

Working Title

November 8, 2024

Contents

1 Toy model: $0d$ GLSM	1
1.1 Setup	1
1.2 Complexification of Real Analytic Manifolds	2
1.2.1 Example: The N -sphere	3

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X , namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z, w) = \beta |\langle \bar{z}, w \rangle|^2 = \beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2} \quad \{\texttt{eq:toy_S}\}$$

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \tag{3} \quad \{\texttt{eq:toy_constr}\}$$

The action enjoys a $U(1) \times U(1)$ gauge freedom (which here is simply a global $U(1) \times U(1)$ symmetry, acting as

$$e^{i\theta} \times e^{i\varphi}: (z, w) \mapsto (e^{i\theta} z, e^{i\varphi} w) \tag{4}$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z = \int \prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2 - 1) \delta(|w|^2 - 1) e^{-S(z,w)} \quad (5) \quad \{\text{eq:toy_Z}\}$$

In order to evaluate (5), we want to embed the space of fields $\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action \tilde{S} is holomorphic in the new variables (fields)
2. when we restrict to \mathcal{F} , \tilde{S} reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [1].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M = m$. Moreover, let $\{U_i, \phi_i\}$ be a real analytic atlas of M , with $U_i \subset \mathbb{R}^m$ and charts $\phi_i: U_i \rightarrow M$ so that the transition functions

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i: U_{ij} \rightarrow U_{ij} \quad (6)$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = m$ and a (real analytic) isomorphism $f: M \rightarrow \tilde{M} \subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}} \subset \mathbb{C}^m$ such that $U_i^{\mathbb{C}} \cap \mathbb{R}^m = U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

- (i) the transition functions $\phi_{ij}^{\mathbb{C}}: U_{ij}^{\mathbb{C}} \rightarrow U_{ij}^{\mathbb{C}}$ are biholomorphic
- (ii) $\phi_{ji}^{\mathbb{C}} = (\phi_{ij}^{\mathbb{C}})^{-1}$ and $\phi_{ii}^{\mathbb{C}} = id$
- (iii) the transition functions $\phi_{ij}^{\mathbb{C}}$ satisfy the usual 2-cocycle condition (gluing condition) on $U_{ijk}^{\mathbb{C}}$: $\phi_{ij}^{\mathbb{C}} \circ \phi_{jk}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U_i^{\mathbb{C}}, \phi_i^{\mathbb{C}}\}$:

$$M^{\mathbb{C}} = \coprod_i U_i^{\mathbb{C}} / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi_{ji}^{\mathbb{C}}(z_i) \text{ on } U_{ij}^{\mathbb{C}} \quad (7)$$

For more details on this construction see Cieliebak and Eliashberg's book [2]

1.2.1 Example: The N -sphere

Consider the N -sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_{\pm} = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in \mathbb{R}^N by $X = (X_1, \dots, X_N)$. The atlas we consider is given by stereographic projection through p_{\pm} : Let $U_{\pm} = \mathbb{R}^N$ and $V_{\pm} = S^N - \{p_{\pm}\}$. Then define charts

$$\phi_{\pm}: U_{\pm} \rightarrow V_{\pm} \subset S^N, \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \quad (8)$$

with inverse

$$\phi_{\pm}^{-1}: x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \quad (9)$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_-^{-1} \circ \phi_+: X \mapsto \frac{X}{\|X\|^2} \quad (10)$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \quad (11)$$

describes an involution at the unit sphere S^{N-1} . On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$) one finds

$$\phi_+ \circ \phi_-: (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \quad (12)$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in \mathbb{R}^N (again, it helps working this out in the case $N = 2$).

Now, since $U_{\pm} = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_{\pm}^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \quad (13) \quad \{\text{eq:toy_quadric}\}$$

The above equation defines a hypersurface (so-called *quadric*) inside \mathbb{C}^{N+1} .

There is a very interesting observation I found in this [stackexchange](#) post: the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N . The diffeomorphism is realised by the following map:

$$\psi: TS^N \rightarrow Q \quad , \quad (x, y) \mapsto z = \psi(x, y) = x\sqrt{1 + \|y\|^2} + iy \quad (14)$$

with inverse

$$\psi^{-1}(x + iy) = \left(\frac{x}{\sqrt{1 + \|y\|^2}}, y \right) \quad (15)$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x, y) \in \mathbb{R}^{N+1}$ such that $\langle x, y \rangle = \sum_i x_i y_i = 0$)

References

- [1] F. Bruhat and H. Whitney, Quelques propriétés fondamentales des ensembles analytiques-réels, Comment. Math. Helv. 33, 132-160 (1959).
- [2] K. Cieliebak and Y. Eliashberg. From Stein to Weinstein and back: symplectic geometry of affine complex manifolds. Vol. 59. American Mathematical Soc., 2012.