

Torus Worksheet

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1 The Bicyclic Case

Set $G \simeq G_1 \times G_2$ where $G_1 = \langle \sigma_1 \rangle \subseteq G$ and $G_2 = \langle \sigma_2 \rangle \subseteq G$ with $n_1 := \#G_1$ and $n_2 := \#G_2$. We define the elements

$$N_i := \sum_{k=0}^{n_i-1} \sigma_i^k \quad \text{and} \quad T_i := (\sigma_i - 1)$$

for $i \in \{1, 2\}$. Additionally, we define

$$N := N_1 N_2 = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} \sigma_1^k \sigma_2^\ell.$$

We will also use the elements N_i, T_i, N to mean the induced multiplication maps $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$.

Remark 1. We will freely use the facts that $\ker T_i = \text{im } N_i$ and $\text{im } T_i = \ker N_i$. These are not too hard to prove and follow similarly as in the cyclic case.

We take a moment to define the maps

$$\begin{aligned} \mathcal{F}_1: \frac{\mathbb{Z}[G]}{\text{im } T_1} \times \frac{\mathbb{Z}[G]}{\text{im } N} \times \frac{\mathbb{Z}[G]}{\text{im } T_2} &\rightarrow \mathbb{Z}[G] \times \mathbb{Z}[G] \\ (x, y, z) &\mapsto (N_2 z + T_1 y, N_1 x - T_2 y) \\ \mathcal{F}_2: \mathbb{Z}[G] \times \mathbb{Z}[G] &\rightarrow \frac{\mathbb{Z}[G]}{\text{im } T_1} \times \frac{\mathbb{Z}[G]}{\text{im } N} \times \frac{\mathbb{Z}[G]}{\text{im } T_2} \\ (a, b) &\mapsto (T_2 a, N_1 a - N_2 b, T_1 b). \end{aligned}$$

Observe that \mathcal{F}_1 is in fact well-defined: if $x - x' = T_1 a \in \text{im } T_1$ and $y - y' = Nb \in \text{im } N$ and $z - z' = T_2 c \in \text{im } T_2$, then

$$\begin{aligned} N_2 z &= N_2 z' + N_2 T_2 c = N_2 z' \\ N_1 x &= N_1 x' + N_1 T_1 a = N_1 x' \\ T_1 y &= T_1 y' + T_1 N b = T_1 y' \\ T_2 y &= T_2 y' + T_2 N b = T_2 y', \end{aligned}$$

so everything is independent of choice of representative.

1.1 The Complex

Now, we can chain \mathcal{F}_1 and \mathcal{F}_2 into an infinite sequence as follows.

$$\dots \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \xrightarrow{\mathcal{F}_2} \frac{\mathbb{Z}[G]}{\text{im } T_1} \times \frac{\mathbb{Z}[G]}{\text{im } N} \times \frac{\mathbb{Z}[G]}{\text{im } T_2} \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \xrightarrow{\mathcal{F}_2} \dots \quad (1.1)$$

We make a few preliminary observations about (1.1).

Lemma 2. The sequence of maps in (1.1) makes a complex.

Proof. We need to show that $\mathcal{F}_1 \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_1 = 0$. This is a matter of force. On one hand,

$$\begin{aligned} (\mathcal{F}_1 \circ \mathcal{F}_2)(a, b) &= \mathcal{F}_1(T_2a, N_1a - N_2b, T_1b) \\ &= (N_2(T_1b) + T_1(N_1a - N_2b), N_1(T_2a) - T_2(N_1a - N_2b)) \\ &= (N_2T_1b + T_1N_1a - T_1N_2b, N_1T_2a - T_2N_1a - T_2N_2b) \\ &= (N_2T_1b - T_1N_2b, N_1T_2a - T_2N_2b) \\ &= (0, 0). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\mathcal{F}_2 \circ \mathcal{F}_1)(x, y, z) &= \mathcal{F}_2(N_2z + T_1y, N_1x - T_2y) \\ &= (T_2(N_2z + T_1y), N_1(N_2z + T_1y) - N_2(N_1x - T_2y), T_1(N_1x - T_2y)) \\ &= (T_2N_2z + T_2T_1y, N_1N_2z + N_1T_1y - N_2N_1x + N_2T_2y, T_1N_1x - T_1T_2y) \\ &= (T_2T_1y, N_1N_2z - N_2N_1x, -T_1T_2y), \end{aligned}$$

which we can see represents 0 in $\mathbb{Z}[G]/\text{im } T_1 \times \mathbb{Z}[G]/\text{im } N \times \mathbb{Z}[G]/\text{im } T_2$. ■

Lemma 3. The complex (1.1) is exact at each $\mathbb{Z}[G]/\text{im } T_1 \times \mathbb{Z}[G]/\text{im } N \times \mathbb{Z}[G]/\text{im } T_2$.

Proof. We need to show that $\ker \mathcal{F}_1 = \text{im } \mathcal{F}_2$. By Lemma 2, we already know that $\text{im } \mathcal{F}_2 \subseteq \ker \mathcal{F}_1$, so it remains to get the reverse inclusion. Well, suppose $(x, y, z) \in \ker \mathcal{F}_1$, which means that

$$N_2z + T_1y = N_1x - T_2y = 0.$$

Now, note that because $x \in \mathbb{Z}[G]/\text{im } T_1$, where $\sigma_1 \equiv 1 \pmod{T_1}$, we have

$$N_1x = \sum_{k=0}^{n_1-1} \sigma_1^k x \equiv n_1x.$$

However,

$$n_1 \cdot N_2x = N_2(n_1x) = N_2N_1x = N_2(N_1x - T_2y) = 0,$$

so $N_2x = 0$. Thus, $x \in \ker N_2 = \text{im } T_2$, so we may write $x = T_2a$ for some $a \in \mathbb{Z}[G]$. A similar argument (replacing 1s with 2s and adjusting some signs) shows that we may write $z = T_1b$ for some $b \in \mathbb{Z}[G]$.

To show $(x, y, z) \in \text{im } \mathcal{F}_2$, it remains to show that $y \equiv N_1a - N_2b \pmod{N}$. Well, notice that

$$\begin{aligned} T_1(N_1a - N_2b - y) &= -(N_2T_1b + T_1y) \\ &= -(N_2z + T_1y) \\ &= 0 \\ T_2(N_1a - N_2b - y) &= N_1T_2a - T_2y \\ &= N_1x - T_2y \\ &= 0, \end{aligned}$$

so $N_1a - N_2b - y \in (\ker T_1) \cap (\ker T_2)$. To finish, we would like $(\ker T_1) \cap (\ker T_2) \subseteq \text{im } N$. Well, if $\alpha \in (\ker T_1) \cap (\ker T_2)$, then we can write

$$\alpha = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} a_{k,\ell} \sigma_1^k \sigma_2^\ell.$$

Now, the condition $T_1\alpha = 0$ implies that $a_{k,\ell} - a_{k-1,\ell} = 0$ for each k , where indices are taken modulo n_1 as necessary; thus, $a_{k,\ell}$ is constant with respect to k . Similarly, $T_2\alpha = 0$ implies that $a_{k,\ell}$ is constant with respect to ℓ , so

$$\alpha = \sum_{k=0}^{n_1-1} \sum_{\ell=0}^{n_2-1} a_{0,0} \sigma_1^k \sigma_2^\ell = N a_{0,0} \in \text{im } N,$$

as desired. ■

1.2 A Cocycle

The benefit to our analysis above is that we have a short exact sequence

$$0 \rightarrow \text{coker } \mathcal{F}_2 \xrightarrow{\mathcal{F}_1} \mathbb{Z}[G] \times \mathbb{Z}[G] \rightarrow \text{coker } \mathcal{F}_1 \rightarrow 1. \quad (1.2)$$

In particular, the (induced) map $\mathcal{F}_1: \text{coker } \mathcal{F}_2 \rightarrow \mathbb{Z}[G] \times \mathbb{Z}[G]$ is injective by [Lemma 3](#).

In the future, we will want a 2-cocycle in $Z^2(G, \text{coker } \mathcal{F}_2)$, which we will create by tracking a boundary morphism through [\(1.2\)](#). Thus, we want to start with a 1-cocycle in $Z^1(G, \text{coker } \mathcal{F}_1)$. For this, we define the notation

$$\sigma_i^{(a_i)} := \sum_{k=0}^{a_i-1} \sigma_i^k$$

for $i \in \{1, 2\}$ and any nonnegative integer $a_i \geq 0$. For example, $\sigma_i^{(n_i)} = N_i$. We also note that

$$\sigma_i^{(a_i+b_i)} = \sigma_i^{(a_i)} + \sigma_i^{a_i} \sigma_i^{(b_i)},$$

which justifies our "almost exponential" notation.

Lemma 4. Define $u \in C^1(G, \mathbb{Z}[G] \times \mathbb{Z}[G])$ by $u(\sigma_1^{a_1} \sigma_2^{a_2}) := (\sigma_1^{a_1} \sigma_2^{(a_2)}, \sigma_1^{(a_1)})$. Then the induced 1-cochain $\bar{u} \in C^1(G, \text{coker } \mathcal{F}_1)$ is a 1-cocycle.

Proof. We need to show that $\overline{du} = d\bar{u} = 0$; i.e., we need to show that $\text{im}(du) \subseteq \text{im } \mathcal{F}_1$. Well, pick up a_i, b_i with $0 \leq a_i, b_i < n_i$, and set

$$a_i + b_i = n_i q_i + r_i$$

by the division algorithm so that $q_i \in \{0, 1\}$ and $0 \leq r_i < n_i$. This lets us compute

$$\begin{aligned} (du) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) &= \sigma_1^{a_1} \sigma_2^{a_2} u \left(\sigma_1^{b_1} \sigma_2^{b_2} \right) - u \left(\sigma_1^{r_1} \sigma_2^{r_2} \right) + u \left(\sigma_1^{a_1} \sigma_2^{a_2} \right) \\ &= \sigma_1^{a_1} \sigma_2^{a_2} \left(\sigma_1^{b_1} \sigma_2^{(b_2)}, \sigma_1^{(b_1)} \right) - \left(\sigma_1^{r_1} \sigma_2^{(r_2)}, \sigma_1^{(r_1)} \right) + \left(\sigma_1^{a_1} \sigma_2^{(a_2)}, \sigma_1^{(a_1)} \right). \end{aligned}$$

Now, note

$$\begin{aligned} \left(\sigma_1^{r_1} \sigma_2^{(r_2)}, \sigma_1^{(r_1)} \right) &= \left(\sigma_1^{a_1+b_1} \sigma_2^{(r_2)}, \sigma_1^{(r_1)} \right) \\ &= \left(\sigma_1^{a_1+b_1} \sigma_2^{(a_2+b_2)} - \sigma_1^{a_1+b_1} \sigma_2^{r_2} \sigma_2^{(n_2 q_2)}, \sigma_1^{(a_1+b_1)} - \sigma_1^{r_1} \sigma_1^{(n_1 q_1)} \right) \\ &= \left(\sigma_1^{a_1+b_1} \sigma_2^{(a_2+b_2)}, \sigma_1^{(a_1+b_1)} \right) - \left(\sigma_1^{a_1+b_1} \sigma_2^{a_2+b_2} \cdot q_2 N_2, \sigma_1^{a_1+b_1} \cdot q_1 N_1 \right). \end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_1^{a_1} \sigma_2^{a_2} \left(\sigma_1^{b_1} \sigma_2^{(b_2)}, \sigma_1^{(b_1)} \right) &= \left(\sigma_1^{a_1+b_1} \sigma_2^{a_2} \sigma_2^{(b_2)}, \sigma_1^{a_1} \sigma_2^{a_2} \sigma_1^{(b_1)} \right) \\ &= \left(\sigma_1^{a_1+b_1} \sigma_2^{a_2} \sigma_2^{(b_2)}, \sigma_1^{a_1} \sigma_1^{(b_1)} \right) + \left(0, \sigma_1^{a_1} (\sigma_2^{a_2} - 1) \sigma_1^{(b_1)} \right) \\ &= \left(\sigma_1^{a_1+b_1} \sigma_2^{a_2} \sigma_2^{(b_2)}, \sigma_1^{a_1} \sigma_1^{(b_1)} \right) + \left(0, \sigma_1^{a_1} \sigma_2^{(a_2)} \sigma_1^{(b_1)} T_2 \right),\end{aligned}$$

and

$$\begin{aligned}\left(\sigma_1^{a_1} \sigma_2^{(a_2)}, \sigma_1^{(a_1)} \right) &= \left(\sigma_1^{a_1+b_1} \sigma_2^{(a_1)}, \sigma_1^{(a_1)} \right) + \left(\sigma_1^{a_1} (1 - \sigma_1^{b_1}) \sigma_2^{(a_2)}, 0 \right) \\ &= \left(\sigma_1^{a_1+b_1} \sigma_2^{(a_1)}, \sigma_1^{(a_1)} \right) + \left(-\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)} T_1, 0 \right).\end{aligned}$$

Synthesizing, we see that $\sigma_2^{(a_2+b_2)} = \sigma_2^{(a_2)} + \sigma_1^{a_2} \sigma_2^{(b_2)}$ causes the “leading” terms to vanish, leaving us with

$$\begin{aligned}(du) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) &= \left(0, \sigma_1^{a_1} \sigma_2^{(a_2)} \sigma_1^{(b_1)} T_2 \right) + \left(q_2 \sigma_1^{a_1+b_1} \sigma_2^{a_2+b_2} N_2, q_1 \sigma_1^{a_1+b_1} N_1 \right) + \left(-\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)} T_1, 0 \right) \\ &= \mathcal{F}_1 \left(q_1 \sigma_1^{a_1+b_1}, \quad -\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)}, \quad q_2 \sigma_1^{a_1+b_1} \sigma_2^{a_2+b_2} \right),\end{aligned}$$

which finishes. ■

In particular, from our above computation of du , we can pull back along \mathcal{F}_1 to find $\delta u \in Z^2(G, \text{coker } \mathcal{F}_2)$ is given by

$$(\delta u) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) = \left(q_1 \sigma_1^{a_1+b_1}, \quad -\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)}, \quad q_2 \sigma_1^{a_1+b_1} \sigma_2^{a_2+b_2} \right).$$

Simplifying a bit, we note that $\sigma_i \equiv 1 \pmod{T_i}$, so we can write this as

$$(\delta u) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) = \left(q_1, \quad -\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)}, \quad q_2 \sigma_1^{a_1+b_1} \right). \quad (1.3)$$

We will use δu again shortly.

1.3 Triples

We have now built enough machinery to be able to talk about triples; for the rest of the article, we will have $G = \text{Gal}(L/K)$ for some bicyclic extension L/K . Let S be the set of triples. The following is the main idea.

Lemma 5. There is a natural isomorphism between morphisms $f \in \text{Hom}_{\mathbb{Z}[G]}(\text{coker } \mathcal{F}_2, L^\times)$ and the set S of triples $(\alpha_1, \beta, \alpha_2)$ given by

$$\varphi: f \mapsto (f(1, 0, 0), f(0, 1, 0), f(0, 0, 1)).$$

Proof. For brevity, set $e_1 := (1, 0, 0)$ and $e_2 := (0, 1, 0)$ and $e_3 := (0, 0, 1)$ to be elements of $\mathbb{Z}[G]$. Now, let $\varphi: \text{Hom}_{\mathbb{Z}[G]}(\text{coker } \mathcal{F}_2, L^\times) \rightarrow S$ be defined by

$$\varphi: f \mapsto (f(e_1), f(e_2), f(e_3)).$$

The main check here is that φ is well-defined. We have the following checks.

- We check that our elements live in the correct fields. Note that $f(e_1), f(e_2), f(e_3) \in L^\times$. Further,

$$\sigma_1(f(e_1)) = f(\sigma_1, 0, 0) = f(1, 0, 0) = f(e_1)$$

where the main step is that $\sigma_1 \equiv 1 \pmod{T_1}$ enforces $(\sigma_1, 0, 0) = (1, 0, 0)$. It follows that $f(e_1) \in L^{\langle \sigma_1 \rangle}$. An analogous argument shows that $f(e_3) \in L^{\langle \sigma_2 \rangle}$.

- We check the relations. The main point is that

$$\begin{aligned}(0, N_1, 0) &\equiv (-T_2, 0, 0) \pmod{\text{im } \mathcal{F}_2}, \\ (0, N_2, 0) &\equiv (0, 0, T_1) \pmod{\text{im } \mathcal{F}_2}.\end{aligned}$$

As such, observe that

$$\begin{aligned}N_{L/L\langle\sigma_1\rangle} f(0, 1, 0) &= f(0, N_1, 0) \\ &= f(-T_2, 0, 0) \\ &= f(1, 0, 0)/\sigma_2 f(1, 0, 0).\end{aligned}$$

The other relation is similar: observe

$$\begin{aligned}N_{L/L\langle\sigma_2\rangle} f(0, 1, 0) &= f(0, N_2, 0) \\ &= f(0, 0, T_1) \\ &= \sigma_1 f(0, 0, 1)/f(0, 0, 1),\end{aligned}$$

which is what we wanted.

We also remark that φ is homomorphic because the operations on both $\text{Hom}_{\mathbb{Z}[G]}(\text{coker } \mathcal{F}_2, L^\times)$ and S are both defined pointwise.

Now, in the other direction, we define $\psi: S \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\text{coker } \mathcal{F}_2, L^\times)$ by

$$\psi: (\alpha_1, \beta, \alpha_2) \mapsto ((z_1, z_2, z_3) \mapsto z_1\alpha_1 \cdot z_2\beta \cdot z_3\alpha_2).$$

Again, the main check is that ψ is well-defined. Well, given a triple $(\alpha_1, \beta, \alpha_2) \in S$, we show that $\psi(\alpha_1, \beta, \alpha_2)$ is well-defined as a G -module homomorphism. To begin, we note that we can start with a function $f: \mathbb{Z}[G] \times \mathbb{Z}[G] \times \mathbb{Z}[G] \rightarrow L^\times$ by

$$f: (z_1, z_2, z_3) \mapsto z_1\alpha_1 \cdot z_2\beta \cdot z_3\alpha_2.$$

This can be quickly checked to be a G -module homomorphism: note that any $\alpha \in L^\times$ makes $z \mapsto z\alpha$ a G -module homomorphism $L^\times \rightarrow L^\times$, and the above is more or less a linear combination of such G -module homomorphisms.

We now investigate $\ker f$. Note that

$$f(T_1, 0, 0) = T_1\alpha = (\sigma_1\alpha_1)/\alpha_1 = 1$$

because $\alpha_1 \in L\langle\sigma_1\rangle$. An identical argument shows that $f(0, 0, T_2) = 1$. Further,

$$f(0, N, 0) = N\beta = N_{L/K}(\beta) = 1.$$

In total, we see that $\text{im } T_1 \times \text{im } N \times \text{im } T_2 \subseteq \ker f$, so we get to induce a function $\bar{f}: \mathbb{Z}[G]/\text{im } T_1 \times \mathbb{Z}[G]/\text{im } N \times \mathbb{Z}[G]/\text{im } T_2$ by

$$\bar{f}: (z_1, z_2, z_3) \mapsto z_1\alpha_1 \cdot z_2\beta \cdot z_3\alpha_2.$$

Further, we note that $(a, b) \in \mathbb{Z}[G] \times \mathbb{Z}[G]$ will have

$$\begin{aligned}\bar{f}(\mathcal{F}_2(a, b)) &= f(T_2a, N_1a - N_2b, T_1b) \\ &= (T_2a)\alpha_1 \cdot (N_1a - N_2b)\beta \cdot (T_1b)\alpha_2 \\ &= a \left(\frac{\sigma_2\alpha_1}{\alpha_1} \cdot N_{L/L\langle\sigma_1\rangle}(\beta) \right) \cdot b \left(\frac{\sigma_1\alpha_2}{\alpha_2} \cdot N_{L/L\langle\sigma_2\rangle}(\beta^{-1}) \right),\end{aligned}$$

which we see goes to 1 by the relations between β and the α s. Thus, we do indeed have a G -module homomorphism $\text{coker } \mathcal{F}_2 \rightarrow S$ defined as $\psi(\alpha_1, \beta, \alpha_2)$ requires.

We now finish by stating that φ and ψ are mutually inverse, essentially by construction. Checking that $(\varphi \circ \psi)(\alpha_1, \beta, \alpha_2) = (\alpha_1, \beta, \alpha_2)$ is a matter of plugging everything through; on the other hand, to show $(\psi \circ \varphi)(f) = f$, we note that

$$f(z_1, z_2, z_3) = z_1f(1, 0, 0) \cdot z_2f(0, 1, 0) \cdot z_3f(0, 0, 1) = \psi(\varphi(f))$$

because f is a G -module homomorphism. ■

In fact, we can even correctly account for the equivalence classes of triples; let S_0 denote the subgroup of triples coming from the trivial gerb.

Proposition 6. The isomorphism φ of Lemma 5 descends to a natural isomorphism

$$\bar{\varphi}: \hat{H}^0(G, \text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times)) \simeq S/S_0.$$

Proof. Observe that

$$\hat{H}^0(G, \text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times)) = \frac{\text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times)^G}{N_G(\text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times))},$$

so to pass through φ appropriately, we have to show that $\varphi(\text{im } N_G) = S_0$.

Well, pick up some $f \in \text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times)$, and we will pass it through N_G . In particular, to compute $\varphi(N_G(f))$, we are only interested in computing $(N_G f)(1, 0, 0)$ and $(N_G f)(0, 1, 0)$ and $(N_G f)(0, 0, 1)$. As usual, we proceed by force. On one hand,

$$\begin{aligned} (N_G f)(1, 0, 0) &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(\sigma_1^{n_1-i} \sigma_2^{n_2-j}, 0, 0 \right) \\ &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(T_2 \sigma_2^{(n_2-j)} + 1, 0, 0 \right) \\ &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(1, -N_1 \sigma_2^{(n_2-j)}, 0 \right) \\ &= \left[\prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f(1, 0, 0) \right] \left[\prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, -N_1 \sigma_2^{(n_2-j)}, 0 \right) \right] \\ &= N f(1, 0, 0) \cdot N_1 \prod_{j=0}^{n_2-1} \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-j-1} \sigma_2^j f(0, \sigma_1^p \sigma_2^q, 0)^{-1} \\ &= N f(1, 0, 0) \cdot N_1 \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{j=0}^{n_2-q-1} \sigma_2^j f(0, \sigma_1^p \sigma_2^q, 0)^{-1}. \end{aligned}$$

As such, we set

$$m_1 := N_2 f(1, 0, 0) \cdot \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{j=0}^{n_2-q-1} \sigma_2^j f(0, \sigma_1^p \sigma_2^q, 0)^{-1}$$

so that $(N_G f)(1, 0, 0) = N_1 m_1$. In particular,

$$\begin{aligned} \frac{m_1}{\sigma_2 m_1} &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{j=0}^{n_2-q-1} \frac{\sigma_2^{j+1} f(0, \sigma_1^p \sigma_2^q, 0)}{\sigma_2^j f(0, \sigma_1^p \sigma_2^q, 0)} \\ &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \frac{\sigma_2^{n_2-q} f(0, \sigma_1^p \sigma_2^q, 0)}{f(0, \sigma_1^p \sigma_2^q, 0)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (N_G f)(0, 0, 1) &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, 0, \sigma_1^{n_1-i} \sigma_2^{n_2-j} \right) \\
 &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, 0, T_1 \sigma_1^{(n_1-i)} + 1 \right) \\
 &= \prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, N_2 \sigma_1^{(n_1-i)}, 1 \right) \\
 &= \left[\prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, 0, 1 \right) \right] \left[\prod_{i=0}^{n_1-1} \prod_{j=0}^{n_2-1} \sigma_1^i \sigma_2^j f \left(0, N_2 \sigma_1^{(n_1-i)}, 0 \right) \right] \\
 &= N f(0, 0, 1) \cdot N_2 \prod_{i=0}^{n_1-1} \prod_{p=0}^{n_1-i-1} \prod_{q=0}^{n_2-1} \sigma_1^i f \left(0, \sigma_1^p \sigma_2^q, 0 \right) \\
 &= N f(0, 0, 1) \cdot N_2 \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{i=0}^{n_1-p-1} \sigma_1^i f \left(0, \sigma_1^p \sigma_2^q, 0 \right).
 \end{aligned}$$

As such, we set

$$m_2 := N_1 f(0, 0, 1) \cdot \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{i=0}^{n_1-p-1} \sigma_1^i \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)$$

so that $(N_G f)(0, 0, 1) = N_2 m_2$. In particular,

$$\begin{aligned}
 \frac{\sigma_1 m_2}{m_2} &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{i=0}^{n_1-p-1} \frac{\sigma_1^{i+1} \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}{\sigma_1^i \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)} \\
 &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \frac{\sigma_1^{n_1-p} \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}{\sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \frac{m_1}{\sigma_2 m_1} \cdot \frac{\sigma_1 m_2}{m_2} &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \left(\frac{\sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}{f \left(0, \sigma_1^p \sigma_2^q, 0 \right)} \cdot \frac{\sigma_1^{n_1-p} \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}{\sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)} \right) \\
 &= \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \left(\frac{\sigma_1^{n_1-p} \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right)}{f \left(0, \sigma_1^p \sigma_2^q, 0 \right)} \right) \\
 &= \left(\prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \sigma_1^{n_1-p} \sigma_2^{n_2-q} f \left(0, \sigma_1^p \sigma_2^q, 0 \right) \right) / f(0, N, 0) \\
 &= (N_G f)(0, 1, 0),
 \end{aligned}$$

where in the last equality we have used that $(0, N, 0) = (0, 0, 0)$. In total, we have computed

$$((N_G f)(1, 0, 0), (N_G f)(0, 1, 0), (N_G f)(0, 0, 1)) = \left(N_1 m_1, \frac{m_1}{\sigma_2 m_1} \cdot \frac{\sigma_1 m_2}{m_2}, N_2 m_2 \right) \in S_0,$$

so we see that $\varphi(\text{im } N_G) \subseteq S_0$.

To get the equality $\varphi(\text{im } N_G) = S_0$, we pick up arbitrary $m_1, m_2 \in L^\times$ and exhibit $f: \text{coker } \mathcal{F}_2 \rightarrow L^\times$ such that f produces m_1 and m_2 as described in the algorithm above.



Warning 7. The following construction is not extremely rigorous. The correct thing to do would be to write down a full formula for f and then verify that it is in fact a morphism $f: \text{coker } \mathcal{F}_2 \rightarrow L^\times$. I am not doing this out of laziness.

We quickly discuss what freedom we are allowed to give to f .

- The fact that $(\sigma_1, 0, 0) = (1, 0, 0)$ corresponds to a relation $f(\sigma_1^i, 0, 0) = f(1, 0, 0)$. Similarly, we require $f(0, 0, \sigma_2^j) = f(0, 0, 1)$.
- Because $(T_2, N_1, 0) = (0, 0, 0)$, we require $(\sigma_2^i, 0, 0) = (1, -N_1\sigma_2^{(i)}, 0)$ (here, $i \geq 1$), as above. However, this is the last relation on the left coordinate, so we may set $f(1, 0, 0)$ however we please and solve for the remaining $f(\sigma_2^i, 0, 0)$ based off of what we'll set the elements $f(0, g, 0)$ equal to.
- Similarly, we may set $f(0, 0, 1)$ however we please and solve for the remaining $f(0, 0, \sigma_1^i)$.
- As for the middle terms, we are unaffected by relations dealing with $\text{im } \mathcal{F}_2$ because we already accounted for them above. So we merely need to vanish on N , which translates into

$$\prod_{g \in G} f(0, g, 0) = 1.$$

So in total, we may set $f(1, 0, 0)$ and $f(0, 0, 1)$ however we please and then set $f(0, g, 0)$ for $g \in G$ however we please so long as

$$\prod_{g \in G} f(0, g, 0) = 1.$$

As such, we do the following.

- Set $f(0, \sigma_2^{n_2-1}, 0) = m_1^{-1}$.
- Set $f(0, \sigma_1^{n_1-1}, 0) = m_2$.
- To correct, set $f(0, 1, 0) = 1 / (f(0, \sigma_2^{n_2-1}, 0) f(0, \sigma_1^{n_1-1}, 0))$ and set $f(0, g, 0) = 1$ for all other $g \in G$. This gives

$$\prod_{g \in G} f(0, g, 0) = 1.$$

- Set $f(1, 0, 0) = f(0, \sigma_1^{n_1-1}, 0) f(0, 1, 0)$.
- Set $f(0, 0, 1) = 1 / (\sigma_2 f(0, \sigma_2^{n_2-1}, 0) \cdot \sigma_2^{n_2} f(0, 1, 0))$.

The above data describe some $f: \text{coker } \mathcal{F}_2 \rightarrow L^\times$, as discussed above. It remains to show that this f produces m_1 and m_2 as needed. Well, we compute

$$\begin{aligned} m_1 &\stackrel{?}{=} N_2 f(1, 0, 0) \cdot \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{j=0}^{n_2-q-1} \sigma_2^j f(0, \sigma_1^p \sigma_2^q, 0)^{-1} \\ &= N_2 f(1, 0, 0) \cdot \frac{1}{f(0, \sigma_2^{n_2-1}, 0) \cdot N_2 f(0, \sigma_1^{n_1-1}, 0) \cdot N_2 f(0, 1, 0)} \\ &= \frac{1}{f(0, \sigma_2^{n_2-1}, 0)} \\ &= m_1, \end{aligned}$$

as needed.

Similarly,

$$\begin{aligned}
m_2 &\stackrel{?}{=} N_1 f(0, 0, 1) \cdot \prod_{p=0}^{n_1-1} \prod_{q=0}^{n_2-1} \prod_{i=0}^{n_1-p-1} \sigma_1^i \sigma_2^{n_2-q} f(0, \sigma_1^p \sigma_2^q, 0) \\
&= N_1 f(0, 0, 1) \cdot \sigma_2^{n_2} f(0, \sigma_1^{n_1-1}, 0) \cdot N_1 \sigma_2 f(0, \sigma_2^{n_2-1}, 0) \cdot N_1 \sigma_2^{n_2} f(0, 1, 0) \\
&= \sigma_2^{n_2} f(0, \sigma_1^{n_1-1}, 0) \\
&= m_2,
\end{aligned}$$

as needed. This finishes the construction. ■

To tie up loose ends, we promised to use δu from [subsection 1.2](#), so here is how.

Proposition 8. Fix a triple $(\alpha_1, \beta, \alpha_2) \in S$. Then the cocycle in $Z^2(G, L^\times)$ corresponding to $(\alpha_1, \beta, \alpha_2)$ is

$$(\delta u) \cup \varphi^{-1}(\alpha_1, \beta, \alpha_2),$$

where we are implicitly passing through the evaluation map $\text{coker } \mathcal{F}_2 \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(\text{coker } \mathcal{F}_2, L^\times) \rightarrow L^\times$.

Proof. The cup product on inhomogeneous cochains is given by

$$\begin{aligned}
((\delta u) \cup \varphi^{-1}(\alpha_1, \beta, \alpha_2)) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) &= (\delta u) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) \otimes_{\mathbb{Z}} \sigma_1^{a_1+b_1} \sigma_2^{a_2+b_2} \varphi^{-1}(\alpha_1, \beta, \alpha_2) \\
&= (\delta u) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) \otimes_{\mathbb{Z}} \varphi^{-1}(\alpha_1, \beta, \alpha_2).
\end{aligned}$$

Passing through evaluation, we will get an element in L^\times which looks like

$$\varphi^{-1}(\alpha_1, \beta, \alpha_2) \left((\delta u) \left(\sigma_1^{a_1} \sigma_2^{a_2}, \sigma_1^{b_1} \sigma_2^{b_2} \right) \right) = q_1 \alpha_1 \cdot -\sigma_1^{a_1} \sigma_1^{(b_1)} \sigma_2^{(a_2)} \beta \cdot q_2 \sigma_1^{a_1+b_1} \alpha_2,$$

which is exactly what we wanted (after a little rearrangement). ■