Working Title

January 24, 2025

Contents

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X, namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z, w) = -\beta |\langle \bar{z}, w\rangle|^2 = -\beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2}$$ {eq:toy_S}

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \tag{3}$$ {eq:toy_constr}

The action enjoys a $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge freedom (which here is simply a global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry, acting as

$$e^{i\theta} \times e^{i\varphi} \colon (z, w) \mapsto (e^{i\theta} z, e^{i\varphi} w) \tag{4}$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z = \int \prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2 - 1)\delta(|w|^2 - 1) e^{-S(z,w)} \tag{5}$$ {eq:toy_Z}

In order to evaluate (5), we want to embed the space of fields $\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action $\tilde{S}$ is holomorphic in the new variables (fields)
2. when we restrict to $\mathcal{F}$, $\tilde{S}$ reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [2].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M = m$. Moreover, let $\{U_i, \phi_i\}$ be a real analytic atlas of M, with $U_i \subset \mathbb{R}^m$ and charts $\phi_i \colon U_i \to M$ so that the transition functions

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i \colon U_{ij} \to U_{ij} \tag{6}$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = m$ and a (real analytic) isomorphism $f \colon M \to \tilde{M} \subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}} \subset \mathbb{C}^m$ such that $U_i^{\mathbb{C}} \cap \mathbb{R}^m = U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

(i) the transition functions $\phi_{ij}^{\mathbb{C}} \colon U_{ij}^{\mathbb{C}} \to U_{ij}^{\mathbb{C}}$ are biholomorphic

(ii) $\phi_{ji}^{\mathbb{C}} = \left(\phi_{ij}^{\mathbb{C}}\right)^{-1}$ and $\phi_{ii}^{\mathbb{C}} = id$

(iii) the transition functions $\phi^{\mathbb{C}}_{ij}$ satisfy the usual 2-cocycle condition (gluing condition) on $U^{\mathbb{C}}_{ijk}$: $\phi^{\mathbb{C}}_{ij} \circ \phi^{\mathbb{C}}_{jk} = \phi^{\mathbb{C}}_{ik}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U^{\mathbb{C}}_i, \phi^{\mathbb{C}}_i\}$:

$$M^{\mathbb{C}} = \coprod_i U^{\mathbb{C}}_i / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi^{\mathbb{C}}_{ji}(z_i) \text{ on } U^{\mathbb{C}}_{ij} \tag{7}$$

For more details on this construction see Cieliebak and Eliashberga's book [3]

1.2.1 Example: The N-sphere

{subsec:cmplxfy_spher

Consider the N-sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_\pm = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in $\mathbb{R}^N$ by $X = (X_1, \dots, X_N)$. The atlas we consider is given by steoreographic projection through $p_\pm$: Let $U_\pm = \mathbb{R}^N$ and $V_\pm = S^N - \{p_\pm\}$. Then define charts

$$\phi_\pm \colon U_\pm \to V_\pm \subset S^N \quad , \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \tag{8}$$

with inverse

$$\phi_\pm^{-1} \colon x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \tag{9}$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_-^{-1} \circ \phi_+ \colon X \mapsto \frac{X}{\|X\|^2} \tag{10}$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \tag{11}$$

describes an involution at the unit sphere S^{N-1}. On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$) one finds

$$\phi_+ \circ \phi_- \colon (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \tag{12}$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in $\mathbb{R}^N$ (again, it helps working this out in the case $N = 2$).

Now, since $U_\pm = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_\pm^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \tag{13}$$ {eq:toy_quadric}

The above equation defines a hypersurface (so-called *quadric*) inside $\mathbb{C}^{N+1}$.

There is a very interesting observation I found in this stackexchange post: the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N. The diffeomorphism is realised by the following map:

$$\Psi\colon TS^N \to Q \quad , \quad (x,y) \mapsto z = \Psi(x,y) = x\sqrt{1+\|y\|^2} + iy \tag{14}$$

with inverse

$$\Psi^{-1}(x+iy) = \left(\frac{x}{\sqrt{1+\|y\|^2}}, y \right) \tag{15}$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that Ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x,y) \in \mathbb{R}^{N+1}$ such that $\langle x,y\rangle = \sum_i x_i y_i = 0$)

There exists another very interesting diffeomorphism (which I have discovered in this stackexchange post

$$\Phi\colon S^N \times S^N/\Delta \to TS^N \quad , \quad (u,v) \mapsto \left(u, \frac{v - \langle u,v\rangle\, u}{1 - \langle u,v\rangle} \right) \tag{16}$$ {eq:diff_SNSN_TSN}

Its inverse is given by

$$\Phi^{-1}\colon TS^N \to S^N \times S^N/\Delta \quad , \quad (x,y) \mapsto \left(x, \frac{x(\|y\|^2 - 1) + 2y}{\|y\|^2 + 1} \right) \tag{17}$$

Remark 3. The map (16) is the sterepgraphic projection of $v \in S^N$ through the "pole" $u \in S^N$.

Finally we have the following commutative diagram

$$\begin{tikzcd} & TS^N \arrow[dr, "\Psi"] & \\ S^N \times S^N/\Delta \arrow[ur, "\Phi"] \arrow[rr, "\Psi \circ \Phi"'] & & Q \end{tikzcd} \tag{18}$$

Remark 4. The original $S^N \subset Q$ was located at $S^N = Q \cap \mathbb{R}^{N+1}$. If we follow this through, we see that this S^N corresponds to the zero section $\{(x,0) \in TS^N\}$ inside TS^N and consequently is defined by $\Phi(u,v) = (u,0)$ inside $S^N \times S^N/\Delta$. Note that this is precisely the anti-diagonal

$$\bar{\Delta} = \{(u,-u) \mid u \in S^N\} \subset S^N \times S^N \tag{19}$$

Indeed, the stereographic projection of $-u$ ("south pole") through u ("north pole") gives zero:

$$-u \mapsto \frac{(-u) - \langle (-u), u\rangle\, u}{1 - \langle (-u), u\rangle} = \frac{-u + \|u\|^2 u}{1 + \|u\|^2} = 0 \tag{20}$$

since $\|u\|^2 = 1$. Hence

$$\Phi(u,-u) = (u,0) \tag{21}$$

and the original S^N is located along the anti-diagonal $\bar{\Delta}$.

1.3 Factor from δ-function Constraint

Recall that we are interested in integrals of the form

$$I = \int_{\mathbb{CP}^n \times \mathbb{CP}^n} d\mu(z) d\mu(w) e^{-S(z,w)} \mathcal{O}(z,w) \tag{22}$$

where $d\mu(z) = \prod_i dz_i d\bar{z}_i$ and we have included some observable $\mathcal{O}\colon \mathbb{CP}^n \to \mathbb{C}$. For our toy model, we consider the action functional (2)

$$S(z,w) = -\beta |\langle \bar{z}, w\rangle|^2 \tag{23}$$

It is instructive pass to real coordinates. If $z_k = a_k + ib_k$, let x_k be the real vector $(a_k\ b_k)^t$. If we collect all components, we can form the real vector $x = (\dots a_k\ b_k \dots)^t \in \mathbb{R}^{2n+2}$. We denote the real vector associated to w by y. In terms of these, the path integral over can be written as

$$vol(S^1)^2 \int_{\mathbb{R}^{2n+2} \times \mathbb{R}^{2n+2}} d^{2n+2}x\ d^{2n+2y}\ \delta(\|x\|^2 - 1)\delta(\|y\|^2 - 1) e^{-S(x,y)} \mathcal{O}(x,y) \tag{24}$$

where the pre-factor stems from the compact part of the gauge group $\mathbb{C}^* \circlearrowleft \mathbb{C}^{n+1}$. Let us focus on the following part of the integral:

$$I \sim \int_{\mathbb{R}^{2n+2}} \delta(\|x\|^2 - 1) = \int_{\mathbb{R}^{2n+2}} \delta(\langle x, x\rangle - 1) \tag{25}$$

Following our idea, we complexify the space of integration, $x \to \zeta$

$$I \sim \int_{\Gamma_0 = \mathbb{R}^{2n+2} \cap \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta\rangle - 1) \tag{26}$$

where

$$\langle\zeta,\zeta\rangle = \sum_k \zeta_k^2 \tag{27}$$

The δ-function constraint restricts the support of the integrand to the quadric Q. Now, we would like to deform the "contour" (domain of integration) inside Q

$$I \sim \int_{\Gamma_a \subset \mathbb{C}^{2n+2}} \delta(\langle\zeta,\zeta\rangle - 1) \tag{28}$$

Inspired by the discussion in 1.2.1, we would like to parametrise Q in terms of TS^{2n+1}. We thus choose a parametrisation of the following form: (by abuse of notation, we will use again x to denote a vector in $\mathbb{R}^{2n+2}$)

$$\zeta(x) = x\sqrt{1+\|y(x)\|^2} + iy(x) \tag{29}$$ {eq:parametrisation}

where $y(x)$ is chosen in such a way that $\langle x, y\rangle = 0$. It follows that

$$I \sim \int_{\mathbb{R}^{2n+2}} d^{2n+2}x \ \det J(x)\delta(\langle\zeta(x),\zeta(x)\rangle - 1) \tag{30}$$

where $J(x)$ denotes the Jacobian of (29). Note that the δ-function constraint can be simplified as follows: Let

$$\lambda(x) = \sqrt{1+\|y\|^2} \tag{31}$$

Then

$$\begin{aligned} C(x) &= \langle\zeta(x),\zeta(x)\rangle - 1 \\ &= \lambda^2(x)\|x\|^2 + 2i\lambda(x)\langle x, y(x)\rangle - \|y\|^2 - 1 \\ &= \lambda^2(x)(\|x\|^2 - 1) \end{aligned} \tag{32}$$

Importantly,

$$\lambda^2(x) > 0 \tag{33}$$

such that the δ-function constraint is of the following form:

$$\begin{aligned} \int \delta(C(x)) = \int d\mu(x)\delta(\underbrace{f(x)g(x)}_{\equiv\phi(x)}) \\ &= \int_{\phi^{-1}(0)} \frac{d\sigma}{\|\nabla\phi(x)\|} \\ &= \int_{f^{-1}(0)} \frac{d\sigma}{\|f(x)\nabla g(x) + g(x)\nabla f(x)\|} \\ &= \int_{f^{-1}(x)} \frac{d\sigma}{\|\nabla f(x)\|}\frac{1}{|g(x)|} \\ &= \int d\mu\frac{\delta(f(x)}{|g(x)|} \end{aligned} \tag{34}$$

where $f(x) = \|x\|^2 - 1$ and $g(x) = \lambda^2(x) > 0$. Hence we find that with the chosen parametrisation, the integral nicely localises to an integral over S^{2n+1}

$$\int_{\mathbb{R}^{2n+2}} \delta(C(x)) = \int_{\mathbb{R}^{2n+2}} \frac{\delta\left(\|x\|^2 - 1\right)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{1}{\lambda^2(x)} \tag{35}$$

Therefore, we are left with an integral of the form

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} \quad , \quad \lambda^2(x) = 1 + \|y(x)\|^2 \tag{36}$$ {eq:toy_schema_I}

1.4 Homogeneous Deformations

1.4.1 The General Case

The $n = 0$ example can be nicely generalised. Tej suggested to consider deformations of the form

$$Y_a(X) = \Omega(a)X \quad , \quad \Omega(a) = \mathrm{diag}\left(\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -a_{n+1} \\ a_{n+1} & 0 \end{pmatrix}\right) \tag{37}$$ {eq:Tej_deformation}

for some $a = (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$. This type of deformation have a very nice geometric origin, as we now explain.

Remark 5. Under the isomorphism

$$\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \quad , \quad \begin{pmatrix} x_k \\ y_k \end{pmatrix} \mapsto z_k = x_k + iy_k \tag{38}$$

the deformation (37) is acting on $\mathbb{C}^{n+1}$ by multiplication with the diagonal matrix $\mathrm{diag}(ia_1, \ldots, ia_{n+1})$

$$\begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ ia_k z_k \\ \vdots \end{pmatrix} \tag{39}$$

The finite diffeomorphism generated by this vector field is given by

$$\begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ e^{ia_k} z_k \\ \vdots \end{pmatrix} \tag{40}$$

The generated finite action is thus the action of the torus

$$U(1)^{n+1} \subset \mathrm{SU}(n+1) \tag{41}$$

Let G be a Lie group and $H < G$ a subgroup with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The orbit space G/H is known as a *homogeneous space*. It is called *reductive* if $\mathfrak{g}$ allows the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad , \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \tag{42}$$

If $\mathfrak{g}$ is equipped with a Killing form B (or if G allows a G-invariant metric as a manifold), then G/H is automatically reductive with the choice

$$\mathfrak{m} = \mathfrak{h}^{\perp} \tag{43}$$

where $\mathfrak{h}^{\perp}$ is the orthogonal to $\mathfrak{h}$ with respect to B. Indeed, since B is G-invariant, schematically

$$0 = B(\mathfrak{h}, \mathfrak{m}) = B([\mathfrak{h}, \mathfrak{h}], \mathfrak{m}) = B(\mathfrak{h}, [\mathfrak{h}, \mathfrak{m}]) \tag{44}$$

so that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

In summary, given a reductive homogeneous space G/H, we get a nice decomposition of the tangent space: first, let $o = [e]$ be the class in G/H at the identity. Then

$$T_o(G/H) \cong \mathfrak{m} \tag{45}$$

This isomorphism is given essentially by the pushforward of the natural projection $\pi \colon G \to G/H$. In fact, $\pi \colon G \to G/H$ defines a principal H-bundle. Its vertical vector fields are given by $\ker(\pi_*) = \mathfrak{h}$ and since π is a submersion (π and π_* are both surjective) we have

$$\pi_* \colon \mathfrak{g}/\ker(\pi_*) = \mathfrak{g}/\mathfrak{h} \cong T_o(G/H) \tag{46}$$

Then, to compute the tangent space at any other point, we use translation by G.

Now, recall that

$$S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n) \tag{47}$$

is a homogeneous space where $H = \mathrm{SU}(n)$ sits inside $G = \mathrm{SU}(n+1)$ as the lower right corner

$$\mathrm{SU}(n) \ni h \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in \mathrm{SU}(n+1) \tag{48}$$

The reductive split is given by

$$\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m} \tag{49}$$ {eq:red_split}

Since $\mathfrak{su}(n+1)$ admits a Killing form $B(x, y) \propto tr(x, y)$, we can choose $\mathfrak{m} = \mathfrak{su}(n)^{\perp}$. In the split (49), we have

$$\mathfrak{su}(n) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{su}(n) \end{pmatrix} \subset \mathfrak{su}(n+1) \tag{50}$$

We then find

$$\mathfrak{m} = \mathfrak{su}(n)^{\perp} = \left\{ \begin{pmatrix} 0 & -\zeta^* \\ \zeta & 0 \end{pmatrix} \right\} \quad , \quad \zeta \in \mathbb{C}^{n+1} \tag{51}$$

Here $\zeta^* = \bar{\zeta}^t$ denotes the hermitian conjugate.

In order to describe the tangent space $T_{[g]}(G/K)$ at any point $[g] \in G/H$, we can use the tangential map of left translations. In fact, left translation by g is defined by the map

$$L_g \colon G \to G \quad , \quad g' \mapsto gg' \tag{52}$$

so that its tangential map

$$\theta_g = (L_{g^{-1}})_* \colon T_g G \to T_e G = \mathfrak{g} \tag{53}$$

This map defines a $\mathfrak{g}$-valued 1-form on G known as the left-invariant Maurer-Cartan form. For a matrix group, it can be written as

$$\theta_g = g^{-1} dg \tag{54}$$

Note that the natural projection $\pi \colon G \to G/H$ is equivariant with respect to left translation: if

$$\tau_g \colon G/H \to G/H \quad , \quad [g'] \mapsto [gg'] \tag{55}$$

then we have

$$\pi \circ L_g = \tau_g \circ \pi \quad , \quad \tau_g \circ \tau_{g'} = \tau_{gg'} \tag{56}$$

Moreover,

$$(\pi \circ L_{g^{-1}})_* = \pi_* \circ (L_{g^{-1}})_* = (\tau_{g^{-1}} \circ \pi)_* = (\tau_{g^{-1}})_* \circ \pi_* \tag{57}$$

Hence, denoting $(\tau_{g^{-1}})_* = \vartheta_g$, we have

$$\pi_* \circ \theta_g = \vartheta_g \circ \pi_* \colon T_g G \to T_o(G/H) = \mathfrak{m} \tag{58}$$

For us, we will be mainly interested in the map

$$(\tau_g)_* \colon \mathfrak{m} = T_{[e]}(G/H) \to T_{[g]}(G/H) \tag{59}$$

which parametrises the tangent space of G/H at a point $[g]$ by $\mathfrak{m}$.

Now, if G is a matrix group the pushforward $(\tau_g)_*$ is essentially just given by left multiplication with g (matrix multiplication is linear) and so in we could describe the tangent space $T_p(G/H)$ at a point $p = g(p)p_0$ as follows

$$T_p(G/H) \cong \{(p, g(p)\xi) \mid \xi \in \mathfrak{m}\} \tag{60}$$

where we implicitly identify $g(p)X$ with $g(p)Xp_0$.

Remark 6. In theory, this is very nice. In practice, I believe it is computationally expensive: for a given $p \in S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, we would need to find $g(p)$ which on top of it is only defined up to a multiplication on the right of $\mathrm{SU}(n)$. We could, for example, choose $p_0 = (1, 0 \dots)^t$. There is a way how to algorithmically compute $g(p)$: A matrix belongs to $\mathrm{SU}(n+1)$ iff its columns (rows) form an orthonormal basis of $\mathbb{C}^{n+1}$. If $p_0 = e_1$ is the first standard basis vector, then we start with the basis $\{p, e_2, \dots, e_{n+1}\}$ and by Gram-Schmidt produce an orthonormal basis $\{p, \tilde{e}_1, \dots, \tilde{e}_{n+1}\}$ and set

$$g(p) = (p \; \tilde{e}_2 \; \dots; \tilde{e}_n) \tag{61}$$

However, I believe that Gram-Schmidt is computationally expensive.

Moreover, it might simply not be necessary. In order to define interesting deformations, we could simply consider any $X \in \mathfrak{g}$. Any such ξ will admit a split into $\xi = \xi_{\mathfrak{m}} + \xi_{\mathfrak{h}}$ and since $\mathfrak{h} = \mathrm{Lie}(H) \cong \mathrm{Stab}(p)$ is in the Lie algebra of the stablizer of p (*note that we actually mean p here, not p_0*), $\xi_h p = 0$ (as $hp = p$ for any h in H). Thus, we could simply not care and just work with X as a whole and define deformations for any $\xi \in \mathfrak{g}$ by

$$Y_\xi(X) = \rho(\xi) \cdot X \quad , \quad \xi \in \mathfrak{g} \quad , \quad X \in S^{2n+1} \tag{62}$$

Here, ρ denotes the appropriate representation of $\mathfrak{g}$ on S^{2n+1}. I believe its easiest description is as follows: $\mathrm{SU}(n+1)$ and hence $\mathfrak{su}(n+1)$ naturally acts (by matrix multiplication) on S^{2n+1} when viewed as a subset of $\mathbb{C}^{n+1}$. Under the isomorphism $\mathbb{C} \cong \mathbb{R}^2$, where $z = x + iy \mapsto (x, y)$, multiplication by a complex number $a + ib$ becomes matrix multiplication:

$$(a + ib)z \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{63}$$

Hence, using our representation of $\mathbb{C}^{n+1}$

$$\begin{pmatrix} \vdots \\ z_k = x_k + iy_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_k \\ y_k \\ \vdots \end{pmatrix} \tag{64}$$

we have to simply replace any complex number $a + ib$ in $\xi \in \mathfrak{su}(n+1)$ by the appropriate matrix:

$$\rho(\xi) \colon a + ib \rightsquigarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{65}$$

In practice, it might be easier to have the following workflow:

$$\begin{array}{ccccccc} S^{2n+1} \subset \mathbb{R}^{2n+2} & \longrightarrow & \mathbb{C}^{n+1} & \xrightarrow{\xi \cdot} & \mathbb{C}^{n+1} & \longrightarrow & \mathbb{R}^{2n+2} \\ X & \longrightarrow & Z & \xrightarrow{\xi \cdot} & \xi \cdot Z & \longrightarrow & Y_\xi(X) \end{array} \tag{66}$$

1.5 Example: $n = 0$

1.5.1 Homogeneous deformation

A particular nice example, which can be computed explicitly and serves as a check that everything works nicely is given for the case $n = 0$. Consider the deformation

$$y(x) = \alpha\Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SO}(2) \quad , \quad \alpha \in \mathbb{R} \tag{67}$$ {eq:toy_cst_def}

so that indeed

$$\langle x, y(x) \rangle = 0 \tag{68}$$

Moreover, note that

$$\|y(x)\|^2 = \langle y(x), y(x) \rangle = \alpha^2 \left\langle x, \Omega^t \Omega x \right\rangle = \alpha^2 \|x\|^2 = \alpha^2 \tag{69}$$

and hence

$$\lambda^2(x) = 1 + \alpha^2 \tag{70}$$

We compute the Jacobian of this parametrisation as follows: let $M_i^{\ j} = \partial_i y^j(x)$. Then

$$J_i^{\ j} = \lambda(x)\delta_i^{\ j} + \frac{\langle y, \partial_i y \rangle\, x^j}{\lambda(x)} + iM_i^{\ j} = \lambda(x)\delta_i^{\ j} + \frac{M_{ik} y^k x^j}{\lambda(x)} + iM_i^{\ j} \tag{71}$$

Using bra-ket notation and trivially lowering indices (considering the flat metric on $\mathbb{R}^2$), we can write J as follows:

$$J = \lambda(x) \cdot id + \lambda^{-1}(x) |My\rangle\langle x| + iM \tag{72}$$

Remark 7. The matrix M has some interesting properties.

1. From $\langle x, y(x) \rangle = 0$ it follows that

   $$0 = \partial_i \langle x, y(x) \rangle = y_i(x) + M_{ik} x_k$$

   so that

   $$y_i(x) = M_{ik} x_k \implies y(x) = Mx$$

   and hence

   $$\langle x, y \rangle = \langle x, Mx \rangle = \left\langle M^t x, x \right\rangle = 0$$

   from which follows (if $y(x) = Mx \neq 0$)

   $$\langle x | M = 0$$

Now, for our choice of

$$y(x) = \alpha \Omega x$$

we have

$$M = \alpha \Omega^t = -\alpha \Omega \tag{73}$$

and hence

$$\begin{aligned} J &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 \Omega^t \Omega |x\rangle\langle x| - i\alpha\Omega \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 |x\rangle\langle x| - i\alpha\Omega \end{aligned} \tag{74}$$

Now, here is a neat trick due to Tej. Consider the unitary matrix

$$U = (x \ \Omega x) \quad , \quad \Omega^* = \begin{pmatrix} x^t \\ x^t \Omega^t \end{pmatrix} \tag{75}$$

where x is a 2×2 column vector with unit norm. Then

$$U^* U = \begin{pmatrix} \langle x, x \rangle & \langle x, \Omega x \rangle \\ \langle \Omega x, x \rangle & \langle x, x \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{76}$$

Moreover,

$$\langle x| U = x^t U = (\langle x, x \rangle \ \ \langle x, \Omega x \rangle) = (1 \ 0) \tag{77}$$

as well as

$$U^* \Omega U = U^* (\Omega x \ \Omega^2 x) = U^* (\Omega x \ \ -x) = \begin{pmatrix} \langle x, \Omega x \rangle & -\langle x, x \rangle \\ \langle x, \Omega^t \Omega x \rangle & -\langle x, \Omega^t x \rangle \end{pmatrix} = -\Omega \tag{78}$$

Then

$$\begin{aligned} U^* J U &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* |x\rangle\langle x| U - i\alpha U^* \Omega U \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i\alpha\Omega \\ &= \begin{pmatrix} \lambda(x) + \lambda^{-1}(x)\alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \end{aligned} \tag{79}$$

It follows that

$$\begin{aligned} \det J(x) &= \det(U^* J(x) U) \\ &= \det \begin{pmatrix} \lambda(x) + \lambda^{-1}(x)\alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \\ &= \lambda^2(x) + \alpha^2 - \alpha^2 = \lambda^2(x) \end{aligned} \tag{80}$$

Finally, we obtain that

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{\lambda^2(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \tag{81}$$

This means that the *constant deformation* (67) comes with a trivial *total* Jacobian (Jacobian + factor from the φ-function constraint).

1.5.2 What it really means

Recall that for $n = 0$ we consider the path integral over S^1. Before, we have considered $x \in S^1 \subset \mathbb{R}^2$. Now, we might think of its complex formulation: let us consider $z = e^{i\theta} \in S^1 \subset \mathbb{C}$. In [**?**], the complexification was chosen in terms of $\theta \to \theta + i\tau$. This means that

$$z \mapsto \tilde{z} = e^{i(\theta+i\tau)} = e^{-\tau} z \tag{82}$$

On the other hand, we can parametrize x by

$$x = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{83}$$

so that

$$\begin{aligned} \tilde{z} &= \begin{pmatrix} \sin(\theta + i\tau) \\ \cos(\theta + i\tau) \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cosh(\tau) + i\cos(\theta)\sinh(\tau) \\ \cos(\theta)\cosh(\tau) - i\sin(\theta)\sinh(\tau) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\tau) & i\sinh(\tau) \\ -i\sinh(\tau) & \cosh(\tau \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} = \lambda x + i \underbrace{\Omega x}_{y(x)} \end{aligned} \tag{84}$$

where

$$\Omega = \begin{pmatrix} 0 & -\sinh(\tau) \\ \sinh(\tau) & 0 \end{pmatrix} \tag{85}$$

and

$$\lambda^2 = 1 + \|y(x)\|^2 = 1 + \sinh^2(\tau) = \cosh^2(\tau) \tag{86}$$

To summarise, if we consider the homogeneous deformation (67),

$$y(x) = \alpha\Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{87}$$

then choosing

$$\boxed{\alpha = \sinh(\tau)} \tag{88}$$

we find that the homogeneous deformation simply scales the complex variable $z = e^{i\theta} \in S^1 \subset \mathbb{C}$,

$$z \to \tilde{z} = e^{-\tau} z \tag{89}$$

1.5.3 Generalisation: Torus deformations

{subsubsec:Torus_defo

Let us now think about $n > 0$. Let $z = [z_0 : \cdots : z_n] \in \mathbb{CP}^n$. Again, we factor out the $\mathrm{U}(1) \subset \mathbb{C}^*$ and fix the $\mathbb{R}_+ \subset \mathbb{C}^*$ by restricting the path integral to S^{2n+1}, i.e.

$$\sum_k |z_k|^2 = 1 \tag{90}$$

Following the logic of the homogeneous deformation, let us consider a deformation parametrised by a generic element in the Cartan subalgebra of $\mathfrak{su}(n+1)$

$$Y = \Omega(\alpha)X = \sum_{k=1}^{n} \rho(i\alpha_k H_k) \tag{91}$$

where we choose the following *hermitian* (hence the factor of i) basis of the Cartan $\mathfrak{h} \subset \mathfrak{su}(n+1)$ subalgebra

$$H_k = E_{kk} - E_{(k+1)(k+1)} \tag{92}$$

where E_{ij} is the $n \times n$ matrix with $(E_{ij})_{ab} = \delta_{ai}\delta_{bj}$. Here, ρ is again the representation of complex multiplication in terms of 2×2 matrices,

$$\rho(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{93}$$

and defined element-wise on a generic complex matrix as

$$\rho(M) = \begin{pmatrix} \rho(m_{11}) & \rho(m_{12}) & \dots \\ \rho(m_{21} & \rho(m_{22}) & \dots \\ \vdots & \vdots & \end{pmatrix} \tag{94}$$

In particular,

$$\rho(iH_k) = \rho(iE_{kk}) - \rho(iE_{(k+1)(k+1)}) = \begin{pmatrix} \ddots & & & & & \\ & 0 & -1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & \ddots \end{pmatrix} \tag{95}$$

Let us choose

$$\alpha_k = \sinh(\tau_k) \quad , \quad \tau_0 \equiv \tau_{n+1} \equiv 0 \tag{96}$$

then, acting on the complex vector $z = (z_0, \dots, z_n)$, we find

$$\boxed{\tilde{z} = (\tilde{z}_0, \dots, \tilde{z}_n) = (e^{-\Delta_{01}} z_0, \dots, e^{-\Delta_{nn+1}} z_n) \quad , \quad \Delta_{kk+1} = \tau_k - \tau_{k+1}} \tag{97}$$

1.6 Correlators

1.6.1 Generalities

We may try to compute correlation functions analytically. For this, let us view $\mathbb{CP}^n$ as the homogeneous space

$$\mathbb{CP}^n = \mathrm{SU}(n+1)/\,\mathrm{SU}(n) \tag{98}$$

with

$$z = g(z)\eta \quad , \quad \eta = [1 : 0 : \cdots : 0] \quad , \quad g(z) \in \mathrm{SU}(n+1) \tag{99}$$

The action can therefore be written as

$$S = -\beta |z^* w|^2 = -\beta |\eta^t U^*(z) U(w) \eta|^2 \tag{100}$$

Note that $U \equiv U(z,w) = U^*(z)U(w) \in \mathrm{SU}(n)$ and so if $\mu(U(z))$ denotes the Haar measure of $\mathrm{SU}(n+1)$, we have

$$\mu(U(w))\mu(U^*(z)U(w)) = \mu(U) \tag{101}$$

Denoting $U(z) \equiv U'$, we can perform a change of variables such that the path integral measure becomes

$$\mu(U(w))\mu(U(z)) = \mu(U(z))\mu(U(z,w)) \equiv \mu(U')\mu(U) \tag{102}$$

Moreover, recall that in our model instead of integrating over $\mathbb{CP}^n \times \mathbb{CP}^n$ directly, we integrate out the $\mathrm{U}(1) \subset \mathbb{C}^*$ first and use the resulting $\mathbb{R}_+ \subset \mathbb{C}^*$ gauge freedom to fix the norm, hence integrating over $S^{2n+1} \times S^{2n+1}$. Now, $S^{2n+1} = \mathrm{SU}(n+1)/SU(n)$ and so

$$\int_{\mathrm{SU}(n+1)} \mu = \int_{\mathrm{SU}(n)} \mu \int_{\mathrm{SU}(n+1)/\,\mathrm{SU}(n)} \hat{\mu} = \int_{S^{2n+1}} \hat{\mu} \tag{103}$$

It is important to remark that we are working with the *normalized* Haar measure μ so that

$$\int_{\mathrm{SU}(n)} \mu = 1 \tag{104}$$

Hence the partition function can be written in terms of a single integration over $\mathrm{SU}(n+1)$

$$\begin{aligned} Z &= \int_{\mathrm{SU}(n+1)} \mu(U') \int_{\mathrm{SU}(n+1)} \mu(U) e^{-S(U)} \\ &= \int_{\mathrm{SU}(n+1)} \mu(U) e^{-S(U)} \end{aligned} \tag{105}$$

The action functional simplifies to

$$S(U) = -\beta |\eta^t U \eta|^2 = -\beta |U_{11}|^2 \tag{106}$$

and any correlation function can be written as

$$\langle \mathcal{O}(z,w) \rangle = \frac{1}{Z} \int_{\mathrm{SU}(n+1)} \mu(U') \int_{\mathrm{SU}(n+1)} \mu(U) e^{-S(U)} \mathcal{O}(U,U') \tag{107}$$

Remark 8. Obviously we could have chosen a different η (having its only non-zero entry sitting at the ith component, say. This choice cannot affect the result and we will see later that we use this freedom to simply certain expressions.

1.6.2 One-point function

It is time to look at interesting operators we would like to compute. We start with the one-point function. Let

$$\mathcal{O}_{ij}(z) = z_i \bar{z}_j \tag{108}$$

Clearly, $\mathcal{O}_{ij}$ is invariant under the U(1) action $z_k \to e^{i\varphi} z_k$ for all k.

It's expectation value is given by

$$\langle \mathcal{O}_{ij}(z) \rangle = \frac{1}{(2\pi)^2} \int_{S^{2n+1} \times S^{2n+1}} \prod_k d^n z_k d^n w_k \; z_i \bar{z}_j e^{-S(z,w)} \tag{109}$$

where

$$S(z,w) = -\beta |\langle \bar{z}, w \rangle|^2 = -\beta \sum_{k,\ell} \bar{z}_\ell z_k w_\ell \bar{w}_k \tag{110}$$

Note that both, the action and the measure is invariant under the diagonal U(1) action

$$(z_k, w_k) \mapsto (e^{i\alpha} z_k, e^{i\alpha} w_k) \quad , \quad \forall k \tag{111}$$

If $i \neq j$, by this change of coordinates, for $k = i$, it follows that

$$\langle \mathcal{O}_{ij}(z) \rangle = e^{-i\alpha} \langle \mathcal{O}_{ij}(z) \rangle \tag{112}$$

for any α and hence we must have

$$\boxed{\langle \mathcal{O}_{ij}(z) \rangle = 0 \quad , \quad \forall z, i \neq j} \tag{113}$$ {eq:toy_one_pt}

We can also see this in terms of the SU$(n+1)$ parametrization. In this parametrization,

$$\mathcal{O}_{ij}(z) = z_i \bar{z}_j = \sum_{k,\ell} U_{ik} \eta_k \bar{U}_{j\ell} \eta_\ell = \sum_{k\ell} \delta_{k1} \delta_{\ell 1} U_{ik} \bar{U}_{j\ell} = U_{i1} U^*_{1j} \tag{114}$$

It turns out to be very useful to make a change of coordinates $U' \equiv U^*(z) U(w)$ for the integration of $U(w)$. Then

$$\begin{aligned} \langle \mathcal{O}_{ij}(U) \rangle &= \frac{1}{Z} \int_{\mathrm{SU}(n+1)} \mu(U) U_{i1} U^*_{1j} \int_{\mathrm{SU}(N)} \mu(U') e^{-S(U')} \\ &= \int_{\mathrm{SU}(n+1)} \mu(U) U_{i1} \bar{U}_{j1} \end{aligned} \tag{115}$$ {eq:toy_one_pt_from_S

Integrals of SU(N) polynomials have been studied in the literature, see for example [1]

$$\begin{aligned} I_N(r,s) &= \int_{\mathrm{SU}(N)} \mu(N) \prod_{k=1}^{r} U_{i_k j_k} \prod_{\ell=1}^{s} \bar{U}_{i'_\ell j'_\ell} \\ &= \sum_{q=0}^{\infty} \delta_{s-r,Nq} \sum_{\sigma\in S_r} \mathrm{Wg}^{N,q}(\sigma) \sum_{\rho\in S_{r+Nq}} \prod_{k=1}^{r} \delta_{i_k i'_{\rho(k)}} \delta_{j_{\sigma(k)} j'_{\rho(k)}} \\ &\cdot \frac{1}{(Nq)!} \sum_{\tau\in S_{Nq}} \chi_{q^N}(\tau) \prod_{k=1}^{Nq} \delta_{i'_{\rho(\tau(k)+r)} j'_{\rho(k+r)}} \end{aligned} \tag{116}$$

This beast simplifies for $r = s$ (which, luckily, is the case we are interested in) to

$$I_N(r) \equiv I_N(r,r) = \sum_{\sigma,\tau\in S_r} \mathrm{Wg}^N(\tau^{-1}\sigma) \prod_{k=1}^{r} \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \tag{117}$$ {eq:SUN_poly_int}

Here, Wg^N denotes the Weingarten map which for $\sigma \in S_r$ is given by

$$\mathrm{Wg}^N(\sigma) = \frac{1}{(r!)^2} \sum_{\substack{\lambda\in irreps(S_r) \\ \ell(\lambda)\leq N}} \frac{d_\lambda^2 \chi_\lambda(\sigma)}{s_\lambda(1^N)} \tag{118}$$

where d_λ is the dimension and $\chi_\lambda(\sigma)$ the character of the irreducible representation λ of S_r. Moreover, $s_\lambda(X)$, $X = (X_1, \dots, X_N)$ denote the Shur functions. Irreps λ of S_r are enumerated by partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ of r such that

$$|\lambda| = \sum_{i=1}^{\ell(l)} \lambda_i = r \quad , \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(l)} \tag{119}$$

The integer $\ell(\lambda)$ is known as the *length* of the partition.

Back to our one-point function. We can rewrite (115) as follows:

$$\begin{aligned} \langle \mathcal{O}_{ij}(U) \rangle &= \int_{\mathrm{SU}(n+1)} \mu(U) U_{i1} \bar{U}_{j1} \\ &= \sum_{\tau,\sigma\in S_1} \mathrm{Wg}^{n+1}(\tau^{-1}\sigma) \prod_{k=1}^{1} \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \\ &= \mathrm{Wg}^{n+1}(id)\delta_{ij} \end{aligned} \tag{120}$$

Let us now have a look at its variance

$$Var = \left\langle |\mathcal{O}_{ij}(z)|^2 \right\rangle = \left\langle |z_i|^2 |z_j|^2 \right\rangle \tag{121}$$

which by the same reasoning can be written as

$$\begin{aligned} Var &= \int_{\mathrm{SU}(n+1)} \mu(U) U_{i1} U_{j1} \bar{U}_{i1} \bar{U}_{j1} \\ &= \sum_{\tau,\sigma \in S_2} \mathrm{Wg}^{n+1}(\tau^{-1}\sigma) \prod_{k=1}^{2} \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \end{aligned} \tag{122}$$

Since $S_2 = \{1^2 \equiv id, 2 \equiv (12)\}$, we find

$$\begin{aligned} Var &= \Big(\mathrm{Wg}^{n+1}(1^2) \left(\delta_{ii}\delta_{11}\delta_{jj}\delta_{11} + \delta_{ij}^2 \delta_{11}^2\right) + \mathrm{Wg}^{n+1}(2) \left(\delta_{ii}\delta_{jj}\delta_{11}^2 + \delta_{ij}^2\delta_{11}^2\right) \Big) \\ &= (1+\delta_{ij}) \Big(\mathrm{Wg}^{n+1}(1^2) + \mathrm{Wg}^{n+1}(2) \Big) \end{aligned} \tag{123}$$

According to Wikipedia,

$$\mathrm{Wg}^d(1^2) = \frac{1}{d^2-1} \quad , \quad \mathrm{Wg}^d(2) = \frac{-1}{d(d^2-1)} \tag{124}$$

so that

$$\boxed{Var(\mathcal{O}_{ij}(z)) = \frac{1+\delta_{ij}}{(n+1)(n+2)}} \tag{125}$$

Remark 9. If we consider a torus / Cartan deformation as outlined in Section 1.5.3, then we find

$$\begin{aligned} \mathrm{Var}(\widetilde{\mathcal{O}}_{ij}(\tilde{z})) &= \langle e^{-2\Delta_{ii+1}} |z_i|^2 e^{-2\Delta_{jj+1}} |z_j|^2 \rangle_{deformed} \\ &= e^{-2\Delta_{ii+1} - 2\Delta_{jj+1}} \left\langle |z_i|^2 |z_j|^2 \ \det(J_{tot}) \ e^{-(\tilde{S}-S)} \right\rangle \end{aligned} \tag{126}$$

1.6.3 Two-point function

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