

Working Title

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1 Toy model: 0d GLSM

1.1 Setup

We start with a 0d GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X , namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 = -\beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2} \quad \{\text{eq:toy_S}\}$$

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \tag{3} \quad \{\text{eq:toy_constr}\}$$

The action enjoys a $U(1) \times U(1)$ gauge freedom (which here is simply a global $U(1) \times U(1)$ symmetry, acting as

$$e^{i\theta} \times e^{i\varphi}: (z, w) \mapsto (e^{i\theta} z, e^{i\varphi} w) \quad (4)$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z = \int \prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2 - 1) \delta(|w|^2 - 1) e^{-S(z, w)} \quad (5) \quad \{\text{eq: toy_Z}\}$$

In order to evaluate (5), we want to embed the space of fields $\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action \tilde{S} is holomorphic in the new variables (fields)
2. when we restrict to \mathcal{F} , \tilde{S} reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [1].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M = m$. Moreover, let $\{U_i, \phi_i\}$ be a real analytic atlas of M , with $U_i \subset \mathbb{R}^m$ and charts $\phi_i: U_i \rightarrow M$ so that the transition functions

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i: U_{ij} \rightarrow U_{ij} \quad (6)$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = m$ and a (real analytic) isomorphism $f: M \rightarrow \tilde{M} \subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}} \subset \mathbb{C}^m$ such that $U_i^{\mathbb{C}} \cap \mathbb{R}^m = U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

- (i) the transition functions $\phi_{ij}^{\mathbb{C}}: U_{ij}^{\mathbb{C}} \rightarrow U_{ij}^{\mathbb{C}}$ are biholomorphic
- (ii) $\phi_{ji}^{\mathbb{C}} = \left(\phi_{ij}^{\mathbb{C}}\right)^{-1}$ and $\phi_{ii}^{\mathbb{C}} = id$
- (iii) the transition functions $\phi_{ij}^{\mathbb{C}}$ satisfy the usual 2-cocycle condition (gluing condition) on $U_{ijk}^{\mathbb{C}}: \phi_{ij}^{\mathbb{C}} \circ \phi_{jk}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U_i^{\mathbb{C}}, \phi_i^{\mathbb{C}}\}$:

$$M^{\mathbb{C}} = \coprod_i U_i^{\mathbb{C}} / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi_{ji}^{\mathbb{C}}(z_i) \text{ on } U_{ij}^{\mathbb{C}} \quad (7)$$

For more details on this construction see [Cieliebak and Eliashberg's book \[2\]](#)

1.2.1 Example: The N -sphere

Consider the N -sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_{\pm} = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in \mathbb{R}^N by $X = (X_1, \dots, X_N)$. The atlas we consider is given by stereographic projection through p_{\pm} : Let $U_{\pm} = \mathbb{R}^N$ and $V_{\pm} = S^N - \{p_{\pm}\}$. Then define charts

$$\phi_{\pm}: U_{\pm} \rightarrow V_{\pm} \subset S^N \quad , \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \quad (8)$$

with inverse

$$\phi_{\pm}^{-1}: x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \quad (9)$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_-^{-1} \circ \phi_+ : X \mapsto \frac{X}{\|X\|^2} \quad (10)$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \quad (11)$$

describes an involution at the unit sphere S^{N-1} . On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$) one finds

$$\phi_+ \circ \phi_- : (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \quad (12)$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in \mathbb{R}^N (again, it helps working this out in the case $N = 2$).

Now, since $U_{\pm} = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_{\pm}^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \quad (13) \quad \{\text{eq:toy_quadric}\}$$

The above equation defines a hypersurface (so-called *quadric*) inside \mathbb{C}^{N+1} .

There is a very interesting observation I found in [this stackexchange post](#): the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N . The diffeomorphism is realised by the following map:

$$\Psi: TS^N \rightarrow Q \quad , \quad (x, y) \mapsto z = \Psi(x, y) = x\sqrt{1 + \|y\|^2} + iy \quad (14)$$

with inverse

$$\Psi^{-1}(x + iy) = \left(\frac{x}{\sqrt{1 + \|y\|^2}}, y \right) \quad (15)$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that Ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x, y) \in \mathbb{R}^{N+1}$ such that $\langle x, y \rangle = \sum_i x_i y_i = 0$)

There exists another very interesting diffeomorphism (which I have discovered in this [stackexchange post](#)

$$\Phi: S^N S^N / \Delta \rightarrow TS^N \quad , \quad (u, v) \mapsto \left(u, \frac{v - \langle u, v \rangle u}{1 - \langle u, v \rangle} \right) \quad (16) \quad \{\text{eq:diff_SNSN_TSN}\}$$

Its inverse is given by

$$\Phi^{-1}: TS^N \rightarrow S^N \times S^N / \Delta \quad , \quad (x, y) \mapsto \left(x, \frac{x(\|y\|^2 - 1) + 2y}{\|y\|^2 + 1} \right) \quad (17)$$

Remark 3. The map (16) is the stereographic projection of $v \in S^N$ through the “pole” $u \in S^N$.

Finally we have the following commutative diagram

$$\begin{array}{ccc} & TS^N & \\ \Phi \nearrow & & \searrow \Psi \\ S^N \times S^N / \Delta & \xrightarrow{\Psi \circ \Phi} & Q \end{array} \quad (18)$$

Remark 4. The original $S^N \subset Q$ was located at $S^N = Q \cap \mathbb{R}^{N+1}$. If we follow this through, we see that this S^N corresponds to the zero section $\{(x, 0) \in TS^N\}$ inside TS^N and consequently is defined by $\Phi(u, v) = (u, 0)$ inside $S^N \times S^N / \Delta$. Note that this is precisely the anti-diagonal

$$\bar{\Delta} = \{(u, -u) \mid u \in S^N\} \subset S^N \times S^N \quad (19)$$

Indeed, the stereographic projection of $-u$ (“south pole”) through u (“north pole”) gives zero:

$$-u \mapsto \frac{(-u) - \langle (-u), u \rangle u}{1 - \langle (-u), u \rangle} = \frac{-u + \|u\|^2 u}{1 + \|u\|^2} = 0 \quad (20)$$

since $\|u\|^2 = 1$. Hence

$$\Phi(u, -u) = (u, 0) \quad (21)$$

and the original S^N is located along the anti-diagonal $\bar{\Delta}$.

1.3 Factor from δ -function Constraint

Recall that we are interested in integrals of the form

$$I = \int_{\mathbb{CP}^n \times \mathbb{CP}^n} d\mu(z) d\mu(w) e^{-S(z, w)} \mathcal{O}(z, w) \quad (22)$$

where $d\mu(z) = \prod_i dz_i d\bar{z}_i$ and we have included some observable $\mathcal{O}: \mathbb{CP}^n \rightarrow \mathbb{C}$. For our toy model, we consider the action functional (2)

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 \quad (23)$$

It is instructive pass to real coordinates. If $z_k = a_k + ib_k$, let x_k be the real vector $(a_k \ b_k)^t$. If we collect all components, we can form the real vector $x = (\dots a_k \ b_k \dots)^t \in \mathbb{R}^{2n+2}$. We denote the real vector associated to w by y . In terms of these, the path integral over can be written as

$$vol(S^1)^2 \int_{\mathbb{R}^{2n+2} \times \mathbb{R}^{2n+2}} d^{2n+2}x \ d^{2n+2}y \ \delta(\|x\|^2 - 1) \delta(\|y\|^2 - 1) e^{-S(x, y)} \mathcal{O}(x, y) \quad (24)$$

where the pre-factor stems from the compact part of the gauge group $\mathbb{C}^* \mathbb{C}^{n+1}$. Let us focus on the following part of the integral:

$$I \sim \int_{\mathbb{R}^{2n+2}} \delta(\|x\|^2 - 1) = \int_{\mathbb{R}^{2n+2}} \delta(\langle x, x \rangle - 1) \quad (25)$$

Following our idea, we complexify the space of integration, $x \rightarrow \zeta$

$$I \sim \int_{\Gamma_0 = \mathbb{R}^{2n+2} \cap \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta \rangle - 1) \quad (26)$$

where

$$\langle \zeta, \zeta \rangle = \sum_k \zeta_k^2 \quad (27)$$

The δ -function constraint restricts the support of the integrand to the quadric Q . Now, we would like to deform the “contour” (domain of integration) inside Q

$$I \sim \int_{\Gamma_a \subset \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta \rangle - 1) \quad (28)$$

Inspired by the discussion in 1.2.1, we would like to parametrise Q in terms of TS^{2n+1} . We thus choose a parametrisation of the following form: (by abuse of notation, we will use again x to denote a vector in \mathbb{R}^{2n+2})

$$\zeta(x) = x \sqrt{1 + \|y(x)\|^2} + iy(x) \quad (29) \quad \{\text{eq:parametrisation}\}$$

where $y(x)$ is chosen in such a way that $\langle x, y \rangle = 0$. It follows that

$$I \sim \int_{\mathbb{R}^{2n+2}} d^{2n+2}x \det J(x) \delta(\langle \zeta(x), \zeta(x) \rangle - 1) \quad (30)$$

where $J(x)$ denotes the Jacobian of (29). Note that the δ -function constraint can be simplified as follows: Let

$$\lambda(x) = \sqrt{1 - \|y\|^2} \quad (31)$$

Then

$$\begin{aligned} C(x) &= \langle \zeta(x), \zeta(x) \rangle - 1 \\ &= \lambda^2(x) \|x\|^2 + 2i\lambda(x) \langle x, y(x) \rangle - \|y\|^2 - 1 \\ &= \lambda^2(x) (\|x\|^2 - 1) \end{aligned} \quad (32)$$

Importantly,

$$\lambda^2(x) > 0 \quad (33)$$

such that the δ -function constraint is of the following form:

$$\begin{aligned} \int \delta(C(x)) &= \int d\mu(x) \delta(\underbrace{f(x)g(x)}_{\equiv \phi(x)}) \\ &= \int_{\phi^{-1}(0)} \frac{d\sigma}{\|\nabla \phi(x)\|} \\ &= \int_{f^{-1}(0)} \frac{d\sigma}{\|f(x)\nabla g(x) + g(x)\nabla f(x)\|} \\ &= \int_{f^{-1}(x)} \frac{d\sigma}{\|\nabla f(x)\|} \frac{1}{|g(x)|} \\ &= \int d\mu \frac{\delta(f(x))}{|g(x)|} \end{aligned} \quad (34)$$

where $f(x) = \|x\|^2 - 1$ and $g(x) = \lambda^2(x) > 0$. Hence we find that with the chosen parametrisation, the integral nicely localises to an integral over S^{2n+1}

$$\int_{\mathbb{R}^{2n+2}} \delta(C(x)) = \int_{\mathbb{R}^{2n+2}} \frac{\delta(\|x\|^2 - 1)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{1}{\lambda^2(x)} \quad (35)$$

Therefore, we are left with an integral of the form

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} \quad , \quad \lambda^2(x) = 1 + \|y(x)\|^2 \quad (36) \quad \{\text{eq:toy_schema_I}\}$$

1.4 Homogeneous Deformations

1.4.1 Example: $n = 0$

A particular nice example, which can be computed explicitly and serves as a check that everything works nicely is given for the case $n = 0$. Consider the deformation

$$y(x) = \alpha \Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SO}(2) \quad , \quad \alpha \in \mathbb{R} \quad (37) \quad \{\text{eq:toy_cst_def}\}$$

so that indeed

$$\langle x, y(x) \rangle = 0 \quad (38)$$

Moreover, note that

$$\|y(x)\|^2 = \langle y(x), y(x) \rangle = \alpha^2 \langle x, \Omega^t \Omega x \rangle = \alpha^2 \|x\|^2 = \alpha^2 \quad (39)$$

and hence

$$\lambda^2(x) = 1 + \alpha^2 \quad (40)$$

We compute the Jacobian of this parametrisation as follows: let $M_i^j = \partial_i y^j(x)$. Then

$$J_i^j = \lambda(x) \delta_i^j + \frac{\langle y, \partial_i y \rangle x^j}{\lambda(x)} + i M_i^j = \lambda(x) \delta_i^j + \frac{M_{ik} y^k x^j}{\lambda(x)} + i M_i^j \quad (41)$$

Using bra-ket notation and trivially lowering indices (considering the flat metric on \mathbb{R}^2), we can write J as follows:

$$J = \lambda(x) \cdot id + \lambda^{-1}(x) |My\rangle \langle x| + iM \quad (42)$$

Remark 5. The matrix M has some interesting properties.

1. From $\langle x, y(x) \rangle = 0$ it follows that

$$0 = \partial_i \langle x, y(x) \rangle = y_i(x) + M_{ik} x_k$$

so that

$$y_i(x) = M_{ik} x_k \implies y(x) = Mx$$

and hence

$$\langle x, y \rangle = \langle x, Mx \rangle = \langle M^t x, x \rangle = 0$$

from which follows (if $y(x) = Mx \neq 0$)

$$\langle x | M = 0$$

Now, for our choice of

$$y(x) = \alpha \Omega x$$

we have

$$M = \alpha \Omega^t = -\alpha \Omega \quad (43)$$

and hence

$$\begin{aligned} J &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 \Omega^t \Omega |x\rangle \langle x| - i\alpha \Omega \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 |x\rangle \langle x| - i\alpha \Omega \end{aligned} \quad (44)$$

Now, here is a neat trick due to Tej. Consider the unitary matrix

$$U = (x \ \Omega x) \quad , \quad \Omega^* = \begin{pmatrix} x^t \\ x^t \Omega^t \end{pmatrix} \quad (45)$$

where x is a 2×2 column vector with unit norm. Then

$$U^* U = \begin{pmatrix} \langle x, x \rangle & \langle x, \Omega x \rangle \\ \langle \Omega x, x \rangle & \langle x, x \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (46)$$

Moreover,

$$\langle x | U = x^t U = (\langle x, x \rangle \ \langle x, \Omega x \rangle) = (1 \ 0) \quad (47)$$

as well as

$$U^* \Omega U = U^* (\Omega x \ \Omega^2 x) = U^* (\Omega x \ -x) = \begin{pmatrix} \langle x, \Omega x \rangle & -\langle x, x \rangle \\ \langle x, \Omega^t \Omega x \rangle & -\langle x, \Omega^t x \rangle \end{pmatrix} = -\Omega \quad (48)$$

Then

$$\begin{aligned} U^* J U &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* |x\rangle \langle x| U - i\alpha U^* \Omega U \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i\alpha \Omega \\ &= \begin{pmatrix} \lambda(x) + \lambda^{-1}(x) \alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \end{aligned} \quad (49)$$

It follows that

$$\begin{aligned} \det J(x) &= \det(U^* J(x) U) \\ &= \det \begin{pmatrix} \lambda(x) + \lambda^{-1}(x) \alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \\ &= \lambda^2(x) + \alpha^2 - \alpha^2 = \lambda^2(x) \end{aligned} \quad (50)$$

Finally, we obtain that

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{\lambda^2(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \quad (51)$$

This means that the *constant deformation* (37) comes with a trivial *total* Jacobian (Jacobian + factor from the φ -function constraint).

1.4.2 The General Case

The $n = 0$ example can be nicely generalised. Tej suggested to consider deformations of the form

$$Y_a(X) = \Omega(a)X \quad , \quad \Omega(a) = \text{diag} \left(\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -a_{n+1} \\ a_{n+1} & 0 \end{pmatrix} \right) \quad (52) \quad \{\text{eq:Tej_deformation}\}$$

for some $a = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$. This type of deformation have a very nice geometric origin, as we now explain.

Remark 6. Under the isomorphism

$$\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \quad , \quad \begin{pmatrix} x_k \\ y_k \end{pmatrix} \mapsto z_k = x_k + iy_k \quad (53)$$

the deformation (52) is acting on \mathbb{C}^{n+1} by multiplication with the diagonal matrix $\text{diag}(ia_1, \dots, ia_{n+1})$

$$\begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ ia_k z_k \\ \vdots \end{pmatrix} \quad (54)$$

Let G be a Lie group and $H < G$ a subgroup with Lie algebras \mathfrak{g} and \mathfrak{h} respectively. The orbit space G/H is known as a *homogeneous space*. It is called *reductive* if \mathfrak{g} allows the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad , \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad (55)$$

If \mathfrak{g} is equipped with a Killing form B (or if G allows a G -invariant metric as a manifold), then G/H is automatically reductive with the choice

$$\mathfrak{m} = \mathfrak{h}^\perp \quad (56)$$

where \mathfrak{h}^\perp is the orthogonal to \mathfrak{h} with respect to B . Indeed, since B is G -invariant, schematically

$$0 = B(\mathfrak{h}, \mathfrak{m}) = B([\mathfrak{h}, \mathfrak{h}], \mathfrak{m}) = B(\mathfrak{h}, [\mathfrak{h}, \mathfrak{m}]) \quad (57)$$

so that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

In summary, given a reductive homogeneous space G/H , we get a nice decomposition of the tangent space: first, let $o = [e]$ be the class in G/H at the identity. Then

$$T_o(G/H) \cong \mathfrak{m} \quad (58)$$

This isomorphism is given essentially by the pushforward of the natural projection $\pi: G \rightarrow G/H$. In fact, $\pi: G \rightarrow G/H$ defines a principal H -bundle. Its vertical vector fields are given by $\ker(\pi_*) = \mathfrak{h}$ and since π is a submersion (π and π_* are both surjective) we have

$$\pi_*: \mathfrak{g}/\ker(\pi_*) = \mathfrak{g}/\mathfrak{h} \cong T_o(G/H) \quad (59)$$

Then, to compute the tangent space at any other point, we use translation by G .

Now, recall that

$$S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n) \quad (60)$$

is a homogeneous space where $H = \mathrm{SU}(n)$ sits inside $G = \mathrm{SU}(n+1)$ as the lower right corner

$$\mathrm{SU}(n) \ni h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in \mathrm{SU}(n+1) \quad (61)$$

The reductive split is given by

$$\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m} \quad (62) \quad \{\text{eq:red_split}\}$$

Since $\mathfrak{su}(n+1)$ admits a Killing form $B(x, y) \propto \mathrm{tr}(x, y)$, we can choose $\mathfrak{m} = \mathfrak{su}(n)^\perp$. In the split (62), we have

$$\mathfrak{su}(n) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{su}(n) \end{pmatrix} \subset \mathfrak{su}(n+1) \quad (63)$$

We then find

$$\mathfrak{m} = \mathfrak{su}(n)^\perp = \left\{ \begin{pmatrix} 0 & -\zeta^* \\ \zeta & 0 \end{pmatrix} \right\}, \quad \zeta \in \mathbb{C}^{n+1} \quad (64)$$

Here $\zeta^* = \bar{\zeta}^t$ denotes the hermitian conjugate. Now, this identifies $\mathfrak{m} \in \mathrm{Mat}(\mathbb{C}, n+1)$ which can of course be described in terms of real matrices $\mathrm{Mat}(\mathbb{R}, 2n+2)$. Let us look at the example $n = 1$; let $\zeta = a + ib$ and we act on $z = (z_1 \ z_2)^t \in \mathbb{C}^2$ with $z_k = x_k + iy_k$. Then

$$\begin{aligned} \begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -\bar{\zeta} z_2 \\ \zeta z_1 \end{pmatrix} = \begin{pmatrix} -(a - ib)(x_2 + iy_2) \\ (a + ib)(x_1 + iy_1) \end{pmatrix} \\ &= \begin{pmatrix} -(ax_2 + by_2 + i(ay_2 - bx_2)) \\ ax_1 - by_1 + i(ay_1 + bx_1) \end{pmatrix} \end{aligned} \quad (65)$$

If we represent $(z_1 \ z_2)^t$ as before by $(x_1 \ y_1 \ x_2 \ y_2)$, then we find

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -(ax_2 + by_2) \\ -(ay_2 - bx_2) \\ ax_1 - by_1 \\ ay_1 + bx_1 \end{pmatrix} \quad (66)$$

so that

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & -a & -b \\ 0 & 0 & b & -a \\ a & -b & 0 & 0 \\ b & a & 0 & 0 \end{pmatrix} \quad (67)$$

This matrix is skew-symmetric and can be diagonalised and admits eigenvalues

$$\lambda_1, \lambda_2 = i\sqrt{a^2 + b^2} \quad , \quad \lambda_3, \lambda_4 = -i\sqrt{a^2 + b^2} \quad (68)$$

Now, every $2k \times 2k$ skew-symmetric matrix can be diagonalised (its eigenvalues can be purely imaginary and come in pairs, $\pm i\lambda_k$) and can be brought into the form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \dots \\ -\lambda_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \lambda_2 & \\ 0 & 0 & -\lambda_2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix} \quad (69)$$

In our case, this means that we can find a representation of $\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}$ acting on \mathbb{R}^{2n+2} such that

$$\rho\left(\begin{pmatrix} 0 & -\bar{\zeta} \\ \zeta & 0 \end{pmatrix}\right) = \|\zeta\|^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (70)$$

which is a natural generalization of the $n = 0$ case but only a special case of Tej's suggestion (52) **question:** what are (52) in general?

References

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