Working Title

November 1, 2024

Contents

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be $X = \mathbb{CP}^1$. The space of fields is then simply given by points on the target manifold X, namely

$$\mathcal{F} = \mathbb{CP}^1 \tag{1}$$

As the action of the model we consider

$$S(z_0, z_1) = \beta\Big(|z_0|^2 - |z_1|^2\Big) \tag{2}$$ {eq:toy_S}

where $[z_0 : z_1] \in \mathbb{CP}^1$. As written, the partition function explicitly shows the $U(1)$ gauge freedom (which here is simply a global $U(1)$ symmetry, acting as

$$e^{i\theta} \colon (z_0, z_1) \mapsto (e^{i\theta} z_0, e^{i\theta} z_1) \tag{3}$$

and defines indeed a function on $\mathbb{CP}^1$ if we implicitly assume the constraint

$$|z_0|^2 + |z_1 = 1 \tag{4}$$ {eq:toy_constr}

The path integral of the model is thus defined by

$$Z = \int dz_0 dz_1 \delta(|z_0|^2 + |z_1 - 1) e^{-S(z_0, z_1)} \tag{5}$$ {eq:toy_Z}

In order to evaluate (5), we want to embed $\mathbb{CP}^1$ into a higher dimensional complex space such that

1. the new action $\tilde{S}$ is holomorphic in the new variables (fields)
2. when we restrict to $\mathbb{CP}^1$, $\tilde{S}$ reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

Going to real fields, $z_k = x_k + iy_k$ for $k = 0, 1$, the action (2) becomes

$$S = \beta \left(x_0^2 + y_0^2 - x_1^2 - y_1^2 \right) \tag{6}$$ {eq:toy_S_real}

while the constraint becomes

$$x_0^2 + y_0^2 + x_1^2 + y_1^2 = 1 \tag{7}$$ {eq:toy_constr_real}

1.2 Manifold Deformation

Suppose that we now complexify in the sense that we consider $x_k, y_k \in \mathbb{C}$. For simplicity, let us rename the variables according to

$$x_0 = u_0 \quad , \quad x_1 = u_1 \quad , \quad y_0 = u_2 \quad , \quad y_1 = u_3 \tag{8}$$ {eq:toy_us}

where now $u_i \in \mathbb{C}$. This slightly unusual renaming is done for later convenience.

The constraint (7) now becomes

$$\sum_{i=0}^{3} u_i^2 = 1 \tag{9}$$

which can be seen as a complex hypersurface inside $\mathbb{CP}^4$, as follows: Let us introduce the complex variable $t \in \mathbb{C}$ and consider the equation

$$\sum_{i=0}^{3} u_i^2 = t^2 \tag{10}$$

This equation describes the zero set of a homogeneous quadratic polynomial

$$P(t, u_i) = t^2 - \sum_{i=0}^{3} u_i^2 \tag{11}$$

Notice that for any $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$,

$$P(\lambda t, \lambda u_i) = \lambda^2 P(t, u_i) \tag{12}$$

Therefore, the solution space of

$$P(u_i, t) = 0 \tag{13}$$

is invariant under the action of $\mathbb{C}^*$ (by multiplication) and hence descends to an equation on $\mathbb{CP}^4$ whose coordinates are $[t : u_0 : u_1 : u_2 : u_3]$

The important observation is that the constraint surface (7) coincides with $P(t, u_i) = 0$ for $t = 1$. But $t = 1$ simply defines the $U_{t\neq 0} \subset \mathbb{CP}^4$ whose coordinates are given by $\left[1 : \frac{u_0}{t} : \frac{u_1}{t} : \frac{u_2}{t}\right]$. Hence, the constraint surface can be embedded into $\mathbb{CP}^4$:

$$[u_0 : u_1 : u_2 : u_3] \mapsto [t(u_i) : u_0 : u_1 : u_2 : u_3] \quad , \quad t(u_i) = \sqrt{\sum_{i=0}^{3} u_i^2} \qquad (14) \quad \{\texttt{eq:toy\_emb}\}$$

Notice that $[t(u_i) : u_0 : u_1 : u_2 : u_3]$ simply describes a point in $P(t, u_i) = 0$ and the original surface is reproduced in the chart $t(u_i) = 1$.

This embedding comes with a natural family of holomorphic deformations parametrised by a vector $\omega \in \mathbb{C}^4 - \{0\}$. Namely, instead of considering zeros of $P(t, u_i)$, one could consider zeros of a general homogeneous quadratic polynomial

$$P_\omega(t, u_i) = t^2 - \sum_{i=0}^{3} \omega_i u_i^2 \qquad (15)$$

which for $\omega = (1, 1, 1, 1) \in \mathbb{C}^4$ coincides with P.

Remark 1. Note that the hypersurface $\mathcal{C}_\omega$ defined by $P_\omega(t, u_i) = 0$ defines a $\mathbb{CP}^3 \subset \mathbb{CP}^4$. Indeed, zeros of $P_\omega(t, u_i)$ are given by points

$$[t(u_i) : u_0 : u_1 : u_2 : u_3] \quad , \quad t(u_i) = \sqrt{\sum_i u_i^2} \qquad (16)$$

and hence parametrised by $(u_0, \ldots, u_3) \in \mathbb{C}^4 - \{0\}$. However, we may freely scale the u_i by the same $\lambda \in \mathbb{C}^*$ simultaneously, since $t(\lambda u_i) = \lambda t(u_i)$. Hence, the hypersurface C_ω is parametrised by $(u_0, \ldots, u_3) \in \mathbb{C}^4 - \{0\}$ only up to the action of $\mathbb{C}^*$, that is by a $\mathbb{CP}^3$ with coordinates $[u_0 : u_1 : u_2 : u_3]$.

1.3 Action along the Deformed Manifold

In the chart $t = 1$, the action (6) becomes

$$S(u) = \beta \left(u_0^2 + u_1^2 - u_2^2 - u_3^2\right) \qquad (17)$$

If we were to reinstate t, we would have to do it in a way that ensures that $S(u)$ is a function on $\mathbb{CP}^4$, i.e. invariant under the $\mathbb{C}^*$ action λ: $(t, u_i) \mapsto (\lambda t, \lambda u_i)$. An obvious candidate is

$$S(t, u) = \frac{\beta \left(u_0^2 + u_1^2 - u_2^2 - u_3^2\right)}{t^2} \qquad (18)$$

On the constraint surface (inside $\mathbb{CP}^4$) we now replace t^2 by $\sum_i \omega_i u_i^2$ for some non-zero $\omega \in \mathbb{C}^4$. We obtain

$$S(t,u)|_{\mathcal{C}_\omega} = \frac{\beta\left(u_0^2 + u_1^2 - u_2^2 - u_3^2\right)}{\sum_i \omega_i u_i^2} \tag{19}$$ {eq:toy_S_deformed}

where we recall that we denote the (generalised) constraint surface by

$$\mathcal{C}_\omega = \{P_\omega(t,u) = 0\} \tag{20}$$

Note that (19) is invariant under simultaneous scaling $u_i \mapsto \lambda u_i$ for any $\lambda \in \mathbb{C}^*$ and thus is indeed a function on $\mathbb{CP}^3 \cong \mathcal{C}_\omega$.

1.4 Integration Domain

So far we have just discussed how the "complexified" space of fields embeds nicely as a $\mathbb{CP}^3$ into $\mathbb{CP}^4$. However, the original integration domain was a $\mathbb{CP}^1$ and the correct deformation domain of the path integral should be a deformation of this $\mathbb{CP}^1$ possibly embedded into $\mathbb{CP}^3 \subset \mathbb{CP}^4$. Here, we present *one way* to do it.

Let us briefly recall the first steps of "complexifying" the space of fields: Starting with $[z_0 : z_1] \in \mathbb{CP}^1$, we rewrite it into real fields, $z_k = x_k + iy_k$ for $k = 0, 1$ and then promoted x_k, y_k to complex fields u_i for $i = 0, \ldots, 3$ and ultimately we found that the resulting space is forms a $\mathbb{CP}^3 \subset \mathbb{CP}^4$ with coordinates $[u_0 : u_1 : u_2 : u_3]$.

It is left to discuss how the $\mathbb{CP}^1$ embeds into this $\mathbb{CP}^3$, i.e. how we have to extend the coordinates x_k, y_k to u_i. One way to do this is by *scaling*: for some $\alpha = (\alpha_1, \ldots, \alpha_3) \in \mathbb{C}^4$ we set

$$u_0 = \alpha_0 x_0 \quad , \quad u_1 = \alpha_1 x_1 \quad , \quad u_2 = \alpha_2 y_0 \quad , \quad u_3 = \alpha_3 y_1 \tag{21}$$ {eq:toy_deform_contou

This scaling has the advantage that it preserves the polynomial degree of x_k, y_k, and hence of the original complex coordinates z_0, z_1, as well as the homogeneity. In turn, this implies that we preserve the action of $\mathbb{C}^*$. Put differently, the deformation (or analytical continuation) of x_k, y_k to u_i is *equivariant*, namely for any $\lambda \in \mathbb{C}^*$

$$u_i(\lambda x, \lambda y) = \lambda u_i(x, y) \tag{22}$$

Due to this equivariance, the deformation $z_k \rightsquigarrow u_i$ can be seen as a deformation of $\mathbb{CP}^1$ *inside* $\mathbb{CP}^3 \subset \mathbb{CP}^4$, since

$$\begin{gathered} u\colon \mathbb{C}^2 \to \mathbb{C}^4 \\ (z_0, z_1) \mapsto u(z_0, z_1) \equiv (u_0(z_0,z_1), u_1(z_0,z_1), u_2(z_0,z_1), u_3(z_0,z_1)) \end{gathered} \tag{23}$$

descends to a map from $\mathbb{CP}^1$ to $\mathbb{CP}^3$ due to its equivariance property

$$u(\lambda z_0, \lambda z_1) = \lambda u(z_0, z_1) \tag{24}$$

which simply means that we can consider the quotient by $\mathbb{C}^*$ on both sides (domain and codomain).

The full path integral domain would then be deformed as follows:

$$\mathcal{E}_{\alpha,\omega} \colon \mathbb{CP}^1 \xhookrightarrow{u_\alpha} \mathbb{CP}^3 \xhookrightarrow{P_\omega} \mathbb{CP}^4 \tag{25}$$ {eq:toy_E}

The first map is given by

$$u_\alpha \colon [z_0 : z_1] = [x_0 + iy_0 : x_1 + iy_1] \mapsto [\alpha_0 u_0 : \cdots : \alpha_3 u_3] \tag{26}$$ {eq:toy_cont_deform1}

with $u_i = u_i(z_0, z_1)$ given by (8). The second map is given by

$$P_\omega \colon [u_0 : u_1 : u_2 : u_3] \mapsto [P_\omega(u_i) : u_0 : u_1 : u_2 : u_3] \tag{27}$$

where $P_\omega(u_i) = \sqrt{\sum_i \omega_i u_i^2}$. The full embedding $\mathcal{E}_{\alpha,\omega}$ is hence parametrised by two complex vectors, $\alpha \in \mathbb{C}^4$ and $\omega \in \mathbb{C}^4$, which gives 8 complex or equivalently 16 real parameters which can be tuned to maximize the StN.

Remark 2. For $x_k + iy_k$ for $k = 0, 1$, the embedding $\mathcal{E}_{\alpha,\omega}$ can be explicitly expressed as

$$\mathcal{E}_{\alpha,\omega}(x, y) = [\sqrt{\omega_0 \alpha_0^2 x_0^2 + \omega_1 \alpha_1^2 x_1^2 + \omega_2 \alpha_2^2 y_0^2 + \omega_3 \alpha_3^2 y_1^2} : \alpha_0 x_0 : \alpha_1 x_1 : \alpha_2 y_0 : \alpha_3 y_1] \tag{28}$$