right) = 1,$$

so $\frac{e}{\prod_{i=1}^m F_i^{a_i}} = \ker \pi = A$. Thus, we can find some $x \in A$ such that

$$e = x \cdot \prod_{i=1}^m F_i^{a_i},$$

which is what we wanted. ■

3 Abelian Group Extensions

3.1 Extensions to Tuples

The above proofs technically don't even require that the group G is abelian. If we want to keep track of the fact our group is abelian, we should extract the elements of A which can do so.

Lemma 78. Let A be a G-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Further, fix some $F_1, F_2 \in \mathcal{E}$ and define $\sigma_i \coloneqq \pi(F_i)$ for $i \in \{1,2\}$, and let $\sigma_i \in G$ have order n_i. Then, setting

$$\alpha_i \coloneqq F_i^{n_i} \qquad \text{and} \qquad \beta \coloneqq F_1 F_2 F_1^{-1} F_2^{-1},$$

we have the following.

(a) $\alpha_i \in A^{\langle \sigma_i \rangle}$ for $i \in \{1,2\}$ and $\beta \in A$.

(b) $N_1(\beta) = \alpha_1/\sigma_2(\alpha_1)$ and $N_2(\beta^{-1}) = \alpha_2/\sigma_1(\alpha_2)$, where $N_i \coloneqq \sum_{p=0}^{n_i-1} \sigma_i^p$.

Proof. These checks are a matter of force. For brevity, we set $A_i \coloneqq A^{\langle \sigma_i \rangle}$ for $i \in \{1,2\}$.

(a) That $\alpha_i \in A_i$ follows from Lemma 72. Lastly, $\beta \in A$ follows from noting

$$\pi(\beta) = \pi(F_1)\pi(F_2)\pi(F_1)^{-1}\pi(F_2)^{-1} = 1,$$

so $\beta \in \ker \pi = A$.

(b) We will check that $\mathrm{N}_{L/L_1}(\beta) = \alpha_1/\sigma_2(\alpha_1)$; the other equality follows symmetrically after switching 1s and 2s because $\beta^{-1} = F_2 F_1 F_2^{-1} F_1^{-1}$. Well, we compute

$$\begin{aligned}
N_1(\beta) &= \sigma_1^{-1}(\beta) \cdot \sigma_1^{-2}(\beta) \cdot \sigma^{-3} \cdot \ldots \cdot \sigma^{-n_1}(\beta) \\
&= F_1^{-1}\left(F_1 F_2 F_1^{-1} F_2^{-1}\right) F_1 \\
&\quad \cdot F_1^{-2}\left(F_1 F_2 F_1^{-1} F_2^{-1}\right) F_1^2 \\
&\quad \cdot F_1^{-3}\left(F_1 F_2 F_1^{-1} F_2^{-1}\right) F_1^3 \cdot \ldots \\
&\quad \cdot F_1^{-n_1}(F_1 F_2 F_1^{-1} F_2^{-1}) F_1^{n_1} \\
&= F_2 F_1^{-1} \\
&\quad \cdot F_1^{-1} \\
&\quad \cdot F_1^{-1} \cdot \ldots \\
&\quad \cdot F_1^{-1} F_2^{-1} F_1^{n_1} \\
&= F_2 F_1^{-n_1} F_2^{-1} F_1^{n_1} \\
&= \alpha_1/\sigma_2(\alpha_1).
\end{aligned}$$

The above computations finish the proof. ■

The proof of (b) above might appear magical, but in fact it comes from a more general idea.

Lemma 79. Fix everything as in Lemma 78. Then, for $x, y \geq 0$, we have

$$F_1^x F_2^y = \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_1^k \sigma_2^\ell(\beta) F_2^y F_1^x.$$

Proof. We induct. We take a moment to write out the case of $x = 1$, for which we induct on y. To be explicit, we will prove

$$F_1 F_2^y = \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y F_1.$$

For $y = 0$, there is nothing to say. So suppose the statement for y (and $x = 1$), and we show $y+1$ (and $x = 1$). Well, we compute

$$\begin{aligned} F_1 F_2^{y+1} &= F_1 F_2^y \cdot F_2 \\ &= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y F_1 \cdot F_2 \\ &= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y \beta F_2 F_1 \\ &= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) \cdot \sigma_2^y(\beta) F_2^y \cdot F_2 F_1 \\ &= \prod_{\ell=0}^{(y+1)-1} \sigma_2^\ell(\beta) \cdot F_2^{y+1} F_1, \end{aligned}$$

which is what we wanted.

We now move on to the general case. We will induct on y. Note that $y = 0$ makes the product empty, leaving us with $F_1^x = F_1^x$, for any x. So suppose that the statement is true for some $y \geq 0$, and we will show $y + 1$. For this, we now turn to inducting on x. For $x = 0$, we note that the product is once again empty, so we are left with showing $F_2^{y+1} = F_2^{y+1}$, which is true.

To finish, we suppose the statement for x and show the statement for $x + 1$. Well, we compute

$$\begin{aligned} F_1^{x+1} F_2^{y+1} &= F_1 \cdot F_1^x F_2^{y+1} \\ &= F_1 \cdot \prod_{k=0}^{x-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot F_2^{y+1} F_1^x \\ &= \sigma_1 \left(\prod_{k=0}^{x-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \right) \cdot F_1 F_2^{y+1} F_1^x \\ &= \prod_{k=1}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot F_1 F_2^{y+1} F_1^x \\ &= \prod_{k=1}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot \prod_{\ell=0}^{(y+1)-1} \sigma_2^\ell(\beta) \cdot \sigma_2^y(\beta) \cdot F_2^{y+1} F_1 \cdot F_1^x \\ &= \prod_{k=0}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) F_2^{y+1} F_1^{x+1}, \end{aligned}$$

which is what we wanted. ■

Remark 80. Setting $x = n_1$ and $y = 1$ recovers $\mathrm{N}_{L/L^{\langle\sigma_1\rangle}}(\beta) = \alpha_1/\sigma_2(\alpha_1)$.

In particular, Remark 80 tells us that coherence of the group law in $\mathcal{E}$ should give rise to relations between our elements of A. Here is a more complex example.

Lemma 81. Let A be a G-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Further, fix some $F_1, F_2, F_3 \in \mathcal{E}$ and define $\sigma_i \coloneqq \pi(F_i)$ for $i \in \{1, 2, 3\}$, and let $\sigma_i \in G$ have order n_i. Then, setting

$$\beta_{ij} \coloneqq F_i F_j F_i^{-1} F_j^{-1}$$

for each pair of indices (i, j) with $i > j$. Then

$$\frac{\sigma_2(\beta_{31})}{\beta_{31}} = \frac{\sigma_1(\beta_{32})}{\beta_{32}} \cdot \frac{\sigma_3(\beta_{21})}{\beta_{21}}.$$

Proof. The point is to turn $F_3F_2F_1$ into $F_1F_2F_3$ in two different ways. On one hand,

$$\begin{aligned}(F_3F_2)F_1 &= \beta_{32}F_2F_3F_1 \\ &= \beta_{32}F_2\beta_{31}F_1F_3 \\ &= \beta_{32}\sigma_2(\beta_{31})(F_2F_1)F_3 \\ &= \beta_{32}\sigma_2(\beta_{31})\beta_{21}F_1F_2F_3.\end{aligned}$$

On the other hand,

$$\begin{aligned}F_3(F_2F_1) &= F_3\beta_{21}F_1F_2 \\ &= \sigma_3(\beta_{21})(F_3F_1)F_2 \\ &= \sigma_3(\beta_{21})\beta_{31}F_1(F_3F_2) \\ &= \sigma_3(\beta_{21})\beta_{31}F_1\beta_{32}F_2F_3 \\ &= \sigma_3(\beta_{21})\beta_{31}\sigma_1(\beta_{32})F_1F_2F_3.\end{aligned}$$

Thus,

$$\beta_{32}\sigma_2(\beta_{31})\beta_{21} = \sigma_3(\beta_{21})\beta_{31}\sigma_1(\beta_{32}),$$

which rearranges into the desired equation. ■

Remark 82. The relation from Lemma 81 may look asymmetric in the β_{ij}, but this is because the definitions of the β_{ij}s themselves are asymmetric in F_i.

3.2 Tuples to Cocycles

3.2.1 The Set-Up

The proceeding lemma is intended to give intuition that the element β is helping to specify the group law on $\mathcal{E}$.

More concretely, we will take the following set-up for the following results: fix a G-module A, and let

$$1 \to A \to \mathcal{E} \to G \to 1$$

be a group extension. Once we choose elements $\{\sigma_i\}_{i=1}^m$ generating G, we know by Proposition 77 that we can generate $\mathcal{E}$ by A and some arbitrarily chosen lifts $\{F_i\}_{i=1}^m$ of the $\{\sigma_i\}_{i=1}^m$. Then, letting n_i be the order of σ_i, we set

$$\alpha_i \coloneqq F_i^{n_i}$$

for each index i and

$$\beta_{ij} \coloneqq F_i F_j F_i^{-1} F_j^{-1}$$

for each index $1 \leq j < i \leq m$. Notably, we will not need more βs: indeed, $\beta_{ii} = 1$ and $\beta_{ij} = \beta_{ji}^{-1}$ for any i and j. Setting $A_i \coloneqq A^{\langle \sigma_i \rangle}$ and $N_i \coloneqq \sum_{p=0}^{n_i - 1} \sigma_i^p$, the story so far is that

$$\alpha_i \in A_i \text{ for each } i \qquad \text{and} \qquad \beta_{ij} \in A \text{ for each } i > j \tag{3.1}$$

and

$$N_i(\beta_{ij}) = \alpha_i / \sigma_j(\alpha_i) \qquad \text{and} \qquad N_j(\beta_{ij}^{-1}) = \alpha_j / \sigma_i(\alpha_j) \qquad \text{for each } i > j \tag{3.2}$$

by Lemma 78, and

$$\frac{\sigma_j(\beta_{ik})}{\beta_{ik}} = \frac{\sigma_k(\beta_{ij})}{\beta_{ij}} \cdot \frac{\sigma_i(\beta_{jk})}{\beta_{jk}} \qquad \text{for each } i > j > k \tag{3.3}$$

by Lemma 81. This data is so important that we will give it a name.

Definition 83. In the above set-up, the data of $(\{\alpha_i\}, \{\beta_{ij}\})$ satisfying (3.1) and (3.2) and (3.3) will be called a $\{\sigma_i\}_{i=1}^m$-*tuple*. When understood, the $\{\sigma_i\}_{i=1}^m$ will be abbreviated. Once G and A are fixed, we will denote the set of $\{\sigma_i\}_{i=1}^m$-tuples by $\mathcal{T}(G, A)$.

Note that this definition is independent of $\mathcal{E}$, but a choice of extension $\mathcal{E}$ and lifts F_i give a $\{\sigma_i\}_{i=1}^m$-tuple as described above.

Remark 84. The $\mathcal{T}(G, A)$ form a group under multiplication in A. Indeed, the conditions (3.1) and (3.2) and (3.3) are closed under multiplication and inversion.

We also know from Lemma 79 that

$$F_i^x F_j^y = \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_i^k \sigma_j^\ell(\beta_{ij}) F_j^y F_i^x$$

for $i > j$ and $x, y \geq 0$. It will be helpful to have some notation for the residue term in A, so we define

$$\beta_{ij}^{(xy)} \coloneqq \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_i^k \sigma_j^\ell(\beta_{ij}).$$

Now, combined with the fact that $F_i x = \sigma_i(x) F_i$ for each F_i and $x \in A$, we have been approximately told how the group operation works in $\mathcal{E}$. Namely, we could conceivably write any element of $\mathcal{E}$ in the form

$$x F_1^{a_1} \cdots F_m^{a_m}$$

for $x \in A$ and $a_i \in \mathbb{Z}/n_i\mathbb{Z}$ because we know how to make these elements commute and generate $\mathcal{E}$. Further, we can multiply out two terms of the form

$$x F_1^{a_1} \cdots F_m^{a_m} \cdot y F_1^{b_1} \cdots F_m^{b_m}$$

into a term of the form $z F_1^{c_1} \cdots F_m^{c_m}$. In fact, it will be helpful for us to see how to do this.

Proposition 85. Fix everything as in the set-up, except drop the assumption that $\{\sigma_i\}_{i=1}^m$ generate G. Then, choosing $a_i, b_i \in \mathbb{N}$ for each i, we have

$$\left(\prod_{i=1}^m F_i^{a_i}\right)\left(\prod_{i=1}^m F_i^{b_i}\right) = \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=1}^m F_i^{a_i+b_i}\right).$$

Proof. The reason that we dropped the assumption on $\{\sigma_i\}_{i=1}^m$ is so that we may induct directly on m. We start by showing that

$$\left(\prod_{i=1}^m F_i^{a_i}\right)F_1^{b_1} = \left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]F_1^{a_1+b_1}\prod_{i=2}^m F_i^{a_i}.$$

We do this by induction on m. When $m = 0$ and even for $m = 1$, there is nothing to say. For the inductive step, we assume

$$\left(\prod_{i=1}^m F_i^{a_i}\right)F_1^{b_1} = \left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]F_1^{a_1+b_1}\prod_{i=2}^m F_i^{a_i}$$

and compute

$$\begin{aligned}
\left(\prod_{i=1}^{m+1} F_i^{a_i}\right)F_1^{b_1} &= \left(\prod_{i=1}^m F_i^{a_i}\right)F_{m+1}^{a_{m+1}}F_1^{b_1}\\
&= \left(\prod_{i=1}^m F_i^{a_i}\right)\beta_{m+1,1}^{(a_{m+1}b_1)}F_1^{b_1}F_{m+1}^{a_{m+1}}\\
&= \left[\left(\prod_{k=1}^m \sigma_k^{a_k}\right)\beta_{m+1,1}^{(a_{m+1}b_1)}\right]\left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]F_1^{a_1+b_1}\left(\prod_{i=2}^m F_i^{a_i}\right)F_{m+1}^{a_{m+1}}\\
&= \left[\prod_{1<i\le m+1}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]F_1^{a_1+b_1}\left(\prod_{i=2}^{m+1} F_i^{a_i}\right),
\end{aligned}$$

which completes our inductive step.

We now attack the statement of the proposition directly, again inducting on m. For $m = 0$ and even for $m = 1$, there is again nothing to say. For the inductive step, take $m > 1$, and we get to assume that

$$\left(\prod_{i=2}^m F_i^{a_i}\right)\left(\prod_{i=2}^m F_i^{b_i}\right) = \left[\prod_{2\le j<i\le m}\left(\prod_{2\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right).$$

From here, we can compute

$$\begin{aligned}\left(\prod_{i=1}^m F_i^{a_i}\right)\left(\prod_{i=1}^m F_i^{b_i}\right) &= \left(\prod_{i=1}^m F_i^{a_i}\right) F_1^{b_1}\left(\prod_{i=2}^m F_i^{b_i}\right)\\ &= \left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\left(\prod_{i=2}^m F_i^{a_i}\right)\left(\prod_{i=2}^m F_i^{b_i}\right)\\ &= \left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\cdot\\ &\qquad \left[\prod_{2\le j<i\le m}\left(\prod_{2\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right)\\ &= \left[\prod_{1<i\le m}\left(\prod_{1\le k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]\cdot\\ &\qquad \sigma_1^{a_1+b_1}\left[\prod_{2\le j<i\le m}\left(\prod_{2\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right).\end{aligned}$$

From here, a little rearrangement finishes the inductive step. ■

The reason we exerted this pain upon ourselves is for the following result.

Proposition 86. Fix everything as in the set-up. Then, if well-defined, we can represent the cohomology class corresponding to $\mathcal{E}$ by the cocycle

$$c(g,h) \coloneqq \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g=\prod_i\sigma_i^{a_i}$ and $h=\prod_i\sigma_i^{b_i}$.

Observe that Proposition 86 has a fairly strong hypothesis that c is well-defined; we will return to this later.

Proof. Very quickly, we use the division algorithm to define

$$a_i+b_i = n_iq_i+r_i$$

where $q_{\in}\{0,1\}$ and $0\le r_i<n_i$. In particular,

$$gh=\prod_{i=1}^m F_i^{r_i}.$$

Now, because the elements σ_i generate G, we see that the lifts $\sigma_i\mapsto F_i$ defines a section $s\colon G\to\mathcal{E}$. As such, we can compute a representing cocycle for our cohomology class as

$$\begin{aligned}c(g,h) &= s(g)s(h)s(gh)^{-1}\\ &= \left(\prod_{i=1}^m F_i^{a_i}\right)\left(\prod_{i=1}^m F_i^{b_i}\right)\left(\prod_{i=1}^m F_i^{r_i}\right)^{-1}\\ &= \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=1}^m F_i^{a_i+b_i}\right)\left(\prod_{i=1}^m F_{m-i+1}^{-r_{m-i+1}}\right).\end{aligned}$$

It remains to deal with the last products; we claim that it is equal to

$$\left(\prod_{i=1}^{m} F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m} F_{m-i+1}^{-r_{m-i+1}}\right) = \prod_{i=1}^{m}\left(\prod_{1\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{q_i},$$

which will finish the proof. We induct on m; for $m = 0$ and $m = 1$, there is nothing to say. For the inductive step, we assume that

$$\left(\prod_{i=2}^{m} F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m-1} F_{m-i+1}^{-r_{m-i+1}}\right) = \prod_{i=2}^{m}\left(\prod_{2\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}$$

and compute

$$\begin{aligned}
\left(\prod_{i=1}^{m} F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m} F_{m-i+1}^{-r_{m-i+1}}\right) &= F_1^{a_1+b_1}\left(\prod_{i=2}^{m} F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m-1} F_{m-i+1}^{-r_{m-i+1}}\right) F_1^{-a_1-b_1} F_1^{a_1+b_1-r_1} \\
&= F_1^{a_1+b_1}\left(\prod_{i=2}^{m}\left(\prod_{2\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}\right) F_1^{-a_1-b_1}\alpha_1^{q_1} \\
&= \left(\prod_{i=2}^{m}\left(\prod_{1\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}\right)\alpha_1^{q_1} \\
&= \prod_{i=1}^{m}\left(\prod_{1\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{q_i},
\end{aligned}$$

finishing. ■

3.2.2 The Modified Set-Up

A priori we have no reason to expect that the c constructed in Proposition 86 is actually a cocycle, especially if the σ_i have nontrivial relations.

To account for this, we modify our set-up slightly. By the classification of finitely generated abelian groups, we may write

$$G \simeq \bigoplus_{k=1}^{m} G_k,$$

where $G_k \subseteq G$ with $G_k \cong \mathbb{Z}/n_k\mathbb{Z}$ and $n_k > 1$ for each n_k. As such, we let σ_k be a generating element of G_k so that we still know that the σ_k generate G. In this case, we have the following result.

Theorem 87. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Then a $\{\sigma_i\}_{i=1}^m$-tuple of $\{\alpha_i\}_{i=1}^m$ and $\{\beta_{ij}\}_{i>j}$ makes

$$c(g,h) \coloneqq \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j} \sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i} \sigma_k^{a_k}\right)\beta_{ij}^{(a_i b_j)}\right]\left[\prod_{i=1}^{m}\left(\prod_{1\le k<i} \sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor \frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g \coloneqq \prod_i \sigma_i^{a_i}$ with $h \coloneqq \prod_i \sigma_j^{a_j}$ and $0 \le a_i, b_i < n_i$, into a cocycle in $Z^2(G, A)$.

Proof. Note that c is now surely well-defined because the elements g and h have unique representations as described. Anyway, we relegate the direct cocycle check to Appendix A because it is long, annoying, and unenlightening. We will also present an alternative proof in section 4, using more abstract theory. ■

Observe that the above construction has now completely forgotten about $\mathcal{E}$! Namely, we have managed to go from tuples straight to cocycles; this is theoretically good because it will allow us to go fully in reverse: we will be able to start with a tuple, build the corresponding cocycle, from which the extension arises. However, equivalence classes of cocycles give the "same" extension, so we will also need to give equivalence classes for tuples as well.

3.3 Building Tuples

We continue in the modified set-up of the previous section. There is already an established way to get from a cocycle to an extension, which means that it should be possible to go straight from the cocycle to a $\{\sigma_i\}_{i=1}^m$-tuple. Again, it will be beneficial to write this out.

Lemma 88. Fix everything as in the modified set-up, but suppose that $\mathcal{E} = \mathcal{E}_c$ is the extension generated from a cocycle $c \in Z^2(G, A)$. Then, if $F_i = (x_i, \sigma_i)$ are our lifts, we have

$$\alpha_i = N_i(x_i) \cdot \prod_{k=0}^{n_i-1} c\left(\sigma_i^k, \sigma_i\right) \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)} \cdot \frac{\sigma_i(x_j)}{x_j} \cdot \frac{c(\sigma_i, \sigma_j)}{c(\sigma_j, \sigma_i)}$$

for each α_i and β_{ij}.

Proof. The equality for the α_i follow from Lemma 73. For the equality about β_{ij}, we simply compute by brute force, writing

$$\begin{aligned} F_iF_j &= (x_i \cdot \sigma_i x_j \cdot c(\sigma_i, \sigma_j), \sigma_i\sigma_j) \\ F_jF_i &= (x_j \cdot \sigma_j x_i \cdot c(\sigma_j, \sigma_i), \sigma_j\sigma_i) \\ (F_jF_i)^{-1} &= \left((\sigma_j\sigma_i)^{-1}(x_j \cdot \sigma_j x_i \cdot c(\sigma_j, \sigma_i))^{-1}, \sigma_i^{-1}\sigma_j^{-1}\right), \end{aligned}$$

which gives

$$\begin{aligned} \beta_{ij} &= (F_iF_j)(F_jF_i)^{-1} \\ &= \left(\frac{x_i}{\sigma_j x_i} \cdot \frac{\sigma_i x_j}{x_j} \cdot \frac{c(\sigma_i, \sigma_j)}{c(\sigma_j, \sigma_i)}, 1\right), \end{aligned}$$

finishing. ∎

Here is a nice sanity check that we are doing things in the right setting: not only can we build tuples from extensions, but we can find an extension corresponding to any tuple.

Corollary 89. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. For any $\{\sigma_i\}_{i=1}^m$-tuple of $\{\alpha_i\}_{i=1}^m$ and $\{\beta_{ij}\}_{i>j}$, there exists an extension $\mathcal{E}$ and lifts F_i of the σ_i so that

$$\alpha_i = F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} = F_iF_jF_i^{-1}F_j^{-1}.$$

Proof. From Theorem 87, we may build the cocycle $c \in Z^2(G, A)$ defined by

$$c(g, h) \coloneqq \left[\prod_{1 \le j < i \le m} \left(\prod_{1 \le k < j} \sigma_k^{a_k + b_k}\right)\left(\prod_{j \le k < i} \sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m \left(\prod_{1 \le k < i} \sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor \frac{a_i+b_i}{n_i} \right\rfloor}\right], \tag{3.4}$$

where $g \coloneqq \prod_i F_i^{a_i}$ and $h \coloneqq \prod_i F_j^{a_j}$ and $0 \le a_i, b_i < n_i$. As such, we use $\mathcal{E} \coloneqq \mathcal{E}_c$ to be the corresponding extension and $F_i \coloneqq (1, \sigma_i)$ as our lifts. We have the following checks.

- To show $\alpha_i = F_i^{n_i}$, we use Lemma 88 to compute $F_i^{n_i}$, which means we want to compute

$$\prod_{k=0}^{n_i-1} c\left(\sigma_i^k, \sigma_i\right).$$

  Well, plugging $c\left(\sigma_i^k, \sigma_i\right)$ into (3.4), we note that all $\beta_{k\ell}^{(a_k b_\ell)}$ terms vanish (either $a_k = 0$ or $b_\ell = 0$ for each $k \neq \ell$), so the big left product completely vanishes.

  As for the right product, the only term we have to worry about is

$$\left(\prod_{1\leq k<i} \sigma_k^{0+0}\right)\alpha_i^{\left\lfloor \frac{k+1}{n_i}\right\rfloor},$$

  which is equal to 1 when $k \leq n_i - 1$ and α_i when $k = n_i - 1$. As such, we do indeed have $\alpha_i = F_i^{n_i}$.

- To show $\beta_{ij} = F_i F_j F_i^{-1} F_j^{-1}$ for $i > j$, we again use Lemma 88 to compute $F_i F_j F_i^{-1} F_j^{-1}$, which means we want to compute

$$\frac{c(\sigma_i, \sigma_j)}{c(\sigma_j, \sigma_i)}.$$

  Plugging into (3.4) once more, there is no way to make $\lfloor (a_k + b_k)/n_k \rfloor$ nonzero (recall we set $n_k > 1$ for each k) in either $c(\sigma_i, \sigma_j)$ or $c(\sigma_j, \sigma_i)$. As such, the right-hand product term disappears.

  As for the left product, we note that it still vanishes for $c(\sigma_j, \sigma_i)$ because $i > j$ implies that either $a_k = 0$ or $b_\ell = 0$ for each $k > \ell$. However, for $c(\sigma_i, \sigma_j)$, we do have $a_i = 1$ and $b_j = 1$ only, so we have to deal with exactly the term

$$\left(\prod_{1\leq k<j} \sigma_k^{a_k+b_k}\right)\left(\prod_{j\leq k<i} \sigma_k^{a_k}\right)\beta_{ij}.$$

  With $i > j$ and $a_k = b_k = 0$ for $k \notin \{i, j\}$, we see that the product of all the σ_ks will disappear, indeed only leaving us with β_{ij}.

The above computations complete the proof. ■

And here is our first taste of (partial) classification.

Corollary 90. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Then the formula of Theorem 87 and the formulae of Lemma 88 (setting $x_i = 1$ for each i) are homomorphisms of abelian groups between tuples in $\mathcal{T}(G, A)$ and cocycles in $Z^2(G, A)$. In fact, the formula of Theorem 87 is a section of the formulae of Lemma 88.

Proof. The formulae in Theorem 87 and Lemma 88 are both large products in their inputs, so they are multiplicative (i.e., homomorphisms). It remains to check that we have a section. Well, starting with a $\{\sigma_i\}_{i=1}^m$-tuple and building the corresponding cocycle c by Theorem 87, the proof of Corollary 89 shows that the formulae of Lemma 88 recovers the correct $\{\sigma_i\}_{i=1}^m$-tuple. ■

3.4 Equivalence Classes of Tuples

We continue in the modified set-up. We would like to make Corollary 90 into a proper isomorphism of abelian groups, but this is not feasible; for example, the cocycle c generated by Theorem 87 will always have $c(\sigma_j, \sigma_i) = 1$ for $i > j$, which is not true of all cocycles in $Z^2(G, A)$.

However, we did have a notion that the data of a $\{\sigma_i\}_{i=1}^m$ should be enough to specify the group law of the extension that the tuple comes from, so we do expect to be able to define all extensions—and hence achieve all cohomology classes—from a specially chosen $\{\sigma_i\}_{i=1}^m$-tuple.

To make this precise, we want to define an equivalence relation on tuples which go to the same cohomology class and then show that the map Theorem 87 is surjective on these equivalence classes. The correct equivalence relation is taken from Lemma 88.

Definition 91. Fix everything as in the modified set-up. We say that two $\{\sigma_i\}_{i=1}^m$-tuples $(\{\alpha_i\},\{\beta_{ij}\})$ and $(\{\alpha_i'\},\{\beta_{ij}'\})$ are *equivalent* if and only if there exist elements $x_1,\dots,x_m \in A$ such that

$$\alpha_i = N_i(x_i)\cdot\alpha_i' \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)}\cdot\frac{\sigma_i(x_j)}{x_j}\cdot\beta_{ij}'$$

for each α_i and β_{ij}. We may notate this by $(\{\alpha_i\},\{\beta_{ij}\}) \sim (\{\alpha_i'\},\{\beta_{ij}'\})$.

Remark 92. It is not too hard to see directly from the definition that this is in fact an equivalence relation. In fact, the set of tuples equivalent to the "trivial" tuple of all 1s is closed under multiplication (and inversion) and hence forms a subgroup of $\mathcal{T}(G,A)$. As such, the set of equivalence classes forms a quotient group of $\mathcal{T}(G,A)$. We will denote this quotient group by $\overline{\mathcal{T}}(G,A)$.

This notion of equivalence can be seen to be the correct one in the sense that it correctly generalizes Proposition 74.

Proposition 93. Fix everything as in the modified set-up with an extension $\mathcal{E}$. As the lifts F_i change, the corresponding values of

$$\alpha_i \coloneqq F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} \coloneqq F_iF_jF_i^{-1}F_j^{-1}$$

go through a full equivalence class of $\{\sigma_i\}_{i=1}^m$-tuples.

Proof. We proceed as in Proposition 74. Given an extension $\mathcal{E}'$, let $S_{\mathcal{E}'}$ be the set of $\{\sigma_i\}_{i=1}^m$-tuples generated as the lifts F_i change. We start by showing that an isomorphism $\varphi\colon \mathcal{E}\simeq\mathcal{E}'$ of extensions implies that $S_{\mathcal{E}} = S_{\mathcal{E}'}$; by symmetry, it will be enough for $S_{\mathcal{E}} \subseteq S_{\mathcal{E}'}$. The isomorphism induces the following diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \mathcal{E} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & \mathcal{E}' & \xrightarrow{\pi'} & G & \longrightarrow & 1 \end{array}$$

To show that $S_{\mathcal{E}} \subseteq S_{\mathcal{E}'}$, pick up some $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\},\{\beta_{ij}\})$ generated from lifts $F_i \in \mathcal{E}$ (i.e., $\pi(F_i)=\sigma_i$), where

$$\alpha_i \coloneqq F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} \coloneqq F_iF_jF_i^{-1}F_j^{-1}.$$

Now, we note that $F_i' \coloneqq \varphi(F_i)$ will have

$$\pi(F_i') = \pi(\varphi(F_i)) = \varphi(\pi(F_i)) = \sigma_i$$

by the commutativity of the diagram, so the F_i' are lifts of the σ_i. Further, we see that

$$(F_i')^{n_i} = \varphi(F_i)^{n_i} = \varphi\left(F_i^{n_i}\right) = \varphi(\alpha_i) = \alpha_i$$

for each i, and

$$F_i'F_j'(F_i')^{-1}(F_j')^{-1} = \varphi\left(F_iF_jF_i^{-1}F_j^{-1}\right) = \varphi(\beta_{ij}) = \beta_{ij}$$

for each $i>j$. Thus, $(\{\alpha_i\},\{\beta_{ij}\})$ is a $\{\sigma_i\}_{i=1}^m$-tuple generated by lifts from $\mathcal{E}'$, implying that $(\{\alpha_i\},\{\beta_{ij}\}) \in S_{\mathcal{E}'}$.

It now suffices to show the statement in the proposition for a specific extension isomorphic to $\mathcal{E}$. Well, the isomorphism class of $\mathcal{E}$ corresponds to some cohomology class in $H^2(G,A)$, for which we let c be a representative; then $\mathcal{E}\simeq\mathcal{E}_c$, so we may show the statement for $\mathcal{E}\coloneqq\mathcal{E}_c$. Indeed, as the lifts $F_i=(x_i,\sigma_i)$ change, we know by Lemma 88 that

$$\alpha_i = N_i(x_i)\cdot\prod_{k=0}^{n_i-1} c\left(\sigma_i^k,\sigma_i\right) \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)}\cdot\frac{\sigma_i(x_j)}{x_j}\cdot\frac{c(\sigma_i,\sigma_j)}{c(\sigma_j,\sigma_i)}$$

for each α_i and β_{ij}. All of these live in the same equivalence class by definition of the equivalence, and as the x_i are allowed to vary over all of A, they will fill up that equivalence class fully. This finishes. ∎

We are now ready to upgrade our section.

Corollary 94. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Fixing a cohomology class $[c] \in H^2(G, A)$, the set of $\{\sigma_i\}_{i=1}^m$-tuples which correspond to $[c]$ (via Theorem 87) forms exactly one equivalence class.

Proof. We show that two tuples are equivalent if and only if their corresponding cocycles (via Theorem 87) to the same cohomology class, which will be enough.

In one direction, suppose $(\{\alpha_i\}, \{\beta_{ij}\}) \sim (\{\alpha_i'\}, \{\beta_{ij}'\})$. By Corollary 89, we can find an extension $\mathcal{E}$ which gives $(\{\alpha_i\}, \{\beta_{ij}\})$ by choosing an appropriate set of lifts. By Proposition 93, we see that $(\{\alpha_i'\}, \{\beta_{ij}'\})$ must also come from choosing an appropriate set of lifts in $\mathcal{E}$. However, the cocycles in $Z^2(G, A)$ generated by Theorem 87 from our two tuples now both represent the isomorphism class of $\mathcal{E}$ by Proposition 86, so these cocycles belong to the same cohomology class.

In the other direction, name the cocycles corresponding to $(\{\alpha_i\}, \{\beta_{ij}\})$ and $(\{\alpha_i'\}, \{\beta_{ij}'\})$ by c and c' respectively, and suppose $[c] = [c']$. Then $\mathcal{E}_c \simeq \mathcal{E}_{c'}$ as extensions, but we know by the proof of Corollary 89 that $(\{\alpha_i\}, \{\beta_{ij}\})$ comes from choosing lifts of $\mathcal{E}_c$ and similar for $(\{\alpha_i'\}, \{\beta_{ij}'\})$. In particular, because $\mathcal{E}_c \simeq \mathcal{E}_{c'}$, we know that $(\{\alpha_i'\}, \{\beta_{ij}'\})$ will also come from choosing some lifts in $\mathcal{E}_c$ (recall the proof of Proposition 93), so $(\{\alpha_i\}, \{\beta_{ij}\}) \sim (\{\alpha_i'\}, \{\beta_{ij}'\})$ follows. ■

Theorem 95. The maps described in Corollary 90 descend to an isomorphism of abelian groups between the equivalence classes in $\overline{\mathcal{T}}(G, A)$ and cohomology classes in $H^2(G, A)$.

Proof. The fact that the maps are well-defined (in both directions) and hence injective is Corollary 94. The fact that we had a section from tuples to cocycles implies that the map from cocycles to tuples was also surjective. Thus, we have a bona fide isomorphism. ■

3.5 Classification of Extensions

We remark that we are now able to classify all extensions up to isomorphism, in some sense. At a high level, an isomorphism class of extensions corresponds to a particular cohomology class in $H^2(G, A)$, so choosing a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$ corresponding to this class, we can write out a representative of this cocycle by Theorem 87, properly corresponding to the original extension by Proposition 86.

In fact, the cocycle in Proposition 86 is generated by the description of the group law in Proposition 85, and the entire computation only needed to use the following relations, for the appropriate choice of lifts F_i.

(a) $F_i x = \sigma_i(x) F_i$ for each i and $x \in A$.

(b) $F_i^{n_i} = \alpha_i$ for each i.

(c) $F_i F_j F_i^{-1} F_j^{-1} = \beta_{ij}$ for each $i > j$; i.e., $F_i F_j = \beta_{ij} F_j F_i$.

As such, the above relations fully describe the extension because they also specify the cocycle, and we know that this cocycle is well-defined. We summarize this discussion into the following theorem.

Theorem 96. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Given a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\},\{\beta_{ij}\})$, define the group $\mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$ as being generated by A and elements $\{F_i\}_{i=1}^n$ having the following relations.

(a) $F_i x = \sigma_i(x) F_i$ for each i and $x \in A$.

(b) $F_i^{n_i} = \alpha_i$ for each i.

(c) $F_i F_j = \beta_{ij} F_j F_i$ for each $i > j$.

Then the natural embedding $A \hookrightarrow \mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$ and projection $\pi\colon \mathcal{E}(\{\alpha_i\},\{\beta_{ij}\}) \twoheadrightarrow G$ by $F_i \mapsto \sigma_i$ makes $\mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$ into an extension. In fact, all extensions are isomorphic to some $\mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$.

Proof. This follows from the preceding discussion, though we will provide a few more words in this proof. The exactness of

$$1 \to A \to \mathcal{E}(\{\alpha_i\},\{\beta_{ij}\}) \xrightarrow{\pi} G \to 1$$

follows quickly. Further, the action of conjugation of $\mathcal{E}$ on A corresponds correctly to the G-action by (a). So we do indeed have an extension.

It remains to show that all extensions are isomorphic to one of this type. Well, note that Proposition 85 and Proposition 86 use only the above relations to write down a cocycle representing the isomorphism class of $\mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$, and it is the cocycle corresponding to the $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\},\{\beta_{ij}\})$ itself as described in Theorem 87.

However, we know that as the equivalence class of $(\{\alpha_i\},\{\beta_{ij}\})$ changes, we will hit all cohomology classes in $H^2(G,A)$ by Theorem 95. Thus, because every extension is represented by some cohomology class, every extension will be isomorphic to some $\mathcal{E}(\{\alpha_i\},\{\beta_{ij}\})$. This completes the proof. ∎

3.6 Change of Group

We continue in the modified set-up, but we will no longer need access to an extension $\mathcal{E}$. In this subsection, we are interested in what happens to tuples when the cocycle operations of $\mathrm{Inf}\colon H^2\left(G/H, A^H\right) \to H^2(G,A)$ and $\mathrm{Res}\colon H^2(G,A) \to H^2(H,A)$ are applied, where $H \subseteq G$ is some subgroup.

In general, this is difficult because the structure of a subgroup $H \subseteq G$ might not be particularly amenable to forming a tuple from a tuple in G. More concretely, H might have generators which look very different from those of G. However, it will be enough for our purposes to restrict our attention to the subgroups of the form

$$H = \langle \sigma_1^{t_1}, \dots, \sigma_m^{t_m} \rangle,$$

where the $\{t_i\}_{i=1}^m$ are some positive integers. With that said, here are our computations. We begin with inflation.

Lemma 97. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Further, let $H \coloneqq \langle \sigma_1^{t_1}, \dots, \sigma_m^{t_m} \rangle$ be a subgroup with $t_\bullet \mid n_\bullet$, and let $\overline{\sigma}_i$ be the image of σ_i in G/H. Consider the inflation map $\mathrm{Inf}\colon H^2\left(G/H, A^H\right) \to H^2(G,A)$.

If the cocycle $\overline{c} \in Z^2\left(G/H, A^H\right)$ gives the $\{\overline{\sigma}_i\}_{i=1}^m$-tuple $(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\})$ (by Corollary 90), then the cocycle $\mathrm{Inf}\,\overline{c} \in Z^2(G,A)$ gives the $\{\sigma_i\}_{i=1}^m$-tuple

$$\mathrm{Inf}(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) \coloneqq (\{\alpha_i\},\{\beta_{ij}\}) = \left(\left\{\overline{\alpha}_i^{n_i/\gcd(t_i,n_i)}\right\},\{\overline{\beta}_{ij}\}\right).$$

Notably, $\gcd(t_i,n_i)$ is the order of $\overline{\sigma}_i \in G/H$.

Proof. The point is to use the explicit formulae for the α_i and β_{ij} of Lemma 88.

More explicitly, the map of Corollary 90 tells us that we can compute the tuple for $\operatorname{Inf}\overline{c}$ by using our explicit formulae for α_i and β_{ij} on the 2-cocycle $\operatorname{Inf}\overline{c} \in Z^2(G,A)$. For some α_i, the computation is

$$\begin{aligned}\alpha_i &= \prod_{k=0}^{n_i-1} (\operatorname{Inf} c)\left(\sigma_i^k, \sigma_i\right) \\ &= \prod_{k=0}^{n_i-1} \overline{c}\left(\overline{\sigma}_i^k, \overline{\sigma}_i\right) \\ &= \left(\prod_{k=0}^{\gcd(n_i,t_i)-1} \overline{c}\left(\overline{\sigma}_i^k, \overline{\sigma}_i\right)\right)^{n_i/\gcd(n_i,t_i)}\end{aligned}$$

where the last equality is because $\overline{\sigma}_i^{\gcd(n_i,t_i)} = 1$ in G/H. In fact, $\gcd(n_i,t_i)$ is the order of $\overline{\sigma}_i$, so the product is just $\overline{\alpha}_i$ by Lemma 88 and how we defined $\overline{\alpha}_i$. It follows

$$\alpha_i = \overline{\alpha}_i^{n_i/\gcd(n_i,t_i)}.$$

Continuing, for some β_{ij}, we have

$$\begin{aligned}\beta_{ij} &= \frac{(\operatorname{Inf}\overline{c})(\sigma_i,\sigma_j)}{(\operatorname{Inf}\overline{c})(\sigma_j,\sigma_i)} \\ &= \frac{\overline{c}(\overline{\sigma}_i,\overline{\sigma}_j)}{\overline{c}(\overline{\sigma}_j,\overline{\sigma}_i)} \\ &= \overline{\beta}_{ij},\end{aligned}$$

where the last equality is by how we defined $\overline{\beta}_{ij}$. These computations complete the proof. ■

Remark 98. We can also the statement of Lemma 97 as asserting that the diagram

$$\begin{array}{ccc} Z^2\left(G/H, A^H\right) & \xrightarrow{\operatorname{Inf}} & Z^2(G,A) \\ \downarrow & & \downarrow \\ \mathcal{T}\left(G/H, A^H\right) & \xrightarrow{\operatorname{Inf}} & \mathcal{T}(G,A) \end{array}$$

commutes, where the vertical morphisms are from Corollary 90.

Remark 99. In light of the fact that the cohomology class of some $\operatorname{Inf}\overline{c} \in Z^2(G,A)$ is only defined up to the cohomology class of $\overline{c} \in Z^2\left(G/H, A^H\right)$, changing an input tuple $(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) \in \mathcal{T}\left(G/H, A^H\right)$ up to equivalence will not change the cohomology class of the associated cocycle in $\overline{c} \in Z^2\left(G/H, A^H\right)$ and hence will not change the cohomology class of $\operatorname{Inf}\overline{c}$ nor the equivalence class of $\operatorname{Inf}(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) \in \mathcal{T}(G,A)$. All this is to say that we have a well-defined map

$$\operatorname{Inf}\colon \overline{\mathcal{T}}\left(G/H, A^H\right) \to \overline{\mathcal{T}}(G,A)$$

and commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{T}}\left(G/H, A^H\right) & \xrightarrow{\operatorname{Inf}} & \overline{\mathcal{T}}(G,A) \\ \downarrow & & \downarrow \\ H^2\left(G/H, A^H\right) & \xrightarrow{\operatorname{Inf}} & H^2(G,A) \end{array}$$

induced by modding out from Remark 98.

Restriction is similar.

Lemma 100. Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Further, let $H \coloneqq \langle \sigma_1^{t_1}, \dots, \sigma_m^{t_m} \rangle$ be a subgroup with $t_\bullet \mid n_\bullet$. Consider the inflation map $\mathrm{Res}\colon H^2(G,A) \to H^2(H,A)$.

If the cohomology class $[c] \in H^2(G,A)$ is represented by the $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$, then the cohomology class $[\mathrm{Res}\, \overline{c}]$ is represented by the $\{\sigma_i\}_{i=1}^m$-tuple

$$(\{\overline{\alpha}_i\}, \{\overline{\beta}_{ij}\}) = \left(\left\{ \alpha_i^{1_{n_i \mid t_i}} \right\}, \left\{ \beta_{ij}^{(\gcd(t_i,n_i) 1_{n_i \mid t_i}, \gcd(t_j,n_j) 1_{n_i \mid t_i})} \right\} \right).$$

Proof. By replacing t_i with $\gcd(t_i, n_i)$ (which does not affect $\langle \sigma_i^{t_i} \rangle$ and hence does not affect H), we may assume that $t_i = \gcd(t_i, n_i)$. As in the previous proof, we will simply define c by Theorem 87, and we will use the formulae of Lemma 88 to retrieve the $\left\{\sigma_i^{t_i}\right\}$-tuple for $\mathrm{Res}\, c$. Indeed, we compute

$$\begin{aligned}
\overline{\alpha}_i &= \prod_{k=0}^{n_i/t_i - 1} (\mathrm{Res}\, c)\left(\sigma_i^{t_i k}, \sigma_i^{t_i}\right) \\
&= \prod_{k=0}^{n_i/t_i - 1} c\left(\sigma_i^{t_i k}, \sigma_i^{t_i}\right) \\
&= \prod_{k=0}^{n_i/t_i - 1} \alpha_i^{\lfloor t_i (k+1)/n_i \rfloor},
\end{aligned}$$

where in the last equality we have used the construction of c. Now, if $n_i \mid t_i$, then $n_i = t_i$, and the product is empty, and we get 1; otherwise, the last term of the product $k = n_i/t_i - 1$ is the only term which does not return 1, and it returns α_i. So this matches the claimed $\alpha_i^{1_{n_i \mid t_i}}$.

Continuing, we compute

$$\begin{aligned}
\overline{\beta}_{ij} &= \frac{(\mathrm{Res}\, c)\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right)}{(\mathrm{Res}\, c)\left(\sigma_j^{t_j}, \sigma_i^{t_i}\right)} \\
&= \frac{c\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right)}{c\left(\sigma_j^{t_j}, \sigma_i^{t_i}\right)} \\
&= c\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right),
\end{aligned}$$

where in the last step we have used the construction of c. Now, if $n_i \mid t_i$ or $n_i \mid t_j$, then we are computing $c\left(1, \sigma_j^{t_j}\right)$ or $c\left(\sigma_i^{t_i}, 1\right)$, which are both 1, as needed. Otherwise, $t_i < n_i$ and $t_j < n_j$, so

$$\overline{\beta}_{ij} = \beta_{ij}^{(t_i t_j)},$$

which again is as claimed. ∎

Thankfully, we will really only care about inflation in the following discussion, but we will say that there are analogues of Remark 98 and Remark 99.

3.7 Profinite Groups

In this subsection, we will use our results on change of group to extend our results a little to allow profinite groups. As such, we will want to slightly modify our set-up; we will call the following set-up the "profinite set-up."

Let $\mathcal{I}$ be a poset category such that any pair of elements has an upper bound (i.e., a directed set), and let the functor $G_\bullet\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{FinAbGrp}$ be an inverse system of finite abelian groups. These will create a profinite group

$$G \coloneqq \varprojlim_{i \in \mathcal{I}} G_i.$$

In order to be able to apply our theory, we will assume that G is a finite direct sum of procyclic groups as

$$G \simeq \bigoplus_{k=1}^{m} \overline{\langle \sigma_k \rangle}$$

for some elements $\{\sigma_k\}_{k=1}^m \subseteq G$. Further, we will require that the kernel N_i of the map $G \twoheadrightarrow G_i$ to take the form

$$N_i := \overline{\left\langle \sigma_1^{t_{i,1}}, \dots, \sigma_m^{t_{i,m}} \right\rangle}.$$

In short, our restriction on the N_i will allow our inflation maps to be computable in the sense of Lemma 97. We quickly remark that, because the topology on G is the coarsest one making the projections $G \twoheadrightarrow G_i$ continuous, the subsets $\{N_i\}_{i\in\mathcal{I}}$ give a fundamental system of open neighborhoods around the identity.

Remark 101. Of course, one could also start with G being a finite direct sum of procyclic groups and then define the N_i and G_i accordingly. We have chosen the above approach because in application one might only have access to select G_is, and it is not obvious how to choose these from such a "top-down" approach.

Example 102. To show that we are still allowing interesting groups, we can set

$$G_{m,\nu} := \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^m-1})\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p\right) \simeq \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^m-1})/\mathbb{Q}_p\right) \oplus \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p\right),$$

which becomes $G = \mathrm{Gal}\left(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p\right) \simeq \widehat{\mathbb{Z}} \oplus \mathbb{Z}_p^\times$ upon taking the inverse limit. It is not very hard to check that the kernels are generated correctly; for example, when p is odd, we have $\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p$, and under our isomorphisms, we will have

$$\mathrm{Gal}\left(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p\right) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p/p^{\nu-1}\mathbb{Z}_p,$$

so the kernel of $G \twoheadrightarrow G_{m,\nu}$ is $m\widehat{\mathbb{Z}} \oplus (\mathbb{Z}/(p-1)\mathbb{Z})^{1_{\nu=0}} \oplus p^{\nu-1}\mathbb{Z}_p$.

Remark 103. I'm not sure if such an explicit construction can be extended to other local fields K (say, via Lubin–Tate theory). Because $K^\times$ is not topologically finitely generated when K is in positive characteristic (see for example [Neu99], Proposition II.5.7) such a construction must do something subtle.

Let A be a discrete G-module. The main goal of this subsection is to be able to provide a notion of a "compatible system" of tuples from each individual $H^2(G_i, A)$ to be able to exactly describe an element of $H^2(G, A)$. To effect this, we have the following somewhat annoying checks.

Lemma 104. Suppose that $\mathcal{P}$ is a directed set, and let $\mathcal{P}' \subseteq \mathcal{P}$ be a subcategory such that any $x \in \mathcal{P}$ has some $x' \in \mathcal{P}'$ such that $x \leq x'$. Then, given a functor $F\colon \mathcal{P} \to \mathcal{C}$, we have

$$\varinjlim_{\mathcal{P}} F \simeq \varinjlim_{\mathcal{P}'} F,$$

provided that both colimits exist.

Proof. For concreteness, if $x \leq y$ in $\mathcal{P}$, we will let $f_{yx}\colon x \to y$ be the corresponding morphism; in particular, $x \leq y \leq z$ has $f_{zx} = f_{zy}f_{yx}$. Now, for brevity, set

$$X := \varinjlim_{\mathcal{P}} F \qquad \text{and} \qquad X' := \varinjlim_{\mathcal{P}'} F.$$

By the Yoneda lemma, it suffices to fix some object $Y \in \mathcal{C}$ and show that $\mathrm{Mor}_{\mathcal{C}}(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X', Y)$. Well, morphsims $X \to Y$ are in (natural) bijection with cones under F with nadir Y, and morphisms $X' \to Y$ are in (natural) bijection with cones under $F' := F|_{\mathcal{P}'}$ with nadir Y.

Thus, it suffices to give a natural bijection between cones under F with nadir Y and cones under F' with nadir Y. Well, given a cone under F with nadir Y, we can simply restrict it to $\mathcal{P}'$ to get a cone under F'. In the other direction, given a cone under F' with nadir Y, we can build a cone under F with nadir Y as follows; let $\varphi_{x'} \colon F(x') \to Y$ for $x' \in \mathcal{P}'$ be the corresponding morphisms in our cone.

For any $x \in \mathcal{P}$, find $x' \in \mathcal{P}'$ such that $x \le x'$. Then set

$$\varphi_x \coloneqq \varphi_{x'} \circ f_{x'x}$$

Note that φ_x is in fact independent of our choice of x': if $x \le x'_1$ and $x \le x'_2$, then because $\mathcal{P}$ is a directed set, we can find $y \in \mathcal{P}$ such that $x'_1, x'_2 \le y$ and then $y' \in \mathcal{P}'$ with $y \le y'$. Then

$$\begin{aligned} \varphi_{x'_\bullet} \circ f_{x'_\bullet x} &= \varphi_{y'} \circ f_{y'x'_\bullet} \circ f_{x'_\bullet x} \\ &= \varphi_{y'} \circ f_{y'x} \end{aligned}$$

for $x'_\bullet \in \{x'_1, x'_2\}$. Anyway, we can check that the morphisms φ do assemble to a cone under F': if $x \le y$ in $\mathcal{P}$, then find $y' \in \mathcal{P}$ with $x \le y \le y'$, and we compute

$$\begin{aligned} \varphi_y \circ f_{yx} &= \varphi_{y'} \circ f_{y'y} \circ f_{yx} \\ &= \varphi_{y'} \circ f_{y'x} \\ &= \varphi_x. \end{aligned}$$

Thus, we do have a natural, well-defined map sending cones under F' with nadir Y to cones under F with nadir Y. It is not too hard to see that these maps are inverse to each other (for example, the cone under F', extended to F, does indeed restrict back to F' properly), which completes the proof. ■

Remark 105. One can remove the hypothesis that the colimits exist and use essentially the same proof.

Proposition 106. Fix everything as in the profinite set-up. Then, given a discrete G-module A,

$$H^2(G, A) \simeq \varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right).$$

Here, the morphisms between the collection of $H^2\left(G_i, A^{N_i}\right)$ are induced by inflation: if $i \to j$ in $\mathcal{I}$, then $G_j \to G_i$ in $\mathrm{FinAbGrp}$, giving an inflation map $\mathrm{Inf} \colon H^2\left(G_i, A^{N_i}\right) \to H^2\left(G_j, A^{N_j}\right)$.

Proof. Let $\mathcal{N}$ be the poset category of open normal subgroups of G, reverse ordered under inclusion; i.e., $N_1 \subseteq N_2$ in G induces a map $N_2 \to N_1$. Then it is already known that

$$H^2(G, A) \simeq \varinjlim_{N \in \mathcal{N}} H^2\left(G/N, A^N\right).$$

On the other hand, observe that $i \le j$ in $\mathcal{I}$ induces $G_j \to G_i$, so $N_j \subseteq N_i$. In other words, $i \mapsto N_i$ will define a functor $\mathcal{I} \to \mathcal{N}$; functoriality follows because $\mathcal{I}$ and $\mathcal{N}$ are poset categories. Letting $\mathcal{N}'$ denote the image of $\mathcal{I}$ in $\mathcal{N}$, we see

$$\varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right) \simeq \varinjlim_{N \in \mathcal{N}'} H^2\left(G/N, A^N\right).$$

Notably, the inflation maps $\mathrm{Inf} \colon H^2\left(G_i, A^{N_i}\right) \to H^2\left(G_j, A^{N_j}\right)$ when $i \le j$ become the inflation maps $\mathrm{Inf} \colon H^2\left(G/N, A^N\right) \to H^2\left(G/N', A^{N'}\right)$ when $N' \subseteq N$. So if we let $F \colon \mathcal{N} \to \mathrm{AbGrp}$ be the functor taking N to $H^2\left(G/N, A^N\right)$ (and $N \subseteq N'$ to the inflation map), we are trying to show

$$\varinjlim_{\mathcal{N}} F = \varinjlim_{\mathcal{N}'} F.$$

For this, we use Lemma 104. Indeed, for a given open normal subgroup $N \in \mathcal{N}$, we need to find some $N' \in \mathcal{N}'$ such that $N \leq N'$, which means $N' \subseteq N$.

However, the elements of $\mathcal{N}'$ are the collection $\{N_i\}_{i\in\mathcal{I}}$, which form a fundamental system of open neighborhoods around the identity. Thus, the fact that N is an open set containing the identity implies there is some $N_i \in \mathcal{N}'$ such that $N_i \subseteq N$. This finishes the proof. ■

Observe that the above proofs did not use the extra hypotheses on G nor N_i to be products of procyclic groups. We use these hypotheses now. To work more concretely, we note that any $i \in \mathcal{I}$ has

$$G_i \simeq \frac{G}{N_i} \simeq \bigoplus_{p=1}^{m} \overline{\langle \sigma_p \rangle} / \overline{\langle \sigma_p^{t_{i,p}} \rangle} \simeq \bigoplus_{p=1}^{m} \langle \sigma_p \rangle / \langle \sigma_p^{t_{i,p}} \rangle \subseteq \bigoplus_{p=1}^{m} \mathbb{Z}/t_{i,p}\mathbb{Z}$$

is a finite abelian group generated by the elements $\sigma_p N_i$. As a warning, the order of $\sigma_p N_i$ might not be $t_{i,p}$, for example if σ_p itself has some small finite order which $t_{i,p}$ is not properly capitalizing on. More concretely, $\mathbb{Z}_5/3\mathbb{Z}_5 = 0$.

Regardless, the main point is that, given a discrete G-module A, we can consider the $\{\sigma_p N_i\}_{p=1}^m$-tuples $\mathcal{T}\left(G_i, A^{N_i}\right)$. Now, as discussed above, $i \leq j$ in $\mathcal{I}$ induces a quotient map $G_j \simeq G/N_j \twoheadrightarrow G_i/N_i$. From this, we have the following coherence check.

Lemma 107. Fix everything as in the profinite set-up, and let A be a discrete G-module. Then, given $i \leq j \leq k$ in $\mathcal{I}$, the diagram

$$\begin{array}{ccc} \mathcal{T}\left(G_i, A^{N_i}\right) & \xrightarrow{\text{Inf}} & \mathcal{T}\left(G_j, A^{N_j}\right) \\ & \searrow^{\text{Inf}} & \downarrow{\scriptstyle \text{Inf}} \\ & & \mathcal{T}\left(G_k, A^{N_k}\right) \end{array}$$

commutes. Here, the Inf maps are defined as in Lemma 97.

Proof. For each $i \in \mathcal{I}$, we let $n_{i,p}$ denote the order of $\sigma_p N_i \in G_i$. Using the definition of Inf from Lemma 97, we just pick up some $\{\sigma_p N_p\}_{p=1}^m$-tuple $(\{\alpha_p\}, \{\beta_{pq}\})$-tuple in $\mathcal{T}\left(G_i, A^{N_i}\right)$ and track through the diagram as follows.

$$\begin{array}{ccc} (\{\alpha_p\}, \{\beta_{pq}\}) & \xrightarrow{\text{Inf}} & \left(\{\alpha_p^{n_{j,p}/n_{i,p}}\}, \{\beta_{pq}\}\right) \\ {\scriptstyle \text{Inf}}\downarrow & & \downarrow{\scriptstyle \text{Inf}} \\ \left(\{\alpha_p^{n_{k,p}/n_{i,p}}\}, \{\beta_{pq}\}\right) & = & \left(\{\alpha_p^{(n_{j,p}/n_{i,p})(n_{k,p}/n_{j,p})}\}, \{\beta_{pq}\}\right) \end{array}$$

This completes the proof. ■

And here is the result.

Theorem 108. Fix everything as in the profinite set-up, and let A be a discrete G-module. Then the isomorphisms of Theorem 95 upgrade into an isomorphism

$$H^2(G, A) \simeq \varinjlim_{i\in\mathcal{I}} \overline{\mathcal{T}}\left(G_i, A^{N_i}\right).$$

Here the morphisms between the $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ are inflation maps of Lemma 97.

Proof. Note that the objects $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ do make a directed system over $\mathcal{I}$ because of the commutativity of Lemma 107. Namely, the lemma checks that $\mathcal{I} \to \mathrm{AbGrp}$ by $i \mapsto \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ is actually functorial; technically we must also check that the maps $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right) \to \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ are the identity, but this follows from the definition.

Now, by Proposition 106, we have

$$H^2(G,A) \simeq \varinjlim_{i\in\mathcal{I}} H^2\left(G_i, A^{N_i}\right),$$

but now the natural isomorphism induced by Remark 99 induces an isomorphism of direct limits

$$\varinjlim_{i\in\mathcal{I}} H^2\left(G_i, A^{N_i}\right) \simeq \varinjlim_{i\in\mathcal{I}} \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$$

given by the isomorphism of Theorem 95 acting pointwise. This completes the proof. ■

Because there are reasonably explicit descriptions of direct limits of abelian groups, and we already have an explicit description of each $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ term in addition to a description of the inflation maps between them, we will be content with our sufficiently explicit description of $H^2(G,A)$. So we call it done here.

4 Studying Tuples

The story so far has been able to generalize the one-variable results from section 2 to results using all generators of an abelian group in section 3. It remains to prove Theorem 87, which is the main goal of this section.

4.1 Set-Up and Overview

The approach here will be to attempt to abstract our data away from the G-module A as much as possible. To set up our discussion, we continue with

$$G \simeq \bigoplus_{i=1}^{m} G_i,$$

where $G_i = \langle\sigma_i\rangle \subseteq G$ and σ_i has order n_k. These variables allow us to define

$$T_i \coloneqq (\sigma_i - 1) \qquad \text{and} \qquad N_i \coloneqq \sum_{p=0}^{n_i-1} \sigma_i^p$$

for each index i. In fact, it will be helpful to also have notation

$$\sigma^{(a)} \coloneqq \sum_{p=0}^{a-1} \sigma^p$$

for any $\sigma \in G$ and nonnegative integer $a \geq 0$; in particular, $\sigma^{(0)} = 0$ and $\sigma_i^{(n_i)} = N_i$. The main benefits to this notation will be the facts that

$$\sigma^{(a+b)} = \sigma^{(a)} + \sigma^a\sigma^{(b)} \qquad \text{and} \qquad \sigma_i^a = T_i\sigma_i^{(a)} + 1,$$

which can be seen by direct expansion. Given $g \in \prod_{p=1}^{n} \sigma_p^{a_p}$, we will also define the notation

$$g_i \coloneqq \prod_{p=1}^{i-1} \sigma_p^{a_p}$$

for $i \geq 0$. In particular $g_0 = g_1 = 1$ and $g_{n+1} = g$.

Now, our tool in the proof of Theorem 87 will be the magical map $\mathcal{F}\colon \mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}} \to \mathbb{Z}[G]^m$ defined by

$$\mathcal{F}\colon \left((x_i)_{i=1}^m, (y_{ij})_{i>j}\right) \mapsto \left(x_iN_i - \sum_{j=1}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j\right)_{i=1}^{m}.$$

This is of course a G-module homomorphism. We will go ahead and state the main results we will prove. Roughly speaking, $\mathcal{F}$ is manufactured to make the following result true.

Proposition 109. Fix everything as in the set-up. Then the function

$$\overline{c}(g) \coloneqq \left(g_i \sigma_i^{(a_i)}\right)_{i=1}^m,$$

where $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$, is a 1-cocycle in $Z^1(G, \operatorname{coker} \mathcal{F})$.

The reason we care about this cocycle is that we can pass it through a boundary morphism induced by the short exact sequence

$$0 \to \underbrace{\frac{\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}}{\ker \mathcal{F}}}_{X \coloneqq} \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0,$$

so we have a 2-cocycle $\delta(\overline{c}) \in Z^2(G, X)$; in fact, we will be able to explicitly compute $\delta(\overline{c})$ as a result of the proof of Proposition 109.

Only now will we bring in tuples. The first result provides an alternate description of tuples.

Proposition 110. Fix everything as in the set-up, and now let A be a G-module. Then $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\operatorname{Hom}_{\mathbb{Z}[G]}(X, A) = H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A))$.

The second result brings in the last ingredient, the cup product.

Theorem 111. Fix everything as in the set-up. Also, fix a G-module A and a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$. Then observe there is a natural cup product map

$$\cup \colon H^2(G, X) \times H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A)) \to H^2(G, A)$$

by using the evaluation map $X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, A) \to A$. Then, using the isomorphism of Proposition 110, the cocycle defined in Theorem 87 is simply the output of $\delta(\overline{c}) \cup (\{\alpha_i\}, \{\beta_{ij}\})$ on cocycles.

Because we know that the cup product sends cocycles to cocycles, this will show that the cocycle of Theorem 87 is in fact well-defined.

4.2 Preliminary Work

We continue in the set-up of the previous subsection. Before jumping into any hard logic, we define some (more) notation which will be useful later on as well. First, in $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$, we define

$$\kappa_p \coloneqq \left((1_{i=p})_i, (0)_{i>j}\right) \in X \qquad \text{and} \qquad \lambda_{pq} \coloneqq \left((0)_i, (1_{(i,j)=(p,q)})_{i>j}\right)$$

for all relevant indices p and q so that the κ_p and λ_{pq} are a basis for $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$ as a $\mathbb{Z}[G]$-module. Secondly, we define

$$\varepsilon_p \coloneqq (1_{i=p})_{i=1}^m$$

for all indices p, again giving a basis for $\mathbb{Z}[G]^m$ as a $\mathbb{Z}[G]$-module. For example, this notation lets us write

$$\mathcal{F}\left(\sum_{i=1}^m x_i \kappa_i + \sum_{i>j} y_{ij} \lambda_{ij}\right) = \sum_{i=1}^m x_i N_i \varepsilon_i + \sum_{i>j} y_{ij}(T_i \varepsilon_j - T_j \varepsilon_i), \tag{4.1}$$

and

$$\overline{c}(g) = \sum_{i=1}^m g_i \sigma_i^{(a_i)} \varepsilon_i$$

where $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$.

Additionally, so that we do not need to interrupt our discussion later, we establish a few lemmas which will aide our proof of Proposition 109.

Lemma 112. Fix everything as in the set-up. Then, for any set of distinct indices $(i_1, \dots, i_k)$, we have

$$\bigcap_{p=1}^{k} \operatorname{im} N_{i_p} = \operatorname{im} \prod_{p=1}^{k} N_{i_p},$$

where we are identifying $x \in \mathbb{Z}[G]$ with its associated multiplication map $x \colon \mathbb{Z}[G] \to \mathbb{Z}[G]$.

Proof. The point is that the elements of $\bigcap_{p=1}^{k} \operatorname{im} N_{i_p}$ and $\operatorname{im} \prod_{p=1}^{k} N_{i_p}$ are both simply the elements whose expansion in the form $\sum_g c_g g \in \mathbb{Z}[G]$ have c_j "constant in σ_p and σ_q." More explicitly, of course, $\prod_{p=1}^{k} N_{i_p} \in \bigcap_{p=1}^{k} \operatorname{im} N_{i_p}$, so

$$\operatorname{im} \prod_{p=1}^{k} N_{i_p} \subseteq \bigcap_{p=1}^{k} \operatorname{im} N_{i_p}.$$

In the other direction, suppose that we have some element

$$z \coloneqq \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \in \bigcap_{p=1}^{k} \operatorname{im} N_{i_p},$$

the sum is over sequences $(a_i)_{i=1}^m$ such that $0 \le a_i < n_i$ for each index i. We will show $z \in \operatorname{im} \prod_{p=1}^{k} N_{i_p}$.

Now, $z \in \operatorname{im} N_r$ for $r \in \{p, q\}$ is equivalent to $z \in \ker T_r$, but upon multiplying by $(\sigma_r - 1)$ we see that we are asking for

$$\sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_{r-1}^{a_{r-1}} \sigma_r^{a_r} \sigma_{r+1}^{a_{r+1}} \cdots \sigma_n^{a_n} = \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_{r-1}^{a_{r-1}} \sigma_r^{a_r+1} \sigma_{r+1}^{a_{r+1}} \cdots \sigma_n^{a_n}.$$

In other words, this is asking for $c_{(a_i)_i} = c_{(a_i)_i + (1_{i=r})_i}$, or more succinctly just that c is constant in the $i = r$ coordinate.

Thus, c is constant in all the $i = i_p$ coordinates for each index i_p. Thus, we let $d_{(a_i)_{i \notin \{i_p\}}}$ be the restricted function equal to $c_{(a_i)_i}$ but forgetting the information input from any of the a_{i_p}. This allows us to write

$$\begin{aligned} z &= \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \\ &= \sum_{(a_i)_{i \notin \{i_p\}}} \sum_{a_{i_1}=0}^{n_{i_1}-1} \cdots \sum_{a_{i_k}=0}^{n_{i_k}-1} d_{(a_i)_{i \notin \{i_p\}}} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \\ &= \left(\sum_{(a_i)_{i \notin \{i_p\}}} d_{(a_i)_{i \notin \{i_p\}}} \prod_{\substack{i=0 \\ i \notin \{i_p\}}}^{m} \sigma_i^{a_i} \right) \left(\sum_{a_{i_1}=0}^{n_{i_1}-1} \sigma_{i_1}^{a_{i_1}} \right) \cdots \left(\sum_{a_{i_k}=0}^{n_{i_k}-1} \sigma_{i_k}^{a_{i_k}} \right), \end{aligned}$$

which is now manifestly in $\operatorname{im} \prod_{p=1}^{k} N_{i_p}$. ∎

Lemma 113. Fix everything as in the set-up. Then, given $g \coloneqq \prod_{i=1}^{m} \sigma_i^{a_i}$, we have

$$g_i = 1 + \sum_{p=1}^{i-1} g_p \sigma_p^{(a_p)} T_p$$

for $i \ge 1$.

Proof. This is by induction. For $i = 1$, there is nothing to say. For the inductive step, we take $i > 1$ where we may assume the statement for $i - 1$. Via some relabeling, we may make our inductive hypothesis assert

$$\prod_{p=2}^{i-1} \sigma_p^{a_p} = 1 + \sum_{p=2}^{i-1} \left(\prod_{q=2}^{p-1} \sigma_q^{a_q} \right) \sigma_p^{(a_p)} T_p.$$

In particular, multiplying through by $\sigma_1^{a_1}$ yields

$$\begin{aligned} g_i &= \sigma_1^{a_1} \cdot \prod_{p=2}^{i-1} \sigma_p^{a_p} \\ &= \sigma_1^{a_1} + \sigma_1^{a_1} \sum_{p=2}^{i-1} \left(\prod_{q=2}^{p-1} \sigma_q^{a_q} \right) \sigma_p^{(a_p)} T_p \\ &= \sigma_1^{a_1} + \sum_{p=2}^{i-1} g_p \sigma_p^{(a_p)} T_p \\ &= 1 + \sigma_1^{(a_1)} T_1 + \sum_{p=2}^{i-1} g_p \sigma_p^{(a_p)} T_p, \end{aligned}$$

which is exactly what we wanted, after a little more rearrangement. ∎

And mostly because we can, we show that our main short exact sequence splits.

Lemma 114. Fix everything as in the set-up. Then consider $\mathbb{Z}$-module map $\rho\colon \mathbb{Z}[G]^m \to \mathbb{Z}[G]^m$ defined by

$$\rho(g\varepsilon_i) \coloneqq g_i\big(\sigma_i^{a_i} - N_i 1_{a_i = n_i - 1}\big)\varepsilon_i + \sum_{j=i+1}^{m} g_j \sigma_j^{(a_j)} T_i \varepsilon_j,$$

where $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$ with $0 \le a_i < n_i$. Then ρ descends to a map $\overline{\rho}\colon \operatorname{coker} \mathcal{F} \to \mathbb{Z}[G]^m$ witnessing the splitting of the short exact sequence

$$0 \to X \to \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0$$

over $\mathbb{Z}$.

Proof. Observe that we have a well-defined map $\rho\colon \mathbb{Z}[G]^m \to \mathbb{Z}[G]^m$ because $\mathbb{Z}[G]^m$ is a free abelian group generated by $g\varepsilon_i$ for $g \in G$ and indices i. It remains to show that $\operatorname{im} \mathcal{F} \subseteq \ker \rho$ to get a map $\overline{\rho}\colon \operatorname{coker} \mathcal{F} \to \mathbb{Z}[G]^m$ and then to show that $\rho(z) \equiv z \pmod{\operatorname{im} \mathcal{F}}$ to get the splitting. We show these individually.

To show that $\operatorname{im} \mathcal{F} \subseteq \ker \rho$, we note from (4.1) that $\operatorname{im} \mathcal{F}$ is generated over $\mathbb{Z}[G]$ by the elements $N_i \varepsilon_i$ and $T_i \varepsilon_j - T_j \varepsilon_i$ for relevant indices i and j. Thus, $\operatorname{im} \mathcal{F}$ is generated over $\mathbb{Z}$ by the elements $gN_i\varepsilon_i$ and $gT_i\varepsilon_j - gT_j\varepsilon_i$ for relevant indices i and j. Thus, we fix any $g \coloneqq \prod_{i=1}^n \sigma_i^{a_i}$ and show that $gN_i\varepsilon_i \in \ker \rho$ and $gT_i\varepsilon_j - gT_j\varepsilon_i \in \ker \rho$ for relevant indices i and j.

- We show $gN_i\varepsilon_i \in \ker \rho$ for any i. Because $gN_i = g\sigma_i N_i$, we may as well as assume that $a_i = 0$. Then

$$\rho\left(g\sigma_i^a \varepsilon_i\right) = g_i\big(\sigma_i^a - N_i 1_{a = n_i - 1}\big)\varepsilon_i + \sum_{j=i+1}^{m} g_j \sigma_i^a \sigma_j^{(a_j)} T_i \varepsilon_j.$$

  As a varies from 0 to $n_i - 1$, we note that the term $g_i\big(\sigma_i^a - N_i 1_{a=n_i-1}\big)\varepsilon_i$ will only get the $-N_i$ contribution exactly once at $a = n_i - 1$. Summing, we thus see that

$$\rho(gN_i\varepsilon_i) = g_i\left(-N_i + \sum_{a=0}^{n_i-1} \sigma_i^a \right)\varepsilon_i + \sum_{a=0}^{n_i-1} \sum_{j=i+1}^{m} g_j \sigma_i^a \sigma_j^{(a_j)} T_i \varepsilon_j.$$

The left term vanishes because $N_i = \sum_{a=0}^{n_i-1} \sigma_i^a$. Additionally, the right term vanishes because we can factor $T_i \sum_{a=0}^{n_i-1} \sigma_i^a = T_i N_i = 0$. So $gN_i\varepsilon_i \in \ker \rho$.

- We show $gT_p\varepsilon_q - gT_q\varepsilon_p \in \ker \rho$ for any $p > q$. Equivalently, we will show that $\rho(g\sigma_p\varepsilon_q) - \rho(g\varepsilon_q) = \rho(g\sigma_q\varepsilon_p) - \rho(g\varepsilon_p)$. On one hand, note

$$\begin{aligned}\rho(g\sigma_p\varepsilon_q) = g_q\big(\sigma_q^{a_q} - N_i 1_{a_q=n_q-1}\big)\varepsilon_q \\ + \sum_{j=q+1}^{p-1} g_j\sigma_j^{(a_j)}T_q\varepsilon_j \\ + g_p\left(\sigma_p^{(a_p+1)} - N_p 1_{a_p=n_p-1}\right)T_q\varepsilon_p \\ + \sum_{j=p+1}^{m} \sigma_p g_j\sigma_j^{(a_j)}T_q\varepsilon_j\end{aligned}$$

because g_j doesn't "see" the extra σ_p term until $j > p$. (For the $j = p$ term, we would like to write $\sigma_p^{(a_p+1)}$ above, but when $a_p = n_p - 1$, we actually end up with $\sigma_p^{(0)} = 0$ and hence have to subtract out $\sigma_p^{(n_p)} = N_p$.) Thus,

$$\rho(g\sigma_p\varepsilon_q) - \rho(g\varepsilon_q) = g_p\left(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\right)T_q\varepsilon_p + \sum_{j=p+1}^{m} g_j\sigma_j^{(a_j)}T_pT_q\varepsilon_j.$$

On the other hand, we have

$$\rho(g\sigma_q\varepsilon_p) = \sigma_q g_p\big(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\big)\varepsilon_p + \sum_{j=p+1}^{m} \sigma_q g_j\sigma_j^{(a_j)}T_p\varepsilon_j$$

where this time all $j > p$ also have $j > q$ and so $(\sigma_q g)_j = \sigma_q g_j$. Thus,

$$\rho(g\sigma_q\varepsilon_p) - \rho(g\varepsilon_p) = g_p\left(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\right)T_q\varepsilon_p + \sum_{j=p+1}^{m} g_j\sigma_j^{(a_j)}T_pT_q\varepsilon_j,$$

as desired.

We now check the splitting. For this, we simply need to check that $\rho(g\varepsilon_i) \equiv g\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$, and we will get the result for all elements of $\mathbb{Z}[G]^m$ by additivity of ρ. Well, using Lemma 113, we write

$$\begin{aligned}g\varepsilon_i &= g_i\sigma_i^{a_i}\left(\prod_{j=i+1}^{m}\sigma_j^{a_j}\right)\varepsilon_i \\ &= g_i\sigma_i^{a_i}\left(1 + \sum_{j=i+1}^{m}\left(\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)\sigma_j^{(a_j)}T_j\right)\varepsilon_i \\ &= g_i\sigma_i^{a_i}\varepsilon_i + \sum_{j=i+1}^{m} g_i\sigma_i^{a_i}\left(\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)\sigma_j^{(a_j)}T_j\varepsilon_i \\ &\equiv g_i\sigma_i^{a_i}\varepsilon_i + \sum_{j=i+1}^{m} g_j\sigma_j^{(a_j)}T_i\varepsilon_j,\end{aligned}$$

where in the last step we have used the fact that $T_j\varepsilon_i \equiv T_j\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$. Lastly, we note that $hN_i\varepsilon_i \equiv h\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$ for any $h \in G$, so in fact

$$g\varepsilon_i \equiv g_i\left(\sigma_i^{a_i} - N_i 1_{a_i=n_i-1}\right)\varepsilon_i + \sum_{j=i+1}^{m} g_j\sigma_j^{(a_j)}T_i\varepsilon_j,$$

and now the right-hand side is $\rho(g\varepsilon_i)$. ■

4.3 Verification of 1-Cocycles

Here we prove Proposition 109. Namely, we show that the 1-cochain $\overline{c} \in C^1(G, \operatorname{coker} \mathcal{F})$ defined by

$$\overline{c}(g) = \sum_{i=1}^{m} g_i \sigma_i^{(a_i)} \varepsilon_i$$

where $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$ is actually a 1-cocycle. It will be beneficial for us to do this by hand, which is a matter of brute force. Set $c \in C^1\left(G, \mathbb{Z}[G]^m\right)$ defined by

$$c(g) \coloneqq \left(g_i \sigma_i^{(a_i)}\right)_{i=1}^m,$$

where $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$. We will show that $\operatorname{im} dc \subseteq \operatorname{im} \mathcal{F}$, which we will mean that $\operatorname{im} \overline{dc} = \operatorname{im} d\overline{c} = 0$, where $f \mapsto \overline{f}$ is the map $C^\bullet\left(G, \mathbb{Z}[G]^m\right) \twoheadrightarrow C^\bullet\left(G, \operatorname{coker} \mathcal{F}\right)$ induced by modding out.

As such, we set $g \coloneqq \prod_{i=1}^m \sigma_i^{a_i}$ and $h \coloneqq \prod_{i=1}^m \sigma_i^{b_i}$ with $0 \le a_i, b_i < n_i$ for each i. Then, using the division algorithm, write

$$a_i + b_i = n_i q_i + r_i$$

where $q_i \in \{0, 1\}$ and $0 \le r_i < n_i$ for each i. Now, we want to show $dc(g, h) \in \operatorname{im} \mathcal{F}$, so we begin by writing

$$\begin{aligned} dc(g,h) &= gc(h) - c(gh) + c(g) \\ &= g\left(h_i \sigma_i^{(b_i)}\right)_{i=1}^m - \left(\prod_{p=0}^{i-1} \sigma_p^{r_p} \cdot \sigma_i^{(r_i)}\right)_{i=1}^m + \left(g_i \sigma_i^{(a_i)}\right)_{i=1}^m \\ &= \left(g h_i \sigma_i^{(b_i)}\right)_{i=1}^m - \left(g_i h_i \sigma_i^{(r_i)}\right)_{i=1}^m + \left(g_i \sigma_i^{(a_i)}\right)_{i=1}^m. \end{aligned} \tag{4.2}$$

We now go term-by-term in (4.2). The easiest is the middle term of (4.2), for which we write

$$\begin{aligned} g_i h_i \sigma_i^{(r_i)} &= g_i h_i \sigma_i^{(a_i+b_i)} - g_i h_i \sigma_i^{r_i} \sigma_i^{(n_i q_i)} \\ &= g_i h_i \sigma_i^{(a_i+b_i)} - g_i h_i \sigma_i^{a_i+b_i} \cdot q_i N_i \\ &= g_i h_i \sigma_i^{(a_i+b_i)} - g_i h_i \cdot q_i N_i, \end{aligned}$$

where the last equality is because $\sigma_i N_i = N_i$. Thus,

$$\begin{aligned} -\left(g_i h_i \sigma_i^{(r_i)}\right)_{i=1}^m &= -\left(g_i h_i \sigma_i^{(a_i+b_i)}\right)_{i=1}^m + \left(g_i h_i \cdot q_i N_i\right)_{i=1}^m \\ &= -\left(g_i h_i \sigma_i^{(a_i+b_i)}\right)_{i=1}^m + \mathcal{F}\big((g_i h_i q_i)_i, (0)_{i>j}\big). \end{aligned}$$

Now, using Lemma 113, the ith coordinate of the left term of (4.2) is

$$\begin{aligned} g h_i \sigma_i^{(b_i)} &= g_i \sigma_i^{a_i} \left(\prod_{j=i+1}^m \sigma_j^{a_j}\right) h_i \sigma_i^{(b_i)} \\ &= g_i \left(1 + \sum_{j=i+1}^m \left(\prod_{q=i+1}^{j-1} \sigma_q^{a_q}\right) \sigma_j^{(a_j)} T_j\right) h_i \sigma_i^{a_i} \sigma_i^{(b_i)} \\ &= g_i h_i \sigma_i^{a_i} \sigma_i^{(b_i)} + \sum_{j=i+1}^m \left(g_i \sigma_i^{a_i} \prod_{q=i+1}^{j-1} \sigma_q^{a_q}\right) h_i \sigma_j^{(a_j)} \sigma_i^{(b_i)} T_j \\ &= g_i h_i \sigma_i^{a_i} \sigma_i^{(b_i)} + \sum_{j=i+1}^m g_j h_i \sigma_j^{(a_j)} \sigma_i^{(b_i)} T_j. \end{aligned}$$

And lastly, for the right term of (4.2), the ith coordinate is

$$\begin{aligned} g_i\sigma_i^{(a_i)} &= g_i\left(h_i - \sum_{j=1}^{i-1} h_j\sigma_j^{(b_j)}T_j\right)\sigma_i^{(a_i)} \\ &= g_ih_i\sigma_i^{(a_i)} - \sum_{j=1}^{i-1} g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j. \end{aligned}$$

So to finish, we continue from (4.2), which gives

$$\begin{aligned} dc(g,h) - \mathcal{F}\big((g_ih_iq_i)_i, (0)_{i>j}\big) &= \left(g_ih_i\sigma_i^{a_i}\sigma_i^{(b_i)}\right)_{i=1}^m - \left(g_ih_i\sigma_i^{(a_i+b_i)}\right)_{i=1}^m + \left(g_ih_i\sigma_i^{(a_i)}\right)_{i=1}^m \\ &\quad + \left(\sum_{j=i+1}^{m} g_jh_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j - \sum_{j=1}^{i-1} g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j\right)_{i=1}^m \\ &= \left(-\sum_{j=1}^{i-1} g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j + \sum_{j=i+1}^{m} g_jh_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j\right)_{i=1}^m \\ &= \mathcal{F}\left((0)_i, \left(g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}\right)_{i>j}\right). \end{aligned}$$

Thus,

$$dc(g,h) = \mathcal{F}\left((g_ih_iq_i)_i, \left(g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}\right)_{i>j}\right) \in \operatorname{im}\mathcal{F}. \tag{4.3}$$

This completes the proof of Proposition 109.

In fact, the above proof has found an explicit element z so that $\mathcal{F}(z) = dc(g,h)$ for each $g,h \in G$. As such, we recall that we set

$$X \coloneqq \frac{\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}}{\ker\mathcal{F}}$$

to give the short exact sequence

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker}\mathcal{F} \to 0.$$

In particular, we can track $\overline{c} \in Z^1(G, \operatorname{coker}\mathcal{F})$ through a boundary morphism: we already have a chosen lift $c \in Z^1(G, \mathbb{Z}[G]^m)$ for $\overline{c}$, and we have also computed $\mathcal{F}^{-1} \circ dc$ from the above work. This gives the following result.

Corollary 115. Fix everything as in the set-up. Then the $\overline{c}$ of Proposition 109 has

$$\delta(c)(g,h) \coloneqq \left((g_ih_iq_i)_i, \left(g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}\right)_{i>j}\right) \in Z^2(G,X)$$

where δ is induced by

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker}\mathcal{F} \to 0.$$

Proof. This follows from tracking how δ behaves, using (4.3). ∎

Remark 116. In some sense, this $\delta(c)$ is exactly the cocycle of Theorem 87, where we have abstracted away everything about A. We will rigorize this notion in our proof of Theorem 111.

4.4 Tuples via Cohomology

We continue in the set-up of the previous subsection. The goal of this subsection is to prove Proposition 110. The main idea is that we will be able to finitely generate $\ker \mathcal{F}$ essentially using the relations of a $\{\sigma_i\}_{i=1}^m$-tuple.

We start with the following basic result.

Lemma 117. Fix everything as in the set-up. Then $\ker \mathcal{F}$ contains the following elements.

(a) $T_p\kappa_p$ for any index p.

(b) $N_pN_q\lambda_{pq}$ for any pair of indices (p,q) with $p > q$.

(c) $T_q\kappa_p + N_p\lambda_{pq}$ for any pair of indices (p,q) with $p > q$.

(d) $T_p\kappa_q - N_q\lambda_{pq}$ for any pair of indices (p,q) with $p > q$.

(e) $T_q\lambda_{pr} - T_r\lambda_{pq} - T_p\lambda_{qr}$ for any triplet of indices (p,q,r) with $p > q > r$.

Proof. We start by showing that all the listed elements are in fact in $\ker \mathcal{F}$.

(a) Note that $\mathcal{F}$ only ever takes the x_i term to x_iN_i, so if $x_i = T_i$, then the effect of x_i vanishes.

(b) Similarly, note that $\mathcal{F}$ only ever takes the y_{ij} term to $y_{ij}T_i$ or $y_{ij}T_j$. As such, if $y_{ij} = N_iN_j$, then the effect of y_{ij} vanishes again.

(c) The only relevant terms are at indices p and q. Here, $i = p$ has $\mathcal{F}$ output

$$T_qN_p - N_pT_q + 0 = 0.$$

For $i = q$, we have no x_q term, so we are left with $N_pT_p = 0$.

(d) Again, the only relevant terms are at indices p and q. This time the interesting term is at $i = q$, where we have

$$T_pN_q - 0 + (-N_q)T_p = 0.$$

Then at $i = p$, we simply have $0N_p - (-N_q)T_q + 0 = 0$.

(e) The relevant terms, as usual, are for $i \in \{p,q,r\}$.

- At $i = p$, we have $0 - (T_qT_r + (-T_r)T_q) + 0 = 0$.
- At $i = q$, we have $0 - (-(T_p)T_r) + ((-T_r)T_p) = 0$.
- At $i = r$, we have $0 - 0 + (T_qT_p + (-T_p)T_q) = 0$.

The above checks complete this part of the proof. ∎

Remark 118. The above elements are intended to encode the relations to be a $\{\sigma_i\}_{i=1}^n$-tuple. We will see this made rigorous in the proof of Proposition 110.

In fact, the following is true.

Lemma 119. Fix everything as in the set-up. Then the elements (a)–(e) of Lemma 117, with (b) removed, generate $\ker \mathcal{F}$.

Proof. We remark that we callously removed (b) because it is implied (c): $T_q\kappa_p + N_p\lambda_{pq} \in \ker \mathcal{F}$ implies that

$$N_q \cdot (T_q\kappa_p + N_p\lambda_{pq}) = N_pN_q\lambda_{pq}$$

is also in $\ker \mathcal{F}$. Anyway, this proof is long and annoying and hence relegated to Appendix B. ∎

Here is the payoff for the hard work in Lemma 119.

Proposition 110. Fix everything as in the set-up, and now let A be a G-module. Then $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]}(X, A) = H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$.

Proof. Let $\mathcal{T}$ denote the set of $\{\sigma_i\}_{i=1}^m$-tuples. We now define the map $\varphi\colon \mathrm{Hom}_{\mathbb{Z}[G]}(X, A) \to \mathcal{T}$ by

$$\varphi\colon f \mapsto \Big(\big(f(\kappa_i)\big)_i, \big(f(\lambda_{ij})\big)_{i>j} \Big).$$

In other words, we simply read off the values of f from indicators on the coordinates of X. It's not hard to see that φ is in fact a G-module homomorphism, but we will have to check that φ is well-defined, for which we have to check the conditions on being a $\{\sigma_i\}_{i=1}^m$-tuple.

Lemma 120. Fix everything as in the set-up, and let A be a G-module. Then, given $f\colon \mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$, we have $\ker \mathcal{F} \subseteq \ker f$ if and only if

$$\Big(\big(f(\kappa_i)\big)_i, \big(f(\lambda_{ij})\big)_{i>j} \Big)$$

is a $\{\sigma_i\}_{i=1}^m$-tuple.

Proof. By Lemma 119, we see $\ker \mathcal{F} \subseteq \ker f$ if and only if f vanishes on the elements given in Lemma 117. As such, we now run the following checks.

1. We discuss (3.1). For one, note that $f(\lambda_{ij}) \in A$ essentially for free. Now, we note

$$\begin{aligned} f(\kappa_i) \in A^{\langle \sigma_i \rangle} &\iff T_i f(\kappa_i) = 0 \\ &\iff f(T_i \kappa_i) = 0 \\ &\iff T_i \kappa_i \in \ker f. \end{aligned}$$

2. We discuss (3.2). On one hand, note that $i > j$ has

$$\begin{aligned} N_i f(\lambda_{ij}) = -T_j f(\lambda_i) &\iff f(N_i \lambda_{ij} + T_j \lambda_i) \\ &\iff N_i \lambda_{ij} + T_j \lambda_i \in \ker f. \end{aligned}$$

   On the other hand,

$$\begin{aligned} -N_j f(\lambda_{ij}) = -T_i f(\lambda_j) &\iff f(N_j \lambda_{ij} + T_i \lambda_j) = 0 \\ &\iff N_j \lambda_{ij} + T_i \lambda_j \in \ker f. \end{aligned}$$

3. We discuss (3.3). Simply note indices $i > j > k$ have

$$\begin{aligned} T_j f(\lambda_{ik}) = T_k f(\lambda_{ij}) + T_i f(\lambda_{jk}) &\iff f(T_j \lambda_{ik} - T_k \lambda_{ij} - T_i \lambda_{jk}) = 0 \\ &\iff T_j \lambda_{ik} - T_k \lambda_{ij} - T_i \lambda_{jk} \in \ker f. \end{aligned}$$

In total, we see that satisfying the relations to be a $\{\sigma_i\}_{i=1}^m$-tuple exactly encodes the data of having the generators of $\ker \mathcal{F}$ live in $\ker f$. ∎

So indeed, given $f\colon X \to A$, the above lemma applied to the composite

$$\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}} \twoheadrightarrow X \xrightarrow{f} A$$

shows that $\varphi(f) \in \mathcal{T}$.

To show that φ is an isomorphism, we exhibit its inverse; fix some $(\{\alpha_i\}, \{\beta_{ij}\}_{i>j}) \in \mathcal{T}$. Well, $\mathbb{Z}[G] \times \mathbb{Z}[G]^{\binom{m}{2}}$ has as a basis the κ_i and λ_{ij}, so we can uniquely define a G-module homomorphism $f\colon X \to A$ by

$$f(\kappa_i) \coloneqq \alpha_i \qquad \text{and} \qquad f(\lambda_{ij}) \coloneqq \beta_{ij}$$

for all relevant indices i, j, and in fact the map $\mathcal{T} \to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}, A\right)$ we can see to be a G-module homomorphism. However, because these outputs are a $\{\sigma_i\}_{i=1}^m$-tuple, we can read Lemma 120 backward to say that f has kernel containing $\ker \mathcal{F}$, so in fact we induce a map $\overline{f}\colon X \to A$.

So in total, we get a G-module homomorphism $\psi\colon \mathcal{T} \to \operatorname{Hom}_{\mathbb{Z}[G]}(X, A)$ by

$$\psi\colon (\{\alpha_i\}, \{\beta_{ij}\}_{i>j}) \mapsto \overline{f},$$

where $\overline{f}$ is defined on the basis elements above. Further, ψ is the inverse of φ essentially because the $\{\kappa_i\}_i \cup \{\lambda_{ij}\}_{i>j}$ form a basis of $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$. This completes the proof. ■

And now because it is so easy, we might as well prove Theorem 111.

Theorem 111. Fix everything as in the set-up. Also, fix a G-module A and a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$. Then observe there is a natural cup product map

$$\cup\colon H^2(G, X) \times H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A)) \to H^2(G, A)$$

by using the evaluation map $X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, A) \to A$. Then, using the isomorphism of Proposition 110, the cocycle defined in Theorem 87 is simply the output of $\delta(\overline{c}) \cup (\{\alpha_i\}, \{\beta_{ij}\})$ on cocycles.

Proof. The main point is that we have a computation of $\delta(\overline{c})$ from Corollary 115, which we merely need to track through. In particular, fix a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}_i, \{\beta_{ij}\}_{i>j})$, and let $f \in H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A))$ be the corresponding morphism. As such, we may compute

$$\delta(\overline{c}) \cup f\colon (g, h) \mapsto \delta(\overline{c})(g, h) \otimes_{\mathbb{Z}} gh \cdot f = \delta(\overline{c})(g, h) \otimes_{\mathbb{Z}} f.$$

To pass through evaluation, we set $g \coloneqq \prod_i \sigma_i^{a_i}$ and $h \coloneqq \prod_i \sigma_i^{b_i}$, from which we get

$$\begin{aligned} f(\delta(\overline{c})(g, h)) &= f\left((g_i h_i q_i)_i, \left(g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)}\right)_{i>j}\right) \\ &= \sum_{i=1}^m g_i h_i \left\lfloor \frac{a_i + b_i}{n_i} \right\rfloor \cdot \alpha_i + \sum_{\substack{i,j=1 \\ i>j}}^m g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)} \cdot \beta_{ij} \\ &= \sum_{\substack{i,j=1 \\ i>j}}^m \left(\prod_{p<i} \sigma_p^{a_p}\right)\left(\prod_{q<j} \sigma_q^{b_q}\right) \sigma_i^{(a_i)} \sigma_j^{(b_j)} \beta_{ij} + \sum_{i=1}^m g_i h_i \alpha_i^{\left\lfloor \frac{a_i + b_i}{n_i} \right\rfloor}. \end{aligned}$$

Doing a little more rearrangement and writing this multiplicatively exactly recovers the cocycle of Theorem 87. This completes the proof. ■

Though we have proven everything we set out to do in subsection 4.1, there is more to discuss with our alternate description of tuples. As a taste, we prove the following extension of Proposition 110.

Proposition 121. Fix everything as in the set-up, and let A be a G-module. Then the isomorphism of Proposition 110 descends to an isomorphism between equivalence classes of $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\widehat{H}^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A))$.

Proof. Recall that the short exact sequence

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0$$

of G-modules splits as $\mathbb{Z}$-modules by Lemma 114, so we have a short exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}, A) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m, A) \xrightarrow{-\circ\mathcal{F}} \operatorname{Hom}_{\mathbb{Z}}(X, A) \to 0.$$

Now, the key trick will be to compare regular group cohomology with Tate cohomology. To begin, we note that our cohomology theories give the following commutative diagram with exact rows.

$$\begin{array}{ccccc} H^0(G,\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m,A)) & \xrightarrow{-\circ\mathcal{F}} & H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A)) & \longrightarrow & H^1(G,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}\,\mathcal{F},A)) \\ & & \downarrow & & \| \\ 0 & \longrightarrow & \widehat{H}^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A)) & \longrightarrow & \widehat{H}^1(G,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}\,\mathcal{F},A)) \end{array} \tag{4.4}$$

Here, the middle vertical map is reduction modulo $\mathrm{im}\,N_G$. The rows are exact from the long exact sequences, and the square commutes by construction of Tate cohomology. Now, the point is that the diagram induces the isomorphism

$$\frac{H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A))}{\mathrm{im}(-\circ\mathcal{F})} \simeq \widehat{H}^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A)), \tag{4.5}$$

which simply sends $[f]\mapsto[f]$.

Thus, the main content here will be to track through the image of $-\circ\mathcal{F}$ in (4.4). Let $\mathcal{T}$ denote the set of $\{\sigma_i\}_{i=1}^m$-triples of A, and let $\mathcal{T}_0$ denote the set (in fact, equivalence class) of triples corresponding to $[0]\in H^2(G,A)$. Letting $\varphi\colon H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A))\to\mathcal{T}$ be defined by

$$\varphi\colon f\mapsto\Big(\big(f(\kappa_i)\big)_i,\big(f(\lambda_{ij})\big)_{i>j}\Big)$$

be the isomorphism of Proposition 110, we claim that the image of $-\circ\mathcal{F}$ in $H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A))$ corresponds under φ to exactly $\mathcal{T}_0$.

Indeed, we take a G-module homomorphism $f\colon\mathbb{Z}[G]^m\to A$ to the G-module homomorphism $(f\circ\mathcal{F})\colon X\to A$. Then we compute

$$\begin{aligned}(f\circ\mathcal{F})(\kappa_i)&=f(N_i\varepsilon_i)\\&=N_if(\varepsilon_i)\\(f\circ\mathcal{F})(\lambda_{ij})&=f(T_i\varepsilon_j-T_j\varepsilon_i)\\&=T_if(\varepsilon_j)-T_jf(\varepsilon_i)\end{aligned}$$

for all relevant indices i and j. Thus,

$$\varphi(f\circ\mathcal{F})=\Big(\big(N_if(\varepsilon_i)\big)_i,\big(T_if(\varepsilon_j)-T_jf(\varepsilon_i)\big)_{i>j}\Big),$$

which we can see lives in $\mathcal{T}_0$ by definition of our equivalence relation (upon using multiplicative notation). In fact, as f varies, we see that the values of $f(\varepsilon_i)$ may vary over all A, so the image of $f\mapsto\varphi(f\circ\mathcal{F})$ is exactly all of $\mathcal{T}_0$. Thus, φ induces an isomorphism

$$\overline{\varphi}\colon\frac{H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,A))}{\mathrm{im}(-\circ\mathcal{F})}\simeq\frac{\mathcal{T}}{\mathcal{T}_0}.$$

Composing this with the "identity" map (4.5) finishes the proof. ■

Remark 122. This proof feels more motivated coming from the perspective that X "should" be a 2-encoding module (for example, $\mathrm{coker}\,\mathcal{F}$ "should" be a 1-encoding module, allowing us to use Proposition 66), so actually the equivalence relation on the tuples from Definition 91 can be seen as falling out of the quotient

$$H^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,-))\Rightarrow\widehat{H}^0(G,\mathrm{Hom}_{\mathbb{Z}}(X,-)).$$

Indeed, the equivalence relations had better match up anyway.

4.5 Algebraic Corollaries

We continue in the set-up of the previous subsection. Observe that Proposition 121 combined with Theorem 95 tells us that we have isomorphisms

$$\delta(\overline{c}) \cup -\colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^2(G, A).$$

In fact, Lemma 11 tells us that these isomorphisms assemble into a natural isomorphism, so we have the following result.

Theorem 123. Fix everything as in the set-up. Then X is a 2-encoding module.

Proof. This follows from the above discussion. ∎

Remark 124. It is perhaps useful to note that we can show that X is a 2-encoding module, without the need to digress to tuples as done in Proposition 121. Indeed, we recall that

$$0 \to X \to \mathbb{Z}[G]^m \to \mathrm{coker}\, \mathcal{F} \to 0$$

splits by Lemma 114, so because $\mathbb{Z}[G]^m \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}^m$ is induced, it suffices to show that $\mathrm{coker}\, \mathcal{F}$ is a 1-encoding module by Proposition 66.

For this, we can use Proposition 44 and manually give x and $x^\vee$; here, $x = [\overline{c}]$ will work, and one can solve for $x^\vee$. Alternatively, one could check the cohomology groups from Proposition 55. One could even solve for $[\delta(\overline{c})]^\vee$ explicitly, though this is harder.

Now that we have a 2-encoding module, we can apply all the theory we built in section 1. For example, it might have felt like magic that the isomorphism sending a tuple to its cohomology class was induced by a cup product, but in fact this must have been true all along by Corollary 35.

Here are some other results.

Corollary 125. Fix everything as in the set-up. Then X is cohomologically equivalent to $I_G \otimes_{\mathbb{Z}} I_G$.

Proof. We know that $I_G \otimes_{\mathbb{Z}} I_G$ is a 2-encoding module by Example 32, from which Proposition 31 finishes. ∎

Corollary 126. Fix everything as in the set-up. Then, for any $i \in \mathbb{Z}$ and subgroup $H \subseteq G$, we have natural isomorphisms

$$\mathrm{Res}[\delta(\overline{c})] \cup -\colon \widehat{H}^i(H, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{i+2}(H, A).$$

Proof. Follow the proof of Corollary 37 to see that we can set $x = [\delta(\overline{c})]$ there. This gives the result for $H = G$, and we get general subgroups by appealing to Corollary 47. ∎

Remark 127. Even though we have some notion of restriction, writing a "tuple" in $\widehat{H}^0(H, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ seems somewhat difficult in general. For example, it is not clear how to (in general) write X as $\mathbb{Z}[H]^m/M$ for an H-module M. In simple cases, we have worked this out in Lemma 100.

Corollary 128. Fix everything as in the set-up. Then $\widehat{H}^2(G, X)$ is cyclic of order $\#G$ generated by $[\delta(\overline{c})]$.

Proof. This follows from Corollary 41. ∎

Remark 129. Fix notation as in subsection 4.1, and take $m = 2$. Then there are natural transformations

$$\widehat{H}^2(G,-) \overset{[\delta(\overline{c})]^\vee \cup -}{\Rightarrow} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X,-)) \Rightarrow \widehat{H}^{-1}(G,-)$$

sending a 2-cocycle to its $\{\sigma_i\}_{i=1}^m$-tuple and then to the (class of) β_{10}. (It turns out that, because G is bicyclic, the equivalence relation on β_{10} is exactly what we need to form a class of $\widehat{H}^{-1}$.) Now, applying Corollary 35, we see that the right natural transformation must be a cup-product map, so by associativity of the cup product, the entire natural transformation is a cup-product map.

Thus, analogously to what Corollary 75 says for αs, we can describe the projection from 2-cocycles to βs purely via (restricted) cup products.

Remark 130. Noting that $\mathcal{F}\colon X \hookrightarrow \mathbb{Z}[G]^m$ implies that X is $\mathbb{Z}$-free, there is a torus $T := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{G}_m)$. It is conceivable that one could realize the approach of Remark 71 for our torus T.

References

[CE56] Henri Cartan and Samuel Eilenberg. *Homological Algebra*. Princeton mathematical series. Princeton: Princeton University Press, 1956.

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin, Heidelberg, 1999.

[AW10] M. F. Atiyah and C. T. C. Wall. "Cohomology of Groups". In: *Algebraic Number Theory: Proceedings of an Instructional Conference*. Ed. by J. W. S. Cassels and A. Fröhlich. 2nd ed. London Mathematical Society, 2010.

[Neu13] Jürgen Neukirch. *Class Field Theory: The Bonn Lectures*. Ed. by Alexander Schmidt. Springer Berlin, Heidelberg, 2013.

A Verification of the Cocycle

In this section, we verify Theorem 87. As such, in this section, we will work under the modified set-up, forgetting about the extension $\mathcal{E}$ but letting $(\{\alpha_i\}, \{\beta_{ij}\})$ be some $\{\sigma_i\}_{i=1}^m$-tuple.

Here the formula looks like

$$c(g,g') \coloneqq \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g=\prod_i\sigma_i^{a_i}$ and $g'=\prod_i\sigma_i^{b_i}$ with $0\le a_i,b_i<n_i$ and $q_i\coloneqq\lfloor(a_i+b_i)/n_i\rfloor$. To make this more digestible, we define

$$g_i\coloneqq\prod_{1\le k<i}\sigma_k^{a_k}$$

for any $g=\prod_i\sigma_i^{a_i}\in G$, so we can write down our formula as

$$c(g,g')\coloneqq\left[\prod_{1\le j<i\le m}g_ig_j'\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m g_ig_i'\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right].$$

Now, given $g,g',g''\in G$, we would like to check

$$gc(g',g'')\cdot c(g,g'g'')\overset{?}{=}c(gg',g'')\cdot c(g,g'),$$

where $g=\prod_i\sigma_i^{a_i}$ and $g'=\prod_i\sigma_i^{b_i}$ and $g''=\prod_i\sigma_i^{c_i}$ with $0\le a_i,b_i,c_i<n_i$.

A.1 Carries

We will begin our verification by dealing with carries; we start with the following lemma, intended to beef up our relation (3.2).

Lemma 131. Given indices $i>j$ with $a_i,a_j,q_i,q_j\ge 0$, we have

$$\beta_{ij}^{(a_ia_j)}=\beta_{ij}^{(a_i+q_in_i,a_j)}\left(\frac{\sigma_j^{a_j}(\alpha_i)}{\alpha_i}\right)^{q_i}\qquad\text{and}\qquad\beta_{ij}^{(a_ia_j)}=\beta_{ij}^{(a_i,a_j+q_jn_j)}\left(\frac{\alpha_j}{\sigma_i^{a_i}(\alpha_j)}\right)^{q_j}.$$

Proof. This is a matter of force. For one, we compute

$$\begin{aligned}\beta_{ij}^{(a_i+n_iq_i,a_j)}&=\prod_{p=0}^{a_i+n_iq_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\\&=\left(\prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\left(\prod_{q=0}^{a_j-1}\prod_{p=a_i}^{a_i+n_iq_i-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\\&=\beta_{ij}^{(a_ia_j)}\left(\prod_{q=0}^{a_j-1}\sigma_j^q\,\mathrm{N}_{L/L_i}(\beta_{ij})\right)^{q_i}.\end{aligned}$$

Now, using the relation $\mathrm{N}_{L/L_i}(\beta_{ij})=\alpha_i/\sigma_j(\alpha_i)$ from (3.2), this becomes

$$\begin{aligned}\beta_{ij}^{(a_i+n_iq_i,a_j)}&=\beta_{ij}^{(a_ia_j)}\left(\prod_{q=0}^{a_j-1}\frac{\sigma_j^q\alpha_i}{\sigma^{j+1}\alpha_i}\right)^{q_i}\\&=\beta_{ij}^{(a_ia_j)}\left(\frac{\alpha_i}{\sigma^{a_j}\alpha_i}\right)^{q_i},\end{aligned}$$

which rearranges into what we wanted.

For the other, we again just compute

$$\begin{aligned}\beta_{ij}^{(a_i,a_j+n_jq_j)} &= \prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j+n_jq_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\\ &= \left(\prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\left(\prod_{p=0}^{a_i-1}\prod_{q=q_j}^{a_j+n_jq_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\\ &= \beta_{ij}^{(a_ia_j)}\left(\prod_{p=0}^{a_i-1}\sigma_i^p\,\mathrm{N}_{L/L_q}(\beta_{ij})\right)^{q_i}.\end{aligned}$$

This time, we use the relation $\mathrm{N}_{L/L_j}(\beta_{ij}) = \sigma_i(\alpha_j)/\alpha_j$, which gives

$$\begin{aligned}\beta_{ij}^{(a_i,a_j+n_jq_j)} &= \beta_{ij}^{(a_ia_j)}\left(\prod_{p=0}^{a_i-1}\frac{\sigma_i^{p+1}(\alpha_j)}{\sigma_i^p(\alpha_j)}\right)^{q_i}\\ &= \beta_{ij}^{(a_ia_j)}\left(\frac{\sigma_i^{a_j}(\alpha_j)}{\alpha_j}\right)^{q_i},\end{aligned}$$

which again rearranges into the desired. ■

We are now ready to begin the computation, dealing with carries to start. Use the division algorithm to write

$$a_i + b_i = n_iu_i + x_i \qquad \text{and} \qquad b_i + c_i = n_iv_i + y_i,$$

where $u_i, v_i \in \{0,1\}$ and $0 \le x_i, y_i < n_i$ for each i. We start by collecting remainder terms on the side of $gc(g',g'')\cdot c(g,g'g'')$.

1. Note

$$gc(g',g'') = g\left[\prod_{1\le j<i\le m} g_i'g_j''\beta_{ij}^{(b_ic_j)}\right]\cdot g\left[\prod_{i=1}^m g_i'g_i''\alpha_i^{v_i}\right],$$

so we set

$$R_1 \coloneqq \prod_{i=1}^m gg_i'g_i''\alpha_i^{v_i}$$

to be our remainder term.

2. Note

$$\begin{aligned}c(g,g'g'') &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_iy_j)}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]\\ &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}\cdot g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]\\ &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}\right]\left[\prod_{1\le j<i\le m} g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right],\end{aligned}$$

so we set

$$R_2 \coloneqq \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]$$

to be our remainder term.

3. Lastly, we collect our remainders. Observe

$$\begin{aligned}
R_2 &= \left[\prod_{j=1}^{m} g'_j g''_j \left(\prod_{i=j+1}^{m} g_i \cdot \frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g'_j g''_j \left(\prod_{i=j+1}^{m} \frac{(\sigma_1^{a_1}\cdots\sigma_{i-1}^{a_{i-1}})\alpha_j}{(\sigma_1^{a_1}\cdots\sigma_{i-1}^{a_{i-1}})\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g'_j g''_j \left(\prod_{i=j+1}^{m} \frac{g_i\alpha_j}{g_{i+1}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g'_j g''_j \cdot \frac{g_{j+1}\alpha_j^{v_j}}{g\alpha_j^{v_j}}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right].
\end{aligned}$$

We now note that $g_{j+1}\alpha_j = g_j\alpha_j$ because α_j is fixed by σ_j. As such,

$$\begin{aligned}
R_1R_2 &= \left[\prod_{i=1}^{m} g g'_i g''_i \alpha_i^{v_i}\right]\left[\prod_{i=1}^{m} g'_i g''_i \cdot \frac{g_i\alpha_i^{v_i}}{g\alpha_i^{v_i}}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{v_i+\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor},
\end{aligned}$$

which is nice enough for us now.

Now, we collect remainder terms from $c(gg', g'') \cdot c(g, g')$.

1. Note

$$\begin{aligned}
c(gg', g'') &= \left[\prod_{1\le j<i\le m} g_i g'_i g''_j \beta_{ij}^{(x_i c_j)}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{1\le j<i\le m} g_i g'_i g''_j \beta_{ij}^{(a_i+b_i,c_j)} \cdot g_i g'_i g''_j \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{1\le j<i\le m} g_i g'_i g''_j \beta_{ij}^{(a_i+b_i,c_j)}\right]\left[\prod_{1\le j<i\le m} g_i g'_i g''_j \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right],
\end{aligned}$$

so we set

$$R_3 \coloneqq \left[\prod_{1\le j<i\le m} g_i g'_i g''_j \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g'_i g''_i \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right].$$

2. Note

$$c(g, g') = \left[\prod_{1\le j<i\le m} g_i g'_j \beta_{ij}^{(a_i b_j)}\right]\left[\prod_{i=1}^{m} g_i g'_i \alpha_i^{u_i}\right],$$

so we set

$$R_4 \coloneqq \left[\prod_{i=1}^{m} g_i g'_i \alpha_i^{u_i}\right].$$

3. Lastly, we collect our remainder terms. Observe

$$\begin{aligned}
R_3 &= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} g_j'' \cdot \frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} \frac{(\sigma_1^{c_1}\cdots\sigma_{j-1}^{c_{j-1}})\sigma_j^{c_j}\alpha_i}{(\sigma_1^{c_1}\cdots\sigma_{j-1}^{c_{j-1}})\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} \frac{g_{j+1}''\alpha_i}{g_j''\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \cdot \frac{g_i''\alpha_i^{u_i}}{\alpha_i^{u_i}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
R_3R_4 &= \left[\prod_{i=1}^{m} g_i g_i' \cdot \frac{g_i''\alpha_i^{u_i}}{\alpha_i^{u_i}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right]\left[\prod_{i=1}^{m} g_i g_i' \alpha_i^{u_i}\right] \\
&= \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{u_i+\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor},
\end{aligned}$$

which is again simple enough for our purposes.

We now note that, for each i,

$$u_i + \left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor = \left\lfloor \frac{a_i+b_i+c_i}{n_i}\right\rfloor = v_i + \left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor$$

by how carried addition behaves. It follows that

$$R_1R_2 = \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{v_i+\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor} = \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{u_i+\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor} = R_3R_4.$$

Thus, it suffices to show that

$$\frac{gc(g',g'')}{R_1}\cdot\frac{c(g,g'')}{R_2} \stackrel{?}{=} \frac{c(gg',g'')}{R_3}\cdot\frac{c(g,g')}{R_4},$$

which is equivalent to

$$g\left[\prod_{1\le j<i\le m} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

by the work above.

A.2 Finishing

We need to verify that

$$g\left[\prod_{1\le j<i\le m} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

as discussed in the previous subsection.

Before beginning the check, we recall the relations on the βs from (3.3) can be written as

$$\frac{\sigma_2(\beta_{31})}{\beta_{31}} = \frac{\sigma_1(\beta_{32})}{\beta_{32}} \cdot \frac{\sigma_3(\beta_{21})}{\beta_{21}},$$

because we only have one triple (i,j,k) of indices with $i > j > k$. This is somewhat difficult to deal with directly, so we quickly show a more general version.

Lemma 132. Fix indices with $i > j > k$, and let $a_i, a_j, a_k \geq 0$. Then

$$\frac{\sigma_j^{a_j}\beta_{ik}^{(a_i a_k)}}{\beta_{ik}^{(a_i a_k)}} = \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} \cdot \frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}}.$$

Proof. We simply compute

$$\begin{aligned}
\frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}} \cdot \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} &= \prod_{r=0}^{a_i-1} \frac{\sigma_i^{r+1}\beta_{jk}^{(a_j a_k)}}{\sigma_i^{r}\beta_{jk}^{(a_j a_k)}} \cdot \prod_{p=0}^{a_k-1} \frac{\sigma_k^{p+1}\beta_{ij}^{(a_i a_j)}}{\sigma_k^{p}\beta_{ij}^{(a_i a_j)}} \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \left(\frac{\sigma_k^p\sigma_j^q\sigma_i^{r+1}\beta_{jk}}{\sigma_k^p\sigma_j^q\sigma_i^r\beta_{jk}} \cdot \frac{\sigma_k^{p+1}\sigma_j^q\sigma_i^r\beta_{ij}}{\sigma_k^p\sigma_j^q\sigma_i^r\beta_{ij}} \right) \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \sigma_k^p\sigma_j^q\sigma_i^r \left(\frac{\sigma_i\beta_{jk}}{\beta_{jk}} \cdot \frac{\sigma_k\beta_{ij}}{\beta_{ij}} \right) \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \sigma_k^p\sigma_j^q\sigma_i^r \left(\frac{\sigma_j\beta_{ik}}{\beta_{ik}} \right),
\end{aligned}$$

where in the last equality we have use the relation on the βs. Continuing,

$$\begin{aligned}
\frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}} \cdot \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} &= \prod_{q=0}^{a_j-1} \left(\prod_{p=0}^{a_k-1}\prod_{r=0}^{a_i-1} \frac{\sigma_j^{q+1}\sigma_k^p\sigma_i^r\beta_{ik}}{\sigma_j^q\sigma_k^p\sigma_i^r\beta_{ik}} \right) \\
&= \prod_{q=0}^{a_j-1} \frac{\sigma_j^{q+1}\beta_{ik}^{(a_i a_k)}}{\sigma_j^{q}\beta_{ik}^{(a_i a_k)}} \\
&= \frac{\sigma_j^{a_j}\beta_{ik}^{(a_i a_k)}}{\beta_{ik}^{(a_i a_k)}},
\end{aligned}$$

which is what we wanted. ∎

We now proceed with the check, by induction. More precisely, we claim that any $m' \leq m$ gives

$$g_{m'+1}\left[\prod_{j<i\leq m'} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\left[\prod_{j<i\leq m'} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{j<i\leq m'} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\left[\prod_{j<i\leq m'} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

which we will show by induction on m'. For $m' = 1$, there is nothing to say because there are no indices $i > j$.

So now suppose we have equality for $m' < m$, and we give equality for $m'' := m' + 1$. That is, we want to show that

$$g_{m'+2}\prod_{j<i\leq m'+1} g_i' g_j'' \beta_{ij}^{(b_i c_j)} \cdot \prod_{j<i\leq m'+1} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)} \stackrel{?}{=} \prod_{j<i\leq m'+1} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)} \cdot \prod_{j<i\leq m'+1} g_i g_j' \beta_{ij}^{(a_i b_j)}$$

but by the inductive hypothesis it suffices for

$$\frac{g_{m''+1}\prod\limits_{j<i\leq m'+1} g_i'g_j''\beta_{ij}^{(b_ic_j)}}{g_{m'+1}\prod\limits_{j<i\leq m'} g_i'g_j''\beta_{ij}^{(b_ic_j)}}\cdot\frac{\prod\limits_{j<i\leq m'+1} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}}{\prod\limits_{j<i\leq m'} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}}\stackrel{?}{=}\frac{\prod\limits_{j<i\leq m'+1} g_ig_i'g_j''\beta_{ij}^{(a_i+b_i,c_j)}}{\prod\limits_{j<i\leq m'} g_ig_i'g_j''\beta_{ij}^{(a_i+b_i,c_j)}}\cdot\frac{\prod\limits_{j<i\leq m'+1} g_ig_j'\beta_{ij}^{(a_ib_j)}}{\prod\limits_{j<i\leq m'} g_ig_j'\beta_{ij}^{(a_ib_j)}}$$

which is collapses to

$$\frac{g_{m''+1}\prod\limits_{j<i\leq m'+1} g_i'g_j''\beta_{ij}^{(b_ic_j)}}{g_{m'+1}\prod\limits_{j<i\leq m'} g_i'g_j''\beta_{ij}^{(b_ic_j)}}\cdot\prod_{j\leq m'} g_{m''}g_j'g_j''\beta_{m''j}^{(a_{m''},b_j+c_j)}\stackrel{?}{=}\prod_{j\leq m'} g_{m''}g_{m''}'g_j''\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}\cdot\prod_{j\leq m'} g_{m''}g_j'\beta_{ij}^{(a_{m''}b_j)}$$

because the terms with $i<m''=m'+1$ got cancelled in the rightmost three products. Rearranging, this is the same as

$$\frac{g_{m''+1}\prod\limits_{j<i\leq m'+1} g_i'g_j''\beta_{ij}^{(b_ic_j)}}{g_{m'+1}\prod\limits_{j<i\leq m'} g_i'g_j''\beta_{ij}^{(b_ic_j)}}\stackrel{?}{=}\frac{\prod\limits_{j<m''} g_{m''}g_{m''}'g_j''\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}\cdot\prod\limits_{j<m''} g_{m''}g_j'\beta_{m''j}^{(a_{m''}b_j)}}{\prod\limits_{j<m''} g_{m''}g_j'g_j''\beta_{m''j}^{(a_{m''},b_j+c_j)}}.$$

Peeling off the $i=m''=m'+1$ terms from the left-hand side numerator, we're showing

$$\frac{g_{m''+1}\prod\limits_{j<i\leq m'} g_i'g_j''\beta_{ij}^{(b_ic_j)}}{g_{m'+1}\prod\limits_{j<i\leq m'} g_i'g_j''\beta_{ij}^{(b_ic_j)}}\stackrel{?}{=}\frac{\prod\limits_{j<m''} g_{m''}g_{m''}'g_j''\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}\cdot\prod\limits_{j<m''} g_{m''}g_j'\beta_{m''j}^{(a_{m''}b_j)}}{\prod\limits_{j<m''} g_{m''+1}g_{m''}'g_j''\beta_{m''j}^{(b_{m''}c_j)}\cdot\prod\limits_{j<m''} g_{m''}g_j'g_j''\beta_{m''j}^{(a_{m''},b_j+c_j)}}.$$

We take a moment to simplify the left-hand side with Lemma 132 by writing

$$\begin{aligned}
g_{m'+1}\prod_{j<i\leq m'} g_i'g_j''\left(\frac{\sigma_{m''}^{a_{m''}}\beta_{ij}^{(b_ic_j)}}{\beta_{ij}^{(b_ic_j)}}\right) &= g_{m''}\prod_{j<i\leq m'} g_i'g_j''\left(\frac{\sigma_i^{b_i}\beta_{m''j}^{(a_{m''}c_j)}}{\beta_{m''j}^{(a_{m''}c_j)}}\cdot\frac{\beta_{m''i}^{(a_{m''}b_i)}}{\sigma_j^{c_j}\beta_{m''i}^{(a_{m''}b_i)}}\right)\\
&= g_{m''}\left[\prod_{j=1}^{m'} g_j''\prod_{i=j+1}^{m'} g_i'\left(\frac{\sigma_i^{b_i}\beta_{m''j}^{(a_{m''}c_j)}}{\beta_{m''j}^{(a_{m''}c_j)}}\right)\cdot\prod_{i=1}^{m'} g_i'\prod_{j=1}^{i-1} g_j''\left(\frac{\beta_{m''i}^{(a_{m''}b_i)}}{\sigma_j^{c_j}\beta_{m''i}^{(a_{m''}b_i)}}\right)\right]\\
&= g_{m''}\left[\prod_{j=1}^{m'}\frac{g_{m'+1}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}{g_{j+1}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}\cdot\prod_{i=1}^{m'}\frac{g_i'\beta_{m''i}^{(a_{m''}b_i)}}{g_i'g_i''\beta_{m''i}^{(a_{m''}b_i)}}\right]\\
&= g_{m''}\left[\prod_{j<m''}\frac{g_{m''}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}{g_{j+1}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}\cdot\prod_{j<m''}\frac{g_j'\beta_{m''j}^{(a_{m''}b_j)}}{g_j'g_j''\beta_{m''j}^{(a_{m''}b_j)}}\right]
\end{aligned}$$

after doing a lot of telescoping. Now, we can remove $g_{m''}$ everywhere to give

$$\prod_{j<m''}\frac{g_{m''}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}{g_{j+1}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}\cdot\prod_{j<m''}\frac{g_j'\beta_{m''j}^{(a_{m''}b_j)}}{g_j'g_j''\beta_{m''j}^{(a_{m''}b_j)}}\stackrel{?}{=}\frac{\prod\limits_{j<m''} g_{m''}'g_j''\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}\cdot\prod\limits_{j<m''} g_j'\beta_{m''j}^{(a_{m''}b_j)}}{\prod\limits_{j<m''} g_{m''+1}'g_j''\beta_{m''j}^{(b_{m''}c_j)}\cdot\prod\limits_{j<m''} g_j'g_j''\beta_{m''j}^{(a_{m''},b_j+c_j)}},$$

or

$$\prod_{j<m''}\frac{g_{m''}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}{g_{j+1}'g_j''\beta_{m''j}^{(a_{m''}c_j)}}\stackrel{?}{=}\frac{\prod\limits_{j<m''} g_{m''}'g_j''\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}\cdot\prod\limits_{j<m''} g_j'g_j''\beta_{m''j}^{(a_{m''}b_j)}}{\prod\limits_{j<m''} g_{m''+1}'g_j''\beta_{m''j}^{(b_{m''}c_j)}\cdot\prod\limits_{j<m''} g_j'g_j''\beta_{m''j}^{(a_{m''},b_j+c_j)}}.$$

Rearranging, we want

$$\prod_{j<m''} \frac{g'_j g''_j \beta_{m''j}^{(a_{m''},b_j+c_j)}}{g'_j g''_j \beta_{m''j}^{(a_{m''}b_j)} \cdot g'_{j+1} g''_j \beta_{m''j}^{(a_{m''}c_j)}} \stackrel{?}{=} \prod_{j<m''} \frac{g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''},c_j)}}{g'_{m''} g''_j \beta_{m''j}^{(a_{m''}c_j)} \cdot g'_{m''+1} g''_j \beta_{m''j}^{(b_{m''}c_j)}},$$

which is

$$\prod_{j<m''} g'_j g''_j \left(\frac{\beta_{m''j}^{(a_{m''},b_j+c_j)}}{\beta_{m''j}^{(a_{m''}b_j)} \cdot \sigma_j^{b_j} \beta_{m''j}^{(a_{m''}c_j)}} \right) \stackrel{?}{=} \prod_{j<m''} g'_{m''} g''_j \left(\frac{\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}}{\beta_{m''j}^{(a_{m''}c_j)} \cdot \sigma_{m''}^{a_{m''}} \beta_{m''j}^{(b_{m''}c_j)}} \right).$$

However, by definition of the $\beta_{ij}^{(xy)}$, we see that

$$\frac{\beta_{m''j}^{(a_{m''},b_j+c_j)}}{\beta_{m''j}^{(a_{m''}b_j)} \cdot \sigma_j^{b_j} \beta_{m''j}^{(a_{m''}c_j)}} = \frac{\beta_{m''j}^{(a_{m''}+b_{m''},c_j)}}{\beta_{m''j}^{(a_{m''}c_j)} \cdot \sigma_{m''}^{a_{m''}} \beta_{m''j}^{(b_{m''}c_j)}} = 1,$$

so everything does indeed cancel out properly. This completes the check.

B Computation of $\ker \mathcal{F}$

In this section we give a proof of Lemma 119. As such, we will use all the context from the statement and proceed directly with the proof; as mentioned earlier, we may add (b) back to our list of generators because it is induced by (c). Pick up some $z := ((x_i)_i, (y_{ij})_{i>j}) \in \ker \mathcal{F}$, which is equivalent to saying

$$x_i N_i - \sum_{j=1}^{i-1} y_{ij} T_j + \sum_{j=i+1}^{m} y_{ji} T_j = 0$$

for each index i. We want to write z as a $\mathbb{Z}[G]$-linear combination of the elements from (a)–(e). The main idea will be to slowly subtract out $\mathbb{Z}[G]$-linear combinations of the above elements (which does not affect $z \in \ker \mathcal{F}$) until we can prove that we have 0 left over. We start with the x_i terms, which we do in two steps.

1. We begin by dealing with the x_i terms. Fix some index p, and we will subtract out a suitable $\mathbb{Z}[G]$-linear combination of the above generators to set $x_p = 0$ while not changing the other x_i terms. Well, using the element
$$\kappa_p T_p, \tag{a}$$
we may assume that x_p has no σ_p terms because $\sigma_p \equiv 1 \pmod{T_p}$. Then for each $q < p$, we can subtract out a suitable multiple of
$$T_q \kappa_p + N_p \lambda_{pq} \tag{c}$$
to make it so that we may assume x_p has no σ_q terms because $\sigma_q \equiv 1 \pmod{T_q}$. Similarly, for each $q > p$, we can subtract out a suitable multiple of
$$T_q \kappa_p - N_p \lambda_{pq} \tag{d}$$
to make it so that we may assume x_p has no σ_q terms because $\sigma_q \equiv 1 \pmod{T_q}$.

2. Thus, the above process allows us to assume that $x_p \in \mathbb{Z}$, and the above linear combinations have not affected any x_i for $i \neq p$. We now use the fact that $z \in \ker \mathcal{F}$. Indeed, we know that
$$x_p N_p - \sum_{j=1}^{p-1} y_{pj} T_j + \sum_{j=p+1}^{m} y_{jp} T_j = 0.$$
Applying the augmentation map $\varepsilon \colon \mathbb{Z}[G] \to \mathbb{Z}$, sending $\varepsilon \colon \sigma_i \mapsto 1$ for each index i, we see that $x_p \in \mathbb{Z}$ implying that x_p remains fixed. On the other hand $\varepsilon \colon T_j \mapsto 0$ for each index j and $\varepsilon \colon N_p \mapsto n_p$, so we are left with
$$n_p x_p = 0.$$
Because $n_p \neq 0$ (it's the order of σ_p), we conclude that $x_p = 0$. Applying this argument to the other x_i terms, we conclude that we may assume $x_i = 0$ for each i.

It remains to deal with the y_{ij} terms, which is a little more involved. For reference, we are showing that

$$-\sum_{j=1}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0$$

for each index i implies that $z = ((0)_i, (y_{ij})_{i>j})$ is a $\mathbb{Z}[G]$-linear combination of the terms from (b) and (e).

We will now more or less proceed with the y_{ij} by induction on m, allowing the group G (in its number of generators m) to be changed in the process. For $m = 1$, there is nothing to say because there is no y_{ij} term at all. For a taste of how we will use Lemma 112, we also work out $m = 2$: our equations read

$$\underbrace{-y_{21}T_1 = 0}_{i=1} \qquad \text{and} \qquad \underbrace{y_{21}T_2 = 0}_{i=2}.$$

Thus, $y_{21} \in (\ker T_1) \cap (\ker T_2) = (\operatorname{im} N_1) \cap (\operatorname{im} N_2)$, which is $\operatorname{im} N_1N_2$ by Lemma 112.

We now proceed with the general case; take $m > 2$. Let $G' := \langle \sigma_2, \dots, \sigma_m \rangle$, which has $m-1$ generators. By the inductive hypothesis, we may assume the statement for G'. Explicitly, we will assume that, if $(y'_{ij})_{i>j\geq 2} \in \mathbb{Z}[G']^{\binom{m-1}{2}}$ are variables satisfying

$$-\sum_{j=2}^{i-1} y'_{ij}T_j + \sum_{j=i+1}^{m} y'_{ji}T_j = 0$$

for each index $i \geq 2$, then y'_{ij} are a linear combination of terms from the elements from (b) and (e) above, only using indices at least 2.

We will again proceed in steps, for clarity.

1. To apply the inductive hypothesis, we need to force $y_{pq} \in \mathbb{Z}[G']$ for each pair of indices (p, q) with $p > q \geq 2$. Well, we use the relation (e) so that we can subtract multiples of

   $$T_q\lambda_{p1} - T_1\lambda_{pq} - T_p\lambda_{q1}.$$

   In particular, this element will subtract out T_1 from y_{pq} while only introducing chaos to the elements y_{p1} and y_{q1} in the process. Thus, subtracting a suitable multiple allows us to assume that y_{pq} has no σ_1 terms while not affecting any other y_{ij} with $i > j \geq 2$.

   Applying this process to all y_{ij} with $i > j \geq 2$, we do indeed get $y_{ij} \in \mathbb{Z}[G']$ for each $i > j \geq 2$.

2. We are now ready to apply the inductive hypothesis. For each index $i \geq 2$, we have the equation

   $$-y_{i1}T_1 - \sum_{j=2}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0.$$

   Because each y_{pq} term with $p > q \geq 2$ features no σ_1, applying the transformation $\sigma_1 \mapsto 1$ will affect no term in the sums while causing $y_{i1}T_1$ to vanish. Thus, we have the equations

   $$-\sum_{j=2}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0$$

   for each index $i \geq 2$. Because $y_{ij} \in \mathbb{Z}[G']$ for $i > j \geq 2$ already, we see that we may apply the inductive hypothesis to assert that the y_{ij} are $\mathbb{Z}[G']$-linear combinations of terms from (b) and (e) (only using indices at least 2).

   Subtracting these linear combinations out, we may assume $y_{ij} = 0$ for each $i > j \geq 2$.

3. To take stock, our equations for $i \geq 2$ now read

   $$-y_{i1}T_1 = 0,$$

which simply tells us that $y_{i1} \in \operatorname{im} N_1$ for each $i \geq 2$. As such, we pick up $w_i \in \mathbb{Z}[G]$ so that $y_{i1} = w_i N_1$ for each $i \geq 2$; because $\sigma_1 N_1 = N_1$, we may assume that $w_i \in \mathbb{Z}[G']$ for each $i \geq 2$.

Now the equation for $i = 1$ reads

$$\sum_{j=2}^{m} y_{j1} T_j = 0,$$

or

$$\sum_{i=2}^{m} w_i N_1 T_i = 0.$$

Sending $\sigma_1 \mapsto 1$, we see that w_i and T_i are both fixed because they feature no σ_1s, so we merely have

$$n_1 \sum_{i=2}^{m} w_i T_i = 0.$$

Dividing out by n_1, we are left with

$$\sum_{i=2}^{m} w_i T_i = 0.$$

4. At this point, we may appear stuck, but we have one final trick: taking indices $p > q \geq 2$, subtracting out multiples of

$$\left(T_q \lambda_{p1} - T_1 \lambda_{pq} - T_p \lambda_{q1}\right) \cdot N_1$$

will not affect the y_{pq} term because $T_1 N_1$. Indeed, subtracting this term out looks like

$$T_q N_1 \lambda_{p1} - T_p N_1 \lambda_{q1},$$

which after factoring out N_1 takes $w_p \mapsto w_p - T_q$ and $w_q \mapsto w_q + T_p$.

In particular, fixing any $q \geq 2$ and then applying this trick for all $p > q$, we may assume that w_q does not feature any σ_p terms for $p > q$. Thus, looking at our equation

$$\sum_{i=2}^{m} w_i T_i = 0,$$

we are now able to show that $w_i \in \ker T_i = \operatorname{im} N_i$ for each $i \geq 2$, which will finish because it shows $y_{i1} \in N_i N_1$. Indeed, starting with $i = 2$, we see that w_2 features no σ_p for $p > 2$, so we may take $\sigma_p \mapsto 1$ for each $p > 2$ safely, giving the equation

$$w_2 T_2 = 0,$$

finishing for w_2. Thus, we are left with the equation

$$\sum_{i=3}^{m} w_i T_i = 0,$$

from which we see we can induct downwards (this has fewer variables) to finish.

The above steps complete the proof, as advertised.