

Working Title

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Contents

1 Toy model: $0d$ GLSM	1
1.1 Setup	1
1.2 Manifold Deformation	2
1.3 Action along the Deformed Manifold	3

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be $X = \mathbb{CP}^1$. The space of fields is then simply given by points on the target manifold X , namely

$$\mathcal{F} = \mathbb{CP}^1 \tag{1}$$

As the action of the model we consider

$$S(z_0, z_1) = \beta \left(|z_0|^2 - |z_1|^2 \right) \tag{2} \quad \{\text{eq:toy_S}\}$$

where $[z_0 : z_1] \in \mathbb{CP}^1$. As written, the partition function explicitly shows the $U(1)$ gauge freedom (which here is simply a global $U(1)$ symmetry, acting as

$$e^{i\theta} : (z_0, z_1) \mapsto (e^{i\theta} z_0, e^{i\theta} z_1) \tag{3}$$

and defines indeed a function on \mathbb{CP}^1 if we implicitly assume the constraint

$$|z_0|^2 + |z_1|^2 = 1 \tag{4} \quad \{\text{eq:toy_constr}\}$$

The path integral of the model is thus defined by

$$Z = \int dz_0 dz_1 \delta(|z_0|^2 + |z_1|^2 - 1) e^{-S(z_0, z_1)} \tag{5} \quad \{\text{eq:toy_Z}\}$$

In order to evaluate (5), we want to embed \mathbb{CP}^1 into a higher dimensional complex space such that

1. the new action \tilde{S} is holomorphic in the new variables (fields)
2. when we restrict to \mathbb{CP}^1 , \tilde{S} reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

Going to real fields, $z_k = x_k + iy_k$ for $k = 0, 1$, the action (2) becomes

$$S = \beta (x_0^2 + y_0^2 - x_1^2 - y_1^2) \quad (6) \quad \{\text{eq:toy_S_real}\}$$

while the constraint becomes

$$x_0^2 + y_0^2 + x_1^2 + y_1^2 = 1 \quad (7) \quad \{\text{eq:toy_constr_real}\}$$

1.2 Manifold Deformation

Suppose that we now complexify in the sense that we consider $x_k, y_k \in \mathbb{C}$. For simplicity, let us rename the variables according to

$$x_0 = u_0 \quad , \quad x_1 = u_1 \quad , \quad y_0 = u_2 \quad , \quad y_1 = u_3 \quad (8)$$

where now $u_i \in \mathbb{C}$. This slightly unusual renaming is done for later convenience.

The constraint (7) now becomes

$$\sum_{i=0}^3 u_i^2 = 1 \quad (9)$$

which can be seen as a complex hypersurface inside \mathbb{CP}^3 , as follows: Let us introduce the complex variable $t \in \mathbb{C}$ and consider the equation

$$\sum_{i=0}^3 u_i^2 = t^2 \quad (10)$$

This equation describes the zero set of a homogeneous quadratic polynomial

$$P(t, u_i) = t^2 - \sum_{i=0}^3 u_i^2 \quad (11)$$

Notice that for any $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$,

$$P(\lambda t, \lambda u_i) = \lambda^2 P(t, u_i) \quad (12)$$

Therefore, the solution space of

$$P(u_i, t) = 0 \quad (13)$$

is invariant under the action of \mathbb{C}^* (by multiplication) and hence descends to an equation on \mathbb{CP}^3 whose coordinates are $[t : u_0 : u_1 : u_2]$

The important observation is that the constraint surface (7) coincides with $P(t, u_i) = 0$ for $t = 1$. But $t = 1$ simply defines the $U_{t \neq 0} \subset \mathbb{CP}^3$ whose coordinates are given by $[1 : \frac{u_0}{t} : \frac{u_1}{t} : \frac{u_2}{t}]$. Hence, the constraint surface can be embedded into \mathbb{CP}^3 :

$$[u_0 : u_1 : u_2] \mapsto [t(u_i) : u_0 : u_1 : u_2] \quad , \quad t(u_i) = \sqrt{\sum_i u_i^2} \quad (14) \quad \{\text{eq:toy_emb}\}$$

Notice that $[t(u_i) : u_0 : u_1 : u_2]$ simply describes a point in $P(t, u_i) = 0$ and the original surface is reproduced in the chart $t(u_i) = 1$.

This embedding comes with a natural family of holomorphic deformations parametrised by a vector $\omega \in \mathbb{C}^3 - \{0\}$. Namely, instead of considering zeros of $P(t, u_i)$, one could consider zeros of a general homogeneous quadratic polynomial

$$P_\omega(t, u_i) = t^2 - \sum_i \omega_i u_i^2 \quad (15)$$

which for $\omega = (1, 1, 1) \in \mathbb{C}^3$ coincides with P .

1.3 Action along the Deformed Manifold

In the chart $t = 1$, the action (6) becomes

$$S(u) = \beta (u_0^2 + u_1^2 - u_2^2 - u_3^2) \quad (16)$$

If we were to reinstate t , we would have to do it in a way that ensures that $S(u)$ is a function on \mathbb{CP}^3 , i.e. invariant under the \mathbb{C}^* action $\lambda: (t, u_i) \mapsto (\lambda t, \lambda u_i)$. An obvious candidate is

$$S(t, u) = \frac{\beta (u_0^2 + u_1^2 - u_2^2 - u_3^2)}{t^2} \quad (17)$$

On the constraint surface (inside \mathbb{CP}^3) we now replace t^2 by $P_\omega(u)$ for some non-zero $\omega \in \mathbb{C}^3$. We obtain

$$S(t, u)|_{\mathcal{C}_\omega} = \frac{\beta (u_0^2 + u_1^2 - u_2^2 - u_3^2)}{P_\omega(u)} = \frac{\beta (u_0^2 + u_1^2 - u_2^2 - u_3^2)}{\sum_i \omega_i u_i^2} \quad (18)$$

where we denote the (generalised) constraint surface by

$$\mathcal{C}_\omega = \{P_\omega(t, u) = 0\} \quad (19)$$