# Classifying Extensions of Abelian Groups

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**Abstract**

We use group cohomology to provide some general theory to classify all group extensions of a $G$-module $A$ in the case of an abelian group $G$. The main idea is to provide a group presentation of the extension using specially chosen elements of $A$.

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# 1 General Group Extensions

Throughout this section, $G$ will be a finite group and $A$ will be a $G$-module; we will write the group operation of $A$ and the group action of $G$ on $A$ multiplicatively. To sketch the idea here, begin with an extension

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1.$$

We know that we can abstractly represent $\mathcal{E}$ as the set $A \times G$ with some group law dictated by a $2$-cocycle in $H^2(G, A)$, so we expect that $\mathcal{E}$ can be presented by $A$ and a choice of lifts from $G$, with some specially chosen relations.

Here are some basic observations realizing this idea. We start by lifting a single element of $G$.

**Lemma 1.** Let $A$ be a $G$-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

denote a group extension. Further, fix some $\sigma \in G$ of order $n_\sigma$, and find $F \in \mathcal{E}$ such that $\sigma := \pi(F)$. Then

$$\alpha := F^{n_\sigma}$$

has $\alpha \in A^{\langle \sigma \rangle}$.

*Proof.* A priori, we only know that $\alpha \in \mathcal{E}$, so we compute

$$\pi(\alpha) = \pi\left(F^{n_\sigma}\right) = \sigma^{n_\sigma} = 1,$$

so $\alpha \in \ker \pi = A$. Thus, we may say that

$$\sigma(\alpha) = F\alpha F^{-1} = F^{n_\sigma} = \alpha,$$

so $\alpha \in A^{\langle \sigma \rangle}$, as desired. ■

We can make the above proof more explicit by specifying the group law of $\mathcal{E}$.

**Lemma 2.** Let $A$ be a $G$-module. Picking up some $2$-cocycle $c \in Z^2(G, A)$, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be the corresponding extension. Fixing $\sigma \in G$ of order $n_\sigma$, let $F := (m, \sigma) \in \mathcal{E}$ be a lift. Then

$$\alpha := F^{n_\sigma} = N_\sigma(m) \prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right),$$

where $N_\sigma := \sum_{i=0}^{n_\sigma - 1} \sigma^i$.

*Proof.* This is a direct computation. By induction, we have that

$$F^k = \left( \prod_{i=0}^{k-1} \sigma^i(m) c\left(\sigma^i, \sigma\right), \sigma^k \right)$$

for $k \in \mathbb{N}$. Indeed, there is nothing to say for $k = 0$, and the inductive step merely expands out $F^k \cdot F$.

It follows that

$$\alpha = F^{n_\sigma} = \left( \prod_{i=0}^{n_\sigma - 1} \sigma^i(m) \cdot \prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right), 1 \right),$$

which is what we wanted. ■

Having this explicit formula lets us say how $\alpha$ changes as we vary the lift.

**Proposition 3.** Let $A$ be a $G$-module. Fixing a cohomology class $u \in H^2(G, A)$, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension whose isomorphism class corresponds to $u$. Further, fix some $\sigma \in G$ of order $n_\sigma$, and let $A_\sigma := A^{\langle \sigma \rangle}$ be the fixed submodule. Then the set

$$S_{\mathcal{E},\sigma} := \{F^{n_\sigma} : \pi(F) = \sigma\}$$

is an equivalence class in $A_\sigma / N_\sigma(A)$, independent of the choice of $\mathcal{E}$. Again, $N_\sigma := \sum_{i=1}^{n_\sigma - 1} \sigma^i$.

*Proof.* Note that $S_{\mathcal{E},\sigma} \subseteq A_\sigma$ already from Lemma 1.

The point is to use Lemma 2. Note the extension $\mathcal{E}$ corresponds to the equivalence class $u \in H^2(G, A)$, so let $c \in Z^2(G, A)$ be a representative. Letting $\mathcal{E}_c$ be the extension constructed from $c$, we are promised an isomorphism $\varphi \colon \mathcal{E} \simeq \mathcal{E}_c$ making the following diagram commute.

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & \mathcal{E}_c \arrow[r, "\pi_c"] & G \arrow[r] & 1
\end{tikzcd}$$

We start by claiming that $S_{\mathcal{E},\sigma} = S_{\mathcal{E}_c,\sigma}$, which will show that $S_{\mathcal{E},\sigma}$ is independent of the choice of representative $\mathcal{E}$. To show $S_{\mathcal{E},\sigma} \subseteq S_{\mathcal{E}_c,\sigma}$, note that $\alpha \in S_{\mathcal{E},\sigma}$ has $F \in \mathcal{E}$ with $\pi(F) = \sigma$ and $\alpha = F^{n_\sigma}$. Pushing this through $\varphi$, we see $\varphi(F) \in \mathcal{E}_c$ has

$$\pi_c(\varphi(F)) = \varphi(\pi(F)) = \sigma \qquad \text{and} \qquad \varphi(F)^{n_\sigma} = \varphi(F^{n_\sigma}) = \alpha,$$

so $\alpha \in S_{\mathcal{E}_c,\sigma}$ follows. An analogous argument with $\varphi^{-1}$ shows the other needed inclusion.

It thus suffices to show that $S_{\mathcal{E}_c,\sigma}$ is an equivalence class in $A_\sigma / N_\sigma(A)$. However, this is exactly what Lemma 2 says as we let the possible lifts $F = (m, \sigma) \in \mathcal{E}_c$ of $\sigma$ vary over $m \in A$. ■

The fact that we are taking elements of $G$ to equivalence classes in $A_\sigma^\times / N_\sigma(A)$ is reminiscent of the (inverse) Artin reciprocity map, and indeed that is exactly what is going on.

**Corollary 4.** Work in the context of Proposition 3. Then

$$S_\sigma := S_{\mathcal{E},\sigma} = [\sigma] \cup [c],$$

where $\cup \colon \widehat{H}^{-2}(G, A) \times \widehat{H}^2(G, A) \to \widehat{H}^0(G, A)$ is the cup product in Tate cohomology.

*Proof.* Using notation as in the proof of Proposition 3, we recall that $S_\sigma = S_{\mathcal{E}_c,\sigma}$, so it suffices to prove the result for $\mathcal{E}_c$. Well, by Lemma 2, $S_\sigma$ is represented by

$$\prod_{i=0}^{n_\sigma - 1} c\left(\sigma^i, \sigma\right).$$

However, this product is exactly the cup product $[\sigma] \cup [c]$. ■

**Corollary 5.** Let $L/K$ be a finite Galois extension of local fields with Galois group $G := \mathrm{Gal}(L/K)$. Further, let

$$1 \to L^\times \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be an $L/K$-gerb bound by $\mathbb{G}_m$ whose isomorphism class corresponds to the fundamental class $u_{L/K} \in H^2(G, L^\times)$. Further, fix some $\sigma \in G$ of order $n_\sigma$, and let $L_\sigma := L^{\langle\sigma\rangle}$ be the fixed field. Then

$$\theta^{-1}_{L/L_\sigma}(\sigma) = \{F^{n_\sigma} : \pi(F) = \sigma\}.$$

*Proof.* Recalling $\theta^{-1}_{L/L_\sigma}$ is a cup product map, note that $\theta^{-1}_{L/L_\sigma}(\sigma)$ is given by $[\sigma] \cup u_{L/K}$. So we are done by Corollary 4. ■

The above results are all interested in lifting single elements of $G$ and studying how they behave on their own. In the discussion that follows, we will need to study how the lifts interact with each other, but for now, we will justify why lifts are adequate to study as follows.

**Proposition 6.** Let $A$ be a $G$-module. Further, let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Given elements $\{\sigma_i\}_{i=1}^m$ which generate $G$, then $\mathcal{E}$ is generated by $A$ and a set of lifts $\{F_i\}_{i=1}^m$ with $\pi(F_i) = \sigma_i$ for each $i$.

*Proof.* Fix some element $e \in \mathcal{E}$, which we need to exhibit as a product of elements in $A$ and $F_i$s. Well, because the $\sigma_i$ generate $G$, we know that $\pi(e) \in G$ can be written as

$$\pi(e) = \prod_{i=1}^m \sigma_i^{a_i}$$

for some sequence of integers $\{a_i\}_{i=1}^m$. It follows that

$$\pi\left(\frac{e}{\prod_{i=1}^m F_i^{a_i}}\right) = 1,$$

so $\frac{e}{\prod_{i=1}^m F_i^{a_i}} = \ker \pi = A$. Thus, we can find some $x \in A$ such that

$$e = x \cdot \prod_{i=1}^m F_i^{a_i},$$

which is what we wanted. ■

# 2 Abelian Group Extensions

## 2.1 Extensions to Tuples

The above proofs technically don't even require that the group $G$ is abelian. If we want to keep track of the fact our group is abelian, we should extract the elements of $A$ which can do so.

**Lemma 7.** Let $A$ be a $G$-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Further, fix some $F_1, F_2 \in \mathcal{E}$ and define $\sigma_i := \pi(F_i)$ for $i \in \{1, 2\}$, and let $\sigma_i \in G$ have order $n_i$. Then, setting

$$\alpha_i := F_i^{n_i} \qquad \text{and} \qquad \beta := F_1 F_2 F_1^{-1} F_2^{-1},$$

we have the following.

(a) $\alpha_i \in A^{\langle \sigma_i \rangle}$ for $i \in \{1, 2\}$ and $\beta \in A$.

(b) $N_1(\beta) = \alpha_1 / \sigma_2(\alpha_1)$ and $N_2(\beta^{-1}) = \alpha_2 / \sigma_1(\alpha_2)$, where $N_i := \sum_{p=0}^{n_i - 1} \sigma_i^p$.

*Proof.* These checks are a matter of force. For brevity, we set $A_i := A^{\langle \sigma_i \rangle}$ for $i \in \{1, 2\}$.

(a) That $\alpha_i \in A_i$ follows from Lemma 1. Lastly, $\beta \in A$ follows from noting

$$\pi(\beta) = \pi(F_1)\pi(F_2)\pi(F_1)^{-1}\pi(F_2)^{-1} = 1,$$

so $\beta \in \ker \pi = A$.

(b) We will check that $\mathrm{N}_{L/L_1}(\beta) = \alpha_1 / \sigma_2(\alpha_1)$; the other equality follows symmetrically after switching $1$s and $2$s because $\beta^{-1} = F_2 F_1 F_2^{-1} F_1^{-1}$. Well, we compute

$$\begin{aligned}
N_1(\beta) &= \sigma_1^{-1}(\beta) \cdot \sigma_1^{-2}(\beta) \cdot \sigma^{-3} \cdot \ldots \cdot \sigma^{-n_1}(\beta) \\
&= F_1^{-1} \left( F_1 F_2 F_1^{-1} F_2^{-1} \right) F_1 \\
&\quad \cdot F_1^{-2} \left( F_1 F_2 F_1^{-1} F_2^{-1} \right) F_1^2 \\
&\quad \cdot F_1^{-3} \left( F_1 F_2 F_1^{-1} F_2^{-1} \right) F_1^3 \cdot \ldots \\
&\quad \cdot F_1^{-n_1} (F_1 F_2 F_1^{-1} F_2^{-1}) F_1^{n_1} \\
&= F_2 F_1^{-1} \\
&\quad \cdot F_1^{-1} \\
&\quad \cdot F_1^{-1} \cdot \ldots \\
&\quad \cdot F_1^{-1} F_2^{-1} F_1^{n_1} \\
&= F_2 F_1^{-n_1} F_2^{-1} F_1^{n_1} \\
&= \alpha_1 / \sigma_2(\alpha_1).
\end{aligned}$$

The above computations finish the proof. ■

The proof of (b) above might appear magical, but in fact it comes from a more general idea.

**Lemma 8.** Fix everything as in Lemma 7. Then, for $x, y \geq 0$, we have

$$F_1^x F_2^y = \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_1^k \sigma_2^\ell(\beta) F_2^y F_1^x.$$

*Proof.* We induct. We take a moment to write out the case of $x = 1$, for which we induct on $y$. To be explicit, we will prove

$$F_1 F_2^y = \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y F_1.$$

For $y = 0$, there is nothing to say. So suppose the statement for $y$ (and $x = 1$), and we show $y+1$ (and $x = 1$). Well, we compute

$$\begin{aligned}
F_1 F_2^{y+1} &= F_1 F_2^y \cdot F_2 \\
&= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y F_1 \cdot F_2 \\
&= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) F_2^y \beta F_2 F_1 \\
&= \prod_{\ell=0}^{y-1} \sigma_2^\ell(\beta) \cdot \sigma_2^y(\beta) F_2^y \cdot F_2 F_1 \\
&= \prod_{\ell=0}^{(y+1)-1} \sigma_2^\ell(\beta) \cdot F_2^{y+1} F_1,
\end{aligned}$$

which is what we wanted.

We now move on to the general case. We will induct on $y$. Note that $y = 0$ makes the product empty, leaving us with $F_1^x = F_1^x$, for any $x$. So suppose that the statement is true for some $y \geq 0$, and we will show $y + 1$. For this, we now turn to inducting on $x$. For $x = 0$, we note that the product is once again empty, so we are left with showing $F_2^{y+1} = F_2^{y+1}$, which is true.

To finish, we suppose the statement for $x$ and show the statement for $x + 1$. Well, we compute

$$\begin{aligned}
F_1^{x+1} F_2^{y+1} &= F_1 \cdot F_1^x F_2^{y+1} \\
&= F_1 \cdot \prod_{k=0}^{x-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot F_2^{y+1} F_1^x \\
&= \sigma_1 \left( \prod_{k=0}^{x-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \right) \cdot F_1 F_2^{y+1} F_1^x \\
&= \prod_{k=1}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot F_1 F_2^{y+1} F_1^x \\
&= \prod_{k=1}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) \cdot \prod_{\ell=0}^{(y+1)-1} \sigma_2^\ell(\beta) \cdot \sigma_2^y(\beta) \cdot F_2^{y+1} F_1 \cdot F_1^x \\
&= \prod_{k=0}^{(x+1)-1} \prod_{\ell=0}^{(y+1)-1} \sigma_1^k \sigma_2^\ell(\beta) F_2^{y+1} F_1^{x+1},
\end{aligned}$$

which is what we wanted. ■

**Remark 9.** Setting $x = n_1$ and $y = 1$ recovers $\mathrm{N}_{L/L^{\langle \sigma_1 \rangle}}(\beta) = \alpha_1 / \sigma_2(\alpha_1)$.

In particular, Remark 9 tells us that coherence of the group law in $\mathcal{E}$ should give rise to relations between our elements of $A$. Here is a more complex example.

**Lemma 10.** Let $A$ be a $G$-module, and let

$$1 \to A \to \mathcal{E} \xrightarrow{\pi} G \to 1$$

be a group extension. Further, fix some $F_1, F_2, F_3 \in \mathcal{E}$ and define $\sigma_i := \pi(F_i)$ for $i \in \{1, 2, 3\}$, and let $\sigma_i \in G$ have order $n_i$. Then, setting

$$\beta_{ij} := F_i F_j F_i^{-1} F_j^{-1}$$

for each pair of indices $(i, j)$ with $i > j$. Then

$$\frac{\sigma_2(\beta_{31})}{\beta_{31}} = \frac{\sigma_1(\beta_{32})}{\beta_{32}} \cdot \frac{\sigma_3(\beta_{21})}{\beta_{21}}.$$

*Proof.* The point is to turn $F_3F_2F_1$ into $F_1F_2F_3$ in two different ways. On one hand,

$$\begin{aligned}(F_3F_2)F_1 &= \beta_{32}F_2F_3F_1 \\ &= \beta_{32}F_2\beta_{31}F_1F_3 \\ &= \beta_{32}\sigma_2(\beta_{31})(F_2F_1)F_3 \\ &= \beta_{32}\sigma_2(\beta_{31})\beta_{21}F_1F_2F_3.\end{aligned}$$

On the other hand,

$$\begin{aligned}F_3(F_2F_1) &= F_3\beta_{21}F_1F_2 \\ &= \sigma_3(\beta_{21})(F_3F_1)F_2 \\ &= \sigma_3(\beta_{21})\beta_{31}F_1(F_3F_2) \\ &= \sigma_3(\beta_{21})\beta_{31}F_1\beta_{32}F_2F_3 \\ &= \sigma_3(\beta_{21})\beta_{31}\sigma_1(\beta_{32})F_1F_2F_3.\end{aligned}$$

Thus,

$$\beta_{32}\sigma_2(\beta_{31})\beta_{21} = \sigma_3(\beta_{21})\beta_{31}\sigma_1(\beta_{32}),$$

which rearranges into the desired equation. ■

**Remark 11.** The relation from Lemma 10 may look asymmetric in the $\beta_{ij}$, but this is because the definitions of the $\beta_{ij}$s themselves are asymmetric in $F_i$.

## 2.2 Tuples to Cocycles

### 2.2.1 The Set-Up

The proceeding lemma is intended to give intuition that the element $\beta$ is helping to specify the group law on $\mathcal{E}$.

More concretely, we will take the following set-up for the following results: fix a $G$-module $A$, and let

$$1 \to A \to \mathcal{E} \to G \to 1$$

be a group extension. Once we choose elements $\{\sigma_i\}_{i=1}^m$ generating $G$, we know by Proposition 6 that we can generate $\mathcal{E}$ by $A$ and some arbitrarily chosen lifts $\{F_i\}_{i=1}^m$ of the $\{\sigma_i\}_{i=1}^m$. Then, letting $n_i$ be the order of $\sigma_i$, we set

$$\alpha_i := F_i^{n_i}$$

for each index $i$ and

$$\beta_{ij} := F_i F_j F_i^{-1} F_j^{-1}$$

for each index $1 \leq j < i \leq m$. Notably, we will not need more $\beta$s: indeed, $\beta_{ii} = 1$ and $\beta_{ij} = \beta_{ji}^{-1}$ for any $i$ and $j$. Setting $A_i := A^{\langle \sigma_i \rangle}$ and $N_i := \sum_{p=0}^{n_i-1} \sigma_i^p$, the story so far is that

$$\alpha_i \in A_i \text{ for each } i \qquad \text{and} \qquad \beta_{ij} \in A \text{ for each } i > j \tag{2.1}$$

and

$$N_i(\beta_{ij}) = \alpha_i/\sigma_j(\alpha_i) \qquad \text{and} \qquad N_j(\beta_{ij}^{-1}) = \alpha_j/\sigma_i(\alpha_j) \qquad \text{for each } i > j \tag{2.2}$$

by Lemma 7, and

$$\frac{\sigma_j(\beta_{ik})}{\beta_{ik}} = \frac{\sigma_k(\beta_{ij})}{\beta_{ij}} \cdot \frac{\sigma_i(\beta_{jk})}{\beta_{jk}} \qquad \text{for each } i > j > k \tag{2.3}$$

by Lemma 10. This data is so important that we will give it a name.

**Definition 12.** In the above set-up, the data of $(\{\alpha_i\}, \{\beta_{ij}\})$ satisfying (2.1) and (2.2) and (2.3) will be called a $\{\sigma_i\}_{i=1}^m$*-tuple*. When understood, the $\{\sigma_i\}_{i=1}^m$ will be abbreviated. Once $G$ and $A$ are fixed, we will denote the set of $\{\sigma_i\}_{i=1}^m$-tuples by $\mathcal{T}(G, A)$.

Note that this definition is independent of $\mathcal{E}$, but a choice of extension $\mathcal{E}$ and lifts $F_i$ give a $\{\sigma_i\}_{i=1}^m$-tuple as described above.

**Remark 13.** The $\mathcal{T}(G, A)$ form a group under multiplication in $A$. Indeed, the conditions (2.1) and (2.2) and (2.3) are closed under multiplication and inversion.

We also know from Lemma 8 that

$$F_i^x F_j^y = \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_i^k \sigma_j^\ell(\beta_{ij}) F_j^y F_i^x$$

for $i > j$ and $x, y \geq 0$. It will be helpful to have some notation for the residue term in $A$, so we define

$$\beta_{ij}^{(xy)} := \prod_{k=0}^{x-1} \prod_{\ell=0}^{y-1} \sigma_i^k \sigma_j^\ell(\beta_{ij}).$$

Now, combined with the fact that $F_i x = \sigma_i(x) F_i$ for each $F_i$ and $x \in A$, we have been approximately told how the group operation works in $\mathcal{E}$. Namely, we could conceivably write any element of $\mathcal{E}$ in the form

$$x F_1^{a_1} \cdots F_m^{a_m}$$

for $x \in A$ and $a_i \in \mathbb{Z}/n_i\mathbb{Z}$ because we know how to make these elements commute and generate $\mathcal{E}$. Further, we can multiply out two terms of the form

$$x F_1^{a_1} \cdots F_m^{a_m} \cdot y F_1^{b_1} \cdots F_m^{b_m}$$

into a term of the form $z F_1^{c_1} \cdots F_m^{c_m}$. In fact, it will be helpful for us to see how to do this.

**Proposition 14.** Fix everything as in the set-up, except drop the assumption that $\{\sigma_i\}_{i=1}^m$ generate $G$. Then, choosing $a_i, b_i \in \mathbb{N}$ for each $i$, we have

$$\left(\prod_{i=1}^m F_i^{a_i}\right)\left(\prod_{i=1}^m F_i^{b_i}\right) = \left[\prod_{1 \leq j < i \leq m} \left(\prod_{1 \leq k < j} \sigma_k^{a_k + b_k}\right)\left(\prod_{j \leq k < i} \sigma_k^{a_k}\right) \beta_{ij}^{(a_i b_j)}\right] \left(\prod_{i=1}^m F_i^{a_i + b_i}\right).$$

*Proof.* The reason that we dropped the assumption on $\{\sigma_i\}_{i=1}^m$ is so that we may induct directly on $m$. We start by showing that

$$\left(\prod_{i=1}^m F_i^{a_i}\right) F_1^{b_1} = \left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\prod_{i=2}^m F_i^{a_i}.$$

We do this by induction on $m$. When $m=0$ and even for $m=1$, there is nothing to say. For the inductive step, we assume

$$\left(\prod_{i=1}^m F_i^{a_i}\right) F_1^{b_1} = \left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\prod_{i=2}^m F_i^{a_i}$$

and compute

$$\begin{aligned}
\left(\prod_{i=1}^{m+1} F_i^{a_i}\right) F_1^{b_1} &= \left(\prod_{i=1}^m F_i^{a_i}\right) F_{m+1}^{a_{m+1}} F_1^{b_1} \\
&= \left(\prod_{i=1}^m F_i^{a_i}\right) \beta_{m+1,1}^{(a_{m+1}b_1)} F_1^{b_1} F_{m+1}^{a_{m+1}} \\
&= \left[\left(\prod_{k=1}^m \sigma_k^{a_k}\right)\beta_{m+1,1}^{(a_{m+1}b_1)}\right]\left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\left(\prod_{i=2}^m F_i^{a_i}\right) F_{m+1}^{a_{m+1}} \\
&= \left[\prod_{1<i\leq m+1}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\left(\prod_{i=2}^{m+1} F_i^{a_i}\right),
\end{aligned}$$

which completes our inductive step.

We now attack the statement of the proposition directly, again inducting on $m$. For $m=0$ and even for $m=1$, there is again nothing to say. For the inductive step, take $m>1$, and we get to assume that

$$\left(\prod_{i=2}^m F_i^{a_i}\right)\left(\prod_{i=2}^m F_i^{b_i}\right) = \left[\prod_{2\leq j<i\leq m}\left(\prod_{2\leq k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\leq k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right).$$

From here, we can compute

$$\begin{aligned}
\left(\prod_{i=1}^m F_i^{a_i}\right)\left(\prod_{i=1}^m F_i^{b_i}\right) &= \left(\prod_{i=1}^m F_i^{a_i}\right) F_1^{b_1}\left(\prod_{i=2}^m F_i^{b_i}\right) \\
&= \left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}\left(\prod_{i=2}^m F_i^{a_i}\right)\left(\prod_{i=2}^m F_i^{b_i}\right) \\
&= \left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right] F_1^{a_1+b_1}. \\
&\qquad \left[\prod_{2\leq j<i\leq m}\left(\prod_{2\leq k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\leq k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right) \\
&= \left[\prod_{1<i\leq m}\left(\prod_{1\leq k<i}\sigma_k^{a_k}\right)\beta_{i1}^{(a_ib_1)}\right]. \\
&\qquad \sigma_1^{a_1+b_1}\left[\prod_{2\leq j<i\leq m}\left(\prod_{2\leq k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\leq k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=2}^m F_i^{a_i+b_i}\right).
\end{aligned}$$

From here, a little rearrangement finishes the inductive step. ∎

The reason we exerted this pain upon ourselves is for the following result.

**Proposition 15.** Fix everything as in the set-up. Then, if well-defined, we can represent the cohomology class corresponding to $\mathcal{E}$ by the cocycle

$$c(g,h) := \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^{m}\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g=\prod_i\sigma_i^{a_i}$ and $h=\prod_i\sigma_i^{b_i}$.

Observe that Proposition 15 has a fairly strong hypothesis that $c$ is well-defined; we will return to this later.

*Proof.* Very quickly, we use the division algorithm to define

$$a_i+b_i=n_iq_i+r_i$$

where $q_\in\{0,1\}$ and $0\le r_i<n_i$. In particular,

$$gh=\prod_{i=1}^{m}F_i^{r_i}.$$

Now, because the elements $\sigma_i$ generate $G$, we see that the lifts $\sigma_i\mapsto F_i$ defines a section $s\colon G\to\mathcal{E}$. As such, we can compute a representing cocycle for our cohomology class as

$$\begin{aligned}c(g,h)&=s(g)s(h)s(gh)^{-1}\\&=\left(\prod_{i=1}^{m}F_i^{a_i}\right)\left(\prod_{i=1}^{m}F_i^{b_i}\right)\left(\prod_{i=1}^{m}F_i^{r_i}\right)^{-1}\\&=\left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left(\prod_{i=1}^{m}F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m}F_{m-i+1}^{-r_{m-i+1}}\right).\end{aligned}$$

It remains to deal with the last products; we claim that it is equal to

$$\left(\prod_{i=1}^{m}F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m}F_{m-i+1}^{-r_{m-i+1}}\right)=\prod_{i=1}^{m}\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{q_i},$$

which will finish the proof. We induct on $m$; for $m=0$ and $m=1$, there is nothing to say. For the inductive step, we assume that

$$\left(\prod_{i=2}^{m}F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m-1}F_{m-i+1}^{-r_{m-i+1}}\right)=\prod_{i=2}^{m}\left(\prod_{2\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}$$

and compute

$$\begin{aligned}\left(\prod_{i=1}^{m}F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m}F_{m-i+1}^{-r_{m-i+1}}\right)&=F_1^{a_1+b_1}\left(\prod_{i=2}^{m}F_i^{a_i+b_i}\right)\left(\prod_{i=1}^{m-1}F_{m-i+1}^{-r_{m-i+1}}\right)F_1^{-a_1-b_1}F_1^{a_1+b_1-r_1}\\&=F_1^{a_1+b_1}\left(\prod_{i=2}^{m}\left(\prod_{2\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}\right)F_1^{-a_1-b_1}\alpha_1^{q_1}\\&=\left(\prod_{i=2}^{m}\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{q_i}\right)\alpha_1^{q_1}\\&=\prod_{i=1}^{m}\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{q_i},\end{aligned}$$

finishing. ■

### 2.2.2 The Modified Set-Up

A priori we have no reason to expect that the $c$ constructed in Proposition 15 is actually a cocycle, especially if the $\sigma_i$ have nontrivial relations.

To account for this, we modify our set-up slightly. By the classification of finitely generated abelian groups, we may write

$$G \simeq \bigoplus_{k=1}^{m} G_k,$$

where $G_k \subseteq G$ with $G_k \cong \mathbb{Z}/n_k\mathbb{Z}$ and $n_k > 1$ for each $n_k$. As such, we let $\sigma_k$ be a generating element of $G_k$ so that we still know that the $\sigma_k$ generate $G$. In this case, we have the following result.

**Theorem 16.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Then a $\{\sigma_i\}_{i=1}^m$-tuple of $\{\alpha_i\}_{i=1}^m$ and $\{\beta_{ij}\}_{i>j}$ makes

$$c(g,h) := \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^{m}\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g := \prod_i \sigma_i^{a_i}$ with $h := \prod_i \sigma_j^{a_j}$ and $0 \le a_i, b_i < n_i$, into a cocycle in $Z^2(G,A)$.

*Proof.* Note that $c$ is now surely well-defined because the elements $g$ and $h$ have unique representations as described. Anyway, we relegate the direct cocycle check to Appendix A because it is long, annoying, and unenlightening. We will also present an alternative proof in section 3, using more abstract theory. ■

Observe that the above construction has now completely forgotten about $\mathcal{E}$! Namely, we have managed to go from tuples straight to cocycles; this is theoretically good because it will allow us to go fully in reverse: we will be able to start with a tuple, build the corresponding cocycle, from which the extension arises. However, equivalence classes of cocycles give the "same" extension, so we will also need to give equivalence classes for tuples as well.

## 2.3 Building Tuples

We continue in the modified set-up of the previous section. There is already an established way to get from a cocycle to an extension, which means that it should be possible to go straight from the cocycle to a $\{\sigma_i\}_{i=1}^m$-tuple. Again, it will be beneficial to write this out.

**Lemma 17.** Fix everything as in the modified set-up, but suppose that $\mathcal{E} = \mathcal{E}_c$ is the extension generated from a cocycle $c \in Z^2(G,A)$. Then, if $F_i = (x_i, \sigma_i)$ are our lifts, we have

$$\alpha_i = N_i(x_i)\cdot\prod_{k=0}^{n_i-1} c\left(\sigma_i^k, \sigma_i\right) \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)}\cdot\frac{\sigma_i(x_j)}{x_j}\cdot\frac{c(\sigma_i,\sigma_j)}{c(\sigma_j,\sigma_i)}$$

for each $\alpha_i$ and $\beta_{ij}$.

*Proof.* The equality for the $\alpha_i$ follow from Lemma 2. For the equality about $\beta_{ij}$, we simply compute by brute force, writing

$$\begin{aligned} F_iF_j &= (x_i\cdot\sigma_ix_j\cdot c(\sigma_i,\sigma_j), \sigma_i\sigma_j) \\ F_jF_i &= (x_j\cdot\sigma_jx_i\cdot c(\sigma_j,\sigma_i), \sigma_j\sigma_i) \\ (F_jF_i)^{-1} &= \left((\sigma_j\sigma_i)^{-1}(x_j\cdot\sigma_jx_i\cdot c(\sigma_j,\sigma_i))^{-1}, \sigma_i^{-1}\sigma_j^{-1}\right), \end{aligned}$$

which gives

$$\begin{aligned}\beta_{ij} &= (F_iF_j)(F_jF_i)^{-1} \\ &= \left(\frac{x_i}{\sigma_j x_i}\cdot\frac{\sigma_i x_j}{x_j}\cdot\frac{c(\sigma_i,\sigma_j)}{c(\sigma_j,\sigma_i)}, 1\right),\end{aligned}$$

finishing. ■

Here is a nice sanity check that we are doing things in the right setting: not only can we build tuples from extensions, but we can find an extension corresponding to any tuple.

**Corollary 18.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. For any $\{\sigma_i\}_{i=1}^m$-tuple of $\{\alpha_i\}_{i=1}^m$ and $\{\beta_{ij}\}_{i>j}$, there exists an extension $\mathcal{E}$ and lifts $F_i$ of the $\sigma_i$ so that

$$\alpha_i = F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} = F_iF_jF_i^{-1}F_j^{-1}.$$

*Proof.* From Theorem 16, we may build the cocycle $c \in Z^2(G, A)$ defined by

$$c(g,h) := \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right], \tag{2.4}$$

where $g := \prod_i F_i^{a_i}$ and $h := \prod_i F_j^{a_j}$ and $0 \le a_i, b_i < n_i$. As such, we use $\mathcal{E} := \mathcal{E}_c$ to be the corresponding extension and $F_i := (1, \sigma_i)$ as our lifts. We have the following checks.

- To show $\alpha_i = F_i^{n_i}$, we use Lemma 17 to compute $F_i^{n_i}$, which means we want to compute

  $$\prod_{k=0}^{n_i-1} c\left(\sigma_i^k, \sigma_i\right).$$

  Well, plugging $c\left(\sigma_i^k, \sigma_i\right)$ into (2.4), we note that all $\beta_{k\ell}^{(a_kb_\ell)}$ terms vanish (either $a_k = 0$ or $b_\ell = 0$ for each $k \ne \ell$), so the big left product completely vanishes.

  As for the right product, the only term we have to worry about is

  $$\left(\prod_{1\le k<i}\sigma_k^{0+0}\right)\alpha_i^{\left\lfloor\frac{k+1}{n_i}\right\rfloor},$$

  which is equal to $1$ when $k \le n_i - 1$ and $\alpha_i$ when $k = n_i - 1$. As such, we do indeed have $\alpha_i = F_i^{n_i}$.

- To show $\beta_{ij} = F_iF_jF_i^{-1}F_j^{-1}$ for $i > j$, we again use Lemma 17 to compute $F_iF_jF_i^{-1}F_j^{-1}$, which means we want to compute

  $$\frac{c(\sigma_i,\sigma_j)}{c(\sigma_j,\sigma_i)}.$$

  Plugging into (2.4) once more, there is no way to make $\lfloor(a_k+b_k)/n_k\rfloor$ nonzero (recall we set $n_k > 1$ for each $k$) in either $c(\sigma_i,\sigma_j)$ or $c(\sigma_j,\sigma_i)$. As such, the right-hand product term disappears.

  As for the left product, we note that it still vanishes for $c(\sigma_j,\sigma_i)$ because $i > j$ implies that either $a_k = 0$ or $b_\ell = 0$ for each $k > \ell$. However, for $c(\sigma_i,\sigma_j)$, we do have $a_i = 1$ and $b_j = 1$ only, so we have to deal with exactly the term

  $$\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}.$$

  With $i > j$ and $a_k = b_k = 0$ for $k \notin \{i, j\}$, we see that the product of all the $\sigma_k$s will disappear, indeed only leaving us with $\beta_{ij}$.

The above computations complete the proof. ■

And here is our first taste of (partial) classification.

**Corollary 19.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Then the formula of Theorem 16 and the formulae of Lemma 17 (setting $x_i = 1$ for each $i$) are homomorphisms of abelian groups between tuples in $\mathcal{T}(G, A)$ and cocycles in $Z^2(G, A)$. In fact, the formula of Theorem 16 is a section of the formulae of Lemma 17.

*Proof.* The formulae in Theorem 16 and Lemma 17 are both large products in their inputs, so they are multiplicative (i.e., homomorphisms). It remains to check that we have a section. Well, starting with a $\{\sigma_i\}_{i=1}^m$-tuple and building the corresponding cocycle $c$ by Theorem 16, the proof of Corollary 18 shows that the formulae of Lemma 17 recovers the correct $\{\sigma_i\}_{i=1}^m$-tuple. ■

## 2.4 Equivalence Classes of Tuples

We continue in the modified set-up. We would like to make Corollary 19 into a proper isomorphism of abelian groups, but this is not feasible; for example, the cocycle $c$ generated by Theorem 16 will always have $c(\sigma_j, \sigma_i) = 1$ for $i > j$, which is not true of all cocycles in $Z^2(G, A)$.

However, we did have a notion that the data of a $\{\sigma_i\}_{i=1}^m$ should be enough to specify the group law of the extension that the tuple comes from, so we do expect to be able to define all extensions—and hence achieve all cohomology classes—from a specially chosen $\{\sigma_i\}_{i=1}^m$-tuple.

To make this precise, we want to define an equivalence relation on tuples which go to the same cohomology class and then show that the map Theorem 16 is surjective on these equivalence classes. The correct equivalence relation is taken from Lemma 17.

**Definition 20.** Fix everything as in the modified set-up. We say that two $\{\sigma_i\}_{i=1}^m$-tuples $(\{\alpha_i\}, \{\beta_{ij}\})$ and $(\{\alpha'_i\}, \{\beta'_{ij}\})$ are *equivalent* if and only if there exist elements $x_1, \ldots, x_m \in A$ such that

$$\alpha_i = N_i(x_i) \cdot \alpha'_i \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)} \cdot \frac{\sigma_i(x_j)}{x_j} \cdot \beta'_{ij}$$

for each $\alpha_i$ and $\beta_{ij}$. We may notate this by $(\{\alpha_i\}, \{\beta_{ij}\}) \sim (\{\alpha'_i\}, \{\beta'_{ij}\})$.

**Remark 21.** It is not too hard to see directly from the definition that this is in fact an equivalence relation. In fact, the set of tuples equivalent to the "trivial" tuple of all 1s is closed under multiplication (and inversion) and hence forms a subgroup of $\mathcal{T}(G, A)$. As such, the set of equivalence classes forms a quotient group of $\mathcal{T}(G, A)$. We will denote this quotient group by $\overline{\mathcal{T}}(G, A)$.

This notion of equivalence can be seen to be the correct one in the sense that it correctly generalizes Proposition 3.

**Proposition 22.** Fix everything as in the modified set-up with an extension $\mathcal{E}$. As the lifts $F_i$ change, the corresponding values of

$$\alpha_i \coloneqq F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} \coloneqq F_i F_j F_i^{-1} F_j^{-1}$$

go through a full equivalence class of $\{\sigma_i\}_{i=1}^m$-tuples.

*Proof.* We proceed as in Proposition 3. Given an extension $\mathcal{E}'$, let $S_{\mathcal{E}'}$ be the set of $\{\sigma_i\}_{i=1}^m$-tuples generated as the lifts $F_i$ change. We start by showing that an isomorphism $\varphi\colon \mathcal{E} \simeq \mathcal{E}'$ of extensions implies that $S_{\mathcal{E}} = S_{\mathcal{E}'}$; by symmetry, it will be enough for $S_{\mathcal{E}} \subseteq S_{\mathcal{E}'}$. The isomorphism induces the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & \mathcal{E}' \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd}$$

To show that $S_{\mathcal{E}} \subseteq S_{\mathcal{E}'}$, pick up some $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$ generated from lifts $F_i \in \mathcal{E}$ (i.e., $\pi(F_i) = \sigma_i$), where

$$\alpha_i := F_i^{n_i} \qquad \text{and} \qquad \beta_{ij} := F_i F_j F_i^{-1} F_j^{-1}.$$

Now, we note that $F_i' := \varphi(F_i)$ will have

$$\pi(F_i') = \pi(\varphi(F_i)) = \varphi(\pi(F_i)) = \sigma_i$$

by the commutativity of the diagram, so the $F_i'$ are lifts of the $\sigma_i$. Further, we see that

$$(F_i')^{n_i} = \varphi(F_i)^{n_i} = \varphi\left(F_i^{n_i}\right) = \varphi(\alpha_i) = \alpha_i$$

for each $i$, and

$$F_i' F_j' (F_i')^{-1} (F_j')^{-1} = \varphi\left(F_i F_j F_i^{-1} F_j^{-1}\right) = \varphi(\beta_{ij}) = \beta_{ij}$$

for each $i > j$. Thus, $(\{\alpha_i\}, \{\beta_{ij}\})$ is a $\{\sigma_i\}_{i=1}^m$-tuple generated by lifts from $\mathcal{E}'$, implying that $(\{\alpha_i\}, \{\beta_{ij}\}) \in S_{\mathcal{E}'}$.

It now suffices to show the statement in the proposition for a specific extension isomorphic to $\mathcal{E}$. Well, the isomorphism class of $\mathcal{E}$ corresponds to some cohomology class in $H^2(G, A)$, for which we let $c$ be a representative; then $\mathcal{E} \simeq \mathcal{E}_c$, so we may show the statement for $\mathcal{E} := \mathcal{E}_c$. Indeed, as the lifts $F_i = (x_i, \sigma_i)$ change, we know by Lemma 17 that

$$\alpha_i = N_i(x_i) \cdot \prod_{k=0}^{n_i - 1} c\left(\sigma_i^k, \sigma_i\right) \qquad \text{and} \qquad \beta_{ij} = \frac{x_i}{\sigma_j(x_i)} \cdot \frac{\sigma_i(x_j)}{x_j} \cdot \frac{c(\sigma_i, \sigma_j)}{c(\sigma_j, \sigma_i)}$$

for each $\alpha_i$ and $\beta_{ij}$. All of these live in the same equivalence class by definition of the equivalence, and as the $x_i$ are allowed to vary over all of $A$, they will fill up that equivalence class fully. This finishes. ■

We are now ready to upgrade our section.

**Corollary 23.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Fixing a cohomology class $[c] \in H^2(G, A)$, the set of $\{\sigma_i\}_{i=1}^m$-tuples which correspond to $[c]$ (via Theorem 16) forms exactly one equivalence class.

*Proof.* We show that two tuples are equivalent if and only if their corresponding cocycles (via Theorem 16) to the same cohomology class, which will be enough.

In one direction, suppose $(\{\alpha_i\}, \{\beta_{ij}\}) \sim (\{\alpha_i'\}, \{\beta_{ij}'\})$. By Corollary 18, we can find an extension $\mathcal{E}$ which gives $(\{\alpha_i\}, \{\beta_{ij}\})$ by choosing an appropriate set of lifts. By Proposition 22, we see that $(\{\alpha_i'\}, \{\beta_{ij}'\})$ must also come from choosing an appropriate set of lifts in $\mathcal{E}$. However, the cocycles in $Z^2(G, A)$ generated by Theorem 16 from our two tuples now both represent the isomorphism class of $\mathcal{E}$ by Proposition 15, so these cocycles belong to the same cohomology class.

In the other direction, name the cocycles corresponding to $(\{\alpha_i\}, \{\beta_{ij}\})$ and $(\{\alpha_i'\}, \{\beta_{ij}'\})$ by $c$ and $c'$ respectively, and suppose $[c] = [c']$. Then $\mathcal{E}_c \simeq \mathcal{E}_{c'}$ as extensions, but we know by the proof of Corollary 18 that $(\{\alpha_i\}, \{\beta_{ij}\})$ comes from choosing lifts of $\mathcal{E}_c$ and similar for $(\{\alpha_i'\}, \{\beta_{ij}'\})$. In particular, because $\mathcal{E}_c \simeq \mathcal{E}_{c'}$, we know that $(\{\alpha_i'\}, \{\beta_{ij}'\})$ will also come from choosing some lifts in $\mathcal{E}_c$ (recall the proof of Proposition 22), so $(\{\alpha_i\}, \{\beta_{ij}\}) \sim (\{\alpha_i'\}, \{\beta_{ij}'\})$ follows. ■

**Theorem 24.** The maps described in Corollary 19 descend to an isomorphism of abelian groups between the equivalence classes in $\overline{\mathcal{T}}(G, A)$ and cohomology classes in $H^2(G, A)$.

*Proof.* The fact that the maps are well-defined (in both directions) and hence injective is Corollary 23. The fact that we had a section from tuples to cocycles implies that the map from cocycles to tuples was also surjective. Thus, we have a bona fide isomorphism. ■

## 2.5 Classification of Extensions

We remark that we are now able to classify all extensions up to isomorphism, in some sense. At a high level, an isomorphism class of extensions corresponds to a particular cohomology class in $H^2(G, A)$, so choosing a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$ corresponding to this class, we can write out a representative of this cocycle by Theorem 16, properly corresponding to the original extension by Proposition 15.

In fact, the cocycle in Proposition 15 is generated by the description of the group law in Proposition 14, and the entire computation only needed to use the following relations, for the appropriate choice of lifts $F_i$.

(a) $F_i x = \sigma_i(x) F_i$ for each $i$ and $x \in A$.

(b) $F_i^{n_i} = \alpha_i$ for each $i$.

(c) $F_i F_j F_i^{-1} F_j^{-1} = \beta_{ij}$ for each $i > j$; i.e., $F_i F_j = \beta_{ij} F_j F_i$.

As such, the above relations fully describe the extension because they also specify the cocycle, and we know that this cocycle is well-defined. We summarize this discussion into the following theorem.

**Theorem 25.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Given a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$, define the group $\mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$ as being generated by $A$ and elements $\{F_i\}_{i=1}^n$ having the following relations.

(a) $F_i x = \sigma_i(x) F_i$ for each $i$ and $x \in A$.

(b) $F_i^{n_i} = \alpha_i$ for each $i$.

(c) $F_i F_j = \beta_{ij} F_j F_i$ for each $i > j$.

Then the natural embedding $A \hookrightarrow \mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$ and projection $\pi\colon \mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\}) \twoheadrightarrow G$ by $F_i \mapsto \sigma_i$ makes $\mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$ into an extension. In fact, all extensions are isomorphic to some $\mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$.

*Proof.* This follows from the preceding discussion, though we will provide a few more words in this proof. The exactness of

$$1 \to A \to \mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\}) \xrightarrow{\pi} G \to 1$$

follows quickly. Further, the action of conjugation of $\mathcal{E}$ on $A$ corresponds correctly to the $G$-action by (a). So we do indeed have an extension.

It remains to show that all extensions are isomorphic to one of this type. Well, note that Proposition 14 and Proposition 15 use only the above relations to write down a cocycle representing the isomorphism class of $\mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$, and it is the cocycle corresponding to the $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$ itself as described in Theorem 16.

However, we know that as the equivalence class of $(\{\alpha_i\}, \{\beta_{ij}\})$ changes, we will hit all cohomology classes in $H^2(G, A)$ by Theorem 24. Thus, because every extension is represented by some cohomology class, every extension will be isomorphic to some $\mathcal{E}(\{\alpha_i\}, \{\beta_{ij}\})$. This completes the proof. ■

## 2.6 Change of Group

We continue in the modified set-up, but we will no longer need access to an extension $\mathcal{E}$. In this subsection, we are interested in what happens to tuples when the cocycle operations of $\mathrm{Inf}\colon H^2\left(G/H, A^H\right) \to H^2(G, A)$ and $\mathrm{Res}\colon H^2(G, A) \to H^2(H, A)$ are applied, where $H \subseteq G$ is some subgroup.

In general, this is difficult because the structure of a subgroup $H \subseteq G$ might not be particularly amenable to forming a tuple from a tuple in $G$. More concretely, $H$ might have generators which look very different from those of $G$. However, it will be enough for our purposes to restrict our attention to the subgroups of the form

$$H = \langle \sigma_1^{t_1}, \ldots, \sigma_m^{t_m} \rangle,$$

where the $\{t_i\}_{i=1}^m$ are some positive integers. With that said, here are our computations. We begin with inflation.

**Lemma 26.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Further, let $H := \langle \sigma_1^{t_1}, \ldots, \sigma_m^{t_m} \rangle$ be a subgroup with $t_\bullet \mid n_\bullet$, and let $\overline{\sigma}_i$ be the image of $\sigma_i$ in $G/H$. Consider the inflation map $\operatorname{Inf}\colon H^2\left(G/H, A^H\right) \to H^2(G, A)$.

If the cocycle $\overline{c} \in Z^2\left(G/H, A^H\right)$ gives the $\{\overline{\sigma}_i\}_{i=1}^m$-tuple $(\{\overline{\alpha}_i\}, \{\overline{\beta}_{ij}\})$ (by Corollary 19), then the cocycle $\operatorname{Inf} \overline{c} \in Z^2(G, A)$ gives the $\{\sigma_i\}_{i=1}^m$-tuple

$$\operatorname{Inf}(\{\overline{\alpha}_i\}, \{\overline{\beta}_{ij}\}) := (\{\alpha_i\}, \{\beta_{ij}\}) = \left(\left\{\overline{\alpha}_i^{n_i/\gcd(t_i, n_i)}\right\}, \{\overline{\beta}_{ij}\}\right).$$

Notably, $\gcd(t_i, n_i)$ is the order of $\overline{\sigma}_i \in G/H$.

*Proof.* The point is to use the explicit formulae for the $\alpha_i$ and $\beta_{ij}$ of Lemma 17.

More explicitly, the map of Corollary 19 tells us that we can compute the tuple for $\operatorname{Inf} \overline{c}$ by using our explicit formulae for $\alpha_i$ and $\beta_{ij}$ on the 2-cocycle $\operatorname{Inf} \overline{c} \in Z^2(G, A)$. For some $\alpha_i$, the computation is

$$\begin{aligned}
\alpha_i &= \prod_{k=0}^{n_i - 1} (\operatorname{Inf} c)\left(\sigma_i^k, \sigma_i\right) \\
&= \prod_{k=0}^{n_i - 1} \overline{c}\left(\overline{\sigma}_i^k, \overline{\sigma}_i\right) \\
&= \left(\prod_{k=0}^{\gcd(n_i, t_i) - 1} \overline{c}\left(\overline{\sigma}_i^k, \overline{\sigma}_i\right)\right)^{n_i/\gcd(n_i, t_i)}
\end{aligned}$$

where the last equality is because $\overline{\sigma}_i^{\gcd(n_i, t_i)} = 1$ in $G/H$. In fact, $\gcd(n_i, t_i)$ is the order of $\overline{\sigma}_i$, so the product is just $\overline{\alpha}_i$ by Lemma 17 and how we defined $\overline{\alpha}_i$. It follows

$$\alpha_i = \overline{\alpha}_i^{n_i/\gcd(n_i, t_i)}.$$

Continuing, for some $\beta_{ij}$, we have

$$\begin{aligned}
\beta_{ij} &= \frac{(\operatorname{Inf} \overline{c})(\sigma_i, \sigma_j)}{(\operatorname{Inf} \overline{c})(\sigma_j, \sigma_i)} \\
&= \frac{\overline{c}(\overline{\sigma}_i, \overline{\sigma}_j)}{\overline{c}(\overline{\sigma}_j, \overline{\sigma}_i)} \\
&= \overline{\beta}_{ij},
\end{aligned}$$

where the last equality is by how we defined $\overline{\beta}_{ij}$. These computations complete the proof. ■

**Remark 27.** We can also the statement of Lemma 26 as asserting that the diagram

$$\begin{tikzcd}
Z^2\left(G/H, A^H\right) \arrow[r, "\operatorname{Inf}"] \arrow[d, two heads] & Z^2(G, A) \arrow[d, two heads] \\
\mathcal{T}\left(G/H, A^H\right) \arrow[r, "\operatorname{Inf}"] & \mathcal{T}(G, A)
\end{tikzcd}$$

commutes, where the vertical morphisms are from Corollary 19.

**Remark 28.** In light of the fact that the cohomology class of some $\operatorname{Inf}\overline{c} \in Z^2(G,A)$ is only defined up to the cohomology class of $\overline{c} \in Z^2\left(G/H, A^H\right)$, changing an input tuple $(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) \in \mathcal{T}\left(G/H, A^H\right)$ up to equivalence will not change the cohomology class of the associated cocycle in $\overline{c} \in Z^2\left(G/H, A^H\right)$ and hence will not change the cohomology class of $\operatorname{Inf}\overline{c}$ nor the equivalence class of $\operatorname{Inf}(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) \in \mathcal{T}(G,A)$. All this is to say that we have a well-defined map

$$\operatorname{Inf}\colon \overline{\mathcal{T}}\left(G/H, A^H\right) \to \overline{\mathcal{T}}(G,A)$$

and commutative diagram

$$\begin{tikzcd} \overline{\mathcal{T}}\left(G/H, A^H\right) \arrow[r, "\operatorname{Inf}"] \arrow[d, hook, two heads] & \overline{\mathcal{T}}(G,A) \arrow[d, hook, two heads] \\ H^2\left(G/H, A^H\right) \arrow[r, "\operatorname{Inf}"] & H^2(G,A) \end{tikzcd}$$

induced by modding out from Remark 27.

Restriction is similar.

**Lemma 29.** Fix everything as in the modified set-up, forgetting about the extension $\mathcal{E}$. Further, let $H := \langle \sigma_1^{t_1}, \ldots, \sigma_m^{t_m} \rangle$ be a subgroup with $t_\bullet \mid n_\bullet$. Consider the inflation map $\operatorname{Res}\colon H^2(G,A) \to H^2(H,A)$.

If the cohomology class $[c] \in H^2(G,A)$ is represented by the $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\},\{\beta_{ij}\})$, then the cohomology class $[\operatorname{Res}\overline{c}]$ is represented by the $\{\sigma_i\}_{i=1}^m$-tuple

$$(\{\overline{\alpha}_i\},\{\overline{\beta}_{ij}\}) = \left(\left\{\alpha_i^{1_{n_i|t_i}}\right\}, \left\{\beta_{ij}^{(\gcd(t_i,n_i)1_{n_i|t_i},\, \gcd(t_j,n_j)1_{n_i|t_i})}\right\}\right).$$

*Proof.* By replacing $t_i$ with $\gcd(t_i,n_i)$ (which does not affect $\langle \sigma_i^{t_i} \rangle$ and hence does not affect $H$), we may assume that $t_i = \gcd(t_i,n_i)$. As in the previous proof, we will simply define $c$ by Theorem 16, and we will use the formulae of Lemma 17 to retrieve the $\{\sigma_i^{t_i}\}$-tuple for $\operatorname{Res} c$. Indeed, we compute

$$\begin{aligned} \overline{\alpha}_i &= \prod_{k=0}^{n_i/t_i-1} (\operatorname{Res} c)\left(\sigma_i^{t_i k}, \sigma_i^{t_i}\right) \\ &= \prod_{k=0}^{n_i/t_i-1} c\left(\sigma_i^{t_i k}, \sigma_i^{t_i}\right) \\ &= \prod_{k=0}^{n_i/t_i-1} \alpha_i^{\lfloor t_i(k+1)/n_i \rfloor}, \end{aligned}$$

where in the last equality we have used the construction of $c$. Now, if $n_i \mid t_i$, then $n_i = t_i$, and the product is empty, and we get $1$; otherwise, the last term of the product $k = n_i/t_i - 1$ is the only term which does not return $1$, and it returns $\alpha_i$. So this matches the claimed $\alpha_i^{1_{n_i|t_i}}$.

Continuing, we compute

$$\begin{aligned} \overline{\beta}_{ij} &= \frac{(\operatorname{Res} c)\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right)}{(\operatorname{Res} c)\left(\sigma_j^{t_j}, \sigma_i^{t_i}\right)} \\ &= \frac{c\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right)}{c\left(\sigma_j^{t_j}, \sigma_i^{t_i}\right)} \\ &= c\left(\sigma_i^{t_i}, \sigma_j^{t_j}\right), \end{aligned}$$

where in the last step we have used the construction of $c$. Now, if $n_i \mid t_i$ or $n_i \mid t_j$, then we are computing $c\left(1, \sigma_j^{t_j}\right)$ or $c\left(\sigma_i^{t_i}, 1\right)$, which are both $1$, as needed. Otherwise, $t_i < n_i$ and $t_j < n_j$, so

$$\overline{\beta}_{ij} = \beta_{ij}^{(t_i t_j)},$$

which again is as claimed. ■

Thankfully, we will really only care about inflation in the following discussion, but we will say that there are analogues of Remark 27 and Remark 28.

## 2.7 Profinite Groups

In this subsection, we will use our results on change of group to extend our results a little to allow profinite groups. As such, we will want to slightly modify our set-up; we will call the following set-up the "profinite set-up."

Let $\mathcal{I}$ be a poset category such that any pair of elements has an upper bound (i.e., a directed set), and let the functor $G_\bullet \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{FinAbGrp}$ be an inverse system of finite abelian groups. These will create a profinite group

$$G := \varprojlim_{i \in \mathcal{I}} G_i.$$

In order to be able to apply our theory, we will assume that $G$ is a finite direct sum of procyclic groups as

$$G \simeq \bigoplus_{k=1}^{m} \overline{\langle \sigma_k \rangle}$$

for some elements $\{\sigma_k\}_{k=1}^m \subseteq G$. Further, we will require that the kernel $N_i$ of the map $G \twoheadrightarrow G_i$ to take the form

$$N_i := \overline{\left\langle \sigma_1^{t_{i,1}}, \ldots, \sigma_m^{t_{i,m}} \right\rangle}.$$

In short, our restriction on the $N_i$ will allow our inflation maps to be computable in the sense of Lemma 26. We quickly remark that, because the topology on $G$ is the coarsest one making the projections $G \twoheadrightarrow G_i$ continuous, the subsets $\{N_i\}_{i \in \mathcal{I}}$ give a fundamental system of open neighborhoods around the identity.

**Remark 30.** Of course, one could also start with $G$ being a finite direct sum of procyclic groups and then define the $N_i$ and $G_i$ accordingly. We have chosen the above approach because in application one might only have access to select $G_i$s, and it is not obvious how to choose these from such a "top-down" approach.

**Example 31.** To show that we are still allowing interesting groups, we can set

$$G_{m,\nu} := \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^m-1})\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p\right) \simeq \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^m-1})/\mathbb{Q}_p\right) \oplus \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p\right),$$

which becomes $G = \mathrm{Gal}\left(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p\right) \simeq \widehat{\mathbb{Z}} \oplus \mathbb{Z}_p^\times$ upon taking the inverse limit. It is not very hard to check that the kernels are generated correctly; for example, when $p$ is odd, we have $\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p$, and under our isomorphisms, we will have

$$\mathrm{Gal}\left(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p\right) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p/p^{\nu-1}\mathbb{Z}_p,$$

so the kernel of $G \twoheadrightarrow G_{m,\nu}$ is $m\widehat{\mathbb{Z}} \oplus (\mathbb{Z}/(p-1)\mathbb{Z})^{1_{\nu=0}} \oplus p^{\nu-1}\mathbb{Z}_p$.

**Remark 32.** I'm not sure if such an explicit construction can be extended to other local fields $K$ (say, via Lubin–Tate theory). Because $K^\times$ is not topologically finitely generated when $K$ is in positive characteristic (see for example [Neu99], Proposition II.5.7) such a construction must do something subtle.

Let $A$ be a discrete $G$-module. The main goal of this subsection is to be able to provide a notion of a "compatible system" of tuples from each individual $H^2(G_i, A)$ to be able to exactly describe an element of $H^2(G, A)$. To effect this, we have the following somewhat annoying checks.

**Lemma 33.** Suppose that $\mathcal{P}$ is a directed set, and let $\mathcal{P}' \subseteq \mathcal{P}$ be a subcategory such that any $x \in \mathcal{P}$ has some $x' \in \mathcal{P}'$ such that $x \leq x'$. Then, given a functor $F \colon \mathcal{P} \to \mathcal{C}$, we have

$$\varinjlim_{\mathcal{P}} F \simeq \varinjlim_{\mathcal{P}'} F,$$

provided that both colimits exist.

*Proof.* For concreteness, if $x \leq y$ in $\mathcal{P}$, we will let $f_{yx} \colon x \to y$ be the corresponding morphism; in particular, $x \leq y \leq z$ has $f_{zx} = f_{zy} f_{yx}$. Now, for brevity, set

$$X := \varinjlim_{\mathcal{P}} F \qquad \text{and} \qquad X' := \varinjlim_{\mathcal{P}'} F.$$

By the Yoneda lemma, it suffices to fix some object $Y \in \mathcal{C}$ and show that $\mathrm{Mor}_{\mathcal{C}}(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X', Y)$. Well, morphsims $X \to Y$ are in (natural) bijection with cones under $F$ with nadir $Y$, and morphisms $X' \to Y$ are in (natural) bijection with cones under $F' := F|_{\mathcal{P}'}$ with nadir $Y$.

Thus, it suffices to give a natural bijection between cones under $F$ with nadir $Y$ and cones under $F'$ with nadir $Y$. Well, given a cone under $F$ with nadir $Y$, we can simply restrict it to $\mathcal{P}'$ to get a cone under $F'$. In the other direction, given a cone under $F'$ with nadir $Y$, we can build a cone under $F$ with nadir $Y$ as follows; let $\varphi_{x'} \colon F(x') \to Y$ for $x' \in \mathcal{P}'$ be the corresponding morphisms in our cone.

For any $x \in \mathcal{P}$, find $x' \in \mathcal{P}'$ such that $x \leq x'$. Then set

$$\varphi_x := \varphi_{x'} \circ f_{x'x}$$

Note that $\varphi_x$ is in fact independent of our choice of $x'$: if $x \leq x'_1$ and $x \leq x'_2$, then because $\mathcal{P}$ is a directed set, we can find $y \in \mathcal{P}$ such that $x'_1, x'_2 \leq y$ and then $y' \in \mathcal{P}'$ with $y \leq y'$. Then

$$\begin{aligned} \varphi_{x'_\bullet} \circ f_{x'_\bullet x} &= \varphi_{y'} \circ f_{y' x'_\bullet} \circ f_{x'_\bullet x} \\ &= \varphi_{y'} \circ f_{y' x} \end{aligned}$$

for $x'_\bullet \in \{x'_1, x'_2\}$. Anyway, we can check that the morphisms $\varphi$ do assemble to a cone under $F'$: if $x \leq y$ in $\mathcal{P}$, then find $y' \in \mathcal{P}$ with $x \leq y \leq y'$, and we compute

$$\begin{aligned} \varphi_y \circ f_{yx} &= \varphi_{y'} \circ f_{y'y} \circ f_{yx} \\ &= \varphi_{y'} \circ f_{y'x} \\ &= \varphi_x. \end{aligned}$$

Thus, we do have a natural, well-defined map sending cones under $F'$ with nadir $Y$ to cones under $F$ with nadir $Y$. It is not too hard to see that these maps are inverse to each other (for example, the cone under $F'$, extended to $F$, does indeed restrict back to $F'$ properly), which completes the proof. ■

**Remark 34.** One can remove the hypothesis that the colimits exist and use essentially the same proof.

**Proposition 35.** Fix everything as in the profinite set-up. Then, given a discrete $G$-module $A$,

$$H^2(G, A) \simeq \varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right).$$

Here, the morphisms between the collection of $H^2\left(G_i, A^{N_i}\right)$ are induced by inflation: if $i \to j$ in $\mathcal{I}$, then $G_j \to G_i$ in $\mathrm{FinAbGrp}$, giving an inflation map $\mathrm{Inf} \colon H^2\left(G_i, A^{N_i}\right) \to H^2\left(G_j, A^{N_j}\right)$.

*Proof.* Let $\mathcal{N}$ be the poset category of open normal subgroups of $G$, reverse ordered under inclusion; i.e., $N_1 \subseteq N_2$ in $G$ induces a map $N_2 \to N_1$. Then it is already known that

$$H^2(G, A) \simeq \varinjlim_{N \in \mathcal{N}} H^2\left(G/N, A^N\right).$$

On the other hand, observe that $i \leq j$ in $\mathcal{I}$ induces $G_j \to G_i$, so $N_j \subseteq N_i$. In other words, $i \mapsto N_i$ will define a functor $\mathcal{I} \to \mathcal{N}$; functoriality follows because $\mathcal{I}$ and $\mathcal{N}$ are poset categories. Letting $\mathcal{N}'$ denote the image of $\mathcal{I}$ in $\mathcal{N}$, we see

$$\varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right) \simeq \varinjlim_{N \in \mathcal{N}'} H^2\left(G/N, A^N\right).$$

Notably, the inflation maps $\mathrm{Inf}\colon H^2\left(G_i, A^{N_i}\right) \to H^2\left(G_j, A^{N_j}\right)$ when $i \leq j$ become the inflation maps $\mathrm{Inf}\colon H^2\left(G/N, A^N\right) \to H^2\left(G/N', A^{N'}\right)$ when $N' \subseteq N$. So if we let $F\colon \mathcal{N} \to \mathrm{AbGrp}$ be the functor taking $N$ to $H^2\left(G/N, A^N\right)$ (and $N \subseteq N'$ to the inflation map), we are trying to show

$$\varinjlim_{\mathcal{N}} F = \varinjlim_{\mathcal{N}'} F.$$

For this, we use Lemma 33. Indeed, for a given open normal subgroup $N \in \mathcal{N}$, we need to find some $N' \in \mathcal{N}'$ such that $N \leq N'$, which means $N' \subseteq N$.

However, the elements of $\mathcal{N}'$ are the collection $\{N_i\}_{i \in \mathcal{I}}$, which form a fundamental system of open neighborhoods around the identity. Thus, the fact that $N$ is an open set containing the identity implies there is some $N_i \in \mathcal{N}'$ such that $N_i \subseteq N$. This finishes the proof. ■

Observe that the above proofs did not use the extra hypotheses on $G$ nor $N_i$ to be products of procyclic groups. We use these hypotheses now. To work more concretely, we note that any $i \in \mathcal{I}$ has

$$G_i \simeq \frac{G}{N_i} \simeq \bigoplus_{p=1}^{m} \overline{\langle \sigma_p \rangle} / \overline{\langle \sigma_p^{t_{i,p}} \rangle} \simeq \bigoplus_{p=1}^{m} \langle \sigma_p \rangle / \langle \sigma_p^{t_{i,p}} \rangle \subseteq \bigoplus_{p=1}^{m} \mathbb{Z}/t_{i,p}\mathbb{Z}$$

is a finite abelian group generated by the elements $\sigma_p N_i$. As a warning, the order of $\sigma_p N_i$ might not be $t_{i,p}$, for example if $\sigma_p$ itself has some small finite order which $t_{i,p}$ is not properly capitalizing on. More concretely, $\mathbb{Z}_5/3\mathbb{Z}_5 = 0$.

Regardless, the main point is that, given a discrete $G$-module $A$, we can consider the $\{\sigma_p N_i\}_{p=1}^m$-tuples $\mathcal{T}\left(G_i, A^{N_i}\right)$. Now, as discussed above, $i \leq j$ in $\mathcal{I}$ induces a quotient map $G_j \simeq G/N_j \twoheadrightarrow G_i/N_i$. From this, we have the following coherence check.

**Lemma 36.** Fix everything as in the profinite set-up, and let $A$ be a discrete $G$-module. Then, given $i \leq j \leq k$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd}
\mathcal{T}\left(G_i, A^{N_i}\right) \arrow[r, "\mathrm{Inf}"] \arrow[dr, "\mathrm{Inf}"'] & \mathcal{T}\left(G_j, A^{N_j}\right) \arrow[d, "\mathrm{Inf}"] \\
& \mathcal{T}\left(G_k, A^{N_k}\right)
\end{tikzcd}$$

commutes. Here, the $\mathrm{Inf}$ maps are defined as in Lemma 26.

*Proof.* For each $i \in \mathcal{I}$, we let $n_{i,p}$ denote the order of $\sigma_p N_i \in G_i$. Using the definition of $\mathrm{Inf}$ from Lemma 26, we just pick up some $\{\sigma_p N_p\}_{p=1}^m$-tuple $(\{\alpha_p\}, \{\beta_{pq}\})$-tuple in $\mathcal{T}\left(G_i, A^{N_i}\right)$ and track through the diagram as follows.

$$\begin{tikzcd}
(\{\alpha_p\}, \{\beta_{pq}\}) \arrow[r, "\mathrm{Inf}"] \arrow[d, "\mathrm{Inf}"'] & \left(\{\alpha_p^{n_{j,p}/n_{i,p}}\}, \{\beta_{pq}\}\right) \arrow[d, "\mathrm{Inf}"] \\
\left(\{\alpha_p^{n_{k,p}/n_{i,p}}\}, \{\beta_{pq}\}\right) \arrow[r, equal] & \left(\{\alpha_p^{(n_{j,p}/n_{i,p})(n_{k,p}/n_{j,p})}\}, \{\beta_{pq}\}\right)
\end{tikzcd}$$

This completes the proof. ■

And here is the result.

**Theorem 37.** Fix everything as in the profinite set-up, and let $A$ be a discrete $G$-module. Then the isomorphisms of Theorem 24 upgrade into an isomorphism

$$H^2(G, A) \simeq \varinjlim_{i \in \mathcal{I}} \overline{\mathcal{T}}\left(G_i, A^{N_i}\right).$$

Here the morphisms between the $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ are inflation maps of Lemma 26.

*Proof.* Note that the objects $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ do make a directed system over $\mathcal{I}$ because of the commutativity of Lemma 36. Namely, the lemma checks that $\mathcal{I} \to \mathrm{AbGrp}$ by $i \mapsto \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ is actually functorial; technically we must also check that the maps $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right) \to \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ are the identity, but this follows from the definition.

Now, by Proposition 35, we have

$$H^2(G, A) \simeq \varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right),$$

but now the natural isomorphism induced by Remark 28 induces an isomorphism of direct limits

$$\varinjlim_{i \in \mathcal{I}} H^2\left(G_i, A^{N_i}\right) \simeq \varinjlim_{i \in \mathcal{I}} \overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$$

given by the isomorphism of Theorem 24 acting pointwise. This completes the proof. ■

Because there are reasonably explicit descriptions of direct limits of abelian groups, and we already have an explicit description of each $\overline{\mathcal{T}}\left(G_i, A^{N_i}\right)$ term in addition to a description of the inflation maps between them, we will be content with our sufficiently explicit description of $H^2(G, A)$. So we call it done here.

# 3 Studying Tuples

The story so far has been able to generalize the one-variable results from section 1 to results using all generators of an abelian group in section 2. It remains to prove Theorem 16, which is the main goal of this section.

## 3.1 Set-Up and Overview

The approach here will be to attempt to abstract our data away from the $G$-module $A$ as much as possible. To set up our discussion, we continue with

$$G \simeq \bigoplus_{i=1}^{m} G_i,$$

where $G_i = \langle \sigma_i \rangle \subseteq G$ and $\sigma_i$ has order $n_k$. These variables allow us to define

$$T_i := (\sigma_i - 1) \qquad \text{and} \qquad N_i := \sum_{p=0}^{n_i - 1} \sigma_i^p$$

for each index $i$. In fact, it will be helpful to also have notation

$$\sigma^{(a)} := \sum_{p=0}^{a-1} \sigma^p$$

for any $\sigma \in G$ and nonnegative integer $a \geq 0$; in particular, $\sigma^{(0)} = 0$ and $\sigma_i^{(n_i)} = N_i$. The main benefits to this notation will be the facts that

$$\sigma^{(a+b)} = \sigma^{(a)} + \sigma^a \sigma^{(b)} \qquad \text{and} \qquad \sigma_i^a = T_i \sigma_i^{(a)} + 1,$$

which can be seen by direct expansion. Given $g \in \prod_{p=1}^n \sigma_p^{a_p}$, we will also define the notation

$$g_i := \prod_{p=1}^{i-1} \sigma_p^{a_p}$$

for $i \geq 0$. In particular $g_0 = g_1 = 1$ and $g_{n+1} = g$.

Now, our tool in the proof of Theorem 16 will be the magical map $\mathcal{F}\colon \mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}} \to \mathbb{Z}[G]^m$ defined by

$$\mathcal{F}\colon \left((x_i)_{i=1}^m, (y_{ij})_{i>j}\right) \mapsto \left(x_i N_i - \sum_{j=1}^{i-1} y_{ij} T_j + \sum_{j=i+1}^{m} y_{ji} T_j\right)_{i=1}^m.$$

This is of course a $G$-module homomorphism. We will go ahead and state the main results we will prove. Roughly speaking, $\mathcal{F}$ is manufactured to make the following result true.

**Proposition 38.** Fix everything as in the set-up. Then the function

$$\overline{c}(g) := \left(g_i \sigma_i^{(a_i)}\right)_{i=1}^m,$$

where $g := \prod_{i=1}^m \sigma_i^{a_i}$, is a $1$-cocycle in $Z^1(G, \operatorname{coker} \mathcal{F})$.

The reason we care about this cocycle is that we can pass it through a boundary morphism induced by the short exact sequence

$$0 \to \underbrace{\frac{\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}}{\ker \mathcal{F}}}_{X:=} \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0,$$

so we have a $2$-cocycle $\delta(\overline{c}) \in Z^2(G, X)$; in fact, we will be able to explicitly compute $\delta(\overline{c})$ as a result of the proof of Proposition 38.

Only now will we bring in tuples. The first result provides an alternate description of tuples.

**Proposition 39.** Fix everything as in the set-up, and now let $A$ be a $G$-module. Then $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\operatorname{Hom}_{\mathbb{Z}[G]}(X, A) = H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A))$.

The second result brings in the last ingredient, the cup product.

**Theorem 40.** Fix everything as in the set-up. Further, fix a $G$-module $A$ and a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$. Then observe there is a natural cup product map

$$\cup\colon H^2(G, X) \times H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A)) \to H^2(G, A)$$

by using the evaluation map $X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, A) \to A$. Then, using the isomorphism of Proposition 39, the cocycle defined in Theorem 16 is simply the output of $\delta(\overline{c}) \cup (\{\alpha_i\}, \{\beta_{ij}\})$ on cocycles.

Because we know that the cup product sends cocycles to cocycles, this will show that the cocycle of Theorem 16 is in fact well-defined.

## 3.2 Preliminary Work

We continue in the set-up of the previous subsection. Before jumping into any hard logic, we define some (more) notation which will be useful later on as well. First, in $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$, we define

$$\kappa_p := \big((1_{i=p})_i, (0)_{i>j}\big) \in X \qquad \text{and} \qquad \lambda_{pq} := \big((0)_i, (1_{(i,j)=(p,q)})_{i>j}\big)$$

for all relevant indices $p$ and $q$ so that the $\kappa_p$ and $\lambda_{pq}$ are a basis for $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$ as a $\mathbb{Z}[G]$-module. Secondly, we define

$$\varepsilon_p := (1_{i=p})_{i=1}^m$$

for all indices $p$, again giving a basis for $\mathbb{Z}[G]^m$ as a $\mathbb{Z}[G]$-module. For example, this notation lets us write

$$\mathcal{F}\left(\sum_{i=1}^m x_i\kappa_i + \sum_{i>j} y_{ij}\lambda_{ij}\right) = \sum_{i=1}^m x_i N_i \varepsilon_i + \sum_{i>j} y_{ij}(T_i\varepsilon_j - T_j\varepsilon_i), \tag{3.1}$$

and

$$\overline{c}(g) = \sum_{i=1}^m g_i \sigma_i^{(a_i)} \varepsilon_i$$

where $g := \prod_{i=1}^m \sigma_i^{a_i}$.

Additionally, so that we do not need to interrupt our discussion later, we establish a few lemmas which will aide our proof of Proposition 38.

**Lemma 41.** Fix everything as in the set-up. Then, for any set of distinct indices $(i_1, \ldots, i_k)$, we have

$$\bigcap_{p=1}^k \operatorname{im} N_{i_p} = \operatorname{im} \prod_{p=1}^k N_{i_p},$$

where we are identifying $x \in \mathbb{Z}[G]$ with its associated multiplication map $x\colon \mathbb{Z}[G] \to \mathbb{Z}[G]$.

*Proof.* The point is that the elements of $\bigcap_{p=1}^k \operatorname{im} N_{i_p}$ and $\operatorname{im} \prod_{p=1}^k N_{i_p}$ are both simply the elements whose expansion in the form $\sum_g c_g g \in \mathbb{Z}[G]$ have $c_j$ "constant in $\sigma_p$ and $\sigma_q$." More explicitly, of course, $\prod_{p=1}^k N_{i_p} \in \bigcap_{p=1}^k \operatorname{im} N_{i_p}$, so

$$\operatorname{im} \prod_{p=1}^k N_{i_p} \subseteq \bigcap_{p=1}^k \operatorname{im} N_{i_p}.$$

In the other direction, suppose that we have some element

$$z := \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \in \bigcap_{p=1}^k \operatorname{im} N_{i_p},$$

the sum is over sequences $(a_i)_{i=1}^m$ such that $0 \le a_i < n_i$ for each index $i$. We will show $z \in \operatorname{im} \prod_{p=1}^k N_{i_p}$.

Now, $z \in \operatorname{im} N_r$ for $r \in \{p, q\}$ is equivalent to $z \in \ker T_r$, but upon multiplying by $(\sigma_r - 1)$ we see that we are asking for

$$\sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_{r-1}^{a_{r-1}} \sigma_r^{a_r} \sigma_{r+1}^{a_{r+1}} \cdots \sigma_n^{a_n} = \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_{r-1}^{a_{r-1}} \sigma_r^{a_r+1} \sigma_{r+1}^{a_{r+1}} \cdots \sigma_n^{a_n}.$$

In other words, this is asking for $c_{(a_i)_i} = c_{(a_i)_i + (1_{i=r})_i}$, or more succinctly just that $c$ is constant in the $i = r$ coordinate.

Thus, $c$ is constant in all the $i = i_p$ coordinates for each index $i_p$. Thus, we let $d_{(a_i)_{i \notin \{i_p\}}}$ be the restricted function equal to $c_{(a_i)_i}$ but forgetting the information input from any of the $a_{i_p}$. This allows us to write

$$\begin{aligned} z &= \sum_{(a_i)_i} c_{(a_i)_i} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \\ &= \sum_{(a_i)_{i \notin \{i_p\}}} \sum_{a_{i_1}=0}^{n_{i_1}-1} \cdots \sum_{a_{i_k}=0}^{n_{i_k}-1} d_{(a_i)_{i \notin \{i_p\}}} \sigma_1^{a_1} \cdots \sigma_m^{a_m} \\ &= \left( \sum_{(a_i)_{i \notin \{i_p\}}} d_{(a_i)_{i \notin \{i_p\}}} \prod_{\substack{i=0 \\ i \notin \{i_p\}}}^{m} \sigma_i^{a_i} \right) \left( \sum_{a_{i_1}=0}^{n_{i_1}-1} \sigma_{i_1}^{a_{i_1}} \right) \cdots \left( \sum_{a_{i_k}=0}^{n_{i_k}-1} \sigma_{i_k}^{a_{i_k}} \right), \end{aligned}$$

which is now manifestly in $\operatorname{im} \prod_{p=1}^{k} N_{i_p}$. ■

**Lemma 42.** Fix everything as in the set-up. Then, given $g := \prod_{i=1}^{m} \sigma_i^{a_i}$, we have

$$g_i = 1 + \sum_{p=1}^{i-1} g_p \sigma_p^{(a_p)} T_p$$

for $i \geq 1$.

*Proof.* This is by induction. For $i = 1$, there is nothing to say. For the inductive step, we take $i > 1$ where we may assume the statement for $i - 1$. Via some relabeling, we may make our inductive hypothesis assert

$$\prod_{p=2}^{i-1} \sigma_p^{a_p} = 1 + \sum_{p=2}^{i-1} \left( \prod_{q=2}^{p-1} \sigma_q^{a_q} \right) \sigma_p^{(a_p)} T_p.$$

In particular, multiplying through by $\sigma_1^{a_1}$ yields

$$\begin{aligned} g_i &= \sigma_1^{a_1} \cdot \prod_{p=2}^{i-1} \sigma_p^{a_p} \\ &= \sigma_1^{a_1} + \sigma_1^{a_1} \sum_{p=2}^{i-1} \left( \prod_{q=2}^{p-1} \sigma_q^{a_q} \right) \sigma_p^{(a_p)} T_p \\ &= \sigma_1^{a_1} + \sum_{p=2}^{i-1} g_p \sigma_p^{(a_p)} T_p \\ &= 1 + \sigma_1^{(a_1)} T_1 + \sum_{p=2}^{i-1} g_p \sigma_p^{(a_p)} T_p, \end{aligned}$$

which is exactly what we wanted, after a little more rearrangement. ■

And mostly because we can, we show that our main short exact sequence splits.

**Lemma 43.** Fix everything as in the set-up. Then consider $\mathbb{Z}$-module map $\rho\colon \mathbb{Z}[G]^m \to \mathbb{Z}[G]^m$ defined by

$$\rho(g\varepsilon_i) := g_i\big(\sigma_i^{a_i} - N_i 1_{a_i=n_i-1}\big)\varepsilon_i + \sum_{j=i+1}^{m} g_j \sigma_j^{(a_j)} T_i \varepsilon_j,$$

where $g := \prod_{i=1}^m \sigma_i^{a_i}$ with $0 \le a_i < n_i$. Then $\rho$ descends to a map $\overline{\rho}\colon \operatorname{coker}\mathcal{F} \to \mathbb{Z}[G]^m$ witnessing the splitting of the short exact sequence

$$0 \to X \to \mathbb{Z}[G]^m \to \operatorname{coker}\mathcal{F} \to 0$$

over $\mathbb{Z}$.

*Proof.* Observe that we have a well-defined map $\rho\colon \mathbb{Z}[G]^m \to \mathbb{Z}[G]^m$ because $\mathbb{Z}[G]^m$ is a free abelian group generated by $g\varepsilon_i$ for $g \in G$ and indices $i$. It remains to show that $\operatorname{im}\mathcal{F} \subseteq \ker\rho$ to get a map $\overline{\rho}\colon \operatorname{coker}\mathcal{F} \to \mathbb{Z}[G]^m$ and then to show that $\rho(z) \equiv z \pmod{\operatorname{im}\mathcal{F}}$ to get the splitting. We show these individually.

To show that $\operatorname{im}\mathcal{F} \subseteq \ker\rho$, we note from (3.1) that $\operatorname{im}\mathcal{F}$ is generated over $\mathbb{Z}[G]$ by the elements $N_i\varepsilon_i$ and $T_i\varepsilon_j - T_j\varepsilon_i$ for relevant indices $i$ and $j$. Thus, $\operatorname{im}\mathcal{F}$ is generated over $\mathbb{Z}$ by the elements $gN_i\varepsilon_i$ and $gT_i\varepsilon_j - gT_j\varepsilon_i$ for relevant indices $i$ and $j$. Thus, we fix any $g := \prod_{i=1}^n \sigma_i^{a_i}$ and show that $gN_i\varepsilon_i \in \ker\rho$ and $gT_i\varepsilon_j - gT_j\varepsilon_i \in \ker\rho$ for relevant indices $i$ and $j$.

- We show $gN_i\varepsilon_i \in \ker\rho$ for any $i$. Because $gN_i = g\sigma_i N_i$, we may as well as assume that $a_i = 0$. Then

  $$\rho\left(g\sigma_i^a\varepsilon_i\right) = g_i\big(\sigma_i^a - N_i 1_{a=n_i-1}\big)\varepsilon_i + \sum_{j=i+1}^{m} g_j \sigma_i^a \sigma_j^{(a_j)} T_i \varepsilon_j.$$

  As $a$ varies from $0$ to $n_i - 1$, we note that the term $g_i\big(\sigma_i^a - N_i 1_{a=n_i-1}\big)\varepsilon_i$ will only get the $-N_i$ contribution exactly once at $a = n_i - 1$. Summing, we thus see that

  $$\rho(gN_i\varepsilon_i) = g_i\left(-N_i + \sum_{a=0}^{n_i-1}\sigma_i^a\right)\varepsilon_i + \sum_{a=0}^{n_i-1}\sum_{j=i+1}^{m} g_j\sigma_i^a\sigma_j^{(a_j)}T_i\varepsilon_j.$$

  The left term vanishes because $N_i = \sum_{a=0}^{n_i-1}\sigma_i^a$. Additionally, the right term vanishes because we can factor $T_i\sum_{a=0}^{n_i-1}\sigma_i^a = T_iN_i = 0$. So $gN_i\varepsilon_i \in \ker\rho$.

- We show $gT_p\varepsilon_q - gT_q\varepsilon_p \in \ker\rho$ for any $p > q$. Equivalently, we will show that $\rho(g\sigma_p\varepsilon_q) - \rho(g\varepsilon_q) = \rho(g\sigma_q\varepsilon_p) - \rho(g\varepsilon_p)$. On one hand, note

  $$\begin{aligned}\rho(g\sigma_p\varepsilon_q) = g_q\big(\sigma_q^{a_q} - N_i 1_{a_q=n_q-1}\big)\varepsilon_q \\ &+ \sum_{j=q+1}^{p-1} g_j\sigma_j^{(a_j)}T_q\varepsilon_j \\ &+ g_p\left(\sigma_p^{(a_p+1)} - N_p 1_{a_p=n_p-1}\right)T_q\varepsilon_p \\ &+ \sum_{j=p+1}^{m}\sigma_p g_j\sigma_j^{(a_j)}T_q\varepsilon_j\end{aligned}$$

  because $g_j$ doesn't "see" the extra $\sigma_p$ term until $j > p$. (For the $j = p$ term, we would like to write $\sigma_p^{(a_p+1)}$ above, but when $a_p = n_p - 1$, we actually end up with $\sigma_p^{(0)} = 0$ and hence have to subtract out $\sigma_p^{(n_p)} = N_p$.) Thus,

  $$\rho(g\sigma_p\varepsilon_q) - \rho(g\varepsilon_q) = g_p\left(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\right)T_q\varepsilon_p + \sum_{j=p+1}^{m} g_j\sigma_j^{(a_j)}T_pT_q\varepsilon_j.$$

On the other hand, we have

$$\rho(g\sigma_q\varepsilon_p) = \sigma_q g_p\big(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\big)\varepsilon_p + \sum_{j=p+1}^{m} \sigma_q g_j \sigma_j^{(a_j)} T_p \varepsilon_j$$

where this time all $j > p$ also have $j > q$ and so $(\sigma_q g)_j = \sigma_q g_j$. Thus,

$$\rho(g\sigma_q\varepsilon_p) - \rho(g\varepsilon_p) = g_p\left(\sigma_p^{a_p} - N_p 1_{a_p=n_p-1}\right) T_q\varepsilon_p + \sum_{j=p+1}^{m} g_j \sigma_j^{(a_j)} T_p T_q \varepsilon_j,$$

as desired.

We now check the splitting. For this, we simply need to check that $\rho(g\varepsilon_i) \equiv g\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$, and we will get the result for all elements of $\mathbb{Z}[G]^m$ by additivity of $\rho$. Well, using Lemma 42, we write

$$\begin{aligned}
g\varepsilon_i &= g_i\sigma_i^{a_i}\left(\prod_{j=i+1}^{m} \sigma_j^{a_j}\right)\varepsilon_i \\
&= g_i\sigma_i^{a_i}\left(1 + \sum_{j=i+1}^{m}\left(\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)\sigma_j^{(a_j)}T_j\right)\varepsilon_i \\
&= g_i\sigma_i^{a_i}\varepsilon_i + \sum_{j=i+1}^{m} g_i\sigma_i^{a_i}\left(\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)\sigma_j^{(a_j)}T_j\varepsilon_i \\
&\equiv g_i\sigma_i^{a_i}\varepsilon_i + \sum_{j=i+1}^{m} g_j\sigma_j^{(a_j)}T_i\varepsilon_j,
\end{aligned}$$

where in the last step we have used the fact that $T_j\varepsilon_i \equiv T_j\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$. Lastly, we note that $hN_i\varepsilon_i \equiv h\varepsilon_i \pmod{\operatorname{im}\mathcal{F}}$ for any $h \in G$, so in fact

$$g\varepsilon_i \equiv g_i\left(\sigma_i^{a_i} - N_i 1_{a_i=n_i-1}\right)\varepsilon_i + \sum_{j=i+1}^{m} g_j\sigma_j^{(a_j)}T_i\varepsilon_j,$$

and now the right-hand side is $\rho(g\varepsilon_i)$. ■

## 3.3 Verification of 1-Cocycles

Here we prove Proposition 38. Namely, we show that the $1$-cochain $\overline{c} \in C^1(G, \operatorname{coker}\mathcal{F})$ defined by

$$\overline{c}(g) = \sum_{i=1}^{m} g_i\sigma_i^{(a_i)}\varepsilon_i$$

where $g := \prod_{i=1}^m \sigma_i^{a_i}$ is actually a $1$-cocycle. It will be beneficial for us to do this by hand, which is a matter of brute force. Set $c \in C^1\left(G, \mathbb{Z}[G]^m\right)$ defined by

$$c(g) := \left(g_i\sigma_i^{(a_i)}\right)_{i=1}^m,$$

where $g := \prod_{i=1}^m \sigma_i^{a_i}$. We will show that $\operatorname{im} dc \subseteq \operatorname{im}\mathcal{F}$, which we will mean that $\operatorname{im}\overline{dc} = \operatorname{im} d\overline{c} = 0$, where $f \mapsto \overline{f}$ is the map $C^\bullet\left(G, \mathbb{Z}[G]^m\right) \twoheadrightarrow C^\bullet\left(G, \operatorname{coker}\mathcal{F}\right)$ induced by modding out.

As such, we set $g := \prod_{i=1}^m \sigma_i^{a_i}$ and $h := \prod_{i=1}^m \sigma_i^{b_i}$ with $0 \le a_i, b_i < n_i$ for each $i$. Then, using the division algorithm, write

$$a_i + b_i = n_i q_i + r_i$$

where $q_i \in \{0,1\}$ and $0 \le r_i < n_i$ for each $i$. Now, we want to show $dc(g,h) \in \operatorname{im}\mathcal{F}$, so we begin by writing

$$\begin{aligned} dc(g,h) &= gc(h) - c(gh) + c(g) \\ &= g\left(h_i\sigma_i^{(b_i)}\right)_{i=1}^m - \left(\prod_{p=0}^{i-1}\sigma_p^{r_p}\cdot\sigma_i^{(r_i)}\right)_{i=1}^m + \left(g_i\sigma_i^{(a_i)}\right)_{i=1}^m \\ &= \left(gh_i\sigma_i^{(b_i)}\right)_{i=1}^m - \left(g_ih_i\sigma_i^{(r_i)}\right)_{i=1}^m + \left(g_i\sigma_i^{(a_i)}\right)_{i=1}^m. \end{aligned} \tag{3.2}$$

We now go term-by-term in (3.2). The easiest is the middle term of (3.2), for which we write

$$\begin{aligned} g_ih_i\sigma_i^{(r_i)} &= g_ih_i\sigma_i^{(a_i+b_i)} - g_ih_i\sigma_i^{r_i}\sigma_i^{(n_iq_i)} \\ &= g_ih_i\sigma_i^{(a_i+b_i)} - g_ih_i\sigma_i^{a_i+b_i}\cdot q_iN_i \\ &= g_ih_i\sigma_i^{(a_i+b_i)} - g_ih_i\cdot q_iN_i, \end{aligned}$$

where the last equality is because $\sigma_iN_i = N_i$. Thus,

$$\begin{aligned} -\left(g_ih_i\sigma_i^{(r_i)}\right)_{i=1}^m &= -\left(g_ih_i\sigma_i^{(a_i+b_i)}\right)_{i=1}^m + (g_ih_i\cdot q_iN_i)_{i=1}^m \\ &= -\left(g_ih_i\sigma_i^{(a_i+b_i)}\right)_{i=1}^m + \mathcal{F}\big((g_ih_iq_i)_i, (0)_{i>j}\big). \end{aligned}$$

Now, using Lemma 42, the $i$th coordinate of the left term of (3.2) is

$$\begin{aligned} gh_i\sigma_i^{(b_i)} &= g_i\sigma_i^{a_i}\left(\prod_{j=i+1}^m\sigma_j^{a_j}\right)h_i\sigma_i^{(b_i)} \\ &= g_i\left(1 + \sum_{j=i+1}^m\left(\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)\sigma_j^{(a_j)}T_j\right)h_i\sigma_i^{a_i}\sigma_i^{(b_i)} \\ &= g_ih_i\sigma_i^{a_i}\sigma_i^{(b_i)} + \sum_{j=i+1}^m\left(g_i\sigma_i^{a_i}\prod_{q=i+1}^{j-1}\sigma_q^{a_q}\right)h_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j \\ &= g_ih_i\sigma_i^{a_i}\sigma_i^{(b_i)} + \sum_{j=i+1}^m g_jh_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j. \end{aligned}$$

And lastly, for the right term of (3.2), the $i$th coordinate is

$$\begin{aligned} g_i\sigma_i^{(a_i)} &= g_i\left(h_i - \sum_{j=1}^{i-1}h_j\sigma_j^{(b_j)}T_j\right)\sigma_i^{(a_i)} \\ &= g_ih_i\sigma_i^{(a_i)} - \sum_{j=1}^{i-1}g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j. \end{aligned}$$

So to finish, we continue from (3.2), which gives

$$\begin{aligned} dc(g,h) - \mathcal{F}\big((g_ih_iq_i)_i, (0)_{i>j}\big) &= \left(g_ih_i\sigma_i^{a_i}\sigma_i^{(b_i)}\right)_{i=1}^m - \left(g_ih_i\sigma_i^{(a_i+b_i)}\right)_{i=1}^m + \left(g_ih_i\sigma_i^{(a_i)}\right)_{i=1}^m \\ &\quad + \left(\sum_{j=i+1}^m g_jh_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j - \sum_{j=1}^{i-1}g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j\right)_{i=1}^m \\ &= \left(-\sum_{j=1}^{i-1}g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}T_j + \sum_{j=i+1}^m g_jh_i\sigma_j^{(a_j)}\sigma_i^{(b_i)}T_j\right)_{i=1}^m \\ &= \mathcal{F}\left((0)_i, \left(g_ih_j\sigma_i^{(a_i)}\sigma_j^{(b_j)}\right)_{i>j}\right). \end{aligned}$$

Thus,

$$dc(g,h) = \mathcal{F}\left((g_i h_i q_i)_i, \left(g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)}\right)_{i>j}\right) \in \operatorname{im} \mathcal{F}. \tag{3.3}$$

This completes the proof of Proposition 38.

In fact, the above proof has found an explicit element $z$ so that $\mathcal{F}(z) = dc(g,h)$ for each $g, h \in G$. As such, we recall that we set

$$X := \frac{\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}}{\ker \mathcal{F}}$$

to give the short exact sequence

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0.$$

In particular, we can track $\overline{c} \in Z^1(G, \operatorname{coker} \mathcal{F})$ through a boundary morphism: we already have a chosen lift $c \in Z^1(G, \mathbb{Z}[G]^m)$ for $\overline{c}$, and we have also computed $\mathcal{F}^{-1} \circ dc$ from the above work. This gives the following result.

**Corollary 44.** Fix everything as in the set-up. Then the $\overline{c}$ of Proposition 38 has

$$\delta(c)(g,h) := \left((g_i h_i q_i)_i, \left(g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)}\right)_{i>j}\right) \in Z^2(G, X)$$

where $\delta$ is induced by

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0.$$

*Proof.* This follows from tracking how $\delta$ behaves, using (3.3). ■

**Remark 45.** In some sense, this $\delta(c)$ is exactly the cocycle of Theorem 16, where we have abstracted away everything about $A$. We will rigorize this notion in our proof of Theorem 40.

## 3.4 Tuples via Cohomology

We continue in the set-up of the previous subsection. The goal of this subsection is to prove Proposition 39. The main idea is that we will be able to finitely generate $\ker \mathcal{F}$ essentially using the relations of a $\{\sigma_i\}_{i=1}^m$-tuple.

We start with the following basic result.

**Lemma 46.** Fix everything as in the set-up. Then $\ker \mathcal{F}$ contains the following elements.

(a) $T_p \kappa_p$ for any index $p$.

(b) $N_p N_q \lambda_{pq}$ for any pair of indices $(p,q)$ with $p > q$.

(c) $T_q \kappa_p + N_p \lambda_{pq}$ for any pair of indices $(p,q)$ with $p > q$.

(d) $T_p \kappa_q - N_q \lambda_{pq}$ for any pair of indices $(p,q)$ with $p > q$.

(e) $T_q \lambda_{pr} - T_r \lambda_{pq} - T_p \lambda_{qr}$ for any triplet of indices $(p,q,r)$ with $p > q > r$.

*Proof.* We start by showing that all the listed elements are in fact in $\ker \mathcal{F}$.

(a) Note that $\mathcal{F}$ only ever takes the $x_i$ term to $x_i N_i$, so if $x_i = T_i$, then the effect of $x_i$ vanishes.

(b) Similarly, note that $\mathcal{F}$ only ever takes the $y_{ij}$ term to $y_{ij} T_i$ or $y_{ij} T_j$. As such, if $y_{ij} = N_i N_j$, then the effect of $y_{ij}$ vanishes again.

(c) The only relevant terms are at indices $p$ and $q$. Here, $i = p$ has $\mathcal{F}$ output

$$T_q N_p - N_p T_q + 0 = 0.$$

For $i = q$, we have no $x_q$ term, so we are left with $N_p T_p = 0$.

(d) Again, the only relevant terms are at indices $p$ and $q$. This time the interesting term is at $i = q$, where we have

$$T_p N_q - 0 + (-N_q) T_p = 0.$$

Then at $i = p$, we simply have $0 N_p - (-N_q) T_q + 0 = 0$.

(e) The relevant terms, as usual, are for $i \in \{p, q, r\}$.

- At $i = p$, we have $0 - (T_q T_r + (-T_r) T_q) + 0 = 0$.
- At $i = q$, we have $0 - (-(T_p) T_r) + ((-T_r) T_p) = 0$.
- At $i = r$, we have $0 - 0 + (T_q T_p + (-T_p) T_q) = 0$.

The above checks complete this part of the proof. ■

**Remark 47.** The above elements are intended to encode the relations to be a $\{\sigma_i\}_{i=1}^n$-tuple. We will see this made rigorous in the proof of Proposition 39.

In fact, the following is true.

**Lemma 48.** Fix everything as in the set-up. Then the elements (a)–(e) of Lemma 46, with (b) removed, generate $\ker \mathcal{F}$.

*Proof.* We remark that we callously removed (b) because it is implied (c): $T_q \kappa_p + N_p \lambda_{pq} \in \ker \mathcal{F}$ implies that

$$N_q \cdot (T_q \kappa_p + N_p \lambda_{pq}) = N_p N_q \lambda_{pq}$$

is also in $\ker \mathcal{F}$. Anyway, this proof is long and annoying and hence relegated to Appendix B. ■

Here is the payoff for the hard work in Lemma 48.

**Proposition 39.** Fix everything as in the set-up, and now let $A$ be a $G$-module. Then $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\operatorname{Hom}_{\mathbb{Z}[G]}(X, A) = H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A))$.

*Proof.* Let $\mathcal{T}$ denote the set of $\{\sigma_i\}_{i=1}^m$-tuples. We now define the map $\varphi\colon \operatorname{Hom}_{\mathbb{Z}[G]}(X, A) \to \mathcal{T}$ by

$$\varphi\colon f \mapsto \Big( \big(f(\kappa_i)\big)_i, \big(f(\lambda_{ij})\big)_{i>j} \Big).$$

In other words, we simply read off the values of $f$ from indicators on the coordinates of $X$. It's not hard to see that $\varphi$ is in fact a $G$-module homomorphism, but we will have to check that $\varphi$ is well-defined, for which we have to check the conditions on being a $\{\sigma_i\}_{i=1}^m$-tuple.

**Lemma 49.** Fix everything as in the set-up, and let $A$ be a $G$-module. Then, given $f\colon \mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$, we have $\ker \mathcal{F} \subseteq \ker f$ if and only if

$$\Big( \big(f(\kappa_i)\big)_i, \big(f(\lambda_{ij})\big)_{i>j} \Big)$$

is a $\{\sigma_i\}_{i=1}^m$-tuple.

*Proof.* By Lemma 48, we see $\ker \mathcal{F} \subseteq \ker f$ if and only if $f$ vanishes on the elements given in Lemma 46. As such, we now run the following checks.

1. We discuss (2.1). For one, note that $f(\lambda_{ij}) \in A$ essentially for free. Now, we note

$$\begin{aligned} f(\kappa_i) \in A^{\langle \sigma_i \rangle} &\iff T_i f(\kappa_i) = 0 \\ &\iff f(T_i \kappa_i) = 0 \\ &\iff T_i \kappa_i \in \ker f. \end{aligned}$$

2. We discuss (2.2). On one hand, note that $i > j$ has

$$\begin{aligned} N_i f(\lambda_{ij}) = -T_j f(\lambda_i) &\iff f(N_i \lambda_{ij} + T_j \lambda_i) \\ &\iff N_i \lambda_{ij} + T_j \lambda_i \in \ker f. \end{aligned}$$

   On the other hand,

$$\begin{aligned} -N_j f(\lambda_{ij}) = -T_i f(\lambda_j) &\iff f(N_j \lambda_{ij} + T_i \lambda_j) = 0 \\ &\iff N_j \lambda_{ij} + T_i \lambda_j \in \ker f. \end{aligned}$$

3. We discuss (2.3). Simply note indices $i > j > k$ have

$$\begin{aligned} T_j f(\lambda_{ik}) = T_k f(\lambda_{ij}) + T_i f(\lambda_{jk}) &\iff f(T_j \lambda_{ik} - T_k \lambda_{ij} - T_i \lambda_{jk}) = 0 \\ &\iff T_j \lambda_{ik} - T_k \lambda_{ij} - T_i \lambda_{jk} \in \ker f. \end{aligned}$$

In total, we see that satisfying the relations to be a $\{\sigma_i\}_{i=1}^m$-tuple exactly encodes the data of having the generators of $\ker \mathcal{F}$ live in $\ker f$. ■

So indeed, given $f \colon X \to A$, the above lemma applied to the composite

$$\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}} \twoheadrightarrow X \xrightarrow{f} A$$

shows that $\varphi(f) \in \mathcal{T}$.

To show that $\varphi$ is an isomorphism, we exhibit its inverse; fix some $(\{\alpha_i\}, \{\beta_{ij}\}_{i>j}) \in \mathcal{T}$. Well, $\mathbb{Z}[G] \times \mathbb{Z}[G]^{\binom{m}{2}}$ has as a basis the $\kappa_i$ and $\lambda_{ij}$, so we can uniquely define a $G$-module homomorphism $f \colon X \to A$ by

$$f(\kappa_i) := \alpha_i \qquad \text{and} \qquad f(\lambda_{ij}) := \beta_{ij}$$

for all relevant indices $i, j$, and in fact the map $\mathcal{T} \to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}, A\right)$ we can see to be a $G$-module homomorphism. However, because these outputs are a $\{\sigma_i\}_{i=1}^m$-tuple, we can read Lemma 49 backward to say that $f$ has kernel containing $\ker \mathcal{F}$, so in fact we induce a map $\overline{f} \colon X \to A$.

So in total, we get a $G$-module homomorphism $\psi \colon \mathcal{T} \to \operatorname{Hom}_{\mathbb{Z}[G]}(X, A)$ by

$$\psi \colon (\{\alpha_i\}, \{\beta_{ij}\}_{i>j}) \mapsto \overline{f},$$

where $\overline{f}$ is defined on the basis elements above. Further, $\psi$ is the inverse of $\varphi$ essentially because the $\{\kappa_i\}_i \cup \{\lambda_{ij}\}_{i>j}$ form a basis of $\mathbb{Z}[G]^m \times \mathbb{Z}[G]^{\binom{m}{2}}$. This completes the proof. ■

And now because it is so easy, we might as well prove Theorem 40.

**Theorem 40.** Fix everything as in the set-up. Further, fix a $G$-module $A$ and a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}, \{\beta_{ij}\})$. Then observe there is a natural cup product map

$$\cup \colon H^2(G, X) \times H^0(G, \operatorname{Hom}_{\mathbb{Z}}(X, A)) \to H^2(G, A)$$

by using the evaluation map $X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X, A) \to A$. Then, using the isomorphism of Proposition 39, the cocycle defined in Theorem 16 is simply the output of $\delta(\overline{c}) \cup (\{\alpha_i\}, \{\beta_{ij}\})$ on cocycles.

*Proof.* The main point is that we have a computation of $\delta(\overline{c})$ from Corollary 44, which we merely need to track through. In particular, fix a $\{\sigma_i\}_{i=1}^m$-tuple $(\{\alpha_i\}_i, \{\beta_{ij}\}_{i>j})$, and let $f \in H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ be the corresponding morphism. As such, we may compute

$$\delta(\overline{c}) \cup f \colon (g,h) \mapsto \delta(\overline{c})(g,h) \otimes_{\mathbb{Z}} gh \cdot f = \delta(\overline{c})(g,h) \otimes_{\mathbb{Z}} f.$$

To pass through evaluation, we set $g := \prod_i \sigma_i^{a_i}$ and $h := \prod_i \sigma_i^{b_i}$, from which we get

$$\begin{aligned} f(\delta(\overline{c})(g,h)) &= f\left((g_i h_i q_i)_i, \left(g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)}\right)_{i>j}\right) \\ &= \sum_{i=1}^m g_i h_i \left\lfloor \frac{a_i + b_i}{n_i} \right\rfloor \cdot \alpha_i + \sum_{\substack{i,j=1 \\ i>j}}^m g_i h_j \sigma_i^{(a_i)} \sigma_j^{(b_j)} \cdot \beta_{ij} \\ &= \sum_{\substack{i,j=1 \\ i>j}}^m \left(\prod_{p<i} \sigma_p^{a_p}\right)\left(\prod_{q<j} \sigma_q^{b_q}\right) \sigma_i^{(a_i)} \sigma_j^{(b_j)} \beta_{ij} + \sum_{i=1}^m g_i h_i \alpha_i^{\left\lfloor \frac{a_i+b_i}{n_i} \right\rfloor}. \end{aligned}$$

Doing a little more rearrangement and writing this multiplicatively exactly recovers the cocycle of Theorem 16. This completes the proof. ■

Though we have proven everything we set out to do in subsection 3.1, there is more to discuss with our alternate description of tuples. As a taste, we prove the following extension of Proposition 39.

**Proposition 50.** Fix everything as in the set-up, and let $A$ be a $G$-module. Then the isomorphism of Proposition 39 descends to an isomorphism between equivalence classes of $\{\sigma_i\}_{i=1}^m$-tuples are canonically isomorphic to $\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$.

*Proof.* Recall that the short exact sequence

$$0 \to X \xrightarrow{\mathcal{F}} \mathbb{Z}[G]^m \to \operatorname{coker} \mathcal{F} \to 0$$

of $G$-modules splits as $\mathbb{Z}$-modules by Lemma 43, so we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}, A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m, A) \xrightarrow{-\circ\mathcal{F}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \to 0.$$

Now, the key trick will be to compare regular group cohomology with Tate cohomology. To begin, we note that our cohomology theories give the following commutative diagram with exact rows.

$$\begin{tikzcd} H^0(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^m, A)) \arrow[r, "-\circ\mathcal{F}"] & H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r] \arrow[d, two heads] & H^1(G, \mathrm{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}, A)) \arrow[d, equal] \\ 0 \arrow[r] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r] & \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(\operatorname{coker} \mathcal{F}, A)) \end{tikzcd} \tag{3.4}$$

Here, the middle vertical map is reduction modulo $\operatorname{im} N_G$. The rows are exact from the long exact sequences, and the square commutes by construction of Tate cohomology. Now, the point is that the diagram induces the isomorphism

$$\frac{H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))}{\operatorname{im}(-\circ\mathcal{F})} \simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)), \tag{3.5}$$

which simply sends $[f] \mapsto [f]$.

Thus, the main content here will be to track through the image of $-\circ\mathcal{F}$ in (3.4). Let $\mathcal{T}$ denote the set of $\{\sigma_i\}_{i=1}^m$-triples of $A$, and let $\mathcal{T}_0$ denote the set (in fact, equivalence class) of triples corresponding to $[0] \in H^2(G, A)$. Letting $\varphi \colon H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \mathcal{T}$ be defined by

$$\varphi \colon f \mapsto \left(\left(f(\kappa_i)\right)_i, \left(f(\lambda_{ij})\right)_{i>j}\right)$$

be the isomorphism of Proposition 39, we claim that the image of $- \circ \mathcal{F}$ in $H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ corresponds under $\varphi$ to exactly $\mathcal{T}_0$.

Indeed, we take a $G$-module homomorphism $f\colon \mathbb{Z}[G]^m \to A$ to the $G$-module homomorphism $(f \circ \mathcal{F})\colon X \to A$. Then we compute

$$\begin{aligned}(f \circ \mathcal{F})(\kappa_i) &= f(N_i \varepsilon_i) \\ &= N_i f(\varepsilon_i) \\ (f \circ \mathcal{F})(\lambda_{ij}) &= f(T_i \varepsilon_j - T_j \varepsilon_i) \\ &= T_i f(\varepsilon_j) - T_j f(\varepsilon_i)\end{aligned}$$

for all relevant indices $i$ and $j$. Thus,

$$\varphi(f \circ \mathcal{F}) = \left( \left(N_i f(\varepsilon_i)\right)_i, \left(T_i f(\varepsilon_j) - T_j f(\varepsilon_i)\right)_{i>j} \right),$$

which we can see lives in $\mathcal{T}_0$ by definition of our equivalence relation (upon using multiplicative notation). In fact, as $f$ varies, we see that the values of $f(\varepsilon_i)$ may vary over all $A$, so the image of $f \mapsto \varphi(f \circ \mathcal{F})$ is exactly all of $\mathcal{T}_0$. Thus, $\varphi$ induces an isomorphism

$$\overline{\varphi}\colon \frac{H^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))}{\mathrm{im}(- \circ \mathcal{F})} \simeq \frac{\mathcal{T}}{\mathcal{T}_0}.$$

Composing this with the "identity" map (3.5) finishes the proof. ■

## 3.5 Some Cup Product Computations

We take a brief intermission to establish a little theory on cup products. In this section, we let $G$ denote a generic finite group (not necessarily assumed to be abelian) and $A$ a $G$-module.

**Lemma 51** ([Neu13], Proposition I.5.3)**.** Let $G$ be a finite group. Given any $G$-modules $A, B, C$ with a $G$-module homomorphism $\varphi\colon B \to C$, the following diagram commutes for any $p, q \in \mathbb{Z}$ and $[a] \in \widehat{H}^p(G, A)$.

$$\begin{tikzcd} \widehat{H}^q(G, B) \arrow[r, "\varphi"] \arrow[d, "{[a]\cup -}"'] & \widehat{H}^q(G, C) \arrow[d, "{[a]\cup -}"] \\ \widehat{H}^{p+q}(G, A \otimes_{\mathbb{Z}} B) \arrow[r, "\mathrm{id}_A \otimes \varphi"] & \widehat{H}^{p+q}(G, A \otimes_{\mathbb{Z}} C) \end{tikzcd}$$

*Proof.* When $p, q \geq 0$, we can argue directly. Indeed, we claim that the diagram commutes on the level of homogeneous cochains: let $[a] \in \widehat{H}^p(G, A)$ and $[b] \in \widehat{H}^q(G, B)$ be cohomology classes represented by the homogeneous cochains $a \in [a]$ and $b \in [b]$. Tracking along the top of the diagram, we see

$$\begin{aligned}(a \cup \varphi(b))(g_0, \ldots, g_{p+q}) &= a(g_0, \ldots, g_p) \otimes \varphi(b)(g_p, \ldots, g_{p+1}) \\ &= a(g_0, \ldots, g_p) \otimes \varphi(b(g_p, \ldots, g_{p+1})).\end{aligned}$$

Tracking along the bottom of the diagram, we see

$$\begin{aligned}(\mathrm{id}_A \otimes \varphi)(a \cup b)(g_0, \ldots, g_{p+q}) &= (\mathrm{id}_A \otimes \varphi)(a(g_0, \ldots, g_p) \otimes b(g_p, \ldots, g_{p+q})) \\ &= a(g_0, \ldots, g_p) \otimes \varphi(b(g_p, \ldots, g_{p+q})),\end{aligned}$$

which is equal. This completes the proof in the case of $p, q \geq 0$.

We will only need the case of $p, q \geq 0$ in the application, but we will go ahead and do the general case now; we dimension-shift $p$ and $q$ downwards. For example, to shift $p$ downwards, we note that the (split) short exact sequence

$$0 \to A \otimes_{\mathbb{Z}} I_G \to A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to A \to 0 \tag{3.6}$$

induces the isomorphism $\delta\colon \widehat{H}^{p-1}(G,A) \to \widehat{H}^p(G, I_G \otimes_{\mathbb{Z}} A)$. As such, given $a \in \widehat{H}^{p-1}(G,A)$, the inductive hypothesis reassures that the following diagram commutes.

$$\begin{tikzcd}
\widehat{H}^q(G,B) \arrow[r, "\varphi"] \arrow[d, "\delta(a)\cup -"'] & \widehat{H}^q(G,C) \arrow[d, "\delta(a)\cup -"] \\
\widehat{H}^{p+q}(G, I_G \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} B) \arrow[r, "\varphi"] & \widehat{H}^{p+q}(G, I_G \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} C)
\end{tikzcd}$$

In other words, all $b \in \widehat{H}^q(G,B)$ have $\varphi(\delta(a) \cup b) = \delta(a) \cup \varphi(b)$.

Now, because (3.6) is split, we can hit it with $- \otimes_{\mathbb{Z}} B$ and $- \otimes_{\mathbb{Z}} C$ to induce the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{\mathbb{Z}} I_G \otimes_{\mathbb{Z}} B \arrow[r] \arrow[d, "\varphi"'] & A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} B \arrow[r] \arrow[d, "\varphi"'] & A \otimes_{\mathbb{Z}} B \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & A \otimes_{\mathbb{Z}} I_G \otimes_{\mathbb{Z}} C \arrow[r] & A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \arrow[r] & A \otimes_{\mathbb{Z}} C \arrow[r] & 0
\end{tikzcd}$$

Letting $\delta_B\colon \widehat{H}^{p-1}(A \otimes_{\mathbb{Z}} B) \to \widehat{H}^p(I_G \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} B)$ and $\delta_C\colon \widehat{H}^{p-1}(A \otimes_{\mathbb{Z}} B) \to \widehat{H}^p(I_G \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} B)$ denote the corresponding isomorphisms (note that the middle terms are induced and hence acyclic), we note that the functoriality of boundary morphisms tells us that $\varphi\delta_B = \delta_C\varphi$. In total, it follows that $b \in \widehat{H}^q(G,B)$ will have

$$\delta_C(\varphi(a \cup b)) = \varphi(\delta_B(a \cup b)) = \varphi(\delta(a) \cup b) \overset{*}{=} \delta(a) \cup \varphi(b) = \delta_C(a \cup \varphi(b)),$$

where we have used the inductive hypothesis at $\overset{*}{=}$. Because $\delta_C$ is an isomorphism, this completes the step to shift $p$ downwards to $p-1$.

Shifting $q$ downwards is similar. This time we start with the following commutative diagram whose rows are (split) short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_G \otimes_{\mathbb{Z}} B \arrow[r] \arrow[d, "\varphi"'] & \mathbb{Z}[G] \otimes_{\mathbb{Z}} B \arrow[r] \arrow[d, "\varphi"'] & B \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & I_G \otimes_{\mathbb{Z}} C \arrow[r] & \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

In particular, we let $\delta'_B\colon \widehat{H}^{q-1}(G,B) \to \widehat{H}^q(G, I_G \otimes_{\mathbb{Z}} B)$ and $\delta'_C\colon \widehat{H}^{q-1}(G,B) \to \widehat{H}^q(G, I_G \otimes_{\mathbb{Z}} C)$ denote the induced isomorphisms, and again functoriality of the boundary morphisms tells us that $\varphi\delta_B = \delta_C\varphi$. Now, the inductive hypothesis tells us that the following diagram commutes for any $a \in \widehat{H}^p(G,A)$.

$$\begin{tikzcd}
\widehat{H}^q(G, I_G \otimes_{\mathbb{Z}} B) \arrow[r, "\varphi"] \arrow[d, "a\cup -"'] & \widehat{H}^q(G, I_G \otimes_{\mathbb{Z}} C) \arrow[d, "a\cup -"] \\
\widehat{H}^{p+q}(G, A \otimes_{\mathbb{Z}} I_G \otimes_{\mathbb{Z}} B) \arrow[r, "\varphi"] & \widehat{H}^{p+q}(G, A \otimes_{\mathbb{Z}} I_G \otimes_{\mathbb{Z}} C)
\end{tikzcd}$$

Namely, any $b \in \widehat{H}^{p-1}(G,B)$ has

$$\begin{aligned}
\delta'_C(a \cup \varphi(b)) &= (-1)^p\big(a \cup \delta'_C(\varphi(b))\big) \\
&= (-1)^p\big(a \cup \varphi(\delta'_B(b))\big) \\
&\overset{*}{=} (-1)^p \varphi(a \cup \delta'_B(b)) \\
&= (-1)^p \cdot (-1)^p \varphi(\delta'_B(a \cup b)) \\
&= \delta'_C(\varphi(a \cup b)),
\end{aligned}$$

where we've applied the inductive hypothesis at $\overset{*}{=}$. Because $\delta'_C$ is an isomorphism, this completes shifting $q$ downwards to $q-1$. ■

**Remark 52.** An analogous argument shows that a $G$-module homomorphism $\psi\colon A \to B$ induces the following commutative diagram, for any $p, q \in \mathbb{Z}$ and $c \in \widehat{H}^q(G, C)$.

$$\begin{tikzcd} \widehat{H}^p(G, A) \arrow[r, "\psi"] \arrow[d, "-\cup c"'] & \widehat{H}^p(G, B) \arrow[d, "-\cup c"'] \\ \widehat{H}^{p+q}(G, A \otimes_{\mathbb{Z}} C) \arrow[r, "\psi"] & \widehat{H}^{p+q}(G, B \otimes_{\mathbb{Z}} C) \end{tikzcd}$$

In a different direction, we will want a duality result. To begin, we recall the following.

**Proposition 53** ([CE56], Corollary XII.6.5)**.** Let $G$ be a finite group and $A$ be any $G$-module. Then the cup-product pairing induces an isomorphism

$$\widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G, A), \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})\right)$$

for all $i \in \mathbb{Z}$. Indeed, this is a duality upon identifying $\widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})$ with $\mathbb{Q}/\mathbb{Z}$.

We will use this to prove the following.

**Proposition 54.** Let $G$ be a finite group, and let $X$ be a finitely generated $\mathbb{Z}$-free $G$-module. Then the cup-product pairing induces an isomorphism

$$\widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G, X), \widehat{H}^{0}(G, \mathbb{Z})\right)$$

for all $i \in \mathbb{Z}$. Indeed, this is a duality upon identifying $\widehat{H}^0(G, \mathbb{Z})$ with $\frac{1}{\#G}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$.

*Proof.* This proof is analogous to [CE56], Theorem XII.6.6. The key to the proof is the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{3.7}$$

The main point is that $X$ being finitely generated and $\mathbb{Z}$-free implies that $X$ is projective (as an abelian group), so we can apply $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ to get out the short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \to 0. \tag{3.8}$$

Now, note that the multiplication-by-$n$ endomorphism on $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})$ is an isomorphism (namely, $\mathbb{Q}$ is a divisible abelian group), so the same will be true of $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}))$ for any $i \in \mathbb{Z}$. However, these cohomology groups must be $\#G$-torsion, so in fact $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})) = 0$ for all $i \in \mathbb{Z}$.

Similarly, we note that we can hit (3.8) with the functor $- \otimes_{\mathbb{Z}} X$ to get another short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} X \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X \to 0. \tag{3.9}$$

Notably, this is exact because $X$ is a finitely generated, torsion-free $\mathbb{Z}$-module and hence flat as a $\mathbb{Z}$-module. Now, $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} X$ is still a divisible abelian group, so again $\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q})) = 0$ for all $i \in \mathbb{Z}$.

The rest of the proof is tracking boundary morphisms around. Fix some $i \in \mathbb{Z}$. Note (3.7) and (3.8) and (3.9) induce boundary isomorphisms

$$\begin{aligned} \delta\colon \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) &\to \widehat{H}^0(G, \mathbb{Z}) \\ \delta_h\colon \widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) &\to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \\ \delta_t\colon \widehat{H}^{-1}(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X) &\to \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} X). \end{aligned}$$

We also note that we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} X \arrow[r] \arrow[d, "\eta_{\mathbb{Z}}"'] & \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}) \otimes_{\mathbb{Z}} X \arrow[r] \arrow[d, "\eta_{\mathbb{Q}}"'] & \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} X \arrow[r] \arrow[d, "\eta_{\mathbb{Q}/\mathbb{Z}}"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the $\eta_\bullet$ are evaluation maps. For peace of mind, we can check that the squares commute by the following lemma.

**Lemma 55.** Let $G$ be a group and $A, B, C$ be $G$-modules with a $G$-module homomorphism $\varphi\colon B \to C$. Then the diagram

$$\begin{tikzcd}
A \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A,B) \arrow[r, "\varphi"] \arrow[d] & A \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A,C) \arrow[d] \\
B \arrow[r, "\varphi"] & C
\end{tikzcd}$$

commutes, where the vertical homomorphisms are evaluation.

*Proof.* We simply pick up some $a \otimes f \in A \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A,B)$ and track through

$$\begin{tikzcd}
a \otimes f \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto] & a \otimes \varphi \circ f \arrow[d, mapsto] \\
f(a) \arrow[r, mapsto, "\varphi"] & \varphi(f(a))
\end{tikzcd}$$

which finishes the proof. ■

Now, Proposition 53 tells us that

$$\begin{array}{ccc}
\widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-i}(G,X), \widehat{H}^{-1}(G,\mathbb{Q}/\mathbb{Z})\right) \\
a & \mapsto & (b \mapsto \eta_{\mathbb{Q}/\mathbb{Z}}(a \cup b))
\end{array}$$

is an isomorphism. Composing this with various other isomorphisms, we can build the isomorphism

$$\begin{array}{ccccccc}
\widehat{H}^{i}(G, X_*) & \to & \widehat{H}^{i-1}(G, X^*) & \to & \mathrm{Hom}\left(\widehat{H}^{-i}(G,X), \widehat{H}^{-1}(G,\mathbb{Q}/\mathbb{Z})\right) & \to & \mathrm{Hom}\left(\widehat{H}^{-i}(G,X), \widehat{H}^{0}(G,\mathbb{Q}/\mathbb{Z})\right) \\
a & \mapsto & \delta_h^{-1} a & \mapsto & (b \mapsto \eta_{\mathbb{Q}/\mathbb{Z}}(\delta_h^{-1} a \cup b)) & \mapsto & (b \mapsto \delta \eta_{\mathbb{Q}/\mathbb{Z}}(\delta_h^{-1} a \cup b))
\end{array}$$

where $X_* := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $X^* := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$, for brevity. This gives an isomorphism between the desired objects, but to prove the result we need to show that the above map is $a \mapsto (b \mapsto \eta_{\mathbb{Z}}(a \cup b))$. Well, given $a \in \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}))$ and $b \in \widehat{H}^{-i}(G, X)$, properties of the boundary morphisms tells us

$$\begin{aligned}
\delta \eta_{\mathbb{Q}/\mathbb{Z}}\left(\delta_h^{-1} a \cup b\right) &= \eta_{\mathbb{Z}} \delta_t \left(\delta_h^{-1} a \cup b\right) \\
&= \eta_{\mathbb{Z}}\left(\delta_h \delta_h^{-1} a \cup b\right) \\
&= \eta_{\mathbb{Z}}(a \cup b),
\end{aligned}$$

which is what we wanted. ■

**Remark 56.** The hypothesis that $X$ be $\mathbb{Z}$-free is necessary: the statement is false for $X = \mathbb{Z}/\#G\mathbb{Z}$ and $i = 0$, for example.

We close this subsection with a dimension-shifting result.

**Lemma 57.** Let $G$ be a finite group and $X$ a $G$-module. If $M$ is an induced $G$-module, then $\mathrm{Hom}_{\mathbb{Z}}(X, M)$ is also an induced $G$-module.

*Proof.* By definition, we can write $M := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ for some $G$-module $A$, where $A$ has perhaps trivial $G$-action. Now, we claim that

$$\begin{array}{lccc} \varphi\colon & \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) & \simeq & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ \varphi\colon & f & \mapsto & \big(z \mapsto (x \mapsto f(x)(z))\big) \end{array}$$

is an isomorphism of $G$-modules. This will finish because the right-hand $G$-module is induced.

Now, $\varphi$ s a homomorphism of abelian groups because

$$\varphi(f + f')(z)(x) = (f + g)(x)(z) = \varphi(f)(z)(x) + \varphi(f')(z)(x)$$

for any $x$ and $z$ and $f, f' \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$. This is a $G$-module homomorphism because any $g \in G$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ has

$$\begin{aligned} \varphi(gf)(z)(x) &= \big(g \cdot \varphi(f)(g^{-1}z)\big)(x) \\ &= g \cdot \varphi(f)(g^{-1}z)(g^{-1}x) \\ &= g \cdot f(g^{-1}x)(g^{-1}z) \\ &= \big(g \cdot f(g^{-1}x)\big)(z) \\ &= (gf)(x)(z) \\ &= \varphi(gf)(x)(z) \end{aligned}$$

for each $x$ and $z$.

Now, we define

$$\begin{array}{lccc} \psi\colon & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(X, A)) & \simeq & \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \\ \psi\colon & f & \mapsto & \big(x \mapsto (z \mapsto f(z)(x))\big) \end{array}$$

to be the inverse morphism. The exact same checks show that this is a $G$-module homomorphism, and it is not hard to see that

$$\varphi\psi(f)(z)(x) = \psi(f)(z)(x) = f(x)(z),$$

so $\varphi \circ \psi$ is the identity; similarly, $\psi \circ \varphi$ is the identity. ■

**Proposition 58.** Let $G$ be a finite group and $X$ a $G$-module. Further, suppose that we have indices $p, q \in \mathbb{Z}$ and $c \in H^p(G, X)$ such that the cup-product map

$$c \cup -\colon \widehat{H}^q(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{p+q}(G, A)$$

is an isomorphism for all $G$-modules $A$. Then the cup-product map

$$c \cup -\colon \widehat{H}^j(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{p+j}(G, A)$$

is an isomorphism for all $G$-modules $A$ and indices $j \in \mathbb{Z}$.

*Proof.* We merely have to shift $q$ up and down. To shift downwards, we suppose that the cup-product map is always an isomorphism for $j$, and we show that it is always an isomorphism $j - 1$. Namely, fix a $G$-module $A$, and we are interested in showing that the cup-product map

$$c \cup -\colon \widehat{H}^{j-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, Z)) \to \widehat{H}^{p+j-1}(G, A)$$

is an isomorphism. To do so, we note the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \tag{3.10}$$

which splits over $\mathbb{Z}$ and thus gives us the short exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, A) \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A) \arrow[r] \arrow[d, "\eta_{I_G}"'] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \arrow[r] \arrow[d, "\eta_{\mathbb{Z}[G]}"'] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \arrow[r] \arrow[d, "\eta_A"'] & 0 \\
0 \arrow[r] & I_G \otimes_{\mathbb{Z}} A \arrow[r] & \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

where the $\eta_\bullet$s are evaluation maps; in particular, the bottom two rows commute by Lemma 55 and thus give a morphism of short exact sequences. These short exact sequences give us boundary morphisms

$$\begin{aligned}
\delta &\colon \widehat{H}^{p+j-1}(G, A) &&\to \widehat{H}^{p+j}(G, I_G \otimes_{\mathbb{Z}} A) \\
\delta_h &\colon \widehat{H}^{j-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) &&\to \widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A)) \\
\delta_t &\colon \widehat{H}^{p+j-1}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)) &&\to \widehat{H}^{p+j}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A)).
\end{aligned}$$

Notably, $\delta$ is an isomorphism because $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ is induced; from this it follows that $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A)$ is also induced by Lemma 57, implying that $\delta_h$ is also an isomorphism.

Now, the key to this dimension-shifting is claiming that the diagram

$$\begin{tikzcd}
\widehat{H}^{j-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "c\cup -"] \arrow[d, "\delta_h"'] & \widehat{H}^{p+j-1}(G, A) \arrow[d, "(-1)^p\delta"'] \\
\widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A)) \arrow[r, "c\cup -"] & \widehat{H}^{p+j}(G, I_G \otimes_{\mathbb{Z}} A)
\end{tikzcd}$$

commutes. Indeed, this will be enough because the bottom row is an isomorphism by the inductive hypothesis, and the left and morphisms are isomorphisms as discussed above, which makes the top row into an isomorphism. Well, to see that the diagram commutes, we expand the diagram as follows.

$$\begin{tikzcd}
\widehat{H}^{j-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "c\cup -"] \arrow[d, "\delta_h"'] & \widehat{H}^{p+j-1}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\eta_A"] \arrow[d, "(-1)^p\delta_t"'] & \widehat{H}^{p+j-1}(G, A) \arrow[d, "(-1)^p\delta"'] \\
\widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A)) \arrow[r, "c\cup -"] & \widehat{H}^{p+j}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X.I_G \otimes_{\mathbb{Z}} A)) \arrow[r, "\eta_{I_G}"] & \widehat{H}^{p+j}(G, I_G \otimes_{\mathbb{Z}} A)
\end{tikzcd}$$

The left square commutes because cup products commute with boundary morphisms; the right square commutes by functoriality of boundary morphisms.

Shifting upwards is similar. Suppose that the cup-product in question is always an isomorphism for $j$, and we show that it is always an isomorphism for $j+1$. Namely, fix a $G$-module $A$, and we are interested in showing that the cup-product map

$$c \cup -\colon \widehat{H}^{j+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{p+j+1}(G, A)$$

is an isomorphism. As before, we use (3.10) to induce the short exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \arrow[r] \arrow[d, "\eta_A"'] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \arrow[r] \arrow[d, "\eta_{\mathbb{Z}[G]}"'] & X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[r] \arrow[d, "\eta_{I_G}"'] & 0 \\
0 \arrow[r] & A \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(I_G, A) \arrow[r] & 0
\end{tikzcd}$$

where the $\eta_\bullet$s are (renamed) evaluation maps. Again, the bottom rows commute by Lemma 55 and hence given a morphism of short exact sequences. As before, we have the boundary morphisms

$$\begin{aligned}
\delta\colon \widehat{H}^{p+j}(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) &\to \widehat{H}^{p+j+1}(G, A)\\
\delta_h\colon \widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) &\to \widehat{H}^{j+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\\
\delta_t\colon \widehat{H}^{p+j}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) &\to \widehat{H}^{p+j+1}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)).
\end{aligned}$$

We again note that $\delta$ is an isomorphism because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ is an induced module; thus, Lemma 57 tells us that $\mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ is also induced, making $\delta_h$ an isomorphism as well.

Once more, the key to the dimension-shifting will be the claim that the diagram

$$\begin{tikzcd}
\widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \arrow[r, "c\cup -"] \arrow[d, "\delta_h"'] & \widehat{H}^{p+j}(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[d, "(-1)^p\delta"'] \\
\widehat{H}^{j+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "c\cup -"] & \widehat{H}^{p+j+1}(G, A)
\end{tikzcd}$$

commutes. This will be enough because the top arrow is an isomorphism by the inductive hypothesis, and the left and right arrows are isomorphisms as discussed above, thus making the bottom arrow also an isomorphism. Now, to see that the diagram commutes, we expand out our cup products as follows.

$$\begin{tikzcd}
\widehat{H}^{j}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \arrow[r, "c\cup -"] \arrow[d, "\delta_h"'] & \widehat{H}^{p+j}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[r, "\eta_{I_G}"] \arrow[d, "(-1)^p\delta_t"'] & \widehat{H}^{p+j}(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[d, "(-1)^p\delta"'] \\
\widehat{H}^{j+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "c\cup -"] & \widehat{H}^{p+j+1}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\eta_A"] & \widehat{H}^{p+j+1}(G, A)
\end{tikzcd}$$

The left square commutes because cup products commute with boundary morphisms, and the right square commutes by functoriality of boundary morphisms. This finishes. ■

And here are some nice corollaries, tying back into our theory.

**Corollary 59.** Fix notation as in subsection 3.1. Then, for any $G$-module $A$ and index $i \in \mathbb{Z}$, the cup-product map

$$[\delta(\overline{c})] \cup -\colon \widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^{i+2}(G, A)$$

is an isomorphism.

*Proof.* Set $p = 2$ and $q = 0$ and $c$ to $[\delta(\overline{c})]$ in Proposition 58; the hypothesis is satisfied by combining the cup-product map of Theorem 40 with Proposition 50. (Namely, the cup-product map is sending an equivalence class of tuples to the corresponding cohomology class, which is an isomorphism by Theorem 24.) Anyway, Proposition 58 does indeed give the result. ■

**Corollary 60.** Fix notation as in subsection 3.1. Then $\widehat{H}^2(G, X) \simeq \mathbb{Z}/\#G\mathbb{Z}$, generated by $[\delta(\overline{c})]$.

*Proof.* For brevity, set $n := \#G$. By Corollary 59, we have the isomorphism

$$[\delta(\overline{c})] \cup -\colon \widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \to \widehat{H}^{0}(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

In particular, $\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \simeq \mathbb{Z}/n\mathbb{Z}$, generated by some element $[\delta(\overline{c})]^\vee$ such that $[\delta(\overline{c})] \cup [\delta(\overline{c})]^\vee = [1]$.

Now, note that the embedding $\mathcal{F}\colon X \hookrightarrow \mathbb{Z}[G]^m$ implies that $X$ is $\mathbb{Z}$-free, so we may apply Proposition 54 to say that the cup-product pairing induces an isomorphism

$$\mathbb{Z}/n\mathbb{Z} \simeq \widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{2}(G, X), \widehat{H}^{0}(G, \mathbb{Z})\right) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{2}(G, X), \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}\right).$$

Because $\widehat{H}^2(G,X)$ is $n$-torsion, homomorphisms $\widehat{H}^2(G,X) \to \mathbb{Q}/\mathbb{Z}$ must have image in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so in fact the rightmost group is the dual of $\widehat{H}^2(G,X)$. Because an abelian group is isomorphic to its dual, we see that $\widehat{H}^2(G,X)$ is in fact cyclic of order $n$.

It remains to show that $[\delta(\overline{c})]$ is a generator; for this, we show that $[\delta(\overline{c})]$ has order at least $n$, which will be enough because $H^2(G,X)$ is cyclic of order $n$. Well, if $k[\delta(\overline{c})] = 0$, then

$$[k] = k\big([\delta(\overline{c})] \cup [\delta(\overline{c})]^\vee\big) = k[\delta(\overline{c})] \cup [\delta(\overline{c})]^\vee = [0] \cup [\delta(\overline{c})]^\vee = [0]$$

in $\widehat{H}^0(G,\mathbb{Z})$, so $n \mid k$. This finishes. ■

## 3.6 A Perfect Pairing

The main goal of this subsection is to prove the following result.

**Theorem 61.** Let $G$ be a finite group, and let $X$ and $A$ be $G$-modules. Then, if there exists an element $c \in H^2(G,X)$ such that the cup-product maps

$$\begin{aligned} c \cup -\colon \widehat{H}^{-2}(G, \operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z})) &\to \widehat{H}^0(G,\mathbb{Z}) \\ c \cup -\colon \widehat{H}^{0}(G, \operatorname{Hom}_{\mathbb{Z}}(X,A)) &\to \widehat{H}^2(G,A) \end{aligned}$$

are isomorphisms, then the cup-product pairing induces an isomorphism

$$\widehat{H}^2(G,A) \to \operatorname{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-2}(G,\operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z})), \widehat{H}^0(G,\operatorname{Hom}_{\mathbb{Z}}(X,A))\right).$$

The main step in the proof is the following lemma.

**Lemma 62.** Let $G$ be a finite group, and let $X$ and $A$ be $G$-modules. Pick up another $G$-module $A$. Then, given any $i \in \mathbb{Z}$ and $c \in \widehat{H}^2(G,X)$ and $u \in \widehat{H}^2(G,A)$, the following diagram commutes, where all arrows are cup-product maps.

$$\begin{tikzcd} \widehat{H}^{i-2}(G,\operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z})) \arrow[r, "-\cup u"] \arrow[d, "c\cup -"'] & \widehat{H}^{i}(G,\operatorname{Hom}_{\mathbb{Z}}(X,A)) \arrow[d, "c\cup -"] \\ \widehat{H}^{i}(G,\mathbb{Z}) \arrow[r, "-\cup u"'] & \widehat{H}^{i+2}(G,A) \end{tikzcd}$$

*Proof.* Formally, our cup-product maps are induced by the following "evaluation morphisms."

- For the left arrow, we have $\eta_L\colon X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) \to \mathbb{Z}$ by evaluation.
- For the top arrow, we have $\eta_T\colon \operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} A \to \operatorname{Hom}_{\mathbb{Z}}(X,A)$ by $f \otimes a \mapsto (x \mapsto f(x)a)$.
- For the bottom arrow, we have $\eta_B\colon \mathbb{Z} \otimes_{\mathbb{Z}} A \to A$ by $k \otimes a \mapsto ka$.
- For the right arrow, we have $\eta_R\colon X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X,A) \to A$ by evaluation.

In particular, these maps are defined so that the following diagram commutes.

$$\begin{tikzcd} X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} A \arrow[r, "\eta_T"] \arrow[d, "\eta_L"'] & X \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(X,A) \arrow[d, "\eta_R"] \\ \mathbb{Z} \otimes_{\mathbb{Z}} A \arrow[r, "\eta_B"'] & A \end{tikzcd} \tag{3.11}$$

Indeed, we can just compute along the following diagram.

$$\begin{tikzcd}
x \otimes f \otimes a \arrow[r, mapsto, "\eta_T"] \arrow[d, mapsto, "\eta_L"'] & x \otimes (x' \mapsto f(x')a) \arrow[d, mapsto, "\eta_R"] \\
f(x) \otimes a \arrow[r, mapsto, "\eta_B"'] & f(x)a
\end{tikzcd}$$

Now, the core of the proof is in drawing the following very large diagram.

$$\begin{tikzcd}
\widehat{H}^{i-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \arrow[r, "-\cup u"] \arrow[d, "c\cup -"'] \arrow[dr, phantom, "(1)"] & \widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A) \arrow[r, "\eta_T"] \arrow[d, "c\cup -"'] \arrow[dr, phantom, "(2)"] & \widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[d, "c\cup -"'] \\
\widehat{H}^{i}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \arrow[r, "-\cup u"] \arrow[d, "\eta_L"'] \arrow[dr, phantom, "(3)"] & \widehat{H}^{i+2}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A) \arrow[r, "\eta_T"] \arrow[d, "\eta_L"'] \arrow[dr, phantom, "(4)"] & \widehat{H}^{i+2}(G, X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[d, "\eta_R"'] \\
\widehat{H}^{i}(G, \mathbb{Z}) \arrow[r, "-\cup u"] & \widehat{H}^{i+2}(G, X \otimes_{\mathbb{Z}} A) \arrow[r, "\eta_B"] & \widehat{H}^{i+2}(G, A)
\end{tikzcd}$$

We are being asked to show that the outer square commutes; we will show that each inner square commutes, which will be enough.

(1) This square commutes by the associativity of the cup product.

(2) This square commutes by Lemma 51.

(3) This square commutes by Lemma 51.

(4) This square commutes by functoriality of $\widehat{H}^{i+2}(G, -)$ applied to (3.11).

The above checks complete the proof. ■

We may now proceed directly with Theorem 61.

*Proof of Theorem 61.* We use the lemma to assert that, for any $u \in H^2(G, A)$, the diagram

$$\begin{tikzcd}
\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})) \arrow[r, "-\cup u"] \arrow[d, "c\cup -"'] & \widehat{H}^{0}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[d, "c\cup -"] \\
\widehat{H}^{0}(G, \mathbb{Z}) \arrow[r, "-\cup u"'] & \widehat{H}^{2}(G, A)
\end{tikzcd}$$

commutes. By hypothesis, the left and right arrows are isomorphisms, so the commutativity means that showing

$$\begin{array}{ccc}
\widehat{H}^2(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})), \widehat{H}^{0}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))\right) \\
u & \mapsto & (a \mapsto (a \cup u))
\end{array}$$

is an isomorphism is the same as showing that

$$\begin{array}{ccc}
\widehat{H}^2(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{0}(G, \mathbb{Z}), \widehat{H}^{2}(G, A)\right) \\
u & \mapsto & (k \mapsto (k \cup u))
\end{array}$$

is an isomorphism. Setting $n := \#G$, we see $\widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and the cup product we are looking at sends $k \in \mathbb{Z}/n\mathbb{Z}$ and $u \in \widehat{H}^2(G, A)$ to $k \cup u = ku$ by how the "evaluation" map $\mathbb{Z} \otimes_{\mathbb{Z}} A \simeq A$ behaves. Thus, we are showing that

$$\begin{array}{ccc}
\widehat{H}^2(G, A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, \widehat{H}^{2}(G, A)\right) \\
u & \mapsto & (k \mapsto ku)
\end{array}$$

is an isomorphism.

However, $\widehat{H}^2(G,A)$ is $n$-torsion, so in fact maps $\mathbb{Z} \to \widehat{H}^2(G,A)$ automatically have $n\mathbb{Z}$ in their kernel and hence reduce to maps $\mathbb{Z}/n\mathbb{Z} \to \widehat{H}^2(G,A)$. Conversely, any map $\mathbb{Z}/n\mathbb{Z} \to \widehat{H}^2(G,A)$ can be extended by $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ to a map $\mathbb{Z} \to \widehat{H}^2(G,A)$, so we have a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, \widehat{H}^2(G,A)\right) & \simeq & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \widehat{H}^2(G,A)\right) \\ f & \mapsto & (k \mapsto f([k])) \\ ([k] \mapsto f(k)) & \leftarrow\!\shortmid & f. \end{array}$$

In particular, it suffices to show that

$$\begin{array}{ccc} \widehat{H}^2(G,A) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \widehat{H}^2(G,A)\right) \\ u & \mapsto & (k \mapsto ku) \end{array}$$

is an isomorphism. But this is a standard fact about the functor $\mathrm{Hom}_{\mathbb{Z}} \colon \mathrm{AbGrp} \to \mathrm{AbGrp}$, so we are done. ■

We now synthesize the theory we have been building.

**Corollary 63.** Fix notation as in subsection 3.1. Then, given a $G$-module $A$, the cup-product pairing induces an isomorphism

$$\widehat{H}^2(G,A) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z})), \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X,A))\right).$$

*Proof.* We apply Theorem 61 to our case; here $X$ is defined as in subsection 3.1, and we take $c$ to be $[\delta(\overline{c})]$. Note $X$ is $\mathbb{Z}$-free because of the embedding $\mathcal{F} \colon X \hookrightarrow \mathbb{Z}[G]^m$. The cup-product maps in question are isomorphisms by Corollary 59. Thus, Theorem 61 kicks in, completing the proof. ■

**Remark 64.** The other side of the pairing

$$\widehat{H}^{-2}(G, \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{H}^2(G,A), \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X,A))\right)$$

need not be an isomorphism; for example, take $A = 0$.

## 3.7 Uniqueness

Let $G$ be a finite group. In this subsection we investigate how unique the $X$ constructed in subsection 3.1. To this end, we introduce the following notion of equivalence.

**Definition 65.** Let $G$ be a finite group. We say that two $G$-modules $X, X'$ are *cohomologically equivalent* if and only if there exist morphisms $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that

$$[\varphi \circ \varphi'] = [\mathrm{id}_X] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X,X)) \qquad \text{and} \qquad [\varphi' \circ \varphi] = [\mathrm{id}_{X'}] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X',X')).$$

**Example 66.** All induced modules $X$ are cohomologically equivalent to $0$. To see this, we set $\varphi \colon 0 \to X$ and $\varphi' \colon X \to 0$ equal to the zero maps (which are our only options). Then note that $\mathrm{Hom}_{\mathbb{Z}}(X,X)$ is induced by Lemma 57 and $\mathrm{Hom}_{\mathbb{Z}}(0,0) = 0$, so

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X,X)) = \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X',X')) = 0,$$

making the checks on $\varphi$ and $\varphi'$ both trivial.

More concretely, $X$ and $X'$ are cohomologically equivalent if and only if we have two $G$-module morphisms $\varphi\colon X' \to X$ and $\varphi'\colon X \to X'$ and two $\mathbb{Z}$-module morphisms $f\colon X \to X$ and $f'\colon X' \to X'$ such that

$$\varphi \circ \varphi' = \mathrm{id}_X + N_G f \qquad \text{and} \qquad \varphi' \circ \varphi = \mathrm{id}_{X'} + N_G f'.$$

As a quick sanity check that this is a reasonable notion of equivalence of modules, we have the following.

**Lemma 67.** Let $G$ be a finite group. If the $G$-modules $X$ and $X'$ are equivalent and $Y$ and $Y'$ are equivalent, then $X \oplus X'$ is equivalent to $Y \oplus Y'$.

*Proof.* We are promised the morphisms

- $\varphi\colon X' \to X$ and $\varphi'\colon X \to X'$ (as morphisms of $G$-modules),
- $f\colon X \to X$ and $f'\colon X' \to X'$ (as morphisms of $\mathbb{Z}$-modules),
- $\psi\colon Y' \to Y$ and $\psi'\colon Y \to Y'$ (as morphisms of $G$-modules),
- $g\colon Y \to Y$ and $g'\colon Y' \to Y'$ (as morphisms of $\mathbb{Z}$-modules),

which are required to satisfy

$$\begin{aligned} \varphi \circ \varphi' &= \mathrm{id}_X + N_G f \qquad \text{and} \qquad \varphi' \circ \varphi = \mathrm{id}_{X'} + N_G f', \\ \psi \circ \psi' &= \mathrm{id}_Y + N_G g \qquad \text{and} \qquad \psi' \circ \psi = \mathrm{id}_{Y'} + N_G g'. \end{aligned}$$

Summing everywhere, we get the $G$-module homomorphisms $\varphi \oplus \psi\colon X \oplus Y \to X' \oplus Y'$ and $\varphi' \oplus \psi'\colon X' \oplus Y' \to X \oplus Y$ satisfying

$$\begin{aligned} (\varphi \oplus \psi) \circ (\varphi' \oplus \psi') &= (\varphi \circ \varphi') \oplus (\psi \circ \psi') \\ &= (\mathrm{id}_X + N_G f) \oplus (\mathrm{id}_Y + N_G g) \\ &= \mathrm{id}_X \oplus \mathrm{id}_Y + N_G(f \oplus g). \end{aligned}$$

The other check is analogous, switching primed and unprimed variables. ■

Anyway, here is the reason why we care about this notion of equivalence.

**Lemma 68.** Let $G$ be a finite group, and let $X$ and $X'$ be $G$-modules. Suppose that there is a natural transformation

$$\Phi_\bullet\colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

Then there exists a $G$-module homomorphism $\varphi\colon X' \to X$ such that $\Phi_A([f]) = [\varphi] \cup [f]$ for any $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(X, A)$.

*Proof.* This is essentially the Yoneda lemma. Choose $\varphi\colon X' \to X$ to represent $[\varphi] = \Phi_X([\mathrm{id}_X])$. Then tracking through the commutativity of

$$\begin{tikzcd} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \arrow[r, "\Phi_X"] \arrow[d, "f"'] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X)) \arrow[d, "f"'] \\ \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\Phi_A"] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \end{tikzcd}$$

reveals that

$$\begin{tikzcd} {[\mathrm{id}_X]} \arrow[r, mapsto, "\Phi_X"] \arrow[d, mapsto, "f"'] & {[\varphi]} \arrow[d, mapsto, "f"'] \\ {[f]} \arrow[r, mapsto, "\Phi_A"] & \Phi_A([f]) = [f \circ \varphi] \end{tikzcd}$$

after plugging in. To finish, we note that $[\varphi] \cup [f] = [\varphi \otimes f]$ goes to $[f \circ \varphi]$ under the canonical evaluation map $\mathrm{Hom}_{\mathbb{Z}}(X', X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A) \to \mathrm{Hom}_{\mathbb{Z}}(X', A)$. This finishes. ■

**Remark 69.** Of course, the converse also holds because the diagram

$$\begin{tikzcd}
A \arrow[d, "f"'] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\varphi\cup -"] \arrow[d, "f"'] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r] \arrow[d, "f"'] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \arrow[d, "f"'] \\
B & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, B)) \arrow[r, "\varphi\cup -"] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, B)) \arrow[r] & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', A))
\end{tikzcd}$$

commutes.

**Proposition 70.** Let $G$ be a finite group, and let $X$ and $X'$ be $G$-modules. Then $X$ and $X'$ are cohomologically equivalent if and only if there is a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

*Proof.* In the forward direction, suppose $X$ and $X'$ are cohomologically equivalent so that we have $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that

$$[\varphi] \cup [\varphi'] = [\varphi \circ \varphi'] = [\mathrm{id}_X] \qquad \text{and} \qquad [\varphi'] \cup [\varphi] = [\varphi' \circ \varphi] = [\mathrm{id}_{X'}],$$

where we are using the canonical evaluation maps for the cup products. Now, we note that, for any $G$-module $A$, we have inverse morphisms

$$\begin{array}{ccc}
\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) & \simeq & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\
{[f]} & \mapsto & {[\varphi] \cup [f]} \\
{[\varphi'] \cup [f']} & \leftarrow\!\shortmid & {[f']}.
\end{array} \tag{3.12}$$

Indeed, these are mutually inverse because

$$[\varphi'] \cup [\varphi] \cup [f] = [f \circ \varphi \circ \varphi'] = [f]$$

and similar for the other inverse check. (Technically, we are using the commutativity of our evaluation maps in the associativity of the cup product.) To finish, we note that the isomorphisms (3.12) assemble into a natural isomorphism by Remark 69.

We now show the backwards direction. Suppose we have a natural isomorphism $\Phi_\bullet$. Then Lemma 68 promises us $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ and $[\varphi'] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X'))$ such that the morphisms

$$\begin{array}{ccc}
\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) & \simeq & \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \\
{[f]} & \mapsto & {[\varphi] \cup [f]} \\
{[\varphi'] \cup [f']} & \leftarrow\!\shortmid & {[f']}
\end{array}$$

are mutually inverse. In particular, we see that

$$[\mathrm{id}_X] = [\varphi'] \cup [\varphi] \cup [\mathrm{id}_X] = [\mathrm{id}_X \circ \varphi \circ \varphi'] = [\varphi \circ \varphi'],$$

so $[\varphi \circ \varphi'] = [\mathrm{id}_X]$. Swapping primed and unprimed variables, we see $[\varphi' \circ \varphi] = [\mathrm{id}_{X'}]$ as well. ■

**Remark 71.** The above result makes it fairly clear that cohomological equivalence actually makes an equivalence relation. In particular, we can invert and compose natural isomorphisms, which gives symmetry and transitivity of cohomological equivalence respectively.

One might hope that we can get more information by using indices away from $0$, but in fact we cannot.

**Proposition 72.** Let $G$ be a finite group, and let $X$ and $X'$ be $G$-modules. Then the following are equivalent.

(a) $X$ and $X'$ are cohomologically equivalent.

(b) There is a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)).$$

(c) For some (fixed) index $p \in \mathbb{Z}$, there is $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ such that

$$[\varphi] \cup - \colon \widehat{H}^p(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^p(G, \mathrm{Hom}_{\mathbb{Z}}(X', -))$$

is a natural isomorphism.

(d) There are natural isomorphisms

$$\Phi_\bullet^{(i)} \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', -))$$

for all $i \in \mathbb{Z}$.

*Proof.* Note that (a) and (b) are equivalent by Proposition 70. Also, there is nothing to say for (d) implies (b), and (b) implies (c) follows from taking $p = 0$ in (b) and noting that Lemma 68 reassures us that this natural transformation (in fact, natural isomorphism) must be a cup-product map. Thus, the interesting direction is (c) implies (d). This follows from dimension-shifting. Indeed, we shift $p$ upwards and downwards from its fixed value to witness our natural isomorphisms.

The proof is similar to Proposition 58, so we will be brief. Indeed, we already know that we have $[\varphi] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', X))$ such that

$$[\varphi] \cup - \colon \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \to \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \tag{3.13}$$

is an isomorphism for all $G$-modules $A$ at $i = p$. To shift downwards, we suppose that we have an isomorphism always at $i$, and we show that we have an isomorphism always at $i - 1$. Well, for any $G$-module $A$, we note the ($\mathbb{Z}$-split) short exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathbb{Z}}(X', I_G \otimes_{\mathbb{Z}} A) \to \mathrm{Hom}_{\mathbb{Z}}(X', \mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \to \mathrm{Hom}_{\mathbb{Z}}(X', A) \to 0 \\ 0 &\to \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \to \mathrm{Hom}_{\mathbb{Z}}(X, A) \to 0 \end{aligned} \tag{3.14}$$

whose boundary morphisms induce the following commutative square.

$$\begin{tikzcd} \widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\varphi \cup -"] \arrow[d, "\delta"'] & \widehat{H}^{i-1}(G, \mathrm{Hom}_{\mathbb{Z}}(X', A)) \arrow[d, "\delta"'] \\ \widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X, I_G \otimes_{\mathbb{Z}} A)) \arrow[r, "\varphi \cup -"] & \widehat{H}^{i}(G, \mathrm{Hom}_{\mathbb{Z}}(X', I_G \otimes_{\mathbb{Z}} A)) \end{tikzcd}$$

(Technically, the commutativity here is using the commutativity of $\delta$s with the cup product in addition to the functoriality of boundary morphisms.) In particular, the inductive hypothesis tells us that the bottom row is an isomorphism, and the fact that both middle terms of (3.14) are induced by Lemma 57 implies that the $\delta$s on either side are also isomorphisms. So the top row is an isomorphism, finishing.

Similarly, to shift upwards, we suppose that (3.13) is always an isomorphism at $i$, and we show that we have an isomorphism always at $i + 1$. Well, for any $G$-module $A$, we note the ($\mathbb{Z}$-split) short exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathbb{Z}}(X', A) \to \mathrm{Hom}_{\mathbb{Z}}(X', \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \to \mathrm{Hom}_{\mathbb{Z}}(X', \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \to 0 \\ 0 &\to \mathrm{Hom}_{\mathbb{Z}}(X, A) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \to 0 \end{aligned} \tag{3.15}$$

whose boundary morphisms induce the following commutative square.

$$\begin{tikzcd}
\widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \arrow[r, "\varphi\cup-"] \arrow[d, "\delta"'] & \widehat{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(X', \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \arrow[d, "\delta"'] \\
\widehat{H}^{i+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\varphi\cup-"] & \widehat{H}^{i+1}(G, \mathrm{Hom}_{\mathbb{Z}}(X', A))
\end{tikzcd}$$

This time around, the top row is an isomorphism by the inductive hypothesis, and the left and row arrows are isomorphisms because the middle terms of (3.15) are induced by Lemma 57. So the bottom row is an isomorphism as well, finishing. ■

Anyway, here are some results.

**Corollary 73.** Let $G$ be a finite group, and let $p \in \mathbb{Z}$ be an index. Then the set of $G$-module $X$ with a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^p(G, -)$$

make up exactly one cohomological equivalence class.

Is there a way to encode change of group here?

*Proof.* Fix some $G$-module $X$ with such a natural isomorphism

$$\Psi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^p(G, -).$$

We would like to show that a $G$-module $X$ has a natural isomorphism $\Phi_\bullet$ between the same functors if and only if $X$ and $X'$ are cohomologically equivalent.

If $X$ and $X'$ are cohomologically equivalent, then we can compose the promised natural isomorphism of Proposition 70 with $\Psi_\bullet$, giving a natural isomorphism

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)) \Rightarrow \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \overset{\Psi_\bullet}{\Rightarrow} \widehat{H}^p(G, -).$$

In the other direction, if we have a natural isomorphism

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)) \Rightarrow \widehat{H}^p(G, -),$$

then we can compose with $\Psi_\bullet^{-1}$ to build a natural isomorphism

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X', -)) \overset{\Phi_\bullet}{\Rightarrow} \widehat{H}^p(G, -) \overset{\Psi_\bullet^{-1}}{\Rightarrow} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)),$$

from which it follows that $X$ and $X'$ are cohomologically equivalent by Proposition 70. ■

**Example 74.** Fix notation as in subsection 3.1. If $M$ is any induced module, then we know $M$ is cohomologically equivalent to $0$. So Lemma 67 reassures us that $X \oplus M$ is cohomologically equivalent to $X \oplus 0 \simeq X$.

**Example 75.** Fix notation as in subsection 3.1. Note that any $G$-module $A$ has

$$\widehat{H}^2(G, A) \simeq \widehat{H}^1(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A)) \simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(I_G, \mathrm{Hom}_{\mathbb{Z}}(I_G, A))) \simeq \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(I_G \otimes_{\mathbb{Z}} I_G, A)),$$

and in fact these isomorphisms are natural by the functoriality of boundary morphisms. It follows that $X$ and $I_G \otimes_{\mathbb{Z}} I_G$ are cohomologically equivalent.

The above two examples should give a feeling for why this uniqueness problem is difficult.

One might object that what makes $X$ of subsection 3.1 special is that the natural isomorphism

$$\widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^2(G, -)$$

is a cup-product map. However, we can show that all such natural transformations come from cup-product maps.

**Proposition 76.** Let $G$ be a finite group, and let $p \in \mathbb{Z}$ be an index. Suppose we have a $G$-module $X$ with a natural transformation

$$\Phi_\bullet \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^p(G, -).$$

Then there exists $[x] \in \widehat{H}^p(G, X)$ such that $\Phi_\bullet$ is the cup-product map $[x] \cup -$.

*Proof.* This is analogous to Lemma 68. Namely, this is again essentially the Yoneda lemma. As such, set $[x] := \Phi_X([\mathrm{id}_X])$. As before, the point is to fix some $G$-module $A$ and $[f] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ in order to track through the commutativity of the following diagram.

$$\begin{tikzcd} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) \arrow[r, "\Phi_X"] \arrow[d, "f"'] & \widehat{H}^p(G, X) \arrow[d, "f"'] \\ \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \arrow[r, "\Phi_A"] & \widehat{H}^p(G, A) \end{tikzcd} \tag{3.16}$$

Because we will need to deal with the cup products with negative indices, we will use the standard resolution of [CF10]. For example, we interpret $f \in [f] \in \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A))$ as a constant function $f \in \mathrm{Hom}_G(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(X, A))$, which means that $f(z)$ is the same $G$-module homomorphism for each $z \in \mathbb{Z}[G]$.

As such, we can track the left arrow of (3.16) as

$$\begin{aligned} f \colon \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, X)) &\to \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, A)) \\ [z \mapsto \mathrm{id}_X] &\to [z \mapsto f(z) \circ \mathrm{id}_X] = [f]. \end{aligned}$$

So, along the bottom of (3.16), we are evaluating $\Phi_A([f])$.

Along the top of (3.16), we immediately send $[x]$ to $\Phi_X([z \mapsto \mathrm{id}_X]) = [x]$, so to finish the proof, we need to show that

$$f([x]) \stackrel{?}{=} [x] \cup [f],$$

which will be enough by the commutativity of (3.16). We have two similar cases to appropriately deal with the cup product.

- Suppose that $p \geq 0$ so that we can interpret $x$ as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[G^{p+1}], X\right)$, using the standard resolution. As such, we compute

  $$(x \cup f)(g_0, \ldots, g_p) = x(g_0, \ldots, g_p) \otimes f(g_p),$$

  where our output is in $X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)$. Applying evaluation, the cup product is outputting

  $$(g_0, \ldots, g_p) \mapsto (f(g_p) \circ x)(g_0, \ldots, g_p)$$

  as our element of $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{p+1}], A)$. Indeed, this morphism represents $f([x])$.

- Analogously, suppose that $p < 0$ so that we interpret $x$ as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^p, \mathbb{Z}), X\right)$. To decrease headaches, we let $g^* \colon \mathbb{Z}[G] \to \mathbb{Z}$ denote the $G$-module homomorphism sending $g \mapsto 1$ and other group elements to $0$. Then $p$-tuples $(g_1^*, \ldots, g_p^*)$ form a $\mathbb{Z}$-basis of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^p, \mathbb{Z})$, so it's enough to specify

  $$(x \cup f)(g_1^*, \ldots, g_p^*) = x(g_1^*, \ldots, g_p^*) \otimes f(g_p),$$

  where the output is in $X \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(X, A)$. Applying evaluation, the cup product is outputting

  $$(g_1^*, \ldots, g_p^*) \mapsto (f(g_p) \circ x)(g_1^*, \ldots, g_p^*)$$

  as an element of $\mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^p, \mathbb{Z}), A\right)$. Indeed, this represents $f([x])$.

The above cases finish tracking through (3.16) and hence finish the proof. ∎

**Example 77.** For $p \geq 0$, standard dimension-shifting arguments give natural isomorphisms

$$\widehat{H}^0\left(G, \mathrm{Hom}_{\mathbb{Z}}(I_G^{\otimes p}, -)\right) \Rightarrow \widehat{H}^p(G, A),$$

so Proposition 76 implies that these isomorphisms are cup products with an element of $\widehat{H}^p(G, I_G^{\otimes p})$. For example, when $p = 0$, we have $[1] \in \widehat{H}^0(G, \mathbb{Z})$; and when $p = 1$, we have $g \mapsto (1 - g)$ in $\widehat{H}^1(G, I_G)$.

**Example 78.** Fix notation as in subsection 3.1, and take $m = 2$. Then there are natural transformations

$$\widehat{H}^2(G, -) \overset{[\delta(\overline{c})]^\vee \cup -}{\Rightarrow} \widehat{H}^0(G, \mathrm{Hom}_{\mathbb{Z}}(X, -)) \Rightarrow \widehat{H}^{-1}(G, -)$$

sending a $2$-cocycle to its $\{\sigma_i\}_{i=1}^m$-tuple and then to the (class of) $\beta_{10}$. (It turns out that, because $G$ is bicyclic, the equivalence relation on $\beta_{10}$ is exactly what we need to form a class of $\widehat{H}^{-1}$.) Now, applying Proposition 76, we see that the right natural transformation must be a cup-product map, so by associativity of the cup product, the entire natural transformation is a cup-product map.

Thus, analogously to what Corollary 4 says for $\alpha$s, we can describe the projection from $2$-cocycles to $\beta$s purely via (restricted) cup products.

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# A Verification of the Cocycle

In this section, we verify Theorem 16. As such, in this section, we will work under the modified set-up, forgetting about the extension $\mathcal{E}$ but letting $(\{\alpha_i\}, \{\beta_{ij}\})$ be some $\{\sigma_i\}_{i=1}^m$-tuple.

Here the formula looks like

$$c(g,g') := \left[\prod_{1\le j<i\le m}\left(\prod_{1\le k<j}\sigma_k^{a_k+b_k}\right)\left(\prod_{j\le k<i}\sigma_k^{a_k}\right)\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m\left(\prod_{1\le k<i}\sigma_k^{a_k+b_k}\right)\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right],$$

where $g=\prod_i\sigma_i^{a_i}$ and $g'=\prod_i\sigma_i^{b_i}$ with $0\le a_i,b_i<n_i$ and $q_i := \lfloor(a_i+b_i)/n_i\rfloor$. To make this more digestible, we define

$$g_i := \prod_{1\le k<i}\sigma_k^{a_k}$$

for any $g=\prod_i\sigma_i^{a_i}\in G$, so we can write down our formula as

$$c(g,g') := \left[\prod_{1\le j<i\le m} g_ig_j'\beta_{ij}^{(a_ib_j)}\right]\left[\prod_{i=1}^m g_ig_i'\alpha_i^{\left\lfloor\frac{a_i+b_i}{n_i}\right\rfloor}\right].$$

Now, given $g,g',g''\in G$, we would like to check

$$gc(g',g'')\cdot c(g,g'g'') \stackrel{?}{=} c(gg',g'')\cdot c(g,g'),$$

where $g=\prod_i\sigma_i^{a_i}$ and $g'=\prod_i\sigma_i^{b_i}$ and $g''=\prod_i\sigma_i^{c_i}$ with $0\le a_i,b_i,c_i<n_i$.

## A.1 Carries

We will begin our verification by dealing with carries; we start with the following lemma, intended to beef up our relation (2.2).

**Lemma 79.** Given indices $i>j$ with $a_i,a_j,q_i,q_j\ge 0$, we have

$$\beta_{ij}^{(a_ia_j)} = \beta_{ij}^{(a_i+q_in_i,a_j)}\left(\frac{\sigma_j^{a_j}(\alpha_i)}{\alpha_i}\right)^{q_i} \qquad \text{and} \qquad \beta_{ij}^{(a_ia_j)} = \beta_{ij}^{(a_i,a_j+q_jn_j)}\left(\frac{\alpha_j}{\sigma_i^{a_i}(\alpha_j)}\right)^{q_j}.$$

*Proof.* This is a matter of force. For one, we compute

$$\begin{aligned}\beta_{ij}^{(a_i+n_iq_i,a_j)} &= \prod_{p=0}^{a_i+n_iq_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\\ &= \left(\prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\left(\prod_{q=0}^{a_j-1}\prod_{p=a_i}^{a_i+n_iq_i-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\\ &= \beta_{ij}^{(a_ia_j)}\left(\prod_{q=0}^{a_j-1}\sigma_j^q\,\mathrm{N}_{L/L_i}(\beta_{ij})\right)^{q_i}.\end{aligned}$$

Now, using the relation $\mathrm{N}_{L/L_i}(\beta_{ij})=\alpha_i/\sigma_j(\alpha_i)$ from (2.2), this becomes

$$\begin{aligned}\beta_{ij}^{(a_i+n_iq_i,a_j)} &= \beta_{ij}^{(a_ia_j)}\left(\prod_{q=0}^{a_j-1}\frac{\sigma_j^q\alpha_i}{\sigma^{j+1}\alpha_i}\right)^{q_i}\\ &= \beta_{ij}^{(a_ia_j)}\left(\frac{\alpha_i}{\sigma^{a_j}\alpha_i}\right)^{q_i},\end{aligned}$$

which rearranges into what we wanted.

For the other, we again just compute

$$\begin{aligned}\beta_{ij}^{(a_i,a_j+n_jq_j)} &= \prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j+n_jq_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\\ &= \left(\prod_{p=0}^{a_i-1}\prod_{q=0}^{a_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\left(\prod_{p=0}^{a_i-1}\prod_{q=q_j}^{a_j+n_jq_j-1}\sigma_i^p\sigma_j^q\beta_{ij}\right)\\ &= \beta_{ij}^{(a_ia_j)}\left(\prod_{p=0}^{a_i-1}\sigma_i^p\,\mathrm{N}_{L/L_q}(\beta_{ij})\right)^{q_i}.\end{aligned}$$

This time, we use the relation $\mathrm{N}_{L/L_j}(\beta_{ij}) = \sigma_i(\alpha_j)/\alpha_j$, which gives

$$\begin{aligned}\beta_{ij}^{(a_i,a_j+n_jq_j)} &= \beta_{ij}^{(a_ia_j)}\left(\prod_{p=0}^{a_i-1}\frac{\sigma_i^{p+1}(\alpha_j)}{\sigma_i^p(\alpha_j)}\right)^{q_i}\\ &= \beta_{ij}^{(a_ia_j)}\left(\frac{\sigma_i^{a_j}(\alpha_j)}{\alpha_j}\right)^{q_i},\end{aligned}$$

which again rearranges into the desired. ■

We are now ready to begin the computation, dealing with carries to start. Use the division algorithm to write

$$a_i + b_i = n_iu_i + x_i \qquad \text{and} \qquad b_i + c_i = n_iv_i + y_i,$$

where $u_i, v_i \in \{0,1\}$ and $0 \le x_i, y_i < n_i$ for each $i$. We start by collecting remainder terms on the side of $gc(g', g'') \cdot c(g, g'g'')$.

1. Note

$$gc(g', g'') = g\left[\prod_{1\le j<i\le m} g_i'g_j''\beta_{ij}^{(b_ic_j)}\right]\cdot g\left[\prod_{i=1}^m g_i'g_i''\alpha_i^{v_i}\right],$$

so we set

$$R_1 \coloneqq \prod_{i=1}^m gg_i'g_i''\alpha_i^{v_i}$$

to be our remainder term.

2. Note

$$\begin{aligned}c(g, g'g'') &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_iy_j)}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]\\ &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}\cdot g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]\\ &= \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\beta_{ij}^{(a_i,b_j+c_j)}\right]\left[\prod_{1\le j<i\le m} g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right],\end{aligned}$$

so we set

$$R_2 \coloneqq \left[\prod_{1\le j<i\le m} g_ig_j'g_j''\left(\frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^m g_ig_i'g_i''\alpha_i^{\left\lfloor\frac{a_i+y_i}{n_i}\right\rfloor}\right]$$

to be our remainder term.

3. Lastly, we collect our remainders. Observe

$$\begin{aligned}
R_2 &= \left[\prod_{j=1}^{m} g_j' g_j'' \left(\prod_{i=j+1}^{m} g_i \cdot \frac{\alpha_j}{\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g_j' g_j'' \left(\prod_{i=j+1}^{m} \frac{(\sigma_1^{a_1}\cdots\sigma_{i-1}^{a_{i-1}})\alpha_j}{(\sigma_1^{a_1}\cdots\sigma_{i-1}^{a_{i-1}})\sigma_i^{a_i}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g_j' g_j'' \left(\prod_{i=j+1}^{m} \frac{g_i\alpha_j}{g_{i+1}\alpha_j}\right)^{v_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{j=1}^{m} g_j' g_j'' \cdot \frac{g_{j+1}\alpha_j^{v_j}}{g\alpha_j^{v_j}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right].
\end{aligned}$$

We now note that $g_{j+1}\alpha_j = g_j\alpha_j$ because $\alpha_j$ is fixed by $\sigma_j$. As such,

$$\begin{aligned}
R_1R_2 &= \left[\prod_{i=1}^{m} g g_i' g_i'' \alpha_i^{v_i}\right]\left[\prod_{i=1}^{m} g_i' g_i'' \cdot \frac{g_i\alpha_i^{v_i}}{g\alpha_i^{v_i}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor}\right] \\
&= \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{v_i+\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor},
\end{aligned}$$

which is nice enough for us now.

Now, we collect remainder terms from $c(gg', g'') \cdot c(g, g')$.

1. Note

$$\begin{aligned}
c(gg', g'') &= \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(x_i c_j)}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)} \cdot g_i g_i' g_j'' \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right],
\end{aligned}$$

so we set

$$R_3 \coloneqq \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \left(\frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right].$$

2. Note

$$c(g, g') = \left[\prod_{1\le j<i\le m} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]\left[\prod_{i=1}^{m} g_i g_i' \alpha_i^{u_i}\right],$$

so we set

$$R_4 \coloneqq \left[\prod_{i=1}^{m} g_i g_i' \alpha_i^{u_i}\right].$$

3. Lastly, we collect our remainder terms. Observe

$$\begin{aligned}
R_3 &= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} g_j'' \cdot \frac{\sigma_j^{c_j}\alpha_i}{\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} \frac{(\sigma_1^{c_1}\cdots\sigma_{j-1}^{c_{j-1}})\sigma_j^{c_j}\alpha_i}{(\sigma_1^{c_1}\cdots\sigma_{j-1}^{c_{j-1}})\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \left(\prod_{j=1}^{i-1} \frac{g_{j+1}''\alpha_i}{g_j''\alpha_i}\right)^{u_i}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right] \\
&= \left[\prod_{i=1}^{m} g_i g_i' \cdot \frac{g_i''\alpha_i^{u_i}}{\alpha_i^{u_i}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
R_3R_4 &= \left[\prod_{i=1}^{m} g_i g_i' \cdot \frac{g_i''\alpha_i^{u_i}}{\alpha_i^{u_i}}\right]\left[\prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor}\right]\left[\prod_{i=1}^{m} g_i g_i' \alpha_i^{u_i}\right] \\
&= \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{u_i+\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor},
\end{aligned}$$

which is again simple enough for our purposes.

We now note that, for each $i$,

$$u_i + \left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor = \left\lfloor \frac{a_i+b_i+c_i}{n_i}\right\rfloor = v_i + \left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor$$

by how carried addition behaves. It follows that

$$R_1R_2 = \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{v_i+\left\lfloor \frac{a_i+y_i}{n_i}\right\rfloor} = \prod_{i=1}^{m} g_i g_i' g_i'' \alpha_i^{u_i+\left\lfloor \frac{x_i+c_i}{n_i}\right\rfloor} = R_3R_4.$$

Thus, it suffices to show that

$$\frac{gc(g',g'')}{R_1}\cdot\frac{c(g,g'')}{R_2} \stackrel{?}{=} \frac{c(gg',g'')}{R_3}\cdot\frac{c(g,g')}{R_4},$$

which is equivalent to

$$g\left[\prod_{1\le j<i\le m} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

by the work above.

## A.2 Finishing

We need to verify that

$$g\left[\prod_{1\le j<i\le m} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{1\le j<i\le m} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\cdot\left[\prod_{1\le j<i\le m} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

as discussed in the previous subsection.

Before beginning the check, we recall the relations on the $\beta$s from (2.3) can be written as

$$\frac{\sigma_2(\beta_{31})}{\beta_{31}} = \frac{\sigma_1(\beta_{32})}{\beta_{32}} \cdot \frac{\sigma_3(\beta_{21})}{\beta_{21}},$$

because we only have one triple $(i, j, k)$ of indices with $i > j > k$. This is somewhat difficult to deal with directly, so we quickly show a more general version.

**Lemma 80.** Fix indices with $i > j > k$, and let $a_i, a_j, a_k \geq 0$. Then

$$\frac{\sigma_j^{a_j}\beta_{ik}^{(a_i a_k)}}{\beta_{ik}^{(a_i a_k)}} = \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} \cdot \frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}}.$$

*Proof.* We simply compute

$$\begin{aligned}
\frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}} \cdot \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} &= \prod_{r=0}^{a_i-1} \frac{\sigma_i^{r+1}\beta_{jk}^{(a_j a_k)}}{\sigma_i^{r}\beta_{jk}^{(a_j a_k)}} \cdot \prod_{p=0}^{a_k-1} \frac{\sigma_k^{p+1}\beta_{ij}^{(a_i a_j)}}{\sigma_k^{p}\beta_{ij}^{(a_i a_j)}} \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \left( \frac{\sigma_k^p\sigma_j^q\sigma_i^{r+1}\beta_{jk}}{\sigma_k^p\sigma_j^q\sigma_i^r\beta_{jk}} \cdot \frac{\sigma_k^{p+1}\sigma_j^q\sigma_i^r\beta_{ij}}{\sigma_k^p\sigma_j^q\sigma_i^r\beta_{ij}} \right) \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \sigma_k^p\sigma_j^q\sigma_i^r \left( \frac{\sigma_i\beta_{jk}}{\beta_{jk}} \cdot \frac{\sigma_k\beta_{ij}}{\beta_{ij}} \right) \\
&= \prod_{p=0}^{a_k-1}\prod_{q=0}^{a_j-1}\prod_{r=0}^{a_i-1} \sigma_k^p\sigma_j^q\sigma_i^r \left( \frac{\sigma_j\beta_{ik}}{\beta_{ik}} \right),
\end{aligned}$$

where in the last equality we have use the relation on the $\beta$s. Continuing,

$$\begin{aligned}
\frac{\sigma_i^{a_i}\beta_{jk}^{(a_j a_k)}}{\beta_{jk}^{(a_j a_k)}} \cdot \frac{\sigma_k^{a_k}\beta_{ij}^{(a_i a_j)}}{\beta_{ij}^{(a_i a_j)}} &= \prod_{q=0}^{a_j-1} \left( \prod_{p=0}^{a_k-1}\prod_{r=0}^{a_i-1} \frac{\sigma_j^{q+1}\sigma_k^p\sigma_i^r\beta_{ik}}{\sigma_j^q\sigma_k^p\sigma_i^r\beta_{ik}} \right) \\
&= \prod_{q=0}^{a_j-1} \frac{\sigma_j^{q+1}\beta_{ik}^{(a_i a_k)}}{\sigma_j^q\beta_{ik}^{(a_i a_k)}} \\
&= \frac{\sigma_j^{a_j}\beta_{ik}^{(a_i a_k)}}{\beta_{ik}^{(a_i a_k)}},
\end{aligned}$$

which is what we wanted. ■

We now proceed with the check, by induction. More precisely, we claim that any $m' \leq m$ gives

$$g_{m'+1}\left[\prod_{j<i\leq m'} g_i' g_j'' \beta_{ij}^{(b_i c_j)}\right]\left[\prod_{j<i\leq m'} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)}\right] \stackrel{?}{=} \left[\prod_{j<i\leq m'} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)}\right]\left[\prod_{j<i\leq m'} g_i g_j' \beta_{ij}^{(a_i b_j)}\right]$$

which we will show by induction on $m'$. For $m' = 1$, there is nothing to say because there are no indices $i > j$.

So now suppose we have equality for $m' < m$, and we give equality for $m'' := m' + 1$. That is, we want to show that

$$g_{m'+2}\prod_{j<i\leq m'+1} g_i' g_j'' \beta_{ij}^{(b_i c_j)} \cdot \prod_{j<i\leq m'+1} g_i g_j' g_j'' \beta_{ij}^{(a_i, b_j+c_j)} \stackrel{?}{=} \prod_{j<i\leq m'+1} g_i g_i' g_j'' \beta_{ij}^{(a_i+b_i, c_j)} \cdot \prod_{j<i\leq m'+1} g_i g_j' \beta_{ij}^{(a_i b_j)}$$

but by the inductive hypothesis it suffices for

$$\frac{g_{m''+1}\prod_{j<i\le m'+1} g'_i g''_j \beta_{ij}^{(b_i c_j)}}{g_{m'+1}\prod_{j<i\le m'} g'_i g''_j \beta_{ij}^{(b_i c_j)}} \cdot \frac{\prod_{j<i\le m'+1} g_i g'_j g''_j \beta_{ij}^{(a_i, b_j+c_j)}}{\prod_{j<i\le m'} g_i g'_j g''_j \beta_{ij}^{(a_i, b_j+c_j)}} \stackrel{?}{=} \frac{\prod_{j<i\le m'+1} g_i g'_i g''_j \beta_{ij}^{(a_i+b_i, c_j)}}{\prod_{j<i\le m'} g_i g'_i g''_j \beta_{ij}^{(a_i+b_i, c_j)}} \cdot \frac{\prod_{j<i\le m'+1} g_i g'_j \beta_{ij}^{(a_i b_j)}}{\prod_{j<i\le m'} g_i g'_j \beta_{ij}^{(a_i b_j)}}$$

which is collapses to

$$\frac{g_{m''+1}\prod_{j<i\le m'+1} g'_i g''_j \beta_{ij}^{(b_i c_j)}}{g_{m'+1}\prod_{j<i\le m'} g'_i g''_j \beta_{ij}^{(b_i c_j)}} \cdot \prod_{j\le m'} g_{m''} g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j+c_j)} \stackrel{?}{=} \prod_{j\le m'} g_{m''} g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''}, c_j)} \cdot \prod_{j\le m'} g_{m''} g'_j \beta_{ij}^{(a_{m''} b_j)}$$

because the terms with $i < m'' = m' + 1$ got cancelled in the rightmost three products. Rearranging, this is the same as

$$\frac{g_{m''+1}\prod_{j<i\le m'+1} g'_i g''_j \beta_{ij}^{(b_i c_j)}}{g_{m'+1}\prod_{j<i\le m'} g'_i g''_j \beta_{ij}^{(b_i c_j)}} \stackrel{?}{=} \frac{\prod_{j<m''} g_{m''} g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''}, c_j)} \cdot \prod_{j<m''} g_{m''} g'_j \beta_{m''j}^{(a_{m''} b_j)}}{\prod_{j<m''} g_{m''} g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j+c_j)}}.$$

Peeling off the $i = m'' = m' + 1$ terms from the left-hand side numerator, we're showing

$$\frac{g_{m''+1}\prod_{j<i\le m'} g'_i g''_j \beta_{ij}^{(b_i c_j)}}{g_{m'+1}\prod_{j<i\le m'} g'_i g''_j \beta_{ij}^{(b_i c_j)}} \stackrel{?}{=} \frac{\prod_{j<m''} g_{m''} g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''}, c_j)} \cdot \prod_{j<m''} g_{m''} g'_j \beta_{m''j}^{(a_{m''} b_j)}}{\prod_{j<m''} g_{m''+1} g'_{m''} g''_j \beta_{m''j}^{(b_{m''} c_j)} \cdot \prod_{j<m''} g_{m''} g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j+c_j)}}.$$

We take a moment to simplify the left-hand side with Lemma 80 by writing

$$\begin{aligned}
g_{m'+1}\prod_{j<i\le m'} g'_i g''_j \left(\frac{\sigma_{m''}^{a_{m''}} \beta_{ij}^{(b_i c_j)}}{\beta_{ij}^{(b_i c_j)}}\right) &= g_{m''} \prod_{j<i\le m'} g'_i g''_j \left(\frac{\sigma_i^{b_i} \beta_{m''j}^{(a_{m''} c_j)}}{\beta_{m''j}^{(a_{m''} c_j)}} \cdot \frac{\beta_{m''i}^{(a_{m''} b_i)}}{\sigma_j^{c_j} \beta_{m''i}^{(a_{m''} b_i)}}\right) \\
&= g_{m''} \left[\prod_{j=1}^{m'} g''_j \prod_{i=j+1}^{m'} g'_i \left(\frac{\sigma_i^{b_i} \beta_{m''j}^{(a_{m''} c_j)}}{\beta_{m''j}^{(a_{m''} c_j)}}\right) \cdot \prod_{i=1}^{m'} g'_i \prod_{j=1}^{i-1} g''_j \left(\frac{\beta_{m''i}^{(a_{m''} b_i)}}{\sigma_j^{c_j} \beta_{m''i}^{(a_{m''} b_i)}}\right)\right] \\
&= g_{m''} \left[\prod_{j=1}^{m'} \frac{g'_{m'+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}}{g'_{j+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}} \cdot \prod_{i=1}^{m'} \frac{g'_i \beta_{m''i}^{(a_{m''} b_i)}}{g'_i g''_i \beta_{m''i}^{(a_{m''} b_i)}}\right] \\
&= g_{m''} \left[\prod_{j<m''} \frac{g'_{m''} g''_j \beta_{m''j}^{(a_{m''} c_j)}}{g'_{j+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}} \cdot \prod_{j<m''} \frac{g'_j \beta_{m''j}^{(a_{m''} b_j)}}{g'_j g''_j \beta_{m''j}^{(a_{m''} b_j)}}\right]
\end{aligned}$$

after doing a lot of telescoping. Now, we can remove $g_{m''}$ everywhere to give

$$\prod_{j<m''} \frac{g'_{m''} g''_j \beta_{m''j}^{(a_{m''} c_j)}}{g'_{j+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}} \cdot \prod_{j<m''} \frac{g'_j \beta_{m''j}^{(a_{m''} b_j)}}{g'_j g''_j \beta_{m''j}^{(a_{m''} b_j)}} \stackrel{?}{=} \frac{\prod_{j<m''} g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''}, c_j)} \cdot \prod_{j<m''} g'_j \beta_{m''j}^{(a_{m''} b_j)}}{\prod_{j<m''} g'_{m''+1} g''_j \beta_{m''j}^{(b_{m''} c_j)} \cdot \prod_{j<m''} g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j+c_j)}},$$

or

$$\prod_{j<m''} \frac{g'_{m''} g''_j \beta_{m''j}^{(a_{m''} c_j)}}{g'_{j+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}} \stackrel{?}{=} \frac{\prod_{j<m''} g'_{m''} g''_j \beta_{m''j}^{(a_{m''}+b_{m''}, c_j)} \cdot \prod_{j<m''} g'_j g''_j \beta_{m''j}^{(a_{m''} b_j)}}{\prod_{j<m''} g'_{m''+1} g''_j \beta_{m''j}^{(b_{m''} c_j)} \cdot \prod_{j<m''} g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j+c_j)}}.$$

Rearranging, we want

$$\prod_{j<m''} \frac{g'_j g''_j \beta_{m''j}^{(a_{m''}, b_j + c_j)}}{g'_j g''_j \beta_{m''j}^{(a_{m''} b_j)} \cdot g'_{j+1} g''_j \beta_{m''j}^{(a_{m''} c_j)}} \stackrel{?}{=} \prod_{j<m''} \frac{g'_{m''} g''_j \beta_{m''j}^{(a_{m''} + b_{m''}, c_j)}}{g'_{m''} g''_j \beta_{m''j}^{(a_{m''} c_j)} \cdot g'_{m''+1} g''_j \beta_{m''j}^{(b_{m''} c_j)}},$$

which is

$$\prod_{j<m''} g'_j g''_j \left( \frac{\beta_{m''j}^{(a_{m''}, b_j + c_j)}}{\beta_{m''j}^{(a_{m''} b_j)} \cdot \sigma_j^{b_j} \beta_{m''j}^{(a_{m''} c_j)}} \right) \stackrel{?}{=} \prod_{j<m''} g'_{m''} g''_j \left( \frac{\beta_{m''j}^{(a_{m''} + b_{m''}, c_j)}}{\beta_{m''j}^{(a_{m''} c_j)} \cdot \sigma_{m''}^{a_{m''}} \beta_{m''j}^{(b_{m''} c_j)}} \right).$$

However, by definition of the $\beta_{ij}^{(xy)}$, we see that

$$\frac{\beta_{m''j}^{(a_{m''}, b_j + c_j)}}{\beta_{m''j}^{(a_{m''} b_j)} \cdot \sigma_j^{b_j} \beta_{m''j}^{(a_{m''} c_j)}} = \frac{\beta_{m''j}^{(a_{m''} + b_{m''}, c_j)}}{\beta_{m''j}^{(a_{m''} c_j)} \cdot \sigma_{m''}^{a_{m''}} \beta_{m''j}^{(b_{m''} c_j)}} = 1,$$

so everything does indeed cancel out properly. This completes the check.

# B Computation of $\ker \mathcal{F}$

In this section we give a proof of Lemma 48. As such, we will use all the context from the statement and proceed directly with the proof; as mentioned earlier, we may add (b) back to our list of generators because it is induced by (c). Pick up some $z := ((x_i)_i, (y_{ij})_{i>j}) \in \ker \mathcal{F}$, which is equivalent to saying

$$x_i N_i - \sum_{j=1}^{i-1} y_{ij} T_j + \sum_{j=i+1}^{m} y_{ji} T_j = 0$$

for each index $i$. We want to write $z$ as a $\mathbb{Z}[G]$-linear combination of the elements from (a)–(e). The main idea will be to slowly subtract out $\mathbb{Z}[G]$-linear combinations of the above elements (which does not affect $z \in \ker \mathcal{F}$) until we can prove that we have $0$ left over. We start with the $x_i$ terms, which we do in two steps.

1. We begin by dealing with the $x_i$ terms. Fix some index $p$, and we will subtract out a suitable $\mathbb{Z}[G]$-linear combination of the above generators to set $x_p = 0$ while not changing the other $x_i$ terms. Well, using the element

$$\kappa_p T_p, \tag{a}$$

we may assume that $x_p$ has no $\sigma_p$ terms because $\sigma_p \equiv 1 \pmod{T_p}$. Then for each $q < p$, we can subtract out a suitable multiple of

$$T_q \kappa_p + N_p \lambda_{pq} \tag{c}$$

to make it so that we may assume $x_p$ has no $\sigma_q$ terms because $\sigma_q \equiv 1 \pmod{T_q}$. Similarly, for each $q > p$, we can subtract out a suitable multiple of

$$T_q \kappa_p - N_p \lambda_{pq} \tag{d}$$

to make it so that we may assume $x_p$ has no $\sigma_q$ terms because $\sigma_q \equiv 1 \pmod{T_q}$.

2. Thus, the above process allows us to assume that $x_p \in \mathbb{Z}$, and the above linear combinations have not affected any $x_i$ for $i \neq p$. We now use the fact that $z \in \ker \mathcal{F}$. Indeed, we know that

$$x_p N_p - \sum_{j=1}^{p-1} y_{pj} T_j + \sum_{j=p+1}^{m} y_{jp} T_j = 0.$$

Applying the augmentation map $\varepsilon \colon \mathbb{Z}[G] \to \mathbb{Z}$, sending $\varepsilon \colon \sigma_i \mapsto 1$ for each index $i$, we see that $x_p \in \mathbb{Z}$ implying that $x_p$ remains fixed. On the other hand $\varepsilon \colon T_j \mapsto 0$ for each index $j$ and $\varepsilon \colon N_p \mapsto n_p$, so we are left with

$$n_p x_p = 0.$$

Because $n_p \neq 0$ (it's the order of $\sigma_p$), we conclude that $x_p = 0$. Applying this argument to the other $x_i$ terms, we conclude that we may assume $x_i = 0$ for each $i$.

It remains to deal with the $y_{ij}$ terms, which is a little more involved. For reference, we are showing that

$$-\sum_{j=1}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0$$

for each index $i$ implies that $z = ((0)_i, (y_{ij})_{i>j})$ is a $\mathbb{Z}[G]$-linear combination of the terms from (b) and (e).

We will now more or less proceed with the $y_{ij}$ by induction on $m$, allowing the group $G$ (in its number of generators $m$) to be changed in the process. For $m = 1$, there is nothing to say because there is no $y_{ij}$ term at all. For a taste of how we will use Lemma 41, we also work out $m = 2$: our equations read

$$\underbrace{-y_{21}T_1 = 0}_{i=1} \qquad \text{and} \qquad \underbrace{y_{21}T_2 = 0}_{i=2}.$$

Thus, $y_{21} \in (\ker T_1) \cap (\ker T_2) = (\operatorname{im} N_1) \cap (\operatorname{im} N_2)$, which is $\operatorname{im} N_1N_2$ by Lemma 41.

We now proceed with the general case; take $m > 2$. Let $G' := \langle \sigma_2, \ldots, \sigma_m \rangle$, which has $m-1$ generators. By the inductive hypothesis, we may assume the statement for $G'$. Explicitly, we will assume that, if $(y'_{ij})_{i>j\geq 2} \in \mathbb{Z}[G']^{\binom{m-1}{2}}$ are variables satisfying

$$-\sum_{j=2}^{i-1} y'_{ij}T_j + \sum_{j=i+1}^{m} y'_{ji}T_j = 0$$

for each index $i \geq 2$, then $y'_{ij}$ are a linear combination of terms from the elements from (b) and (e) above, only using indices at least $2$.

We will again proceed in steps, for clarity.

1. To apply the inductive hypothesis, we need to force $y_{pq} \in \mathbb{Z}[G']$ for each pair of indices $(p, q)$ with $p > q \geq 2$. Well, we use the relation (e) so that we can subtract multiples of

   $$T_q\lambda_{p1} - T_1\lambda_{pq} - T_p\lambda_{q1}.$$

   In particular, this element will subtract out $T_1$ from $y_{pq}$ while only introducing chaos to the elements $y_{p1}$ and $y_{q1}$ in the process. Thus, subtracting a suitable multiple allows us to assume that $y_{pq}$ has no $\sigma_1$ terms while not affecting any other $y_{ij}$ with $i > j \geq 2$.

   Applying this process to all $y_{ij}$ with $i > j \geq 2$, we do indeed get $y_{ij} \in \mathbb{Z}[G']$ for each $i > j \geq 2$.

2. We are now ready to apply the inductive hypothesis. For each index $i \geq 2$, we have the equation

   $$-y_{i1}T_1 - \sum_{j=2}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0.$$

   Because each $y_{pq}$ term with $p > q \geq 2$ features no $\sigma_1$, applying the transformation $\sigma_1 \mapsto 1$ will affect no term in the sums while causing $y_{i1}T_1$ to vanish. Thus, we have the equations

   $$-\sum_{j=2}^{i-1} y_{ij}T_j + \sum_{j=i+1}^{m} y_{ji}T_j = 0$$

   for each index $i \geq 2$. Because $y_{ij} \in \mathbb{Z}[G']$ for $i > j \geq 2$ already, we see that we may apply the inductive hypothesis to assert that the $y_{ij}$ are $\mathbb{Z}[G']$-linear combinations of terms from (b) and (e) (only using indices at least $2$).

   Subtracting these linear combinations out, we may assume $y_{ij} = 0$ for each $i > j \geq 2$.

3. To take stock, our equations for $i \geq 2$ now read

   $$-y_{i1}T_1 = 0,$$

which simply tells us that $y_{i1} \in \operatorname{im} N_1$ for each $i \geq 2$. As such, we pick up $w_i \in \mathbb{Z}[G]$ so that $y_{i1} = w_i N_1$ for each $i \geq 2$; because $\sigma_1 N_1 = N_1$, we may assume that $w_i \in \mathbb{Z}[G']$ for each $i \geq 2$.

Now the equation for $i = 1$ reads

$$\sum_{j=2}^{m} y_{j1} T_j = 0,$$

or

$$\sum_{i=2}^{m} w_i N_1 T_i = 0.$$

Sending $\sigma_1 \mapsto 1$, we see that $w_i$ and $T_i$ are both fixed because they feature no $\sigma_1$s, so we merely have

$$n_1 \sum_{i=2}^{m} w_i T_i = 0.$$

Dividing out by $n_1$, we are left with

$$\sum_{i=2}^{m} w_i T_i = 0.$$

4. At this point, we may appear stuck, but we have one final trick: taking indices $p > q \geq 2$, subtracting out multiples of

$$\left(T_q \lambda_{p1} - T_1 \lambda_{pq} - T_p \lambda_{q1}\right) \cdot N_1$$

will not affect the $y_{pq}$ term because $T_1 N_1$. Indeed, subtracting this term out looks like

$$T_q N_1 \lambda_{p1} - T_p N_1 \lambda_{q1},$$

which after factoring out $N_1$ takes $w_p \mapsto w_p - T_q$ and $w_q \mapsto w_q + T_p$.

In particular, fixing any $q \geq 2$ and then applying this trick for all $p > q$, we may assume that $w_q$ does not feature any $\sigma_p$ terms for $p > q$. Thus, looking at our equation

$$\sum_{i=2}^{m} w_i T_i = 0,$$

we are now able to show that $w_i \in \ker T_i = \operatorname{im} N_i$ for each $i \geq 2$, which will finish because it shows $y_{i1} \in N_i N_1$. Indeed, starting with $i = 2$, we see that $w_2$ features no $\sigma_p$ for $p > 2$, so we may take $\sigma_p \mapsto 1$ for each $p > 2$ safely, giving the equation

$$w_2 T_2 = 0,$$

finishing for $w_2$. Thus, we are left with the equation

$$\sum_{i=3}^{m} w_i T_i = 0,$$

from which we see we can induct downwards (this has fewer variables) to finish.

The above steps complete the proof, as advertised.