Working Title

November 14, 2024

Contents

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X, namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z,w) = \beta |\langle \bar{z}, w\rangle|^2 = \beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2}$$ {eq:toy_S}

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \tag{3}$$ {eq:toy_constr}

The action enjoys a $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge freedom (which here is simply a global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry, acting as

$$e^{i\theta} \times e^{i\varphi} \colon (z,w) \mapsto (e^{i\theta} z, e^{i\varphi} w) \tag{4}$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z = \int \prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2 - 1)\delta(|w| - 1)e^{-S(z,w)} \tag{5}$$ {eq:toy_Z}

In order to evaluate (5), we want to embed the space of fields $\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action $\tilde{S}$ is holomorphic in the new variables (fields)
2. when we restrict to $\mathcal{F}$, $\tilde{S}$ reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [1].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M = m$. Moreover, let $\{U_i, \phi_i\}$ be a real analytic atlas of M, with $U_i \subset \mathbb{R}^m$ and charts $\phi_i \colon U_i \to M$ so that the transition functions

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i \colon U_{ij} \to U_{ij} \tag{6}$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = m$ and a (real analytic) isomorphism $f \colon M \to \tilde{M} \subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}} \subset \mathbb{C}^m$ such that $U_i^{\mathbb{C}} \cap \mathbb{R}^m = U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

(i) the transition functions $\phi_{ij}^{\mathbb{C}} \colon U_{ij}^{\mathbb{C}} \to U_{ij}^{\mathbb{C}}$ are biholomorphic

(ii) $\phi_{ji}^{\mathbb{C}} = \left(\phi_{ij}^{\mathbb{C}}\right)^{-1}$ and $\phi_{ii}^{\mathbb{C}} = id$

(iii) the transition functions $\phi_{ij}^{\mathbb{C}}$ satisfy the usual 2-cocycle condition (gluing condition) on $U_{ijk}^{\mathbb{C}}$: $\phi_{ij}^{\mathbb{C}} \circ \phi_{jk}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U_i^{\mathbb{C}}, \phi_i^{\mathbb{C}}\}$:

$$M^{\mathbb{C}} = \coprod_i U_i^{\mathbb{C}} / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi_{ji}^{\mathbb{C}}(z_i) \text{ on } U_{ij}^{\mathbb{C}} \tag{7}$$

For more details on this construction see Cieliebak and Eliashberga's book [2]

1.2.1 Example: The N-sphere

Consider the N-sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_\pm = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in $\mathbb{R}^N$ by $X = (X_1, \dots, X_N)$. The atlas we consider is given by steoreographic projection through $p_\pm$: Let $U_\pm = \mathbb{R}^N$ and $V_\pm = S^N - \{p_\pm\}$. Then define charts

$$\phi_\pm \colon U_\pm \to V_\pm \subset S^N \quad , \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \tag{8}$$

with inverse

$$\phi_\pm^{-1} \colon x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \tag{9}$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_-^{-1} \circ \phi_+ \colon X \mapsto \frac{X}{\|X\|^2} \tag{10}$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \tag{11}$$

describes an involution at the unit sphere S^{N-1}. On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$) one finds

$$\phi_+ \circ \phi_- \colon (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \tag{12}$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in $\mathbb{R}^N$ (again, it helps working this out in the case $N = 2$).

Now, since $U_\pm = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_\pm^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \tag{13}$$ {eq:toy_quadric}

The above equation defines a hypersurface (so-called *quadric*) inside $\mathbb{C}^{N+1}$.

There is a very interesting observation I found in this stackexchange post: the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N. The diffeomorphism is realised by the following map:

$$\Psi\colon TS^N \to Q \quad , \quad (x, y) \mapsto z = \Psi(x, y) = x\sqrt{1 + \|y\|^2} + iy \tag{14}$$

with inverse

$$\Psi^{-1}(x + iy) = \left(\frac{x}{\sqrt{1 + \|y\|^2}}, y \right) \tag{15}$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that Ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x, y) \in \mathbb{R}^{N+1}$ such that $\langle x, y\rangle = \sum_i x_i y_i = 0$)

There exists another very interesting diffeomorphism (which I have discovered in this stackexchange post

$$\Phi\colon S^N S^N/\Delta \to TS^N \quad , \quad (u, v) \mapsto \left(u, \frac{v - \langle u, v\rangle\, u}{1 - \langle u, v\rangle} \right) \tag{16}$$

{eq:diff_SNSN_TSN}

Its inverse is given by

$$\Phi^{-1}\colon TS^N \to S^N \times S^N/\Delta \quad , \quad (x, y) \mapsto \left(x, \frac{x(\|y\|^2 - 1) + 2y}{\|y\|^2 + 1} \right) \tag{17}$$

Remark 3. The map (16) is the sterepgraphic projection of $v \in S^N$ through the "pole" $u \in S^N$.

Finally we have the following commutative diagram

$$\begin{tikzcd}
 & TS^N \arrow[dr, "\Psi"] & \\
S^N \times S^N/\Delta \arrow[ur, "\Phi"] \arrow[rr, "\Psi \circ \Phi"'] & & Q
\end{tikzcd} \tag{18}$$

Remark 4. The original $S^N \subset Q$ was located at $S^N = Q \cap \mathbb{R}^{N+1}$. If we follow this through, we see that this S^N corresponds to the zero section $\{(x, 0) \in TS^N\}$ inside TS^N and consequently is defined by $\Phi(u, v) = (u, 0)$ inside $S^N \times S^N/\Delta$. Note that this is precisely the anti-diagonal

$$\bar{\Delta} = \{(u, -u) \mid u \in S^N\} \subset S^N \times S^N \tag{19}$$

Indeed, the stereographic projection of $-u$ ("south pole") through u ("north pole") gives zero:

$$-u \mapsto \frac{(-u) - \langle (-u), u\rangle\, u}{1 - \langle (-u), u\rangle} = \frac{-u + \|u\|^2 u}{1 + \|u\|^2} = 0 \tag{20}$$

since $\|u\|^2 = 1$. Hence

$$\Phi(u, -u) = (u, 0) \tag{21}$$

and the original S^N is located along the anti-diagonal $\bar{\Delta}$.

1.3 Application to the Model

References

[1] F. Bruhat and H. Whitney, Quelques propriétés fondamentales des ensembles analytiques-réels, Comment. Math. Helv. 33, 132-160 (1959).

[2] K. Cieliebak and Y. Eliashberg. From Stein to Weinstein and back: symplectic geometry of affine complex manifolds. Vol. 59. American Mathematical Soc., 2012.