

Working Title

January 24, 2025

Contents

1 Toy model: $0d$ GLSM	1
1.1 Setup	1
1.2 Complexification of Real Analytic Manifolds	2
1.2.1 Example: The N -sphere	3
1.3 Factor from δ -function Constraint	5
1.4 Homogeneous Deformations	7
1.4.1 The General Case	7
1.5 Example: $n = 0$	11
1.5.1 Homogeneous deformation	11
1.5.2 What it really means	13
1.5.3 Generalisation: Torus deformations	13
1.6 Correlators	14
1.6.1 Generalities	14
1.6.2 One-point function	16
1.6.3 Two-point function	18

1 Toy model: $0d$ GLSM

1.1 Setup

We start with a $0d$ GLSM toy model, i.e. we consider the source manifold $\Sigma = \{pt\}$ to be an abstract point and the target manifold to be

$$X = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$$

The space of fields is then simply given by points on the target manifold X , namely

$$\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \tag{1}$$

As the action of the model we consider

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 = -\beta \sum_{j,k=0}^{N-1} \bar{z}_j z_k \bar{w}_j w_k \tag{2} \quad \{\text{eq:toy_S}\}$$

where $z, w \in \mathbb{C}^N$ subject to the condition

$$|z|^2 = \sum_j |z_j|^2 = 1 \quad , \quad |w|^2 = \sum_k |w_k|^2 = 1 \quad (3) \quad \{\text{eq:toy_constr}\}$$

The action enjoys a $U(1) \times U(1)$ gauge freedom (which here is simply a global $U(1) \times U(1)$ symmetry, acting as

$$e^{i\theta} \times e^{i\varphi}: (z, w) \mapsto (e^{i\theta} z, e^{i\varphi} w) \quad (4)$$

Under the assumption of the condition (3), the action (2) does define a function on $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$.

The path integral of the model is defined by

$$Z = \int \prod_j dz_j d\bar{z}_j dw_j d\bar{w}_j \delta(|z|^2 - 1) \delta(|w|^2 - 1) e^{-S(z, w)} \quad (5) \quad \{\text{eq:toy_Z}\}$$

In order to evaluate (5), we want to embed the space of fields $\mathcal{F} = \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into a higher dimensional complex space such that

1. the new action \tilde{S} is holomorphic in the new variables (fields)
2. when we restrict to \mathcal{F} , \tilde{S} reduces to S

We think of this embedding as an analytic continuation of the space of fields in an appropriate sense.

1.2 Complexification of Real Analytic Manifolds

Let us first recall a basic construction of complexification of real analytic manifolds due to Bruhat and Whitney [2].

Let M be a real analytic manifold of dimension $\dim_{\mathbb{R}} M = m$. Moreover, let $\{U_i, \phi_i\}$ be a real analytic atlas of M , with $U_i \subset \mathbb{R}^m$ and charts $\phi_i: U_i \rightarrow M$ so that the transition functions

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i: U_{ij} \rightarrow U_{ij} \quad (6)$$

are real analytic diffeomorphisms. The idea of complexifying M is to find a complex manifold $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = m$ and a (real analytic) isomorphism $f: M \rightarrow \tilde{M} \subset M^{\mathbb{C}}$ of M onto a submanifold of $M^{\mathbb{C}}$. (Fancy way to say that M should be a real analytic submanifold of $M^{\mathbb{C}}$ up to isomorphism) Now, find opens $U_i^{\mathbb{C}} \subset \mathbb{C}^m$ such that $U_i^{\mathbb{C}} \cap \mathbb{R}^m = U_i$ and extend the charts ϕ_i charts $\phi_i^{\mathbb{C}}$ such that

(i) the transition functions $\phi_{ij}^{\mathbb{C}}: U_{ij}^{\mathbb{C}} \rightarrow U_{ij}^{\mathbb{C}}$ are biholomorphic

(ii) $\phi_{ji}^{\mathbb{C}} = \left(\phi_{ij}^{\mathbb{C}}\right)^{-1}$ and $\phi_{ii}^{\mathbb{C}} = id$

- (iii) the transition functions $\phi_{ij}^{\mathbb{C}}$ satisfy the usual 2-cocycle condition (gluing condition) on $U_{ijk}^{\mathbb{C}}$: $\phi_{ij}^{\mathbb{C}} \circ \phi_{jk}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$

These conditions ensure that we can glue $M^{\mathbb{C}}$ from the local data $\{U_i^{\mathbb{C}}, \phi_i^{\mathbb{C}}\}$:

$$M^{\mathbb{C}} = \coprod_i U_i^{\mathbb{C}} / \sim \quad , \quad z_i \sim z_j \text{ iff } z_j = \phi_{ji}^{\mathbb{C}}(z_i) \text{ on } U_{ij}^{\mathbb{C}} \quad (7)$$

For more details on this construction see [Cieliebak and Eliashberg's book \[3\]](#)

1.2.1 Example: The N -sphere

Consider the N -sphere $S^N \subset \mathbb{R}^{N+1}$. First, consider the following atlas: let $p_{\pm} = (0, \dots, 0, \pm 1) \in S^N$ be the north and south pole respectively. We denote points on the sphere by $x = (x_1, \dots, x_{N+1})$, $\|x\|^2 = 1$ and points in \mathbb{R}^N by $X = (X_1, \dots, X_N)$. The atlas we consider is given by stereographic projection through p_{\pm} : Let $U_{\pm} = \mathbb{R}^N$ and $V_{\pm} = S^N - \{p_{\pm}\}$. Then define charts

$$\phi_{\pm}: U_{\pm} \rightarrow V_{\pm} \subset S^N \quad , \quad X \mapsto \left(\frac{2X}{1 + \|X\|^2}, \pm \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right) \quad (8)$$

with inverse

$$\phi_{\pm}^{-1}: x \mapsto \left\{ \frac{x_i}{1 \mp x_{N+1}} \right\} \quad (9)$$

From this, one finds the transition functions

$$\phi_{+-} = (\phi_{-+})^{-1} = \phi_{-}^{-1} \circ \phi_{+}: X \mapsto \frac{X}{\|X\|^2} \quad (10)$$

Remark 1. There is a nice geometric interpretation of these transition functions. Note that the map

$$X \mapsto \frac{X}{\|X\|^2} \quad (11)$$

describes an involution at the unit sphere S^{N-1} . On the sphere, the maps differ merely by a sign switch in the x_{N+1} component and indeed, if one computes the transition functions directly on the embedded sphere (that is as maps from $\mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$) one finds

$$\phi_{+} \circ \phi_{-}: (x_i, x_{N+1}) \mapsto (x_i, -x_{N+1}) \quad (12)$$

which corresponds to a reflection of x about the equator! (It helps visualising this for the case 2-sphere) But under stereographic projection, a reflection about the equator becomes an inversion on the unit sphere S^{N-1} in \mathbb{R}^N (again, it helps working this out in the case $N = 2$).

Now, since $U_{\pm} = \mathbb{R}^N$ there exist obvious candidates for a complexification, namely $U_{\pm}^{\mathbb{C}} = \mathbb{C}^N$. We thus promote every X to a complex variable $Z = X + iY$. Conversely, we can promote any $x \in \mathbb{R}^{N+1}$ satisfying $\|x\|^2 = \sum_j x_j^2 = 1$ to complex variables z satisfying

$$\sum_j z_j^2 = 1 \quad (13) \quad \{\text{eq:toy_quadric}\}$$

The above equation defines a hypersurface (so-called *quadric*) inside \mathbb{C}^{N+1} .

There is a very interesting observation I found in [this stackexchange post](#): the quadric Q defined by (13) is *diffeomorphic* to the tangent space TS^N . The diffeomorphism is realised by the following map:

$$\Psi: TS^N \rightarrow Q \quad , \quad (x, y) \mapsto z = \Psi(x, y) = x\sqrt{1 + \|y\|^2} + iy \quad (14)$$

with inverse

$$\Psi^{-1}(x + iy) = \left(\frac{x}{\sqrt{1 + \|y\|^2}}, y \right) \quad (15)$$

where $\|y\|^2 = \sum_i y_i^2$.

Remark 2. Verification that Ψ does indeed define a diffeomorphism goes by a straight forward calculation. (Recall in particular that the tangent space TS^N can be described by pairs $(x, y) \in \mathbb{R}^{N+1}$ such that $\langle x, y \rangle = \sum_i x_i y_i = 0$)

There exists another very interesting diffeomorphism (which I have discovered in this [stackexchange post](#)

$$\Phi: S^N \times S^N / \Delta \rightarrow TS^N \quad , \quad (u, v) \mapsto \left(u, \frac{v - \langle u, v \rangle u}{1 - \langle u, v \rangle} \right) \quad (16) \quad \{\text{eq:diff_SNSN_TSN}\}$$

Its inverse is given by

$$\Phi^{-1}: TS^N \rightarrow S^N \times S^N / \Delta \quad , \quad (x, y) \mapsto \left(x, \frac{x(\|y\|^2 - 1) + 2y}{\|y\|^2 + 1} \right) \quad (17)$$

Remark 3. The map (16) is the stereographic projection of $v \in S^N$ through the “pole” $u \in S^N$.

Finally we have the following commutative diagram

$$\begin{array}{ccc} & TS^N & \\ \Phi \nearrow & & \searrow \Psi \\ S^N \times S^N / \Delta & \xrightarrow{\Psi \circ \Phi} & Q \end{array} \quad (18)$$

Remark 4. The original $S^N \subset Q$ was located at $S^N = Q \cap \mathbb{R}^{N+1}$. If we follow this through, we see that this S^N corresponds to the zero section $\{(x, 0) \in TS^N\}$ inside TS^N and consequently is defined by $\Phi(u, v) = (u, 0)$ inside $S^N \times S^N / \Delta$. Note that this is precisely the anti-diagonal

$$\bar{\Delta} = \{(u, -u) \mid u \in S^N\} \subset S^N \times S^N \quad (19)$$

Indeed, the stereographic projection of $-u$ (“south pole”) through u (“north pole”) gives zero:

$$-u \mapsto \frac{(-u) - \langle (-u), u \rangle u}{1 - \langle (-u), u \rangle} = \frac{-u + \|u\|^2 u}{1 + \|u\|^2} = 0 \quad (20)$$

since $\|u\|^2 = 1$. Hence

$$\Phi(u, -u) = (u, 0) \quad (21)$$

and the original S^N is located along the anti-diagonal $\bar{\Delta}$.

1.3 Factor from δ -function Constraint

Recall that we are interested in integrals of the form

$$I = \int_{\mathbb{CP}^n \times \mathbb{CP}^n} d\mu(z) d\mu(w) e^{-S(z, w)} \mathcal{O}(z, w) \quad (22)$$

where $d\mu(z) = \prod_i dz_i d\bar{z}_i$ and we have included some observable $\mathcal{O}: \mathbb{CP}^n \rightarrow \mathbb{C}$. For our toy model, we consider the action functional (2)

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 \quad (23)$$

It is instructive pass to real coordinates. If $z_k = a_k + ib_k$, let x_k be the real vector $(a_k \ b_k)^t$. If we collect all components, we can form the real vector $x = (\dots a_k \ b_k \dots)^t \in \mathbb{R}^{2n+2}$. We denote the real vector associated to w by y . In terms of these, the path integral over can be written as

$$vol(S^1)^2 \int_{\mathbb{R}^{2n+2} \times \mathbb{R}^{2n+2}} d^{2n+2}x \ d^{2n+2}y \ \delta(\|x\|^2 - 1) \delta(\|y\|^2 - 1) e^{-S(x, y)} \mathcal{O}(x, y) \quad (24)$$

where the pre-factor stems from the compact part of the gauge group $\mathbb{C}^* \mathbb{C}^{n+1}$. Let us focus on the following part of the integral:

$$I \sim \int_{\mathbb{R}^{2n+2}} \delta(\|x\|^2 - 1) = \int_{\mathbb{R}^{2n+2}} \delta(\langle x, x \rangle - 1) \quad (25)$$

Following our idea, we complexify the space of integration, $x \rightarrow \zeta$

$$I \sim \int_{\Gamma_0 = \mathbb{R}^{2n+2} \cap \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta \rangle - 1) \quad (26)$$

where

$$\langle \zeta, \zeta \rangle = \sum_k \zeta_k^2 \quad (27)$$

The δ -function constraint restricts the support of the integrand to the quadric Q . Now, we would like to deform the “contour” (domain of integration) inside Q

$$I \sim \int_{\Gamma_a \subset \mathbb{C}^{2n+2}} \delta(\langle \zeta, \zeta \rangle - 1) \quad (28)$$

Inspired by the discussion in 1.2.1, we would like to parametrise Q in terms of TS^{2n+1} . We thus choose a parametrisation of the following form: (by abuse of notation, we will use again x to denote a vector in \mathbb{R}^{2n+2})

$$\zeta(x) = x \sqrt{1 + \|y(x)\|^2} + iy(x) \quad (29) \quad \{\text{eq:parametrisation}\}$$

where $y(x)$ is chosen in such a way that $\langle x, y \rangle = 0$. It follows that

$$I \sim \int_{\mathbb{R}^{2n+2}} d^{2n+2}x \det J(x) \delta(\langle \zeta(x), \zeta(x) \rangle - 1) \quad (30)$$

where $J(x)$ denotes the Jacobian of (29). Note that the δ -function constraint can be simplified as follows: Let

$$\lambda(x) = \sqrt{1 + \|y\|^2} \quad (31)$$

Then

$$\begin{aligned} C(x) &= \langle \zeta(x), \zeta(x) \rangle - 1 \\ &= \lambda^2(x) \|x\|^2 + 2i\lambda(x) \langle x, y(x) \rangle - \|y\|^2 - 1 \\ &= \lambda^2(x) (\|x\|^2 - 1) \end{aligned} \quad (32)$$

Importantly,

$$\lambda^2(x) > 0 \quad (33)$$

such that the δ -function constraint is of the following form:

$$\begin{aligned} \int \delta(C(x)) &= \int d\mu(x) \delta(\underbrace{f(x)g(x)}_{\equiv \phi(x)}) \\ &= \int_{\phi^{-1}(0)} \frac{d\sigma}{\|\nabla \phi(x)\|} \\ &= \int_{f^{-1}(0)} \frac{d\sigma}{\|f(x)\nabla g(x) + g(x)\nabla f(x)\|} \\ &= \int_{f^{-1}(x)} \frac{d\sigma}{\|\nabla f(x)\|} \frac{1}{|g(x)|} \\ &= \int d\mu \frac{\delta(f(x))}{|g(x)|} \end{aligned} \quad (34)$$

where $f(x) = \|x\|^2 - 1$ and $g(x) = \lambda^2(x) > 0$. Hence we find that with the chosen parametrisation, the integral nicely localises to an integral over S^{2n+1}

$$\int_{\mathbb{R}^{2n+2}} \delta(C(x)) = \int_{\mathbb{R}^{2n+2}} \frac{\delta(\|x\|^2 - 1)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{1}{\lambda^2(x)} \quad (35)$$

Therefore, we are left with an integral of the form

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} \quad , \quad \lambda^2(x) = 1 + \|y(x)\|^2 \quad (36) \quad \{\text{eq:toy_schema_I}\}$$

1.4 Homogeneous Deformations

1.4.1 The General Case

The $n = 0$ example can be nicely generalised. Tej suggested to consider deformations of the form

$$Y_a(X) = \Omega(a)X \quad , \quad \Omega(a) = \text{diag} \left(\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -a_{n+1} \\ a_{n+1} & 0 \end{pmatrix} \right) \quad (37) \quad \{\text{eq:Tej_deformation}\}$$

for some $a = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$. This type of deformation have a very nice geometric origin, as we now explain.

Remark 5. Under the isomorphism

$$\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \quad , \quad \begin{pmatrix} x_k \\ y_k \end{pmatrix} \mapsto z_k = x_k + iy_k \quad (38)$$

the deformation (37) is acting on \mathbb{C}^{n+1} by multiplication with the diagonal matrix $\text{diag}(ia_1, \dots, ia_{n+1})$

$$\begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ ia_k z_k \\ \vdots \end{pmatrix} \quad (39)$$

The finite diffeomorphism generated by this vector field is given by

$$\begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ e^{ia_k} z_k \\ \vdots \end{pmatrix} \quad (40)$$

The generated finite action is thus the action of the torus

$$U(1)^{n+1} \subset \text{SU}(n+1) \quad (41)$$

Let G be a Lie group and $H < G$ a subgroup with Lie algebras \mathfrak{g} and \mathfrak{h} respectively. The orbit space G/H is known as a *homogeneous space*. It is called *reductive* if \mathfrak{g} allows the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad , \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad (42)$$

If \mathfrak{g} is equipped with a Killing form B (or if G allows a G -invariant metric as a manifold), then G/H is automatically reductive with the choice

$$\mathfrak{m} = \mathfrak{h}^\perp \quad (43)$$

where \mathfrak{h}^\perp is the orthogonal to \mathfrak{h} with respect to B . Indeed, since B is G -invariant, schematically

$$0 = B(\mathfrak{h}, \mathfrak{m}) = B([\mathfrak{h}, \mathfrak{h}], \mathfrak{m}) = B(\mathfrak{h}, [\mathfrak{h}, \mathfrak{m}]) \quad (44)$$

so that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

In summary, given a reductive homogeneous space G/H , we get a nice decomposition of the tangent space: first, let $o = [e]$ be the class in G/H at the identity. Then

$$T_o(G/H) \cong \mathfrak{m} \quad (45)$$

This isomorphism is given essentially by the pushforward of the natural projection $\pi: G \rightarrow G/H$. In fact, $\pi: G \rightarrow G/H$ defines a principal H -bundle. Its vertical vector fields are given by $\ker(\pi_*) = \mathfrak{h}$ and since π is a submersion (π and π_* are both surjective) we have

$$\pi_*: \mathfrak{g} / \ker(\pi_*) = \mathfrak{g} / \mathfrak{h} \cong T_o(G/H) \quad (46)$$

Then, to compute the tangent space at any other point, we use translation by G .

Now, recall that

$$S^{2n+1} = \mathrm{SU}(n+1) / \mathrm{SU}(n) \quad (47)$$

is a homogeneous space where $H = \mathrm{SU}(n)$ sits inside $G = \mathrm{SU}(n+1)$ as the lower right corner

$$\mathrm{SU}(n) \ni h \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in \mathrm{SU}(n+1) \quad (48)$$

The reductive split is given by

$$\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m} \quad (49) \quad \{\text{eq:red_split}\}$$

Since $\mathfrak{su}(n+1)$ admits a Killing form $B(x, y) \propto \mathrm{tr}(x, y)$, we can choose $\mathfrak{m} = \mathfrak{su}(n)^\perp$. In the split (49), we have

$$\mathfrak{su}(n) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{su}(n) \end{pmatrix} \subset \mathfrak{su}(n+1) \quad (50)$$

We then find

$$\mathfrak{m} = \mathfrak{su}(n)^\perp = \left\{ \begin{pmatrix} 0 & -\zeta^* \\ \zeta & 0 \end{pmatrix} \right\}, \quad \zeta \in \mathbb{C}^{n+1} \quad (51)$$

Here $\zeta^* = \bar{\zeta}^t$ denotes the hermitian conjugate.

In order to describe the tangent space $T_{[g]}(G/H)$ at any point $[g] \in G/H$, we can use the tangential map of left translations. In fact, left translation by g is defined by the map

$$L_g: G \rightarrow G, \quad g' \mapsto gg' \quad (52)$$

so that its tangential map

$$\theta_g = (L_{g^{-1}})_*: T_g G \rightarrow T_e G = \mathfrak{g} \quad (53)$$

This map defines a \mathfrak{g} -valued 1-form on G known as the left-invariant Maurer-Cartan form. For a matrix group, it can be written as

$$\theta_g = g^{-1} dg \quad (54)$$

Note that the natural projection $\pi: G \rightarrow G/H$ is equivariant with respect to left translation: if

$$\tau_g: G/H \rightarrow G/H, \quad [g'] \mapsto [gg'] \quad (55)$$

then we have

$$\pi \circ L_g = \tau_g \circ \pi, \quad \tau_g \circ \tau_{g'} = \tau_{gg'} \quad (56)$$

Moreover,

$$(\pi \circ L_{g^{-1}})_* = \pi_* \circ (L_{g^{-1}})_* = (\tau_{g^{-1}} \circ \pi)_* = (\tau_{g^{-1}})_* \circ \pi_* \quad (57)$$

Hence, denoting $(\tau_{g^{-1}})_* = \vartheta_g$, we have

$$\pi_* \circ \theta_g = \vartheta_g \circ \pi_*: T_g G \rightarrow T_o(G/H) = \mathfrak{m} \quad (58)$$

For us, we will be mainly interested in the map

$$(\tau_g)_*: \mathfrak{m} = T_{[e]}(G/H) \rightarrow T_{[g]}(G/H) \quad (59)$$

which parametrises the tangent space of G/H at a point $[g]$ by \mathfrak{m} .

Now, if G is a matrix group the pushforward $(\tau_g)_*$ is essentially just given by left multiplication with g (matrix multiplication is linear) and so in we could describe the tangent space $T_p(G/H)$ at a point $p = g(p)p_0$ as follows

$$T_p(G/H) \cong \{(p, g(p)\xi) \mid \xi \in \mathfrak{m}\} \quad (60)$$

where we implicitly identify $g(p)X$ with $g(p)Xp_0$.

Remark 6. In theory, this is very nice. In practice, I believe it is computationally expensive: for a given $p \in S^{2n+1} = \text{SU}(n+1)/\text{SU}(n)$, we would need to find $g(p)$ which on top of it is only defined up to a multiplication on the right of $\text{SU}(n)$. We could, for example, choose $p_0 = (1, 0 \dots)^t$. There is a way how to algorithmically compute $g(p)$: A matrix belongs to $\text{SU}(n+1)$ iff its columns (rows) form an orthonormal basis of \mathbb{C}^{n+1} . If $p_0 = e_1$ is the first standard basis vector, then we start with the basis $\{p, e_2, \dots, e_{n+1}\}$ and by Gram-Schmidt produce an orthonormal basis $\{p, \tilde{e}_1, \dots, \tilde{e}_n\}$ and set

$$g(p) = (p \ \tilde{e}_2 \ \dots; \tilde{e}_n) \quad (61)$$

However, I believe that Gram-Schmidt is computationally expensive.

Moreover, it might simply not be necessary. In order to define interesting deformations, we could simply consider any $X \in \mathfrak{g}$. Any such ξ will admit a split into $\xi = \xi_{\mathfrak{m}} + \xi_{\mathfrak{h}}$ and since $\mathfrak{h} = \text{Lie}(H) \cong \text{Stab}(p)$ is in the Lie algebra of the stablizer of p (*note that we actually mean p here, not p_0*), $\xi_{\mathfrak{h}} p = 0$ (as $h p = p$ for any h in H). Thus, we could simply not care and just work with X as a whole and define deformations for any $\xi \in \mathfrak{g}$ by

$$Y_{\xi}(X) = \rho(\xi) \cdot X \quad , \quad \xi \in \mathfrak{g} \quad , \quad X \in S^{2n+1} \quad (62)$$

Here, ρ denotes the appropriate representation of \mathfrak{g} on S^{2n+1} . I believe its easiest description is as follows: $\text{SU}(n+1)$ and hence $\mathfrak{su}(n+1)$ naturally acts (by matrix multiplication) on S^{2n+1} when viewed as a subset of \mathbb{C}^{n+1} . Under the isomorphism $\mathbb{C} \cong \mathbb{R}^2$, where $z = x + iy \mapsto (x, y)$, multiplication by a complex number $a + ib$ becomes matrix multiplication:

$$(a + ib)z \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (63)$$

Hence, using our representation of \mathbb{C}^{n+1}

$$\begin{pmatrix} \vdots \\ z_k = x_k + iy_k \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ x_k \\ y_k \\ \vdots \end{pmatrix} \quad (64)$$

we have to simply replace any complex number $a + ib$ in $\xi \in \mathfrak{su}(n+1)$ by the appropriate matrix:

$$\rho(\xi): a + ib \rightsquigarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad (65)$$

In practice, it might be easier to have the following workflow:

$$\begin{array}{ccccccc} S^{2n+1} \subset \mathbb{R}^{2n+2} & \longrightarrow & \mathbb{C}^{n+1} & \xrightarrow{\xi} & \mathbb{C}^{n+1} & \longrightarrow & \mathbb{R}^{2n+2} \\ X & \longrightarrow & Z & \xrightarrow{\xi} & \xi \cdot Z & \longrightarrow & Y_{\xi}(X) \end{array} \quad (66)$$

1.5 Example: $n = 0$

1.5.1 Homogeneous deformation

A particular nice example, which can be computed explicitly and serves as a check that everything works nicely is given for the case $n = 0$. Consider the deformation

$$y(x) = \alpha \Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SO}(2) \quad , \quad \alpha \in \mathbb{R} \quad (67) \quad \{\text{eq:toy_cst_def}\}$$

so that indeed

$$\langle x, y(x) \rangle = 0 \quad (68)$$

Moreover, note that

$$\|y(x)\|^2 = \langle y(x), y(x) \rangle = \alpha^2 \langle x, \Omega^t \Omega x \rangle = \alpha^2 \|x\|^2 = \alpha^2 \quad (69)$$

and hence

$$\lambda^2(x) = 1 + \alpha^2 \quad (70)$$

We compute the Jacobian of this parametrisation as follows: let $M_i^j = \partial_i y^j(x)$. Then

$$J_i^j = \lambda(x) \delta_i^j + \frac{\langle y, \partial_i y \rangle x^j}{\lambda(x)} + i M_i^j = \lambda(x) \delta_i^j + \frac{M_{ik} y^k x^j}{\lambda(x)} + i M_i^j \quad (71)$$

Using bra-ket notation and trivially lowering indices (considering the flat metric on \mathbb{R}^2), we can write J as follows:

$$J = \lambda(x) \cdot id + \lambda^{-1}(x) |My\rangle \langle x| + iM \quad (72)$$

Remark 7. The matrix M has some interesting properties.

1. From $\langle x, y(x) \rangle = 0$ it follows that

$$0 = \partial_i \langle x, y(x) \rangle = y_i(x) + M_{ik} x_k$$

so that

$$y_i(x) = M_{ik} x_k \implies y(x) = Mx$$

and hence

$$\langle x, y \rangle = \langle x, Mx \rangle = \langle M^t x, x \rangle = 0$$

from which follows (if $y(x) = Mx \neq 0$)

$$\langle x | M = 0$$

Now, for our choice of

$$y(x) = \alpha \Omega x$$

we have

$$M = \alpha \Omega^t = -\alpha \Omega \quad (73)$$

and hence

$$\begin{aligned} J &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 \Omega^t \Omega |x\rangle \langle x| - i\alpha \Omega \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 |x\rangle \langle x| - i\alpha \Omega \end{aligned} \quad (74)$$

Now, here is a neat trick due to Tej. Consider the unitary matrix

$$U = (x \ \Omega x) \quad , \quad \Omega^* = \begin{pmatrix} x^t \\ x^t \Omega^t \end{pmatrix} \quad (75)$$

where x is a 2×2 column vector with unit norm. Then

$$U^* U = \begin{pmatrix} \langle x, x \rangle & \langle x, \Omega x \rangle \\ \langle \Omega x, x \rangle & \langle x, x \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (76)$$

Moreover,

$$\langle x | U = x^t U = (\langle x, x \rangle \ \langle x, \Omega x \rangle) = (1 \ 0) \quad (77)$$

as well as

$$U^* \Omega U = U^* (\Omega x \ \Omega^2 x) = U^* (\Omega x \ -x) = \begin{pmatrix} \langle x, \Omega x \rangle & -\langle x, x \rangle \\ \langle x, \Omega^t \Omega x \rangle & -\langle x, \Omega^t x \rangle \end{pmatrix} = -\Omega \quad (78)$$

Then

$$\begin{aligned} U^* J U &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* |x\rangle \langle x| U - i\alpha U^* \Omega U \\ &= \lambda(x) \cdot id + \lambda^{-1}(x) \alpha^2 U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i\alpha \Omega \\ &= \begin{pmatrix} \lambda(x) + \lambda^{-1}(x) \alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \end{aligned} \quad (79)$$

It follows that

$$\begin{aligned} \det J(x) &= \det(U^* J(x) U) \\ &= \det \begin{pmatrix} \lambda(x) + \lambda^{-1}(x) \alpha^2 & i\alpha \\ -i\alpha & \lambda(x) \end{pmatrix} \\ &= \lambda^2(x) + \alpha^2 - \alpha^2 = \lambda^2(x) \end{aligned} \quad (80)$$

Finally, we obtain that

$$I \sim \int_{S^{2n+1}} \frac{\det J(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \frac{\lambda^2(x)}{\lambda^2(x)} = \int_{S^{2n+1}} \quad (81)$$

This means that the *constant deformation* (67) comes with a trivial *total* Jacobian (Jacobian + factor from the φ -function constraint).

1.5.2 What it really means

Recall that for $n = 0$ we consider the path integral over S^1 . Before, we have considered $x \in S^1 \subset \mathbb{R}^2$. Now, we might think of its complex formulation: let us consider $z = e^{i\theta} \in S^1 \subset \mathbb{C}$. In [?], the complexification was chosen in terms of $\theta \rightarrow \theta + i\tau$. This means that

$$z \mapsto \tilde{z} = e^{i(\theta+i\tau)} = e^{-\tau} z \quad (82)$$

On the other hand, we can parametrize x by

$$x = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (83)$$

so that

$$\begin{aligned} \tilde{z} &= \begin{pmatrix} \sin(\theta + i\tau) \\ \cos(\theta + i\tau) \end{pmatrix} = \begin{pmatrix} \sin(\theta) \cosh(\tau) + i \cos(\theta) \sinh(\tau) \\ \cos(\theta) \cosh(\tau) - i \sin(\theta) \sinh(\tau) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\tau) & i \sinh(\tau) \\ -i \sinh(\tau) & \cosh(\tau) \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} = \lambda x + i \underbrace{\Omega x}_{y(x)} \end{aligned} \quad (84)$$

where

$$\Omega = \begin{pmatrix} 0 & -\sinh(\tau) \\ \sinh(\tau) & 0 \end{pmatrix} \quad (85)$$

and

$$\lambda^2 = 1 + \|y(x)\|^2 = 1 + \sinh^2(\tau) = \cosh^2(\tau) \quad (86)$$

To summarise, if we consider the homogeneous deformation (67),

$$y(x) = \alpha \Omega x \quad , \quad \|x\|^2 = 1 \quad , \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (87)$$

then choosing

$$\boxed{\alpha = \sinh(\tau)} \quad (88)$$

we find that the homogeneous deformation simply scales the complex variable $z = e^{i\theta} \in S^1 \subset \mathbb{C}$,

$$z \rightarrow \tilde{z} = e^{-\tau} z \quad (89)$$

1.5.3 Generalisation: Torus deformations

{subsubsec:Torus_defc

Let us now think about $n > 0$. Let $z = [z_0 : \dots : z_n] \in \mathbb{CP}^n$. Again, we factor out the $U(1) \subset \mathbb{C}^*$ and fix the $\mathbb{R}_+ \subset \mathbb{C}^*$ by restricting the path integral to S^{2n+1} , i.e.

$$\sum_k |z_k|^2 = 1 \quad (90)$$

Following the logic of the homogeneous deformation, let us consider a deformation parametrised by a generic element in the Cartan subalgebra of $\mathfrak{su}(n+1)$

$$Y = \Omega(\alpha)X = \sum_{k=1}^n \rho(i\alpha_k H_k) \quad (91)$$

where we choose the following *hermitian* (hence the factor of i) basis of the Cartan $\mathfrak{h} \subset \mathfrak{su}(n+1)$ subalgebra

$$H_k = E_{kk} - E_{(k+1)(k+1)} \quad (92)$$

where E_{ij} is the $n \times n$ matrix with $(E_{ij})_{ab} = \delta_{ai}\delta_{bj}$. Here, ρ is again the representation of complex multiplication in terms of 2×2 matrices,

$$\rho(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad (93)$$

and defined element-wise on a generic complex matrix as

$$\rho(M) = \begin{pmatrix} \rho(m_{11}) & \rho(m_{12}) & \dots \\ \rho(m_{21}) & \rho(m_{22}) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad (94)$$

In particular,

$$\rho(iH_k) = \rho(iE_{kk}) - \rho(iE_{(k+1)(k+1)}) = \begin{pmatrix} \ddots & & & & \\ & 0 & -1 & & \\ & 1 & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \\ & & & & \ddots \end{pmatrix} \quad (95)$$

Let us choose

$$\alpha_k = \sinh(\tau_k) \quad , \quad \tau_0 \equiv \tau_{n+1} \equiv 0 \quad (96)$$

then, acting on the complex vector $z = (z_0, \dots, z_n)$, we find

$$\boxed{\tilde{z} = (\tilde{z}_0, \dots, \tilde{z}_n) = (e^{-\Delta_{01}} z_0, \dots, e^{-\Delta_{nn+1}} z_n) \quad , \quad \Delta_{kk+1} = \tau_k - \tau_{k+1}} \quad (97)$$

1.6 Correlators

1.6.1 Generalities

We may try to compute correlation functions analytically. For this, let us view \mathbb{CP}^n as the homogeneous space

$$\mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n) \quad (98)$$

with

$$z = g(z)\eta \quad , \quad \eta = [1 : 0 : \dots : 0] \quad , \quad g(z) \in \text{SU}(n+1) \quad (99)$$

The action can therefore be written as

$$S = -\beta|z^*w|^2 = -\beta|\eta^t U^*(z)U(w)\eta|^2 \quad (100)$$

Note that $U \equiv U(z, w) = U^*(z)U(w) \in \text{SU}(n)$ and so if $\mu(U(z))$ denotes the Haar measure of $\text{SU}(n+1)$, we have

$$\mu(U(w))\mu(U^*(z)U(w)) = \mu(U) \quad (101)$$

Denoting $U(z) \equiv U'$, we can perform a change of variables such that the path integral measure becomes

$$\mu(U(w))\mu(U(z)) = \mu(U(z))\mu(U(z, w)) \equiv \mu(U')\mu(U) \quad (102)$$

Moreover, recall that in our model instead of integrating over $\mathbb{CP}^n \times \mathbb{CP}^n$ directly, we integrate out the $U(1) \subset \mathbb{C}^*$ first and use the resulting $\mathbb{R}_+ \subset \mathbb{C}^*$ gauge freedom to fix the norm, hence integrating over $S^{2n+1} \times S^{2n+1}$. Now, $S^{2n+1} = \text{SU}(n+1)/\text{SU}(n)$ and so

$$\int_{\text{SU}(n+1)} \mu = \int_{\text{SU}(n)} \mu \int_{\text{SU}(n+1)/\text{SU}(n)} \hat{\mu} = \int_{S^{2n+1}} \hat{\mu} \quad (103)$$

It is important to remark that we are working with the *normalized* Haar measure μ so that

$$\int_{\text{SU}(n)} \mu = 1 \quad (104)$$

Hence the partition function can be written in terms of a single integration over $\text{SU}(n+1)$

$$\begin{aligned} Z &= \int_{\text{SU}(n+1)} \mu(U') \int_{\text{SU}(n+1)} \mu(U) e^{-S(U)} \\ &= \int_{\text{SU}(n+1)} \mu(U) e^{-S(U)} \end{aligned} \quad (105)$$

The action functional simplifies to

$$S(U) = -\beta|\eta^t U \eta|^2 = -\beta|U_{11}|^2 \quad (106)$$

and any correlation function can be written as

$$\langle \mathcal{O}(z, w) \rangle = \frac{1}{Z} \int_{\text{SU}(n+1)} \mu(U') \int_{\text{SU}(n+1)} \mu(U) e^{-S(U)} \mathcal{O}(U, U') \quad (107)$$

Remark 8. Obviously we could have chosen a different η (having its only non-zero entry sitting at the i th component, say. This choice cannot affect the result and we will see later that we use this freedom to simplify certain expressions.

1.6.2 One-point function

It is time to look at interesting operators we would like to compute. We start with the one-point function. Let

$$\mathcal{O}_{ij}(z) = z_i \bar{z}_j \quad (108)$$

Clearly, \mathcal{O}_{ij} is invariant under the $U(1)$ action $z_k \rightarrow e^{i\varphi} z_k$ for all k .

It's expectation value is given by

$$\langle \mathcal{O}_{ij}(z) \rangle = \frac{1}{(2\pi)^2} \int_{S^{2n+1} \times S^{2n+1}} \prod_k d^n z_k d^n w_k z_i \bar{z}_j e^{-S(z,w)} \quad (109)$$

where

$$S(z, w) = -\beta |\langle \bar{z}, w \rangle|^2 = -\beta \sum_{k,\ell} \bar{z}_\ell z_k w_\ell \bar{w}_k \quad (110)$$

Note that both, the action and the measure is invariant under the diagonal $U(1)$ action

$$(z_k, w_k) \mapsto (e^{i\alpha} z_k, e^{i\alpha} w_k) \quad , \quad \forall k \quad (111)$$

If $i \neq j$, by this change of coordinates, for $k = i$, it follows that

$$\langle \mathcal{O}_{ij}(z) \rangle = e^{-i\alpha} \langle \mathcal{O}_{ij}(z) \rangle \quad (112)$$

for any α and hence we must have

$$\boxed{\langle \mathcal{O}_{ij}(z) \rangle = 0 \quad , \quad \forall z, i \neq j} \quad (113) \quad \{\text{eq:toy_one_pt}\}$$

We can also see this in terms of the $SU(n+1)$ parametrization. In this parametrization,

$$\mathcal{O}_{ij}(z) = z_i \bar{z}_j = \sum_{k,\ell} U_{ik} \eta_k \bar{U}_{j\ell} \eta_\ell = \sum_{k\ell} \delta_{k1} \delta_{\ell 1} U_{ik} \bar{U}_{j\ell} = U_{i1} U_{1j}^* \quad (114)$$

It turns out to be very useful to make a change of coordinates $U' \equiv U^*(z)U(w)$ for the integration of $U(w)$. Then

$$\begin{aligned} \langle \mathcal{O}_{ij}(U) \rangle &= \frac{1}{Z} \int_{SU(n+1)} \mu(U) U_{i1} U_{1j}^* \int_{SU(N)} \mu(U') e^{-S(U')} \\ &= \int_{SU(n+1)} \mu(U) U_{i1} \bar{U}_{j1} \end{aligned} \quad (115) \quad \{\text{eq:toy_one_pt_from_S}\}$$

Integrals of $SU(N)$ polynomials have been studied in the literature, see

for example [1, 4]

$$\begin{aligned}
I_N(r, s) &= \int_{\text{SU}(N)} \mu(N) \prod_{k=1}^r U_{i_k j_k} \prod_{\ell=1}^s \bar{U}_{i'_\ell j'_\ell} \\
&= \sum_{q=0}^{\infty} \delta_{s-r, Nq} \sum_{\sigma \in S_r} \text{Wg}^{N, q}(\sigma) \sum_{\rho \in S_{r+Nq}} \prod_{k=1}^r \delta_{i_k i'_{\rho(k)}} \delta_{j_{\sigma(k)} j'_{\rho(k)}} \\
&\quad \cdot \frac{1}{(Nq)!} \sum_{\tau \in S_{Nq}} \chi_{q^N}(\tau) \prod_{k=1}^{Nq} \delta_{i'_{\rho(\tau(k)+r)} j'_{\rho(k+r)}}
\end{aligned} \tag{116}$$

This beast simplifies for $r = s$ (which, luckily, is the case we are interested in) to

$$I_N(r) \equiv I_N(r, r) = \sum_{\sigma, \tau \in S_r} \text{Wg}^N(\tau^{-1}\sigma) \prod_{k=1}^r \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \tag{117} \quad \{\text{eq:SUN_poly_int}\}$$

Here, Wg^N denotes the Weingarten map which for $\sigma \in S_r$ is given by

$$\text{Wg}^N(\sigma) = \frac{1}{(r!)^2} \sum_{\substack{\lambda \in \text{irreps}(S_r) \\ \ell(\lambda) \leq N}} \frac{d_\lambda^2 \chi_\lambda(\sigma)}{s_\lambda(1^N)} \tag{118}$$

where d_λ is the dimension and $\chi_\lambda(\sigma)$ the character of the irreducible representation λ of S_r . Moreover, $s_\lambda(X)$, $X = (X_1, \dots, X_N)$ denote the Shur functions. Irreps λ of S_r are enumerated by partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ of r such that

$$|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i = r \quad , \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} \tag{119}$$

The integer $\ell(\lambda)$ is known as the *length* of the partition.

Back to our one-point function. We can rewrite (115) as follows:

$$\begin{aligned}
\langle \mathcal{O}_{ij}(U) \rangle &= \int_{\text{SU}(n+1)} \mu(U) U_{i1} \bar{U}_{j1} \\
&= \sum_{\tau, \sigma \in S_1} \text{Wg}^{n+1}(\tau^{-1}\sigma) \prod_{k=1}^1 \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \\
&= \text{Wg}^{n+1}(id) \delta_{ij}
\end{aligned} \tag{120}$$

Let us now have a look at its variance

$$\text{Var} = \langle |\mathcal{O}_{ij}(z)|^2 \rangle = \langle |z_i|^2 |z_j|^2 \rangle \tag{121}$$

which by the same reasoning can be written as

$$\begin{aligned} Var &= \int_{\text{SU}(n+1)} \mu(U) U_{i1} U_{j1} \bar{U}_{i1} \bar{U}_{j1} \\ &= \sum_{\tau, \sigma \in S_2} \text{Wg}^{n+1}(\tau^{-1}\sigma) \prod_{k=1}^2 \delta_{i_k i'_{\sigma(k)}} \delta_{j_k j'_{\tau(k)}} \end{aligned} \quad (122)$$

Since $S_2 = \{1^2 \equiv id, 2 \equiv (12)\}$, we find

$$\begin{aligned} Var &= \left(\text{Wg}^{n+1}(1^2) (\delta_{ii} \delta_{11} \delta_{jj} \delta_{11} + \delta_{ij}^2 \delta_{11}^2) + \text{Wg}^{n+1}(2) (\delta_{ii} \delta_{jj} \delta_{11}^2 + \delta_{ij}^2 \delta_{11}^2) \right) \\ &= (1 + \delta_{ij}) \left(\text{Wg}^{n+1}(1^2) + \text{Wg}^{n+1}(2) \right) \end{aligned} \quad (123)$$

According to [Wikipedia](#),

$$\text{Wg}^d(1^2) = \frac{1}{d^2 - 1} \quad , \quad \text{Wg}^d(2) = \frac{-1}{d(d^2 - 1)} \quad (124)$$

so that

$$\boxed{Var(\mathcal{O}_{ij}(z)) = \frac{1 + \delta_{ij}}{(n+1)(n+2)}} \quad (125)$$

Remark 9. If we consider a torus / Cartan deformation as outlined in Section 1.5.3, then we find

$$\begin{aligned} \text{Var}(\tilde{\mathcal{O}}_{ij}(\tilde{z})) &= \langle e^{-2\Delta_{ii+1}} |z_i|^2 e^{-2\Delta_{jj+1}} |z_j|^2 \rangle_{deformed} \\ &= e^{-2\Delta_{ii+1} - 2\Delta_{jj+1}} \left\langle |z_i|^2 |z_j|^2 \det(J_{tot}) e^{-(\tilde{S}-S)} \right\rangle \end{aligned} \quad (126)$$

1.6.3 Two-point function

References

- [1] O. Borisenko, S. Voloshyn and V. Chelnokov. SU(N) Polynomial Integrals and Some Applications. Rept. Math. Phys. 85, 129–145. arXiv: [1812.06069 \[hep-lat\]](#) (2020).
- [2] F. Bruhat and H. Whitney, Quelques propriétés fondamentales des ensembles analytiques-réels, Comment. Math. Helv. 33, 132-160 (1959).
- [3] K. Cieliebak and Y. Eliashberg. From Stein to Weinstein and back: symplectic geometry of affine complex manifolds. Vol. 59. American Mathematical Soc., 2012.
- [4] B. Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. International Mathematics Research Notices. 2003;2003(17):953-82. arXiv: [math-ph/0205010](#)