Contents

1 Cohomology of $\mathbb{CP}^n$

1.1 algebraic approach

We know: $\mathbb{CP}^n$ is Kähler with Kähler form given by the Fubini-Study form (in a chart $z_0 \neq 0$

$$\omega_{FS} = \partial\bar{\partial} \log \left(1 + \sum_{i=1}^{n} \left| \frac{z_i}{z_0} \right|^2 \right)$$

Note that ω_{FS} is $SU(n+1)$ invariant.

Remark 1. Locally ($z_0 \neq 0$) one can consider

$$\omega_{FS} = \partial\bar{\partial}\log\left(|z_0|^2 + \sum_{i=1}^{n}|z_i|^2\right).$$

The argument of the log is just the $SU(n+1)$-invariant metric on $\mathbb{C}^{n+1}$. However, this metric is *not* $\mathbb{C}^*$-invariant and therefore does not descent to $\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$. $\triangleleft$

Example 1 ($n = 1$)**.** For $n = 1$ one finds

$$\omega_{FS} = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

◀

Now,

$$\left.\begin{array}{r} \omega_{FS} \in H^2(\mathbb{CP}^n) \\ \omega_{FS}^2 \in H^4(\mathbb{CP}^n) \\ \vdots \\ \omega_{FS}^n \in H^{2n}(\mathbb{CP}^n) \end{array}\right\} \implies \int \omega_{FS}^n = 1$$

In fact, we know that the cohomology ring $H^\bullet(\mathbb{CP}^n)$ has exactly *one* generator: $\sigma = [\omega_{FS}]$:

$$H^\bullet(\mathbb{CP}^n) = span\left(1, \sigma, \sigma^2, \ldots, \sigma^n\right). \tag{1}$$

Moreover, there exists an integration map

$$\int : H^{2n}(\mathbb{CP}^n) \to \mathbb{C}$$
$$\sigma^k \mapsto \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

This integration map can be written as a *residue formula*:

$$\sigma^k \mapsto \frac{1}{2\pi i}\oint \frac{d\sigma}{\sigma^{n+1}}\sigma^k.$$

Any element of $H^\bullet(\mathbb{CP}^n) = span\{1, \sigma, \ldots, \sigma^n\}$ is a polynomial in σ, and thus the integration map for a general element of $H^\bullet(\mathbb{CP}^n)$ is given by the residue formula

$$P(\sigma) \mapsto \frac{1}{2\pi i}\oint \frac{d\sigma}{\sigma^{n+1}}P(\sigma) \tag{2}$$

1.2 geometric approach

We can calculate the cohomology of $\mathbb{CP}^n$ also in a more geometric way. Recall that $\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$. Consider a hyperplane $\tilde{H} \subset \mathbb{C}^{n+1}$ given by some linear equation

$$\tilde{H}_\alpha : \quad \sum_{i=0}^{n} \alpha_i z_i = \alpha \cdot z = 0.$$

Since the defining equation is homogeneous in z_i, $\tilde{H}_\alpha$ is invariant under the $\mathbb{C}^*$-action $z_i \mapsto \lambda z_i$ and thus descents to $\mathbb{CP}^n$:

$$H_\alpha = \tilde{H}_\alpha/\mathbb{C}^* \subseteq \mathbb{CP}^n.$$

Example 2 ($\mathbb{CP}^1$)**.** Consider $n = 1$.

$$\tilde{H}_\alpha : \quad \alpha_0 z_0 + \alpha_1 z_1 = 0 \implies \text{a line}$$

Remark 2. For example, let $\alpha_0 \neq 0$, then $z_0 = \frac{\alpha_1}{\alpha_0} z_1$ which describes indeed a line. ◁

Thus $H_\alpha = \tilde{H}_\alpha/\mathbb{C}^* = pt.$ ◀

Example 3 ($\mathbb{CP}^2$)**.**

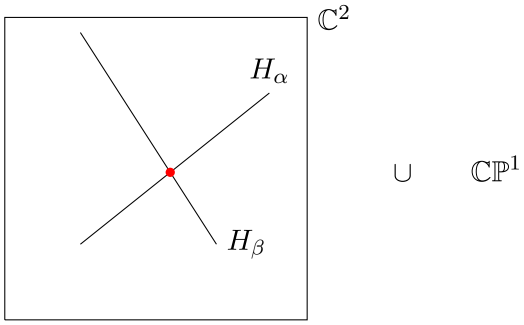
$$\tilde{H}_\alpha : \qquad \alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0 \implies \text{a plane}$$

Then

$$H_\alpha = \tilde{H}_\alpha/\mathbb{C}^* = \text{ (projective) line}$$

Remark 3. This can be understood best in an example: suppose $\alpha_0 \neq 0, \alpha_1 = \alpha_2 = 0$. Then $\tilde{H} = \{(0, z_1, z_2)\} \to \{(z_1, z_2) \mid (z_1, z_2) \neq (0,0)\} = \mathbb{C}^2 - \{0\}$. If we thus mod out by the $\mathbb{C}^*$ action, we obtain $H = \tilde{H}/\mathbb{C}^* = \mathbb{CP}^1$. More generally, we can always solve z_0 as a function of z_1, z_2 which then again gives an identification of $\tilde{H}$ with $\mathbb{C}^2 - \{0\}$. ◁

Question: How many intersection points do any two hyperplanes H_α and H_β have in $\mathbb{CP}^2$? The answer to this question is: any two (distinct not parallel) hyperplanes intersect in a *unique* point. Note that $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$ (think of $\mathbb{C}^2$ with the point at infinity blown up). That two hyperplanes in general position intersect at all, is best understood in teerms of a picture:



The two hyperplanes (in general position) intersect in $\mathbb{C}^2$ in a unique point. The question is thus, if they intersect at "∞". However, since in $\mathbb{CP}^2$ infinity is blown up to a $\mathbb{CP}^1$, ∞ is not just a point, but *a point with a direction.* Since the two hyperplanes H_α and H_β approach ∞ with two different directions, they *do not* intersect at infinity. ◀

How do we pass from intersection theory to cohomology? Suppose that Φ is any figure. Consider the singular (smeared) differential form $\delta^\varepsilon(\Phi)$ defined as a smeared delta-function with support in Φ.

Example 4 (a point in $\mathbb{R}$)**.** Let $\Phi = 0 \in \mathbb{R}$.

$$\delta^\varepsilon(\Phi) = e^{-|x|^2/\varepsilon}\frac{dx}{\sqrt{\varepsilon}} \underset{\varepsilon\to 0}{\longrightarrow} \delta(x)dx$$

Thus the Gaussian exponential localizes to the support to the "hyperplane" $x = 0$. ◀

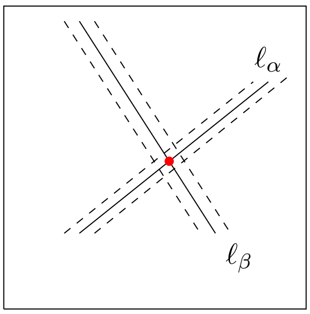
Example 5 (a line in $\mathbb{C}$)**.** Consider a line $\ell_\alpha : f_\alpha(x) = \alpha_1 x_1 + \alpha_2 x_2 = 0$ in $\mathbb{C}$. Then

$$\delta^\varepsilon(\ell) = e^{-|f_\alpha(x)|^2/\varepsilon}\frac{df_\alpha}{\sqrt{\varepsilon}}.$$

Given two lines, what is their intersection? Consider the integral

$$\int_{\mathbb{C}=\mathbb{R}^2} \delta^\varepsilon(\ell_\alpha) \wedge \delta^{\varepsilon'}(\ell_\beta) \tag{3}$$

For finite $\varepsilon, \varepsilon'$, the support of the integrand localizes in a small area around the intersection point:



the support of $\delta^\varepsilon(\ell_\alpha) \wedge \delta^{\varepsilon'}(\ell_\beta)$ is proportional to the area $\sim \sqrt{\varepsilon\varepsilon'}$ which, in the integral (3) is canceled by the denominator of $df_\alpha \wedge df_\beta/\sqrt{\varepsilon\varepsilon'}$. Now, if one considers the change of coordinates $f_\alpha(x) = y$ and $f_\beta = y'$, then the integral becomes

$$\int_{R^2} \delta^\varepsilon(\ell_\alpha) \wedge \delta^{\varepsilon'}(\ell_\beta) = \int_{\mathbb{R}^2} e^{-|y|^2} e^{-|y'|^2} \frac{dydy'}{\sqrt{\varepsilon\varepsilon'}} \sim 1 \pmod{\pi' s}$$

Therefore, in the limit $\varepsilon, \varepsilon' \to 0$ one has

$$\int_{R^2} \delta^\varepsilon(\ell_\alpha) \wedge \delta^{\varepsilon'}(\ell_\beta) = \#(\ell_\alpha \cap \ell_\beta)$$

Remark 4. The $\delta^\varepsilon(\ell)$ is nothing but the smeared Poincaré dual of the (homology) class $[\ell]$ of the line, that is

$$\int_\ell \alpha = \int_{\mathbb{R}^2} \delta^\varepsilon(\ell) \wedge \alpha.$$

Then it is clear that

$$\#(\ell_\alpha \cap \ell_\beta) = \int_{\ell_\alpha \cap \ell_\beta} 1 = \int_{\ell_\alpha} \delta^{\varepsilon'}(\ell_\beta) = \int_{\mathbb{R}^2} \delta^\varepsilon(\ell_\alpha) \wedge \delta^{\varepsilon'}(\ell_\beta).$$

$\triangleleft$

Remark 5 (complex vs real)**.** Note that the support of the delta-forms is given near the vicinity of

$$f_\alpha(x) = f_\beta(x) = 0.$$

If we consider x as a real variable, then the above set of equations might not have a solution. However, there exists always a solution if we consider x to be complex-valued. $\triangleleft$

$\blacktriangleleft$

Now, the crucial observation for the calculation of $H^\bullet(\mathbb{CP}^n)$ is that

$$[\delta^\varepsilon(H)] = [\omega_{FS}]. \tag{4}$$

This can be seen as follows: note that outside of the hyperplane $H : z_0 = 0$, $\omega_{FS} = \partial\bar{\partial}\log(1 + |z_1/z_0|^2 + \dots)$ is exact. Therefore,

$$\omega_{FS} = d\xi + \eta$$

where ξ is supported on the hyperplane $H : z_0 = 0$. Supposing that log is replaced by $\log^\varepsilon$, some smeared version of the logarithm, which is modified near $z_0 = 0$, one shows $[\omega_{FS}] = [\delta^\varepsilon(H)]$ by the following idea:

$$\delta^{\varepsilon_1}(H_1) \wedge \delta^{\varepsilon_2}(H_2) \sim \delta(H_1 \cap H_2).$$

Thus

$$\delta^{\varepsilon_1}(H_1) \wedge \dots \wedge \delta^{\varepsilon_n}(H_n) \sim \delta(H_1 \cap \dots \cap H_n).$$

Now, n-hyperplanes in general position have a unique fixed point. It follows that

$$\int_{\mathbb{CP}^n} \delta^{\varepsilon_1}(H_1) \wedge \dots \wedge \delta^{\varepsilon_n}(H_n) \sim \int_{\mathbb{CP}^n} \delta(H_1 \cap \dots \cap H_n) = 1 = \int_{\mathbb{CP}^n} \omega_{FS}^n.$$

However, notice that one can continuously deform the hyperplanes (as long as they stay in general position), which shows that the class $[\delta(H)]$ does not depend on H. **true? the above does not show that** $[\omega_{FS}] = [\delta(H)]$

2 equivaraiant cohomology of $\mathbb{CP}^n$

2.1 recollection of equivariant cohomology

Consider a manifold X with an $U(1)$-action $U(1)\circlearrowright X$ generated by the vector field v. Consider the linear operator

$$d_\varepsilon = d + \varepsilon\iota_v, \qquad d_\varepsilon^2 = \varepsilon\mathcal{L}_v, \qquad \deg(\varepsilon) = 2$$

One thus has a complex

$$(\Omega_{inv}(X), d_\varepsilon),$$

where $\Omega_{inv}(X) \subseteq \Omega(X)$ is given by invariant forms, that is $\mathcal{L}_v\omega = 0$ for all $\omega \in \Omega_{inv}$. We want to compare this complex with the usual de Rham complex (Ω, d).

(i) any $[\omega] \in H^\bullet(\Omega^\bullet, d)$ has a representative $\omega_{inv} \in \Omega^\bullet_{inv}$.

Proof. Set

$$\omega_{inv} = \int_{S^1} d\theta e^{i\theta\mathcal{L}_v}\omega = \int_{S^1} d\theta e^{i\theta\{d,\iota_v\}}\omega = \int_{S^1} d\theta e^{i\theta d\iota_v}\omega.$$

Then

$$d\omega_{inv} \propto \int_{S^1} d\theta d(\omega + i\theta d\iota_v\omega + \dots) = 0.$$

$\square$

(ii) $(\Omega^\bullet, d)$ is a dga. In fact, so is $(\Omega^\bullet_{inv}, d_\varepsilon)$

(iii) There exists an integration map $\int_X : \Omega^\bullet \to \mathbb{C}$ which satisfies in the case of compact X

$$\int_X d\omega = 0.$$

This endows the ring $\Omega^\bullet$ with a Frobenius structure, where a *Frobenius structure* on a ring R is a pairing $\langle\cdot,\cdot\rangle$, satisfying $\langle ab, c\rangle = \langle a, bc\rangle$. The Frobenius structure for $\Omega^\bullet$ is given simply by

$$\langle\omega_1, \omega_2\rangle = \int_X \omega_1 \wedge \omega_2.$$

Clearly one has $\langle\omega_1 \wedge \omega_2, \omega_3\rangle = \langle\omega_1, \omega_2 \wedge \omega_3\rangle$. Moreover, $(\Omega^\bullet, d)$ is a *differential Frobenius algebra*, that is one has $\langle d\omega_1, \omega_2\rangle = \pm\langle\omega_1, d\omega_2\rangle$. Does $(\Omega^\bullet_{inv}, d_\varepsilon)$ has a similar "differential Frobenius structure"?

Consider the map

$$av\colon \Omega^\bullet \to \Omega^\bullet_{inv}$$

which is defined as

$$av(\omega) = \int_{S^1} d\theta e^{i\theta \mathcal{L}_v}\omega.$$

Note that av is a map of complexes, i.e. it commutes with the differential: $[av, d] = 0$.

Claim: The kernel $ker(av)$ is big, but one does not loose any cohomology. Put differently, the claim is that the complement of $\Omega^\bullet_{inv}$ inside $\Omega^\bullet$ is contractible, that is there exists a chain homotopy h such that

$$[d, h] = 1 - \pi_{inv}$$

where π_{inv} denotes the projection to Ω_{inv}. We therefore have the following picture

$$h \circlearrowright \ \begin{tikzcd} \Omega^\bullet \arrow[r, bend left, "\pi_{inv}"] & \Omega^\bullet_{inv} \arrow[l, bend left, "\iota"] \end{tikzcd}$$

How do we build the homotopy h? Suppose one can diagonalize the $U(1)$-action. Then $\Omega^\bullet = \oplus_\lambda \Omega_\lambda$ decomposes into weights (charges): $\mathcal{L}_v\omega_\lambda = i\lambda\omega_\lambda$. The invariant space Ω_{inv} is therefore given by the zero modes Ω_0. On $\Omega_{\lambda\neq 0}$, thus the complement of Ω_{inv} one has $h = (i\lambda)^{-1}\iota_v$. Indeed, one finds

$$\{d, h\} = (i\lambda)^{-1}\mathcal{L}_v = 1 - \pi_{inv}.$$

To actually prove the claim, one still would have to show that $\mathcal{L}_v$ is actually diagonalizable. The idea here is to show that $i\mathcal{L}_v$ is actually a symmetric linear operator. The claim thus shows that the cohomology of $(\Omega^\bullet_{inv}, d_\varepsilon)$ is the same as $(\Omega^\bullet, d)$. However, as a ring with Forbenius structure, one can allow more general spaces than compact X.

2.2 the equivariant integral

Let us now come back to the question of an integration map. What we want:

a) for compact X (denoted by X_c in the following):

$$\int_{X_c}^{eqvr} \omega := \int_{X_c} \omega$$

In this case, it is indeed true that

$$\int_{X_c}^{eqvr} d_\varepsilon\omega = \int_{X_c} (d\omega + \varepsilon\iota_v\omega) = 0$$

where the first part vanishes due to Stokes and the second because $\iota_v\omega$ is of lesser dimension.

b) for X not necessarily compact, but equipped with a $U(1)$-invariant metric (any metric can be made invariant by averaging) one defines

$$\int_{X_g}^{eqvr} \omega := \int_{X_g} e^{-\Lambda\{d_\varepsilon,\rho\}}\omega$$

where ρ is some regulator function.

Remark 6 (regularization by inclusion of cohomological 1)**.** The inclusion of a cohomological 1, $e^{\{Q,reg\}}$ is a generally good way to regularize. $\lhd$

Properties:

1. for $X = X_c$, one reduces to $\int_{X_c}\omega$.
2. this form of the equivariant integration map can be localized to the zeros of v.

What could happen for non-compact X? For non-compact spaces X, the integral diverges as $\Lambda \to 0$. In order to regularize the integral, one would like to constrain the support of the integrand in such a way that for $\Lambda \to \infty$ the integral localizes around zeros of v. To do so, one chooses a regulator function ρ in such a way that one obtains $|v|^2$ in the exponent (to get a Gaussian integral). Note that identifying $dx^\mu = \psi^\mu$, one has $\iota_v = v^\mu \partial/\partial\psi^\mu$. If one thus considers

$$v^m g_{mn}\psi^n = g^\flat(v)$$

one finds

$$\{\iota_v, g^\flat(v)\} = g(v,v) = \|v\|^2.$$

Remark 7 (regulator function is Hodge dual of ι_v)**.** One can identify $g^\flat(v)$ with the Hodge adjoint of the linear operator ι_v:

$$g_{mn}v^m dx^n \propto \iota_v^* = \pm * \iota_v * .$$

$\lhd$

A good choice of regulator is

$$\rho = \iota_v^* = (-1)^{n(k-1)+1} * \iota_v *,$$

seen as an operator acting on k-forms ($n = \dim X$). With this choice, one obtains

$$\int^{eqvr} \omega = \int_{X_g} e^{-\Lambda\{d_\varepsilon,\iota_v^*\}}\omega = \int_{X_g} e^{-\Lambda\varepsilon\|v\|^2 - \Lambda\{d,\iota_v^*\}}\omega.$$

Remark 8. In general, one can always choose the regularization function to be

$$\rho = f\iota_v^*$$

where f is some *amplification function* whose sole purpose is to cut off the support at ∞. $\lhd$

Example 6. Consider $\mathbb{R}^2 = \mathbb{C}$ endowed with the $U(1)$-invariant metric

$$ds^2 = \frac{dzd\bar{z}}{(1-|z|^2)^m}.$$

This could naively not be integrated. However, it can be integrated in the equivariant setup. ◀

Example 7 ($\mathbb{R}^2$ with standard metric)**.** Consider $\mathbb{R}^2$ with the standard metric

$$ds^2 = dx^2 + dy^2.$$

Consider the $U(1)$-action given by (ccw) rotations around the origin. It is generated by the vector field

$$v = x\partial_y - y\partial_x, \qquad \|v\|^2 = x^2 + y^2.$$

Then one can show by direct calculation (simply act on $1, dx, dy, dx \wedge dy$), that as an operator one has

$$\iota_v^* = - * \iota_v * = (xdy - ydx)\wedge$$

and hence

$$\{\iota_v, \iota_v^*\} = \{\iota_v, (xdy - ydx)\wedge\} = x^2 + y^2 = \|v\|^2.$$

Moreover,

$$\{d, \iota_v^*\} = 2dx \wedge dy.$$

Therefore, for any $\omega \in \Omega^\bullet_{inv}$ one has

$$\begin{aligned}\int_{\mathbb{R}^2}^{eqvr} \omega &= \int_{\mathbb{R}^2} e^{-\Lambda\varepsilon(x^2+y^2) - 2\Lambda dx\wedge dy}\omega \\ &= \int_{\mathbb{R}^2} e^{-\Lambda\varepsilon(x^2+y^2)}(1 - 2\Lambda dx \wedge dy) \wedge \omega\end{aligned}$$

In polar coordinates, let

$$\omega = f(r) + f_\theta(r)d\theta + f_r(r)dr + f_{r\theta}(r)dr \wedge d\theta \in \Omega_{inv}$$

(note that one has indeed $\mathcal{L}_v\omega = 0$ for the above parametrization of ω) one thus has

$$\int_{\mathbb{R}^2}^{eqvr} \omega = \int_{\mathbb{R}^2} e^{-\Lambda\varepsilon r^2} \left(f_{r\theta}(r) - 2\Lambda r f(r)\right) dr \wedge d\theta.$$

One is the above integral independent of Λ? This happens precisely when the integrand is *equivariantly closed*, that is

$$d_\varepsilon(f(r) + f_{r\theta} dr \wedge d\theta) = (f'(r) - \varepsilon f_{r\theta}(r))dr = 0 \iff f'(r) = \varepsilon f_{r\theta}(r).$$

Under this condition, the integral becomes (one can directly integrate over θ)

$$\int_{\mathbb{R}^2}^{eqvr} \omega = 2\pi \int_0^\infty e^{-\Lambda\varepsilon r^2} \left(\frac{f'(r)}{\varepsilon} - 2\Lambda r f(r) \right) = 2\pi \int_0^\infty d \left(\frac{f(r) e^{-\Lambda\varepsilon r^2}}{\varepsilon} \right) = -2\pi \frac{f(0)}{\varepsilon},$$

which is indeed independent of Λ. Even more is true:

(i) one only picks up a contribution form the zero of v, namely from $(0,0) \in \mathbb{R}^2$

(ii) the only interesting part of ω is the degree zero part. In particular, one can integrate $\omega = 1$, which leads to the notion of *equivariant volume*

The equivariant volume (integrating $\omega = 1$) in this case is

$$\int_{(\mathbb{R}^2, ds_0^2)}^{eqvr} 1 = -\frac{2\pi}{\varepsilon}.$$

Remark 9 (importance of choice of metric in regulator function)**.** In fact, ∞ *is not* a zero of v because a) what is $\infty \in \mathbb{R}^2$ (note that we are not working with the one-point compactification here) and b) what can we evaluate at ∞? We can only evaluate the norm $|v|$ at a point, but not the vector field itself. However, here the importance of the *choice of metric* shows! For different choices of metrics, one obtains different answers (see examples below). The answer thus depends on the choice of regulator. $\triangleleft$

$\blacktriangleleft$

Example 8 ($\mathbb{C}^2$ with standard metric)**.** Consider now $\mathbb{C}^2$ with the standard metric

$$ds_0^2 = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 = dr_1^2 + r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2.$$

We orient $\mathbb{C}^2$ as the product of orientations of the two $\mathbb{C}$ factors, that is the volume form on $\mathbb{C}^2$ is taken to be

$$dvol(\mathbb{C}^2) = dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 = (r_1 dr_1 \wedge d\theta_1) \wedge (r_2 dr_2 \wedge d\theta_2).$$

Consider the $U(1)$ action on $\mathbb{C}^2$ given by simultaneous rotation:

$$(z_0, z_1) \mapsto (\lambda z_0, \lambda z_1), \qquad \lambda \in U(1)$$

Remark 10 ($U(1) \subseteq \mathbb{C}^*$)**.** This $U(1)$-action will be important later when we discuss equivariant integration formulas on $\mathbb{CP}^n$. In this case, it is precisely the $U(1) \subseteq \mathbb{C}^*$-action. $\triangleleft$

It is generated by the vector field

$$v = \sum_{i=1}^{2} x_i \partial_{y_i} - y_i \partial_{x_i} = \partial_{\theta_1} + \partial_{\theta_2},$$

of norm

$$\|v\|^2 = r_1^2 + r_2^2.$$

The regulator ι_v^* is computed for example by its action on 1: let

$$\begin{aligned}
\iota_v^* 1 &= - * \iota_v (r_1 r_2 dr_1 \wedge d\theta_1 \wedge dr_2 \wedge d\theta_2) \\
&= - * (-r_1 r_2 dr_1 \wedge dr_2 \wedge d\theta_2 - r_1 r_2 dr_1 \wedge d\theta_1 \wedge dr_2) \\
&= (r_1^2 d\theta_1 + r_2^2 d\theta_2) \wedge = \sum_{i=1}^{2} (x_i dy_i - y_i dx_i) \wedge
\end{aligned}$$

Therefore,

$$\{d_\varepsilon, \iota_v^*\} = \{d, \iota_v^*\} + \varepsilon \{\iota_v, \iota_v^*\} = 2 \underbrace{\sum_i r_i dr_i \wedge d\theta_i}_{\equiv \nu} + r_1^2 + r_2^2.$$

The equivariant integral thus takes the form

$$\begin{aligned}
\int_{(\mathbb{C}^2, ds_0^2)}^{eqvr} \omega &= \int_{\mathbb{C}^2} e^{-\Lambda \{d_\varepsilon, \iota_v^*\}} \omega = \int_{\mathbb{C}^2} e^{-\Lambda \varepsilon (r_1^2 + r_2^2) - 2\Lambda \nu} \omega \\
&= \int_{\mathbb{C}^2} e^{-\Lambda \varepsilon (r_1^2 + r_2^2)} \left(1 - 2\Lambda \nu + \frac{1}{2} (2\Lambda \nu)^2\right) \omega.
\end{aligned}$$

As we expand ω in its degree parts (recall that ω is to be seen as an inhomogeneous form on $\mathbb{C}^2$)

$$\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \omega^{(3)} + \omega^{(4)},$$

we see that for dimensional reasons we can neglect the odd-degree parts, which we thus set to zero for brevity. The remaining form of $\omega = \omega^{(0)} + \omega^{(2)} + \omega^{(4)}$ must be

a) invariant: $\mathcal{L}_v \omega = 0$,

b) equivariantly closed: $d_\varepsilon \omega = 0$.

Before we study the implications of the above conditions, note that since $\nu = \sum_i r_i dr_i d\theta_i$, i.e. ν is the sum of the volume forms on each factor $\mathbb{C} \subset \mathbb{C}^2$, the only contribution of the degree-two part $\omega^{(2)}$ comes from the $(2,0)$ and $(0,2)$ parts. We thus set

$$\omega^{(2)} = f_{11} dr_1 \wedge d\theta_1 + f_{22} dr_2 \wedge d\theta_2.$$

Now, the invariance condition states that we have an expansion of the form

$$\omega = f(r_1,r_2)+f_{11}(r_1,r_2)dr_1\wedge d\theta_2+f_{22}(r_1,r_2)dr_2\wedge d\theta_2+g(r_1,r_2)dr_1\wedge d\theta_1\wedge dr_2\wedge d\theta_2.$$

The second condition, equivariant closedness, gives us a relation among the coefficient functions, which is solved degree by degree:

$$\begin{aligned}
0 &= df + \varepsilon\iota_v(f_{11}dr_1\wedge d\theta_2 + f_{22}dr_2\wedge d\theta_2)+\\
&\quad + d(f_{11}dr_1\wedge d\theta_2 + f_{22}dr_2\wedge d\theta_2) + \varepsilon\iota_v(g dr_1\wedge d\theta_1\wedge dr_2\wedge d\theta_2)\\
&= (\partial_1 f - \varepsilon f_{11})\, dr_1 + (\partial_2 f - \varepsilon f_{22})\, dr_2\\
&\quad + (\partial_2 f_{11} - \varepsilon g)\, dr_1\wedge d\theta_1\wedge dr_2 + (\partial_1 f_{22} - \varepsilon g)\, dr_1\wedge dr_2\wedge d\theta_1
\end{aligned}$$

where $\partial_i \equiv \partial/\partial r_i$. This implies that

$$f_{11} = \frac{\partial_1 f}{\varepsilon}, \qquad f_{22} = \frac{\partial_2 f}{\varepsilon},$$

and thus

$$g = \frac{\partial_1\partial_2 f}{\varepsilon^2}.$$

We can thus write the equivariant integral of ω over $\mathbb{C}^2$ solely in terms of $f = \omega^{(0)}$:

$$\begin{aligned}
\int\limits_{(\mathbb{C}^2, ds_0^2)}^{eqvr} \omega &= \int_{\mathbb{C}^2} e^{-\varepsilon\Lambda(r_1^2+r_2^2)} \left(4\Lambda^2 r_1 r_2 f - 2\Lambda(r_2 f_{11} + r_1 f_{22}) + g\right) dr_1 d\theta_1 dr_2 d\theta_2\\
&= (2\pi)^2 \int dr_1 dr_2 e^{-2\varepsilon\Lambda(r_1^2+r_2^2)} \left(4\Lambda r_1 r_2 f - 2\Lambda\left(\frac{r_2\partial_1 f + r_1\partial_2 f}{\varepsilon}\right) + \frac{\partial_1\partial_2 f}{\varepsilon^2}\right)\\
&= (2\pi)^2 \int dr_1 dr_2 \frac{d}{dr_1}\frac{d}{dr_2}\left(\frac{f}{\varepsilon^2} e^{-\varepsilon\Lambda(r_1^2+r_2^2)}\right)\\
&= \left(-\frac{2\pi}{\varepsilon}\right)^2 f(0).
\end{aligned}$$

In particular, the equivariant volume of $\mathbb{C}^2$ is the product of the equivariant volumes of the factors:

$$\int\limits_{(\mathbb{C}^2, ds_0^2)}^{eqvr} 1 = \left(-\frac{2\pi}{\varepsilon}\right)^2.$$

◀

Example 9 ($\mathbb{C}^n$ with diagonal $U(1)$-action and standard metric)**.** The example of the equivariant integration of $\mathbb{C}^2$ (endowed with the standard metric) with respect to the diagonal $U(1)$-action generalizes to the case of $\mathbb{C}^n$

(again equipped with the standard metric and the diagonal $U(1)$-action). In this case one gets again that the equivariant volume of $\mathbb{C}^n$ is the produt of equivariant volumes of the factors:

$$\int_{(\mathbb{C}, ds_0^2)}^{eqvr} 1 = \left(\frac{2\pi}{\varepsilon}\right)^n .$$

◀

Example 10 ($\mathbb{R}^2$ with Fubini-Study metric)**.** Let us endow $\mathbb{R}^2 = \mathbb{C}$ with the Fubini-Study metric

$$ds_{FS}^2 = \frac{dz d\bar{z}}{(1+|z|^2)^2}.$$

The vector field generating the $U(1)$-action can be written in complex coordinates as

$$v = x\partial_y - y\partial_x = i(z\partial - \bar{z}\bar{\partial}).$$

Then

$$\|v\|_{FS}^2 = \frac{|z|^2}{(1+|z|^2)^2}$$

which shows that both, $z = 0$ and $z = \infty$ are zeros of v, while the Euclidean metric has only one zero at $z = 0$:

$$\|v\|_{Eucl} = |z|^2.$$

The equivariant volume in this case is calculated as follows: again, we choose the regulator function to be $\rho = \iota_v^*$, that is

$$\iota_v^* = *_{FS}\iota_v *_{FS} .$$

Now, however, the Hodge star is calculated with respect to the Fubini-Study metric. On the other hand, since we are still working in two dimensions, one finds

$$\begin{aligned} *_{FS}1 = K &= \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \\ *_{FS}K &= 1 \\ *_{FS}dz &= -idz \\ *_{FS}d\bar{z} &= id\bar{z}. \end{aligned}$$

Therefore, as an operator one finds (after a brute force calculation, i.e. acting with ι_v^* on 1)

$$\iota_v^* = \frac{i}{2}\frac{zd\bar{z} - \bar{z}dz}{(1+|z|^2)^2}\wedge = \frac{xdy - ydx}{(1+x^2+y^2)^2}\wedge .$$

It follows that,

$$\{\iota_v, \iota_v^*\} = \frac{|z|^2}{(1+|z|^2)} = \frac{x^2+y^2}{(1+x^2+y^2)^2} = \frac{r^2}{(1+r^2)^2}$$
$$\{d, \iota_v^*\} = \frac{2dx\wedge dy}{(1+r^2)^2} - 2\frac{2(xdx+ydy)\wedge(xdy-ydx)}{(1+r^2)^3} = 2\frac{(1-r^2)dx\wedge dy}{(1+r^2)^3}$$

Then, using again polar coordinates

$$\int\limits_{(\mathbb{CP}^1, ds^2_{FS})}^{eqvr} \omega = \int_{\mathbb{CP}^1} e^{-\Lambda\varepsilon\frac{r^2}{(1+r^2)^2} - 2\Lambda\frac{(1-r^2)r}{(1+r^2)^2}dr\wedge d\theta}\omega$$
$$= \int_{\mathbb{CP}^1} e^{-\Lambda\varepsilon\frac{r^2}{(1+r^2)^2}}\left(1 - 2\Lambda\frac{(1-r^2)r}{(1+r^2)^3}dr\wedge d\theta\right)\omega.$$

We again expand $\omega \in \Omega^\bullet_{inv}$ and impose the equivariantly closedness condition, such that as before

$$\omega = f(r) + \cdots + f_{r\theta}(r)dr\wedge d\theta, \qquad f'(r) = \varepsilon f_{r\theta}(r).$$

Then one has

$$\int\limits_{(\mathbb{CP}^1, ds^2_{FS})}^{eqvr} \omega = \int_{\mathbb{CP}^1} e^{-\frac{\Lambda\varepsilon r^2}{(1+r^2)^2}}\left(1 - 2\Lambda\frac{(1-r^2)r}{(1+r^2)^3}dr\wedge d\theta\right)(f(r) + f_{r\theta}(r)dr\wedge d\theta)$$
$$= \int_{\mathbb{CP}^1} e^{-\frac{\Lambda\varepsilon r^2}{(1+r^2)^2}}\left(f_{r\theta}(r) - 2\Lambda\frac{(1-r^2)r}{(1+r^2)^3}f(r)\right)dr\wedge d\theta$$
$$= \int_{\mathbb{CP}^1} e^{-\frac{\Lambda\varepsilon r^2}{(1+r^2)^2}}\left(\frac{f'(r)}{\varepsilon} - 2\Lambda\frac{(1-r^2)r}{(1+r^2)^3}f(r)\right)dr\wedge d\theta$$
$$= 2\pi\int_0^\infty d\left(e^{-\frac{\Lambda\varepsilon r^2}{(1+r^2)^2}}\frac{f(r)}{\varepsilon}\right)$$
$$= 2\pi\left(\frac{f(\infty)}{\varepsilon} - \frac{f(0)}{\varepsilon}\right)$$

We integral localizes around the now two fixed points at 0 and ∞. It follows imediately that the equivariant volume is zero, sicne the two contributions from the two fixed points cancel each other.

$$\int\limits_{(\mathbb{CP}^1, ds^2_{FS})}^{eqvr} 1 = \frac{2\pi}{\varepsilon} - \frac{2\pi}{\varepsilon} = 0.$$

Remark 11 (DH, eqvr extension and 1)**.** Note that the integrand, 1, *is not* an equivariant extension of anything but merely an equivariant closed form itself. Indeed, if 1 would be the equivariant extension, than it should come with acompanied by the equivariant parameter ε and one would not have to think about what it means to devide by ε. Another way to see this is to consider the Duistermaat-Heckman integral

$$\overset{eqvr}{\underset{\mathbb{CP}^1}{\int}} e^{t\hat{\omega}_{FS}} \underset{\mathbb{CP}^1}{\int} e^{t(\varepsilon\mu+\omega_{FS})} \sim \frac{2\pi}{\varepsilon} \sum_{\text{fixed pts } p} \frac{e^{\varepsilon\mu(p)}}{\prod w_i}$$

where w_i are the weights of the $U(1)$-action and $\hat{\omega}_{FS}$ is the equivariant extension of the Fubini-Study form, i.e.

$$(d+\varepsilon\iota_v)(\varepsilon\mu+\omega_{FS}) = 0 \iff \mu \text{ moment map}.$$

Then one can consider an expansion in t which gives at order $\mathcal{O}(1)$ the equivariant volume

$$\overset{eqvr}{\underset{\mathbb{CP}^1}{\int}} 1 = \lim_{t\to 0} \overset{eqvr}{\underset{\mathbb{CP}^1}{\int}} e^{t\hat{\omega}_{FS}} = \sum \pm \frac{2\pi}{\varepsilon}.$$

◁

Remark 12. Notice that by our definition of the equivariant integral, it must coincide with the ordinary integral for any compact space X_c. In particular, this is consistent with what we have shown in the example above:

$$0 = \overset{eqvr}{\underset{\mathbb{CP}^1}{\int}} 1 = \int_{\mathbb{CP}^1} 1$$

where the last integral vanishes for dimensional reasons. In fact, the equivariant volume for any compact manifold is zero. ◁

◀

Example 11 (Cylinder $\mathbb{C}^*$ with metric $dzd\bar{z}/|z|^2$)**.** Consider the cylinder $\mathbb{C}^* \simeq S^1 \times \mathbb{R}$ with metric

$$ds^2_{cyl} = \frac{|dz|^2}{|z|^2} = \frac{dx^2+dy^2}{x^2+y^2} = \frac{dr^2+r^2d\theta^2}{r^2}.$$

This time, we will work in polar coordinates form the beginning. The $U(1)$-action given by roatation around the axis of the cylinder is generated by the vector field

$$v = x\partial_y - y\partial_x = \partial_\theta, \qquad \|v\|_{cyl} = 1.$$

Remark 13. Note that

$$\iota_v \frac{xdy-ydx}{r^2} = 1 \implies \frac{xdy-ydx}{r^2} = d\theta.$$

◁

We have

$$\iota_v^* 1 = - * \iota_v * 1 = - * \iota_{\partial_\theta} \frac{r dr \wedge d\theta}{r^2} = - * \left(- \frac{dr}{r} \right)$$
$$= * \frac{x dx + y dy}{r^2} = \frac{x dy - y dx}{r^2} = d\theta \wedge 1$$

and hence, as an operator,

$$\iota_v^* = d\theta \wedge .$$

Thus

$$\{\iota_v, \iota_v^*\} = 1$$
$$\{d, \iota_v^*\} = 0.$$

It follows that

$$\int\limits_{(\mathbb{C}^*, ds^2_{cyl})}^{eqvr} \omega = \int_{\mathbb{C}^*} e^{-\Lambda \{d_\varepsilon, \iota_v^*\}} \omega = \int_{\mathbb{C}^*} e^{-\Lambda \varepsilon} f_{r\theta}(r) dr \wedge d\theta$$
$$= 2\pi \int_0^\infty e^{-\Lambda \varepsilon} \frac{f'(r)}{\varepsilon} = 2\pi \int_0^\infty d \left(e^{-\Lambda \varepsilon} \frac{f(r)}{\varepsilon} \right) = 0$$

since f is a function on the cylinder $\mathbb{C}^*$ and thus vanishes (fast enought) at 0 and ∞. Here we have once again expanded $\omega \in \Omega^\bullet_{inv}$ as correct argument?

$$\omega = f(r) + \cdots + f_{r\theta}(r) dr \wedge d\theta$$

and imposed the equivariantly closedness condition

$$f(r) = \varepsilon f_{r\theta}.$$

The vanishing of the equivariant integral of *any* equivariantly closed form is consistent with the observation that there is no fixed point of the $U(1)$-action on the cylinder (i.e. there v is everywhere non-vanishing). ◀

2.3 equivariant cohomology of $\mathbb{CP}^n$ from localization

Warm up: $\mathbb{CP}^1$. To start, consider $\mathbb{CP}^1$ endowed with the Fubini-Study metric. In homogeneous coordinates, z_0, z_1, there are two actions of $\mathbb{C}^*$:

1. the "gauge action" $\mathbb{C}^*_{gauge}\colon (z_0, z_1) \mapsto (\lambda z_0, \lambda z_1)$
2. the "external action" $U(1) \subset \mathbb{C}^*_{ext}\colon (z_0, z_1) \mapsto (z_0, \mu z_1)$

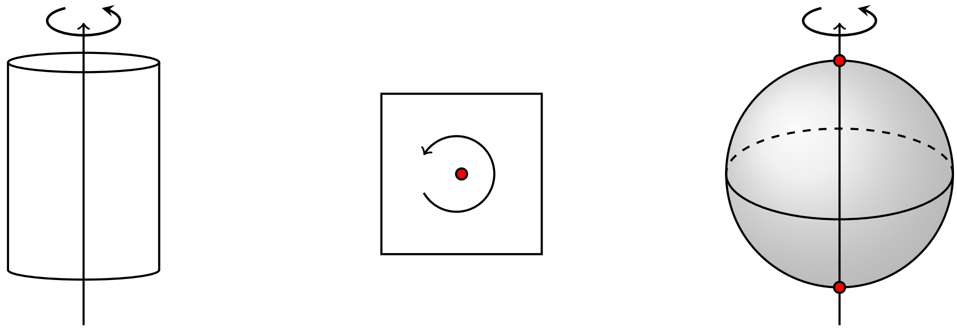


Figure 1: Fixed points of the $U(1)$-action on the sphere (left), the plane (middle), and the cylinder (right).

Let us consider the fixed points (better: fixed gauge orbits): First, there is the gauge orbit through $z_1 = 0$:

$$\mathbb{C}^*_{ext} \colon (z_0, 0) \mapsto (z_0, 0) \ni (1, 0),$$

next, there is the gauge orbit through $z_0 = 0$:

$$\mathbb{C}^*_{ext} \colon (0, z_1) \mapsto (0, \xi z_1) \sim_{gauge} (0, z_1) \ni (0, 1).$$

It turns out that these are the only two orbits (modulo gauge) which are fixed under $\mathbb{C}^*_{ext}$.

Remark 14. The special representatives correspond to the north and south pole of $\mathbb{C}^1$. $\triangleleft$

That there are only two fixed (gauge) orbits, hints to the fact that the equivariant cohomology of $\mathbb{CP}^1$ is only two dimensional. This can be made more precise as follows: suppose that ω is a $U(1)$-invariant closed two form: $d\omega = 0$. Let us try to extend ω to an equivariantly closed form. We thus set $\hat{\omega} = \omega + \varepsilon\mu$ with $\deg \varepsilon = 2$. The condition on μ turns out to be the moment map condition:

$$d_\varepsilon \hat{\omega} = (d + \varepsilon \iota_v)(\omega + \varepsilon\mu) = \varepsilon(\iota_v \omega + d\mu) = 0.$$

Notice that the existence of μ is given at least locally. Since ω is taken to be $U(1)$-invariant, that is $\mathcal{L}_v \omega = 0$, one has

$$d\iota_v \omega = \mathcal{L}_v \omega = 0$$

which implies that $\iota_v \omega = \pm d\mu$ at least locally. Recall that the equivariant integration map is non-degenerate and depends only on the 0-form part of the equivariantly closed integrand. It then follows that

$$\int_{\mathbb{CP}^1}^{eqvr} \hat{\mu} = \frac{2\pi}{\varepsilon} \left(\varepsilon\mu(\infty) - \varepsilon\mu\left(0\right)\right) = 2\pi(\mu(\infty) - \mu(0)).$$

The non-degeneracy of the equivariant integration map implies that the equivariant cohomology of $\mathbb{CP}^1$ is only two dimensional, since one expect that the integral, seen as a linear functional, maps any basis element to a certain number. Non-degeneracy ensures that the kernel of the map is empty. We can deduce more:

i) $\dim H_{eqvr}(\mathbb{CP}^1)) = 2$

ii) the linear functional written in an "obvious" basis (namely the basis dual to the fixed points) is given by the pairing with $\frac{2\pi}{\varepsilon}(1,-1)$

iii) the ring structure is given by $\hat{\alpha} \wedge \hat{\beta} \mapsto \hat{\alpha}^{(0)} \cdot \hat{\beta}^{(0)}$

There exist, in particular, two "natural basis" in $H_{eq}(\mathbb{CP}^1)$:

1. $1, \hat{\omega}_{FS} \equiv \sigma$
2. the basis dual to the fixed point of the action

Let us compute σ: recall that

$$\omega_{FS} = \frac{r dr \wedge d\theta}{(1+r^2)^2}.$$

Then

$$\iota_v \omega_{FS} = -\frac{r dr}{(1+r^2)^2} \stackrel{!}{=} d\mu \implies \mu = \frac{1}{2}\frac{1}{(1+r^2)} + cst.$$

Hence

$$\sigma = \hat{\omega}_{FS} = cst. + \frac{\varepsilon}{2}\frac{1}{(1+r^2)} + \omega_{FS}.$$

The first basis elements are then mapped via the integration map to

$$\int_{\mathbb{CP}^1}^{eqvr} 1 \mapsto \frac{2\pi}{\varepsilon} - \frac{2\pi}{\varepsilon} = 0$$
$$\int_{\mathbb{CP}^1}^{eqvr} \sigma \mapsto \frac{2\pi}{\varepsilon}\left(0 - \frac{\varepsilon}{2}\right) = -\pi$$

Therefore, the integration map can be written again as a contour integral (up to factors of π):

$$\int_{\mathbb{CP}^1}^{eqvr} P(\sigma) = \oint_{\mathcal{C}} \frac{d\sigma}{2\pi i}\frac{P(\sigma)}{(\sigma-\varepsilon)\sigma}$$

where the contour $\mathcal{C}$ encloses the two contributions of the fixed points

$$\hat{\omega}_{FS}(0) \sim \varepsilon \qquad \text{and} \qquad \hat{\omega}_{FS}(\infty) \sim 0.$$

Notice that if one expands the denominator in a fractional linear combination,

$$\oint \frac{d\sigma}{2\pi i} \frac{P(\sigma)}{(\sigma - \varepsilon)\sigma} = \oint \frac{d\sigma}{2\pi i} P(\sigma) \left(\frac{1}{\sigma - \varepsilon} - \frac{1}{\sigma} \right)$$

One sees explicitly that one has to evaluate $P(\sigma)$ at the two (fixed) points.

Generalization to $\mathbb{CP}^n$. Consider the (external) torus action $T = \underbrace{U(1) \times \ldots U(1)}_{n \text{ times}}$ on $\mathbb{CP}^n$ acting by weights λ_k^i. The generating vector fields is given by

$$v_k = \sum_i \lambda_k^i (z^i \partial_i - \bar{z}^i \bar{\partial}_i) \sim \sum_i \lambda_k^i \partial_{\theta_i}.$$

The k-th copy of $U(1)$ inside the torus T thus acts by the weight λ_k^i on the i-th coordinate (in a coordinate chart $z_0 \neq 0$. The regulator function must then be taken, such that one obtains $\sum \|v_k\|^2$ in the action (in order to localize the support around the common zeros of the v_k. A convenient choice is to take (as an operator)

$$\boxed{\rho = \sum_i z^i d\bar{z}^i - \bar{z}^i dz^i.}$$

Remark 15 (is $\rho = \iota^*$?)**.** In homogeneous coordinates $[z_0, \ldots, z_n]$ on $\mathbb{CP}^n$, the Fubini-Study metric is given by

$$\omega_{FS} = \frac{i}{2} \frac{1}{(|z_0|^2 + \ldots |z_n|^2)^2} \sum_{j \neq k} |z_j|^2 dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k$$

Example 12 ($\mathbb{CP}^1$)**.** To build up intuition, consider the case of $\mathbb{CP}^1$. In a local chart $z_0 \neq 0$ we set $z = z_1/z_0$. It follows that

$$dz \wedge d\bar{z} = \frac{1}{|z_0|^2} \left(dz_1 \wedge d\bar{z}_1 + |z|^2 dz_0 \wedge d\bar{z}_0 - z dz_0 \wedge d\bar{z}_1 - \bar{z} dz_1 \wedge d\bar{z}_0 \right).$$

Furthermore, we have

$$\begin{aligned}
\omega_{FS} &= \frac{i}{2}\frac{|z_0|^2 dz_1\wedge d\bar z_1 + |z_1|^2 dz_0\wedge d\bar z_0 - \bar z_0 z_1 dz_0\wedge d\bar z_1 - \bar z_1 z_0 dz_1\wedge d\bar z_0}{(|z_0|^2+|z_1|^2)^2}\\
&= \frac{i}{2}\frac{|z_0|^2 dz_1\wedge d\bar z_1 + |z_1|^2 dz_0\wedge d\bar z_0 - \bar z_0 z_1 dz_0\wedge d\bar z_1 - \bar z_1 z_0 dz_1\wedge d\bar z_0}{|z_0|^4(1+|z_1/z_0|^2)^2}\\
&= \frac{i}{2}\frac{\frac{1}{|z_0|^2}\left(dz_1\wedge d\bar z_1 + \left|\frac{z_1}{z_0}\right|^2 dz_0\wedge d\bar z_0 - \frac{z_1}{z_0} dz_0\wedge d\bar z_1 - \frac{\bar z_1}{\bar z_0} dz_1\wedge d\bar z_0\right)}{\left(1+\left|\frac{z_1}{z_0}\right|^2\right)^2}\\
&= \frac{i}{2}\frac{\frac{1}{|z_0|^2}\left(dz_1\wedge d\bar z_1 + |z|^2 dz_0\wedge d\bar z_0 - z dz_0\wedge d\bar z_1 - \bar z dz_1\wedge d\bar z_0\right)}{(1+|z|^2)^2}\\
&= \frac{i}{2}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}
\end{aligned}$$

which is the standard form in a chart. ◀

In a chart, $z_0 \neq 0$ one has with coordinates $\zeta_i = z_i/z_0$

$$\omega_{FS} = \frac{\sum_i d\zeta_i\wedge d\bar\zeta_i}{(1+\sum_i |\zeta_i|^2)^2}.$$

is thus $\rho = \sum_k \iota^*_{v_k}$?? ◁

With the above choice of v_k and ρ, the equivariant integration map then becomes

$$\int\limits_{\mathbb{CP}^n}^{eqvr} \omega = \int_{\mathbb{CP}^n} e^{-\Lambda\{d_\varepsilon,\rho\}}\omega = \int_{\mathbb{CP}^n} e^{-\Lambda(2dz^i\wedge d\bar z^i - \sum_{k,i}\varepsilon^k\lambda^i_k|z^i|^2)}\omega \sim \frac{\omega(0)}{\prod_{i=1}^n\sum_k \varepsilon^k\lambda_k}$$

where now

$$d_\varepsilon = d + \varepsilon^k\iota_{v_k}.$$

question: what is the fubini-study metric on $\mathbb{CP}^n$?

spell out: how many fixed points are there? $\hat H(\mathbb{CP}^n)$ should be ?-dimensional. The integral should then look like

$$\int\limits_{\mathbb{CP}^n}^{eqvr} \hat\omega = \frac{\sum_k \varepsilon^k\mu_k}{\prod_i \varepsilon^a\lambda^i_a}$$

How to write it as contour integral?

$$\oint d\sigma \frac{P(\sigma)}{(\sigma-\varepsilon_1)(\sigma-\varepsilon_2)\dots(\sigma-\varepsilon_n)\sigma}$$

2.4 equivariant cohomology of $\mathbb{CP}^n$ from factorization

Here we present another viewpoint on how to obtain the equivariant cohomology of $\mathbb{CP}^n$. The motivation is that in the formula we found for the equivariant integral of $\mathbb{CP}^1$

$$\oint \frac{d\sigma}{2\pi i} \frac{P(\sigma)}{\sigma(\sigma - \varepsilon)}$$

one can interpret the factor σ^{-1} and $(\sigma - \varepsilon)^{-1}$ as equivariant volumes themselves. In this interpretation, one has to assume that σ is the equivariant factor corresponding to the $\mathbb{C}^*_{gauge}$ action on $\mathbb{C}^2$ and ε corresponds to the equivariant parameter associated to the $\mathbb{C}^*_{ext}$ action on $\mathbb{C}^2$. In more detail: in order to integrate over $\mathbb{CP}^1$ one can consider the integration over $\mathbb{C}^2 - \{0\}$ and treat the $\mathbb{C}^*_{gauge}$ action on $\mathbb{C}^2 - \{0\}$ as a gauge action. To fix the gauge, one proceeds in steps. Recall that $\mathbb{C}^*_{gauge} = U(1)_{gauge} \times \mathbb{R}_{gauge}$. The $\mathbb{R}_{gauge}$ can be fixed by forcing the condition

$$\mu = |z_0|^2 + |z_1|^2 - R^2 = 0. \tag{5}$$

Note that μ is a moment map for the remaining $U(1)$-action (when $\mathbb{C}^2$ is endowed with the standard symplectic form $\omega = dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1$. The condition is introduced via a Lagrange multiplier for μ and $d\mu$. Note that geometrically, one considers the hypersurface $\{(z_0, z_1) \in \mathbb{C}^2 \mid \mu(z_0, z_1) = 0\} \subset \mathbb{C}^2$. To localize the integral on this hypersurface, one considers its Poincaré dual, constructed via (smeared) δ-functions:

$$\delta^\varepsilon(\mu) := e^{-|\mu|^2/2\varepsilon} \frac{d\mu}{\sqrt{\varepsilon}} = (2\pi)^{-1/2} \int d\lambda dc \; e^{-i\lambda\mu + cd\mu - \varepsilon\lambda^2/2}.$$

where c is an *odd* Lagrangian multiplier (ghost) and λ an even one. Notice that

$$\int dc \; e^{cd\mu} = \int dc \; (1 + cd\mu) = d\mu$$

and

$$\int d\lambda e^{-i\lambda\mu - \varepsilon\lambda^2/2} = \sqrt{\frac{2\pi}{\varepsilon}} e^{-\mu^2/2\varepsilon}.$$

What we have achieved so far:

$$\int_{\mathbb{CP}^1} \pi^*\omega \sim \int_{\mathbb{C}^2} \delta^\varepsilon(\mu) \pi^*\omega,$$

where $\pi\colon \mathbb{C}^2 - \{0\} \twoheadrightarrow \mathbb{CP}^1$ is the canonical projection.

After having gauged the $\mathbb{R}_{gauge}$ action by introducing the moment map, we still have to gauge the $U(1)_{gauge}$-action, in order to pass to the quotient. We thus would like to work equivariantly. The natural question is then if

$\delta^\varepsilon(\mu)$, which we have to introduce under integral is actually equivariantly closed.

For future use, let us introduce the following differential

$$Q = d + i\lambda\tfrac{\partial}{\partial c}.$$

Then we can write the exponent of $\delta^\varepsilon(\mu)$ as

$$QG, \qquad G = -c\mu + i\varepsilon c\lambda/2.$$

Since μ is the moment map for the $U(1)_{gauge}$-action, one has

$$\iota_v d\mu = \iota_v\iota_v\omega = 0$$

or equivalently

$$\iota_v d\mu = \mathcal{L}_v\mu = 0$$

since μ is invariant under the $U(1)_{gauge}$-action (c.f. (5)). The derivative of the moment map $d\mu$ is thus horizontal: $\iota_v d\mu = 0$ and hence $\delta^\varepsilon(\mu)$ is basic, that is horizontal and invariant:

$$\iota_v\delta^\varepsilon(\mu) = \mathcal{L}_v\delta^\varepsilon(\mu) = 0.$$

We may now try to pass to the quotient (modding out the $U(1)_{gauge}$-action).

the math way Suppose that we have an S^1 bundle $\pi\colon X \twoheadrightarrow B = X/U(1)$. Suppose further that we want to understand the integral over the base in terms of an integral over the total space. The naive approach, simply replacing B by X and consider only basic forms, does not work for dimensional reasons:

$$\int_B \omega_1\wedge\cdots\wedge\omega_n \overset{?}{=} \int_X \pi^*\omega_1\wedge\cdots\wedge\pi^*\omega_n = 0.$$

Indeed, the basic forms $\pi^*\omega_i$ do not have any component along the fibers since by definition they are horizontal: $\iota_v\pi^*\omega_i = 0$ (here v is a vector field along/tangent to the fiber). The second approach is to introduce a connection form A along the fiber. A must be invariant

$$\mathcal{L}_v A = 0.$$

Let $Lie(U(1)) = \mathbb{R}\langle 1\rangle$ such that one usually demands

$$\iota_v A = 1.$$

Remark 16. A connection 1-form A on a principal G-bundle $P \to B$, satisfies

$$\iota(X^\sharp)A = X$$

where $X \in \mathfrak{g}$ and $X^\sharp$ is the corresponding fundamental vector field. In the example $G = U(1)$, the fundamental vector field of 1 (the generator of $Lie(U(1)) = \mathbb{R}\langle 1\rangle$ is denoted by v. $\triangleleft$

The condition $\iota_v A = 1$ is, however, best seen as a *normalization*. For a connection 1-form it would suffice that $\iota_v A \neq 0$. The normalization is thus a new condition, which we will implement using (odd) Lagrange multipliers.

Let $\tilde{A}$ be the unnormalized connection, i.e. $\iota_v \tilde{A} \neq 0$. It will be convenient to work in a super-manifold setting, i.e. $\tilde{A} = \tilde{A}_\mu \psi^\mu$ and $\iota_v = v^\mu \frac{\partial}{\partial \psi^\mu}$ where $\psi^\mu \sim dx^\mu$. We introduce fermionic (super) ghosts $\bar{\eta}_\alpha$, and bosonic (super) ghosts σ^a and $\bar{\sigma}_\alpha$ and introduce

$$\left(\prod_{\alpha,a,\beta} d\bar{\eta}_\beta d\sigma^a d\bar{\sigma}_\alpha\right) e^{\bar{\eta}_\alpha \tilde{A}^\alpha + \sigma^a \iota_{v^a} \tilde{A}^\alpha \bar{\sigma}_\alpha}$$

in the integral. Note that the above insertion, when integrated over the $\bar{\eta}, \sigma$ and $\bar{\sigma}$, yields an insertion of $\tilde{A}/\iota_v \tilde{A}$, which is equivalent to an insertion of the *normalized connection* A.

Remark 17 (normalization matrix)**.** The matrix

$$N_a{}^\alpha := \iota_{v^a} \tilde{A}^\alpha$$

is called the normalization matrix. The integrand

$$e^{\sigma^a \iota_{v^a} \tilde{A}^\alpha \bar{\sigma}_\alpha} = e^{\sigma^a N_a{}^\alpha \bar{\sigma}_\alpha}$$

integrated over σ and $\bar{\sigma}$, then gives nothing but $\det N_a{}^\alpha$. $\triangleleft$

the physics way one treats the $U(1)$-action of the bundle $\pi\colon X \twoheadrightarrow B$ as a gauge symmetry. The $\bar{\eta}$ und $\bar{\sigma}$ play the role of the ghosts needed for the gauge fixing of the super group generated by ι_v. The gauge fixing condition is simply $\tilde{A} = 0$, which is achieved by inclusion of a delta function

$$\int d\bar{\eta} e^{\bar{\eta}_\alpha \tilde{A}^\alpha} = \prod_\alpha \tilde{A}^\alpha = \delta(\tilde{A})$$

since $\tilde{A} = \tilde{A}_\mu \psi^\mu$ is an *odd* object. The corresponding Faddeev-Poppov determinant is then given by

$$\Delta = \int d\sigma d\bar{\sigma} e^{-\sigma^a \iota_{v^a} \tilde{A}^\alpha \bar{\sigma}_\alpha}.$$

Back to $\mathbb{CP}^1$: The full integral over $\mathbb{CP}^1$, where we have gauged the $\mathbb{R}_{gauge}$-action due to the inclusion of the moment map and the $U(1)_{gauge}$-action due to the inclusion of the (un-/normalized) connection, reads

$$\int_{\mathbb{CP}^1} \pi^* \omega \sim \int_{\mathbb{C}^2} dx d\psi d\lambda dc d\bar{\eta} d\sigma d\bar{\sigma} \ \pi^* \omega \ e^{-i\lambda\mu + cd\mu - \varepsilon\lambda^2/2 + \bar{\eta}\tilde{A} + \sigma \iota_v \tilde{A} \bar{\sigma}}.$$

Let us introduce the differential

$$d^{tot} = \psi^\mu \partial_{x^\mu} + \lambda \partial_c + \sigma \iota_v + \bar{\eta} \partial_{\bar{\sigma}},$$

where we have written the deRham differential in terms of super-coordinates: $d = dx^\mu \partial_{x^\mu} = \psi^\mu \partial_{x^\mu}$. We can then write the exponent as

$$d^{tot} G_1, \qquad G_1 = -c\mu + i\varepsilon c\lambda/2 + \tilde{A}\bar{\sigma}.$$

Hence

$$\int_{\mathbb{CP}^1} \pi^* \omega \sim \int_{\mathbb{C}^2} dx d\psi d\lambda dc d\bar{\eta} d\sigma d\bar{\sigma} \ \pi^* \omega \ e^{d^{tot} G_1} \tag{6}$$

Recall that we aimed to understand the integral

$$\oint \frac{P(\sigma)}{\sigma(\sigma - \varepsilon)}.$$

Here, σ corresponds to the equivariant parameter of the internal $U(1)_{gauge}$-action. This can be seen as follows: in the equivariant cohomology calculation of $\mathbb{CP}^1$, there were secretely two $U(1)$-actions: the internal $U(1)_{gauge}$-action on $\mu^{-1}(0) = \{|z_0|^2 + |z_1|^2 - R^2 = 0\} \subset \mathbb{C}^2$ and the $U(1)_{ext}$-action. The former acted as $(z_0, z_1) \mapsto (\lambda z_0, \lambda z_1)$ while the latter acted as $(z_0, z_1) \mapsto (z_0, \lambda z_1)$. Therefore, one could consider an equivariant differential not on $\mathbb{CP}^1$ but on $\mathbb{C}^2$, where now one would have two equivariant parameters σ and ε:

$$d_{eq} = d + \sigma \iota_{v_{int}} + \varepsilon \iota_{v_{ext}}.$$

Note that the $U(1)_{guage}$-action has only one fixed-point in $\mathbb{C}^2$, namely the origin. However, this is cut out, due to the moment map (the space of integration is actually $\mu^{-1}(0) \not\ni 0$).

We are therefore aiming to keep the integral over σ in (6). Moreover, the integral over $dx d\psi$ corresponds to the integral over $\mathbb{C}^2$, which we also wish to keep. That means that we should somehow perform the integral over $d\lambda dc d\bar{\eta} d\bar{\sigma}$. In order to do so, notice that we can always introduce a term of the form $d^{tot} G_2$ in the exponent without changing the integral. Let us put

$$G_2 = i\Lambda c \bar{\sigma}$$

such that

$$d^{tot} G_2 = i\Lambda(\bar{\eta}\lambda + \lambda\bar{\sigma}).$$

Here Λ is some parameter, which will be taken to infinity in order to localize the integral to zeros of $\bar{\eta} c + \lambda \bar{\sigma}$. Hence, in the limit $\Lambda \to \infty$, we see that $\bar{\eta}, \lambda, c$ and $\bar{\sigma}$ go to zero and thus drop out form the integral. At the same time, however, we see that also G_1 goes to zero and that d^{tot} reduces to the $U(1)_{gauge}$-equivariant differential

$$d^{tot} \to \psi^\mu \partial_{x^\mu} + \sigma \iota_v = d_{eq}^{gauge}.$$

Remark 18 (homotopy)**.** Note that the piece $\bar{\eta}\partial_{\bar{\sigma}} + \lambda\partial_c$ in d^{tot} does not contribute to any cohomology of d^{tot}. This suggests that there exists a homotopy for this piece of the complex. The introduction of G_2 then contracts this piece such that it drops out from the integral. $\triangleleft$

Assuming that $\pi^*\omega = P(\sigma)$, the integral becomes

$$\int_{\mathbb{CP}^1} \pi^*\omega = \int_{\mathbb{C}^2} dx d\psi \int_{\mathbb{R}} d\sigma P(\sigma).$$

To regularize the intergal over $\mathbb{C}^2$, one works $U(1)_{gauge}$-equivariently, that is one introduces a regulator of the form $d^{tot}\iota_v^*$ in the exponential, such that

$$\begin{aligned} \int_{\mathbb{CP}^1} \pi^*\omega &\sim \int_{\mathbb{C}^2} dx d\psi \int_{\mathbb{R}} d\sigma \; P(\sigma) e^{d^{tot}\iota_v^*} \\ &= \int_{\mathbb{C}^2} dx d\psi \int_{\mathbb{R}} d\sigma \; P(\sigma) e^{d_{eq}^{gauge}\iota_v^*} \\ &= \int d\sigma \; 2\pi\sigma^{-2} P(\sigma), \end{aligned}$$

where the integration over $\mathbb{C}^2$ gives its equivariant volume σ^{-2}.

Note, however, that we have not yet considerd an *external* $U(1)$-action. If we would inlcude the external $U(1)$-action in our considerations, then we would have to add a piece $\varepsilon\iota_{v_{ext}}$ to d^{tot}. Here we have introduced an equivariant parameter ε for the $U(1)_{ext}$-action. In the limit $\Lambda \to \infty$, the total differential then becomes

$$d^{tot} = \psi^\mu \partial_{x^\mu} + \sigma\iota_{v_{guage}} + \varepsilon\iota_{v_{ext}}.$$

And working equivariantly (now also with respect to the $U(1)_{ext}$-action, one has

make precise

$$\int\limits_{\mathbb{CP}^1}^{eqvr} \pi^*\omega = \int_{\mathbb{C}^2} dx d\psi \int_{\mathbb{R}} d\sigma \; P(\sigma) e^{d^{tot}(\iota_{v_{gauge}}^* + \iota_{v_{ext}}^*)} = \int_{\mathbb{R}} d\sigma \; \frac{P(\sigma)}{\sigma(\sigma - \varepsilon)}.$$

This is almost what we wanted to show. The problem is that the original euivariant integral formula is a contour integration, where here we integrate σ over the line (recall that σ was a bosonic ghost variable). However, one has to be more careful: when introducing G_2 and taking the limit $\Lambda \to \infty$, one encounters fast oscillating intgrals. In order to make these integrals well-defined, one must integrate over a contour which is deformed away from the real line.

Remark 19 (hypersurfaces in toric varieties)**.** Hypersurfaces in toric varieties are cut out by homogeneous polynomials: $F(\lambda x) = \lambda^p F(x)$ and

$\Sigma = \{F(x) = 0\} \subseteq \mathbb{C}^n$. In order to integrate over them, one proceeds as before but now considers the differential

$$d \to d + F(x)\partial_\pi$$

where one augments $\mathbb{C}^n$ by an odd coordinate: $\mathbb{C}^n \times \mathbb{C}[1]$ with $\pi \in \mathbb{C}[1]$. The new part of the differential is called the *Koszul differential.* In the integral equations, all things then localize to $F(x) = 0$, thus to the hypersurface. $\lhd$

2.5 Cohomology of complete intersections

Consider $\mathbb{CP}^n$ (or any other toric variety) and a set of equations $F_i = 0$.

Remark 20 (sections vs holomorphic functions)**.** We know that on $\mathbb{CP}^n$ there exist no holomorphic functions, only (holomorphic) sections of line bundles. However, for $\mathbb{CP}^n$, sections of degree d are degree d (homogeneous) polynomials in the (homogeneous) coordinates $z_0, \dots, z_n$. The set of equations $F_i = 0$ are then intersections of those sections with the zero section. $\lhd$

We want to study the submanifold of $\mathbb{CP}^n$ given by $\bigcap\{F_i(z_0, \dots, z_n) = 0\}/\mathbb{C}^*$. The strategy is to work on $\mathbb{C}^{n+1}$ and inforce $F_i = 0$ via a delta function. Finally we mod out by the $\mathbb{C}^*$ action. The idea is then always to

a) construct $\delta(F_i)$,

b) interpret $\delta(F_i)$ as e^{QG} for some differential Q.

Remark 21. If we would be interested in the equivariant volume of the submanifolds $F_i = 0$, we would need an external $\mathbb{C}^*_{ext}$-action on the space of solutions of $F_i = 0$. $\lhd$

Remark 22 (complete intersection)**.** With *complete intersection* we mean that one has to consider the variety build from *all* components. For example, the varity $xy = 0$ is the union of two lines (the x-axis and the y-axis). One may now be interested in only one component, say the x-axis. This is itself a variety. However, when we speak of *complete intersections* we mean that all components (in this case the x- and y-axis) have to be considered. $\lhd$

Let us now construct $\delta(F_i)$: as in the example of the moment map, it is given (up to factors of 2π's) by

$$\delta^\varepsilon(F) \equiv \prod_i \delta^\varepsilon(F_i) = \int dp d\bar{p} d\pi d\bar{\pi}\; e^{p^a F_a + \bar{p}^{\bar{a}} \bar{F}_{\bar{a}} + \pi^a dF_a + \bar{\pi}^{\bar{a}} d\bar{F}_{\bar{a}} - \varepsilon p^a \bar{p}^{\bar{a}}}$$

$$\sim \prod_a e^{-|F_a|^2/\varepsilon} \frac{dF_a d\bar{F}_{\bar{a}}}{\varepsilon^2}.$$

In order to check that $\delta^\varepsilon(F)$ is closed, we introduce the differential

$$Q = d + p^a \partial_{\pi^a} + \bar{p}^{\bar{a}} \partial_{\bar{\pi}^{\bar{a}}}.$$

Then the exponent can be written as

$$Q(\pi^a F_a + \bar{\pi}^{\bar{a}} \bar{F}_{\bar{a}} + \varepsilon \pi^a \bar{p}_{\bar{a}}).$$

Note that the second part of Q can be seen as a deRham differential acting on an odd space with coordinates $\pi^a, \bar{\pi}^{\bar{a}}$. In the study of integrals over complete intersections, one is thus naturally led to consider the the super-manifold with coordinates $(x, \bar{x}, \pi, \bar{\pi})$ and $\psi = Qx, p = Q\pi$.

Remark 23 (generalization to toric varieties)**.** The generalization to toric varieties is then given by the following presrciption:

$$\mathbb{C}^n /\!\!/ (\mathbb{C}^*)^k \to \left(\mathbb{C}^n \times (\Pi\mathbb{C})^\ell\right) /\!\!/ (\mathbb{C}^*)^k$$

with Q being the super deRham differential. $\triangleleft$

In order to define the equivariant integral over the complete intersections, one proceeds schematically as follows:

$$\int\limits_{\{s_a=0\}}^{eqvr} \pi^*\omega \to \int_{\mathbb{C}^{n+1}\times(\Pi\mathbb{C})^\ell} P(\sigma) \to \int d\sigma \frac{P(\sigma)\dots}{\sigma(\sigma-\varepsilon_1)\dots}$$

The difference to before is now that the integration over $(\Pi\mathbb{C})^\ell$ can give contributions to the numerator.

Example 13 (fundamental line bundle over $\mathbb{CP}^n$)**.** Consider $\mathbb{CP}^n$. We can construct a holomorphic line bundle $\mathcal{O}(k)$ of degree k as follows: Consider charts U_i where $z_i \neq 0$. Then we define the transition functions of the line bundle $\mathcal{O}(k)$ by

$$g_{\alpha\beta}([z]) = \left(\frac{z_\alpha}{z_\beta}\right)^k,$$

defined on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ where $z_\alpha, z_\beta \neq 0$. In particular, for $n = 1$ we have

$$g_{01}(z_0, z_1) = \left(\frac{z_0}{z_1}\right)^k,$$

defined on $U_{01} = \mathbb{C}^*$. If we use the standard coordinate $z = z_1/z_0$ on U_0 and $w = z_0/z_1$ on U_1, then we see that the transition function is given by

$$w = z^{-k}.$$

Note that if we define $\mathcal{O}(1) \equiv \mathcal{L}$, then $\mathcal{O}(k) = \mathcal{L}^k$.

write difference to tauto- logical bdle

Let now $s \in \Gamma\mathcal{L}$ and consider the equation $s = 0$. This correspinds to elimenating one coordinate, we are thus led to the study of $\mathbb{CP}^{n-1}$.

The equivariant integration gives then

$$\int\limits_{\{s=0\}\subseteq\mathbb{CP}^n}^{eqvr} \to \int_{\mathbb{C}^{n+1}\times\Pi\mathbb{C}} \to \int d\sigma \frac{P(\sigma)\sigma}{\sigma^{n+1}}$$

where the denominator $\sigma^{-(n+1)}$ is the equivariant volume of $\mathbb{C}^{n+1}$ and the *numerator* σ comes from the integration of the odd fiber $\Pi\mathbb{C}$. We observe that the integral is independent of the section and that only its degree matters. For example, if we would have taken $s \in \mathcal{L}^2$ and considered $s = 0$ (amounts to the equation $z_j^2 = 0$) then we would have encountered

$$\int\limits_{\{s=0\}\subseteq\mathbb{CP}^n}^{eqvr} \to \int d\sigma \frac{P(\sigma)(2\sigma)}{\sigma^{n+1}}.$$

The factor of 2 in the denominator can be interpreted as the weight of the $U(1)$-action on the super-variables $\pi, \bar{\pi} \in \Pi\mathbb{C}$. ◀

3 Quasi maps $\mathbb{CP}^1 \to \mathbb{CP}^n$

3.1 Quasi maps and frackles

We are interested in the study of holomorphic maps discussion follows [1, 2]

$$\mathbb{CP}^1 \to \mathbb{CP}^n.$$

We denote homogeneous coordinates on the source $\mathbb{CP}^1$ by $[z_0, z_1]$ and on the target $\mathbb{CP}^n$ by $[\phi_0, \ldots, \phi_n]$. Given a map $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ we thus have components

$$\phi_i = F^i(z_0, z_1).$$

If the F^i would be arbitrary functions, however, we would get simply a map into $\mathbb{C}^{n+1}$. In order to get a map into $\mathbb{CP}^n$ we would like to quotient by an $\mathbb{C}^*$-action. The maps F^i therefore have to be taken equivariantly with respect to the usual $\mathbb{C}^*$ actions on $\mathbb{C}^2$ and $\mathbb{C}^{n+1}$, that is

$$F^i(\lambda z_0, \lambda z_1)) = \lambda^d F^i(z_0, z_1).$$

We therefore will take the F^i to be homogeneous polynomials in $[z_0, z_1]$ of degree d. The F^i therefore admit an expansion of the form

$$F^i = \sum_{k=0}^{d} A^i_k z_0^k z_1^{d-k}. \tag{7}$$

Those maps, however, do not quite define a map from $\mathbb{CP}^1 \to \mathbb{CP}^n$: the problem, which may arise, is best understood geometrically. A map from $\mathbb{CP}^1 \to \mathbb{CP}^n$ is a map between lines in $\mathbb{C}^2$ and $\mathbb{C}^{n+1}$. As follows from the discussion above, a line $\ell = t[z_0, z_1] \in \mathbb{C}^2$ is sent to a line $L = F(\ell) = t^d[F^0(z_0, z_1), \ldots, F^n(z_0, z_1)] \in \mathbb{C}^{n+1}$. Suppose now that $F(\ell) = 0$. Then $F(\ell)$ does not define a point in $\mathbb{CP}^n$ because $0 \in \mathbb{C}^{n+1}$ *does not* define a line in $\mathbb{C}^{n+1}$. We would like to discard those bad cases. Therefore, a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^n$ of degree d is given by a set of homogeneous polynomials $F^i(z_0, z_1)$ such that

$$\forall\ (z_0, z_1)\ \exists\ i \text{ such that } F^i(z_0, z_1) \neq 0. \tag{8}$$

Let $\mathcal{M}_d$ be the space of all such maps, i.e. the space of holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^n$. Notice that by (7) such a map $F \in \mathcal{M}_d$ is completely defined by the coefficients $A^i_k \in \mathbb{C} - \{0\}$ up to an action of $\mathbb{C}^*$. Therefore, if we recall that $i = 0, \ldots, n$ and $k = 0, \ldots, d$, we see that $\mathcal{M}_d$ is an *open subspace* of the projective space $\mathbb{CP}^{(n+1)(d+1)-1}$. It is only a *subspace*, because not all points $\{A^i_k\} \in \mathbb{CP}^{(n+1)(d+1)-1}$ give rise to such a map.

Counter example. We consider the case $n = d = 1$, that is we consider degree 1 maps $F = (F^1, F^2)\colon \mathbb{CP}^1 \to \mathbb{CP}^1$. Given a matrix

$$A = \begin{pmatrix} A^0_0 & A^0_1 \\ A^1_0 & A^1_1 \end{pmatrix} \leftrightarrow (A^0_0, A^0_1, A^1_0, A^1_1) \equiv [w_0, w_1, w_2, w_3] \in \mathbb{CP}^3$$

we define the map

$$F_A(z_0, z_1) = \begin{pmatrix} F^1(z_0, z_1) \\ F^2(z_0, z_1) \end{pmatrix} = \begin{pmatrix} w_0 z_0 + w_1 z_1 \\ w_2 z_0 + w_3 z_1 \end{pmatrix}.$$

Now, in order to construct a counter example we need to choose $[w_0, \ldots, w_3] \in \mathbb{CP}^3$ in such a way that there exists a point $[z_0, z_1] \in \mathbb{CP}^1$ such that both components of F, that is F^1 and F^2, vanish simultaneously. This is achieved for example for the choice

$$[w_0, w_1, w_2, w_3] = [1, w_1, w_2, w_1 w_2].$$

Indeed, at the point $[z_0, z_1] = [-w_1 z_1, z_1] \in \mathbb{CP}^1$, we have

$$F(-w_1 z_1, z_1) = \begin{pmatrix} -w_1 z_1 + w_1 z_1 \\ -w_1 w_2 z_1 + w_1 w_2 z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $F_A \notin \mathcal{M}_d$.

Note that the essential property of F_A is that the F^i share a common factor: (which turns out to be F^0)

$$\begin{aligned} F^0(z_0, z_1) &= z_0 + w_1 z_1, \\ F^1(z_0, z_1) &= w_2 z_0 + w_3 z_1 = w_2 F^0(z_0, z_1), \end{aligned}$$

where we recall that A was chosen such that $w_0 = 1$ and $w_3 = w_1 w_2$. Therefore, at the zero of F^0, $(z_0, z_1) = (-w_1 z_1, z_1)$ both F^i vanish simultaneously. ◀

More generally, suppose that $A \in \mathbb{CP}^{(n+1)(d+1)-1}$ gives rise to a map $F_A = (F^0, \ldots, F^n)$ in such a way that the F^i share a common factor P, say a degree k polynomial:

$$F^i(z_0, z_1) = P(z_0, z_1)\tilde{F}^i(z_0, z_1), \qquad \forall\, i$$

where the $\tilde{F}^i$ do not share a common factor among them. Then, the zeros of P are points where *all* the F^i vanish. Therefore, F_A does not satisfy the condition (8) and hence is not an element of $\mathcal{M}_d$.

One can compactify $\mathcal{M}_d$ by simply relaxing the condition (8), i.e. by allowing the F^i to share a common factor. We then come to the following

Definition 3.1 (quasi-maps)**.** The space of *quasi-maps* $\mathcal{QM}$ is the set of all maps

$$\mathcal{QM}_d := \{F = (F^0, \ldots, F^i) \mid F^i = \text{cmplx. hom. degree } d \text{ polynomial}\}/\sim$$

where the equivalence relation is given by $F \sim \lambda F$ for $\lambda \in \mathbb{C}^*$

This gives a compactification of $\mathcal{M}_d$ to $\overline{\mathcal{M}_d} = \mathbb{CP}^{(n+1)(d+1)-1}$. In particular, the space of quasi-maps is a complex projective space and hence we can calculate use the techniques developed earlier to calculate its equivariant cohomology, equivariant volume and other interesting quantities like its intersection theory.

3.2 Freckles

Recall that in $\mathcal{QM}_d$ there exists maps F such that the F^i share a common factor P. Suppose that P has degree k. Since it is a complex-valued polynomial, it can be factorized:

$$P = c\prod_{j=1}^{k}(z_1 - b_j z_0),$$

where c is some overall constant.

Remark 24 (b_k as roots of P)**.** In the chart $z_0 \neq 0$, then

$$P = cz_0^k \prod_{j=1}^{k}\left(\frac{z_1}{z_0} - b_j\right) = cz_0^k \prod_{j=1}^{k}(z - b_j)\,.$$

Therefore, the b_j are just the roots of the polynomial P. ◁

The roots b_k are precisely the points where F is not-well defined as a map.

Definition 3.2. The roots $b^i \in \mathbb{CP}^1$ of P are called *freckles*. A map F such that the F^i share a common factor is called a *freckled map*. non-std notation

Note that if $\mathcal{P}_k$ denotes the space of homogeneous polynomials of degree k, then $\mathcal{QM}_d$ admits a stratification

$$\mathcal{QM}_d = \mathcal{M}_d \cup (\mathcal{M}_{d-1} \times P_1) \cup \cdots \cup (\mathcal{M}_0 \times P_d)$$

where maps in $\mathcal{M}_0 \times P_d$ are of the form $F^i = cP$ for all $i = 0, \ldots, n$.

claim: The space of freckled maps $\mathcal{F} \subseteq \mathcal{QM}_d$ has $\mathrm{codim}_{\mathbb{C}} = (n+1)(d+1) - 1 - d(n+1) = n$.

Example 14 (low n)**.** If $n = 0$, and we are studying quasi-maps form $\mathbb{CP}^1$ into $\mathbb{CP}^0 = pt$, then the codimension of the space of freckles is 0, which means that every point in the source $\mathbb{CP}^1$ is a freckle.

If $n = 1$, that and we are studying quasi-maps form $\mathbb{CP}^1$ into $\mathbb{CP}^1$, then the (complex) codimension of the space of freckles is 1 thus its dimension is 0 and hence it is a collection of a bunch of points, that is a divisor on the source $\mathbb{CP}^1$. ◀

3.3 Evaluation maps and quantum cohomology

There exists evaluation maps

$$\begin{aligned} ev_i \colon \mathcal{QM} \setminus \mathcal{F} \times \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{k-\text{times}} &\to \mathbb{CP}^n \\ (F, (p_1, \ldots, p_k)) &\mapsto F(z_0(p_i), z_1(p_i)) \end{aligned}$$

which evaluates the quasi-map at the i-th factor of $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$.

Now, we can take k cohomology classes $\Omega_1, \ldots, \Omega_k \in H^\bullet(\mathbb{CP}^n)$ and pull them back via the evaluation maps. One can then define

$$\left\langle \mathcal{O}^{(0)}(\Omega_1) \ldots \mathcal{O}^{(0)}(\Omega_k) \right\rangle_q := \sum_{d=0}^{\infty} q^d \int_{\mathcal{M}_d = \mathcal{QM}_d \setminus \mathcal{F}} \prod_{j=1}^{k} ev_j^*(\Omega_j). \tag{9}$$

Note that $ev_j^*(\Omega_j)$ defines a differential form on $\mathcal{QM} \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$. The notation $\mathcal{O}^{(0)}(\Omega_j)$ means that one considers only the $(\bullet, 0, \ldots, 0)$-part of the differential form, i.e. one treats $\mathcal{O}^{(0)}(\Omega_j)$ as a 0-form on the factor $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$.

Now, we know that the Ω_j are given by polynomials in the Fubini-Study form $\omega_{FS} \equiv \sigma$: $\Omega_j \sim P_j(\sigma)$. By the operation (9), we therefore get map

$$\text{polynomials} \longrightarrow \mathbb{C}[\![q]\!].$$

This can be seen as a *deformed integral formula* of $H^\bullet(\mathbb{CP}^n)$, c.f. (2): for $q=0$, the only contribution comes from $d=0$ quasi-maps, that is from constant functions $\mathbb{CP}^1 \to \mathbb{CP}^n$, which are thus just points on $\mathbb{CP}^n$. Indeed, $\mathcal{QM}_0 = \mathbb{CP}^n$. Therefore, (9) reduces for $q=0$ to

$$\left\langle \mathcal{O}^{(0)}(\Omega_1)\dots\mathcal{O}^{(0)}(\Omega_j)\right\rangle_q \underset{\lim q\to 0}{\longrightarrow} \int_{\mathbb{CP}^n} P_1(\omega_{FS})\dots P_k(\omega_{FS}) \sim \oint \frac{d\sigma}{2\pi i}\frac{P_1(\sigma)\dots P_k(\sigma)}{\sigma^{n+1}}.$$

In order to compute $\langle\dots\rangle_q$, one must first understand the classes $ev_j^*\Omega_j \in H^\bullet(\mathcal{QM}_d) = H^\bullet(\mathbb{CP}^{(n+1)(d+1)-1})$. In particular, in the expression of $\langle\dots\rangle_q$ one formally integrates over the open manifolds $\mathcal{QM}_d\backslash\mathcal{F} = \mathcal{M}_d$. One would like to replace the integration domain by its compactification $\mathcal{QM}$. In order to do so, one has to ensure that given a top form in $H^{top}(\mathbb{CP}^{(n+1)(d+1)-1})$, one can neglect freckles,that is that one can always find a representative which avoids freckles.

A geometric argument in favor, goes as follows: at the i-th point, consider a representation of $\omega_{FS}^{n_i}$ (for some $n_i \in \mathbb{Z}_+$) by a product of delta functions supported on some hyperplanes H_{n_i}:

$$\omega_{FS}^{n_i} \sim \delta^{\varepsilon_1}(H_1)\wedge\dots\wedge\delta^{\varepsilon_{n_i}}(H_{n_i}).$$

If one pulls $\omega_{FS}^{n_i}$ back by the evaluation map, one still ends up with a delta function supported on some hyperplane in $\mathcal{QM}$. In fact, if the j-th hyperplane $H_j \subseteq \mathbb{CP}^n$ is given by $\alpha_\ell^{(j)}$, (we denote by Z_j the homogeneous coordinates in $\mathbb{CP}^n$)

$$H_j : \sum_{\ell=0}^n \alpha_\ell^{(j)} Z_\ell = 0,$$

then one finds that $ev^*\omega_{FS}^{n_i}$ is a delta function supported at the space of solutions of

$$X^{(j)} : \sum_{\ell=0}^n \alpha_\ell^{(j)} F^\ell(z_0(p_i), z_1(p_i)) = \sum_{\ell=0}^n \alpha_\ell^{(j)} \sum_k A_k^\ell z_0^k(p_i) z_1^{d-k}(p_i) = 0,$$

where $i=1,\dots,k$ runs through all k-factors of $\mathbb{CP}^1$ in $\mathcal{QM}\times\mathbb{CP}^1\times\dots\times\mathbb{CP}^1$ and $j=1,\dots,n_i$ runs trough all n_i hyperplanes H_j. It is important to note that all of those equations are *linear in the* $\{A_k^\ell\}$ i.e. linear in $\mathcal{QM}$. They are, however, non-linear in z_0 and z_1. The hyperplanes $X^{(j)} \subset \mathcal{QM} = \mathbb{CP}^{(n+1)(d+1)-1}$ intersect in a point in $\mathcal{QM}$ (or in a line in the space of the $\{A_k^i\}$ where one than has to factor out $\mathbb{C}^*$). More importantly, they can be freely moved around in such a way that one always avoids freckled maps. It follows that there exists a selection rule: why?

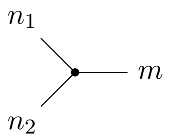
$$\langle\sigma^{n_1}\dots\sigma^{n_k}\rangle_q = q^d \quad \text{iff} \quad n_1+\dots+n_k = (n+1)(d+1)-1. \tag{10}$$

It turns out that (10) defines a commutative associative ring structure such that

$$\langle \sigma^{n_1} \dots \sigma^{n_k} \rangle = q^d = \sum_{m_1,\dots,m_{k-1}} f_{m_1}^{n_1 n_2} f_{m_2}^{m_1 n_3} \dots f_{m_{k-1}}^{m_{k-2} n_k} e^{m_{k-1}}.$$

Here, the $f_m^{n_1 n_2}$ should be seen as multiplication maps

$$f_m^{n_1 n_2} =$$

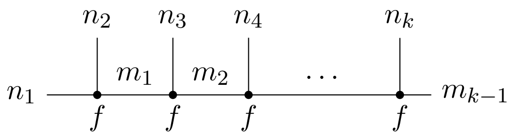


For example, one has

$$\sigma^{n_1} \cdot \sigma^{n_2} = \sum_m f_m^{n_1 n_2} \sigma^m$$

Pictorially, one then has

$$\langle \sigma^{n_1} \dots \sigma^{n_k} \rangle =$$



and one finds

$$f_m^{n_1 n_2} = \delta_{n_1+n_2,m} + q\delta_{n_1+n_2-n-1,m}$$

4 Quasi maps $\Sigma_g \to \mathbb{CP}^n$

4.1 dimension and index theorem

We are interested in the dimension of the space of quasi-maps form a higher genus Riemann surface Σ_g into $\mathbb{CP}^n$. Consider the space $\mathcal{QM}_d(\Sigma_g) = \{X \colon \Sigma_g \to \mathbb{CP}^n \mid X \text{ hol }, \deg(X) = d\}$. Since $X \in \mathcal{QM}_d(\Sigma_g)$ is holomorphic, it satisfies $\bar{\partial} X^i = 0$, for all $i = 0, \dots, n$. The only parameter in the game are

- the degree of the map d,
- the geneus g.

We therefore make the following ansatz:

$$\dim \mathcal{QM}_d(\Sigma_g) = \gamma + \alpha d + \beta(g-1).$$

Suppose that $d = 0, g = 0$, that is we are studying holomorphic maps $X \colon \mathbb{CP}^1 \to \mathbb{CP}^n$ of degree zero. Those maps are constant maps and hence the dimension of the space $\mathcal{QM}_0(\Sigma_0)$ is equal to the dimension of $\mathbb{CP}^n$:

$$\dim \mathcal{QM}_0(\Sigma_0) = \gamma - \beta = n.$$

Next, suppose we study the case $g = 1, d = 0$, that is holomorphic maps $X \colon S^1 \times S^1 \to \mathbb{CP}^n$.

A toric manifolds: geometric construction using symplectic geometry

Consider again $\mathbb{CP}^n$. Let $\{z_0, \ldots, z_n\}$ be coordinates on $\mathbb{C}^{n+1}$ and

$$\omega = \sum_{i=0}^{n} dz_i \wedge d\bar{z}_i.$$

Consider the $U(1)$-action (here all coordinates transform with the same weight / charge)

$$z_k \mapsto e^{i\varphi} z_k$$

whose fundamental vector field is given by

$$v = i \sum_k (z_k \partial_k - \bar{z}_k \bar{\partial}_k).$$

The Hamiltonian for this action is given by

$$\iota_v \omega = d\left(\frac{i}{2} \sum_k |z_k|^2\right) = dH(z).$$

Study the hyperplane

$$H_R(z) = H(z) - R^2 = 0, \qquad R \in \mathbb{R}_+$$

Then the preimage $H_R^{-1}(0)$ are spheres S^{2n+1} of radius R, and

$$H_R^{-1}(0)/U(1) = S^{2n+1}/U(1).$$

In the case $n = 1$ one has

$$S^3/U(1) = S^2 \qquad \text{Hopf fibration.}$$

However, for $n > 1$, one does in general not get spheres S^{2n} after the (symplectic) reduction.

Claim: After reduction one gets $\mathbb{CP}^n$, i.e.

$$\{\sum_{i=0}^{n} |z_i|^2 = 1\}/U(1) \cong \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$$

Note that on the left hand side one quotients by a compact group, while on the right hand side on quotients by a *non*-compact group.

c.f. Donaldson. In consider $H_1^{-1}(0) \subseteq \mathbb{C}^{n+1}$. This hyperplane is preserved by the flow of the vector field

$$v = \sum_k z_k \partial_k - \bar{z}_k \bar{\partial}_k$$

which generates the action of $U(1)$ on $\mathbb{C}^{n+1}$. Now, consider the vector field

$$u = \sum_k z_k \partial_k + \bar{z}_k \bar{\partial}_k$$

which is the Euler vector field which is (together with v) generating the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$. Consider the flow of $\alpha v + \beta u$: since for $H(z) = \sum |z_k|^2$

$$\mathcal{L}_v H = 0, \qquad \mathcal{L}_u H = H > 0$$

the function $H_R(z)$ grows monotonically under the flow of u. However, for $z = 0$, H_R is negative, while for big enough $|z_k|$, H_R is positive. Due to the monotonic growths, $H_R(z)$ must cross zero in exactly one point, i.e. there exists a unique intersection point for any non-trivial $\mathbb{C}^*$-orbit (any but the one through $0 \in \mathbb{C}^{n+1}$) with $H_R^{-1}(0)$:

picture

Therefore

$$\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^* \cong \{H(z) = 1\}/U(1)$$

□

This establishes in particular $\mathbb{CP}^n$ as the symplectic reduction of $\mathbb{C}^{n+1}$ under the (diagonal) $U(1)$-action.

Definition A.1. A *toric manifold* is the manifold obtained by a symplectic reduction $\mathbb{C}^n /\!\!/ U(1)^k$, where $U(1)^k$ acts on $\mathbb{C}^n$ with different weights / charges.

The Hamiltonians of the $U(1)^k$ action look like

$$H_i = \sum_{a=1}^{n} q_i^a |z_a|^2 - D_i, \qquad i = 1, \dots, k, \quad D_i \in \mathbb{R}_{\geq 0}$$

where for $D = 0$ one obtains a singular space.

Example 15 ($k = 1 : \ \mathbb{C}^{n+1} /\!\!/ U(1)$)**.** Consider $\mathbb{C}^{n+1} /\!\!/ U(1)$, where $U(1)$ acts with weights all equal to 1.

For $n = 1$, one has with $z_a = r_a e^{i\varphi_a}$

$$H(z) = |z_0|^2 + |z_1|^2 - D = r_0^2 + r_1^2 - D.$$

We want to study $\{H(z) = 0\}/U(1)$. Here $U(1)$ acts (diagonally) by shifting the angles: $\varphi_a \to \varphi_a + \psi$. Pictorially, $H(z) = 0$ looks like in Figure 2. Note that in addition to the coordinates r_0, r_1 one has two angle coordinates φ_0, φ_1 which have to be considered modulo $U(1)$, which acts by simultaneous shifts: $\varphi_a \mapsto \varphi_a + \psi$. In the region where $r_0, r_1 \neq 0$, this reduces the number of "free

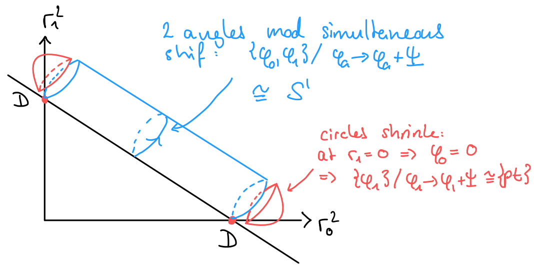


Figure 2: $\mathbb{CP}^1$ as a toric manifold

angle coordinates" to one. One therefore has a S^1 over each point $(r_0, r_1) \neq (0, 0)$ which gives the shape of an cylinder, c.f. Figure 2. At the special points $r_0 = 0$ resp. $r_1 = 0$, however, the angle φ_0 resp. φ_1 degenerates (is ill-defined) and hence one has only one angle coordinate which can always be fixed by the $U(1)$ action. Therefore, the circle degenerates to a point. That is (topologically) the same as attaching a disk, c.f. Figure 2. Hence, topologically one finds

$$\mathbb{C}^2/\!\!/U(1) = \mathbb{C}^* \cup \{pt\} \cup \{pt\} \simeq \mathbb{CP}^1.$$

◀

Example 16 $(k = 1 : \ \mathbb{C}^2/\!\!/U(1))$**.** Consider the $U(1)$ action on $\mathbb{C}^2$ with charges $\{\pm 1\}$, that is the Hamiltonian takes the form

$$H(z) = |z_0|^2 - |z_1|^2 - D = r_0^2 - r_1^2 - D.$$

Then,

$$H^{-1}(0) = \{r_0^2 - r_1^2 = D\}.$$

The same analysis as before, c.f. Figure 3 shows that in this case there is just one vanishing cycle and hence topologically one has

$$\mathbb{C}^2/\!\!/U(1)_{(+1,-1)} = \mathbb{C}^* \cup \{0\} = \mathbb{C}.$$

◀

Example 17 $(k = 1 : \ \mathbb{C}^3/\!\!/U(1))$**.** Consider now $\mathbb{C}^3/\!\!/U(1)$ where $U(1)$ acts with weights $(1, 1, 1)$. The Hamiltonian thus looks like

$$H(z) = r_1^2 + r_2^2 + r_3^2$$

and (setting $\rho_i = r_i^2$)

$$H^{-1}(0) = \{\rho_1 + \rho_2 + \rho_3 = D\}.$$

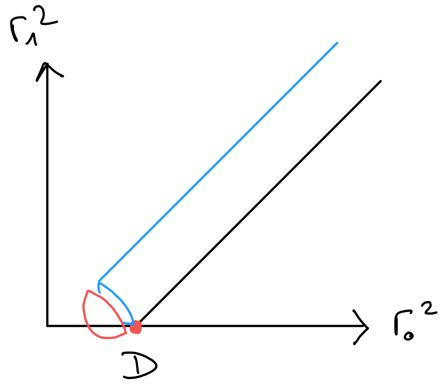


Figure 3: The toric manifold $\mathbb{C}^2/U(1)$ where $U(1)$ acts with weights $(+1,-1)$.

bla

Now consider the weight vector $(1,1,-1)$, such that the Hamiltonian reads

$$H(z) = r_1^2 + r_2^2 - r_3^2$$

and

$$H^{-1}(0) = \{\rho_1 + \rho_2 - \rho_3 = D\}.$$

There are now two cases: $D > 0$ and $D < 0$. ◀

B toric manifolds: combinatorial construction using fans

For an excellent review see these lecture notes.

B.1 basic definitions

Let $\Lambda \cong \mathbb{Z}^m$ a lattice and $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ the vector space over $\mathbb{R}$ generated by generators of Λ.

Definition B.1 (cone)**.** A *strongly convex rational polyhedral cone* $\sigma \subset \Lambda_{\mathbb{R}}$ is a set

$$\sigma = \left\{ \sum_i a_i v_i \mid a_i \geq 0 \right\}, \qquad \sigma \cap (-\sigma) = \{0\} \text{ (strong convexity)}$$

generated by a finite set of vectors $\{v_i\}_{i=1}^N \subset \Lambda$. A *face* τ is a cone generated by a subset $\{v_i\}_{i=1}^k$. We write $\tau < \sigma$.

Definition B.2. A *fan* is a collection Δ of cones in $\Lambda_{\mathbb{R}}$ such that

1. each face of a cone in Δ is also a cone in Δ

2. if $\tau = \sigma \cap \sigma'$ then $\tau < \sigma$ and $\tau < \sigma'$, i.e. the intersection of two cones in Δ is a face of each.

We call $\Delta(1)$ the set of one-dimensional cones in $\Lambda_{\mathbb{R}}$.

Let Δ be a fan. We denote the vectors $v_1, \ldots, v_n \in \Lambda$ corresponding to the edges (one-dimensional cones) in $\Delta(1)$. Now, to each v_i we associate a *homogeneous coordinate* z_i in $\mathbb{C}^n$. Recall that $\Lambda \cong \mathbb{Z}^m$ and note that we always will have $n \geq m$. One can produce a $m \times n$ matrix (by putting the vectors $v_i \in \mathbb{Z}^m = \Lambda$ next to each other)

$$A = \begin{pmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & & \vdots \\ v_1^m & \cdots & v_n^m \end{pmatrix} = (v_1, \ldots, v_n), \qquad v_i = \begin{pmatrix} v_1^1 \\ \vdots \\ v_1^m \end{pmatrix}. \tag{11}$$

This gives a map

$$\begin{aligned} \phi \colon \mathbb{C}^n &\to \mathbb{C}^m \\ (z_1, \ldots, z_n) &\mapsto \left(\prod_{i=1}^n z_i^{v_i^1}, \ldots, \prod_{i=1}^n z_i^{v_i^m} \right). \end{aligned} \tag{12}$$

We set

$$G = \ker(\phi) \cong (\mathbb{C}^*)^{n-m} \tag{13}$$

Remark 25 (about $\ker\phi$ and charge vectors)**.** If we set $Q_i = \log z_i$ then $\ker\phi$ can be computed as follows: we compute AQ as a matrix-vector product

$$\begin{aligned} AQ &= \left(\sum_{i=1}^n v_i^1 Q_i, \ldots, \sum_{i=1}^n v_i^m Q_i \right)^t \\ &= \left(\sum_{i=1}^n v_i^1 \log z_i, \ldots, \sum_{i=1}^n v_i^m \log z_i \right)^t \\ &= \left(\log \prod_i z_i^{v_i^1}, \ldots, \log \prod_i z_i^{v_i^m} \right)^t \end{aligned}$$

Thus $Q \in \ker A$ iff $\log \prod_i z_i^{v_i^1} = \cdots = \log \prod_i z_i^{v_i^m} = 0$ iff $\prod_i z_i^{v_i^1} = \cdots = \prod_i z_i^{v_i^m} = 1$ iff $(z_1, \ldots, z_n) \in \ker\phi$. Hence $\ker\phi \cong \ker A \cong (\mathbb{C}^*)^{n-m}$. $\lhd$

Example 18 (charge vector I)**.** Let $\Lambda \cong \mathbb{Z}^2$ and consider the three vectors

$$v_1 = (1,0), \qquad v_2 = (0,1), \qquad v_3 = (-1,-1).$$

Then

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Let $Q = (Q_1, Q_2, Q_3)$ then

$$AQ = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} Q_1 - Q_3 \\ Q_2 - Q_3 \end{pmatrix}.$$

Thus $Q \in \ker A$ iff $Q_1 = Q_2 = Q_3$ (not all zero). Hence $\ker A = \mathbb{C}^* \langle (1,1,1) \rangle$. ◀

Example 19 (charge vector I)**.** Let again $\Lambda \cong \mathbb{Z}^2$ and consider the four vectors

$$v_1 = (1,0), \qquad v_2 = (0,1), \qquad v_3 = (-1,-1), \qquad v_3 = (1,1).$$

Then

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

Let $Q = (Q_1, Q_2, Q_3, Q_3)$ then

$$AQ = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} Q_1 - Q_3 + Q_4 \\ Q_2 - Q_3 + Q_4 \end{pmatrix}$$

such that $Q \in \ker A$ iff $Q_1 = Q_2$ and $Q_3 = Q_1 + Q_4$. This means that $\ker A = \{(q_1, q_1, q_1 + q_2, q_2)\} = \mathbb{C}^* \langle (1,1,1,0), (0,0,1,1) \rangle$. ◀

Given a basis $\{Q^a\}$ of $\ker A$ (which is to give a basis of $\ker \phi = G$) we define an action of G on $\mathbb{C}^n$ as follows:

Definition B.3.

$$(\mathbb{C}^*)_a \colon (z_1, \ldots, z_n) \mapsto (\lambda^{Q_1^a} z_1, \ldots, \lambda^{Q_n^a} z_n) \tag{14}$$

where the a-th factor $(\mathbb{C}^*)_a \subset G = (\mathbb{C}^*)^{n-m}$ is the coefficient of the a-th basis vector Q^a.

Finally, define the zero set $Z(\Delta)$ as follows: for any subset $S \subset \Delta(1)$ which does not span a cone in Δ. Then one sets $V(S) = \{z_{i_1} = \cdots = z_{i_\ell} = 0\}$ and sets $Z(\Delta) = \bigcup_S V(S)$.

Definition B.4 (toric variety from a fan)**.**

$$X(\Delta) := (\mathbb{C}^n - Z(\Delta))/G \tag{15}$$

Remark 26 (orbifold singularities)**.** One also defines a discrete group $\Gamma = \Lambda/\mathbb{Z}\langle v_i \rangle$ and takes the quotient by $G \times \Gamma$ instead of only G. The quotient by Γ gives rise to so-called orbifold singularities. ◁

B.2 fan of $\mathbb{CP}^1$

Consider $\Lambda = \mathbb{Z}$ and the fan generated by the two vectors

$$v_1 = 1 \qquad v_2 = -1, \qquad A = \begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{16}$$

Note that

$$\Delta = \{\{0\}, v_1, v_2\} \tag{17}$$

since $\sigma = \{v_1, v_2\}$ is *not* a cone since the strong convexity condition fails: $\sigma \cap (-\sigma) \neq \{0\}$ since $v_1 = -v_2$ and hence $v_1 \in \sigma$ and $v_1 \in (-\sigma)$. From here we see readily that there is only one subset of $\Delta(1)$ which does not span a cone, namely $\{v_1, v_2\} \subset \Delta(1)$. Therefore $Z(\Delta) = \{(0,0)\}$.

Next, we compute $\ker A$: for $Q = (Q_1, Q_2)$

$$AQ = 0 \iff Q_1 - Q_2 = 0.$$

Therefore, $\ker A = \mathbb{C}^*\{(1,1)\} \cong \mathbb{C}^*$ which induces the usual diagonal $\mathbb{C}^*$ action on $\mathbb{C}^2$:

$$(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2). \tag{18}$$

Therefore

$$X(\Delta) = (\mathbb{C}^2 - (0,0))/\mathbb{C}^* = \mathbb{CP}^1. \tag{19}$$

B.3 fan of $\mathbb{CP}^1 \times \mathbb{CP}^1$

Consider $\Lambda = \mathbb{Z}^2$ and the fan generated by the four vectors

$$v_1 = (1,0) \quad v_2 = (-1,0), \quad v_3 = (0,1) \quad v_4 = (0,-1) \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{20}$$

Then

$$\Delta = \{\{0\}, v_i, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\} \tag{21}$$

Let us construct the zero set $Z(\Delta)$. We already know that the sets $\sigma_1 = \{v_1, v_2\}$ and $\sigma_2 = \{v_3, v_4\}$ do not span a cone since they fail to satisfy the strong convexity condition $\sigma \cap (-\sigma) = \{0\}$. Now any other subset of $\Delta(1)$ containing σ_1 or σ_2 will hence also fail to span a cone. Therefore one can conclude that $Z(\Delta) = \{(0,0,*,*)\} \cup \{(*,*,0,0\}$.

Next, we find $\ker A$: let $Q = (Q_1, Q_2, Q_3, Q_4)$. Then

$$AQ = (Q_1 - Q_2, Q_3 - Q_4) = 0 \iff Q_1 = Q_2 \text{ and } Q_3 = Q_4$$

which implies that $\ker A = \mathbb{C}^* \langle (1,1,0,0), (0,0,1,1) \rangle$. We thus have two basis vectors Q^a of $\ker A$ which define the action

$$\begin{aligned} Q^1 &: (z_1, z_2, z_3, z_4) \mapsto (\lambda z_1, \lambda z_2, z_3, z_4), \\ Q^2 &: (z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, \lambda z_3, \lambda z_4) \end{aligned} \tag{22}$$

which give each a diagonal action of $\mathbb{C}^*$ on one of the $\mathbb{C}^2$ factors of $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. It now follows that

$$\begin{aligned} X(\Delta) &= (\mathbb{C}^4 - \{(0,0,*,*)\} \cup \{(*,*,0,0)\})/G \\ &= \Big[(\mathbb{C}^2 - \{(0,0)\}) \times (\mathbb{C}^2 - \{(0,0)\})\Big]/\mathbb{C}^* \times \mathbb{C}^* \\ &= \big[(\mathbb{C}^2 - \{(0,0)\})/\mathbb{C}^*\big] \times \big[(\mathbb{C}^2 - \{(0,0)\})/\mathbb{C}^*\big] \\ &= \mathbb{CP}^1 \times \mathbb{CP}^1. \end{aligned} \tag{23}$$

B.4 fan of $\mathbb{CP}^2$

Consider $\Lambda = \mathbb{Z}^2$ and the fan generated by the three vectors of Example 19

$$v_1 = (1,0), \qquad v_2 = (0,1), \qquad v_3 = (-1,-1), \qquad v_3 = (1,1). \tag{24}$$

Then

$$\Delta = \{\{0\}, v_1, v_2, v_3, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}. \tag{25}$$ picture

Note that $\sigma = \{v_1, v_2, v_3\}$ is not a cone since the strong convexity condition fails: $\sigma \cap (-\sigma) \neq \{0\}$. This is seen by considering $\tau = v_1 + v_3 = (0,-1) = -v_2$. Then $\tau \neq \{0\}$ and $\tau \in \sigma$ and $\tau \in (-\sigma)$.

Then by Example (19) we find that $\ker A = \mathbb{C}^* \langle (1,1,1) \rangle$ such that there is a single basis element $Q = (1,1,1)$ and hence G acts on $\mathbb{C}^3$ by

$$(z_1, z_2, z_3) \mapsto (\lambda z_1, \lambda z_2, \lambda z_3) \tag{26}$$

which is the usual diagonal $\mathbb{C}^*$ action. Moreover, since there is just one subset of $\Delta(1)$ which does not span a cone, namely $\{v_1, v_2, v_3\} \subset \Delta(1)$, the zero set $Z(\Delta)$ is given by $(0,0,0)$. Hence the toric variety associated to Δ is

$$X(\Delta) = (\mathbb{C}^3 - (0,0,0))/\mathbb{C}^* = \mathbb{CP}^2. \tag{27}$$

C toric manifolds: algebrogeometric constructions

Let N be a lattice (isomorphic to $\mathbb{Z}^n$ for some n) and $M = N^*$ the dual lattice. For

$$\sigma = \sum_i a_i v_i, \qquad a_i \geq 0, \{v_i\} \subset N_{\mathbb{R}} = N \otimes \mathbb{R} \tag{28}$$

a cone define the dual cone by

$$\sigma^\vee = \{aw \mid a \in \mathbb{R}_{\geq 0},\ \langle w, u_i \rangle \geq 0 \forall u_i \in \sigma\}. \tag{29}$$

Lemma C.1. *$S_\sigma := \sigma^\vee \cap M$ is a* finitely generated semigroup[1]*.*

[1] A *semigroup* is a set with an associative binary operation

For any semigroup S one can define the group ring $\mathbb{C}[S]$ which is a commutative algebra over $\mathbb{C}$. As a vector space $\mathbb{C}[S]$ has a basis

$$\{\chi^u \mid u \in S\}$$

and multiplication law

$$\chi^u \chi^{u'} = \chi^{u+u'}, \qquad \chi^0 = 1.$$

For any commutative algebra over $\mathbb{C}$ one can define a space $X_A = \mathrm{Spec}(A)$.
Note: if A is generated by a set of generators $\{X_i\}$ plus some relations (given by some ideal $I \subset A$), i.e.

$$A = \mathbb{C}[X_1, \ldots, X_m]/I$$

then

$$\mathrm{Spec}(A) = \{V(I) = \text{common zeros of polynomials in } I\}. \tag{30}$$

Remark 27 (coordinate free description of $\mathrm{Spec}(A)$)**.** For all $\varphi \in \mathrm{Hom}(A, B)$ one has a morphisim $\varphi^* \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We define closed points x in A by homomorphisms $\mathrm{Hom}(A, \mathbb{C})$. Thus *a point* $x \in A$ induces a map $x^* \colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$, where $\mathrm{Spec}(\mathbb{C})$ is a point since the only ideals of $\mathbb{C}$ (which is a field) are $\{0\}$ and $\mathbb{C}$ itself, since any ideal necessarily contains 1[2]. Now, $\mathrm{Spec}(A)$ is defined to be the set of all prime ideals (an ideal $\mathfrak{p} \neq (1) \subset A$ such that $A/\mathfrak{p}$ is an integral domain (no zero divisors)). Hence $\mathrm{Spec}(\mathbb{C}) = \{0\}$. The upshot is that points of $\mathrm{Spec}(A)$ are defined by homomorphisms $\mathrm{Hom}(A, \mathbb{C})$. In particular, for S a semigroup and $A = \mathbb{C}[S]$ it group ring, points are given by homomorphisms $\mathrm{Hom}(S, \mathbb{C})$ where $\mathbb{C} = \mathbb{C}^* \cup \{0\}$ is a semigroup with respect to multiplication: for $u \in S$ and $x \in \mathrm{Hom}(S, \mathbb{C})$ we define a map $\chi^u \in \mathrm{Hom}(\mathbb{C}[S], \mathbb{C})$ by

$$\chi^u(x) = x(u). \tag{31}$$

Therefore, $\mathrm{Hom}(S, \mathbb{C})$ describes points in $\mathrm{Spec}(\mathbb{C}[S])$. $\triangleleft$

Example 20 (Obtaining the torus $(\mathbb{C}^*)^n$ from the trivial cone $\sigma = \{0\}$.)**.** Let $N = \mathbb{Z}\langle e_1, \ldots, e_n\rangle$ where e_i is the standard i-th basis vector of $\mathbb{R}^n$. Let $M = N^* = \mathbb{Z}\langle e_1^*, \ldots, e_n^*\rangle$ be the dual lattice. Consider the cone $\sigma = \{0\}$ such that $S_\sigma = \sigma^\vee \cap M = M$ (since $\sigma^\vee = M$). Note that as a semigroup $S_{\{0\}}$ is generated by $\pm e_i^*$ such that

$$\mathbb{C}[S_{\{0\}}] = \mathbb{C}[X_1^{\pm 1} = \chi^{\pm e_1^*}, \ldots, X_n^{\pm 1} = \chi^{\pm e_n^*}] \tag{32}$$

which is the ring of Laurent polynomials. Here the ideal of relations is trivial: $I = 0$. Therefore,

$$U_{\{0\}} := \mathrm{Spec}(\mathbb{C}[S_{\{0\}}]) = (\mathbb{C}^*)^n. \tag{33}$$

[2]if $z \in I \subset \mathbb{C}$ then $z^{-1} \cdot z = 1 \in I$ thus $1 \in I$.

The last equality is seen as follows: if we introduce coordinates $X_i = \chi^{e_i^*}$ and $Y_i = \chi^{-e_i^*}$ then

$$\mathbb{C}[S_{\{0\}}] = \mathbb{C}[X_i, Y_i] / \langle X_i Y_I - 1 \rangle . \tag{34}$$

We thus have a set of generators and relations and the common zeros of $I = \langle X_i Y_i - 1 \rangle$ are equivalent to $X_i Y_i = 1$ which means that $X_i \neq 0$ and $Y_i = X_i^{-1}$. Thus $\mathrm{Spec}(\mathbb{C}[S_{\{0\}}]) \cong (\mathbb{C}^*)^n$. ◀

Example 21 (getting $\mathbb{C}^n$)**.** Let $N = \mathbb{Z}^n$ and

$$\sigma = \mathbb{Z} \langle e_1, \dots, e_n \rangle .$$

Then

$$\sigma^\vee = \mathbb{Z}_{\geq 0} \langle e_1^*, \dots, e_n^* \rangle$$

such that

$$S_\sigma = \sigma^\vee \cap M = \mathbb{Z}_{\geq 0} \langle e_1^*, \dots, e_n^* \rangle .$$

Therefore, with $X_i = \chi^{e_i^*}$ we find

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \dots, X_n]$$

which implies that

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathbb{C}^n .$$

Note that here we used the fact that the prime ideals of $\mathbb{C}[X]$ are of the form $x - a$ for $a \in \mathbb{C}$ and hence $\mathrm{Spec}(\mathbb{C}[X]) = \mathbb{C}$. ◀

C.1 gluing coordinate patches

Every face, $\tau < \sigma$ implies that $S_\sigma \subset S_\tau$. This in turn implies that $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$ is a subalgebra and hence we have an inclusion $U_\tau = \mathrm{Spec}(\mathbb{C}[S_\tau]) \subset U_\sigma$. This means that we can glue the U_σ to a variety!

Example 22 (gluing $\mathbb{CP}^1$)**.** Consider the fan $\Delta = \{0, e_1, -e_1\}$ and the cones $\sigma_\pm = \langle \pm e_1 \rangle$ and $\sigma_0 = \{0\}$. We already have seen that

$$U_{\{0\}} = \mathbb{C}^* \tag{35}$$

$$U_{\sigma_\pm} = \mathbb{C}. \tag{36}$$

Now since $\{0\}$ is (trivially) a face of all cones in a fan, we have $\{0\} < \sigma_\pm$ and hence $U_{\{0\}} \subset U_{\sigma_\pm}$ as is clear from (35) and (36). As coordinates, we have $z^{\pm 1} = \chi^{\pm e_1}$ on $U_{\sigma_\pm}$. Now let $\tau = \sigma_+ \cap \sigma_-$ be a common face. Then we can glue U_{σ_+} and U_{σ_-} along U_τ. Here $\tau = \{0\}$ and hence we glue U_{σ_+} and U_{σ_-} along $U_{\{0\}} = \mathbb{C}^*$. The transition function is simply given by

$$U_{\{0\}} = \mathbb{C}^* \ni z = \chi^{e_1} \mapsto w = \chi^{-e_1} = z^{-1} \in \mathbb{C}^* = U_{\{0\}}. \tag{37}$$

This is but the well-known gluing of coordinate patches of $S^2 = \mathbb{CP}^1$. ◀

Example 23 (gluing $\mathbb{CP}^2$)**.** Let $N = \mathbb{Z}\langle e_1, e_2\rangle$ and $M = N^* = \langle e_1^*, e_2^*\rangle$ Consider the fan

$$\Delta = \langle 0, e_1, e_2, -e_1 - e_2\rangle . \tag{38}$$

Consider the cones

$$\sigma_1 = \langle e_1, e_2\rangle , \quad \sigma_2 = \langle e_1, -e_1 - e_2\rangle , \quad \sigma_3 = \langle e_2, -e_1 - e_2\rangle \tag{39}$$

To determine U_{σ_1} we note that $\sigma_1^\vee = \mathbb{Z}_{\geq 0}\langle e_1^*, e_2^*\rangle$ such that

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}\langle e_1^*, e_2^*\rangle . \tag{40}$$

Hence

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X_1, X_2] \implies U_{\sigma_1} = \mathbb{C}^2(z_1, z_2), \tag{41}$$

where $\mathbb{C}^2(z, w)$ denotes $\mathbb{C}^2$ with coordinates (z, w). Next, note that

$$\sigma_2^\vee = \mathbb{Z}_{\geq 0}\langle -e_2, e_1 - e_2\rangle . \tag{42}$$

Then

$$S_{\sigma_2} = \mathbb{Z}_{\geq 0}\langle -e_2, e_1 - e_2\rangle \tag{43}$$

such that

$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[X_1X_2^{-1}, X_2^{-1}] \implies U_{\sigma_2} = \mathbb{C}^2(z_1z_2^{-1}, z_2^{-1}). \tag{44}$$

Finally, for U_{σ_3} we find the same as for U_{σ_2} but with z_1 and z_2 exchanged, that is

$$U_{\sigma_3} = \mathbb{C}^2(z_1^{-1}z_2, z_1^{-1}). \tag{45}$$

We can now glue U_{σ_1} and U_{σ_2} along U_τ where $\tau = \sigma_1 \cap \sigma_2 = \langle e_1\rangle$. Then

$$\tau^\vee = \langle e_1, \pm e_2\rangle \implies S_\tau = \mathbb{Z}_{\geq 0}\langle e_1, \pm e_2\rangle \implies U_\tau = \mathbb{C}(z_1) \times \mathbb{C}^*(z_2). \tag{46}$$

The transition function is thus given by

$$(z_1, z_2) \mapsto (z_1z_2^{-1}, z_2^{-1}) \tag{47}$$

Likewise one can compute the transition functions between U_{σ_1} and U_{σ_3} and between U_{σ_2} and U_{σ_3}.

Compare this to the well-known construction of $\mathbb{CP}^2$ with homogeneous coordinates $[t_0 : t_1 : t_2]$ and charts U_i for $z_i \neq 0$ with affine coordinates

$$\begin{aligned}
U_0 : &\qquad (t_1/t_0, t_2/t_0) = (z_1, z_2)\\
U_1 : &\qquad (t_0/t_1, t_2/t_1) = (z_1^{-1}, z_2z_1^{-1})\\
U_2 : &\qquad (t_0/t_2, t_1/t_2) = (z_2^{-1}, z_1z_2^{-1})
\end{aligned}$$

◀

D About the notion of section

The "analytic" definition. Let $E \to M$ be a vector bundle over M with fiber V. A *section* of E is a set of functions $s = \{s_\alpha : U_\alpha \to V \mid s_\alpha = g_{\alpha\beta} s_\beta\}$ where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are the transition functions of E. Here and henceforth, $U_{\alpha\beta} = U_\alpha \cap U_\beta$. For line bundles, $V = \mathbb{C}$, if $s^{(1)}$ and $s^{(2)}$ are two sections, then

$$\frac{s_\alpha^{(1)}}{s_\alpha^{(2)}} = \frac{g_{\alpha\beta} s_\beta^{(1)}}{g_{\alpha\beta} s_\beta^{(2)}} = \frac{s_\beta^{(1)}}{s_\beta^{(2)}}.$$

Therefore, the ratio of two sections agree on any overlap $U_{\alpha\beta}$ and therefore form a *function.*

The "algebraic geometry" definition. Let $P_d(z_0, \dots, z_n)$ be a homogeneous polynomial of degree d. We may ask ourselves what is the *geometric meaning of P_d?* How is the polynomial related to $\mathbb{CP}^n$?

P_d is of course not invariant under the (diagonal) $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$. If it were, then it would descent to a function on $\mathbb{C}^{n+1}$. However, since P_d is homogeneous, it is an equivariant object, namely

$$P_d(\lambda z_0, \dots, \lambda z_n) = \lambda^d P_d(z_0, \dots, z_n).$$

Let $U_i = \mathbb{CP}^n - \{z_i = 0\} = \{(z_0, \dots, z_n) \mid z_i \neq 0\}$ a standard chart of $\mathbb{CP}^n$ with non-homogeneous coordinates $x_{(i)}^j = z^j / z^i$. On U_i we can rewrite P_d as follows:

$$P_d(z_0, \dots, z_n) = (z^i)^d P_d^{(i)}(x_{(i)}^j)$$

where $P_d^{(i)}(x_{(i)}^j)$ is a *function* on U_i, while $(z^i)^d$ *is not.* The collection $\{(z^i)^d\}$, however, form a *free module* for the ring of polynomials on U_i, $\mathbb{C}[x_{(i)}^j]$. On the overlap U_{ij}, we have

$$P_d(z_0, \dots, z_n) = (z^j)^d P_d^{(j)}(x_{(j)}^k) = (z^i)^d P_d^{(i)}(x_{(i)}^\ell).$$

Therefore, the ratio $(z^i/z^j)^d$ defines a *function* on the overlap U_{ij}. Moreover, since on U_{ij}, $z^i, z^j \neq 0$, we know that $(z^j/z_i)^d \in \mathbb{C}^*$ and hence can be seen as transition function of a certain degree d line bundle over $\mathbb{CP}^n$ whose sections are given by homogeneous degree d polynomials. This line bundle is denoted $\mathcal{O}(d)$. More about the line bundle $\mathcal{O}(d)$ will be explained in Appendix E.

Note that any chart U_i comes with a preferred section, namely $(z^i)^d$. Indeed, on the overlap U_{ij} one has

$$(z^i)^d = \left(\frac{z^i}{z^j}\right)^d (z^j)^d \implies g_{ij} = \left(\frac{z^i}{z^j}\right)^d (z^j)^d.$$

However, the section $(z^i)^d$ is not the only choice. Given a smooth non-vanishing function $f(x,\bar{x})$ on U_i (with coordinates $x^k = z^k/z^i$, one could have chosen $\sigma^i = f(x,\bar{x})(z^i)^d$ as a section. Then

$$P_d(z_0,\dots,z_n) = \sigma^i \frac{P_d^{(i)}(x,\bar{x})}{f(x,\bar{x})}.$$

An important property of the section $(z^i)^d$ is that it is holomorphic on U_i:

$$d\bar{x}\bar{\partial}_{\bar{x}}\,(z^i)^d = 0.$$

If one would have chosen the section $\sigma^i = (z^i)^d f(x\bar{x})$ instead, then σ^i is holomorphic with respect to the connection $\bar{A} = -\bar{\partial}\log f(x,\bar{x})$:

$$(\bar{\partial} + \bar{A})\sigma^i(x,\bar{x}) = 0.$$

The advantage of this definition is that the covariant derivative $\bar{\partial} + \bar{A}$ can be defined *globally.*

local sections. Given two polynomials P_k and P_ℓ with $k = \ell + d$, one can form their ratio $Q_d = P_k/P_\ell$ which satisfies

$$Q_d(\lambda z_0,\dots,\lambda z_n) = \frac{P_k(\lambda z_0,\dots,\lambda z_n)}{P_\ell(\lambda z_0,\dots,\lambda z_n)} = \lambda^d \frac{P_k(z_0,\dots,z_n)}{P_\ell(z_0,\dots,z_n)} = \lambda^d Q_d(z_0,\dots,z_n).$$

This ratio is called a *meromorphic section.* Intuitively, Q_d has poles at the zeros of P_ℓ which are not a root of P_k. However, since neither P_ℓ nor P_k is a function, one can only talk about the position of the zeros and poles of Q_d. If P_k and P_ℓ have no common factor, one defines

(i) *divisor of zeros* D_0 as the set (manifold) of points where $P_k = 0$,

(ii) *divisor of poles* D_∞ as the set (manifold) of points where P_ℓ is zero.

Riemann surfaces. The same considerations as above also apply to Riemann surfaces Σ. However, the discussion here is more subtle since there may exists many non-equivalent connections $\bar{A}$. This means that one can have *different* holomorphic (line) bundles of the same degree. In particular, one can have non-trivial holomorphic bundles of degree zero.

Another question is, how these sections are related to maps to $\mathbb{CP}^n$. Suppose that one is given a holomorphic function $\varphi\colon \Sigma \to \mathbb{CP}^n$. Note that on $\mathbb{CP}^n$ there exists a canonical line bundle (see Appendix E) $\mathcal{O}(1) \to \mathbb{CP}^n$. As in the discussion of $\mathcal{O}(d)$ above, holomorphic sections of $\mathcal{O}(1)$ are homogeneous polynomials of degree 1, i.e. linear in $z_0,\dots z_n$. Therefore, $\mathcal{O}(1)$

comes equipped with $n+1$ canonical holomorphic sections $z_0, \dots, z_n$. Now, one may consider the pullback bundle $\varphi^*\mathcal{O}(1)$

$$\begin{tikzcd} \varphi^*\mathcal{O}(1) \arrow[r] \arrow[d] & \mathcal{O}(1) \arrow[d] \\ \Sigma \arrow[r, "\varphi \text{ hol}"'] & \mathbb{CP}^n \end{tikzcd}$$

This gives a holomorphic line bundle over Σ together with holomorphic sections $s_i = \varphi^* z_i$. If $\mathcal{O}(1)$ is equipped with a connection $\bar{A}$ and associated holomorphic sections σ^i satisfying $(\bar\partial + \bar{A})\sigma^i = 0$, then the pullback constructions yields a holomorphic line bundle with holomorphic sections $s^i = \varphi^*\sigma^i$ satisfying $(\bar\partial + \bar{a})s^i = 0$, where $\bar{a} = \varphi^*\bar{A}$.

1st Chern Class Recall that line bundles $\mathcal{L} \to X$ are classified by their first Chern class $c_1(\mathcal{L}) \in H^2(X)$. Now, one may show that the Poincaré dual of $c_1(\mathcal{O}(1))$ defines a hyperplane make precise!

$$c_1(\mathcal{O}(1)) \simeq \delta(z_{(i)} = 0).$$

This hyperplane corresponds to the zeros of the canonical sections $z_0, \dots, z_n$ of $\mathcal{O}(1)$. The *degree* of the line bundle $\varphi^*\mathcal{O}(1)$ is then precisely the number of intersection points of this hyperplane with the embedded surface $\varphi(\Sigma) \subseteq \mathbb{CP}^n$:

$$\deg(\varphi^*\mathcal{O}(1)) = \#\varphi(\sigma) \cap H(c_1(\mathcal{O}(1))).$$

meromorphic sections of $\mathcal{L} \to \sigma$

E Holomorphic line bundles over $\mathbb{CP}^n$

There exists a canonical line bundle over $\mathbb{CP}^n$, called the *tautological line bundle*. It is standard to denote it by $\mathcal{O}(-1)$. Its fibers over every point $\zeta \in \mathbb{CP}^n$ is exactly the line $\ell(\zeta) = \{\lambda\zeta \mid \lambda \in \mathbb{C}^*\} \subset \mathbb{C}^{n+1}$ determined by ζ. Schematically, one writes

$$\mathcal{O}(-1) = \{(\zeta, z) \in \mathbb{CP}^n \times \mathbb{C} \mid z \in \ell(\zeta)\}.$$

Remark 28 ($\mathcal{O}(-1)$ is a holomorphic line bundle)**.** Recall that a vector bundle is *holomorphic* if it allows local trivializations such that their transition functions are holomorphic. In the case at hand, let U_α be a coordinate chart of $\mathbb{CP}^n$ and z_α the corresponding coordinates. Over U_α, $\mathcal{O}(-1)$ is trivialized by functions $\varphi_\alpha \colon U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha)$, where $\pi \colon \mathcal{O}(-1) \to \mathbb{CP}^n$, where

$$\varphi_\alpha^{-1}(z_\alpha, \lambda_\alpha) = \lambda_\alpha z_\alpha \qquad \lambda_\alpha \neq 0.$$

Note that $\lambda_\alpha z_\alpha$ indeed lies in the line $\ell(z_\alpha)$. Over the intersection $U_\alpha \cap U_\beta$ one therefore has transition function

$$t_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1} \colon z_\beta \mapsto z_\alpha = \lambda_\alpha \lambda_\beta^{-1} z_\beta,$$

which is clearly holomorphic. $\lhd$

Its dual, denoted by $\mathcal{O}(1)$ is called the *hyperplane bundle*. The fibers of $\mathcal{O}(1)$ are given by linear maps from the fibers of $\mathcal{O}(-1)$ to $\mathbb{C}$, i.e. $\Gamma(\mathcal{O}(1)) \cong \{\ell(\zeta)^* \mid \zeta \in \mathbb{CP}^n\}$.

Let us discuss holomorphic sections of $\mathcal{O}(1)$. An element $\alpha \in (\mathbb{C}^{n+1})^*$ is simply a linear map from $\mathbb{C}^{n+1} \to \mathbb{C}$

$$\alpha(z_0, \ldots, z_n) = \sum_{i=0}^{n} \alpha^i z_i, \qquad \alpha^i \in \mathbb{C}.$$

Since the fiber $\ell(\zeta)$ over $\zeta \in \mathbb{CP}^n$ is a linear subspace of $\mathbb{C}^{n+1}$, α defines, by restriction, a linear map $s_\alpha \colon \ell(\zeta) \to \mathbb{C}$ and thus a section of $\mathcal{O}(1)$. We therefore have

$$\Gamma(\mathcal{O}(1)) \sim (\mathbb{C}^{n+1})^*.$$

The linear map $\alpha \colon \mathbb{C}^{n+1} \to \mathbb{C}$ might have a non-zero kernel, which defines a hyperplane $\tilde{H}_\alpha \subseteq \mathbb{C}^{n+1}$. Since α is linear, it is a degree 1 polynomial in $(z_0, \ldots, z_1) \in \mathbb{CP}^{n+1}$. Its kernel,

$$\ker \alpha = \{z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \alpha(z) = 0\}$$

is invariant under the $\mathbb{C}^*$-action $(z_0, \ldots, z_n) \mapsto \lambda(z_0, \ldots, z_n)$ and hence descends to $\mathbb{CP}^n$. Hence, the map α defines a hyperplane $H_\alpha = \tilde{H}_\alpha / \mathbb{C}^* \subset \mathbb{CP}^n$.

Now, the zero locus of the section σ_α is by definition given by points $[z_0, \ldots, z_n] \in \mathbb{CP}^n$ such that $s_\alpha([z_0, \ldots, z_n]) = 0$. If s_α comes form α restricted to some fiber of $\mathcal{O}(-1)$, then

$$s_\alpha([z_0, \ldots, z_n]) = s_\alpha(\ell(z_0, \ldots, z_n)) = \alpha\big|_{\ell(z_0, \ldots, z_n))} = 0$$

if and only if

$$\forall (z_0, \ldots z_n) \in \ell(z_0, \ldots, z_n) = [z_0, \ldots, z_n] \in \mathbb{CP}^n \colon (z_0, \ldots z_n) \in \ker(\alpha) = \tilde{H}_\alpha$$

and thus $[z_0, \ldots, z_n] \in H_\alpha$. The zero locus of the section $s_\alpha \in \Gamma\mathcal{O}(1)$ therefore determines hyperplanes in $\mathbb{CP}^n$.

Now, let $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. We can build sections of $\mathcal{O}(n)$ by taking tensor products of sections of $\mathcal{O}(1)$:

$$\Gamma\mathcal{O}(n) \ni s_{\alpha_1}^{k_1} \otimes \cdots \otimes s_{\alpha_n}^{k_n}, \qquad k_1 + \ldots k_n = n.$$

Since the $s_{\alpha_j}^{k_j}$ are powers of restrictions of linear functions α_j to a certain linear subspace of $\mathbb{C}^{n+1}$, the above tensor product can be viewed as a homogeneous polynomial of degree n in the variables $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. The restriction of the polynomial to the linear subspace $\ell(\zeta) \subset \mathbb{C}^{n+1}$ defines a degree n-map from $\ell(\zeta) \to \mathbb{C}$ or, equivalently, a linear map from $\ell^n(\zeta) \to \mathbb{C}$.

Take-home-message: holomorphic sections of $\mathcal{O}(n)$ are homogeneous polynomials of degree n. [3]

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