Working Title

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1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X, namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 = -\beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2}$$ {eq:toy_S}

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \tag{3}$$ {eq:toy_constr}

The action enjoys a $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge freedom (which here is simply a global $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry, acting as

$$e^{i\theta}\times e^{i\varphi}\colon (z,w)\mapsto (e^{i\theta}z, e^{i\varphi}w) \tag{4}$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1}\times\mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z=\int\prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2-1)\delta(|w|^2-1)e^{-S(z,w)} \tag{5}$$ {eq:toy_Z}

In order to evaluate (5), we want to embed the space of fields $\mathcal{F}=\mathbb{CP}^{N-1}\times\mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action $\tilde{S}$ is holomorphic in the new variables (fields)
2. when we restrict to $\mathcal{F}$, $\tilde{S}$ reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [1].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M=m$. Moreover, let $\{U_i,\phi_i\}$ be a real analytic atlas of M, with $U_i\subset\mathbb{R}^m$ and charts $\phi_i\colon U_i\to M$ so that the transition functions

$$\phi_{ij}=\phi_j^{-1}\circ\phi_i\colon U_{ij}\to U_{ij} \tag{6}$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}}=m$ and a (real analytic) isomorphism $f\colon M\to\tilde{M}\subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}}\subset\mathbb{C}^m$ such that $U_i^{\mathbb{C}}\cap\mathbb{R}^m=U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

(i) the transition functions $\phi_{ij}^{\mathbb{C}}\colon U_{ij}^{\mathbb{C}}\to U_{ij}^{\mathbb{C}}$ are biholomorphic

(ii) $\phi_{ji}^{\mathbb{C}}=\left(\phi_{ij}^{\mathbb{C}}\right)^{-1}$ and $\phi_{ii}^{\mathbb{C}}=id$

(iii) the transition functions $\phi_{ij}^{\mathbb{C}}$ satisfy the usual 2-cocycle condition (gluing condition) on $U_{ijk}^{\mathbb{C}}$: $\phi_{ij}^{\mathbb{C}}\circ\phi_{jk}^{\mathbb{C}}=\phi_{ik}^{\mathbb{C}}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U_i^{\mathbb{C}}, \phi_i^{\mathbb{C}}\}$:

$$M^{\mathbb{C}} = \coprod_i U_i^{\mathbb{C}} / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi_{ji}^{\mathbb{C}}(z_i) \text{ on } U_{ij}^{\mathbb{C}} \tag{7}$$

For more details on this construction see Cieliebak and Eliashberga's book [2]

1.2.1 Example: The N-sphere

{subsec:cmplxfy_spher

Consider the N-sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_\pm = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in $\mathbb{R}^N$ by $X = (X_1, \dots, X_N)$. The atlas we consider is given by steoreographic projection through $p_\pm$: Let $U_\pm = \mathbb{R}^N$ and $V_\pm = S^N - \{p_\pm\}$. Then define charts

$$\phi_\pm \colon U_\pm \to V_\pm \subset S^N \quad , \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \tag{8}$$

with inverse

$$\phi_\pm^{-1} \colon x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \tag{9}$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_-^{-1} \circ \phi_+ \colon X \mapsto \frac{X}{\|X\|^2} \tag{10}$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \tag{11}$$

describes an involution at the unit sphere S^{N-1}. On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$) one finds

$$\phi_+ \circ \phi_- \colon (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \tag{12}$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in $\mathbb{R}^N$ (again, it helps working this out in the case $N = 2$).

Now, since $U_\pm = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_\pm^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \tag{13}$$ {eq:toy_quadric}

The above equation defines a hypersurface (so-called *quadric*) inside $\mathbb{C}^{N+1}$.

There is a very interesting observation I found in this stackexchange post: the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N. The diffeomorphism is realised by the following map:

$$\Psi\colon TS^N \to Q \quad , \quad (x,y) \mapsto z = \Psi(x,y) = x\sqrt{1+\|y\|^2} + iy \tag{14}$$

with inverse

$$\Psi^{-1}(x+iy) = \left(\frac{x}{\sqrt{1+\|y\|^2}}, y\right) \tag{15}$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that Ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x,y) \in \mathbb{R}^{N+1}$ such that $\langle x,y\rangle = \sum_i x_i y_i = 0$)

There exists another very interesting diffeomorphism (which I have discovered in this stackexchange post

$$\Phi\colon S^N S^N/\Delta \to TS^N \quad , \quad (u,v) \mapsto \left(u, \frac{v - \langle u,v\rangle\, u}{1 - \langle u,v\rangle}\right) \tag{16}$$ {eq:diff_SNSN_TSN}

Its inverse is given by

$$\Phi^{-1}\colon TS^N \to S^N \times S^N/\Delta \quad , \quad (x,y) \mapsto \left(x, \frac{x(\|y\|^2 - 1) + 2y}{\|y\|^2 + 1}\right) \tag{17}$$

Remark 3. The map (16) is the sterepgraphic projection of $v \in S^N$ through the "pole" $u \in S^N$.

Finally we have the following commutative diagram

$$\begin{tikzcd}
& TS^N \arrow[dr, "\Psi"] & \\
S^N \times S^N/\Delta \arrow[ur, "\Phi"] \arrow[rr, "\Psi \circ \Phi"'] & & Q
\end{tikzcd} \tag{18}$$

Remark 4. The original $S^N \subset Q$ was located at $S^N = Q \cap \mathbb{R}^{N+1}$. If we follow this through, we see that this S^N corresponds to the zero section $\{(x, 0) \in TS^N\}$ inside TS^N and consequently is defined by $\Phi(u, v) = (u, 0)$ inside $S^N \times S^N/\Delta$. Note that this is precisely the anti-diagonal

$$\bar{\Delta} = \{(u, -u) \mid u \in S^N\} \subset S^N \times S^N \tag{19}$$

Indeed, the stereographic projection of $-u$ ("south pole") through u ("north pole") gives zero:

$$-u \mapsto \frac{(-u) - \langle(-u), u\rangle\, u}{1 - \langle(-u), u\rangle} = \frac{-u + \|u\|^2 u}{1 + \|u\|^2} = 0 \tag{20}$$

since $\|u\|^2 = 1$. Hence

$$\Phi(u, -u) = (u, 0) \tag{21}$$

and the original S^N is located along the anti-diagonal $\bar{\Delta}$.

1.3 Factor from δ-function Constraint

Recall that we are interested in integrals of the form

$$I = \int_{\mathbb{CP}^n \times \mathbb{CP}^n} d\mu(z) d\mu(w) e^{-S(z,w)} \mathcal{O}(z, w) \tag{22}$$

where $d\mu(z) = \prod_i dz_i d\bar{z}_i$ and we have included some observable $\mathcal{O} \colon \mathbb{CP}^n \to \mathbb{C}$. For our toy model, we consider the action functional (2)

$$S(z, w) = -\beta |\langle \bar{z}, w\rangle|^2 \tag{23}$$

It is instructive pass to real coordinates. If $z_k = a_k + ib_k$, let x_k be the real vector $(a_k\ b_k)^t$. If we collect all components, we can form the real vector $x = (\dots a_k\ b_k \dots)^t \in \mathbb{R}^{2n+2}$. We denote the real vector associated to w by y. In terms of these, the path integral over can be written as

$$vol(S^1)^2 \int_{\mathbb{R}^{2n+2} \times \mathbb{R}^{2n+2}} d^{2n+2}x\ d^{2n+2y}\ \delta(\|x\|^2 - 1)\delta(\|y\|^2 - 1) e^{-S(x,y)} \mathcal{O}(x, y) \tag{24}$$

where the pre-factor stems from the compact part of the gauge group $\mathbb{C}^* \circlearrowleft \mathbb{C}^{n+1}$. Let us focus on the following part of the integral:

$$I \sim \int_{\mathbb{R}^{2n+2}} \delta(\|x\|^2 - 1) = \int_{\mathbb{R}^{2n+2}} \delta(\langle x, x\rangle - 1) \tag{25}$$

Following our idea, we complexify the space of integration, $x \to \zeta$

$$I \sim \int_{\Gamma_0 = \mathbb{R}^{2n+2} \cap \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta\rangle - 1) \tag{26}$$

where

$$\langle \zeta, \zeta \rangle = \sum_k \zeta_k^2 \tag{27}$$

The δ-function constraint restricts the support of the integrand to the quadric Q. Now, we would like to deform the "contour" (domain of integration) inside Q

$$I \sim \int_{\Gamma_a \subset \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta \rangle - 1) \tag{28}$$

Inspired by the discussion in 1.2.1, we would like to parametrise Q in terms of TS^{2n+1}. We thus choose a parametrisation of the following form: (by abuse of notation, we will use again x to denote a vector in $\mathbb{R}^{2n+2}$)

$$\zeta(x) = x\sqrt{1 + \|y(x)\|^2} + iy(x) \tag{29}$$ {eq:parametrisation}

where $y(x)$ is chosen in such a way that $\langle x, y \rangle = 0$. It follows that

$$I \sim \int_{\mathbb{R}^{2n+2}} d^{2n+2}x \ \det J(x) \delta(\langle \zeta(x), \zeta(x) \rangle - 1) \tag{30}$$

where $J(x)$ denotes the Jacobian of (29). Note that the δ-function constraint can be simplified as follows: Let

$$\lambda(x) = \sqrt{1 - \|y\|^2} \tag{31}$$

Then

$$\begin{aligned} C(x) &= \langle \zeta(x), \zeta(x) \rangle - 1 \\ &= \lambda^2(x)\|x\|^2 + 2i\lambda(x) \langle x, y(x) \rangle - \|y\|^2 - 1 \\ &= \lambda^2(x)(\|x\|^2 - 1) \end{aligned} \tag{32}$$

Importantly,

$$\lambda^2(x) > 0 \tag{33}$$

such that the δ-function constraint is of the following form:

$$\begin{aligned} \int \delta(C(x)) &= \int d\mu(x) \delta(\underbrace{f(x)g(x)}_{\equiv \phi(x)}) \\ &= \int_{\phi^{-1}(0)} \frac{d\sigma}{\|\nabla \phi(x)\|} \\ &= \int_{f^{-1}(0)} \frac{d\sigma}{\|f(x)\nabla g(x) + g(x)\nabla f(x)\|} \\ &= \int_{f^{-1}(x)} \frac{d\sigma}{\|\nabla f(x)\|} \frac{1}{|g(x)|} \\ &= \int d\mu \frac{\delta(f(x)}{|g(x)|} \end{aligned} \tag{34}$$

where $f(x) = \|x\|^2 - 1$ and $g(x) = \lambda^2(x) > 0$. Hence we find that with the chosen parametrisation, the integral nicely localises to an integral over S^{2n+1}

$$\int_{\mathbb{R}^{2n+2}} \delta(C(x)) = \int_{\mathbb{R}^{2n+2}} \frac{\delta\left(\|x\|^2 - 1\right)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{1}{\lambda^2(x)} \tag{35}$$

Therefore, we are left with an integral of the form

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} \quad , \quad \lambda^2(x) = 1 + \|y(x)\|^2 \tag{36}$$ {eq:toy_schema_I}

1.4 Homogeneous Deformations

1.4.1 Example: $n = 0$

A particular nice example, which can be computed explicitly and serves as a check that everything works nicely is given for the case $n = 0$. Consider the deformation

$$y(x) = \alpha\Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SO}(2) \quad , \quad \alpha \in \mathbb{R} \tag{37}$$ {eq:toy_cst_def}

so that indeed

$$\langle x, y(x) \rangle = 0 \tag{38}$$

Moreover, note that

$$\|y(x)\|^2 = \langle y(x), y(x) \rangle = \alpha^2 \left\langle x, \Omega^t \Omega x \right\rangle = \alpha^2 \|x\|^2 = \alpha^2 \tag{39}$$

and hence

$$\lambda^2(x) = 1 + \alpha^2 \tag{40}$$

We compute the Jacobian of this parametrisation as follows: let $M_i^{\ j} = \partial_i y^j(x)$. Then

$$J_i^{\ j} = \lambda(x)\delta_i^{\ j} + \frac{\langle y, \partial_i y \rangle x^j}{\lambda(x)} + iM_i^{\ j} = \lambda(x)\delta_i^{\ j} + \frac{M_{ik} y^k x^j}{\lambda(x)} + iM_i^{\ j} \tag{41}$$

Using bra-ket notation and trivially lowering indices (considering the flat metric on $\mathbb{R}^2$), we can write J as follows:

$$J = \lambda(x) \cdot id + \lambda^{-1}(x) |My\rangle\langle x| + iM \tag{42}$$

Remark 5. The matrix M has some interesting properties.

1. From $\langle x, y(x) \rangle = 0$ it follows that

$$0 = \partial_i \langle x, y(x) \rangle = y_i(x) + M_{ik} x_k$$

so that

$$y_i(x) = M_{ik} x_k \implies y(x) = Mx$$

and hence

$$\langle x, y\rangle = \langle x, Mx\rangle = \left\langle M^t x, x\right\rangle = 0$$

from which follows (if $y(x) = Mx \neq 0$)

$$\langle x| M = 0$$

Now, for our choice of

$$y(x) = \alpha\Omega x$$

we have

$$M = \alpha\Omega^t = -\alpha\Omega \tag{43}$$

and hence

$$\begin{aligned} J &= \lambda(x)\cdot id + \lambda^{-1}(x)\alpha^2\Omega^t\Omega|x\rangle\langle x| - i\alpha\Omega \\ &= \lambda(x)\cdot id + \lambda^{-1}(x)\alpha^2|x\rangle\langle x| - i\alpha\Omega \end{aligned} \tag{44}$$

Now, here is a neat trick due to Tej. Consider the unitary matrix

$$U = (x\ \Omega x) \quad , \quad \Omega^* = \begin{pmatrix} x^t \\ x^t\Omega^t \end{pmatrix} \tag{45}$$

where x is a 2×2 column vector with unit norm. Then

$$U^*U = \begin{pmatrix} \langle x, x\rangle & \langle x, \Omega x\rangle \\ \langle \Omega x, x\rangle & \langle x, x\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{46}$$

Moreover,

$$\langle x|U = x^t U = (\langle x, x\rangle\ \ \langle x, \Omega x\rangle) = (1\ 0) \tag{47}$$

as well as

$$U^*\Omega U = U^*(\Omega x\ \Omega^2 x) = U^*(\Omega x\ \ -x) = \begin{pmatrix} \langle x, \Omega x\rangle & -\langle x, x\rangle \\ \left\langle x, \Omega^t\Omega x\right\rangle & -\left\langle x, \Omega^t x\right\rangle \end{pmatrix} = -\Omega \tag{48}$$

Then

$$\begin{aligned} U^*JU &= \lambda(x)\cdot id + \lambda^{-1}(x)\alpha^2 U^*|x\rangle\langle x|U - i\alpha U^*\Omega U \\ &= \lambda(x)\cdot id + \lambda^{-1}(x)\alpha^2 U^*\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i\alpha\Omega \\ &= \begin{pmatrix} \lambda(x) + \lambda^{-1}(x)\alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \end{aligned} \tag{49}$$

It follows that

$$\begin{aligned} \det J(x) &= \det(U^* J(x) U) \\ &= \det\begin{pmatrix} \lambda(x) + \lambda^{-1}(x)\alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \\ &= \lambda^2(x) + \alpha^2 - \alpha^2 = \lambda^2(x) \end{aligned} \tag{50}$$

Finally, we obtain that

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{\lambda^2(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \tag{51}$$

This means that the *constant deformation* (37) comes with a trivial *total* Jacobian (Jacobian + factor from the φ-function constraint).

1.4.2 The General Case

The $n = 0$ example can be nicely generalised. Tej suggested to consider deformations of the form

$$Y_a(X) = \Omega(a)X \quad , \quad \Omega(a) = \mathrm{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & a_{n+1} \\ -a_{n+1} & 0 \end{pmatrix}\right) \tag{52}$$

{eq:Tej_deformation}

for some $a \in \mathbb{R}^{n+1}$. This type of deformation have a very nice geometric origin, as we now explain.

Let G be a Lie group and $H < G$ a subgroup with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The orbit space G/H is known as a *homogeneous space*. It is called *reductive* if $\mathfrak{g}$ allows the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad , \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \tag{53}$$

If $\mathfrak{g}$ is equipped with a Killing form B (or if G allows a G-invariant metric as a manifold), then G/H is automatically reductive with the choice

$$\mathfrak{m} = \mathfrak{h}^\perp \tag{54}$$

where $\mathfrak{h}^\perp$ is the orthogonal to $\mathfrak{h}$ with respect to B. Indeed, since B is G-invariant, schematically

$$0 = B(\mathfrak{h}, \mathfrak{m}) = B([\mathfrak{h}, \mathfrak{h}], \mathfrak{m}) = B(\mathfrak{h}, [\mathfrak{h}, \mathfrak{m}]) \tag{55}$$

so that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

In summary, given a reductive homogeneous space G/H, we get a nice decomposition of the tangent space: first, let $o = [e]$ be the class in G/H at the identity. Then

$$T_o(G/H) \cong \mathfrak{m} \tag{56}$$

This isomorphism is given essentially by the pushforward of the natural projection $\pi\colon G \to G/H$. In fact, $\pi\colon G \to G/H$ defines a principal H-bundle. Its vertical vector fields are given by $\ker(\pi_*) = \mathfrak{h}$ and since π is a submersion (π and π_* are both surjective) we have

$$\pi_*\colon \mathfrak{g}/\ker(\pi_*) = \mathfrak{g}/\mathfrak{h} \cong T_o(G/H) \tag{57}$$

Then, to compute the tangent space at any other point, we use translation by G.

Now, recall that

$$S^{2n+1} = \mathrm{SU}(n+1)/\,\mathrm{SU}(n) \tag{58}$$

is a homogeneous space where $H = \mathrm{SU}(n)$ sits inside $G = \mathrm{SU}(n+1)$ as the lower right corner

$$\mathrm{SU}(n) \ni h \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in \mathrm{SU}(n+1) \tag{59}$$

The reductive split is given by

$$\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m} \tag{60}$$ {eq:red_split}

Since $\mathfrak{su}(n+1)$ admits a Killing form $B(x,y) \propto tr(x,y)$, we can choose $\mathfrak{m} = \mathfrak{su}(n)^{\perp}$. In the split (60), we have

$$\mathfrak{su}(n) \equiv \begin{pmatrix} 0 & 0 \\ \mathfrak{su}(n) & 0 \end{pmatrix} \subset \mathfrak{su}(n+1) \tag{61}$$

We then find

$$\mathfrak{m} = \mathfrak{su}(n)^{\perp} = \left\{ \begin{pmatrix} 0 & -\zeta^* \\ \zeta & 0 \end{pmatrix} \right\} \quad , \quad \zeta \in \mathbb{C}^{n+1} \tag{62}$$

Here $\zeta^* = \bar{\zeta}^t$ denotes the hermitian conjugate. Now, this identifies $\mathfrak{m} \in \mathrm{Mat}(\mathbb{C}, n+1)$ which can of course be described in terms of real matrices $\mathrm{Mat}(\mathbb{R}, 2n+2)$. Let us look at the example $n = 1$; let $\zeta = a + ib$ and we act on $z = (z_1\ z_2)^t \in \mathbb{C}^2$ with $z_k = x_k + iy_k$. Then

$$\begin{aligned} \begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{\zeta} z_2 \\ \zeta z_1 \end{pmatrix} &= \begin{pmatrix} -(a-ib)(x_2+iy_2) \\ (a+ib)(x_1+iy_1) \end{pmatrix} \\ &= \begin{pmatrix} -(ax_2+by_2+i(ay_2-bx_2)) \\ ax_1-by_1+i(ay_1+bx_1) \end{pmatrix} \end{aligned} \tag{63}$$

If we represent $(z_1\ z_2)^t$ as before by $(x_1\ y_1\ x_2\ y_2\ )$, then we find

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -(ax_2+by_2) \\ -(ay_2-bx_2) \\ ax_1-by_1 \\ ay_1+bx_1 \end{pmatrix} \tag{64}$$

so that

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & -a & -b \\ 0 & 0 & b & -a \\ a & -b & 0 & 0 \\ b & a & 0 & 0 \end{pmatrix} \tag{65}$$

This matrix is skew-symmetric and can be diagonalised and admits eigenvalues

$$\lambda_1, \lambda_2 = i\sqrt{a^2+b^2} \quad , \quad \lambda_3, \lambda_4 = -i\sqrt{a^2+b^2} \tag{66}$$

Now, every $2k \times 2k$ skew-symmetric matrix can be diagonalised (its eigenvalues can are purely imaginary and come in pairs, $\pm i\lambda_k$) and can be braught into the form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \dots \\ -\lambda_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \lambda_2 & \\ 0 & 0 & -\lambda_2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{67}$$

In our case, this means that we can find a representation of $\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}$ acting on $\mathbb{R}^{2n+2}$ such that

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right) = \|\zeta\|^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{68}$$

which is a natural generalization of the $n = 0$ case but only a special case of Tej's suggestion (52) **question:** what are (52) in general?

References

[1] F. Bruhat and H. Whitney, Quelques propriétés fondamentales des ensembles analytiques-réels, Comment. Math. Helv. 33, 132-160 (1959).

[2] K. Cieliebak and Y. Eliashberg. From Stein to Weinstein and back: symplectic geometry of affine complex manifolds. Vol. 59. American Mathematical Soc., 2012.