INTERTWINING OPERATORS FOR SIEGEL PARABOLICS OVER FINITE FIELDS

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Abstract. We consider degenerate principal series $\operatorname{Ind}_P^G \chi$ representations over finite fields, where G is a classical group of even rank, and P is the Siegel parabolic subgroup of matrices of the form $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$. For example, we show that this representation is multiplicity-free and irreducible for most χ. We then discuss a particular intertwining operator I on $\operatorname{Ind}_P^G \chi$ and related combinatorics. A basis of $\operatorname{Ind}_P^G \chi$ provides a matrix representation, leading to families of diagonalizable antitriangular matrices with extraordinarily well-behaved eigenvalues. Lastly, applying I to a special vector in $\operatorname{Ind}_P^G \chi$ leads us to various matrix Gauss sums, whose evaluations imply an explicit equidistribution result of the trace and determinant of symmetric and alternating invertible matrices.

Contents

1. Introduction

Let q be a prime-power not divisible by 2 or 3, and let $2n$ be a positive even integer. For the purposes of the introduction, we work with the group $G \coloneqq \mathrm{SL}_{2n}(\mathbb{F}_q)$, and we let $P \subseteq G$ denote the subgroup of matrices of the form $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$ where $A, B, D \in \mathbb{F}_q^{n\times n}$. This paper is interested in the degenerate principal series representations $\operatorname{Ind}_P^G \chi$, where $\chi\colon P \to \mathbb{C}^\times$ is some character, and the combinatorics attached to these representations.

For example, we are able to show that the representation $\operatorname{Ind}_P^G \chi$ tends to be irreducible.

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Theorem 1.0.1. *The representation* $\operatorname{Ind}_P^G \chi$ *is multiplicity-free. Furthermore, the number of irreducible components equals*

$$\begin{cases} 1 & \text{if } \chi^2 \neq 1, \\ 2 & \text{if } \chi^2 = 1 \text{ but } \chi \neq 1, \\ n+1 & \text{if } \chi = 1. \end{cases}$$

The result follows by combining Propositions 2.3.1 and 2.3.7. In short, we use Gelfand pairs to show that the representation is multiplicity-free, and we use Mackey theory to compute the number of irreducible components.

Anyway, the point is that this representation is reasonably simple. The combinatorics attached to this representation comes from defining a special intertwining operator $I\colon \operatorname{Ind}_P^G \chi \to \operatorname{Ind}_P^G \chi'$ by

$$If(g) \coloneqq \sum_{B \in \mathbb{F}_q^{n\times n}} f\left(\begin{bmatrix} & 1_n \\ -1_n & \end{bmatrix} \begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix} g \right),$$

where $\chi'\colon P \to \mathbb{C}^\times$ is some other explicitly defined character. When $\chi^2 \neq 1$, one has that $I \circ I$ is an operator on $\operatorname{Ind}_P^G \chi$; otherwise, when $\chi^2 = 1$, it turns out that $\chi' = \chi$ so that I is an operator on $\operatorname{Ind}_P^G \chi$.

As such, we see that there are two cases of interest for our operator I. On one hand, in the exceptional cases where $\chi^2 = 1$, we can ask how I acts on the irreducible components of $\operatorname{Ind}_P^G \chi$. For simplicity, we will consider $\chi = 1$ for the time being. By choosing a basis of $\operatorname{Ind}_P^G 1$, we show the following in Proposition 2.4.6 and Theorem 3.2.1.

Theorem 1.0.2. *One can give* $\operatorname{Ind}_P^G 1$ *an ordered basis so that the operator* I *on* $\operatorname{Ind}_P^G 1$ *has matrix given by*

$$\left[(-1)^{r+s-n} \frac{(q;q)_s^2}{(q;q)_{n-r}^2 (q;q)_{r+s-n}} q^{n^2-s^2+\binom{r+s-n}{2}} \right]_{0\le s,r\le n},$$

where $(a;q)_n \coloneqq \prod_{i=1}^n \left(1 - aq^i\right)$ *is the q-Pochhammer symbol. This matrix is diagonalizable and has eigenvalues given by*

$$\left\{ (-1)^r q^{\binom{n}{2}+\binom{n-r+1}{2}} : 0 \le r \le n \right\}.$$

Example 1.0.3. *For* $n = 4$*, this matrix is*

$$\begin{bmatrix} & & & & q^{16} \\ & & & q^{15} & \frac{(q-1)}{1} q^{15} \\ & & q^{12} & \frac{(q^2-1)^2}{(q-1)} q^{12} & \frac{(q-1)(q^2-1)}{1} q^{13} \\ & q^7 & \frac{(q^3-1)^2}{(q-1)} q^7 & \frac{(q^2-1)(q^3-1)^2}{(q-1)} q^8 & \frac{(q-1)(q^2-1)(q^3-1)}{1} q^{10} \\ 1 & \frac{(q^4-1)^2}{(q-1)} & \frac{(q^3-1)^2(q^4-1)^2}{(q-1)(q^2-1)} q^1 & \frac{(q^2-1)(q^3-1)(q^4-1)^2}{(q-1)} q^3 & \frac{(q-1)(q^2-1)(q^3-1)(q^4-1)}{1} q^6 \end{bmatrix},$$

and the eigenvalues are $\left\{q^{16}, -q^{12}, q^9, -q^7, q^6\right\}$.

Remark 1.0.4. *Considerations with other classical groups G produces other families of diagonalizable*

What is remarkable is that we have produced a family of diagonalizable "antitriangular" matrices. We are not aware of any general method to handle such diagonalization problems, and it does not appear clear a priori that the eigenvalues listed above should be so well-behaved. Diagonalizing certain antitriangular (satisfying a "global antidiagonal property") matrices have combinatorial applications in [BW22], and some aspects of our methods can be considered q-analogues of their arguments, but the analogy is somewhat weak. Notably, the above family of matrices does not satisfy the global antidiagonal property considered there.

On the other hand, we may still be interested in the generic case $\chi^2 \neq 1$. Here, a matrix representation as above can describe the behavior of $I \circ I$. Another approach would be to choose a particularly special vector of $\operatorname{Ind}_G^P \chi$ and then evaluate $I \circ I$ on this vector; this will determine the entire behavior of $I \circ I$ because $\operatorname{Ind}_P^G \chi$ is irreducible. A particular choice of vector leads us to consider the matrix Gauss sums of [Kim97]; the following result is Proposition 2.5.6.

Theorem 1.0.5. *Fix a nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. For each character $\chi\colon P \to \mathbb{C}^\times$, we define a vector $f_\chi \in \operatorname{Ind}_P^G \chi$ as supported on matrices of the form $P\left[\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1_n & B \\ & 1_n \end{smallmatrix}\right]$ with value*

$$f_\chi\left(p\begin{bmatrix} & -1_n \\ 1_n & \end{bmatrix}\begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix}\right) := \chi(p)\psi(\operatorname{tr} B).$$

Then

$$If_\chi = \left(\sum_{B\in \mathrm{GL}_n(\mathbb{F}_q)} \chi(\det B)\psi(\operatorname{tr} B)\right) f_{\chi'}.$$

Thus, we are motivated to evaluate these Gauss sums. In the case of $G = \mathrm{SL}_{2n}(\mathbb{F}_q)$, the corresponding Gauss sums given as above have been evaluated in [Kim97]. However, considerations of other groups G lead to different sums. For example, $G = \mathrm{Sp}_{2n}(\mathbb{F}_q)$ leads to a sum over invertible symmetric matrices considered in [Sai91], and $G = \mathrm{O}_{4n}(\mathbb{F}_q)$ leads to a sum over invertible alternating matrices (which appears to be new).

We provide evaluations for all of these matrix Gauss sums. Our methods are based on an explicit row-reduction analogous to the Bruhat decomposition considerations of [Kim97], but the explicit nature of our exposition allows our proofs over all the various kinds of sums to be rather uniform. For example, even though the sum over invertible symmetric matrices has already been considered in [Sai91], our method seems to be easier to visualize.

Evaluating these Gauss sums also has a combinatorial application: we are able to provide a reasonably explicit formula for the number of invertible matrices (symmetric, alternating, or neither) with given trace and determinant. For general invertible matrices, this application is essentially implicit in [Kim97, Theorem 6.2], so we state results for symmetric and alternating matrices, which appear to be new. The following results follow from Corollaries 4.3.6 and 4.4.7, and they provide a rather explicit equidistribution result for the trace and determinant.

Theorem 1.0.6. *Fix $d \in \mathbb{F}_q^\times$ and $t \in \mathbb{F}_q$. For odd integers $2m+1$, the number $N(d,t)$ of symmetric $A \in \mathrm{GL}_{2m+1}(\mathbb{F}_q)$ is bounded by*

$$-q^{m(m+1)}(q-1)^{m+1}\cdot\frac{1}{q(q-1)} \le N(d,t) - \frac{N}{q(q-1)} \le q^{m(m+1)}(q-1)^{m+1}\cdot\frac{q(q-1)-1}{q(q-1)},$$

where N is the total number of invertible symmetric $(2m+1)\times(2m+1)$ matrices.

Remark 1.0.7. *There is analogous result for even integers $2m$, but it is slightly more complicated to state.*

Theorem 1.0.8. *Fix $d \in \mathbb{F}_q^{\times 2}$ and $t \in \mathbb{F}_q$. For even integers $2m$, the number $N(d,t)$ of alternating $A \in \mathrm{GL}_{2n}(\mathbb{F}_q)$ is bounded by*

$$-q^{m(m-1)}(q-1)^{m}\cdot\frac{2}{q(q-1)} \le N(d,t) - \frac{N}{q(q-1)/2} \le q^{m(m-1)}(q-1)^{m}\cdot\frac{q(q-1)-2}{q(q-1)},$$

where N is the total number of invertible alternating $2m\times 2m$ matrices.

1.1. **Layout.** We quickly explain the outline of the paper. In section 2, we examine the representation theory of $\operatorname{Ind}_P^G \chi$ and explain where the combinatorial applications arise. In section 3, we provide the diagonalization of our intertwining operator. Lastly, in section 4, we evaluate our matrix Gauss sums and provide the combinatorial applications.

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2. Group-Theoretic Set-Up

In this section, we set up the necessary representation theory to proceed with the results in the rest of the paper.

2.1. **Groups and Subgroups.**

Let q be an odd prime-power, and let $2n$ be a positive even integer; for convenience, we will take $3 \nmid q$, but this is used infrequently. Throughout, G will be one of the groups $\{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}, \mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$ over the finite field $\mathbb{F}_q$. To explicate our orthogonal and symplectic groups, we fix

$$\varepsilon \coloneqq \begin{cases} +1 & \text{if } G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}, \\ -1 & \text{if } G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}, \end{cases} \qquad \text{and} \qquad J \coloneqq \begin{bmatrix} & \varepsilon 1_n \\ 1_n & \end{bmatrix}$$

so that G is defined to preserve the quadratic form J. In the cases where $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, it will be convenient to define $\varepsilon \coloneqq -1$ as well. Here, the blank entries in J indicate zeroes, a convention that will stay in place for the rest of the article. Throughout, when there are multiple groups G involved, we will use a superscript $(\cdot)^G$ to distinguish between multiple elements; for example, $\varepsilon^{\mathrm{GL}_{2n}} = -1$.

Note that G has split maximal torus T given by the diagonal matrices. Anyway, the benefit $2n$ being even is that we may use the Siegel parabolic subgroup

$$P \coloneqq \left\{ \begin{bmatrix} A & B \\ & D \end{bmatrix} \in G \right\},$$

where A, B, C, D are implicitly in $\mathbb{F}_q^{n \times n}$, a convention that will remain in place for any expression in block matrix form as above. We let $U \subseteq P$ be the unipotent radical of P, and we let $M \subseteq P$ be the Levi subgroup so that $P = M \ltimes U$. Explicitly,

$$U = \left\{ \begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix} \in G \right\} \qquad \text{and} \qquad M = \left\{ \begin{bmatrix} A & \\ & D \end{bmatrix} \in G \right\}.$$

The various cases of G provide more constraints on these subgroups. For example, if $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$, then B above must be alternating; if $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, then B above must be symmetric. Similarly, if $G = \mathrm{SL}_{2n}$, then $\det D = (\det A)^{-1}$; if $G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$, then $D = A^{-\intercal}$; and if $G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$, then $D = \lambda A^{-\intercal}$ for some $\lambda \in \mathbb{F}_q^\times$. A quick computation with the definition of G in the various cases reveals that these are only the constraints.

It will be helpful in the sequel to understand characters of P. In all cases, we are able to define a "Siegel determinant" $\chi_{\det} \colon P \to \mathbb{F}_q^\times$ given by

$$\chi_{\det}\left(\begin{bmatrix} A & B \\ & D \end{bmatrix} \right) \coloneqq (\det D)^{-1}.$$

In the cases $G \in \{\mathrm{GL}_{2n}, \mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$, there is an additional "multiplier" $m \colon P \to \mathbb{F}_q^\times$ given by

$$m\left(\begin{bmatrix} A & B \\ & D \end{bmatrix} \right) = \det AD \text{ if } G = \mathrm{GL}_{2n}, \text{ and } m\left(\begin{bmatrix} \lambda A & B \\ & A^{-\intercal} \end{bmatrix} \right) \coloneqq \lambda \text{ else.}$$

For the remaining cases of G, we will define m to just be the trivial character. Both $\chi_{\det}$ and m are characters by a direct computation. It turns out that these are essentially the only characters.

Lemma 2.1.1. *Let $\chi \colon P \to \mathbb{C}$ be a character. Then $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$ for some characters $\alpha, \beta \colon \mathbb{F}_q^\times \to \mathbb{C}$.*

Proof. We will do casework on G, but before going any further, we argue (directly) that χ vanishes on U; we will actually show that U is contained in the commutator subgroup $[P, P]$. Well, choose some B such that $u \coloneqq \left[\begin{smallmatrix} 1_n & B \\ & 1_n \end{smallmatrix}\right]$ is in U. Choosing some $a \in \mathbb{F}_q^\times$, we note $\left[\begin{smallmatrix} a1_n & \\ & a^{-1}1_n \end{smallmatrix}\right] \in M$, so we consider the commutator

$$\begin{bmatrix} a1_n & \\ & a^{-1}1_n \end{bmatrix} \begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix} \begin{bmatrix} a1_n & \\ & a^{-1}1_n \end{bmatrix}^{-1} \begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix}^{-1} = \begin{bmatrix} 1_n & \left(a^2 - 1\right) B \\ & 1_n \end{bmatrix}.$$

As long as we can choose a such that $a^2 - 1 \in \mathbb{F}_q^\times$, we can replace B with $\left(a^2 - 1\right)^{-1} B$ in the above computation to conclude that $u \in [P, P]$. Well, because $3 \nmid q$, we may choose $a \coloneqq 2$ so that $a^2 - 1 = 3$.

The point of the previous paragraph is that χ now factors through $P/U = M$, so we may as well consider χ as a character on M. We now proceed with our casework.

- If $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, we claim that χ is also trivial on the subgroup
$$\left\{ \begin{bmatrix} A & \\ & D \end{bmatrix} : \det A = \det D = 1 \right\}.$$
Well, fix any such $\left[\begin{smallmatrix} A & \\ & D \end{smallmatrix}\right]$. Indeed, we are given that $A, D \in \mathrm{SL}_n$, so we appeal to the fact that $[\mathrm{SL}_n, \mathrm{SL}_n] = \mathrm{SL}_n$ for our $q > 3$. Thus, A and D can be expressed as commutators $A = A_1 A_2 A_1^{-1} A_2^{-1}$ and $D = D_1 D_2 D_1^{-1} D_2^{-1}$ so that
$$\begin{bmatrix} A & \\ & D \end{bmatrix} = \begin{bmatrix} A_1 & \\ & D_1 \end{bmatrix} \begin{bmatrix} A_2 & \\ & D_2 \end{bmatrix} \begin{bmatrix} A_1 & \\ & D_1 \end{bmatrix}^{-1} \begin{bmatrix} A_2 & \\ & D_2 \end{bmatrix}^{-1},$$
so our element is a commutator.
Thus, χ on M now factors through $M/(\mathrm{SL}_n \times \mathrm{SL}_n)$. If $G = \mathrm{GL}_{2n}$, this quotient is isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ by $\left[\begin{smallmatrix} A & \\ & D \end{smallmatrix}\right] \mapsto \left(\det AD, \det D^{-1}\right)$, so χ indeed factors through a product of m and $\chi_{\det}$. On the other hand, if $G = \mathrm{SL}_{2n}$, this quotient is isomorphic to $\mathbb{F}_q^\times$ just by $\left[\begin{smallmatrix} A & \\ & D \end{smallmatrix}\right] \mapsto \det D^{-1}$ because we have the condition $\det AD = 1$ already, so we again factor through $\chi_{\det}$.
- If $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, we claim that χ is also trivial on the subgroup
$$\left\{ \begin{bmatrix} A & \\ & A^{-\intercal} \end{bmatrix} : \det A = 1 \right\}.$$
Well, fix any such $\left[\begin{smallmatrix} A & \\ & A^{-\intercal} \end{smallmatrix}\right]$. Again, we are given that $A \in \mathrm{SL}_n$, so we appeal to the fact that $[\mathrm{SL}_n, \mathrm{SL}_n] = \mathrm{SL}_n$ for our $q > 3$, meaning we may write $A = A_1 A_2 A_1^{-1} A_2^{-1}$ so that
$$\begin{bmatrix} A & \\ & A^{-\intercal} \end{bmatrix} = \begin{bmatrix} A_1 & \\ & A_1^{-\intercal} \end{bmatrix} \begin{bmatrix} A_2 & \\ & A_2^{-\intercal} \end{bmatrix} \begin{bmatrix} A_1 & \\ & A_1^{-\intercal} \end{bmatrix}^{-1} \begin{bmatrix} A_2 & \\ & A_2^{-\intercal} \end{bmatrix}^{-1},$$
so our element is a commutator.
Thus, χ now factors through M/SL_n, where SL_n is embedded into M as the above subgroup. If $G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$, then elements of M look like $\left[\begin{smallmatrix} A & \\ & A^{-\intercal} \end{smallmatrix}\right]$, so this quotient is isomorphic to $\mathbb{F}_q^\times$ via $\chi_{\det}$, meaning χ indeed factors through $\chi_{\det}$. Otherwise, $G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$, so elements of M look like $\left[\begin{smallmatrix} \lambda A & \\ & A^{-\intercal} \end{smallmatrix}\right]$, so this quotient is isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ via $(m, \chi_{\det})$, allowing us to conclude again. ■

The lemma now allows us to take any character $\chi \colon P \to \mathbb{C}^\times$ and define $\alpha_\chi, \beta_\chi \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ such that $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$. When m is trivial, we will take α_χ to be trivial as well; otherwise, m and $\chi_{\det}$ are surjective, so α_χ and β_χ are uniquely determined by χ.

2.2. **Some Weyl Group Computations.** An argument similar to [Mil17, Example 17.88] verifies that the diagonal subgroup T of G is always a maximal torus; namely, one can check that $C_G(T) = T$. Then an argument similar to [Mil17, Example 17.42] verifies that $N_G(T)$ consists of permutation matrices (up to torus elements); alternatively, one can study the Weyl group of the relevant root system and then convert this back into permutation matrices by hand. In any case, we let W denote the Weyl group of G, and we let W_P denote the Weyl group of the Siegel parabolic subgroup P.

It will be useful to explicitly compute these Weyl groups explicitly. If $G \in \{\mathrm{GL}_n, \mathrm{SL}_n\}$, then W consists of the permutation matrices up to a sign. For each $w \in W$, we let $\sigma_w \in N_G(T)$ denote the corresponding permutation matrix, and we let $d_w \in T$ be a diagonal matrix with entries in ± 1 such that $\det d_w \sigma_w = 1$. (The choice of d_w will not matter too much in the sequel.) The point is that $\{d_w \sigma_w\}_{w \in W}$ provides a set of representatives for W in G.

We would like a similar description for $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$. The main point of the computation here is to ask which permutation matrices σ actually belong to G, and then we can again use d_w to correct for the determinant condition.

Lemma 2.2.1. *Suppose $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$. Let Σ be the set of permutations $\sigma \in S_{2n}$ such that $\sigma(i+n) \equiv \sigma(i) + n \pmod{2n}$ for each i.*

(a) *For each w representing a class in W, there exists a unique permutation $\sigma \in \Sigma$ such that $w = d\sigma$ for some diagonal matrix d.*
(b) *For each $\sigma \in \Sigma$, there exists some diagonal matrix d with entries in $\{\pm 1\}$ such that $d\sigma \in G$.*

Proof. We will show the parts independently.

(a) Recalling that the diagonal matrices of G make up a maximal torus in B, we note that diagonal matrices are normalized by the semidirect product of permutation matrices and diagonal matrices (this is even true in GL_{2n}), so we can view elements of W as permutation matrices with elements adjusted by a diagonal element to lie in G.

In particular, we may write $w = d\sigma$ for some diagonal matrix d, and this σ is unique. It remains to show $\sigma \in \Sigma$. Well, the main point is that $d\sigma \in G$ requires

$$d\sigma J \sigma^\intercal d^\intercal = \lambda J$$

for some scalar λ, possibly forced equal to 1 if $G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$. Setting $d := \mathrm{diag}(d_1, \ldots, d_{2n})$, we now pass through a basis vector $e_{\sigma(i)}$ to compute

$$\varepsilon^{1_{i>n}} d_{\sigma(i+n)} d_{\sigma(i)} e_{\sigma(i+n)} = \varepsilon^{1_{\sigma(i)>n}} e_{\sigma(i)+n}, \tag{2.1}$$

where indices live in $\{1, 2, \ldots, 2n\}$ but are considered $\pmod{2n}$. Because the diagonal elements of d are nonzero, we must have $\sigma(i+n) \equiv \sigma(i) + n \pmod{2n}$, meaning $\sigma \in \Sigma$.

(b) We need a diagonal matrix $d = \mathrm{diag}(d_1, \ldots, d_{2n})$ such that $d\sigma \in G$, so it is enough for $d\sigma J\sigma^\intercal d^\intercal = J$. Well, it suffices to check this on basis vectors $e_{\sigma(i)}$, for which we see it is enough (2.1). But because $\sigma \in \Sigma$, it is equivalent to require

$$\varepsilon^{1_{i>n}} d_{\sigma(i)+n} d_{\sigma(i)} = \varepsilon^{1_{i>n}} d_{\sigma(i+n)} d_{\sigma(i)} = \varepsilon^{1_{\sigma(i)>n}}$$

for each index i. Observe $\varepsilon^{1_{(i+n)>n}} = -\varepsilon^{1_{i>n}}$ and $\varepsilon^{1_{\sigma(i+n)>n}} = -\varepsilon^{1_{\sigma(i)>n}}$ (indices are still taken $\pmod{2n}$), so if the above equation is satisfied at index i, then it is satisfied at index $i+n$.

As such, given signs $\{d_{\sigma(1)}, \ldots, d_{\sigma(n)}\}$, we must set $d_{\sigma(i)+n} := \varepsilon^{1_{\sigma(i)>n}} d_{\sigma(i)}$ for each $i \in \{1, 2, \ldots, 2\}$ to satisfy the equation at the indices $i \in \{1, 2, \ldots, n\}$, and this choice of signs will work. ■

Remark 2.2.2. *Let's provide a convenient choice of signs d_w for $w \in W$. If $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$, then $\varepsilon = 1$, so we see that $d_w := 1_{2n}$ will always work. If $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, then we claim that we can put signs d_w on the top-right quadrant of σ_w. Explicitly, we take $d_{\sigma(i)} = -1$ if $i \le n$ and $\sigma(i) > n$, and we take $d_{\sigma(i)} = 1$ otherwise. The computation of the above lemma shows that this works (note $\varepsilon = -1$).*

Our benefit to having explicit representatives of W is that we get explicit representatives of certain double quotients. For example, W itself provides representatives of $B\backslash G/B$ by the Bruhat decomposition, where $B \subseteq G$ is a Borel subgroup containing T. We will be interested in $P\backslash G/P$.

Lemma 2.2.3. *For each $r \in \{0, 1, \ldots, n\}$, define*

$$\eta_r := \begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix}.$$

Then $\{\eta_0, \ldots, \eta_r\} \subseteq G$ provides a set of representatives of the double quotients $P\backslash G/P$.

Proof. Even though the statement is rather uniform in G, the proof will require some moderate casework. Intuitively, what is going on here is that the Weyl group W_P of P consists of permutation matrices in the top-left and bottom-right quadrants, so to compute $P\backslash G/P \cong W_P\backslash W/W_P$, one can correct an arbitrary Weyl element $w \in W$ into the above form by row-reduction into the above form. We will argue slightly more directly.

Before doing anything serious, we check that $\eta_r \in G$ in all cases. Observe that

$$\eta_r^\intercal J \eta_r = \begin{bmatrix} 1_{n-r} & & & \\ & & & 1_r \\ & & 1_{n-r} & \\ & \varepsilon 1_r & & \end{bmatrix} \begin{bmatrix} & & \varepsilon 1_{n-r} & \\ & & & \varepsilon 1_r \\ 1_{n-r} & & & \\ & 1_r & & \end{bmatrix} \begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix} = J$$

by a direct computation. So if $\varepsilon = 1$, we see $\eta_r \in \mathrm{O}_{2n}$; and if $\varepsilon = -1$, we see $\eta_r \in \mathrm{Sp}_{2n} \subseteq \mathrm{SL}_{2n}$.

It remains to show that $\{\eta_0, \ldots, \eta_n\}$ provides a set of representatives. We define a function $\rho\colon G \to \{0, \ldots, n\}$ by $\rho\left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\right) := \mathrm{rank}\, C$. Note that ρ is surjective because $\rho(\eta_r) = r$ for each $r \in \{0, \ldots, n\}$. We will show that ρ descends to a bijection $P\backslash G/P \to \{0, \ldots, n\}$, from which the result follows because $\rho(\eta_r) = r$.

To begin, we show that ρ descends to a surjection $P\backslash G/P \to \{0,\ldots,n\}$. Because ρ is already a surjection, we just need to check that ρ is defined up to double cosets. Well, we compute that

$$\rho\left(\begin{bmatrix} A' & B' \\ & D' \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A'' & B'' \\ & D'' \end{bmatrix}\right) = \rho\left(\begin{bmatrix} * & * \\ D'CA'' & * \end{bmatrix}\right) = \operatorname{rank} D'CA'',$$

where $*$ indicates some value we have not bothered to compute. Now, multiplication by an invertible matrix does not adjust rank, so $\operatorname{rank} D'CA'' = \operatorname{rank} C = \rho\left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\right)$.

It remains to show that the function $\rho\colon P\backslash G/P \to \{0,\ldots,n\}$ is injective. Unwinding definitions, it is enough to show that we must show that $\rho(g) = r$ implies that $g \in P\eta_r P$. Choosing a Borel subgroup $B \subseteq P$ containing T, we may use the Bruhat decomposition to see that $B\backslash G/B$ is represented by the Weyl group W. Thus, we may assume that $g = w = d_w\sigma_w$ where $d_w \in T$ and σ_w is a permutation matrix. Now, to use that $\rho(d_w\sigma_w) = r$, we note that σ_w being a permutation matrix means that the rank of the bottom-left quadrant is just

$$r = \rho(d_w\sigma_w) = \#\{i \le n : \sigma(i) > n\}.$$

Now, we may choose a permutation σ of $\{1,\ldots,n\}$ so that

$$\{i \in \{1,\ldots,n\} : \sigma_w\sigma(i) > n\} = \{1,\ldots,r\}.$$

If $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, then we extend σ to S_{2n} by requiring $\{i > n : \sigma_w\sigma(i) \le n\} = \{n+1,\ldots,n+r\}$. Otherwise, σ may be extended to a permutation in Σ (from Lemma 2.2.1), and we can see that $\sigma \in D^{\mathrm{sp}}_{2n}$. Thus, σ belongs to some Weyl element in $W \cap P$, so multiplication on the right of w by this Weyl element, we may assume that

$$\begin{aligned} \{i \le n : \sigma_w(i) > n\} &= \{1,\ldots,r\} \\ \{i > n : \sigma_w(i) > n\} &= \{n+1,\ldots,n+r\} \end{aligned}$$

on the nose. A similar argument by multiplying on the left of w is able to rearrange the actual values of σ_w to show that w has the same underlying permutation matrix as η_r, so $w = \eta_r \in P\eta_r P$ follows. ■

Remark 2.2.4. *A benefit of the above proof with ρ (instead of arguing with $W_P\backslash W/W_P$) is that we see that the double cosets*

$$P\eta_r P = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G : \operatorname{rank} C = r\right\}$$

are all (Zariski) locally closed. In fact, $P\eta_0 P$ is (Zariski) closed, and $P\eta_r P$ is the only (Zariski) open double coset (defined by $\det C \neq 0$).

2.3. **Parabolic Induction.** In the sequel, we will be interested in the representations $\operatorname{Ind}_P^G \chi$ where $\chi\colon P \to \mathbb{C}^\times$ is a character. We spend this subsection collecting a few facts about these representations. In particular, we will show that these representations are multiplicity-free and irreducible for "general" χ.

Let's begin by showing our generic irreducibility.

Proposition 2.3.1. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$. Then*

$$\dim \operatorname{End}_G \operatorname{Ind}_P^G \chi = \begin{cases} n+1 & \text{if } \beta = 1, \\ 2 & \text{if } \beta^2 = 1, \beta \neq 1 \text{ and } G = \mathrm{SL}_{2n}, \\ n+1 & \text{if } \beta^2 = 1, \beta \neq 1 \text{ and } G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}, \\ \left\lfloor \frac{1}{2}(n+1) \right\rfloor & \text{if } \beta^2 = 1, \beta \neq 1 \text{ and } G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}, \\ 1 & \text{else.} \end{cases}$$

In particular, $\operatorname{Ind}_P^G \chi$ is irreducible provided $\beta^2 \neq 1$.

Proof. We use Mackey theory in the form of [Bum13, Theorem 32.1]. Namely, we are interested in computing the dimension of the space

$$\mathcal{H} := \{f \in \operatorname{Mor}(G,\mathbb{C}) : f(p_1 g p_2) = \chi(p_1)\chi(p_2)f(g) \text{ for } p_1, p_2 \in P, g \in G\}.$$

Thus, we see that $f \in \mathcal{H}$ is uniquely determined by its values on representatives of the double cosets $P\backslash G/P$. As such, we set $f_r \in \mathcal{H}$ to be supported on $P\eta_r P$ defined by $f_r(\eta_r) \in \{0,1\}$, where we take $f_r(\eta_r) = 1$ provided that this gives a well-defined function in $\mathcal{H}$. Then our computation of $P\backslash G/P$ in Lemma 2.2.3 tells us that $\{f_r : f_r \neq 0\}$ is a basis of $\mathcal{H}$.

Thus, we are left computing the number of r such that $f_r \in \mathcal{H}$ is well-defined with $f_r(\eta_r) = 1$. So fix some r, and we check if $f_r \in \mathcal{H}$ is well-defined with $f_r(\eta_r) = 1$. Namely, if $p_1\eta_r p_2 = p_1'\eta_r p_2'$ for $p_1, p_1', p_2, p_2' \in P$, we must check that $\chi(p_1)\chi(p_2) = \chi(p_1')\chi(p_2')$. Rearranging, it is enough to check that $p_1 = \eta_r p_2 \eta_r^{-1}$ implies that $\chi(p_1) = \chi(p_2)$. In other words, if $p \in P$ has $\eta_r p \eta_r^{-1} \in P$, we need $\chi(p) = \chi\left(\eta_r p \eta_r^{-1}\right)$. Well, we write

$$p \coloneqq \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ & & D_1 & D_2 \\ & & D_3 & D_4 \end{bmatrix}$$

to have the same dimensions as η_r, and then we see that

$$\begin{aligned}\eta_r p \eta_r^{-1} &= \begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix} \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ & & D_1 & D_2 \\ & & D_3 & D_4 \end{bmatrix} \begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix}^{-1} \\ &= \begin{bmatrix} A_1 & \varepsilon B_2 & B_1 & A_2 \\ & D_4 & \varepsilon D_3 & \\ & \varepsilon D_2 & D_1 & \\ A_3 & \varepsilon B_4 & B_3 & A_4 \end{bmatrix},\end{aligned}$$

so this is in P if and only if $A_3 = B_4 = D_2 = 0$. Thus, $\chi(p) = \chi\left(\eta_r p \eta_r^{-1}\right)$ is equivalent to always having

$$\chi\left(\begin{bmatrix} A_1 & \varepsilon B_2 & B_1 & A_2 \\ & D_4 & \varepsilon D_3 & \\ & & D_1 & \\ & & B_3 & A_4 \end{bmatrix}\right) \stackrel{?}{=} \chi\left(\begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ & A_4 & B_3 & \\ & & D_1 & \\ & & D_3 & D_4 \end{bmatrix}\right).$$

The multiplier of the left-hand side is $m\left(\eta_r p \eta_r^{-1}\right) = m(p)$, which is also the multiplier of the right-hand side, so we are allowed to ignore α for the rest of the proof. (The point is that m is defined as a character on G.) As for β, we go ahead and compute $\chi_{\det}$ on both sides to see that we must have

$$\beta(\det D_1 \cdot \det A_4)^{-1} \stackrel{?}{=} \beta(\det D_1 \cdot \det D_4)^{-1},$$

where we take the convention that the "empty" matrix has determinant 1. Equivalently, we are asking for

$$\beta(\det A_4) \stackrel{?}{=} \beta(\det D_4).$$

We now work in cases on G and r.

- If $r = 0$, then A_4 and D_4 are empty, so the condition holds. Thus, we will take $r > 1$ in the rest of our casework.
- If $\beta = 1$, then the condition holds. Thus, we will take $\beta \neq 1$ in the rest of our casework.
- Take $G = \mathrm{GL}_{2n}$. Because $r > 0$, det is always surjective, and here there are no conditions on how $\det A_4$ and $\det D_4$ should relate to each other, so the condition never holds.
- Take $G = \mathrm{SL}_{2n}$. Because $r > 0$, det will always be surjective. If $r = n$, then the condition $\det p = 1$ becomes $\det A_4 = \det D_4^{-1}$, so we get a contribution in this case only when $\beta^2 = 1$. Otherwise, $r \notin \{0, n\}$, so $\det A_4$ and $\det D_4$ can be arbitrary elements of $\mathbb{F}_q^\times$ (our condition $\det p = 1$ only requires $\det A_1 D_4 D_1 A_4 = 1$), so the condition never holds.
- Take $G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$. Then $A_4 = D_4^{-\intercal}$, so we are requiring $\beta(\det A_4)^2 = 1$. Because det is surjective when $r > 0$, nonzero r contribute in this case exactly when $\beta^2 = 1$.
- Take $G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$. Then $A_4 = m(p) D_4^{-\intercal}$, so we are requiring
$$\beta(\det A_4)^2 = \beta(m(p))^r.$$
With $r > 0$, the values $\det A_4$ and $m(p)$ are arbitrary elements of $\mathbb{F}_q^\times$, so we would like for $\beta(x)^2 = \beta(y)^r$ for any $x, y \in \mathbb{F}_q^\times$. Taking $y = 1$ shows that we will only get contributions in this case when $\beta^2 = 1$, and taking $x = 1$ shows that we will only get contributions when $\beta^r = 1$ too. However, with $\beta \neq 1$, we see that $\beta^r = 1$ only happens when r is even.

Tallying the above cases completes the proof. ■

Remark 2.3.2. *In the sequel, we will make frequent use of the basis of $f_\bullet$s of $\mathcal{H}$.*

Even though it is not currently relevant to our discussion, we will want a similar Mackey theory computation in the future, so we will get it out of the way now. We require a definition.

Definition 2.3.3. *Note that the element J normalizes M: for any $\left[\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right] \in M$, we see that $J^{-1}\left[\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right]J = \left[\begin{smallmatrix} D & \\ & A\end{smallmatrix}\right] \in M$. Thus, for any character $\chi\colon P \to \mathbb{C}^\times$, we define the character χ^J as the following composite.*

$$\begin{array}{ccccccc} P & \twoheadrightarrow & M & \overset{J}{\cong} & M & \overset{\chi}{\to} & \mathbb{C}^\times \\ \left[\begin{smallmatrix} A & B \\ & D\end{smallmatrix}\right] & \mapsto & \left[\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right] & \mapsto & \left[\begin{smallmatrix} D & \\ & A\end{smallmatrix}\right] & \mapsto & \chi\left(\left[\begin{smallmatrix} D & \\ & A\end{smallmatrix}\right]\right)\end{array}$$

Remark 2.3.4. *The importance of this definition arises in Lemma 2.4.1.*

Remark 2.3.5. *Write $\chi = (\alpha\circ m)(\beta\circ\chi_{\det})$. Then we will evaluate χ^J on $p := \left[\begin{smallmatrix} A & B \\ & D\end{smallmatrix}\right]$ as*

$$\chi^J\left(\begin{bmatrix} A & B \\ & D\end{bmatrix}\right) = \chi\left(\begin{bmatrix} D & \\ & A\end{bmatrix}\right).$$

If $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, then this is $\alpha(\det AD)\beta(\det A)^{-1} = (\alpha/\beta)(\det AD)\beta(\det D)$, so we see that $\chi^J = \left(\alpha\beta^{-1}\circ m\right)\left(\beta^{-1}\circ\chi_{\det}\right)$. Otherwise, if $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, then we rewrite p as $\left[\begin{smallmatrix} \lambda A & B \\ & A^{-\intercal}\end{smallmatrix}\right]$ so that we get $\chi^J(p) = \alpha(\lambda)\beta(\det\lambda A)^{-1}$, so $\chi^J = \left(\alpha\beta^{-n}\circ m\right)\left(\beta^{-1}\circ\chi_{\det}\right)$.

In particular, all cases find that $\left(\chi^J\right)^J = \chi$. Further, if $\beta = 1$, then $\chi^J = \chi$; alternatively, if we only have $\beta^2 = 1$ but $G \in \{\mathrm{SL}_{2n}, \mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$ so that $m = 1$, then we still have $\chi^J = \chi$.

Proposition 2.3.6. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha\circ m)(\beta\circ\chi_{\det})$. Then we compute a basis for $\left(\operatorname{Ind}_P^G\chi\right)^{\chi^J}$. In particular, we find*

$$\dim\left(\operatorname{Ind}_P^G\chi\right)^{\chi^J} = \dim\left(\operatorname{Ind}_P^G\chi\right)^{\chi}.$$

Proof. We proceed as in Proposition 2.3.1. For brevity, set $\mathcal{H}_J := \left(\operatorname{Ind}_P^G\chi\right)^{\chi^J}$. Again, $f \in \mathcal{H}_J$ is uniquely determined by its values on representatives of $P\backslash G/P$, so we set $f_r \in \mathcal{H}_J$ to be supported on $P\eta_r P$ defined by $f_r(\eta_r) \in \{0,1\}$ where we take $f_r(\eta_r) = 1$ whenever possible; thus, $\{f_r : f_r \neq 0\}$ is a basis of $\mathcal{H}_J$.

Continuing, as in Proposition 2.3.1, we are checking which $f_r \in \mathcal{H}_J$ are well-defined with $f_r(\eta_r) = 1$. Namely, for $p_1, p_2, p_1', p_2' \in P$, we must have $\chi(p_1)\chi^J(p_2) = f_r(p_1\eta_r p_2) = f_r(p_1'\eta_r p_2') = \chi(p_1')\chi^J(p_2)$. Rearranging, it is enough to check that $p_1 = \eta_r p_2\eta_r^{-1}$ implies that $\chi(p_1) = \chi^J(p_2)$. In other words, if $p \in P$ has $\eta_r p\eta_r^{-1} \in P$, we need $\chi(p) = \chi\left(\eta_r p\eta_r^{-1}\right)$. Writing

$$p := \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ & & D_1 & D_2 \\ & & D_3 & D_4\end{bmatrix}$$

to have the same dimensions as η_r, and then we see that

$$\eta_r p\eta_r^{-1} = \begin{bmatrix} A_1 & \varepsilon B_2 & B_1 & A_2 \\ & D_4 & \varepsilon D_3 & \\ & \varepsilon D_2 & D_1 & \\ A_3 & \varepsilon B_4 & B_3 & A_4\end{bmatrix},$$

so this is in P if and only if $A_3 = B_4 = D_2 = 0$. Thus, $\chi(p) = \chi^J\left(\eta_r p\eta_r^{-1}\right)$ is equivalent to always having

$$\chi\left(\begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ & A_4 & B_3 & \\ & & D_1 & \\ & & D_3 & D_4\end{bmatrix}\right) \overset{?}{=} \chi^J\left(\begin{bmatrix} A_1 & \varepsilon B_2 & B_1 & A_2 \\ & D_4 & \varepsilon D_3 & \\ & & D_1 & \\ & & B_3 & A_4\end{bmatrix}\right).$$

We now do casework on G and r.

- If $G \in \{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$, then $\chi^J = \beta^{-1}\circ\chi_{\det}$, so we are asking for

$$\beta(\det D_1\det D_4)^{-1} \overset{?}{=} \beta(\det D_1\det A_4).$$

In this case, $A_4 = D_4^{-\intercal}$, so we see that the above is equivalent to $\beta(\det D_1)^2 = 1$. If $r = n$, then we always get a contribution because D_1 is empty; otherwise, det is surjective, so we get contributions only when $\beta^2 = 1$.

- If $G = \mathrm{SL}_{2n}$, then $\chi^J = \beta^{-1} \circ \chi_{\det}$, so we are still asking for
$$\beta(\det D_1 \det D_4)^{-1} \stackrel{?}{=} \beta(\det D_1 \det A_4).$$
This simplifies to
$$\beta\left(\det A_4 \det D_1^2 \det D_4\right) = \beta(\det A_1)^{-1}\beta(\det D_1) \stackrel{?}{=} 1.$$
Now, if $r = n$, then A_1 and D_1 is empty, so we are asking for $\beta(\det A_4 D_4) = 1$, which is true because $\det A_4 D_4 = 1$. If $r = 0$, then we must have $\det A_1 D_1 = 1$, so we get a contribution provided $\beta^2 = 1$. Otherwise, with $r \notin \{0, n\}$, the determinants $\det A_1$ and $\det D_1$ are arbitrary, so we only get a contribution when $\beta = 1$.
- If $G = \mathrm{GL}_{2n}$, then $\chi^J = \left(\alpha\beta^{-1} \circ m\right)\left(\beta^{-1} \circ \chi_{\det}\right)$, so we are asking for
$$\alpha\left(\det A_1 D_1 \det A_4 D_4\right)\beta(\det D_1 \det D_4)^{-1} = \alpha\beta^{-1}\left(\det A_1 D_1 \det A_4 D_4\right)\beta(\det D_1 \det A_4).$$
Here, we see that α cancels on both sides, so we may ignore it. Rearranging, this is equivalent to
$$\beta(\det D_1) = \beta\left(\det A_1\right).$$
If $r = n$, then D_1 and A_4 are empty, so this condition holds. Otherwise, with $r < n$, these determinants are basically arbitrary, so we only get a contribution when $\beta = 1$.
- If $G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$, then $\chi^J = (\alpha\beta^{-n} \circ m)\left(\beta^{-1} \circ \chi_{\det}\right)$. Additionally, letting $\lambda \in \mathbb{F}_q^\times$ be the multiplier, we see that $A_4 = \lambda D_4^\intercal$, so we are asking for
$$\alpha(\lambda)\beta(\det D_1 \det D_4)^{-1} = \alpha\beta^{-n}(\lambda)\beta\left(\det D_1 \det \lambda D_4^{-\intercal}\right).$$
Once again, α cancels on both sides, so we ignore it. Now, this rearranges to
$$\beta(\det D_1)^{-2} = \beta^{r-n}(\lambda).$$
As usual, if $r = n$, then the right-hand side is 1, and the left-hand side is 1 because D_1 is empty, so we will get a contribution. Otherwise, det is surjective, so we will get a contribution only when $\beta^2 = 1$ and $\beta^{r-n} = 1$. So $\beta = 1$ is always okay, and the case where $\beta^2 = 1$ while $\beta \neq 1$ only takes f_r where $r \equiv n \pmod 2$.

Tallying the above cases and comparing with Proposition 2.3.1 completes the proof. ■

We now show that $\mathrm{Ind}_P^G \chi$ is multiplicity-free.

Proposition 2.3.7. *For any character $\chi\colon P \to \mathbb{C}^\times$, the representation $\mathrm{Ind}_P^G \chi$ is multiplicity-free.*

Proof. Write $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$, as usual. If $\beta^2 \neq 1$, then Proposition 2.3.1 tells us that $\mathrm{Ind}_P^G \chi$ is irreducible. It remains to handle the case where $\beta^2 = 1$. By [Bum13, Theorem 45.1], it suffices for the Hecke algebra $\mathcal{H}$ of functions
$$\mathcal{H} \coloneqq \{f \in \mathrm{Mor}(G, \mathbb{C}) : f(p_1 g p_2) = \chi(p_1)\chi(p_2)f(g) \text{ for } p_1, p_2 \in P, g \in G\},$$
with product given by the convolution, to be commutative. We will split this into two cases.

- Take $G = \mathrm{SL}_{2n}$ where $\chi \neq 1$. We will apply force. Here, $\alpha = 1$, so we still have $\chi^2 = 1$. Then the computation of Proposition 2.3.1 tells us that $\mathcal{H}$ has $\mathbb{C}$-basis by the functions $f_0, f_n\colon G \to \mathbb{C}$ where f_r is supported on $P\eta_r P$ with $f_r(\eta_r) = 1$. Thus, to check that $\mathcal{H}$ is commutative, it is enough to verify that $f_0 * f_n = f_n * f_0$. We will do this by explicit computation. Note that $f \in \mathcal{H}$ satisfies $f = f(\eta_0)f_0 + f(\eta_n)f_n$, so it is enough to check that
$$(f_0 * f_n)(\eta_r) \stackrel{?}{=} (f_n * f_0)(\eta_r)$$
for $r \in \{0, n\}$. We do this by explicit computation. For $\eta_0 = 1_{2n}$, we note
$$(f_0 * f_n)(\eta_0) = \sum_{h \in P\backslash G} f_0\left(h^{-1}\right) f_n(h).$$
Now, f_0 and f_n have disjoint supports, so this sums to 0. A symmetric argument shows $(f_n * f_0)(\eta_0) = 0$.

On the other hand, we see

$$(f_0 * f_n)(\eta_n) = \sum_{h \in P\backslash G} f_0\left(\eta_n h^{-1}\right) f_n(h) = \sum_{h \in P\backslash G} f_0\left(h^{-1}\right) f_n(h\eta_n).$$

Now, f_0 is supported on P, so the only nonzero term of the sum is at the identity coset of $P\backslash G$, so this evaluates to $f_n(\eta_n)$. A similar argument shows that

$$(f_n * f_0)(\eta_n) = \sum_{h \in P\backslash G} f_n\left(\eta_n h^{-1}\right) f_0(h)$$

equals $f_n(\eta_n)$ again, completing the proof.

- Take $G \in \{\mathrm{SL}_{2n}, \mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$, except the above case. Again, $\alpha = 1$, so $\chi^2 = 1$; namely, $\chi = \chi^{-1}$. We will apply an argument similar to [Bum13, Theorem 45.2]. In particular, define $\iota\colon G \to G$ by $\iota(g) := g^{-1}$. For example, we see that $\iota(\iota(g)) = g$ and $\iota(gh) = \iota(h)\iota(g)$ for $g \in G$ and $\chi(\iota(p)) = \chi(p)^{-1} = \chi(p)$ for $p \in P$, which can be checked directly. Thus, we may define an operator $(\cdot)^\iota\colon \mathcal{H} \to \mathcal{H}$ by

$$f^\iota(g) := f(\iota(g)).$$

To see that $f^\iota \in \mathcal{H}$, we simply must check that

$$f^\iota(p_1 g p_2) = f\left(p_2^{-1} g^{-1} p_1^{-1}\right) = \chi(\iota(p_2))\chi(\iota(p_1))f(\iota(g)) = \chi(p_1)\chi(p_2)f^\iota(g).$$

Now, $(\cdot)^\iota$ is of course $\mathbb{C}$-linear, and we claim that it is anti-commutative: for $f, f' \in \mathcal{H}$, we compute

$$(f * f')^\iota(g) = \sum_{h \in P\backslash G} f\left(g^{-1}h^{-1}\right) f'(h) = \sum_{h \in P\backslash G} f\left(hg^{-1}\right) f'\left(h^{-1}\right) = (f'^\iota * f^\iota)(g).$$

However, we claim that $(\cdot)^\iota$ is in fact the identity map on $\mathcal{H}$, from which it follows that $\mathcal{H}$ is commutative. Well, fix some $f \in \mathcal{H}$, and we must show that $f^\iota = f$. By Lemma 2.2.3, we see that f is uniquely determined by its values on the η_r for $r \in \{0, \ldots, n\}$, so it is enough to check that $f\left(\eta_r^{-1}\right) = f(\eta_r)$. Well,

$$\eta_r^{-1} = \begin{bmatrix} 1_{n-r} & & & \\ & & & 1_r \\ & & 1_{n-r} & \\ & \varepsilon 1_r & & \end{bmatrix} = \begin{bmatrix} 1_{n-r} & & & \\ & \varepsilon 1_r & & \\ & & 1_{n-r} & \\ & & & \varepsilon 1_r \end{bmatrix} \begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix},$$

so

$$f\left(\eta_r^{-1}\right) = \chi\left(\begin{bmatrix} 1_{n-r} & & & \\ & \varepsilon 1_r & & \\ & & 1_{n-r} & \\ & & & \varepsilon 1_r \end{bmatrix}\right) f(\eta_r).$$

If $\chi = 1$, then the extra factor here goes away, so we are safe. Otherwise, $\chi \neq 1$. Continuing, if $G = \mathrm{O}_{2n}$, then $\varepsilon = 1$, so the extra factor here still goes away. Otherwise, $\varepsilon = -1$. If $G = \mathrm{Sp}_{2n}$ with $\chi \neq 1$, then the proof of Proposition 2.3.1 tells us that we only have to pay attention to the case where r is even, and here the Siegel determinant of the matrix in question is 1, so we are okay. Lastly, we have dealt with the case where $G = \mathrm{SL}_{2n}$ and $\chi \neq 1$ in the previous point.
- Take $G \in \{\mathrm{GL}_{2n}, \mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$. Let $S := \ker m$ so that $S \in \{\mathrm{SL}_{2n}, \mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$. Now, let $\mathcal{H}^S$ denote the Hecke algebra corresponding to the group S and character χ^S (which is the restriction of χ), and we will set $\mathcal{H}^G := \mathcal{H}$ and $\chi^G := \chi$. (In short, χ^S basically ignores α.) We will show that $\mathcal{H}^S$ surjects onto $\mathcal{H}^G$, which shows that $\mathcal{H}_G$ is commutative.[1]

  For each $r \in \{0, \ldots, n\}$, let $f_r^G \in \mathcal{H}^G$ and $f_r^S \in \mathcal{H}^S$ denote the functions on the corresponding group supported on the double coset of η_r with $f_r^\bullet(\eta_r) = 1$ whenever possible. Then the set of nonzero $f_r^\bullet$ forms a basis of $\mathcal{H}^\bullet$ as discussed in the proof of Proposition 2.3.1. In fact, a careful reading of the computation in Proposition 2.3.1 shows that $f_r^G \neq 0$ implies that $f_r^S \neq 0$ for each r, so we may construct a $\mathbb{C}$-linear surjection $\pi\colon \mathcal{H}^S \to \mathcal{H}^G$ by sending $\pi\colon f_r^S \mapsto f_r^G$ (for $f_r^S \neq 0$, which is our basis).

[1] The argument of the previous point does not directly apply to this case because it requires $\chi^2 = 1$, which is not true when α is allowed to be general.

It remains to show that this map is multiplicative. Fix indices $r, s, t \in \{0, \ldots, n\}$ with $f_r^G, f_s^G, f_t^G \neq 0$, and we claim that

$$\left(f_r^G * f_s^G\right)(\eta_t) \stackrel{?}{=} \left(f_r^S * f_s^S\right)(\eta_t).$$

To see why this is enough, we let t vary to note that this would imply

$$\begin{aligned} f_r^G * f_s^G &= \sum_{t=0}^n \left(f_r^G * f_s^G\right)(\eta_t) f_t^G \\ &= \sum_{t=0}^n \left(f_r^S * f_s^S\right)(\eta_t) \pi\left(f_t^S\right) \\ &= \pi\left(\sum_{t=0}^n \left(f_r^S * f_s^S\right)(\eta_t) f_t^S\right) \\ &= \pi\left(f_r^S * f_s^S\right). \end{aligned}$$

It remains to show the claim. Expanding out the convolution, we are being asked to show that

$$\sum_{h \in P^G \backslash G} f_r^G\left(\eta_t h^{-1}\right) f_s^G(h) \stackrel{?}{=} \sum_{h \in P^S \backslash S} f_r^S\left(\eta_t h^{-1}\right) f_s^S(h),$$

where $P^G \subseteq G$ and $P^S \subseteq S$ are the Siegel parabolic subgroups. We will show that these two sums are equal term-wise.

For the claim that the two sums are equal term-wise, we quickly verify that the inclusion $S \subseteq G$ induces a bijection $P^S \backslash S \to P^G \backslash G$. Well, this map is certainly well-defined: if $s, s' \in S$ are in the same class (mod P^S), they will be in the same class (mod P^G). Continuing, this map is injective because having $s = ps'$ for $s, s' \in S$ and $p \in P^G$ implies $p \in P^G \cap S$, so s and s' are in the same class in $P^S \backslash S$. Lastly, to see that this map is surjective, we must show that any $g \in G$ can be written as ps where $p \in P^G$ and $s \in S$. Equivalently, we want to show that the composite $P^G \subseteq G \twoheadrightarrow G/S$ is surjective, but $m\colon G/S \to \mathbb{F}_q^\times$ is an isomorphism, so we want to show that $m\colon P^G \to \mathbb{F}_q^\times$ is surjective. This can be seen by explicit example in all cases of G.

We now show that our sums are equal "term-wise." In light of the previous paragraph, it is enough to show that

$$f_r^G\left(\eta_t h^{-1}\right) f_s^G(h) \stackrel{?}{=} f_r^S\left(\eta_t h^{-1}\right) f_s^S(h)$$

for any $h \in S$. We quickly claim that $f_s^G(h) \neq 0$ if and only if $f_s^S(h) \neq 0$, and an analogous argument is able to show that $f_r^G\left(\eta_t h^{-1}\right) \neq 0$ if and only if $f_r^S\left(\eta_t h^{-1}\right)$. Looking at the support of f_s^G and f_s^S, we see that we are basically trying to show $S \cap P^G \eta_s P^G = P^S \eta_s P^S$. Well, for concreteness, write $h := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then $h \in P^G \eta_s P^G$ if and only if $\operatorname{rank} C = s$ by the proof of Lemma 2.2.3, which is equivalent to $h \in P^S \eta_s P^S$, as required.

In light of the previous paragraph, we now may assume that $f_r^S\left(\eta_t h^{-1}\right) f_s^S(h) \neq 0$. Then $h = p_1 \eta_s p_2$ and $\eta_t h^{-1} = p_1' \eta_s p_2'$ for $p_1, p_2, p_1', p_2' \in P_S$. Then we see

$$f_r^G\left(\eta_t h^{-1}\right) f_s^G(h) = \chi^G(p_1' p_2' p_1 p_2) = \chi^S(p_1' p_2' p_1 p_2) = f_r^S\left(\eta_t h^{-1}\right) f_s^S(h),$$

as desired. ■

In the sequel, we will be interested in G-invariant operators on $\operatorname{Ind}_P^G \chi$, so it will be worth our time to provide a basis of sorts for this space. The main idea is as follows.

Lemma 2.3.8. *Fix a character $\chi\colon P \to \mathbb{C}^\times$. For each irreducible subrepresentation π of $\operatorname{Ind}_P^G \chi$, there exists exactly one dimension of χ-eigenvectors in π.*

Proof. We are being asked to show that $\dim \operatorname{Hom}_P\left(\chi, \operatorname{Res}_P^G \pi\right) = 1$. By Frobenius reciprocity, this is just

$$\dim \operatorname{Hom}_P\left(\chi, \operatorname{Res}_P^G \pi\right) = \dim \operatorname{Hom}_G\left(\pi, \operatorname{Ind}_P^G \chi\right).$$

Because π is an irreducible subrepresentation, this dimension is at least 1, but $\operatorname{Ind}_P^G \chi$ is multiplicity-free by Proposition 2.3.7, so this dimension is at most 1. This completes the proof. ■

Thus, we note that we can understand operators on $\operatorname{Ind}_P^G \chi$ by merely understanding where they send a vector from each irreducible subrepresentation. Each irreducible subrepresentation contributes a basis element to $\left(\operatorname{Ind}_P^G \chi\right)^\chi$, so we may just understand how the operator behaves on $\left(\operatorname{Ind}_P^G \chi\right)^\chi$. Now, $\left(\operatorname{Ind}_P^G \chi\right)^\chi$ is exactly the underlying vector space of the corresponding Hecke algebra $\mathcal{H}$, so the computation of Proposition 2.3.1 provides a basis for this space. Explicitly, we are told the set of $r \in \{0, \ldots, n\}$ such that we can define a nonzero basis vector $f_r \in \mathcal{H}$ supported on the double coset $P\eta_r P$ defined by $f_r(\eta_r) := 1$.

2.4. **The Intertwining Operator.** We are now ready to introduce the main character of our story, which is an operator I on the space $\operatorname{Mor}(G, \mathbb{C}) = \operatorname{Ind}^G 1$ defined by

$$(If)(g) := \sum_{u \in U} f\left(J^{-1}ug\right).$$

Note that $I\colon \operatorname{Ind}^G 1 \to \operatorname{Ind}^G 1$ is G-invariant. In more typical notation, I is the intertwining operator M_J, where we view J as representing a Weyl group element. The space $\operatorname{Ind}^G 1$ is a little too big to be useful, so we will work the spaces $\operatorname{Ind}_P^G \chi$ instead, where $\chi\colon P \to \mathbb{C}^\times$ is some character. As such, we should explain that I behaves nicely on these spaces.

Lemma 2.4.1. *Fix a character $\chi\colon P \to \mathbb{C}^\times$. Then I restricts to a G-invariant map $\operatorname{Ind}_P^G \chi \to \operatorname{Ind}_P^G \chi^J$.*

Proof. We already know that I is G-invariant, so the main point is to check that $If \in \operatorname{Ind}_P^G \chi^J$. Namely, for any $p \in P$ and $g \in G$, we must show that $If(pg) = \chi^J(p)If(g)$. We may decompose p as $p = d_p u_p$ where $u_p \in U$ and $d_p \in M$. Because the sum in If is U-invariant, we see that $If(d_p u_p g) = If(d_p g)$, and we know $\chi^J(u_p) = 1$ by its construction. Thus, we may safely ignore u_p. As for d_p, we write

$$If(d_p g) = \sum_{u \in U} f\left(J^{-1}d_pJ \cdot J^{-1}d_p^{-1}ud_pg\right) = \sum_{u \in U} \chi\left(J^{-1}d_pJ\right) f(J^{-1}ug) = \chi^J(d_p)If(g),$$

as required. ■

Our end goal is to understand the linear transformation $I\colon \operatorname{Ind}_P^G \chi \to \operatorname{Ind}_P^G \chi^J$. Some aspects of this operator are not so hard to see: I is invertible because of its description as the (restriction of the) intertwining operator $M_J\colon \operatorname{Ind}_B^G \chi \to \operatorname{Ind}_B^G \chi^J$ (where $B \subseteq P$ is a suitable Borel subgroup containing T), which is known to be invertible.

Another way we would like to understand I is to expand I out as a matrix using the bases of Lemma 2.3.8, which we see makes I into a linear transformation

$$\left(\operatorname{Ind}_P^G \chi\right)^\chi \to \left(\operatorname{Ind}_P^G \chi^J\right)^\chi,$$

both of which have explicit bases by the computations of Propositions 2.3.1 and 2.3.6. Lastly, we will want to understand I is through its eigenvalues. Notably, $\operatorname{Ind}_P^G \chi$ is multiplicity-free by Proposition 2.3.7, so I must be diagonalizable; we will be able to see this explicitly. In general, $\chi \neq \chi^J$ while $\left(\chi^J\right)^J = \chi$, so only $I \circ I$ will be an operator (on $\operatorname{Ind}_P^G \chi$) that could possibly have eigenvalues. Thus, to compose matrix representations, we will also want to expand I as a linear transformation

$$\left(\operatorname{Ind}_P^G \chi^J\right)^\chi \to \left(\operatorname{Ind}_P^G \chi\right)^\chi,$$

where we again have explicit bases.

In this section, we will not build enough tools to precisely describe the eigenvalues of I, but we will be able to provide the matrix representations. We begin with the easier generic case.

Proposition 2.4.2. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$. Suppose $\beta^2 \neq 1$. Then let $\{f_0\}$ be the basis of $\left(\operatorname{Ind}_P^G \chi\right)^\chi$ described in Proposition 2.3.1, and let $\left\{f_n^J\right\}$ be the basis of $\left(\operatorname{Ind}_P^G \chi^J\right)^\chi$ described in Proposition 2.3.6. Then*

$$If_0 = f_n^J \qquad \text{and} \qquad If_n^J = \beta(\varepsilon)^n\, |U|\, f_0.$$

In particular, $I \circ I$ is the scalar $\beta(\varepsilon)^n\, |U|$.

Proof. Certainly $If_0 \in \operatorname{span}\left\{f_n^J\right\}$ and $If_n^J \in \operatorname{span}\{f_0\}$ because we have given bases. Anyway, we do our computations separately.

For If_0, we see that $If_0 = If_0(\eta_n)f_n$, so we want to compute

$$If_0(\eta_n) = \sum_{u\in U} f_0\left(J^{-1}u\eta_n\right).$$

To compute the sum, we see $\eta_n = J = \left[\begin{smallmatrix} & \varepsilon 1_n \\ 1_n & \end{smallmatrix}\right]$, so we write $u := \left[\begin{smallmatrix} 1 & B \\ & 1\end{smallmatrix}\right]$ so that $J^{-1}uJ = \left[\begin{smallmatrix} 1 & \\ \varepsilon B & 1\end{smallmatrix}\right]$. Now, f_0 is nonzero $J^{-1}uJ$ only when $J^{-1}uJ \in P$, which we see only happens when $B = 0$ so that $u = 1_{2n}$ and $J^{-1}uJ = 1_{2n}$. Thus, $If_0(\eta_n) = 1$, as required.

For If_n^J, we see that $If_n^J = If_n^J(\eta_0)f_0$, so we want to compute

$$If_n^J(\eta_0) = \sum_{u\in U} f_n^J\left(J^{-1}u\eta_0\right) = \sum_{u\in U}\chi(u)f_n^J\left(J^{-1}\right) = |U|\, f_n^J\left(J^{-1}\right).$$

Now, $J^{-1} = \left[\begin{smallmatrix} & 1_n \\ \varepsilon 1_n & \end{smallmatrix}\right] = \varepsilon J$, so $f_n^J\left(J^{-1}\right) = \chi(\varepsilon 1_{2n})$. Plugging in for χ completes the proof. ■

We now turn towards the case $\beta^2 = 1$. We begin by stating a general result which will help us with our subsequent computations.

Lemma 2.4.3. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha\circ m)(\beta\circ\chi_{\det})$. Given $r, s \in \{0,\ldots,n\}$*
Check sign *where the usual basis vector $f_r \in \left(\operatorname{Ind}_P^G \chi\right)^\chi$ (of Proposition 2.3.1) is nonzero, we compute*

$$If_r(\eta_s) = \beta(\varepsilon)^{n-s} Q \sum_{\substack{D\in\mathbb{F}_q^{s\times s} \\ \left[\begin{smallmatrix} 1_n & \operatorname{diag}(D,0_{n-s}) \\ & 1_n\end{smallmatrix}\right]\in G \\ \operatorname{rank} D = r+s-n}} \beta(\det E)^{-1},$$

where

$$Q := \begin{cases} q^{n^2-s^2} & \text{if } G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}, \\ q^{\binom{n}{2}-\binom{s}{2}} & \text{if } G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}, \\ q^{\binom{n+1}{2}-\binom{s+1}{2}} & \text{if } G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}, \end{cases}$$

and $E \in \mathrm{GL}_{r+s-n}(\mathbb{F}_q)$ is some matrix defined from D determined as follows: we always have $\left[\begin{smallmatrix} E & \\ & 0\end{smallmatrix}\right] = D_1DD_2$ for $D_1, D_2 \in \mathrm{SL}_s(\mathbb{F}_q)$. Then if $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, we require $D_2 = D_1^\intercal$. Lastly, if $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$, we require $E = \left[\begin{smallmatrix} & -1_{(r+s-n)/2} \\ 1_{(r+s-n)/2} & \end{smallmatrix}\right]$; and if $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, we require E to be diagonal.

Proof. We are asked to compute $If_r(\eta_s) = \sum_{u\in U} f_r\left(J^{-1}u\eta_s\right)$. For this, we want to compute $J^{-1}u\eta_s$ and in particular want to ask when it lives in $P\eta_rP$. As such, we write u in a block matrix form and compute

$$\begin{aligned} J^{-1}u\eta_s &= \begin{bmatrix} & & 1_{n-s} & \\ & & & 1_s \\ \varepsilon 1_{n-s} & & & \\ & \varepsilon 1_s & & \end{bmatrix} \begin{bmatrix} 1_{n-s} & & A & B \\ & 1_s & C & D \\ & & 1_{n-s} & \\ & & & 1_s \end{bmatrix} \begin{bmatrix} 1_{n-s} & & & \\ & & & \varepsilon 1_s \\ & & 1_{n-s} & \\ & 1_s & & \end{bmatrix} \\ &= \varepsilon \begin{bmatrix} & & \varepsilon 1_{n-s} & \\ & \varepsilon 1_s & & \\ 1_{n-s} & B & A & \\ & D & C & \varepsilon 1_s \end{bmatrix} \\ &= \varepsilon \begin{bmatrix} & & \varepsilon 1_{n-s} & \\ & \varepsilon 1_s & & \\ 1_{n-s} & & & \\ & D & & \varepsilon 1_s \end{bmatrix} \begin{bmatrix} 1_{n-s} & B & A & \\ & 1_s & & \\ & & 1_{n-s} & \\ & & \varepsilon C & 1_s \end{bmatrix}. \end{aligned}$$

Now, χ vanishes on the last rightmost matrix (it has Siegel determinant and multiplier both equal to 1, so Lemma 2.1.1 finishes), so to compute $f_r\left(J^{-1}u\eta_s\right)$, we see that A, B, C in u do not matter, so

$$If_r(\eta_s) = Q \sum_{D\in\mathbb{F}_q^{s\times s}} f_r\left(\begin{bmatrix} & & 1_{n-s} & \\ & 1_s & & \\ \varepsilon 1_{n-s} & & & \\ & \varepsilon D & & 1_s \end{bmatrix}\right).$$

(Here, Q counts the number of ways to choose A, B, C.) Of course, we may replace D with εD, so in fact this sum is

$$If_r(\eta_s) = Q \sum_{D\in\mathbb{F}_q^{s\times s}} f_r\left(\begin{bmatrix} & & 1_{n-s} & \\ & 1_s & & \\ \varepsilon 1_{n-s} & & & \\ & D & & 1_s \end{bmatrix}\right).$$

Now, f_r is supported on $P\eta_r P$, so by Lemma 2.2.3, we see that D gives a nonzero contribution if and only if $\operatorname{rank}\left[\begin{smallmatrix} 1_{n-s} & \\ & D \end{smallmatrix}\right] = r$, which is equivalent to requiring $\operatorname{rank} D = r + s - n$, which we will assume from now on. Set $d \coloneqq \operatorname{rank} D$ for brevity.

We now place D into a normal form. This requires some casework on G.

- If $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, then we use row-reduction to find matrices $D_1, D_2 \in \mathrm{SL}_s(\mathbb{F}_q)$ such that $D_1 D D_2$ takes the form $\left[\begin{smallmatrix} E & \\ & 0 \end{smallmatrix}\right]$ where $E \in \mathrm{GL}_d(\mathbb{F}_q)$. (The exact choice of D_1 and D_2 will not matter.)
- If $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$ so that D is symmetric, finding an orthogonal basis grants $D_1 \in \mathrm{SL}_s(\mathbb{F}_q)$ such that $D_2 \coloneqq D_1^\intercal$ has $D_1 D D_2 = \left[\begin{smallmatrix} E & \\ & 0 \end{smallmatrix}\right]$ where $E \in \mathrm{GL}_d(\mathbb{F}_q)$ is diagonal.
- If $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$ so that D is alternating, theory about alternating forms grants a normal form with respect to a chosen basis (namely, we can make D into the standard symplectic form together with some maximal isotropic subspace). As before, we are really being granted $D_1 \in \mathrm{SL}_s(\mathbb{F}_q)$ such that $D_2 \coloneqq D_1^\intercal$ has $D_1 D D_2 = \left[\begin{smallmatrix} E & \\ & 0 \end{smallmatrix}\right]$ where $E = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] \in \mathbb{F}_q^{d\times d}$.

Then we see that the matrix of interest is

$$\underbrace{\begin{bmatrix} 1_{n-s} & & & \\ & D_2 & & \\ & & 1_{n-s} & \\ & & & D_1^{-1} \end{bmatrix}}_{\in G} \begin{bmatrix} & & 1_{n-s} & \\ & 1_s & & \\ \varepsilon 1_{n-s} & & & \\ & D_1 D D_2 & & 1_s \end{bmatrix} \underbrace{\begin{bmatrix} 1_{n-s} & & & \\ & D_2^{-1} & & \\ & & 1_{n-s} & \\ & & & D_1 \end{bmatrix}}_{\in G},$$

reducing ourselves from D to $D_1 D D_2 = \left[\begin{smallmatrix} E & \\ & 0 \end{smallmatrix}\right]$: once again, χ will be trivial on the left and right matrices, so they do not matter for the computation for f_r. We now note that $\left[\begin{smallmatrix} 1 & \\ E & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -\varepsilon E^{-1} & 1 \\ & E \end{smallmatrix}\right]\left[\begin{smallmatrix} & \varepsilon 1_d \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & E^{-1} \\ & 1 \end{smallmatrix}\right]$, so

$$\begin{bmatrix} & & & 1_{n-s} & & \\ & 1_d & & & & \\ & & 1_{n-r} & & & \\ \varepsilon 1_{n-s} & & & & & \\ & E & & & 1_d & \\ & & 0_{n-r} & & & 1_{n-r} \end{bmatrix}$$

equals

$$\begin{bmatrix} 1_{n-s} & & & & & \\ & -\varepsilon E^{-1} & & & 1_d & \\ & & 1_{n-r} & & & \\ & & & 1_{n-s} & & \\ & & & & E & \\ & & & & & 1_{n-r} \end{bmatrix} \begin{bmatrix} & & & 1_{n-s} & & \\ & & & & \varepsilon 1_d & \\ & & 1_{n-r} & & & \\ \varepsilon 1_{n-s} & & & & & \\ & 1_d & & & & \\ & & 0_{n-r} & & & 1_{n-r} \end{bmatrix} \begin{bmatrix} 1_{n-s} & & & & & \\ & 1_d & & & E^{-1} & \\ & & 1_{n-r} & & & \\ & & & 1_{n-s} & & \\ & & & & 1_d & \\ & & & & & 1_{n-r} \end{bmatrix}.$$

Quickly, we note that the right matrix is in U (in all cases for G, essentially by construction of E), and so it is trivial under χ. We also note the middle matrix is in G, so the left matrix is in G too, and we can see visually that it is in P. Note that it is possible that the left matrix is nontrivial when passed through χ; in fact, we will receive a contribution of $\beta(\det E)^{-1}$. (The multiplier is still trivial.)

To compute the contribution of this element, it remains to transform the middle matrix into η_r. To begin, we note that this middle matrix equals

$$\begin{bmatrix} & & & \varepsilon 1_{n-s} & & \\ & & & & \varepsilon 1_d & \\ & & 1_{n-r} & & & \\ 1_{n-s} & & & & & \\ & 1_d & & & & \\ & & & & & 1_{n-r} \end{bmatrix} \begin{bmatrix} \varepsilon 1_{n-s} & & & & & \\ & 1_d & & & & \\ & & 1_{n-r} & & & \\ & & & \varepsilon 1_{n-s} & & \\ & & & & 1_d & \\ & & & & & 1_{n-r} \end{bmatrix},$$

so this right matrix produces a contribution of $\beta(\varepsilon)^{n-s}$. Lastly, the left matrix equals

$$\begin{bmatrix} & & \varepsilon 1_r & \\ & 1_{n-r} & & \\ 1_r & & & \\ & & & 1_{n-r} \end{bmatrix} = \begin{bmatrix} & 1_r & & \\ 1_{n-r} & & & \\ & & & 1_r \\ & & 1_{n-r} & \end{bmatrix} \underbrace{\begin{bmatrix} 1_{n-r} & & & \\ & & & \varepsilon 1_r \\ & & 1_{n-r} & \\ & 1_r & & \end{bmatrix}}_{\eta_r} \begin{bmatrix} & 1_{n-r} & & \\ 1_r & & & \\ & & & 1_{n-r} \\ & & 1_r & \end{bmatrix}.$$

The left and right matrices are inverses of each other, so their contributions via χ will cancel out. Totaling our contributions from this $u \in U$, we achieve $\beta(\varepsilon)^{n-s}\beta(\det E)^{-1}$; summing over all u completes the proof. ■

Remark 2.4.4. *Consider $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$ with $\beta \neq 1$. In this case, we can see that the value of $\det E$, even up to squares, fails to be well-defined given D for most values of r and s, so the sum doesn't even make sense! This corresponds to the fact that we tend to have $f_r = 0$ for most r. A similar phenomenon can be seen for the other groups.*

Remark 2.4.5. *We explain why we will essentially ignore what happens if we want to compute $If_r^J(\eta_s)$, where $f_r^J \in \left(\mathrm{Ind}_P^G \chi^J\right)^\chi$ is the usual basis vector (of Proposition 2.3.6). If $\beta^2 \neq 1$, then we appeal to Proposition 2.4.2. Otherwise, $\beta^2 = 1$. Now, one could compute as in the above proof, but observe that the entire proof only works with matrices with multiplier 1, so there is no real chance for α to have any effect. But then χ and χ^J are equal on matrices with multiplier 1 because $\beta = \beta^{-1}$, so we may as well have $f_r = f_r^J$! In particular, the answer is the same!*

We are now in a position to write down some matrices when $\beta^2 = 1$. We begin with the case where $\beta = 1$ because it is a little simpler.

Proposition 2.4.6. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$. Assume that $\beta = 1$ so that $\chi = \chi^J$. Give $\left(\mathrm{Ind}_P^G \chi\right)^\chi$ the standard basis $\{f_0, \ldots, f_n\}$ of Proposition 2.3.1. For each $r, s \in \{0, \ldots, n\}$, define*

$$I(r,s) := \begin{cases} (-1)^{r+s-n} \dfrac{(q;q)_s^2}{(q;q)_{n-r}^2 (q;q)_{r+s-n}} q^{n^2 - s^2 + \binom{r+s-n}{2}} & \text{if } G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}, \\ (-1)^{(r+s-n)/2} \dfrac{(q;q)_s}{(q;q)_{n-r}(q^2;q^2)_{(r+s-n)/2}} q^{\binom{n-1}{2} - \binom{s-1}{2} + 2\binom{(r+s-n)/2}{2}} & \text{if } G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}, \\ (-1)^{r+s-n-\lfloor (r+s-n)/2 \rfloor} \dfrac{(q;q)_s}{(q;q)_{n-r}(q^2;q^2)_{\lfloor (r+s-n)/2 \rfloor}} q^{\binom{n+1}{2} - \binom{s+1}{2} + 2\binom{\lfloor (r+s-n)/2 \rfloor}{2}} & \text{if } G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}, \end{cases}$$

where we implicitly take $I(r,s) = 0$ unless $r + s - n$ is nonnegative (and $I(r,s) = 0$ unless $r + s - n$ is also even when $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$). Then $[I(r,s)]_{0 \le s, r \le n}$ is the matrix representation of I.

Proof. We use Lemma 2.4.3, where the point is that $\beta = 1$, so we are now just counting the number of possible D. Namely, the (s,r) matrix coefficient is given by $If_r(\eta_s)$ because f_r is the rth basis vector of the source, and plugging in η_s detects the value of the coefficient for the sth target basis vector f_s. We now do our casework.

- Take $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$. Then we are counting matrices $D \in \mathbb{F}_q^{s \times s}$ of rank $r + s - n$, which is

$$q^{\binom{r+s-n}{2}} \cdot \frac{(q;q)_s^2}{(q;q)_{n-r}^2} \cdot \frac{(-1)^{r+s-n}}{(q;q)_{r+s-n}}$$

  by [HJ20, Theorem 7.1.5]. Plugging this in and simplifying completes the computation.

- Take $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$. Then we are counting alternating matrices $D \in \mathbb{F}_q^{s\times s}$ of rank $r+s-n$, which [HJ20, Theorem 7.5.5] explains equals
$$q^{2\binom{(r+s-n)/2}{2}} \cdot \frac{1}{(-1)^{(r+s-n)/2}(q^2;q^2)_{r+s-n}} \cdot \frac{(-1)^s(q;q)_s}{(-1)^{n-r}(q;q)_{n-r}}$$
provided that $r+s-n$ is even and nonnegative (else is 0). Plugging this in and simplifying completes.
- Take $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$. Then we are counting symmetric matrices $D \in \mathbb{F}_q^{s\times s}$ of rank $r+s-n$, which [HJ20, Theorem 7.5.2] explains equals
$$q^{2\binom{\lfloor (r+s-n)/2\rfloor}{2}} \cdot \frac{1}{(-1)^{\lfloor (r+s-n)/2\rfloor}(q^2;q^2)_{\lfloor (r+s-n)/2\rfloor}} \cdot \frac{(-1)^s(q;q)_s}{(-1)^{n-r}(q;q)_{n-r}}.$$
Plugging this in and simplifying completes. ■

Lastly, we address the case where $\beta^2 = 1$ but $\beta \neq 1$. Again, because it is simplest, we handle $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$ first.

Proposition 2.4.7. *Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha\circ m)(\beta\circ\chi_{\det})$. Assume that $\beta^2 = 1$ but $\beta \neq 1$.*

- *If $G = \mathrm{GL}_{2n}$, then we give $\left(\operatorname{Ind}_P^G \chi\right)^\chi$ the basis $\{f_0\}$ of Proposition 2.3.1 and give $\left(\operatorname{Ind}_P^G \chi^J\right)^\chi$ the basis $\{f_n^J\}$ of Proposition 2.3.6. Then we compute*
$$If_0 = f_n^J \qquad \text{and} \qquad If_n^J = \beta(-1)^n q^{n^2} f_0.$$
- *If $G = \mathrm{SL}_{2n}$, then we see $\chi = \chi^J$, so we give $\left(\operatorname{Ind}_P^G \chi\right)^\chi$ the basis $\{f_0, f_n\}$ of Proposition 2.3.1. Then I has the matrix representation*
$$\begin{bmatrix} & \beta(-1)q^{n^2} \\ 1 & \end{bmatrix}.$$

Proof. It suffices to compute $If_0 \in \operatorname{span}\left\{f_0^J, f_n^J\right\}$ and $If_n^J \in \operatorname{span}\{f_0, f_n\}$ in both cases. Note $If_0(\eta_0) = 0$ because the sum in Lemma 2.4.3 vanishes; as a small trick, the fact that f_n vanishes for $G = \mathrm{GL}_{2n}$, we see that the sum for $If_n^J(\eta_n)$ must vanish. (One could also compute this sum by hand in the case $G = \mathrm{SL}_{2n}$: it is $\sum_{D\in\mathrm{GL}_s(\mathbb{F}_q)} \beta(\det D)$, which vanishes because $\beta\circ\det$ is a nontrivial character on the group $\mathrm{GL}_s(\mathbb{F}_q)$.)

So it remains to compute $If_0 = If_0(\eta_n)f_n^J$ and $If_n^J = If_n^J(\eta_0)f_0$, which are the expected computations from the $G = \mathrm{GL}_{2n}$ case. We handle the computations separately.

- We compute $If_0(\eta_n)$. Here, Lemma 2.4.3 wants us to sum $D \in \mathbb{F}_q^{n\times n}$ of rank 0. As such, we are only looking at $D = 0$, for which the sum returns 1. Because $\beta(\varepsilon)^{n-n} = Q = 1$ too, we see $If_0(\eta_n) = 1$.
- We compute $If_n^J(\eta_0)$. We may still use Lemma 2.4.3 for our computation by Remark 2.4.5. This time, we are summing $D \in \mathbb{F}_q^{0\times 0}$ of rank 0, of which there is exactly one invertible matrix of determinant 1 (by convention), so the sum still returns 1. Thus, we find $If_n^J(\eta_0) = \beta(\varepsilon)^n q^{n^2}$.

Using the above computations to build matrices completes the proof. ■

Remark 2.4.8. *Proposition 2.4.7 provides our first nontrivial example where we can see that I is diagonalizable. Namely, for $G = \mathrm{SL}_{2n}$, the characteristic polynomial of I is $X^2 - \beta(-1)q^{n^2}$, so I has distinct eigenvalues $\pm\sqrt{\beta(-1)}q^{n^2/2}$.*

It remains to handle $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$. Though it was a little troublesome for $G = \mathrm{GL}_{2n}$, this will be the first time when we will really have to deal with the complication $\chi \neq \chi^J$ for $G \in \{\mathrm{GL}_{2n}, \mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$. Luckily, our bases are well-behaved enough so that this is not really a problem.

Lemma 2.4.9. *Take $G \in \{\mathrm{GO}_{2n}, \mathrm{GSp}_{2n}\}$ so that $S \coloneqq \ker m$ is in $\{\mathrm{O}_{2n}, \mathrm{Sp}_{2n}\}$. Fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha\circ m)(\beta\circ\chi_{\det})$. Assume that $\beta^2 = 1$ but $\beta \neq 1$. Let $\left\{f_r^G\right\}_{r\equiv_2 0}$ be the usual basis of $\left(\operatorname{Ind}_P^G \chi\right)^\chi$, let $\left\{f_r^{JG}\right\}_{r\equiv_2 n}$ be the usual basis of $\left(\operatorname{Ind}_P^G \chi^J\right)^\chi$, and define f_r^S similarly for $\chi|_S$. Let $\left[I^S(r,s)\right]_{0\le s,r\le n}$ be the matrix representation of I on $\left(\operatorname{Ind}_P^G \chi\right)^\chi$.*

- *The matrix representation of* $I\colon \left(\operatorname{Ind}_G^P \chi\right)^\chi \to \left(\operatorname{Ind}_G^P \chi^J\right)^\chi$ *is*

$$\left[I^S(r,s)\right]_{\substack{0\le s,r\le n\\ r\equiv_2 0, s\equiv_2 n}}.$$

- *The matrix representation of* $I\colon \left(\operatorname{Ind}_G^P \chi^J\right)^\chi \to \left(\operatorname{Ind}_G^P \chi\right)^\chi$ *is*

$$\left[I^S(r,s)\right]_{\substack{0\le s,r\le n\\ r\equiv_2 n, s\equiv_2 0}}.$$

Proof. The proofs of the two points are essentially the same (though one should use Remark 2.4.5 for the second point), so we focus on the first one. As in Proposition 2.4.6, we see that the (s,r) coefficient of I will be the coefficient of f_s^J in If_r. So we note that the computation of Lemma 2.4.3 explains that the value of $If_r(\eta_s)$ is the same no matter if we work with G or S, so our answer is $I^S(r,s)$, as required. ■

Remark 2.4.10. *Similarly, one can see that the matrix representation of I acting on the spaces $\left(\operatorname{Ind}_G^P \chi^J\right)^{\chi^J}$ and $\left(\operatorname{Ind}_G^P \chi\right)^{\chi^J}$ are the corresponding submatrices of $\left[I^S(r,s)\right]_{0\le s,r\le n}$.*

Remark 2.4.11. *One can see a version of the above result still hold for $\mathrm{SL}_{2n}\subseteq\mathrm{GL}_{2n}$ in the sense that the matrices for GL_{2n} are submatrices of SL_{2n} essentially dictated by which basis vectors are in play.*

Remark 2.4.12. *Here is a cute application of the above result, akin to the argument that $If_n^J(\eta_n)=0$ in Proposition 2.4.7. Using the notation of the lemma above, we will show that $I^S(r,s)=0$ if $r+s\not\equiv n \pmod 2$ and one of r or $n-r$ is even. In the case that r is even, we simply note that f_r^G is nonzero, so Lemma 2.4.3 informs us that*

$$I^S(r,s)=I^S f_r^S(\eta_s)=I^G f_r^G(\eta_s)=0,$$

where the last equality holds because f_s^{JG} is zero. In the case that $n-r$ is even, we run the same argument but replace f_r^G with $f_r^{JG}\ne 0$ and replace f_s^{JG} with f_s.

Lemma 2.4.9 tells us that we can essentially focus on $G\in\{\mathrm{O}_{2n},\mathrm{Sp}_{2n}\}$. For example, the eigenvalues of $I\circ I$ in the case that $G\in\{\mathrm{GO}_{2n},\mathrm{GSp}_{2n}\}$ can be read off the eigenvalues by taking suitable submatrices everywhere. Anyway, we now handle $G\in\{\mathrm{O}_{2n},\mathrm{Sp}_{2n}\}$. The approaches are rather different, so we will handle them separately.

Proposition 2.4.13. *Take $G=\mathrm{O}_{2n}$. Fix a character $\chi\colon P\to\mathbb{C}^\times$, which we write as $\chi=\beta\circ\chi_{\det}$. Assume that $\beta^2=1$ but $\beta\ne 1$ so that $\chi=\chi^J$. Let $\{f_0,\ldots,f_n\}$ be the standard basis of $\left(\operatorname{Ind}_P^G \chi\right)^\chi$. Then the matrix representation of I is*

$$\left[(-1)^{(r+s-n)/2}q^{\binom{n-1}{2}-\binom{s-1}{2}+2\binom{(r+s-n)/2}{2}}\frac{(q;q)_s}{(q;q)_{n-r}(q^2;q^2)_{(r+s-n)/2}}\right]_{0\le s,r\le n},$$

where the coefficient implicitly vanishes unless $r+s-n$ is a nonnegative even integer.

Proof. Arguing as in Proposition 2.4.6, we use Lemma 2.4.3. In Proposition 2.4.6, evaluating the sum in Lemma 2.4.3 was relatively easy because β was trivial, implying $\beta(\det E)^{-1}=1$ always. But in the case that $G=\mathrm{O}_{2n}$, we see that $\det E=1$ by definition of E, so we still have $\beta(\det E)^{-1}=1$. Thus, the argument of Proposition 2.4.6 goes through, with the caveat that we must consider the sign $\beta(\varepsilon)^{n-s}=\beta(1)^{n-s}=1$ which occurs when evaluating $If_r(\eta_s)$. ■

Proposition 2.4.14. *Take $G=\mathrm{Sp}_{2n}$. Fix a character $\chi\colon P\to\mathbb{C}^\times$, which we write as $\chi=\beta\circ\chi_{\det}$. Assume that $\beta^2=1$ but $\beta\ne 1$ so that $\chi=\chi^J$. Let $\{f_0,\ldots,f_n\}$ be the standard basis of $\left(\operatorname{Ind}_P^G \chi\right)^\chi$. If $\beta(-1)=1$, then the matrix representation of I is*

$$\left[\beta(-1)^{n-s+(r+s-n)/2}(-1)^{(r+s-n)/2}q^{\binom{n+1}{2}-\binom{s+1}{2}+2\binom{(r+s-n)/2}{2}-(r+s-n)/2}\frac{(q;q)_s}{(q;q)_{n-r}(q^2;q^2)_{(r+s-n)/2}}\right]_{0\le s,r\le n},$$

where the coefficient implicitly vanishes unless $r+s-n$ is a nonnegative even integer.

Proof. Arguing as in Proposition 2.4.6, we use Lemma 2.4.3. In Proposition 2.4.6, evaluating the sum in Lemma 2.4.3 was relatively easy because β was trivial. This time around, we see that we are taking the difference between the number of symmetric $D \in \mathbb{F}_q^{s\times s}$ of rank $r+s-n$ with $\det E \in \mathbb{F}_q^{\times 2}$ and the number of D with $\det E \notin \mathbb{F}_q^{\times 2}$. The formulae of [Mac69] tell us that the number of such D with $\det E \in \mathbb{F}_q^{\times 2}$ is

$$\begin{cases} \frac{1}{2}N & \text{if } r+s-n \equiv 1 \pmod 2, \\ \frac{1}{2}N \cdot \frac{q^{(r+s-n)/2}+\beta(-1)^{(r+s-n)/2}}{q^{(r+s-n)/2}} & \text{if } r+s-n \equiv 0 \pmod 2, \end{cases}$$

where N is the total number of symmetric matrices $D \in \mathbb{F}_q^{s\times s}$ of rank $r+s-n$. Thus, we see that the desired difference is 0 if $r+s-n$ is odd and $\beta(-1)^{(r+s-n)/2}q^{-(r+s-n)/2}N$ otherwise. Arguing as in Proposition 2.4.6 to extract the desired matrix coefficient from Lemma 2.4.3 completes the proof. ■

2.5. **A Multiplicity One Result.** Our understanding of I so far has relied on eigenvectors of $\operatorname{Ind}_P^G \chi$ with eigenvalue χ (or χ^J). The eigenvector we are interested in constructing will be an eigenvector for the smaller subgroup $U \subseteq P$. As such, we are interested in building some characters for U.

Definition 2.5.1. *Fix $T \in \mathbb{F}_q^{n\times n}$ and a character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Then we define the character $\psi_T\colon U \to \mathbb{C}$ by*

$$\psi_T\left(\begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix}\right) := \psi(\operatorname{tr} BT).$$

We are interested in ψ_T-eigenvectors of $\operatorname{Ind}_P^G \chi$. As such, we begin by showing that such eigenvectors exist.

Example 2.5.2. *Fix $T \in \mathbb{F}_q^{n\times n}$ and a character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Given a character $\chi\colon P \to \mathbb{C}^\times$, define $f_T \in \operatorname{Ind}_G^P \chi$ to be supported on $P\eta_n P$ and defined by*

$$f_{\chi,T}(p\eta_n u) := \chi(p)\psi_T(u).$$

This will be a perfectly fine definition of a nonzero element of $\left(\operatorname{Ind}_G^P \chi\right)^{\psi_T}$ as soon as we can show that any $g \in P\eta_n P$ can be uniquely expressed as $p\eta_n u$ where $p \in P$ and $u \in U$. On one hand, recall that $\eta_n = J$ normalizes M (see Definition 2.3.3), so

$$P\eta_n P = P\eta_n MU = P\eta_n M\eta_n^{-1}\eta_n U = PM\eta_n U = P\eta_n U,$$

so any $g \in P\eta_n P$ can be expressed as $p\eta_n u$ for $p \in P$ and $u \in U$. It remains to show that this expression is unique: if $p_1\eta_n u_1 = p_2\eta_n u_2$, then $\eta_n u\eta_n^{-1} \in P$ for $u := u_2u_1^{-1}$, so writing $u := \left[\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right]$, we see

$$\begin{bmatrix} & \varepsilon 1_n \\ 1_n & \end{bmatrix}\begin{bmatrix} 1_n & B \\ & 1_n \end{bmatrix}\begin{bmatrix} & 1_n \\ \varepsilon 1_n & \end{bmatrix} = \begin{bmatrix} 1_n & \\ \varepsilon B & 1_n \end{bmatrix}.$$

For this to be in P, we see that we must have $u = 1_{2n}$, which then implies $u_1 = u_2$ and so $p_1 = p_2$.

Here is our main result of the present subsection.

Proposition 2.5.3. *Fix $T \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\left[\begin{smallmatrix} 1 & T \\ & 1 \end{smallmatrix}\right] \in G$ and a nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. For any character $\chi\colon P \to \mathbb{C}^\times$, we have*

$$\dim \operatorname{Hom}_U\left(\psi_T, \operatorname{Ind}_P^G \chi\right) = 1.$$

In other words, $\operatorname{Hom}_U\left(\psi_T, \operatorname{Ind}_P^G \chi\right)$ is spanned by the $f_{\chi,T}$ of Example 2.5.2.

Proof. We use Mackey theory. By Frobenius reciprocity, we are computing the dimension of the space $\operatorname{Hom}_G\left(\operatorname{Ind}_U^G \psi_T, \operatorname{Ind}_P^G \chi\right)$, which [Bum13, Theorem 32.1] explains equals

$$\mathcal{H} := \{f \in \operatorname{Mor}(G,\mathbb{C}) : f(pgu) = \chi(p)f(g)\psi_T(u) \text{ for } p \in P, g \in G, u \in U\}.$$

We proceed in steps.

(1) We see that we are interested in the double coset space $P\backslash G/U$. Lemma 2.2.3 tells us that any $g \in G$ can be written as $p_1\eta_r p_2$ for some $p_1, p_2 \in P$ and $r \in \{0,\ldots,n\}$. Because $P/U \cong M$, we thus see that any $g \in G$ can be written as $p\eta_r du$ for $p_1 \in P$ and $r \in \{0,\ldots,n\}$ and $d \in M$ and $u \in U$. Thus, $f \in \mathcal{H}$ is determined by its values on $f(\eta_r d)$ for $r \in \{0,\ldots,n\}$ and $d \in M$. (We are not claiming that these are unique double coset classes, but it will not usually be significant.)

(2) For the remainder of the proof, our goal will be to show that $f \in \mathcal{H}$ will have $f(\eta_r d) = 0$ for any $d \in M$ whenever $r \neq n$. This will complete the proof because it shows that any $f \in \mathcal{H}$ is supported on $P\eta_n P = P\eta_n U$, meaning that $f = f(\eta_n) f_{\chi,T}$, so $\{f_{\chi,T}\}$ is a basis of $\mathcal{H}$. (Here, $f_{\chi,T}$ is defined from Example 2.5.2.)

The basic sketch is that we will find various $u \in U$ such that $\eta_r du = p\eta_r d$ for some $p \in P$, which will allow us to show that $f(\eta_r d) = f(\eta_r du)$, but then $f(\eta_r d) \neq 0$ requires $\psi_T(u) = 1$. Having enough u will allow us to force a full column of T to vanish, violating the hypothesis that T is invertible.

(3) Fix some η_r and $d \in M$. If $\eta_r du = p\eta_r d$ for some $u \in U$ and $p \in P$, then we claim $\chi(p) = 1$. In other words, we are showing that χ is trivial on any $p \in P \cap \eta_r dUd^{-1}\eta_r^{-1}$. Quickly, note that M normalizes U, so $dUd^{-1} = U$, so we may as well assume that $d = 1_{2n}$.

Now, we are given $u \in U$ such that $p := \eta_r u \eta_r^{-1}$ is in P, and we want to show that $\chi(p) = 1$. Well, we expand u using block matrices to see that

$$\begin{aligned}\eta_r u \eta_r^{-1} &= \begin{bmatrix} 1_{n-r} &&& \\ &&& \varepsilon 1_r \\ && 1_{n-r} & \\ & 1_r && \end{bmatrix}\begin{bmatrix} 1_{n-r} && A & B \\ & 1_r & C & D \\ && 1_{n-r} & \\ &&& 1_r \end{bmatrix}\begin{bmatrix} 1_{n-r} &&& \\ &&& 1_r \\ && 1_{n-r} & \\ & \varepsilon 1_r && \end{bmatrix} \\ &= \begin{bmatrix} 1_{n-r} & \varepsilon B & A & \\ & 1_r && \\ && 1_{n-r} & \\ & \varepsilon D & C & 1_r \end{bmatrix}.\end{aligned}$$

For this to be in P, we see that $D = 0$. We then see that the resulting matrix has multiplier 1 and Siegel determinant 1, so χ is trivial on it.

(4) Fix some η_r and $d \in M$ such that $r < n$. We claim that there exists $u \in U$ such that $\psi_T(u) \neq 1$ and $\eta_r du = p\eta_r d$ for some $p \in P$. We will proceed more or less by contraposition: we show that having $\psi_T(u) = 1$ for all such u will imply that T fails to be invertible. Observe that this is the only step of the proof which will use that T is supposed to be invertible and that ψ is nontrivial.

The condition on $u \in U$ is that $\eta_r dud^{-1}\eta_r^{-1} \in P$. Again, because M normalizes U, our hypothesis is simply that $\eta_r u \eta_r^{-1} \in P$ implies $\psi_T\left(d^{-1}ud\right) = 1$; by replacing T with dTd^{-1} (which does not change whether T is invertible), we see that we may assume $d = 1_{2n}$. Using the computation in the previous step, we see that $\eta_r u \eta_r^{-1} \in P$ is equivalent to $D = 0$. But with $r > 0$, we thus see that we are permitted to use many $u \in U$; we will use the subgroup

$$U_1 := \left\{ \begin{bmatrix} 1 & B \\ & 1 \end{bmatrix} : B_{ij} = 0 \text{ for } i, j > 1 \right\}.$$

Namely, it remains to show that ψ_T being trivial on U_1 implies that T fails to invertible. In fact, we will show that $Te_1 = 0$, which will finish the proof. Quickly, we note that $\psi_T(u) = 1$ for u in the above form is simply asserting

$$1 \stackrel{?}{=} \psi(\operatorname{tr} BT) = \sum_{i=1}^n \psi((TB)_{ii}) = \sum_{i,j=1}^n \psi(T_{ij}B_{ji}) = \psi(T_{11}B_{11}) + \sum_{i=2}^n \psi(T_{i1}B_{1i}) + \sum_{j=2}^n \psi(T_{1j}B_{j1}).$$

To continue, we will do some casework on G.

- Take $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$. Then we may set the B_{ij} arbitrarily, provided that $i = 1$ or $j = 1$. We would like to show that $T_{i1} = 0$ for all i, so fixing some i, we set all coordinates except B_{1i} to zero so that we know $\psi(T_{i1}B_{1i}) = 1$ for all $B_{1i} \in \mathbb{F}_q$. Now, if T_{i1} were nonzero, then left multiplication by T_{i1} would be a surjective map $\mathbb{F}_q \to \mathbb{F}_q$, so we would be able to find some B_{1i} such that $\psi(T_{i1}B_{1i}) \neq 1$ because ψ is nontrivial. Thus, we must instead have $T_{1i} = 0$.
- Take $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$. Then both T and B must be alternating; for example, we get $T_{11} = B_{11} = 0$ for free. Thus, it remains to show that $T_{i1} = 0$ for all $i \geq 2$. Again, we fix some i, and we set all coordinates except B_{1i} and $B_{1i} = -B_{i1}$ to zero, so we see that $\psi(2T_{i1}B_{1i}) = 1$ for all $B_{1i} \in \mathbb{F}_q$. Arguing as in the previous point, this implies $T_{i1} = 0$, as required.
- Take $G \in \{\mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$. Then both T and B must be symmetric. We want to show that $T_{i1} = 0$ for all $i \geq 1$. Again, we fix some i, and we set all coordinates except B_{1i} and $B_{1i} = -B_{i1}$ to

zero, so we see that $\psi(cT_{i1}B_{1i}) = 1$ for all $B_{1i} \in \mathbb{F}_q$, where $c = 1$ if $i = 1$ and $c = 2$ if $i > 1$. Arguing as in the previous point, this still implies $T_{i1} = 0$, as required.

(5) We now complete the proof. Given some $f \in \mathcal{H}$, we would like to show that $f(\eta_r d) = 0$ whenever $r < 0$. Well, the previous step provides $p \in P$ and $u \in U$ such that $p\eta_r d = \eta_r du$ and $\psi_T(u) \neq 1$. But we know that any such p must have $\chi(p) = 1$, so the equation

$$f(\eta_r d) = \chi(p)f(\eta_r d) = f(p\eta_r d) = f(\eta_r du) = \psi_T(u)f(\eta_r d)$$

forces $f(\eta_r d) = 0$, as claimed. ■

Remark 2.5.4. *It is possible for no T satisfying the hypotheses of Proposition 2.5.3 to exist! Namely, suppose n is odd and $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$. Then we are asking for T to be an invertible $n\times n$ alternating matrix, but any alternating matrix has odd rank, so no such T exists! However, we can find some T in all other cases. For example, $T = 1_{2n}$ works for $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$ for any n, and $T \coloneqq \operatorname{diag}\left(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\right)$ works for $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$ when n is even.*

This multiplicity-one result means that we can gain insight into $I\colon \operatorname{Ind}_P^G \chi \to \operatorname{Ind}_P^G \chi^J$ by plugging in $f_{\chi,T}$. This will lead us to evaluate certain matrix Gauss sums. We begin by defining the relevant sums.

Definition 2.5.5. *Fix $T \in \mathbb{F}_q^{n\times n}$ and characters $\beta\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Then we define the "Gauss sum"*

$$g^G(\beta, \psi, T) \coloneqq \sum_{\substack{B\in \mathrm{GL}_n(\mathbb{F}_q) \\ \left[\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right] \in G}} \beta(\det B)\psi(\operatorname{tr} BT).$$

And here is our result.

Proposition 2.5.6. *Fix $T \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\left[\begin{smallmatrix} 1 & T \\ & 1 \end{smallmatrix}\right] \in G$ and a nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Further, fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$. Then*

$$If_{\chi,T} = g^G(\beta, \psi, T) f_{\chi^J,T}.$$

Proof. For brevity, let $\overline{U}$ denote the subgroup of $B \in \mathbb{F}_q^{n\times n}$ such that $\left[\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right] \in G$, and we let $\overline{U}^\times$ denote the invertible subset. Note that I carries ψ_T-eigenvectors to ψ_T-eigenvectors, so Proposition 2.5.3 tells us that

$$If_{\chi,T} = \left(If_{\chi,T}(\eta_n)\right) f_{\chi^J,T}.$$

So it remains to evaluate $If_{\chi,T}(\eta_n)$. We will do this by direct computation. We write

$$If_{\chi,T}(\eta_n) = \sum_{u\in U} f_{\chi,T}\left(\eta_n^{-1}u\eta_n\right).$$

Now, writing $u = \left[\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right]$, we see that $\eta_n^{-1}u\eta_n = \left[\begin{smallmatrix} 1 & \\ \varepsilon B & 1 \end{smallmatrix}\right]$, so because $\overline{U}$ is an additive group, we see

$$If_{\chi,T}(\eta_n) = \sum_{B\in\overline{U}} f_{\chi,T}\left(\begin{bmatrix} 1_n & \\ B & 1_n \end{bmatrix}\right).$$

Now, $f_{\chi,T}$ is supported on $P\eta_n U = P\eta_n P$, so the proof of Lemma 2.2.3 tells us that $B \in \overline{U}$ produces a nonzero contribution if and only if B is invertible. To compute this contribution, we note

$$\begin{bmatrix} 1_n & \\ B & 1_n \end{bmatrix} = \begin{bmatrix} -\varepsilon B^{-1} & 1_n \\ & B \end{bmatrix} \begin{bmatrix} & \varepsilon 1_n \\ 1_n & \end{bmatrix} \begin{bmatrix} 1_n & B^{-1} \\ & 1_n \end{bmatrix},$$

and we see that the left and right matrices here do in fact live in G. For the right, we simply need to note that $B^{-1} \in \overline{U}$ still (namely, if B is symmetric or alternating, then so is B^{-1}). Additionally, the middle matrix is in G, so the left matrix is also in G.[2] Thus, we conclude

$$If_{\chi,T} = \sum_{B\in\overline{U}^\times} f_{\chi,T}\left(\begin{bmatrix} -\varepsilon B^{-1} & 1_n \\ & B \end{bmatrix} \begin{bmatrix} & \varepsilon 1_n \\ 1_n & \end{bmatrix} \begin{bmatrix} 1_n & B^{-1} \\ & 1_n \end{bmatrix}\right) = \sum_{B\in\overline{U}^\times} \beta\left(\det B^{-1}\right)\psi_T\left(B^{-1}\right).$$

Replacing B with B^{-1} completes the proof. ■

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[2]Alternatively, in the interesting cases when $G \notin \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$, we can directly compute $B^{-\intercal} = -\varepsilon B^{-1}$ so that $\left[\begin{smallmatrix} -\varepsilon B^{-1} & 1 \\ & B \end{smallmatrix}\right] = \left[\begin{smallmatrix} B^{-\intercal} & \\ & B \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & B^\intercal \\ & 1 \end{smallmatrix}\right]$ is in P.

Thus, we see that the values of $g^G(\omega, \psi, T)$ will be interesting to us. For example, in the cases where $\chi = \chi^J$, the above proposition tells us that $g^G(\beta, \psi, T)$ is an eigenvalue of I. In the general case when merely $I \circ I$ is an operator on $\operatorname{Ind}_G^P \chi$, we get the following.

Corollary 2.5.7. *Fix $T \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\left[\begin{smallmatrix} 1 & T \\ & 1 \end{smallmatrix}\right] \in G$ and a nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Further, fix a character $\chi\colon P \to \mathbb{C}^\times$, which we write as $\chi = (\alpha \circ m)(\beta \circ \chi_{\det})$. Then*

$$(I \circ I) f_{\chi,T} = \beta(-1)^n \left| g^G(\beta, \psi, T) \right|^2 f_{\chi,T}.$$

Proof. Applying Proposition 2.5.6 twice, we see that

$$(I \circ I) f_{\chi,T} = g^G(\beta, \psi, T) g^G\left(\beta^{-1}, \psi, T\right) f_{\chi,T}$$

because $\left(\chi^J\right)^J = \chi$. In order to relate the above scalar to $\left|g^G(\beta, \psi, T)\right|^2$, we compute

$$\begin{aligned}
\overline{g^G(\beta, \psi, T)} &= \sum_B \overline{\beta}(\det B) \overline{\psi}(\operatorname{tr} BT) \\
&= \sum_B \beta^{-1}(\det B)\, \psi(\operatorname{tr} -BT) \\
&= \sum_B \beta^{-1}(\det -B) \psi(\operatorname{tr} BT) \\
&= \beta(-1)^n g^G\left(\beta^{-1}, \psi, T\right).
\end{aligned}$$

Thus, $g^G(\beta, \psi, T) g^G\left(\beta^{-1}, \psi, T\right) = \beta(-1)^n \left|g^G(\beta, \psi, T)\right|^2$. ■

Remark 2.5.8. *We take $\beta^2 \neq 1$, and we compare the above computation with Proposition 2.4.2. When $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}, \mathrm{GSp}_{2n}, \mathrm{Sp}_{2n}\}$, we see that $\varepsilon = -1$, so it follows that*

$$\left|g^G(\beta, \psi, T)\right|^2 = |U|. \tag{2.2}$$

When $G \in \{\mathrm{GO}_{2n}, \mathrm{O}_{2n}\}$, it may appear that our signs may disagree, but recall from Remark 2.5.4 that T does not even exist when n is odd, so we will always have $\beta(-1)^n = \beta(\varepsilon)^n$; thus, the above equation still holds. As such, we see that the sum in the definition of $g^G(\beta, \psi, T)$ obeys the expected "square root" cancellation generically.

Remark 2.5.9. *Later on, we will compute the values of $g^G(\beta, \psi, T)$ in terms of usual Gauss sums; for example, we will be able to check (2.2) directly. Do note that (2.2) cannot hold in the non-generic cases where $\beta^2 = 1$ because $\left|g^G(\beta, \psi, T)\right|^2$ is (up to sign) an eigenvalue of $I \circ I$, but there is no such eigenvalue when $\beta^2 = 1$.*

Check this.

3. q-Combinatorial Inputs

In this section, we discuss some combinatorial inputs into our main results. We forget about all the notation we set in the previous section in the first few subsections. Instead, q will be treated as a free variable until stated otherwise.

3.1. A Couple q-Identities.

In this quick subsection, we pick up a couple q-identities which will be useful in the sequel. Throughout, we freely use the packages `qZeil` and `qMultiSum` developed by Axel Riese; see [Rie97; Rie03] for a description of these packages. The following identity is used for the symplectic and orthogonal groups.

Proposition 3.1.1. *For any nonnegative integers $m, n \in \mathbb{Z}$, we have*

$$q^{\binom{m-n}{2}} \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(q;q)_n}{(q^2;q^2)_i\, (q;q)_{n-2i}} q^{\binom{n-m-2i}{2}} = \sum_{j=0}^{m} (-1)^{m-j} \frac{\left(q^2;q^2\right)_m}{(q;q)_j \left(q^2;q^2\right)_{m-j}} q^{\binom{j-n}{2}}.$$

Proof. For technical reasons, set

$$L_{m,n}(q) \coloneqq q^{\frac{1}{2}\left(n^2+2m^2-4mn-n\right)} \sum_{i=0}^{\lfloor n/2\rfloor} \frac{(q;q)_n}{(q^2;q^2)_i\,(q;q)_{n-2i}} q^{2i^2-2ni+i}\left(q^m\right)^{2i}$$
$$R_{m,,n}(q) \coloneqq \sum_{j=0}^{m} (-1)^{m-j} \frac{\left(q^2;q^2\right)_m}{(q;q)_j\,(q^2;q^2)_{m-j}} q^{\frac{1}{2}\left(j^2-j\right)}\left(q^n\right)^{-j}$$

so that we want to show that $L_{m,n}(q) = R_{m,n}(q)$. We will show that $L_{m,n}(q)$ and $R_{m,n}(q)$ satisfy the same recurrence in m and then check that $L_{m,n,}(q) = R_{m,n}(q)$ for some small m. With this outline in mind, we have the following steps.

(1) The package `qZeil` shows that $m \geq 2$ has
$$R_{m,n}(q) = q^{-2-n}\left(q^{2m}+q^{1+2m}-q^{2+n}\right)R_{m-1,n}(q) + q^{-3+2m-2n}\left(1-q^{-2+2m}\right)R_{m-2,n}(q).$$
We would like to show that $L_{m,n}(q)$ satisfies the same recurrence in m. Well, define $\widetilde{L}_{m,n}(q)$ to be
$$L_{m,n}(q) - q^{-2-n}\left(q^{2m}+q^{1+2m}-q^{2+n}\right)L_{m-1,n}(q) - q^{-3+2m-2n}\left(1-q^{-2+2m}\right)L_{m-2,n}(q),$$
and we want to show that $\widetilde{L}_{m,n}(q)$ vanishes. Some elementary rearrangement shows that $\widetilde{L}_{m,n}(q)$ is
$$\sum_{i=0}^{\lfloor n/2\rfloor} q^{-1+2m(-1+i)+2i^2-i(3+2n)}$$
$$\cdot\left(q^{1+2m+4i}+q^{2(m+n)}+q^{2(1+i+n)}-q^{2m+2i+n}-q^{1+2m+2i+n}-q^{2+2n}\right)$$
$$\cdot\frac{(q;q)_n}{(q;q)_{n-2i}\,(q^2;q^2)_i}$$
up to some powers of q that we have divided out. The package `qZeil` is able to show that this sum vanishes.

(2) It remains to check that $L_{m,n}(q) = R_{m,n}(q)$ for $m \in \{0,1\}$. For $m = 0$, `qZeil` shows that $L_{0,n}(q) = 1$, which agrees with $R_{0,n}(q)$. For $m = 1$, `qZeil` shows that
$$L_{1,n}(q) = \frac{1+q-q^n}{q+q^2-q^n} L_{1,n-1}(q)$$
and checks that $R_{1,n}$ satisfies the same recurrence. So it is enough to check that $L_{1,0}(q) = R_{1,0}(q) = q$. ■

The following identity is used for the linear groups.

Proposition 3.1.2. *For any nonnegative integers $m,n \in \mathbb{Z}$, we have*

$$q^{-nm+\binom{m}{2}} \sum_{i=0}^{n} \frac{(q;q)_n^2}{(q;q)_i^2\,(q;q)_{n-i}} q^{i^2-im}$$
$$= \sum_{0\leq j_2\leq j_1\leq m} (-1)^{m-j_1} \frac{(q;q)_m}{(q;q)_{m-j_1}\,(q;q)_{j_1-j_2}\,(q;q)_{j_2}} q^{\frac{1}{2}\left(j_1^2-j_1+2j_2^2-2j_1j_2-2nj_1\right)}.$$

Proof. Let the left-hand side be $L_{m,n}(q)$ and the right-hand side be $R_{m,n}(q)$ so that we want to show that $L_{m,n}(q) = R_{m,n}(q)$. As with the last proof, we will show that $L_{m,n}(q)$ and $R_{m,n}(q)$ satisfy the same recurrence in m and then check that $L_{m,n,}(q) = R_{m,n}(q)$ for some small m. With this outline in mind, we have the following steps.

(1) After some rearranging, `qMultiSum` shows that $m \geq 2$ has
$$R_{m,n}(q) = q^{-n}\left(2q^{(1+m-2)}-q^n\right)R_{m-1,n}(q) - q^{m-2n-2}\left(q^{m-1}-1\right)R_{m-2,n}(q).$$
We would like to show that $L_{m,n}(q)$ satisfies the same recurrence in m. Well, define $\widetilde{L}_{m,n}(q)$ to be
$$\widetilde{L}_{m,n}(q) \coloneqq L_{m,n}(q) - q^{-n}\left(2q^{(1+m-2)}-q^n\right)L_{m-1,n}(q) + q^{m-2n-2}\left(q^{m-1}-1\right)L_{m-2,n}(q),$$

Some elementary rearranging shows that $\widetilde{L}_{m,n}(q)$ is

$$\sum_{i=0}^{n} \frac{(q;q)_n^2}{(q;q)_i^2\, p\, (q;q)_{n-iover}} q^{i^2-im} \left(-1 + qq^m - 2q^{1-i}q^m + q^{n-i} + q^{1-2i}q^m\right)$$

up to some factors that we have divided out. Now, `qZeil` provides the recurrence

$$\widetilde{L}_{m,n}(q) = -q^{-2-m}\left(-q^{2n} - q^{2+m} - q^{3+m} + 2q^{2+n+m}\right)\widetilde{L}_{m,n-1}(q) - q\left(1-q^{n-1}\right)^2 \widetilde{L}_{m,n-2}(q),$$

so it suffices to show that $\widetilde{L}_{m,0}(q) = \widetilde{L}_{m,1}(q) = 0$. Both of these small cases are readily checked by hand.

(2) It remains to check that $L_{m,n}(q) = R_{m,n}(q)$ for $m \in \{0,1\}$. For $m = 0$, `qZeil` shows $L_{0,n}(q) = 1$, which agrees with $R_{0,n}(q)$. For $m = 1$, `qZeil` shows

$$L_{1,n}(q) = \frac{q^n - 2}{q\left(q^{n-1} - 2\right)} L_{1,n-1}(q)$$

and checks that $R_{1,n}(q)$ satisfies the same recurrence. So it is enough to check that $L_{1,0}(q) = R_{1,0}(q) = 1$. ■

3.2. **Some Antitriangular Matrices.** In this subsection, we discuss the eigenvalues of some antitriangular matrices. Essentially the only method in the literature to access the eigenvalues of an antitriangular matrix is to do some educated guessing in order to make the give matrix upper-triangular. See [BW22] for a thorough discussion of a special case; the work in this subsection can be seen as a q-analogue for some of their results.

Theorem 3.2.1. *Eigenvalues for* $G \in \{\mathrm{GL}_{2n}, \mathrm{SL}_{2n}\}$.

4. Computation of Matrix Gauss Sums

As before, let $\mathbb{F}_q$ denote the finite field with q elements, where q is an odd prime-power. For characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}$, we are interested in computing sums of the form

$$\sum_{A} \omega(\det A)\psi(\operatorname{tr} AT),$$

where A and T are possibly subject to certain constraints (e.g., symmetric or alternating). To be explicit, our sums will be done in three cases.

- $\mathrm{GL}_n(\mathbb{F}_q)$.
- $\mathrm{Sym}_n^\times(\mathbb{F}_q)$, the set of invertible $n \times n$ symmetric matrices with coefficients in $\mathbb{F}_q$.
- $\mathrm{Alt}_{2n}^\times(\mathbb{F}_q)$, the set of invertible $2n \times 2n$ alternating matrices with coefficients in $\mathbb{F}_q$. (Note that there are no invertible alternating matrices of odd dimension.)

For brevity, we will abbreviate $\mathbb{F}_q$ from our notation as much as possible.

Note that the sum over $A \in \mathrm{GL}_n$ has already been considered by [Kim97] and many authors before; see [Kim97, Section 1]. Additionally, the sum over symmetric matrices was considered in [Sai91], but the method there is based on a rather lengthy consideration with the Bruhat decomposition. We are under the impression that the sum over alternating matrices is new.

Our method is rather uniform over all kinds of sums considered. We will induct on the size of A via an explicit row-reduction. As such, the arguments are essentially the same as the spirit of the arguments in [Kim97] in the case of GL_n. However, we believe that there is gain to the case of sums of symmetric matrices because the arguments presented are somewhat more direct.

4.1. **Miscellaneous Computations.** We take a moment to discuss a few sums which will be used frequently in the sequel. For characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we denote the usual Gauss sum by

$$g(\omega,\psi) \coloneqq \sum_{a \in \mathbb{F}_q^\times} \omega(a)\psi(a).$$

It will be helpful to have the following well-known fact about the quadratic Gauss sum. Because the proof is so quick, we include the proof.

Proposition 4.1.1. *Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ denote nontrivial characters. Then*
$$g(\omega,\psi)g\left(\omega^{-1},\psi^{-1}\right) = q.$$
Thus, if $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denotes the nontrivial quadratic character, then $g(\chi,\psi)^2 = \chi(-1)q$.

Proof. For the first claim, we want to show
$$\sum_{a,b\in\mathbb{F}_q^\times} \omega(a/b)\psi(a-b) \stackrel{?}{=} q.$$
Well, set $c \coloneqq a/b$ so that the sum is
$$\sum_{c\in\mathbb{F}_q^\times}\left(\omega(c)\sum_{a\in\mathbb{F}_q^\times}\psi(a-ac)\right).$$
If $c \neq 1$, then the inner sum is $-\psi(0) + \sum_{a\in\mathbb{F}_q}\psi(a-ac) = -1$. Otherwise, if $c = 1$, then the inner sum is $q-1$. In total, we are left with
$$(q-1) + \sum_{c\in\mathbb{F}_q^\times\setminus\{1\}} -\omega(c) = q - \sum_{c\in\mathbb{F}_q^\times}\omega(c) = q,$$
which is what we wanted.

For the second claim, we see
$$g\left(\chi^{-1},\psi^{-1}\right) = \sum_{a\in\mathbb{F}_q^\times}\chi(a)\psi(-a) = \chi(-1)\sum_{a\in\mathbb{F}_q^\times}\chi(a)\psi(a) = \chi(-1)g(\chi,\psi),$$
so the second claim follows from the first. ■

The computation of the Gauss sums over $\mathrm{Sym}_n^\times$ will use the following fact.

Proposition 4.1.2. *Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ be characters, and let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character. Then*
$$\omega(4)g(\omega,\psi)g(\omega\chi,\psi) = g\left(\omega^2,\psi\right)g(\chi,\psi).$$

Proof. Expanding out the Gauss sums, we are trying to show that
$$\sum_{a,b\in\mathbb{F}_q^\times}\omega(4ab)\chi(b)\psi(a+b) \stackrel{?}{=} \sum_{a,b\in\mathbb{F}_q^\times}\omega\left(a^2\right)\chi(b)\psi(a+b).$$
Fixing some $d\in\mathbb{F}_q^\times$ and $t\in\mathbb{F}_q$, it is enough to show that
$$\sum_{\substack{a+b=t\\4ab=d}}\chi(b) \stackrel{?}{=} \sum_{\substack{a+b=t\\a^2=d}}\chi(b) \tag{4.1}$$
and then sum over all possible values of d and t. At this point, the proof has become combinatorial number theory. For convenience, extend χ to $\mathbb{F}_q$ by $\chi(0) \coloneqq 0$, and allow $a,b\in\mathbb{F}_q$ in the right-hand sum above; this will not change its value.

For example, suppose that d is not a square. Then the right-hand side of (4.1) is empty and hence zero. On the other hand, we claim that the left-hand side is zero. Let $(a_1,b_1),\ldots,(a_m,b_m)$ denote the solutions to the system of equations $a+b=t$ and $4ab=d$. Because d is not a square, $a_k\neq b_k$ for each k—in fact, if a_k is a square, then b_k is not a square (and vice versa). Thus, if (a,b) is a solution, then (b,a) is a distinct solution with $\{\chi(a),\chi(b)\} = \{1,-1\}$, so the two pairs (a,b) and (b,a) contribute $1-1=0$ to the left-hand side of (4.1). It follows that the left-hand side vanishes.

Thus, in the rest of the proof, we may assume that $d = x^2$ where $x\in\mathbb{F}_q^\times$, so the right-hand side of (4.1) reads
$$\chi(t+x) + \chi(t-x).$$
To continue, observe that solving the system of equations $a+b=t$ and $4ab=d$ is equivalent to having $a = t-b$ and
$$(2b-t)^2 = t^2-d.$$

As such, for our next case, suppose that t^2-d fails to be a square. Then the left-hand side of (4.1) is empty and hence vanishes, so we want to show that the right-hand side also vanishes. Well, $t^2-d=(t+x)(t-x)$ is then not a square, so both are nonzero, and one is a square while the other is not a square. Thus, $\chi(t+x)+\chi(t-x)=0$, as needed.

Thus, in the rest of the proof, we may assume that $t^2-d=y^2$ for some $y\in\mathbb{F}_q$. Quickly, we deal with the case where $y=0$. On one hand, we have $t^2=d$, so $t=\pm x$, so the right-hand side of (4.1) is $\chi(2t)$. On the other hand, we see the left-hand side of (4.1) is $\chi(t/2)$, so we finish by noting $\chi(2t)=\chi(t/2)$.

At the current point, we can now say that $t^2=x^2+y^2$ where $x,y\in\mathbb{F}_q^\times$, and the left-hand side of (4.1) is $\chi\left(\frac{t+y}{2}\right)+\chi\left(\frac{t-y}{2}\right)$, so we are trying to show that

$$\chi\left(\frac{t+y}{2}\right)+\chi\left(\frac{t-y}{2}\right)\stackrel{?}{=}\chi(t+x)+\chi(t-x). \tag{4.2}$$

Because $(t-x)(t+x)=y^2$ and $\left(\frac{t+y}{2}\right)\left(\frac{t-y}{2}\right)=\frac{1}{4}x^2$, we see that all values above are nonzero, and $\chi\left(\frac{t+y}{2}\right)=\chi\left(\frac{t-y}{2}\right)$ and $\chi(t+x)=\chi(t-x)$. Because, these values are in $\{\pm1\}$, we see that it is enough to show that $\chi(t+x)=1$ if and only if $\chi\left(\frac{t+y}{2}\right)=1$.

The main claim, now, is that $\chi(t+x)=1$ implies that $\chi\left(\frac{t+y}{2}\right)=1$. This approximately boils down to the enumeration of Pythagorean triples. The above logic grants that $\chi(t+x)=\chi(t-x)=1$, so both $t+x$ and $t-x$ are squares; write $t+x=x_1^2$ and $t-x=x_2^2$ for $x_1,x_2\in\mathbb{F}_q^\times$. Adjusting signs, we may assume that $y=x_1x_2$. Thus,

$$\frac{t+y}{2}=\frac{1}{2}\left(\frac{x_1^2+x_2^2}{2}+x_1x_2\right)=\left(\frac{x_1+x_2}{2}\right)^2$$

is a square, and we know $\frac{t+y}{2}$ is nonzero from the above logic, so $\chi\left(\frac{t+y}{2}\right)=1$, as desired.

To finish the proof, we must show the reverse implication: we claim that $\chi\left(\frac{t+y}{2}\right)=1$ implies $\chi(t+x)=1$. Well, we see that $\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2=\left(\frac{t}{2}\right)^2$, so the argument of the previous paragraph tells us that $\chi\left(\frac{t}{2}+\frac{y}{2}\right)=1$ implies

$$\chi(t+x)=\chi\left(\frac{t+x}{4}\right)=\chi\left(\frac{\frac{t}{2}+\frac{x}{2}}{2}\right)=1,$$

as desired. ■

All of our computations will frequently sum over vectors in some way, so we pick up the following fact.

Lemma 4.1.3. *Fix a character $\psi\colon\mathbb{F}_q\to\mathbb{C}^\times$ and some $A\in\mathbb{F}_q^{n\times m}$. Then*

$$\sum_{B\in\mathbb{F}_q^{m\times n}}\psi(\operatorname{tr}AB)=\begin{cases}0 & \text{if } A\neq0 \text{ and } \psi\neq1,\\ q^{mn} & \text{if } A=0 \text{ or } \psi=1.\end{cases}$$

Proof. Note that

$$\operatorname{tr}AB=\sum_{j=1}^n(AB)_{jj}=\sum_{i=1}^m\sum_{j=1}^nA_{ji}B_{ij},$$

so

$$\sum_{B\in\mathbb{F}_q^{m\times n}}\psi(\operatorname{tr}AB)=\prod_{i=1}^m\prod_{j=1}^n\sum_{B_{ij}}\psi(A_{ji}B_{ij}).$$

If $A=0$ or $\psi=1$, then all terms are 1, so we total to q^{mn}. Otherwise, say $A_{ij}\neq0$ for some given (i,j). Then the corresponding term $\sum\psi(A_{ji}B_{ij})$ in the above product will vanish, as desired. ■

4.2. **The Sum Over GL_n.** For the purposes of this subsection, we define

$$g_n(\omega,\psi,T)\coloneqq\sum_{A\in\mathrm{GL}_n}\omega(\det A)\psi(\operatorname{tr}AT)$$

where $\omega\colon\mathbb{F}_q^\times\to\mathbb{C}^\times$ and $\psi\colon\mathbb{F}_q\to\mathbb{C}^\times$ are characters, and $T\in\mathrm{GL}_n$. Even though our method to compute $g_n(\omega,\psi,T)$ is essentially equivalent to the one presented in [Kim97], we present it here because it provides a reasonable background to the approach.

The following general results will be helpful.

Lemma 4.2.1. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and some $T \in \mathrm{GL}_n$.*

(a) *For any $g, h \in \mathrm{GL}_n$, we have*
$$g_n(\omega, \psi, gTh) = \omega(\det gh)^{-1} g_n(\omega, \psi, T).$$

(b) *If $\psi = 1$, then $g_n(\omega, \psi, T) = 0$ unless $\omega = 1$.*

Proof. Here we go.

(a) We compute
$$\begin{aligned} g_n(\omega, \psi, gTh) &= \sum_{A \in \mathrm{GL}_n} \omega(\det A)\psi(\operatorname{tr} AgTh) \\ &= \sum_{A \in \mathrm{GL}_n} \omega\left(\det h^{-1}Ag^{-1}\right)\psi(\operatorname{tr} AT) \\ &= \omega(\det gh)^{-1} g_n(\omega, \psi, T). \end{aligned}$$

(b) With $\psi = 1$, we see that $g_n(\omega, \psi, T) = \sum_{A \in \mathrm{GL}_n} \omega(\det A)$ is the sum of a character ω over a group GL_n, from which the result follows. ■

Our explicit row-reduction is based on two cases: $A_{nn} \neq 0$ and $A_{nn} = 0$. We begin with the case $A_{nn} = 0$ because it is easier.

Lemma 4.2.2. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Then*
$$\sum_{\substack{A \in \mathrm{GL}_{n+1} \\ A_{n+1,n+1} \neq 0}} \omega(\det A)\psi(\operatorname{tr} A) = \begin{cases} q^n g(\omega, \psi) g_n(\omega, \psi, 1_n) & \text{if } \psi \neq 1, \\ q^{2n}(q-1) g_n(1, 1, 1_n) & \text{if } \omega = \psi = 1. \end{cases}.$$

Proof. The main point is that
$$\begin{array}{ccccccccc} \mathrm{GL}_n & \times & \mathbb{F}_q^n & \times & \mathbb{F}_q^n & \times & \mathbb{F}_q^\times & \to & \mathrm{GL}_{n+1} \\ (B & , & v & , & w & , & c) & \mapsto & \left[\begin{smallmatrix} 1_n & v \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} B & \\ & c \end{smallmatrix}\right]\left[\begin{smallmatrix} 1_n & \\ w^\intercal & 1 \end{smallmatrix}\right] \end{array}$$
is a bijection onto elements of $A \in \mathrm{GL}_{n+1}$ with nonzero entry $A_{n+1,n+1}$. Indeed, we can compute
$$\begin{bmatrix} 1_n & v \\ & 1 \end{bmatrix} \begin{bmatrix} B & \\ & c \end{bmatrix} \begin{bmatrix} 1_n & \\ w^\intercal & 1 \end{bmatrix} = \begin{bmatrix} B + cvw^\intercal & cv \\ cw^\intercal & c \end{bmatrix},$$
so we can see that any $A \in \mathrm{GL}_{n+1}$ with nonzero $c \coloneqq A_{n+1,n+1}$ is uniquely given by some choice (B, v, w). Thus, our total is
$$\underbrace{\sum_{B \in \mathrm{GL}_n} \omega(\det B)\psi(\operatorname{tr} B)}_{g_n(\omega,\psi,1_n)} \sum_{c,v,w} \omega(c)\psi(c)\psi(\operatorname{tr} cvw^\intercal).$$
If $(\omega, \psi) = (1, 1)$, then our term is always 1, so we total to $q^{2n}(q-1)g_n(1, 1, 1_n)$. Otherwise, $\psi = 1$, so we can get cancellation by summing over w: this produces an internal sum of the form
$$\sum_{w \in \mathbb{F}_q^n} \psi(\operatorname{tr} v \cdot cw^\intercal) = \begin{cases} 0 & \text{if } v \neq 0, \\ q^n & \text{if } v = 0, \end{cases}$$
by Lemma 4.1.3. Thus, our total is
$$q^n g_n(\omega, \psi, 1_n) \underbrace{\sum_{c \in \mathbb{F}_q^\times} \omega(c)\psi(c)}_{g(\omega,\psi)}$$
after summing over c as well. ■

We now handle $A_{nn} = 0$. It is mildly more technical because row-reduction still requires some nonzero term in the right column, so we want to rearrange the right column suitably.

Lemma 4.2.3. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Choosing some nonzero $\overline{v}, \overline{w} \in \mathbb{F}_q^{n+2}$ such that $\overline{v}_{n+2} = \overline{w}_{n+2} = 0$, we have*

$$\sum_{\substack{A \in \mathrm{GL}_{n+2} \\ Ae_{n+2} = \overline{v} \\ A^\intercal e_{n+2} = \overline{w}}} \omega(\det A)\psi(\operatorname{tr} A) = \begin{cases} 0 & \text{if } \psi \neq 1, \\ q^{2n+1} g_n(1,1,1_n) & \text{if } (\psi,\omega) = (1,1). \end{cases}$$

Proof. By allowing $T = 1_{n+2}$ to be a permutation matrix, we argue that we may assume $\overline{v}_{n+1}, \overline{w}_{n+1} \neq 0$. We want to rearrange the coordinates of $\overline{v}$ and $\overline{w}$ so that $\overline{v}_{n+1}, \overline{w}_{n+1} \neq 0$ by mapping $A \mapsto \sigma A \tau$ for suitable permutation matrices σ and τ. (One can use τ to achieve $\overline{v}_{n+1} \neq 0$ and use σ to achieve $\overline{w}_{n+2} \neq 0$.) This does not change $\det A$, but it transforms $\operatorname{tr} AT$ into $\operatorname{tr} A\sigma\tau$. For brevity, we rewrite $\sigma\tau$ as σ. Note the construction promises that $\sigma(n+2) = n+2$.

The rest of the argument proceeds as before. Write $\overline{v} = (cv, c, 0)$ and $\overline{w} = (dw, w, 0)$ (as a column vector). Then the point is that

$$\begin{array}{ccc} \mathrm{GL}_n \times \mathbb{F}_q^n \times \mathbb{F}_q^n \times \mathbb{F}_q & \to & \mathrm{GL}_{n+1} \\ (B \quad , \quad v' \quad , \quad w' \quad , \quad e) & \mapsto & \begin{bmatrix} 1_n & v & v' \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & e & c \\ & d & \end{bmatrix} \begin{bmatrix} 1_n & & \\ w^\intercal & 1 & \\ (w')^\intercal & & 1 \end{bmatrix} \end{array}$$

is a bijection onto $A \in \mathrm{GL}_{n+2}$ satisfying $Ae_{n+2} = \overline{v}$. Indeed, we compute

$$\begin{bmatrix} 1_n & v & v' \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & e & c \\ & d & \end{bmatrix} \begin{bmatrix} 1_n & & \\ w^\intercal & 1 & \\ (w')^\intercal & & 1 \end{bmatrix} = \begin{bmatrix} B + cv'w^\intercal + dv(w')^\intercal + ecdvw^\intercal & ev + cv' & cv \\ ew^\intercal + d(w')^\intercal & e & c \\ dw^\intercal & d & 0 \end{bmatrix},$$

so we can see that any $A \in \mathrm{GL}_{n+2}$ with $Ae_{n+2} = \overline{v}$ and $A^\intercal e_{n+2} = \overline{w}$ is uniquely given by some choice of (B, v', w', e). Now, if $\omega = \psi = 1$, then we are just counting, so we total to $q^{2n+1} g_n(1,1,1_n)$.

Otherwise, we take $\psi \neq 1$; we have two cases on σ.

- Suppose that $\sigma(n+1) = n+1$. Then we may write σ as $\left[\begin{smallmatrix} T_n & \\ & 1_2 \end{smallmatrix}\right]$. Here, our sum looks like

  $$\sum_{B \in \mathrm{GL}_n} \omega(\det B)\psi(\operatorname{tr} AT_n) \sum_{v', w', d} \omega(-cd)\psi(\operatorname{tr} cv'w^\intercal T_n)\psi(\operatorname{tr} dv(w')^\intercal T_n)\psi(\operatorname{tr} ecdvw^\intercal T_n)\psi(e).$$

  By Lemma 4.1.3, we see that the sum over w' will only produce nonzero contribution if $v = 0$. But in this case, the sum over e is just $\sum \psi(e) = 0$, so the total sum vanishes.
- Suppose $\sigma(n+1) < n+1$; say $\sigma(i_0) = n+1$. Here, we hold sum over v' while holding all other variables fixed. The determinant does not depend on v', so we are left summing over the ψ terms. Only paying attention to v', we see that we are computing

  $$\sum_{v' \in \mathbb{F}_q^n} \prod_{i=1}^{n+2} \psi\left(e_i^\intercal \begin{bmatrix} cv'w^\intercal & ev + cv' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e_{\sigma(i)} \right).$$

  We now hold sum over v'_{i_0} and hold the remaining coordinates $v'_\bullet$ constant. Indeed, the only i in the product which will ever use this coordinate is when $i = i_0$ so that $\sigma(i) = n+1$. This means that the sum over v'_{i_0} contains the factor $\sum \psi(cv'_{i_0})$, which we see vanishes as v'_{i_0} varies over $\mathbb{F}_q$. ■

We now synthesize our cases to evaluate our Gauss sums.

Theorem 4.2.4. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and some $T \in \mathrm{GL}_n$.*

(a) Suppose $\psi \neq 1$. Then

$$g_n(\omega, \psi, T) = \frac{q^{n(n-1)/2}}{\omega(\det T)} \cdot g(\omega, \psi)^n.$$

(b) Suppose $\psi = 1$ and $\omega = 1$. Then

$$g_n(\omega, \psi, T) = \prod_{i=0}^{n-1} \left(q^n - q^i\right).$$

For any (ψ, ω) not in the above list, the sum vanishes.

Proof. Note the last sentence follows by Lemma 4.2.1. Quickly, we note that both (a) and (b) reduce to the case where $T = 1_n$ by Lemma 4.2.1: both sides of the equalities in both statements are invariant under replacing T by gT for some $g \in \mathrm{GL}_n$, so taking $g = T^{-1}$ allows us to take $T = 1_n$. We now proceed with our cases separately.

(a) For $n = 0$, there is nothing to prove. Thus, by induction, it is enough to show that

$$g_{n+1}(\omega, \psi, 1_{n+1}) \stackrel{?}{=} q^n g_n(\omega, \psi, T),$$

which follows by summing Lemmas 4.2.2 and 4.2.3.

(b) For $n \in \{0, 1\}$, there is not much to say. We now induct, noting that summing Lemmas 4.2.2 and 4.2.3 produces

$$g_{n+2}(1,1,1_{n+2}) = q^{2n+2}(q-1)g_{n+1}(1,1,1_{n+1}) + q^{2n+1}\left(q^{n+2}-1\right)^2 g_n(1,1,1_n),$$

so it is enough to show that our right-hand side also satisfies this recurrence relation. Well, the right-hand side of the above equation is

$$\begin{aligned}
&q^{2n+2}(q-1)\prod_{i=0}^{n+1}\left(q^{n+1}-q^i\right) + q^{2n+1}\left(q^{n+2}-1\right)^2\prod_{i=0}^{n}\left(q^n-q^i\right) \\
&= \left(q^{n+1}(q-1)\left(q^{n+2}-q\right) + q\left(q^{n+1}-1\right)^2\right)\prod_{i=2}^{n+1}\left(q^{n+2}-q^i\right) \\
&= \left(q^{n+1}(q-1) + \left(q^{n+1}-1\right)\right)\prod_{i=1}^{n+1}\left(q^{n+2}-q^i\right) \\
&= \prod_{i=0}^{n+1}\left(q^{n+2}-q^i\right),
\end{aligned}$$

as required. ■

We close this subsection with a combinatorial application; note there is a similar result in [Kim97, Theorem 6.2].

Corollary 4.2.5. *Let n be a nonnegative integer, and fix some $T \in \mathrm{GL}_n$. Further, fix $d \in \mathbb{F}_q^\times$ and $t \in \mathbb{F}_q$. Then the number of $A \in \mathrm{GL}_n$ such that* $\det A = d$ *and* $\operatorname{tr} AT = t$ *is*

$$\begin{aligned}
&\frac{1}{q(q-1)}\left(\prod_{i=0}^{n-1}\left(q^n-q^i\right) - q^{n(n-1)/2}(q-1)^n\right) \\
&+ q^{n(n-1)/2}\cdot\#\left\{(y_1,\ldots,y_n) : (y_1+\cdots+y_n) = t, \frac{y_1\cdots y_n}{\det T} = d\right\}.
\end{aligned}$$

Proof. For any characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we claim that

$$g_n(\omega,\psi,T) \stackrel{?}{=} \frac{q^{n(n-1)/2}}{\omega(\det T)}\cdot g(\omega,\psi)^n + \frac{1}{q(q-1)}\left(\prod_{i=0}^{n-1}\left(q^n-q^i\right) - q^{n(n-1)/2}(q-1)^n\right)\sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q}\omega(a)\psi(b).$$

If $\psi \neq 1$, then this is (a) of Theorem 4.2.4; if $\psi = 1$, then both sides vanish unless $\omega = 1$, in which case this is Theorem 4.2.4. Now, we notice that full expansion gives

$$\frac{1}{\omega(\det T)}\cdot g(\omega,\psi)^n = \sum_{y_1,\ldots,y_n\in\mathbb{F}_q^\times}\omega\left(\frac{y_1\cdots y_n}{\det T}\right)\psi(y_1+\cdots+y_n),$$

so get the result by summing appropriately over all ω and ψ. ■

4.3. **The Sum Over** $\mathrm{Sym}_n^\times$**.** For the purposes of this subsection, we define

$$g_n(\omega,\psi,T) := \sum_{A\in\mathrm{Sym}_n^\times} \omega(\det A)\psi(\operatorname{tr} AT)$$

where $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ are characters, and $T\in\mathrm{Sym}_n^\times$. Additionally, throughout we let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character.

Anyway, we follow the outline of the previous subsection on GL_n.

Lemma 4.3.1. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and some $T\in\mathrm{Sym}_n^\times$.*

(a) For any $g\in\mathrm{GL}_n$, we have

$$g_n(\omega,\psi,gTg^\intercal) = \omega(\det g)^{-2} g_n(\omega,\psi,T).$$

(b) If $\psi=1$, then $g_n(\omega,\psi,T)=0$ unless $\omega^2=1$ and n is even.

Proof. Here we go.

(a) We directly compute

$$\begin{aligned} g_n(\omega,\psi,gTg^\intercal) &= \sum_{A\in\mathrm{Sym}_n^\times} \omega(\det A)\psi(\operatorname{tr} AgTg^\intercal) \\ &= \sum_{A\in\mathrm{Sym}_n^\times} \omega(\det A)\psi(\operatorname{tr} g^\intercal AgT) \\ &= \sum_{A\in\mathrm{Sym}_n^\times} \omega(\det g^{-\intercal}Ag^{-1})\psi(\operatorname{tr} AT) \\ &= \omega(\det g)^{-2} g_n(\omega,\psi,T). \end{aligned}$$

(b) We have two cases.

- Suppose $\omega^2\ne 1$. Then for any $g\in\mathrm{GL}_n$, we see that $A\in\mathrm{Sym}_n^\times$ if and only if $gAg^\intercal\in\mathrm{Sym}_n^\times$, so

$$g_n(\omega,1,T) = \sum_{A\in\mathrm{Sym}_n^\times}\omega(\det A) = \sum_{A\in\mathrm{Sym}_n^\times}\omega(\det gAg^\intercal) = \omega(\det g)^2 g_n(\omega,1,T).$$

  Thus, to conclude $g_n(\omega,1,T)=0$, it suffices to find $g\in\mathrm{GL}_n$ with $\omega(\det g)^2\ne 1$. Well, $\omega^2\ne1$, so find $c\in\mathbb{F}_q^\times$ such that $\omega(c)^2\ne 1$ and then set $g:=\operatorname{diag}(c,1,\ldots,1)$.
- Suppose n is odd. By the previous case, we may assume that $\omega^2=1$. Now, for any $c\in\mathbb{F}_q^\times$, we see that $A\in\mathrm{Sym}_n^\times$ if and only if $cA\in\mathrm{Sym}_n^\times$, so

$$g_n(\omega,1,T) = \sum_{A\in\mathrm{Sym}_n^\times}\omega(\det A) = \sum_{A\in\mathrm{Sym}_n^\times}\omega(c\det A) = \omega(c)^n g_n(\omega,1,T).$$

  Now, if we did have $g_n(\omega,1,T)\ne 0$, then we would have $\omega(c)^n=1$ for all $c\in\mathbb{F}_q^\times$ and hence $\omega^n=1$; however, n is odd and $\omega^2=1$ already, so it would follow $\omega=1$. However, $\omega\ne1$ by hypothesis. ■

We now handle $A_{nn}\ne 0$.

Lemma 4.3.2. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and diagonal $T_{n+1}\in\mathrm{Sym}_{n+1}^\times$. Letting $T_n := \operatorname{diag}(T_{11},\ldots,T_{nn})$, we have*

$$\sum_{\substack{A\in\mathrm{Sym}_{n+1}^\times\\ A_{n+1,n+1}\ne 0}} \omega(\det A)\psi(\operatorname{tr} AT_{n+1}) = \begin{cases} g_n(\omega,\psi,T_n)\frac{\chi(\det T_n)\chi(T_{n+1,n+1})^n}{\omega(T_{n+1,n+1})} g(\omega\chi^n,\psi)g(\chi,\psi)^n & \text{if } \psi\ne 1,\\ 0 & \text{if } (\omega,\psi)=(\chi,1),\\ q^n(q-1)g_n(1,1,1_n) & \text{if } (\omega,\psi)=1. \end{cases}$$

Proof. The main point is that

$$\begin{array}{ccc} \mathrm{Sym}_n^\times \times \mathbb{F}_q^n \times \mathbb{F}_q^\times & \to & \mathrm{Sym}_{n+1}^\times \\ (B\quad,\quad v\quad,\quad c) & \mapsto & \left[\begin{smallmatrix}1 & v\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}B & \\ & c\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ v^\intercal & 1\end{smallmatrix}\right] \end{array}$$

is a bijection onto $A \in \operatorname{Sym}_{n+1}^\times$ with $A_{n+1,n+1} \neq 0$. Indeed, we can compute

$$\begin{bmatrix} 1 & v \\ & 1 \end{bmatrix} \begin{bmatrix} B & \\ & c \end{bmatrix} \begin{bmatrix} 1 & \\ v^\intercal & 1 \end{bmatrix} = \begin{bmatrix} B + cvv^\intercal & cv \\ cv^\intercal & c \end{bmatrix},$$

from which the bijection follows. Thus, our sum in question is

$$\underbrace{\sum_{B \in \operatorname{Sym}_n^\times} \omega(\det B)\psi(\operatorname{tr} BT_n)}_{g_n(\omega,\psi,T_n)} \sum_{v,c} \omega(c)\psi(\operatorname{tr} cvv^\intercal T_n)\psi(cT_{n+1,n+1}).$$

If $\psi = 1$ and $\omega = \chi$, then the right sum is $\sum_{v,c} \chi(c) = 0$ because χ is nontrivial; if $\psi = 1$ and $\omega = 1$, then the right sum is $q^n(q-1)$ because all terms equal 1.

For the rest of the proof, we may assume that $\psi \neq 1$. For brevity, we set $T := \operatorname{diag}(d_1, \ldots, d_{n+1})$. The main point is to compute our sum over v and c, which we see is

$$\sum_{c \in \mathbb{F}_q^\times} \omega(c)\psi(cd_{n+1}) \prod_{i=1}^{n} \left(\sum_{a \in \mathbb{F}_q} \psi\left(cd_i a^2\right) \right)$$

after some expansion (of $v \in \mathbb{F}_q^n$). Quickly, we claim that

$$\sum_{a \in \mathbb{F}_q} \psi\left(cd_k a^2\right) \stackrel{?}{=} \sum_{a \in \mathbb{F}_q} (1 + \chi(cd_k a))\psi(a),$$

where we have extended χ to $\mathbb{F}_q$ by $\chi(0) := 0$. Indeed, for any $b \in \mathbb{F}_q$, we see that $\psi(b)$ appears on the left-hand side 0 times if b does not have the form $cd_k a^2$, appears 1 time if $b = 0$, and appears 2 times if b is nonzero and has the form $cd_k a^2$; these values are exactly $1 + \chi(cd_k a)$ in all cases. As such, the claim follows, and because ψ is nontrivial, we actually have

$$\sum_{a \in \mathbb{F}_q} \psi\left(cd_k a^2\right) = \sum_{a \in \mathbb{F}_q} \chi(cd_k a)\psi(a) = \chi(cd_k) g(\chi, \psi).$$

Plugging this in, we see that our sum is

$$\begin{aligned} & g_n(\omega, \psi, T_n) \sum_{c \in \mathbb{F}_q^\times} \omega(c)\chi(c)^n \psi(cd_{n+1})\chi(d_1 \cdots d_n) g(\chi, \psi)^n \\ &= g_n(\omega, \psi, T_n) \cdot \frac{\chi(d_1 \cdots d_n)\chi(d_{n+1})^n}{\omega(d_{n+1})} \sum_{c \in \mathbb{F}_q^\times} \omega(c)\chi(c)^n \psi(c) g(\chi, \psi)^n \\ &= g_n(\omega, \psi, T_n) \cdot \frac{\chi(d_1 \cdots d_n)\chi(d_{n+1})^n}{\omega(d_{n+1})} \cdot g\left(\omega\chi^n, \psi\right) g(\chi, \psi)^n, \end{aligned}$$

as desired. ∎

Next, we handle $A_{nn} = 0$.

Lemma 4.3.3. *Fix characters $\omega \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$ and some diagonal $T_{n+2} \in \operatorname{Sym}_{n+2}^\times$, and let $T_n := \operatorname{diag}(T_{11}, \ldots, T_{nn})$. Choosing some nonzero $\overline{v} \in \mathbb{F}_q^{n+2}$ such that $\overline{v}_{n+2} = 0$, we have*

$$\sum_{\substack{A \in \operatorname{Sym}_{n+2}^\times \\ Ae_{n+2} = \overline{v}}} \omega(\det A)\psi(\operatorname{tr} AT) = \begin{cases} 0 & \text{if } \psi \neq 1, \\ \omega(-1)q^{n+1} g_n(\omega, 1, T_n) & \text{if } \psi = 1 \text{ and } \omega^2 = 1. \end{cases}$$

Proof. We begin by arguing that we may assume $\overline{v}_{n+1} \neq 0$. We want to rearrange the coordinates of $\overline{v}$ so that $v_{n+1} \neq 0$ by mapping $A \mapsto \sigma^{-1} A \sigma$ for suitable permutation matrix σ. This does not change $\det A$, but it transforms $\operatorname{tr} AT$ into $\operatorname{tr} A\sigma T\sigma^{-1}$, effectively rearranging the rows and columns of T into a different diagonal matrix. However, the conclusion is independent of T, so this rearrangement is legal.

The rest of the argument proceeds as before. Write $\overline{v} = (cv, c, 0)$ (as a column vector). Now, once again, the main point is that there is a bijection

$$\begin{array}{ccccccc} \mathrm{Sym}_n^\times & \times & \mathbb{F}_q^n & \times & \mathbb{F}_q & \to & \mathrm{Sym}_{n+2}^\times \\ (B & , & w & , & d) & \mapsto & \begin{bmatrix} 1 & v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & d & c \\ & c & \end{bmatrix} \begin{bmatrix} 1 & & \\ v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} \end{array}$$

onto the set of $A \in \mathrm{Sym}_{n+2}^\times$ such that $Ae_{n+2} = \overline{v}$. To see that this is a bijection, we expand out the matrix product as

$$\begin{bmatrix} B + dvv^\intercal + c(vw^\intercal + wv^\intercal) & cdv + cw & cv \\ cdv^\intercal + cw^\intercal & d & c \\ cv^\intercal & c & 0 \end{bmatrix},$$

from which the bijection follows. Thus, we see that our sum is

$$\sum_{B \in \mathrm{Sym}_n^\times} \omega\left(-c^2 \det B\right) \psi(\operatorname{tr} BT_n) \left(\sum_{d \in \mathbb{F}_q} \psi\left(dT_{nn} + d \operatorname{tr} vv^\intercal T_n\right) \sum_{w \in \mathbb{F}_q^n} \psi(2c \operatorname{tr} vw^\intercal T_n) \right).$$

If $\psi \neq 1$, then we see that the sum over w vanishes by Lemma 4.1.3 unless $v = 0$. But in the case where $v = 0$, we see that the sum over d will vanish, so the total sum continues to vanish. Otherwise, $\psi = 1$ with $\omega^2 = 1$. Then our sums over d and w have all terms equal to 1, so they produce contributions of $\omega(-1)q^{n+1}g_n(\omega, 1, T_n)$, as required. ■

We now synthesize our cases to evaluate our Gauss sums.

Theorem 4.3.4. *Fix characters* $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ *and* $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ *and some* $T \in \mathrm{Sym}_n^\times$. *Let* $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ *denote the quadratic character.*

(a) Suppose $\psi \neq 1$.
- *If* $n = 2m$ *is an even nonnegative integer, then*

$$g_{2m}(\omega, \psi, T) = \frac{\chi(-1)^m \chi(\det T) q^{m^2}}{\omega(4^m \det T)} \cdot g\left(\omega^2, \psi\right)^m.$$

- *If* $n = 2m+1$ *is an odd nonnegative integer, then*

$$g_{2m+1}(\omega, \psi, T) = \frac{q^{m(m+1)}}{\omega(4^m \det T)} \cdot g(\omega, \psi) g\left(\omega^2, \psi\right)^m.$$

(b) Suppose $\psi = 1$ *and* $\omega = \chi$. *If* $n = 2m$ *is even, then*

$$g_{2m}(\chi, 1, T) = \chi(-1)^m q^{m^2} \prod_{k=0}^{m-1} \left(q^{2k+1} - 1\right).$$

(c) Suppose $\psi = 1$ *and* $\omega = 1$.
- *If* $n = 2m$ *is even, then*

$$g_{2m}(1, 1, T) = q^{m^2+m} \prod_{k=0}^{m-1} \left(q^{2k+1} - 1\right).$$

- *If* $n = 2m+1$ *is odd, then*

$$g_{2m+1}(1, 1, T) = q^{m^2+m} \prod_{k=0}^{m} \left(q^{2k+1} - 1\right).$$

For any (ψ, ω) *not in the above list, the sum vanishes.*

Proof. Note the last sentence follows by Lemma 4.3.1. Anyway, we quickly reduce to the case where T is diagonal. Indeed, by choosing an orthogonal basis for the symmetric bilinear form given by T, we receive some $g \in \mathrm{GL}_n$ such that $D \coloneqq gTg^\intercal$ is diagonal. As such, Lemma 4.3.1 yields

$$g_n(\omega, \psi, T) = \omega(\det g)^2 g_n(\omega, \psi, D).$$

Now, suppose we have proven the theorem for diagonal matrices. In this case, we see $g_n(\omega,\psi,D) = (\det D)^{-1}g_n(\omega,\psi,1)$, so $\det D = (\det g)^2(\det T)$ implies that

$$g_n(\omega,\psi,T) = (\det T)^{-1}g_n(\omega,\psi,1),$$

which is the theorem for T, as desired.

Thus, we may assume that $T_n \coloneqq \operatorname{diag}(d_1,\dots,d_n)$; set $T_{n-1} \coloneqq \operatorname{diag}(d_1,\dots,d_{n-1})$ and define T_{n-2} analogously. We now induct on n in cases.

(a) We split our arguments by parity.
- Suppose that $n = 2m$ is an even positive integer. In this case, Lemmas 4.3.2 and 4.3.3 and induction yields

$$\begin{aligned} g_{2m}(\omega,\psi,T) &= g_{2m-1}(\omega,\psi,T_{2m-1}) \cdot \frac{\chi(\det T)}{\omega(d_{n+1})} \cdot g(\omega\chi,\psi)g(\chi,\psi)^{2m-1} \\ &= \frac{\chi(\det T)q^{(m-1)m}}{\omega(4^{m-1}\det T)} \cdot g(\omega,\psi)g\left(\omega^2,\psi\right)^{m-1} g(\omega\chi,\psi)g(\chi,\psi)^{2m-1}. \end{aligned}$$

  By Proposition 4.1.2, this is

$$g_{2m}(\omega,\psi,T) = \frac{\chi(\det T)q^{m^2-m}}{\omega(4^m\det T)} \cdot g\left(\omega^2,\psi\right)^m g(\chi,\psi)^{2m}.$$

  Lastly, Proposition 4.1.1 yields

$$g_{2m}(\omega,\psi) = \frac{\chi(-1)^m\chi(\det T)q^{m^2}}{\omega(4^m\det T)} \cdot g\left(\omega^2,\psi\right)^m.$$

- Suppose $n = 2m+1$ is an odd positive integer with $m \geq 1$. In this case, Lemmas 4.3.2 and 4.3.3 and induction yields

$$\begin{aligned} g_{2m+1}(\omega,\psi) &= g_{2m}(\omega,\psi)g(\omega,\psi) \cdot \frac{\chi(\det T_{2m-1})}{\omega(d_{n+1})} \cdot g(\chi,\psi)^{2m} \\ &= \frac{\chi(-1)^m q^{m^2}}{\omega(4^m\det T)} \cdot g\left(\omega^2,\psi\right)^m g(\omega,\psi)g(\chi,\psi)^{2m}. \end{aligned}$$

  From here, Proposition 4.1.1 implies

$$g_{2m+1}(\omega,\psi) = \frac{q^{m^2+m}}{\omega(4^m\det T)} \cdot g(\omega,\psi)g\left(\omega^2,\psi\right)^m.$$

(b) Because $\psi = 1$, the value of T is irrelevant, so we will not adjust it in our notation. By induction, it is enough to show

$$g_{2m+2}(\chi,1,T) \overset{?}{=} g_{2m}(\chi,1,T) \cdot \chi(-1)q^{2m+1}\left(q^{2m+1}-1\right),$$

for all m, which is immediate upon summing Lemmas 4.3.2 and 4.3.3.

(c) Again, because $\psi = 1$, the value of T is irrelevant, so we will not adjust it in our notation. As with previous parts, we induct. For $n \in \{0,1\}$, there is not much to say. The main point is that

$$g_{n+2}(1,1,T) = q^{n+1}(q-1)g_{n+1}(1,1,T_{n+1}) + q^{n+1}\left(q^{n+1}-1\right)g_n(1,1,T_n)$$

for any n, which we see follows from summing Lemmas 4.3.2 and 4.3.3. Now, to synthesize cases, we note that

$$q^{m^2+m}\prod_{k=0}^{m}\left(q^{2k+1}-1\right) = q^{\frac{1}{2}(2m+1)(2m+2)} \prod_{\substack{1\leq k\leq 2m+1 \\ k \text{ odd}}} \left(1-\frac{1}{q^k}\right)$$

and analogously for the even case. Thus, for our induction, we take $n \geq 0$ and use the recursive formula to see $g_{n+2}(1,1,T)$ is

$$\begin{aligned}
&q^{n+1}(q-1)g_{n+1}(1,1,T) + q^{n+1}\left(q^{n+1}-1\right)g_n(1,1,T)\\
&= q^{\frac{1}{2}(n+2)(n+1)}\left(q^{n+1}(q-1)\prod_{\substack{n<k\leq n+1\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right) + \left(q^{n+1}-1\right)\right)\prod_{\substack{1\leq k\leq n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right).
\end{aligned}$$

If n is odd, we have

$$q^{\frac{1}{2}(n+2)(n+1)}\left(q^{n+2}-1\right)\prod_{\substack{1\leq k\leq n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right),$$

which simplifies correctly. If n is even, we have

$$q^{\frac{1}{2}(n+2)(n+1)}\underbrace{\left(q^{n+1}(q-1)\left(1-\frac{1}{q^{n+1}}\right)+\left(q^{n+1}-1\right)\right)}_{q(q^{n+1}-1)}\prod_{\substack{1\leq k\leq n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right),$$

which still simplifies correctly. This completes the induction. ■

Remark 4.3.5. *Part (c) above has recovered the computation of the number of invertible symmetric matrices; part (b) even recovers the number of invertible symmetric matrices with square and non-square determinant. We have included an inductive proof above to show that our methods are capable of this.*

We conclude this subsection with a combinatorial application.

Corollary 4.3.6. *Let n be a nonnegative integer, and fix some $T \in \operatorname{Sym}_n^\times$. Further, fix $d \in \mathbb{F}_q^\times$ and $t \in \mathbb{F}_q$.*

(a) Suppose that $n = 2m+1$ is odd. Then the number of $A \in \operatorname{Sym}_{2m+1}^\times$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\begin{aligned}
&\frac{q^{m^2+m}}{q(q-1)}\left(\prod_{k=0}^{m}\left(q^{2k+1}-1\right)-(q-1)^{m+1}\right)\\
&+q^{m^2+m}\cdot\#\left\{(x,y_1,\ldots,y_m): x+(y_1+\cdots+y_m)=t, \frac{x(y_1\cdots y_m)^2}{4^m\det T}=d\right\}.
\end{aligned}$$

(b) Suppose that $n = 2m$ is even. Let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character. Then the number of $A \in \operatorname{Sym}_{2m}^\times$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\begin{aligned}
&\frac{q^{m^2}}{q(q-1)}\left(\left(q^m+\chi(-1)^m\chi(d)\right)\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right)-\chi(-1)^m\left(\chi(d)+\chi(\det T)\right)(q-1)^m\right)\\
&+\chi(-1)^m\chi(\det T)q^{m^2}\cdot\#\left\{(y_1,\ldots,y_m): y_1+\cdots+y_m=t, \frac{(y_1\cdots y_m)^2}{4^m\det T}=d\right\}.
\end{aligned}$$

Proof. We prove these separately.

(a) For any characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we claim that

$$\begin{aligned}
g_n(\omega,\psi,T) &\stackrel{?}{=} \frac{q^{m(m+1)}}{\omega\left(4^m\det T\right)}\cdot g(\omega,\psi)g\left(\omega^2,\psi\right)^m\\
&\quad+\frac{g_n(1,1,T)-q^{m(m+1)}(q-1)^{m+1}}{q(q-1)}\sum_{a\in\mathbb{F}_q^\times,b\in\mathbb{F}_q}\omega(a)\psi(b).
\end{aligned}$$

This is by casework. If ψ is nontrivial, the second sum on the right-hand side vanishes, so the claim follows from Theorem 4.3.4. If ψ is trivial and ω is nontrivial, then the right-hand side vanishes, and left-hand side vanishes by Theorem 4.3.4. Lastly, if both ψ and ω are trivial, then both sides are $g_n(1,1,T)$.

Now, we notice that full expansion gives

$$\frac{1}{\omega\left(4^m \det T\right)} \cdot g(\omega,\psi) g\left(\omega^2,\psi\right)^m = \sum_{x,y_1,\ldots,y_m\in\mathbb{F}_q^\times} \omega\left(\frac{x(y_1\cdots y_m)^2}{4\det T}\right)\psi(x+(y_1+\cdots+y_m)),$$

so by summing appropriately over all ω and ψ, we see that the number of $A \in \mathrm{Sym}_n^\times$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\begin{aligned}&\frac{g_n(1,1,T) - q^{m(m+1)}(q-1)^{m+1}}{q(q+1)}\\ &+ q^{m^2+m}\cdot \#\left\{(x,y_1,\ldots,y_m) : x+(y_1+\cdots+y_m) = t, \frac{x(y_1\cdots y_m)^2}{4^m\det T} = d\right\}.\end{aligned}$$

To finish, we note that we can simplify the first term with from Theorem 4.3.4.

(b) For any characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we claim that

$$\begin{aligned}g_n(\omega,\psi,T) \overset{?}{=} &\frac{\chi(-1)^m\chi(\det T)q^{m^2}}{\omega\left(4^m\det T\right)}\cdot g\left(\omega^2,\psi\right)^m\\ &+ \frac{g_n(\chi,1,T) - \chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)} \sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q} \chi(a)\omega(a)\psi(b)\\ &+ \frac{g_n(1,1,T) - \chi(-1)^m\chi(\det T) q^{m^2}(q-1)^m}{q(q-1)} \sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q} \omega(a)\psi(b).\end{aligned}$$

Again, this is by casework. If ψ is nontrivial, this is Theorem 4.3.4; otherwise, ψ is trivial. Then if $\omega^2 \neq 1$ (i.e., $\omega \notin \{1,\chi\}$) the right-hand side vanishes, and the left-hand side vanishes by Theorem 4.3.4. Lastly, if $\omega \in \{1,\chi\}$, then both sides are equal by construction.

Now, as in (a), by full expansion of $\omega\left(4^m\det T\right)^{-1} g\left(\omega^2,\psi\right)^m$ and summing the claim over all ω and ψ appropriately, we see that the number of $A \in \mathrm{Sym}_n^\times$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\begin{aligned}&\frac{g_n(\chi,1,T) - \chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)}\cdot\chi(d) + \frac{g_n(1,1,T) - \chi(-1)^m\chi(\det T)q^{m^2}(q-1)^m}{q(q-1)}\\ &+ \chi(-1)^m\chi(\det T)q^{m^2}\cdot\#\left\{(y_1,\ldots,y_m) : y_1+\cdots+y_m = t, \frac{(y_1\cdots y_m)^2}{4^m\det T} = d\right\}.\end{aligned}$$

It remains to simplify the first two terms. On one hand, we note Theorem 4.3.4 gives

$$\frac{g_n(\chi,1,T) - \chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)}\cdot\chi(d) = \frac{\chi(-1)^m q^{m^2}}{q(q-1)}\left(\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right) - (q-1)^m\right)\chi(d).$$

On the other hand, Theorem 4.3.4 gives

$$\frac{g_n(1,1,T) - \chi(-1)^m\chi(\det T)q^{m^2}(q-1)^m}{q(q-1)} = \frac{q^{m^2}}{q(q-1)}\left(q^m\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right) - \chi(-1)^m\chi(\det T)(q-1)^m\right).$$

Summing the above two equalities completes the simplification. ■

4.4. **The Sum Over $\mathrm{Alt}_{2n}^\times$.** For the purposes of this subsection, we define

$$g_{2n}(\omega,\psi,T) \coloneqq \sum_{A\in\mathrm{Alt}_{2n}^\times} \omega(\det A)\psi(\operatorname{tr} AT)$$

where $\omega\colon \mathbb{F}_q^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ are characters, and $T \in \mathrm{Alt}_{2n}^\times$. Additionally, throughout we let $J \coloneqq \left[\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right]$.

We would like to follow the outline established for GL_n, but we can immediately tell that something will be different here because $A \in \mathrm{Alt}_{2n}^\times$ will automatically have $A_{2n,2n} = 0$. Instead, our row-reduction will be based on subdividing A into 2×2 minors. As such, our casework is based on $A_{2n,2n-1} = -A_{2n-1,2n}$.

Otherwise, our outline remains the same.

Lemma 4.4.1. *Fix characters $\omega\colon \mathbb{F}_q^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and some $T \in \mathrm{Alt}_{2n}^\times$.*

(a) *For any* $g \in \mathrm{GL}_n$*, we have*
$$g_{2n}(\omega, \psi, gTg^\intercal) = \omega(\det g)^{-2} g_{2n}(\omega, \psi, T).$$
(b) *If* $\psi = 1$*, then* $g_{2n}(\omega, \psi, T) = 0$ *unless* $\omega^2 = 1$.

Proof. Here we go.

(a) We directly compute
$$\begin{aligned} g_{2n}(\omega, \psi, gTg^\intercal) &= \sum_{A \in \mathrm{Alt}_{2n}^\times} \omega(\det A)\psi(\operatorname{tr} AgTg^\intercal) \\ &= \sum_{A \in \mathrm{Alt}_{2n}^\times} \omega(\det A)\psi(\operatorname{tr} g^\intercal AgT) \\ &= \sum_{A \in \mathrm{Alt}_{2n}^\times} \omega(\det g^{-\intercal} Ag^{-1})\psi(\operatorname{tr} AT) \\ &= \omega(\det g)^{-2} g_{2n}(\omega, \psi, T). \end{aligned}$$

(b) For any $g \in \mathrm{GL}_n$, we see that $A \in \mathrm{Alt}_{2n}^\times$ if and only if $gAg^\intercal \in \mathrm{Alt}_{2n}^\times$, so
$$g_{2n}(\omega, 1, T) = \sum_{A \in \mathrm{Alt}_{2n}^\times} \omega(\det A) = \sum_{A \in \mathrm{Alt}_{2n}^\times} \omega(\det gAg^\intercal) = \omega(\det g)^2 g_{2n}(\omega, 1, T).$$
Thus, to conclude $g_{2n}(\omega, 1, T) = 0$, it suffices to find $g \in \mathrm{GL}_n$ with $\omega(\det g)^2 \neq 1$. Well, $\omega^2 \neq 1$, so find $c \in \mathbb{F}_q^\times$ such that $\omega(c)^2 \neq 1$ and then set $g \coloneqq \operatorname{diag}(cJ, J, \ldots, J)$. ■

We now handle $A_{2n,2n-1} \neq 0$.

Lemma 4.4.2. *Fix characters* $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ *and* $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$*, and set* $T_{2i} \coloneqq \operatorname{diag}(J, \ldots, J) \in \mathrm{Alt}_{2i}^\times$ *for each* i*. Then*
$$\sum_{\substack{A \in \mathrm{Alt}_{2n+2}^\times \\ A_{2n,2n-1} \neq 0}} \omega(\det A)\psi(\operatorname{tr} AT_{2n+2}) = \begin{cases} q^{2n} g(\omega^2, \psi) g_{2n}(\omega, \psi, T_{2n}) & \text{if } \psi \neq 1, \\ q^{4n}(q-1) g_{2n}(1, 1, T_n) & \text{if } \psi = 1 \text{ and } \omega^2 = 1. \end{cases}$$

Proof. The point is that the bottom-right 2×2 minor of our considered $A \in \mathrm{Alt}_{2n+2}^\times$ is invertible. Thus, the main point is that
$$\begin{array}{ccccccc} \mathrm{Alt}_{2n}^\times & \times & \mathbb{F}_q^{2n \times 2} & \times & \mathbb{F}_q^\times & \to & \mathrm{Alt}_{2n+2}^\times \\ (B & , & V & , & c) & \mapsto & \left[\begin{smallmatrix} 1_{2n} & V \\ & 1_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} B & \\ & cJ \end{smallmatrix}\right]\left[\begin{smallmatrix} 1_{2n} & \\ V^\intercal & 1_2 \end{smallmatrix}\right] \end{array}$$
is a bijection onto $A \in \mathrm{Alt}_{2n+2}^\times$ with $A_{2n,2n-1} \neq 0$. Indeed, letting $V = [\, v \; w \,]$, we can expand
$$\begin{bmatrix} 1_{2n} & v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & & -c \\ & c & \end{bmatrix} \begin{bmatrix} 1_{2n} & & \\ v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} = \begin{bmatrix} B + cwv^\intercal - cvw^\intercal & cw & -cv \\ -cw^\intercal & & -c \\ cv^\intercal & c & \end{bmatrix},$$
from which the bijection follows: $A \in \mathrm{Alt}_{2n+2}^\times$ with nonzero $c \coloneqq A_{2n,2n-1}$ is uniquely determined by a choice of (B, v, w). Thus, the sum in question is
$$\sum_{B,v,w,c} \omega(\det B)\omega\left(c^2\right)\psi(\operatorname{tr} BT_{2n})\psi(\operatorname{tr}(cwv^\intercal - cvw^\intercal)T_{2n})\psi(\underbrace{\operatorname{tr} cJJ}_{-2c}).$$
If $\psi = 1$ and $\omega^2 = 1$, then every term in the sum equals 1, so we total to $q^{2n}(q-1)g_{2n}(1, 1, T_{2n})$. Otherwise, we take $\psi \neq 1$. In this case, we quickly compute $-\operatorname{tr} vw^\intercal T_{2n} = -\operatorname{tr} T_{2n}^\intercal wv^\intercal = \operatorname{tr} wv^\intercal T_{2n}$, so we may look at the sum
$$\sum_{v,w} \psi(\operatorname{tr}(cwv^\intercal - cvw^\intercal)T_{2n}) = \sum_{v,w} \psi(-2c \operatorname{tr} vw^\intercal T_{2n}).$$
Fixing v and summing over w, Lemma 4.1.3 tells us that we only get a nonzero contribution when $v = 0$, where we see the sum will evaluate to q^{2n}. Thus, in this case, the sum in question compresses down to $q^{4n} g\left(\omega^2, \psi^2\right) g_{2n}(\omega, \psi, T_{2n})$. ■

Next, we handle $A_{2n-1,2n} = 0$.

Lemma 4.4.3. *Take $n \geq 1$. Fix characters $\omega\colon \mathbb{F}_q^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, and set $T_{2i} := \operatorname{diag}(J, \ldots, J) \in \operatorname{Alt}_{2i}^\times$ for each i. Choosing some nonzero vector $\overline{v} \in \mathbb{F}_q^{2n+2}$ such that $\overline{v}_{2n+1} = \overline{v}_{2n+2} = 0$, we have*

$$\sum_{\substack{A \in \operatorname{Alt}_{2n+2}^\times \\ Ae_{n+2} = \overline{v}}} \omega(\det A)\psi(\operatorname{tr} AT_{2n+2}) = \begin{cases} 0 & \text{if } \psi = 1, \\ q^{2n} g_{2n}(\psi, \omega, T_{2n}) & \text{if } \psi = 1 \text{ and } \omega^2 = 1. \end{cases}$$

Proof. We would like to use the same bijection as Lemma 4.4.2, but of course something must change because the conclusion is different. Because $\overline{v}$ is nonzero, we may find an index $i_0 \notin \{2n+1, 2n+2\}$ such that $\overline{v}_{i_0}$ is nonzero. By mapping $A \mapsto \sigma A \sigma$ where σ is the permutation matrix associated to some permutation of the form $(2i, 2j)(2i+1, 2j+1)$, we see that the sum will not change (because $\det \sigma A \sigma = \det A$ and $\sigma T_{2n+2} \sigma = T_{2n+2}$); thus, we may apply such a permutation to assume that $i_0 \in \{2n-1, 2n\}$. We now set $\sigma := (i_0, 2n+1)$ and apply $A \mapsto \sigma A \sigma$ to our sum, which does adjust $\overline{v}$ (so that $\overline{v}_{2n+1} \neq 0$ while $\overline{v}_{i_0} = 0$) as well as T_{2n+2} so that

$$\sigma T_{2n+2} \sigma \in \left\{ \begin{bmatrix} \operatorname{diag}(J, \ldots, J) & & & & \\ & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \end{bmatrix}, \begin{bmatrix} \operatorname{diag}(J, \ldots, J) & & & & \\ & & & -1 & \\ & & & & -1 \\ & 1 & & & \\ & & 1 & & \end{bmatrix} \right\}$$

(The left happens when $i_0 = 2n-1$, and the right happens when $i_0 = 2n$.) With our now adjusted $\overline{v}$, we write $\overline{v} = (-cv, -c, 0)$ where $v \in \mathbb{F}_q^{2n}$ and $c \in \mathbb{F}_q^\times$.

Now, as in Lemma 4.4.2, the main point is that

$$\begin{aligned} \operatorname{Alt}_{2n}^\times \times \mathbb{F}_q^n &\to \operatorname{Alt}_{2n+2}^\times \\ (B \quad , \ w) &\mapsto \begin{bmatrix} 1_{2n} & v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & & -c \\ & c & \end{bmatrix} \begin{bmatrix} 1_{2n} & & \\ v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} \end{aligned}$$

is a bijection onto $A \in \operatorname{Alt}_{2n+2}^\times$ with $Ae_{2n+2} = (-cv, -c, 0)$ because we can compute

$$\begin{bmatrix} 1_{2n} & v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} B & & \\ & & -c \\ & c & \end{bmatrix} \begin{bmatrix} 1_{2n} & & \\ v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} = \begin{bmatrix} B + cwv^\intercal - cvw^\intercal & cw & -cv \\ -cw^\intercal & & -c \\ cv^\intercal & c & \end{bmatrix}$$

as before. Quickly, if $\psi = 1$ and $\omega^2 = 1$, then the whole business with ψ does not matter, so our sum looks like $\sum_{B,w} \omega(\det B)\omega\left(c^2\right) = q^{2n} g_{2n}(\psi, \omega, T_{2n})$, as required.

Otherwise, we take $\psi \neq 1$, and we need to produce cancellation in the sum. For this, we hold B constant and let w vary; in fact, we only need to let $w_{2n-1}, w_{2n} \in \mathbb{F}_q$ vary. In particular, after some rearrangement, we see that the sum in question contains the factor

$$\sum_{w \in \mathbb{F}_q^n} \psi\left(\operatorname{tr} \begin{bmatrix} wv^\intercal - cvw^\intercal & cw & -cv \\ -cw^\intercal & & -c \\ cv^\intercal & c & \end{bmatrix} \sigma T_{2n+2} \sigma \right),$$

which we will show vanishes. In fact, we will look at a factor of this sum. Because $\sigma T_{2n+2} \sigma$ takes the form $\operatorname{diag}(T_{2n-2}, T')$ for some $T' \in \operatorname{Alt}_4^\times$, we see that the sum above contains the factor

$$\sum_{w_{2n-1}, w_{2n} \in \mathbb{F}_q} \psi\left(\operatorname{tr} \begin{bmatrix} & cw_{2n-1}v_{2n} - cv_{2n-1}w_{2n} & cw_{2n-1} & -cv_{2n-1} \\ cw_{2n}v_{2n-1} - cv_{2n}w_{2n-1} & & cw_{2n} & -cv_{2n} \\ -cw_{2n-1} & -cw_{2n} & & -c \\ cv_{2n-1} & cv_{2n} & c & \end{bmatrix} T' \right)$$

by using the bottom-right 4×4 minors of our matrices. We now compute this sum in two casework on i_0.

- If $i_0 = 2n-1$, our trace is

$$\operatorname{tr} \begin{bmatrix} & cw_{2n-1}v_{2n} - cv_{2n-1}w_{2n} & cw_{2n-1} & -cv_{2n-1} \\ cw_{2n}v_{2n-1} - cv_{2n}w_{2n-1} & & cw_{2n} & -cv_{2n} \\ -cw_{2n-1} & -cw_{2n} & & -c \\ cv_{2n-1} & cv_{2n} & c & \end{bmatrix} \begin{bmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix},$$

  so our sum is $\sum \psi(-2cv_{2n-1})\psi(-2cw_{2n})$, which vanishes as $w_{2n} \in \mathbb{F}_q$ varies.

- If $i_0 = 2n$, our trace is
$$\operatorname{tr}\begin{bmatrix} & cw_{2n-1}v_{2n} - cv_{2n-1}w_{2n} & cw_{2n-1} & -cv_{2n-1} \\ cw_{2n}v_{2n-1} - cv_{2n}w_{2n-1} & & cw_{2n} & -cv_{2n} \\ -cw_{2n-1} & -cw_{2n} & & -c \\ cv_{2n-1} & cv_{2n} & c & \end{bmatrix}\begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix},$$
so our sum is $\sum \psi(2cw_{2n-1} - 2cv_{2n})$, which vanishes as $w_{2n-1} \in \mathbb{F}_q$ varies. ■

Remark 4.4.4. *In the case where $\psi \neq 1$, it may appear suspicious that we induced cancellation in the sum of Lemma 4.4.3 but not in Lemma 4.4.2. Namely, if $\overline{v} \in \mathbb{F}_q^{2n+2}$ with $\overline{v}_{2n+1} = \overline{v}_{2n+2} = 0$ can swap a coordinate with $\overline{v}_{2n+1}$ to cause the sum to vanish, then it looks like the sum should still vanish in the case where $\overline{v}_{2n+1} \neq 0$. The way out of this apparent contradiction is to note that Lemma 4.4.2 shows, for fixed v with $\overline{v}_{2n+1} \neq 0$, the sum will in fact produce no contribution unless all the other coordinates vanish.*

We now synthesize our cases.

Theorem 4.4.5. *Fix characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ and some $T \in \operatorname{Alt}_{2n}^\times$.*

(a) Suppose $\psi \neq 1$. Then
$$g_{2n}(\omega, \psi, T) = \frac{q^{n(n-1)}}{\omega(\det T)} \cdot g\left(\omega^2, \psi^2\right)^n.$$
(b) Suppose $\psi = 1$ and $\omega^2 = 1$. Then
$$g_{2n}(\omega, \psi, T) = q^{n(n-1)} \prod_{i=1}^{n} \left(q^{2i-1} - 1\right).$$

For any (ψ, ω) not in the above list, the sum vanishes.

Proof. Note the last sentence follows from Lemma 4.4.1. Anyway, we quickly reduce to the case where $T = \operatorname{diag}(J, \ldots, J)$ using Lemma 4.4.1. For (b), T does not even matter, so we may as well take this value of T. For (a), we note that we can find $g \in \mathrm{GL}_{2n}$ such that $\operatorname{diag}(J, \ldots, J) = gTg^\intercal$. Now, Lemma 4.4.1 and inspection tells us that both sides of our equalities adjust by the same factor of $\omega(\det g)^{-2}$ upon sending $T \mapsto gTg^\intercal$ in (a). Thus, we may as well assume $T = \operatorname{diag}(J, \ldots, J)$.

We now use induction to show both (a) and (b). Write $T_{2i} := \operatorname{diag}(J, \ldots, J) \in \mathrm{GL}_{2i}$ for each i.

(a) For $n = 0$, there is nothing to say. Now, for our induction, we note that summing Lemmas 4.4.2 and 4.4.3 implies
$$g_{2n+2}(\omega, \psi, T_{2n+2}) = q^{2n} g_{2n}(\omega, \psi, T_{2n}),$$
so the result follows by induction.

(b) For $n \in \{0, 1\}$, there is not much to say. Now, for our induction, we note that summing Lemmas 4.4.2 and 4.4.3 implies
$$g_{2n+2}(\omega, \psi, T) = \left(q^{4n}(q-1) + q^{2n}\left(q^{2n} - 1\right)\right) g_{2n}(\omega, \psi, T)$$
for $n \geq 1$. Thus, it is enough to show that our right-hand side satisfies the same recurrence. Namely, we would like to show
$$q^{(n+1)n} \prod_{i=1}^{n+1} \left(q^{2i-1} - 1\right) \stackrel{?}{=} \left(q^{4n}(q-1) + q^{2n}\left(q^{2n} - 1\right)\right) q^{n(n-1)} \prod_{i=1}^{n} \left(q^{2i-1} - 1\right),$$
which we note simplifies down to
$$q^{2n}\left(q^{2n+1} - 1\right) \stackrel{?}{=} \left(q^{4n}(q-1) + q^{2n}\left(q^{2n} - 1\right)\right),$$
which is true after a little more rearrangement. ■

Remark 4.4.6. *A careful extraction of the row-reductions involved in our proof is able to show that any $T \in \operatorname{Alt}_{2n}^\times$ can be written as $g \operatorname{diag}(J, \ldots, J) g^\intercal$ for some $g \in \mathrm{GL}_{2n}$.*

As usual, we close with a combinatorial application.

Corollary 4.4.7. *Fix some even nonnegative integer* $2n$ *and some* $T \in \mathrm{Alt}_{2n}^{\times}$. *Further, fix* $d \in \mathbb{F}_q^{\times 2}$ *and* $t \in \mathbb{F}_q$. *Then the number of* $A \in \mathrm{Alt}_{2n}^{\times}$ *such that* $\det A = d$ *and* $\operatorname{tr} AT = t$ *is*

$$\frac{2}{q(q-1)}\left(q^{n(n-1)}\prod_{i=1}^{n}\left(q^{2i-1}-1\right)-q^{n(n-1)}(q-1)^n\right)$$
$$+\,q^{n(n-1)}\cdot\#\left\{(y_1,\ldots,y_n):2(y_1+\cdots+y_n)=t,\frac{(y_1\cdots y_n)^2}{\det T}=d\right\}.$$

Proof. For any characters $\omega\colon \mathbb{F}_q^{\times}$ and $\psi\colon \mathbb{F}_q\to\mathbb{C}^{\times}$, we claim that

$$g_{2n}(\omega,\psi,T)\stackrel{?}{=}\frac{q^{n(n-1)}}{\omega(\det T)}\cdot g\left(\omega^2,\psi^2\right)^n$$
$$+\frac{2}{q(q-1)}\left(q^{n(n-1)}\prod_{i=1}^{n}\left(q^{2i-1}-1\right)-q^{n(n-1)}(q-1)^n\right)\sum_{a\in\mathbb{F}_q^{\times 2},b\in\mathbb{F}_q}\omega^2(a)\psi(b).$$

If $\psi\neq 1$, then this follows immediately from Theorem 4.4.5. If $\psi=1$ and $\omega^2\neq 1$, then everything vanishes. Lastly, if $\psi=1$ and $\omega^2=1$, then this follows from Theorem 4.4.5 again.

We now note that a direct expansion implies

$$\frac{1}{\omega(\det T)}\cdot g\left(\omega^2,\psi^2\right)=\sum_{y_1,\ldots,y_n}\omega\left(\frac{(y_1\cdot y_n)^2}{\det T}\right)\psi(2(y_1+\cdots+y_n)),$$

so the result follows from summing the claim suitably over all ω and ψ. ∎

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