Torus Worksheet

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1 The Bicyclic Case

Set $G \simeq G_1 \times G_2$ where $G_1 = \langle \sigma_1 \rangle \subseteq G$ and $G_2 = \langle \sigma_2 \rangle \subseteq G$ with $n_1 := \#G_1$ and $n_2 := \#G_2$. We define the elements

$$N_i := \sum_{k=0}^{n_i-1} \sigma_i^k \qquad \text{and} \qquad T_i := (\sigma_i - 1)$$

for $i \in \{1,2\}$. Additionally, we define

$$N := N_1 N_2 = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} \sigma_1^k \sigma_2^\ell.$$

We will also use the elements N_i, T_i, N to mean the induced multiplication maps $\mathbb{Z}[G] \to \mathbb{Z}[G]$.

> **Remark 1.** We will freely use the facts that $\ker T_i = \operatorname{im} N_i$ and $\operatorname{im} T_i = \ker N_i$. These are not too hard to prove and follow similarly as in the cyclic case.

We take a moment to define the maps

$$\begin{aligned}
\mathcal{F}_1 \colon \frac{\mathbb{Z}[G]}{\operatorname{im} T_1} \times \frac{\mathbb{Z}[G]}{\operatorname{im} N} \times \frac{\mathbb{Z}[G]}{\operatorname{im} T_2} &\to \mathbb{Z}[G] \times \mathbb{Z}[G] \\
(x,y,z) &\mapsto (N_2 z + T_1 y, N_1 x - T_2 y) \\
\mathcal{F}_2 \colon \mathbb{Z}[G] \times \mathbb{Z}[G] &\to \frac{\mathbb{Z}[G]}{\operatorname{im} T_1} \times \frac{\mathbb{Z}[G]}{\operatorname{im} N} \times \frac{\mathbb{Z}[G]}{\operatorname{im} T_2} \\
(a,b) &\mapsto (T_2 a, N_1 a - N_2 b, T_1 b).
\end{aligned}$$

Observe that $\mathcal{F}_1$ is in fact well-defined: if $x - x' = T_1 a \in \operatorname{im} T_1$ and $y - y' = Nb \in \operatorname{im} N$ and $z - z' = T_2 c \in \operatorname{im} T_2$, then

$$\begin{aligned}
N_2 z &= N_2 z' + N_2 T_2 c = N_2 z' \\
N_1 x &= N_1 x' + N_1 T_1 a = N_1 x' \\
T_1 y &= T_1 y' + T_1 N b = T_1 y' \\
T_2 y &= T_2 y' + T_2 N b = T_2 y',
\end{aligned}$$

so everything is independent of choice of representative.

1.1 The Complex

Now, we can chain $\mathcal{F}_1$ and $\mathcal{F}_2$ into an infinite sequence as follows.

$$\cdots \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \xrightarrow{\mathcal{F}_2} \frac{\mathbb{Z}[G]}{\operatorname{im} T_1} \times \frac{\mathbb{Z}[G]}{\operatorname{im} N} \times \frac{\mathbb{Z}[G]}{\operatorname{im} T_2} \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \xrightarrow{\mathcal{F}_2} \cdots \tag{1.1}$$

We make a few preliminary observations about (1.1).

Lemma 2. The sequence of maps in (1.1) makes a complex.

Proof. We need to show that $\mathcal{F}_1 \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_1 = 0$. This is a matter of force. On one hand,

$$\begin{aligned}
(\mathcal{F}_1 \circ \mathcal{F}_2)(a,b) &= \mathcal{F}_1(T_2 a, N_1 a - N_2 b, T_1 b) \\
&= \big(N_2(T_1 b) + T_1(N_1 a - N_2 b), N_1(T_2 a) - T_2(N_1 a - N_2 b)\big) \\
&= \big(N_2 T_1 b + T_1 N_1 a - T_1 N_2 b, N_1 T_2 a - T_2 N_1 a - T_2 N_2 b\big) \\
&= \big(N_2 T_1 b - T_1 N_2 b, N_1 T_2 a - T_2 N_2 b\big) \\
&= (0,0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathcal{F}_2 \circ \mathcal{F}_1)(x,y,z) &= \mathcal{F}_2(N_2 z + T_1 y, N_1 x - T_2 y) \\
&= \big(T_2(N_2 z + T_1 y), N_1(N_2 z + T_1 y) - N_2(N_1 x - T_2 y), T_1(N_1 x - T_2 y)\big) \\
&= \big(T_2 N_2 z + T_2 T_1 y, N_1 N_2 z + N_1 T_1 y - N_2 N_1 x + N_2 T_2 y, T_1 N_1 x - T_1 T_2 y\big) \\
&= \big(T_2 T_1 y, N_1 N_2 z - N_2 N_1 x, -T_1 T_2 y\big),
\end{aligned}$$

which we can see represents 0 in $\mathbb{Z}[G]/\operatorname{im} T_1 \times \mathbb{Z}[G]/\operatorname{im} N \times \mathbb{Z}[G]/\operatorname{im} T_2$. ■

Lemma 3. The complex (1.1) is exact at each $\mathbb{Z}[G]/\operatorname{im} T_1 \times \mathbb{Z}[G]/\operatorname{im} N \times \mathbb{Z}[G]/\operatorname{im} T_2$.

Proof. We need to show that $\ker \mathcal{F}_1 = \operatorname{im} \mathcal{F}_2$. By Lemma 2, we already know that $\operatorname{im} \mathcal{F}_2 \subseteq \ker \mathcal{F}_2$, so it remains to get the reverse inclusion. Well, suppose $(x,y,z) \in \ker \mathcal{F}_1$, which means that

$$N_2 z + T_1 y = N_1 x - T_2 y = 0.$$

Now, note that because $x \in \mathbb{Z}[G]/\operatorname{im} T_1$, where $\sigma_1 \equiv 1 \pmod{T_1}$, we have

$$N_1 x = \sum_{k=0}^{n_1 - 1} \sigma_1^k x \equiv n_1 x.$$

However,

$$n_1 \cdot N_2 x = N_2(n_1 x) = N_2 N_1 x = N_2(N_1 x - T_2 y) = 0,$$

so $N_2 x = 0$. Thus, $x \in \ker N_2 = \operatorname{im} T_2$, so we may write $x = T_2 a$ for some $a \in \mathbb{Z}[G]$. A similar argument (replacing 1s with 2s and adjusting some signs) shows that we may write $z = T_1 b$ for some $b \in \mathbb{Z}[G]$.

To show $(x,y,z) \in \operatorname{im} \mathcal{F}_2$, it remains to show that $y \equiv N_1 a - N_2 b \pmod{N}$. Well, notice that

$$\begin{aligned}
T_1(N_1 a - N_2 b - y) &= -(N_2 T_1 b + T_1 y) \\
&= -(N_2 z + T_1 y) \\
&= 0 \\
T_2(N_1 a - N_2 b - y) &= N_1 T_2 a - T_2 y \\
&= N_1 x - T_2 y \\
&= 0,
\end{aligned}$$

so $N_1a - N_2b - y \in (\ker T_1) \cap (\ker T_2)$. To finish, we would like $(\ker T_1) \cap (\ker T_2) \subseteq \operatorname{im} N$. Well, if $\alpha \in (\ker T_1) \cap (\ker T_2)$, then we can write

$$\alpha = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} a_{k,\ell}\sigma_1^k\sigma_2^\ell.$$

Now, the condition $T_1\alpha = 0$ implies that $a_{k,\ell} - a_{k-1,\ell} = 0$ for each k, where indices are taken modulo n_i as necessary; thus, $a_{k,\ell}$ is constant with respect to k. Similarly, $T_2\alpha = 0$ implies that $a_{k,\ell}$ is constant with respect to ℓ, so

$$\alpha = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} a_{0,0}\sigma_1^k\sigma_2^\ell = Na_{0,0} \in \operatorname{im} N,$$

as desired. ■

1.2 A Cocycle

The benefit to our analysis above is that we have a short exact sequence

$$0 \to \operatorname{coker} \mathcal{F}_2 \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \to \operatorname{coker} \mathcal{F}_1 \to 1. \tag{1.2}$$

In particular, the (induced) map $\mathcal{F}_1\colon \operatorname{coker} \mathcal{F}_2 \to \mathbb{Z}[G] \times \mathbb{Z}[G]$ is injective by Lemma 3.

In the future, we will want a 2-cocycle in $Z^2(G, \operatorname{coker} \mathcal{F}_2)$, which we will create by tracking a boundary morphism through (1.2). Thus, we want to start with a 1-cocycle in $Z^2(G, \operatorname{coker} \mathcal{F}_1)$. For this, we define the notation

$$\sigma_i^{(a_i)} := \sum_{k=0}^{a_i-1} \sigma_i^k$$

for $i \in \{1, 2\}$ and any nonnegative integer $a_i \geq 0$. For example, $\sigma_i^{(n_i)} = N_i$. We also note that

$$\sigma_i^{(a_i+b_i)} = \sigma_i^{(a_i)} + \sigma_i^{a_i}\sigma_i^{(b_i)},$$

which justifies our "almost exponential" notation.

Lemma 4. Define $u \in C^1(G, \mathbb{Z}[G] \times \mathbb{Z}[G])$ by $u\left(\sigma_1^{a_1}\sigma_2^{a_2}\right) := \left(\sigma_1^{a_1}\sigma_2^{(a_2)}, \sigma_1^{(a_1)}\right)$. Then the induced 1-cochain $\overline{u} \in C^1(G, \operatorname{coker} \mathcal{F}_1)$ is a 1-cocycle.

Proof. We need to show that $\overline{du} = d\overline{u} = 0$; i.e., we need to show that $\operatorname{im}(du) \subseteq \operatorname{im} \mathcal{F}_1$. Well, pick up a_i, b_i with $0 \leq a_i, b_i < n_i$, and set

$$a_i + b_i = n_iq_i + r_i$$

by the division algorithm so that $q_i \in \{0, 1\}$ and $0 \leq r_i < n_i$. This lets us compute

$$\begin{aligned}
(du)\left(\sigma_1^{a_1}\sigma_2^{a_2}, \sigma_1^{b_1}\sigma_2^{b_2}\right) &= \sigma_1^{a_1}\sigma_2^{a_2}u\left(\sigma_1^{b_1}\sigma_2^{b_2}\right) - u\left(\sigma_1^{r_1}\sigma_2^{r_2}\right) + u\left(\sigma_1^{a_1}\sigma_2^{a_2}\right) \\
&= \sigma_1^{a_1}\sigma_2^{a_2}\left(\sigma_1^{b_1}\sigma_2^{(b_2)}, \sigma_1^{(b_1)}\right) - \left(\sigma_1^{r_1}\sigma_2^{(r_2)}, \sigma_1^{(r_1)}\right) + \left(\sigma_1^{a_1}\sigma_2^{(a_2)}, \sigma_1^{(a_1)}\right).
\end{aligned}$$

Now, note

$$\begin{aligned}
\left(\sigma_1^{r_1}\sigma_2^{(r_2)}, \sigma_1^{(r_1)}\right) &= \left(\sigma_1^{a_1+b_1}\sigma_2^{(r_2)}, \sigma_1^{(r_1)}\right) \\
&= \left(\sigma_1^{a_1+b_1}\sigma_2^{(a_2+b_2)} - \sigma_1^{a_1+b_1}\sigma_2^{r_2}\sigma_2^{(n_2q_2)}, \sigma_1^{(a_1+b_1)} - \sigma_1^{r_1}\sigma_1^{(n_1q_1)}\right) \\
&= \left(\sigma_1^{a_1+b_1}\sigma_2^{(a_2+b_2)}, \sigma_1^{(a_1+b_1)}\right) - \left(\sigma_1^{a_1+b_1}\sigma_2^{a_2+b_2} \cdot q_2N_2, \sigma_1^{a_1+b_1} \cdot q_1N_1\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_1^{a_1}\sigma_2^{a_2}\left(\sigma_1^{b_1}\sigma_2^{(b_2)},\sigma_1^{(b_1)}\right) &= \left(\sigma_1^{a_1+b_1}\sigma_2^{a_2}\sigma_2^{(b_2)},\sigma_1^{a_1}\sigma_2^{a_2}\sigma_1^{(b_1)}\right) \\ &= \left(\sigma_1^{a_1+b_1}\sigma_2^{a_2}\sigma_2^{(b_2)},\sigma_1^{a_1}\sigma_1^{(b_1)}\right) + \left(0,\sigma_1^{a_1}\left(\sigma_2^{a_2}-1\right)\sigma_1^{(b_1)}\right) \\ &= \left(\sigma_1^{a_1+b_1}\sigma_2^{a_2}\sigma_2^{(b_2)},\sigma_1^{a_1}\sigma_1^{(b_1)}\right) + \left(0,\sigma_1^{a_1}\sigma_2^{(a_2)}\sigma_1^{(b_1)}T_2\right),\end{aligned}$$

and

$$\begin{aligned}\left(\sigma_1^{a_1}\sigma_2^{(a_2)},\sigma_1^{(a_1)}\right) &= \left(\sigma_1^{a_1+b_1}\sigma_2^{(a_1)},\sigma_1^{(a_1)}\right) + \left(\sigma_1^{a_1}\left(1-\sigma_1^{b_1}\right)\sigma_2^{(a_2)},0\right) \\ &= \left(\sigma_1^{a_1+b_1}\sigma_2^{(a_1)},\sigma_1^{(a_1)}\right) + \left(-\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)}T_1,0\right).\end{aligned}$$

Synthesizing, we see that $\sigma_2^{(a_2+b_2)} = \sigma_2^{(a_2)} + \sigma_1^{a_2}\sigma_2^{(b_2)}$ causes the "leading" terms to vanish, leaving us with

$$\begin{aligned}(du)\left(\sigma_1^{a_1}\sigma_2^{a_2},\sigma_1^{b_1}\sigma_2^{b_2}\right) &= \left(0,\sigma_1^{a_1}\sigma_2^{(a_2)}\sigma_1^{(b_1)}T_2\right) + \left(q_2\sigma_1^{a_1+b_1}\sigma_2^{a_2+b_2}N_2,q_1\sigma_1^{a_1+b_1}N_1\right) + \left(-\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)}T_1,0\right) \\ &= \mathcal{F}_1\left(q_1\sigma_1^{a_1+b_1},\quad -\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)},\quad q_2\sigma_1^{a_1+b_1}\sigma_2^{a_2+b_2}\right),\end{aligned}$$

which finishes. ■

In particular, from our above computation of du, we can pull back along $\mathcal{F}_1$ to find $\delta u \in Z^2(G,\operatorname{coker}\mathcal{F}_2)$ is given by

$$(\delta u)\left(\sigma_1^{a_1}\sigma_2^{a_2},\sigma_1^{b_1}\sigma_2^{b_2}\right) = \left(q_1\sigma_1^{a_1+b_1},\quad -\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)},\quad q_2\sigma_1^{a_1+b_1}\sigma_2^{a_2+b_2}\right).$$

Simplifying a bit, we note that $\sigma_i \equiv 1 \pmod{T_i}$, so we can write this as

$$(\delta u)\left(\sigma_1^{a_1}\sigma_2^{a_2},\sigma_1^{b_1}\sigma_2^{b_2}\right) = \left(q_1,\quad -\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)},\quad q_2\sigma_1^{a_1+b_1}\right). \tag{1.3}$$

We will use δu again shortly.

1.3 Triples

We have now built enough machinery to be able to talk about triples; for the rest of the article, we will have $G = \operatorname{Gal}(L/K)$ for some bicyclic extension L/K. Let S be the set of triples. The following is the main idea.

Lemma 5. There is a natural isomorphism between morphisms $f \in \operatorname{Hom}_{\mathbb{Z}[G]}\left(\operatorname{coker}\mathcal{F}_2, L^\times\right)$ and the set S of triples $(\alpha_1,\beta,\alpha_2)$ given by

$$\varphi\colon f \mapsto \big(f(1,0,0),f(0,1,0),f(0,0,1)\big).$$

Proof. For brevity, set $e_1 := (1,0,0)$ and $e_2 := (0,1,0)$ and $e_3 := (0,0,1)$ to be elements of $\mathbb{Z}[G]$. Now, let $\varphi\colon \operatorname{Hom}_{\mathbb{Z}[G]}(\operatorname{coker}\mathcal{F}_2,L^\times)\to S$ be defined by

$$\varphi\colon f\mapsto \big(f(e_1),f(e_2),f(e_3)\big).$$

The main check here is that φ is well-defined. We have the following checks.

- We check that our elements live in the correct fields. Note that $f(e_1),f(e_2),f(e_3)\in L^\times$. Further,

  $$\sigma_1(f(e_1)) = f(\sigma_1,0,0) = f(1,0,0) = f(e_1)$$

  where the main step is that $\sigma_1 \equiv 1 \pmod{T_1}$ enforces $(\sigma_1,0,0)=(1,0,0)$. It follows that $f(e_1)\in L^{\langle\sigma_1\rangle}$. An analogous argument shows that $f(e_3)\in L^{\langle\sigma_2\rangle}$.

- We check the relations. The main point is that
$$\begin{aligned}(0, N_1, 0) &\equiv (-T_2, 0, 0) \pmod{\operatorname{im} \mathcal{F}_2},\\ (0, N_2, 0) &\equiv (0, 0, T_1) \pmod{\operatorname{im} \mathcal{F}_2}.\end{aligned}$$
  As such, observe that
$$\begin{aligned}\mathrm{N}_{L/L^{\langle\sigma_1\rangle}} f(0,1,0) &= f(0, N_1, 0)\\ &= f(-T_2, 0, 0)\\ &= f(1,0,0)/\sigma_2 f(1,0,0).\end{aligned}$$
  The other relation is similar: observe
$$\begin{aligned}\mathrm{N}_{L/L^{\langle\sigma_2\rangle}} f(0,1,0) &= f(0, N_2, 0)\\ &= f(0,0,T_1)\\ &= \sigma_1 f(0,0,1)/f(0,0,1),\end{aligned}$$
  which is what we wanted.

We also remark that φ is homomorphic because the operations on both $\operatorname{Hom}_{\mathbb{Z}[G]}(\operatorname{coker} \mathcal{F}_2, L^\times)$ and S are both defined pointwise.

Now, in the other direction, we define $\psi\colon S \to \operatorname{Hom}_{\mathbb{Z}[G]}(\operatorname{coker} \mathcal{F}_2, L^\times)$ by
$$\psi\colon (\alpha_1, \beta, \alpha_2) \mapsto \big((z_1, z_2, z_3) \mapsto z_1\alpha \cdot z_2\beta \cdot z_3\alpha_2\big).$$
Again, the main check is that ψ is well-defined. Well, given a triple $(\alpha_1, \beta, \alpha_2) \in S$, we show that $\psi(\alpha_1, \beta, \alpha_2)$ is well-defined as a G-module homomorphism. To begin, we note that we can start with a function $f\colon \mathbb{Z}[G] \times \mathbb{Z}[G] \times \mathbb{Z}[G] \to L^\times$ by
$$f\colon (z_1, z_2, z_3) \mapsto z_1\alpha_1 \cdot z_2\beta \cdot z_3\alpha_2.$$
This can be quickly checked to be a G-module homomorphism: note that any $\alpha \in L^\times$ makes $z \mapsto z\alpha$ is a G-module homomorphism $L^\times \to L^\times$, and the above is more or less a linear combination of such G-module homomorphisms.

We now investigate $\ker f$. Note that
$$f(T_1, 0, 0) = T_1\alpha = (\sigma_1\alpha_1)/\alpha_1 = 1$$
because $\alpha_1 \in L^{\langle\sigma_1\rangle}$. An identical argument shows that $f(0,0,T_2) = 1$. Further,
$$f(0, N, 0) = N\beta = \mathrm{N}_{L/K}(\beta) = 1.$$
In total, we see that $\operatorname{im} T_1 \times \operatorname{im} N \times \operatorname{im} T_2 \subseteq \ker f$, so we get to induce a function $\overline{f}\colon \mathbb{Z}[G]/\operatorname{im} T_1 \times \mathbb{Z}[G]/\operatorname{im} N \times \mathbb{Z}[G] \operatorname{im} T_2$ by
$$\overline{f}\colon (z_1, z_2, z_3) \mapsto z_1\alpha_1 \cdot z_2\beta \cdot z_3\alpha_2.$$
Further, we note that $(a, b) \in \mathbb{Z}[G] \times \mathbb{Z}[G]$ will have
$$\begin{aligned}\overline{f}(\mathcal{F}_2(a,b)) &= f(T_2a, N_1a - N_2b, T_1b)\\ &= (T_2a)\alpha_1 \cdot (N_1a - N_2b)\beta \cdot (T_1b)\alpha_2\\ &= a\left(\frac{\sigma_2\alpha_1}{\alpha_1} \cdot \mathrm{N}_{L/L^{\langle\sigma_1\rangle}}(\beta)\right) \cdot b\left(\frac{\sigma_1\alpha_2}{\alpha_2} \cdot \mathrm{N}_{L/L^{\langle\sigma_2\rangle}}(\beta^{-1})\right),\end{aligned}$$
which we see goes to 1 by the relations between β and the αs. Thus, we do indeed a G-module homomorphism $\operatorname{coker} \mathcal{F}_2 \to S$ defined as $\psi(\alpha_1, \beta, \alpha_2)$ requires.

We now finish by stating that φ and ψ are mutually inverse, essentially by construction. Checking that $(\varphi \circ \psi)(\alpha_1, \beta, \alpha_2) = (\alpha_1, \beta, \alpha_2)$ is a matter of plugging everything through; on the other hand, to show $(\psi \circ \varphi)(f) = f$, we note that
$$f(z_1, z_2, z_3) = z_1 f(1,0,0) \cdot z_2 f(0,1,0) \cdot z_3 f(0,0,1) = \psi(\varphi(f))$$
because f is a G-module homomorphism. ∎

In fact, we can even correctly account for the equivalence classes of triples; let S_0 denote the subgroup of triples coming from the trivial gerb.

Proposition 6. The isomorphism φ of Lemma 5 descends to a natural isomorphism

$$\overline{\varphi}\colon \widehat{H}^0(G, \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}_2, L^\times)) \simeq S/S_0.$$

Proof. Observe that

$$\widehat{H}^0(G, \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}_2, L^\times)) = \frac{\operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}_2, L^\times)^G}{N_G(\operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}_2, L^\times))},$$

Show this.

so to pass through φ appropriately, we have to show that $\varphi(\operatorname{im} N_G) = S_0$. ■

To tie up loose ends, we promised to use δu from subsection 1.2, so here is how.

Proposition 7. Fix a triple $(\alpha_1, \beta, \alpha_2) \in S$. Then the cocycle in $Z^2(G, L^\times)$ corresponding to $(\alpha_1, \beta, \alpha_2)$ is

$$(\delta u) \cup \varphi^{-1}(\alpha_1, \beta, \alpha_2),$$

where we are implicitly passing through the evaluation map $\operatorname{coker} \mathcal{F}_2 \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}_2, L^\times) \to L^\times$.

Proof. The cup product on inhomogeneous cochains is given by

$$\begin{aligned} \left((\delta u) \cup \varphi^{-1}(\alpha_1, \beta, \alpha_2)\right)\left(\sigma_1^{a_1}\sigma_2^{a_2}, \sigma_1^{b_1}\sigma_2^{b_2}\right) &= (\delta u)\left(\sigma_1^{a_1}\sigma_2^{a_2}, \sigma_1^{b_1}\sigma_2^{b_2}\right) \otimes_{\mathbb{Z}} \sigma_1^{a_1+b_1}\sigma_2^{a_2+b_2}\varphi^{-1}(\alpha_1, \beta, \alpha_2) \\ &= (\delta u)\left(\sigma_1^{a_1}\sigma_2^{a_2}, \sigma_1^{b_1}\sigma_2^{b_2}\right) \otimes_{\mathbb{Z}} \varphi^{-1}(\alpha_1, \beta, \alpha_2). \end{aligned}$$

Passing through evaluation, we will get an element in $L^\times$ which looks like

$$q_1\alpha_1 \cdot -\sigma_1^{a_1}\sigma_1^{(b_1)}\sigma_2^{(a_2)}\beta \cdot q_2\sigma_1^{a_1+b_1}\alpha_2,$$

which is exactly what we wanted (after a little rearrangement). ■