Solutions to hand-in problems in the course Geodynamics at Uppsala university VT 2008

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March 9, 2009

Contents

2-4. A sedimentary basin has a thickness of 4 km. Assuming that the crustal stretching model is applicable and that h_{cc} = 35 km, ρ_m = 3300 kg m^{-3}, ρ_{cc} = 2750 kg m^{-3}, and ρ_s = 2550 kg m^{-3}, determine the stretching factor.

The crustal stretching model considers a section of width w_o of the continental crust of original thickness h_{cc} under tension. The tension stretches the segment to a new width w_b with thickness h_{cb}, thus creating a basin of depth h_{sb}, which is assumed to subsequently be filled by a sediment of density ρ_s (see figure 1).

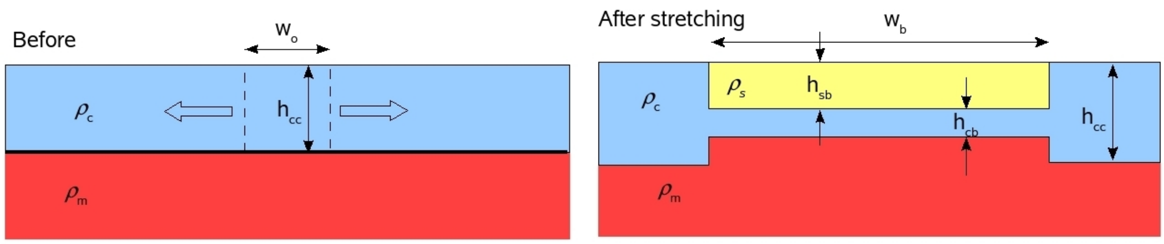

Figure 1: The crustal stretching model.

The stretching factor α for the crustal stretching model is defined as

$$\alpha = \frac{w_b}{w_o} \tag{2-6}$$

If we assume that the segment is stretched at constant volume it must hold that

$$w_b h_{cb} = w_o h_{cc} \tag{2-7}$$

$$\Rightarrow \quad \alpha = \frac{h_{cc}}{h_{cb}} \tag{2-8}$$

Assuming that the region after the stretching and the fill-in of sediments is at isostatic equilibrium, we find that the equilibrium equation becomes

$$\rho_c h_{cc} = \rho_s h_{sb} + \rho_c h_{cb} + \rho_m \left(h_{cc} - h_{sb} - h_{cb}\right) \tag{2-9}$$

Using equation (2-8) we then find that

$$\rho_c h_{cc} = \rho_s h_{sb} + \rho_c \frac{h_{cc}}{\alpha} + \rho_m \left(h_{cc} - h_{sb} - \frac{h_{cc}}{\alpha}\right)$$

$$\Rightarrow \quad \frac{h_{cc}}{\alpha}\left(\rho_c - \rho_m\right) = h_{cc}\left(\rho_c - \rho_m\right) + h_{sb}\left(\rho_m - \rho_s\right)$$

$$\Rightarrow \quad \alpha = \frac{h_{cc}\left(\rho_c - \rho_m\right)}{h_{cc}\left(\rho_c - \rho_m\right) + h_{sb}\left(\rho_m - \rho_s\right)}$$

$$\Rightarrow \quad \alpha = \frac{35000 \cdot (2750 - 3300)}{35000 \cdot (2750 - 3300) + 4000 \cdot (3300 - 2550)} = 1.18$$

2-7. Consider a continental block to have a thickness of 70 km corresponding to a major mountain range. If the continent has a density of 2800 kg m^{-3}, determine the tensional stress in the continental block.

Consider a continental block ”floating” in the mantle as displayed in figure 2.

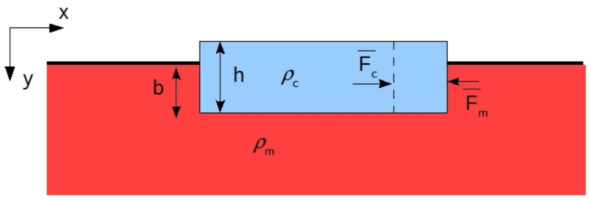

Figure 2

The horizontal force, per unit width, acting on the edge of the block F_m is then given by the lithostatic pressure p_L in the mantle, integrated over the edge of the block (note that since we are considering the force per unit width we only need to integrate over depth)

$$F_m = \int_0^b p_L dy = \rho_m g \int_0^b y dy = \frac{1}{2}\rho_m g b^2 \qquad \textbf{(2-14)}$$

The horizontal force F_c, per unit width, on a vertical cross-section inside the block can then be found by integrating the horizontal normal stress σ_{xx} over the cross-section.

Let us assume that σ_{xx} is made up of two parts:

1. The lithostatic contribution: $\rho_c g y$
2. A constant tectonic contribution: $\Delta\sigma_{xx}$
   (i.e. the deviatoric stress we wish to find)

I.e. we have

$$F_c = \int_0^h \sigma_{xx} dy = \int_0^h (\rho_c g y + \Delta\sigma_{xx})\, dy = \frac{1}{2}\rho_c g h^2 + \Delta\sigma_{xx} h \qquad \textbf{(2-16)}$$

From isostatic considerations it then holds that

$$\rho_c h = \rho_m b \qquad \textbf{(*)}$$

Equating equations (2-14), (2-16), and using (*)we than find that

$$\frac{1}{2}\rho_c g h^2 + \Delta\sigma_{xx} h = \frac{1}{2}\rho_m g b^2 = \frac{1}{2}\frac{\rho_c^2}{\rho_m} g h^2$$

$$\Rightarrow \quad \Delta\sigma_{xx} = \frac{1}{2}\rho_c g h \left(\frac{\rho_c}{\rho_m} - 1\right)$$

And putting in the numerics we find

$$\Delta\sigma_{xx} = \frac{1}{2} \cdot 2800 \cdot 10 \cdot 70000 \cdot \left(\frac{2800}{3300} - 1\right) \approx -1.48 \times 10^8 \text{ Pa} = -148 \text{ MPa}$$

2-9. Assume that the friction law given in Equation (2-23) is applicable to the strike-slip fault illustrated in Figure 2-10 with f = 0.3. Also assume that the normal stress σ_{xx} is lithostatic with ρ_c = 2750 kg m^{-3}. If the fault is 10 km deep, what is the force (per unit length of the fault) resisting motion on the fault? What is the mean tectonic shear stress over this depth $\bar{\sigma}_{zx}$ required to overcome this frictional resistance?

Let the length of the fault be L_z, and consider a strip along the fault plane of height δy, i.e. of area $A = L_z\delta y$. Let the shear stress on the fault plane be σ_{xz}, the force resisting motion on the strip ΔF is then simply

$$\Delta F = A\sigma_{xz} = L_z\delta y\sigma_{xz}$$

Or expressed per unit length of the fault

$$\frac{\Delta F}{L_z} = \delta y\sigma_{xz}$$

which is equal to the situation displayed in figure 2-10 (assuming $\delta A = \delta y \cdot 1$)

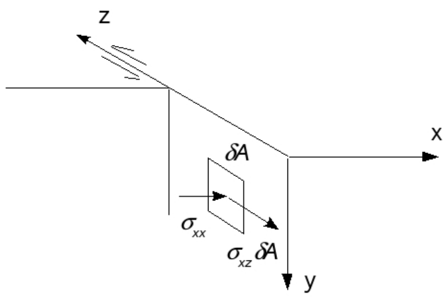

Figure 3: Figure 2-10

Assuming that the shear stress on the fault plane is given by

$$\sigma_{xz} = f\sigma_{xx} \qquad \textbf{(2-23)}$$

and that the normal stress on the fault plane is lithostatic (i.e. due to the overburden only)

$$\sigma_{xx} = \rho g y \qquad \textbf{(2-24)}$$

we have that the force per unit length resisting motion on the strip is

$$\frac{\Delta F}{L_z} = \delta y f\rho g y$$

If we then let the width of the strip become infinitesimal we find that

$$\frac{dF}{L_z} = f\rho g y dy$$

Which can be integrated over the depth of the fault to yield

$$\frac{F}{L_z} = \int_0^y f\rho g y' dy' = \frac{1}{2}f\rho g y^2$$

Putting in the numerics we find

$$\frac{F}{L_z} = \frac{1}{2} \cdot 0.3 \cdot 2750 \cdot 10 \cdot 10000^2 = 4.152 \times 10^{11} \text{ N m}^{-1} = 412.5 \text{ GN m}^{-1}$$

The mean tectonic shear stress $\bar{\sigma}_{xz}$ over the depth of the fault, required to overcome this frictional resistance then needs to be of at least the same magnitude as this force divided by the depth of the fault (to get from force to stress), hence

$$\bar{\sigma}_{xz} = \frac{F/L_z}{y} = \frac{4.152 \times 10^{11}}{10000} = 4.152 \times 10^7 \text{ Pa} = 41.52 \text{ MPa}$$

2-11. Consider a rock mass resting on an inclined bedding plane as shown in Figure 2-12. By balancing the forces acting on the block parallel to the inclined plane, show that the tangential force per unit area $\sigma_{x'y'}$ on the plane supporting the block is $\rho g h \sin\theta$ (ρ is the density and h is the thickness of the block). Show that the sliding condition is

$$\theta = \tan^{-1} f \qquad \textbf{(2-26)}$$

The only force acting on the rock mass is the gravitational force, i.e.

$$F_y = -mg$$

where the minus sign comes from the gravitational force acting in the opposite direction to the y-direction. Consider then a vertical column through the rock mass of width $\delta x'$ against the inclined plane. The width of this column in the x-direction is then (see figure 4)

$$\delta x' \cos\theta$$

and the height is

$$\frac{h}{\cos\theta}$$

hence the volume of the column is

$$V = \delta z' \cdot \delta x' \cos\theta \cdot \frac{h}{\cos\theta} = \delta z' \delta x' h$$

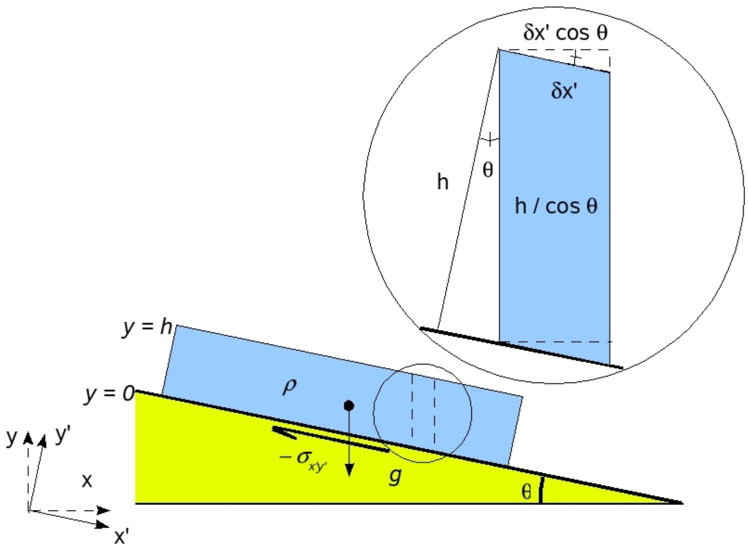

Figure 4: Modified from figure 2-12 of Turcotte and Schubert

Let the density of the rock mass be ρ, then the mass of the column is simply given by

$$m = \rho V = \rho \delta x' \delta z' h$$

And hence the gravitational force on the column

$$F_y = -\rho \delta x' \delta z' h g$$

or alternatively expressed per unit area of the inclined plane

$$\frac{F_y}{\delta x' \delta z'} = -\rho h g$$

If we then choose a coordinate system, $x'y'$ rotated clockwise by an angle θ, i.e. such that the x'-axis is parallel to inclined plane, we find that the force F_y in this coordinate system has the components

$$\begin{aligned}\frac{F_{x'}}{\delta x' \delta z'} &= -\frac{F_y}{\delta x' \delta z'} \sin\theta = \rho h g \sin\theta = \sigma_{x'y'} \\ \frac{F_{y'}}{\delta x' \delta z'} &= \frac{F_y}{\delta x' \delta z'} \cos\theta = \rho h g \cos\theta = \sigma_{y'y'}\end{aligned}$$

where we have recognized that the force components per unit area is nothing else than the normal and shear stress on the inclined plane in the $x'y'$-coordinate system.

Assuming a friction law of the form

$$\sigma^{(f)}_{x'y'} = -f\sigma_{y'y'} \qquad \textbf{(2-23)}$$

where the minus sign comes from the fact that the friction force acts in the negative x'-direction, and using the condition of sliding is

$$\sigma_{x'y'} \geq \sigma^{(f)}_{x'y'}$$

we find that at the onset of sliding

$$\begin{aligned} & f = \frac{\sigma_{x'y'}}{-\sigma_{y'y'}} = \frac{-h\rho g \sin\theta}{-h\rho g \cos\theta} = \tan\theta \\ \Rightarrow \quad & \theta = \tan^{-1} f \qquad \textbf{(2-26)}\end{aligned}$$

2-14. The state of stress at a point on a fault plane is σ_{yy} = 150 MPa, σ_{xx} = 200 MPa, σ_{xy} = 0 (y is depth and x points westward). What are the normal stress and tangential stress on the fault plane if the fault strikes N-S and dips 35^o to the west?

The geometry of the problem can be seen in figure 5

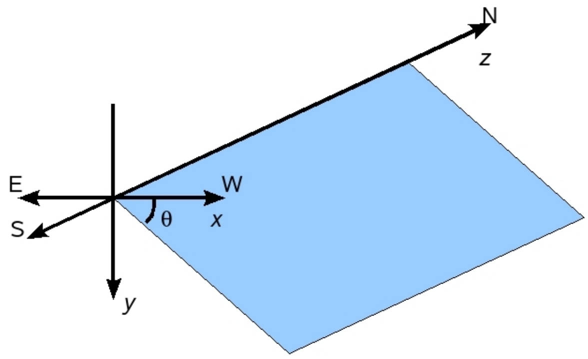

Figure 5

We are given σ_{yy} and σ_{xx} and want to find the normal and tangential stresses to the plane. As the strike of the fault is aligned to the z-axis we can consider the problem in the xy-plane only, i.e. as a 2D-problem instead.

Thus by rotating the coordinate system by an angle θ to the $x'y'$-system (see figure 6), we can find the sought for stresses as the normal stress $\sigma_{y'y'}$ and the tangential stress $\sigma_{x'y'}$

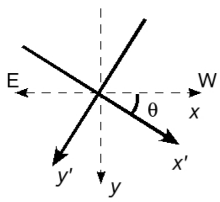

Figure 6

Using tensor notation the rotation is given by

$$\boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is the rotation matrix for a *clockwise* rotation where as the rotation matrix for an *anti-clockwise* rotation has opposite signs on the sines, and

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{xx} \end{bmatrix}$$

is the 2D-stress tensor, hence

$$\begin{aligned}
\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{x'x'} \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta & -\sigma_{xx}\sin\theta + \sigma_{xy}\cos\theta \\ \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta & -\sigma_{yx}\sin\theta + \sigma_{yy}\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta & (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) \\ (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) & \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \end{bmatrix}
\end{aligned}$$

Hence the sought for stresses are

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin 2\theta \qquad \textbf{(2-41)}$$

$$\sigma_{x'y'} = (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) = \frac{1}{2}(\sigma_{yy} - \sigma_{xx})\sin 2\theta + \sigma_{xy}\cos 2\theta \qquad \textbf{(2-40)}$$

Putting in the numerics we find that

$$\begin{aligned}
\sigma_{y'y'} &= 200 \cdot \sin^2(35^o) + 150 \cdot \cos^2(35^o) = 166 \text{ MPa} \\
\sigma_{x'y'} &= \frac{(150 - 200)}{2} \cdot \sin(2 * 35^o) = -23.5 \text{ MPa}
\end{aligned}$$

2-15. Show that the sum of the normal stresses on any two orthogonal planes is a constant. Evaluate the constant.

Actually this is generally not true. For the statement to hold true we need to demand that the intersection of the planes is parallel for all of the possible configurations (a simple example of a case in which the statement does not hold can be found below). I.e. the orientation of the planes are only allowed to change in 2D, whilst being fixed in the third dimension.

We can relate the normal stresses on the planes via a finite rotation around the intersection axis. Choose a coordinate system such that the intersection is parallel to the z-axis, we can then have that

$$\boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is the rotation matrix,and

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{xx} \end{bmatrix}$$

is the 2D-stress tensor, hence

$$\begin{aligned}
\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{x'x'} \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{xx} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta & -\sigma_{xx}\sin\theta + \sigma_{xy}\cos\theta \\ \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta & -\sigma_{yx}\sin\theta + \sigma_{yy}\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta & (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) \\ (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) & \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \end{bmatrix}
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_{x'x'} + \sigma_{y'y'} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \\
&= \sigma_{xx}(\cos^2\theta + \sin^2\theta) + \sigma_{yy}(\sin^2\theta + \cos^2\theta) \\
&= \sigma_{xx} + \sigma_{yy}
\end{aligned}$$

Hence the sum of the normal stresses for a particular choice of orientation of the two orthogonal planes is simply the sum of the normal stresses for any orientation of the planes such that the direction of the intersection between the planes is preserved.

Note that since we are only rotating in the xy-plane we could as well have written

$$\sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} = ... + \sigma_{zz} = ... = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

as $\sigma_{z'z'} = \sigma_{zz}$ for this choice of coordinate system. Hence we find that the sum of the three normal stresses are constant. This however is true in general (i.e. for all rotations) and the sum of the normal components of the stress tensor is generally referred to as the first invariant of the stress tensor.

As a simple example of a case in which the statement does not hold true consider the stress state: $\sigma_{xx}, \sigma_{yy} \neq 0, \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$. Consider two orthogonal planes such that their intersection is parallel to the z-axis. According to our derivation above the sum of the normal stresses on the planes are then simply $\sigma_{xx} + \sigma_{yy}$. Next consider two orthogonal planes such that the their intersection is parallel to the x-axis, and the normals to the planes points in the y- and z-direction, the sum of the the normal stresses on these planes are then σ_{yy}.

2-20. The measured horizontal principal stresses at a depth of 200 m are given in Table 2-1 as a function of distance from the San Andreas fault. What are the values of maximum shear stress at each distance?

Distance from fault [km]	Maximum principal stress [MPa]	Minimum principal stress [MPa]
2	9	8
4	14	8
22	18	8
34	22	11

Table 1: Table 2-1 of Turcotte and Schubert: Stress measurements at 200 m depth vs. distance from the San Andreas fault.

The maximum shear stress in 2D as a function of the principal stresses is given by

$$\sigma_{xy}\big|_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) \qquad \textbf{(2-60)}$$

Hence

Distance from fault [km]	$\sigma_{xy}\big\|_{max}$
2	$\frac{1}{2}$(9-8) = 0.5 MPa
4	$\frac{1}{2}$(14-8) = 3.0 MPa
22	$\frac{1}{2}$(18-8) = 5.0 MPa
34	$\frac{1}{2}$(22-11) = 5.5 MPa

2-21. Uplift and subsidence of large areas are also accompanied by horizontal or lateral strain because of the curvature of the Earth's surface. Show that the lateral strain ϵ accompanied by an uplift Δy is given by

$$\epsilon = \frac{\delta y}{R} \qquad \textbf{(2-74)}$$

where R is the radius of the Earth

Given that the radius of the Earth is much larger than any possible uplift or subsidence of any surface area we can consider the lateral strain as a normal strain. I.e. as the ratio of lateral change divided by the original size.

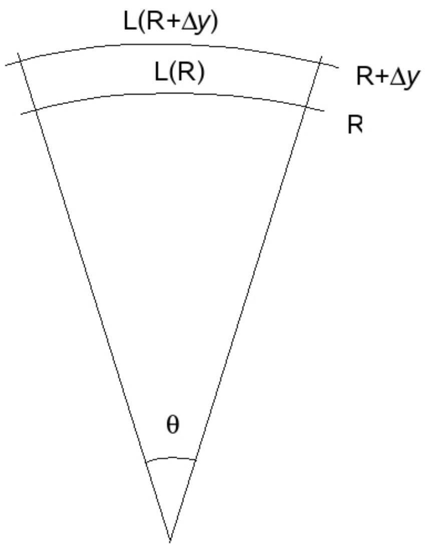

Figure 7

Let the horizontal distance for an area at a radius R be $L(R)$ and that of the same area uplifted a distance Δy be $L(R + \Delta y)$ (see Figure 7). We have that

$$L(R) = R\theta$$
$$L(R + \Delta y) = (R + \Delta y)\theta$$

Hence

$$\epsilon = \frac{L(R + \Delta y) - L(R)}{L(R)} = \frac{(R + \Delta y)\theta - R\theta}{R\theta} = \frac{\Delta y}{R} \qquad \textbf{(2-74)}$$

whereas for subsidence we have that

$$\epsilon = \frac{L(R - \Delta y) - L(R)}{L(R - \Delta y)} = \frac{(R - \Delta y)\theta - R\theta}{(R - \Delta y)\theta} = -\frac{\Delta y}{R - \Delta y} = -\frac{\Delta y}{R}$$

where we have used that $\Delta y \ll R$

An alternative derivation is to express $L(R+\Delta y)$ as $L(R)(1+\epsilon) = R\theta(1+\epsilon) = (R+\Delta y)\theta \Rightarrow 1+\epsilon = 1+\Delta y/R$ $\Rightarrow \epsilon = \Delta y/R$

Given

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}$$

what is the inner matrix product $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$? Is the inner product $\mathbf{D} = \mathbf{B} \cdot \mathbf{A}$ defined?

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 12 & 1 \\ 3 & 5 \end{bmatrix}$$

Since the size of $\mathbf{A}^T$ is equal to the size of $\mathbf{B}$ both $\mathbf{C}$ and $\mathbf{D}$ are defined.

$$\mathbf{D} = \mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -1 \\ 3 & 9 & 3 \\ 0 & 5 & 3 \end{bmatrix}$$

3-1. Determine the surface stress after the erosion of 10 km of granite. Assume that the initial state of stress is lithostatic and that ρ = 2700 kg m^{-3} and ν = 0.25

Assuming that the stress initially is lithostatic we have that the stresses at a depth h are

$$\sigma_1 = \sigma_2 = \sigma_3 = \rho g h \qquad \textbf{(3-28)}$$

After the erosion of h km of overburden, the vertical stress at the centre is $\sigma_1' = 0$ (where the prime indicates stress after erosion). If we assume no horizontal strain due to the erosion, i.e. only $\varepsilon_1 \neq 0$, the situation is that of uniaxial strain. Hence, from the equation

$$\sigma_2 = \sigma_3 = \left(\frac{\nu}{1-\nu}\right)\sigma_1 \qquad \textbf{(3-21)}$$

which can be rewritten as

$$\Delta\sigma_2 = \Delta\sigma_3 = \left(\frac{\nu}{1-\nu}\right)\Delta\sigma_1 \qquad \textbf{(3-29)}$$

where

$$\Delta\sigma_n = \sigma_n' - \sigma_n$$

Hence, for the vertical stress

$$\Delta\sigma_1 = -\rho g h$$

And so for the horizontal stresses we find

$$\sigma_2' = \sigma_3' = \sigma_2 + \Delta\sigma_2 = \rho g h - \left(\frac{\nu}{1-\nu}\right)\rho g h = \left(\frac{1-2\nu}{1-\nu}\right)\rho g h \qquad \textbf{(3-30)}$$

Putting in the numerics we find that

$$\sigma_2' = \sigma_3' = \left(\frac{1-2\cdot 0.25}{1-0.25}\right)\cdot 2700\cdot 10\cdot 10000 = 1.8\times 10^8 Pa = 180 \text{ MPa}$$

3-2. An unstressed surface is covered with sediments with a density of 2500 kg m^{-3} to a depth of 5 km. If the surface is laterally constrained and has a Poisson's ratio of 0.25, what are the three components of stress at the original surface?

Since the surface is assumed to be laterally constrained, this is a case of uniaxial strain, i.e. only $\varepsilon_1 \neq 0$. Hence we have that the vertical stress after the deposition of sediments, simply is lithostatic, i.e.

$$\sigma_1 = \rho g h \qquad \textbf{(3-28)}$$

and for the horizontal stresses we find

$$\sigma_2 = \sigma_3 = \frac{\nu}{1-\nu}\sigma_1 \qquad \textbf{(3-21)}$$

Putting in the numerics we find

$$\sigma_1 = 2500 \cdot 10 \cdot 5000 = 1.25 \times 10^8 \text{ Pa} = 125 \text{ MPa}$$
$$\sigma_2 = \sigma_3 = \frac{0.25}{1-0.25} \cdot 125 = 41.7 \text{ Mpa}$$

3-7. What is the displacement of a plate pinned at both ends (w = 0 at x = 0, L) with equal and opposite bending moments applied at the ends? The problem is illustrated in Figure 3-16.

An equivalent of Figure (3-16) of Turcotte and Schubert canbe seen in figure 8 below

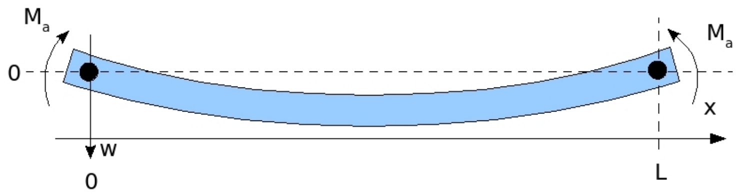

Figure 8: Modified from Figure (3-16) of Turcotte and Schubert

The general equation for the deflection w of an elastic plate is

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2} \quad \textbf{(3-74)}$$

where $q(x)$ and P are the downward (vertical) and horizontal forces acting on the plate, and D is the flexural rigidity of the plate.

In the given problem no horizontal forces are acting on the plate, i.e. $P = 0$, and for simplicity we shall assume the plate to be massless, i.e. $q(x) = 0$. The only action on the plate are then the bending moments of magnitude M applied to the ends. And as far as we are given information, these are constant. We then have that

$$M = -D\frac{d^2w}{dx^2} \quad \textbf{(3-73)}$$

Integration yields

$$\begin{aligned}\frac{d^2w}{dx^2} &= -\frac{M}{D}\\ \frac{dw}{dx} &= -\frac{M}{D}x + c_1\\ w &= -\frac{M}{2D}x^2 + c_1x + c_2\end{aligned}$$

Since the plate is pinned at its ends we know that the deflection here is zero, i.e. the boundary conditions of the problem is

$$\text{BC1: } w(x=0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\text{BC1: } w(x=L) = 0 \quad \Rightarrow \quad -\frac{M}{2D}L^2 + c_1L = 0 \quad \Rightarrow \quad c_1 = \frac{ML}{2D}$$

Hence the deflection of the plate is given by

$$w = \frac{M}{2D}x^2 + \frac{M}{2D}Lx = \frac{M}{2D}(L-x)\,x$$

3-16. Show that the cross-sectional area of a two-dimensional laccolith is given by $(p - \rho g h)L^5/720D$

A laccolith is when an intruding magma instead of reaching the surface lifts up the surface layer, forming a sheet structure beneath. An expression for deflection w of the surface layer due to the intruding laccolith is then given by

$$w = -\frac{(p-\rho g h)}{24D}\left(x^4 - \frac{L^2x^2}{2} + \frac{L^4}{16}\right) \qquad \textbf{(3-98)}$$

over the extent of the laccolith $[-L/2 \; +L/2]$ (from equation (3-98) we have that $w(\pm L/2) = 0$), where w is defined positive downwards. The cross-sectional area A of the laccolith is then found by integration of equation (3-98) over the width of the laccolith (where the minus-sign in front of the integral comes from the deflection being defined positive downwards)

$$\begin{aligned}
A &= -\int_{-L/2}^{L/2} w dx = \frac{(p-\rho g h)}{24D}\int_{-L/2}^{L/2}\left(x^4 - \frac{L^2x^2}{2} + \frac{L^4}{16}\right)dx \\
&= \frac{(p-\rho g h)}{24D}\left(\frac{x^5}{5} - \frac{L^2x^3}{6} + \frac{L^4x}{16}\right)\Bigg|_{-L/2} L/2 \\
&= \frac{(p-\rho g h)}{24D}\left(2\frac{L^5}{5\cdot 32} - 2\frac{L^5}{6\cdot 8} + 2\frac{L^4}{16\cdot 2}\right) \\
&= \frac{(p-\rho g h)}{24D}\left(\frac{L^5}{80} - \frac{L^5}{24} + \frac{L^5}{16}\right) \\
&= \frac{(p-\rho g h)}{24D\cdot 8}\left(\frac{L^5}{10} - \frac{L^5}{3} + \frac{L^5}{2}\right) \\
&= \frac{(p-\rho g h)}{192D}\left(\frac{3L^5}{30} - \frac{10L^5}{30} + \frac{15L^5}{30}\right) \\
&= \frac{(p-\rho g h)}{192D}\frac{4}{15}L^5 \\
&= \frac{(p-\rho g h)}{720D}L^5
\end{aligned}$$

3-19. (a) Consider a lithospheric plate under a line load. Show that the absolute value of the bending moment is a maximum at

$$x_m = \alpha \cos^{-1} 0 = \frac{\pi}{2}\alpha \qquad \textbf{(3-137)}$$

and that its value is

$$M_m = -\frac{2Dw_0}{\alpha^2}e^{-\pi/2} = -0.416\frac{Dw_0}{\alpha^2} \qquad \textbf{(3-138)}$$

(b) Refraction studies show that the Moho is depressed about 10 km beneath the center of the Hawaiian Islands. Assuming that this is the value of w_0 and that h = 34 km, E = 70 Gpa, ν = 0.25, $\rho_m - \rho_w$ = 2300 kg m^{-3}, and g = 10 m s^{-2}, determine the maximum bending stress in the lithosphere.

---

a)

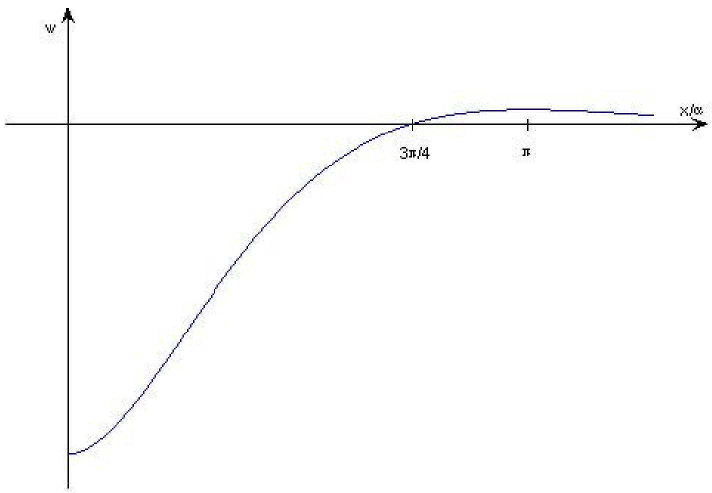

Figure 9: The deflection of an elastic plate under a line load from Equation (3-132)

The deflection w of a lithospheric plate under a line load V_o is given by (see figure 9)

$$w = w_o e^{-x/\alpha}\left[\cos(x/\alpha) + \sin(x/\alpha)\right] \qquad \textbf{(3-132)}$$

where the maximum amplitude of the deflection is given by

$$w_o = \frac{V_o \alpha^3}{8D} \qquad \textbf{(3-131)}$$

and

$$\alpha = \left[\frac{4D}{(\rho_m - \rho_w)g}\right]^{1/4} \qquad \textbf{(3-127)}$$

is known as the flexural parameter. The bending moment in turn is defined by

$$M = -D\frac{d^2w}{dx^2} \qquad \textbf{(3-73)}$$

where D is the flexural rigidity, defined as

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \textbf{(3-72)}$$

Hence we find that

$$M = -D\frac{d^2}{dx^2}\left\{w_o e^{-x/\alpha}\left[\cos(x/\alpha) + \sin(x/\alpha)\right]\right\}$$

$$\begin{aligned}
M &= -D\frac{d}{dx}\left\{-w_o\frac{1}{\alpha}e^{-x/\alpha}\left[\cos(x/\alpha) + \sin(x/\alpha)\right] + w_o e^{-x/\alpha}\left[-\frac{1}{\alpha}\sin(x/\alpha) + \frac{1}{\alpha}\cos(x/\alpha)\right]\right\} \\
&= -\frac{Dw_o}{\alpha}\left\{\frac{1}{\alpha}e^{-x/\alpha}\left[\cos(x/\alpha) + \sin(x/\alpha)\right] - e^{-x/\alpha}\left[-\frac{1}{\alpha}\sin(x/\alpha) + \frac{1}{\alpha}\cos(x/\alpha)\right]\right. \\
&\qquad \left. -\frac{1}{\alpha}e^{-x/\alpha}\left[-\sin(x/\alpha) + \cos(x/\alpha)\right] + e^{-x/\alpha}\left[-\frac{1}{\alpha}\cos(x/\alpha) - \frac{1}{\alpha}\sin(x/\alpha)\right]\right\} \\
&= 2\frac{Dw_o}{\alpha^2}e^{-x/\alpha}\left[\cos(x/\alpha) - \sin(x/\alpha)\right] \qquad \textbf{(*)}
\end{aligned}$$

To find $M = M_{max}$ we search for the zero-points of the derivative of the bending moment, i.e.

$$\begin{aligned}
\frac{dM}{dx} &= 2\frac{Dw_o}{\alpha^3}e^{-x/\alpha}\left[\cos(x/\alpha) - \sin(x/\alpha) + \sin(x/\alpha) + \cos(x/\alpha)\right] \\
&= 4\frac{Dw_o}{\alpha^3}e^{-x/\alpha}\cos(x/\alpha) = 0
\end{aligned}$$

For this to hold true it has to hold that

$$\begin{aligned}
&\cos(x/\alpha) = 0 \\
\Rightarrow \qquad & x_{Max} = \alpha\cos^{-1}0 = \alpha\frac{\pi}{2} \qquad \textbf{(3-137)}
\end{aligned}$$

Inserting x_{Max} into equation (*) we find that

$$\begin{aligned}
M_{max} &= 2\frac{Dw_o}{\alpha^2}e^{-\pi/2}\left[\cos(\pi/2) - \sin(\pi/2)\right] \\
&= -2\frac{Dw_o}{\alpha^2}e^{-\pi/2} \\
\Rightarrow \qquad M_{max} &\approx -0.416\frac{Dw_o}{\alpha^2} \qquad \textbf{(3-138)}
\end{aligned}$$

b)

The bending (or fiber) stress is given by

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\varepsilon_{xx} \qquad \textbf{(3-64)}$$

where the horizontal strain in the plate can be shown to be

$$\varepsilon_{xx} = -y\frac{d^2w}{dx^2} \qquad \textbf{(3-70)}$$

where $y = 0$ at the center of the plate. Hence for a plate of thickness h we the maximum bending stress at $y = \pm h/2$ as

$$\sigma_{xx}^{max} = \mp\frac{Eh}{2(1-\nu^2)}\frac{d^2w}{dx^2} \qquad \textbf{(3-86)}$$

Which can be compared to the bending moment

$$M = -\frac{Eh^3}{12(1-\nu^2)}\frac{d^2w}{dx^2} \qquad \textbf{(3-71)}$$

Where we explicitly have written out the definition of the flexural rigidity D. Hence

$$\begin{aligned}
\sigma_{xx}^{max} &= \frac{6}{h^2}M_{max} \approx 2.496\frac{Dw_o}{\alpha^2 h^2} \\
&= 2.496 w_o \left[\frac{D(\rho_m - \rho_w)g}{4h^4}\right]^{1/2} \\
&= 2.496 w_o \left[\frac{E(\rho_m - \rho_w)g}{48h(1-\nu^2)}\right]^{1/2} \\
&= 2.496 \cdot 10000 \left[\frac{70 \times 10^9 \cdot 2300 \cdot 10}{48 \cdot 34000 \cdot (1 - 0.25^2)}\right]^{1/2} \\
&= 8.1 \times 10^8 \text{ Pa} = 810 \text{ MPa}
\end{aligned}$$

Extra)

Inserting $x = 0$ into the equation (*) we find that

$$M(0) = 2\frac{Dw_o}{\alpha^2}e^{-0/\alpha}\left[\cos(0/\alpha) - \sin(0/\alpha)\right] = \left|2\frac{Dw_o}{\alpha^2}\right| > \left|-0.416\frac{Dw_o}{\alpha^2}\right|$$

I.e. The bending moment directly underneath the load is actually larger than the maximum bending moment we in part b) (which was what we set out to prove). The reason for this comes from the fact that the bending moment displays a cusp at the origin (see figure 10), hence the above scheme of finding the max value from the zero-points of the derivative misses this extrema.

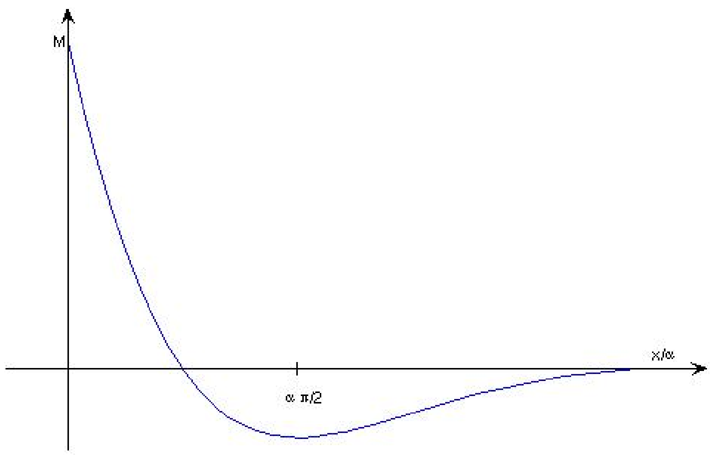

Figure 10: The bending moment over an elastic plate under a line load, from Equationn (*)

3-21. An ocean basin has a depth of 5.5 km. If it is filled to sea level with sediments of density 2600 kg m^{-3}, what is the maximum depth of the resulting sedimentary basin? assume ρ_m = 3300 kg m^{-3}.

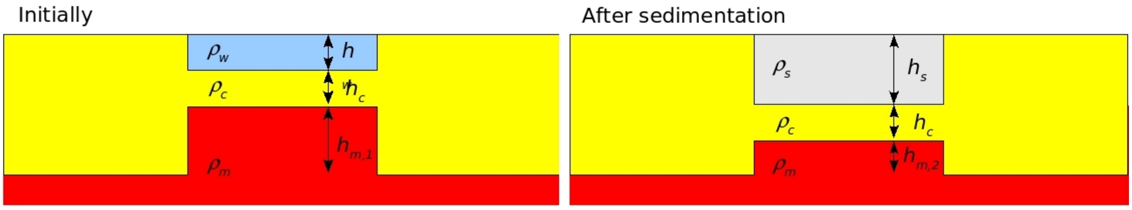

Figure 11: Geometry of problem (3-21) of Turcotte and Schubert

This is really not a flexure problem but an isostacy problem. From Figure 11 We set up the isostatic equilibrium equation as

$$\begin{aligned} & h_w\rho_w + h_c\rho_c + h_{m,1}\rho_m = h_s\rho_s + h_c\rho_c + h_{m,2}\rho_m \\ \Rightarrow\quad & h_w\rho_w + h_{m,1}\rho_m = h_s\rho_s + h_{m,2}\rho_m \\ \Rightarrow\quad & h_s\rho_s = h_w\rho_w + (h_{m,1} - h_{m,2})\rho_m \end{aligned}$$

However

$$\begin{aligned} & h_w + h_c + h_{m,1} = h_s + h_c + h_{m,2} \\ \Rightarrow\quad & h_{m,1} - h_{m,2} = h_s - h_w \end{aligned}$$

Hence we have that

$$\begin{aligned} & h_s\rho_s = h_w\rho_w + (h_s - h_w)\rho_m + h_s\rho_s \\ \Rightarrow\quad & h_s = h_w\left(\frac{\rho_m - \rho_s}{\rho_m - \rho_w}\right) \end{aligned}$$

Using $\rho_w \approx$ 1000 kg m^{-3} we find that

$$h_s = 5500 \cdot \left(\frac{3300 - 1000}{3300 - 2600}\right) \approx 18.1 \text{ km}$$

3-22. The Amazon River in Brazil has a width of 400 km. Assuming that the basin is caused by a line load at its center and that the elastic lithosphere is not broken, determine the thickness of the elastic lithosphere. Assume E = 70 GPa, ν = 0.25, and $\rho_m - \rho_s$ = 700 kg m^{-3}.

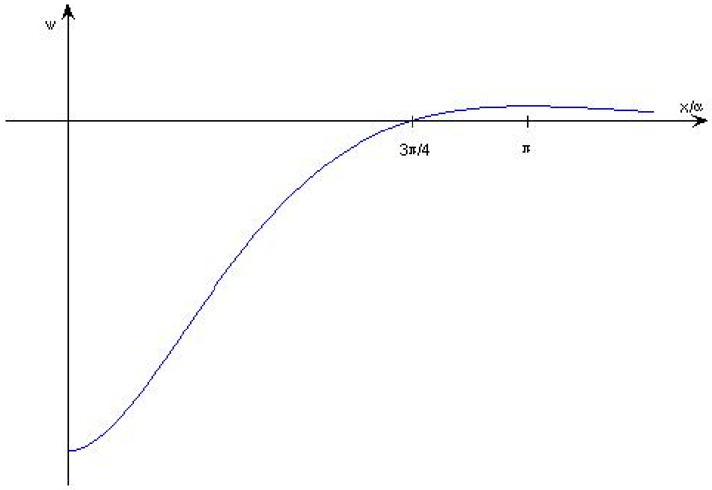

Figure 12: The deflection of an elastic plate under a line load from Equation (3-132)

The forebulge in the deflection of a plate under a line load has a maximum at (see figure 12)

$$x_b = \pi\alpha \qquad \textbf{(3-135)}$$

and a zero-value (i.e. no deflection) at (see figure 12)

$$x_0 = \frac{3\pi}{4}\alpha \qquad \textbf{(3-133)}$$

Hence if x_b = 400/2 = 200 km $\Rightarrow$ $\alpha = 200/\pi$ km. From the definition of the flexural parameter

$$\alpha = \left[\frac{4D}{(\rho_m - \rho_s)g}\right]^{1/4} \qquad \textbf{(3-127)}$$

and the definition of flexural rigidity

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \textbf{(3-72)}$$

we find that

$$\begin{aligned}
\alpha^4 &= \frac{\frac{Eh^3}{12(1-\nu^2)}}{(\rho_m - \rho_s)g} \\
\Rightarrow \quad h^3 &= \frac{3\alpha^4(1-\nu^2)(\rho_m - \rho_s)g}{E} \\
\Rightarrow \quad h &= \left[\frac{3\alpha^4(1-\nu^2)(\rho_m - \rho_s)g}{E}\right]^{1/3} \\
&= \left[\frac{3 \cdot (2.0 \times 10^5)^4 \cdot (1 - 0.25^2) \cdot 700 \cdot 10}{\pi^4 \cdot 7 \times 10^{10}}\right]^{1/3} \approx 16700 \text{ m} = 16.7\text{km}
\end{aligned}$$

Alternatively, using x_0 we find $h \approx 24.2$ km.

4-8 Consider one-dimensional steady-state heat conduction in a half-space with heat production that decreases exponentially with depth. The surface heat flow-heat production relation is $q_o = q_m + \rho H_o h_r$. What is the heat flow - heat production relation at depth $y = h^*$? Let q^* and H^* be the upward heat flux and heat production at $y = h^*$.

The equation given in the problem formulation is valid for the surface heat flow (i.e. y = 0), hence the depth variable y is not present in the equation. Therefore, we need to g back to the basic equations to find the appropriate form of the depth dependent solution. The heat equation for the given scenario takes the form

$$k\frac{d^2T}{dy^2} + \rho H_o e^{-y/h_r} = 0 \qquad \textbf{(4-25)}$$

where H_o is surface heat production and h_r a length scale. Equation (4-25) can be integrated once to yield an expression for the heat flux (given that we adopt the assumption that the density does not change with depth)

$$k\frac{dT}{dy} = -q = C_1 + \rho H_o h_r e^{-y/h_r} \qquad \textbf{(4-26)}$$

Assuming that h_r is such that the source term effectively vanishes before reaching the base of the mantle, it follows that C_1 is equal to the heat flux from the mantle q_m, hence

$$q = -q_m - \rho H_o h_r e^{-y/h_r} \qquad \textbf{(4-28)}$$

At the surface ($y = 0$) we have that the outward heat flux is

$$q_o = -q = q_m + \rho H_o h_r \qquad \textbf{(4-29)}$$

At a depth $y = h^*$ we have

$$q^* = -q(h^*) = q_m + \rho H_o h_r e^{-h^*/h_r}$$

From the source term of the governing Equation (4-25) we have that

$$H(h^*) = H^* = H_o e^{-h^*/h_r}$$

Hence

$$q^* = q_m + \rho h_r H^*$$

4-31 Derive an expression for the thickness of the thermal boundary layer if we define it to be the distance where θ = 0.01

This problem relates to the instantaneous cooling of a body from the surface. I.e. a body initially of uniform temperature that at an instance (t $>$ 0) comes in contact with a heat-resorvoir, of different temperature, sufficiently large that the temperature of the reservoir does not change.

To solve the problem with its boundary conditions we introduce the two dimensionless variables

$$\theta = \frac{T - T_1}{T_o - T_1} \qquad \textbf{(4-93)}$$

$$\eta = \frac{y}{s\sqrt{\kappa t}} \qquad \textbf{(4-96)}$$

For which the solution can be found to be:

$$\theta = \frac{T - T_1}{T_o - T_1} = \text{erfc}\left(\frac{y}{s\sqrt{\kappa t}}\right) = \text{erfc}\eta \qquad \textbf{(4-113)}$$

In the near-surface regions of the body there will at t $>$ 0 be a significant temperature change, this region is what is referred to as a thermal boundary layer. The thickness of of the thermal boundary layer is defined by a specific temperature, or more generally a specific value of $\}theta$. From Equation (4-133) we have that

$$\theta = 0.01 = \text{erfc}\left(\frac{y}{s\sqrt{\kappa t}}\right)$$

$$\Rightarrow \qquad y = 2\sqrt{\kappa t}\text{erfc}^{-1}0.01$$

Using table 4-5 we find

η	erfη	erfcη
1.8	0.989091	0.010909
1.9	0.992790	0.007210

$$\Rightarrow \qquad \text{erfc}^{-1}0.001 = 1.8 + 0.1\frac{0.010909 - 0.01}{0.010909 - 0.007210} \simeq 1.82$$

Hence

$$y \simeq 3.64\sqrt{\kappa t}$$

4-33 One way of determining the effects of erosion on subsurface temperatures is to consider the instantaneous removal of a thickness l of ground. Prior to the removal $T = T_o + \beta y$ where y is the depth, β is the geothermal gradient, and T_o is the surface temperature. After removal, the new surface is maintained at temperature T_o. Show that the subsurface temperature after the removal of the surface layer is given by

$$T = T_o + \beta y + \beta l \text{erf}\left(\frac{y}{2\sqrt{\kappa t}}\right)$$

How is the surface heat flow affected by the removal of surface material.

The expected temperature profile at given times can be seen in figure 13

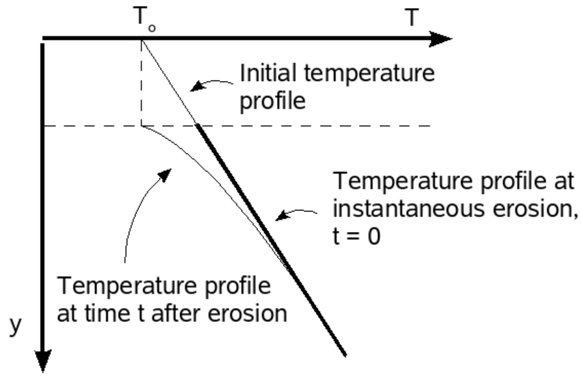

Figure 13

For the problem the governing Equation is

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2} \qquad \textbf{(4-68)}$$

Since Equation (4-68) is linear we can simply superpose (add) solutions as long as we make shure that the boundary conditions are astisfied. Let us start at time $t = 0$, assuming the erosion to be instantaneous the new subsurface temperature is simply that of before erosion, adjusted by the thickness l of the eroded layer times the geothermal gradient β. For simplicity we introduce a new depth varible $y' = y - l$, i.e. such that $y' = 0$ at the surface of the eroded surface. At $t = 0$ the temperature of the sub-surface beneath the eroded surface is then given by:

$$T(t = 0, y') = T_o + \beta y' + l\beta$$

For times $t > 0$ the problem is equivalent to that of instantaneous cooling, we then seek a solution T' to add to our solution above, i.e.

$$T(t, y') = T_o + \beta y' + l\beta + T'$$

To be able to do so we demand that T' is 0 for all depths at $t = 0$ (i.e. the initial temperature pertubation T_1' = 0), $-l\beta$ at the surface for all $t > 0$, and approaches the initial temperature pertubation (i.e. 0) at depth for all $t > 0$ (i.e. no difference at depth), to summarise our boundary conditions are

$$\begin{array}{lll} T' = T_1' = 0 & \text{at} & t = 0; \;\; y' > 0 \\ T' = T_0' = -l\beta & \text{at} & y' = 0; \;\; t > 0 \\ T' \to T_1' = 0 & \text{at} & y \to \infty; \;\; t > 0 \end{array}$$

The solution satifying these boundary conditions is

$$\theta = \frac{T' - T_1'}{T_0' - T_1'} = \frac{T'}{-l\beta} = \text{erfc}\left(\frac{y'}{2\sqrt{\kappa t}}\right) = \text{erfc}\eta \qquad \textbf{(4-113)}$$

Hence

$$T' = -l\beta\text{erfc}\left(\frac{y'}{2\sqrt{\kappa t}}\right)$$

and

$$T(t, y') = T_o + \beta y' + l\beta - l\beta\text{erfc}\left(\frac{y'}{2\sqrt{\kappa t}}\right) = T_o + \beta y' + l\beta\text{erf}\left(\frac{y'}{2\sqrt{\kappa t}}\right) \qquad \textbf{(*)}$$

Since we are only interested in how surface heat flow is *changed* by the removal of material, we only need to consider the last term of equation (*), we have that

$$\Delta q(y' = 0) = -k\frac{\partial T}{\partial y'} = -kl\beta\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\kappa t}}e^{y'^2/(4\kappa t)}\Big|_{y'=0} = -\frac{k\beta l}{\sqrt{\pi\kappa t}} = \begin{cases} \infty & \text{at} \quad t = 0 \\ 0 & \text{at} \quad t \to \infty \end{cases}$$

4-39 One of the estimates for the age of the Earth given by Lord Kelvin in the 1860's assumed that Earth was initially molten at a constant temperature T_m and that it subsequently cooled by conduction with a constant surface temperature T_o. The age of the Earth could then be determined from the present surface thermal gradient $(dT/dy)_o$. Reproduce Kelvin's results assuming $T_m - T_o$ = 1700 K, C = 1 kJ kg^{-1} K^{-1}, L = 400 kJ kg^{-1}, κ = 1 mm^2 s^{-1}, and $(dT/dy)_o$ = 25 K km^{-1}. In addition, determine the thickness of the solidified lithosphere.

This is a Stefan problem, and the expected solution at time t can be seen in figure 14

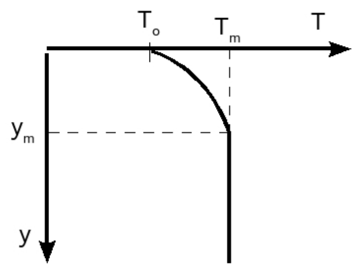

Figure 14

Under the assumption that the molten Earth was initially at constant temperature T_m - equal to the melting temperature of the Earth, this is the Stefan problem and the solution is given by

$$\theta = \frac{T - T_o}{T_m - T_o} = \frac{\text{erf}\eta}{\text{erf}\lambda_1} = \frac{\text{erf}\left(\frac{y}{2\sqrt{\kappa t}}\right)}{\text{erf}\lambda_1} \qquad \textbf{(4-137)}$$

where λ_1 is the solution to the transcendental equation

$$\frac{L\sqrt{\pi}}{c(T_m - T_o)} = \frac{e^{-\lambda_1^2}}{\lambda_1 \text{erf}\lambda_1} \qquad \textbf{(4-141)}$$

putting in the numerics we find

$$\frac{e^{-\lambda_1^2}}{\lambda_1 \text{erf}\lambda_1} = \frac{400 \times 10^3 \cdot \sqrt{\pi}}{1 \times 10^3 \cdot 1700} \simeq 0.417$$

we can then either solve this using some iterative scheme, e.g.

$$\begin{aligned}
\lambda_1^{(0)} &= 1 \\
\lambda_1^{(1)} &= \left[-\ln\left(0.417 \cdot \lambda_1^{(0)}\text{erf}\lambda_1^{(0)}\right)\right]^{1/2} \simeq 1.0226 \\
\lambda_1^{(2)} &= \left[-\ln\left(0.417 \cdot \lambda_1^{(1)}\text{erf}\lambda_1^{(1)}\right)\right]^{1/2} \simeq 1.0063 \\
\lambda_1^{(3)} &= \left[-\ln\left(0.417 \cdot \lambda_1^{(2)}\text{erf}\lambda_1^{(2)}\right)\right]^{1/2} \simeq 1.0181 \\
&\vdots \\
\lambda_1^{(n+1)} &= \left[-\ln\left(0.417 \cdot \lambda_1^{(n)}\text{erf}\lambda_1^{(n)}\right)\right]^{1/2} \simeq 1.0131
\end{aligned}$$

(this scheme converges for n = 18 in this problem). However, since most pocket calculators does not support the error function (as a built-in function) we can also use figure 4-31 of Turcotte and Schubert to find λ_1.

We want to express the age of the Earth in terms of the present surface thermal gradient $(dT/dy)_{y=0}$, we can either do this directly or by using the coordinate transformation (since the derivatives are easier)

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \left[\frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}\right]_{y=0}$$

we find using Equation (4-137)

$$\begin{aligned}
\theta &= \frac{T - T_o}{T_m - T_o} \quad &\Rightarrow \quad & \left.\frac{\partial T}{\partial \theta}\right|_{y=0} = T_m - T_o \\
\theta &= \frac{\mathrm{erf}\eta}{\mathrm{erf}\lambda_1} \quad &\Rightarrow \quad & \left.\frac{\partial \theta}{\partial \eta}\right|_{y=0} = \frac{1}{\mathrm{erf}\lambda_1}\frac{2}{\sqrt{\pi}}e^{-\eta^2}\Big|_{y=0} = \frac{2}{\mathrm{erf}\lambda_1\sqrt{\pi}} \\
\eta &= \frac{y}{2\sqrt{\kappa t}} \quad &\Rightarrow \quad & \left.\frac{\partial \eta}{\partial y}\right|_{y=0} = \frac{1}{2\sqrt{\kappa t}}
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{\partial T}{\partial y}\right)_{y=0} &= \frac{T_m - T_o}{\sqrt{\pi\kappa t}\mathrm{erf}\lambda_1} \\
\Rightarrow \quad t &= \left[\frac{T_m - T_o}{\sqrt{\pi\kappa}\mathrm{erf}\lambda_1}\frac{1}{\left(\frac{\partial T}{\partial y}\right)_{y=0}}\right]^2
\end{aligned}$$

Using $\mathrm{erf}(1.013) \simeq 0.85$ (from table 4-5) we find that

$$t = \left[\frac{1700}{\sqrt{\pi 10^{-6}}} \cdot 0.85\frac{1}{0.025}\right]^2 \simeq 2.037 \times 10^{15}\ \mathrm{s} \simeq 64.5\ \mathrm{Myr}$$

(according to Turcotte and Schubert $t = 65.9$ Myr). The thickness of the layer is then found from

$$\begin{aligned}
y_m &= 2\lambda_1\sqrt{\kappa t} \qquad \textbf{(4-136)} \\
&= 2 \cdot 1.013 \cdot \sqrt{10^{-6} \cdot 2.037 \times 10^{15}} = 91400\ \mathrm{m} = 91.4\ \mathrm{km}
\end{aligned}$$

(92 km in Turcotte and Schubert)

4-57 If petroleum formation requires temperatures between 380 and 430 K, how deep would you drill in a sedimentary basin 20 Ma old? Assume T_o = 285 K, T_1 = 1600 K, κ_m = 1 mm^2 s^{-1}, k$_s$ = 2 W m^{-1} K^{-1}, and k$_m$ = 3.3 W m^{-1} K^{-1}

Assuming the sedimentary basin to have formed in accordance with the cooling lithosphere model presented in Turcotte and Schubert (2^{nd} ed. pp.179-180), the thermal profile of the basin is given by

$$T_s = T_o + \frac{k_m}{k_s}\frac{T_1 - T_o}{\sqrt{\pi \kappa_m t}} y \qquad \textbf{(4-225)}$$

where s-subscripts refer to physical parameters of the sedimentary basin, and m-subscripts to mantle properties (we are considering oceanic lithosphere), hence

$$y = \frac{T_s - T_o}{T_1 - T_o}\frac{k_s}{k_m}\sqrt{\pi \kappa_m t} = \frac{T_s - 285}{1600 - 285}\frac{2}{3.3}\sqrt{\pi \cdot 10^6 \cdot 6.31 \times 10^{14}}$$

(where we have used 20 Myr $\simeq$ 6.31 x 10^{14} s). Hence

$$\begin{aligned} T_s &= 380 \text{ K} \quad \Rightarrow \quad y = 1.95 \text{ km} \\ T_s &= 430 \text{ K} \quad \Rightarrow \quad y = 2.98 \text{ km} \end{aligned}$$

6-1 Show that the mean velocity in the channel is given by

$$\bar{u} = -\frac{h^2}{12\mu}\frac{dp}{dx} + \frac{u_o}{2} \qquad \textbf{(6-17)}$$

The channel in question is a model of 1D flow (in the horizontal (x-) direction in a channel of vertical extent $y \in [0\ h]$, due to a horizontal pressure gradient dp/dx and a moving boundary at $y = 0$ (stick slip, i.e. a prescribed velocity u_o at $y = 0$).

The velocity distribution in the channel is given by

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right) - \frac{u_o y}{h} + u_o \qquad \textbf{(6-12)}$$

where h is the width of the channel (in the y-direction) and $\frac{dp}{dx}$ the pressure gradient (the channel has no slip boundaries, i.e. $u(0) = u_o; u(h) = 0$). The mean velocity is then

$$\begin{aligned}
\bar{u} &= \frac{1}{h}\int_0^h u(y)dy = \frac{1}{h}\int_0^h \left(\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right) - \frac{u_o y}{h} + u_o\right) dy \\
&= \frac{1}{h}\left[\frac{1}{2\mu}\frac{dp}{dx}\left(\frac{y^3}{3} - \frac{hy^2}{2}\right) - \frac{u_o y^2}{2} + u_o y\right]_0^h \\
&= \frac{1}{2\mu}\frac{dp}{dx}\left(\frac{h^2}{3} - \frac{h^2}{2}\right) - \frac{u_o}{2} + u_o \\
&= -\frac{h^2}{12\mu}\frac{dp}{dx} + \frac{u_o}{2}
\end{aligned}$$

6-2 Derive a general expression for the shear stress τ at any location y in the channel. What are the simplified forms of τ for Couette flow and for the case $u_o = 0$?

The channel in question is a model of 1D flow (in the horizontal (x-) direction in a channel of vertical extent $y \in [0\ h]$, due to a horizontal pressure gradient dp/dx and a moving boundary at $y = 0$ (stick slip, i.e. a prescribed velocity u_o at $y = 0$).

The shear stress in a Newtonian fluid is given by

$$\tau = \mu \frac{du}{dy} \quad \textbf{(6-1)}$$

Hence from the velocity distribution in the channel

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right) - \frac{u_o y}{h} + u_o \quad \textbf{(6-12)}$$

we find a general expression for the shear stress as

$$\tau = \frac{1}{2}\frac{dp}{dx}(2y - h) - \frac{u_o \mu}{h}$$

For Couette flow there is no pressure gradient along the channel i.e. $\frac{dp}{dx} = 0$, and so

$$\tau = -\frac{u_o \mu}{h}$$

I.e. constant.

And for the case $u_o = 0$ we find

$$\tau = \frac{dp}{dx}\left(y - \frac{h}{2}\right)$$

I.e. zero at the center of the channel and at maximum at the channel walls.

6-3 Find the point in the channel at which the velocity is a maximum.

The maximum velocity in the channel is found at the point where the shear stress is 0. Using the general expression for the shear stress in the channel derived in problem 6-2

$$\tau = \frac{1}{2}\frac{dp}{dx}(2y - h) - \frac{u_o\mu}{h} = 0$$

we find that

$$y = \frac{u_o\mu}{h\,(dp/dx)} + \frac{h}{2}$$

Hence if $u_o = 0$

$$y = \frac{h}{2}$$

I.e. at the center of the channel.

However, for the Couette flow we cannot use this expression since the shear stress is constant, and hence we cannot solve for y, instead we must start from the velocity distribution for the Couette flow

$$u = u_o\left(1 - \frac{y}{h}\right) \qquad \textbf{(6-13)}$$

which is linear, and so we find that the maximum velocity for the Couette flow is at $y = 0$, i.e. at the moving boundary

6-4 Consider the steady, unidirectional flow of a viscous fluid down the upper face of an inclined plane. Assume that the flow occurs in a layer of constant thickness h, as shown in Figure 6-3. Show that the velocity profile is given by

$$u = \frac{\rho g \sin\alpha}{2\mu}\left(h^2 - y^2\right) \tag{6-18}$$

where y is the coordinate measured perpendicular to the inclined plane (y=h is the surface of the plane), α is the inclination of the plane to the horizontal, and g is the acceleration of gravity. First show that

$$\frac{d\tau}{dy} = -\rho g \sin\alpha \tag{6-19}$$

and then apply the no-slip condition at $y = h$ and the free-surface condition, $\tau = 0$, at $y = 0$. What is the mean velocity in the layer? What is the thickness of a layer whose rate of flow down the incline (per unit width in the direction perpendicular to the plane in figure 6-3) is Q?

Consider the force balance on a small element inside the channel as displayed in figure 15

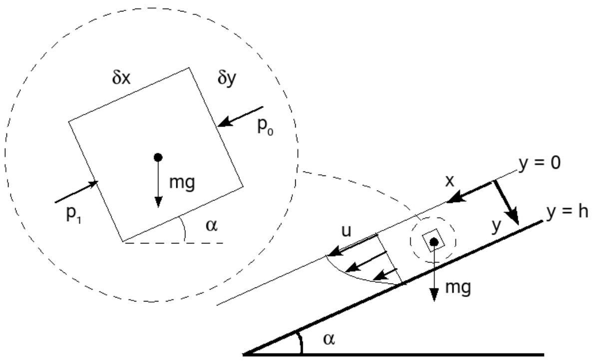

Figure 15

In equilibrium we have that

$$p_1 = p_0 + \frac{F_x}{\delta y}$$

where from figure 15 we find

$$F_x = mg\sin\alpha = \delta x \delta y \rho g \sin\alpha$$

Hence

$$\delta p = p_0 - p_1 = p_0 - \left[p_0 + \frac{\delta x \delta y \rho g \sin\alpha}{\delta y}\right] = -\delta x \rho g \sin\alpha$$

$$\Rightarrow \quad \frac{\delta p}{\delta x} = -\rho g \sin\alpha$$

$$\Rightarrow \quad \frac{dp}{dx} = -\rho g \sin\alpha$$

So from the equation of motion in the channel

$$\frac{dp}{dx} = \frac{d\tau}{dy} \tag{6-8}$$

we find that

$$\frac{d\tau}{dy} = -\rho g \sin\alpha \tag{6-19}$$

Integration then yields

$$\tau = -\int_0^y \rho g \sin \alpha dy' = -\rho g y \sin \alpha + C_1$$

And from the free-surface boundary condition $\tau(0) = 0$ we find that $C_1 = 0$. Assuming the fluid to be Newtonian we then have that the shear stress is related to the velocity gradient as

$$\frac{du}{dy} = \frac{\tau}{\mu} \tag{6-1}$$

Integration the yields

$$u = \int_0^y \frac{\tau}{\mu} dy' = -\int_0^y \frac{\rho g y \sin \alpha}{\mu} dy' = -\frac{\rho g \sin \alpha}{\mu} \frac{y^2}{2} + C_2$$

Applying the no-slip boundary condition at the base, i.e. $u(h) = 0$ we find that

$$C_2 = \frac{\rho g \sin \alpha}{\mu} \frac{h^2}{2}$$

$$\Rightarrow \quad u = \frac{\rho g \sin \alpha}{2\mu} \left(h^2 - y^2\right) \tag{6-18}$$

Yielding the mean velocity

$$\bar{u} = \frac{1}{h} \int_0^h \frac{\rho g \sin \alpha}{2\mu} \left(h^2 - y^2\right) dy = \frac{1}{h} \left[\frac{\rho g \sin \alpha}{2\mu} \left(h^2 y - \frac{y^3}{3} \right) \right]_0^h = \frac{\rho g h^2 \sin \alpha}{3\mu}$$

The rate of flow can then be found as

$$Q = \bar{u} \cdot h = \frac{\rho g h^3 \sin \alpha}{3\mu}$$

yielding the thickness of a layer with a flow rate Q as

$$h = \left(\frac{3\mu Q}{\rho g \sin \alpha} \right)^{1/3}$$

6-5 For an asthenosphere with a viscosity $\mu = 4 \times 10^{19}$ Pa s and a thickness $h = 200$ km, what is the shear stress on the base of the lithosphere if there is no counterflow ($\partial p/\partial x = 0$)? Assume $u_o = 50$ mm yr^{-1} and that the base of the asthenosphere has zero velocity.

If no counterflow exist (i.e. $dp/dx = 0$) the flow in the asthenosphere is simply linearly varying with a velocity equal to the velocity of the lithosphere at top and zero at the base (see figure 16)

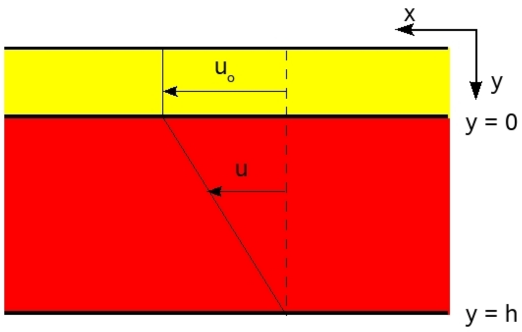

Figure 16

I.e. this is just a case of Couette flow, hence he velocity in the asthenosphere is then given by

$$u = u_o \left(1 - \frac{y}{h}\right) \qquad \textbf{(6-13)}$$

And so the shear stress is given by

$$\tau = \mu \frac{du}{dy} = -\frac{\mu u_o}{h} \qquad \textbf{(6-1)}$$

I.e. Constant everywhere and especially at the base of the lithosphere. Putting in the numerics we find

$$\tau = -\frac{4 \times 10^{19} \cdot (5 \times 10^{-2})/(365.25 \cdot 24 \cdot 3600)}{2 \times 10^5} \simeq 3.169 \times 10^5 \text{ Pa} = 316.9 \text{ kPa}$$

6-6 Assume that the base stress obtained in probelm 6-5 is acting on 6000 km of lithosphere with a thickness of 100 km. What tensional stress in the lithosphere ($h_L = 100$ km) must be applied at a trench to overcome this basal drag?

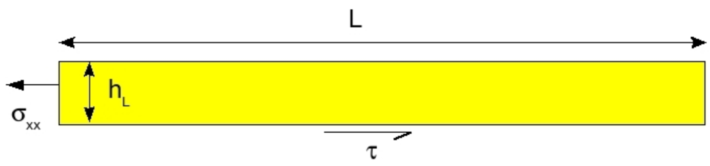

Figure 17

To overcome the basal drag the horizontal force F_x at the trench must be larger (see figure 17), hence

$$
\begin{aligned}
& \sigma_{xx} h_L > \tau L \\
\Rightarrow \quad & \sigma_{xx} > \tau \frac{L}{h_L} \\
\Rightarrow \quad & \sigma_{xx} > 316.9 \frac{6000}{100} \text{ kPa} \simeq 1.90 \times 10^4 \text{ kPa} = 19.0 \text{ MPa}
\end{aligned}
$$

6-9 Determine the rate at which magma flows up a two-dimensional channel of width d under the buoyant pressure gradient $-(\rho_s - \rho_l)g$. Assume laminar flow.

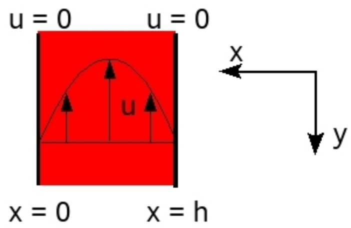

Figure 18

The velocity profile in the channel is given by (see figure 18)

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(x^2 - xd\right) \qquad \textbf{(6-14)}$$

i.e. using the buoyant pressure gradient $-(\rho_s - \rho_l)g$ we find

$$u = -\frac{(\rho_s - \rho_l)g}{2\mu}\left(x^2 - xd\right)$$

And so the flow rate Q becommes

$$\begin{aligned} Q &= \int_0^d u dx \\ &= -\frac{(\rho_s - \rho_l)g}{2\mu}\int_0^d \left(x^2 - xd\right) dx \\ &= -\frac{(\rho_s - \rho_l)g}{2\mu}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^d \\ &= \frac{(\rho_s - \rho_l)gd^3}{12\mu} \end{aligned}$$

6-11 Show that the constant of integration A in the above post glacial rebound solution is given by

$$A = -\left(\frac{\lambda}{2\pi}\right)^2 \frac{\rho g w_{mo}}{2\mu} e^{-t/\tau_r} \qquad \textbf{(6-106)}$$

The model for the post-glacial rebound considers flow in a semi-infinite, viscous fluid half-space, subjected to an initial periodic surface displacement

$$w_m = w_{mo} \cos \frac{2\pi x}{\lambda} \qquad \textbf{(6-79)}$$

The evolution of the deflection of the surface is then given by

$$\left.\frac{\partial w}{\partial t}\right|_{y=0} = A\frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda} \qquad \textbf{(6-101)}$$

where λ is the wavelength of the pertubation (x is the horizontal and y the vertical coordinate). The solution to this differential equation at $y = 0$ can then be argued to be

$$w(y = 0) = w_m e^{-t/\tau_r} \qquad \textbf{(6-104)}$$

where w_m is the initial deflection of the surface at $t = 0$, and τ_r is the characteristic time for relaxation, given by

$$\tau_r = \frac{4\pi\mu}{\rho g \lambda} \qquad \textbf{(6-105)}$$

Taking the derivative of Equation (6-104) yields

$$\frac{\partial w}{\partial t} = -\frac{w_m}{\tau_r} e^{-t/\tau_r}$$

And combining this with Equation (6-101) then gives

$$-\frac{w_m}{\tau_r} e^{-t/\tau_r} = A\frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda}$$

inserting Equation (6-79) we then find

$$- \frac{w_{mo}}{\tau_r} e^{-t/\tau_r} \cos \frac{2\pi x}{\lambda} = A\frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda}$$

$$\Rightarrow \quad A = \frac{\lambda w_{mo}}{2\pi\tau_r} e^{-t/\tau_r} = \left(\frac{\lambda}{2\pi}\right)^2 \frac{\rho g w_{mo}}{2\mu} e^{-t/\tau_r}$$

where we in the last step have substituted in the expression for τ_r

Alternatively we could start from the expression for the deflection of the solution at the surface

$$w_{y=0} = -\frac{2\mu A}{\rho g} \left(\frac{2\pi}{\lambda}\right)^2 \cos \frac{2\pi x}{\lambda} \qquad \textbf{(6-97)}$$

Which canbe rewritten using Equation (6-104) as

$$w_m e^{-t/\tau_r} = -\frac{2\mu A}{\rho g} \left(\frac{2\pi}{\lambda}\right)^2 \cos \frac{2\pi x}{\lambda}$$

And again using equation (6-79) as

$$w_{mo} e^{-t/\tau_r} \cos \frac{2\pi x}{\lambda} = -\frac{2\mu A}{\rho g} \left(\frac{2\pi}{\lambda}\right)^2 \cos \frac{2\pi x}{\lambda}$$

Some algebraic manipulations then yields the desired A.

6-12 The ice sheet over Hudson Bay, Canada, had an estimated thickness of 2 km. At the present time there is a negative free-air gravity anomaly in this region of 0.3 mm s^{-2}

- **a) Assuming that the ice (density of 1000 kg m^{-3}) was in isostatic equilibrium and displaced mantle rock with a density of 3300 kg m^{-3}, determine the depression of the land surface w_{mo}**
- **b) Assuming that the negative free-air gravity anomaly is due to incomplete rebound, determine w at the present time**
- **c) Applying the periodic analysis given above, determine the mantle viscosity. Assume that the ice-sheet melted 10000 years ago and that the appropriate wave length for the Hudson Bay ice sheet was 500 km**
- **d) Dicuss the difference between the viscosity obtained in c) and that obtained for Scandinavia**

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a)

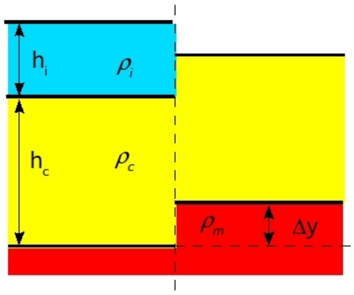

Figure 19

The geometry of the problem can be seen in figure 19, we have that

$$h_i\rho_i + h_c\rho_c = h_c\rho_c + \Delta y \rho_m$$
$$\Rightarrow \quad \Delta y = h_i\frac{\rho_i}{\rho_m} = 2000 \cdot \frac{1000}{3300} \simeq 610 \text{ m}$$

b)

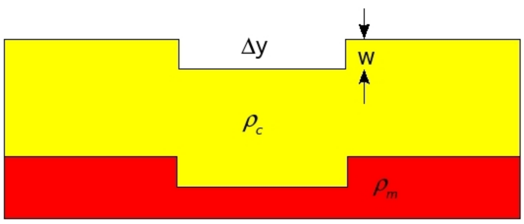

Figure 20

The geometry of the problem can be seen in figure 20, we have that the free-air anomaly is given by

$$\Delta g = -2\pi G \rho_c w \qquad \textbf{(5-111)}$$

hence

$$w = -\frac{\Delta g}{2\pi G\rho_c} = \frac{3 \times 10^{-4}}{2\pi \cdot 6.673 \times 10^{-11} \cdot 3300} \simeq 220 \text{ m}$$

c) The model for the post-glacial rebound considers flow in a semi-infinite, viscous fluid half-space, subjected to an initial periodic surface displacement

$$w_m = w_{mo} \cos \frac{2\pi x}{\lambda} \qquad \textbf{(6-79)}$$

The deflection of the surface is then given by

$$\left.\frac{\partial w}{\partial t}\right|_{y=0} = A\frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda} \qquad \textbf{(6-101)}$$

where λ is the wavelength of the pertubation (x is the horizontal and y the vertical coordinate). The solution to this differential equation at $y = 0$ can then be argued to be

$$w(y = 0) = w_m e^{-t/\tau_r} \qquad \textbf{(6-104)}$$

which can be rewritten as an expression for the characteristic time of relaxation τ_r

$$\tau_r = -\frac{t}{\ln\left(\frac{w}{w_m}\right)}$$

Combining this with the definition of τ_r we find

$$\tau_r = \frac{4\pi\mu}{\rho g\lambda} \qquad \textbf{(6-105)}$$

$$\Rightarrow \qquad \mu = -\frac{t\rho_m g\lambda}{4\pi \ln\left(\frac{w}{w_m}\right)}$$

and putting in the numerics we find

$$\mu = -\frac{10^4 \cdot (365.25 \cdot 24 \cdot 3600) \cdot 3300 \cdot 10 \cdot 5 \times 10^6}{4\pi \ln\left(\frac{220}{610}\right)} \simeq 4.1 \times 10^{21} \text{ Pa s}$$

d) For Scandinavia $\mu \simeq 1.1 \times 10^{21}$ Pa s, this is due to different scales of the sites, as Scandinavia is smaller the ice-sheet was also smaller, hence the deflection ”senses” a shallower part of the mantle.

N.B. Flexure is unimportant for scales $\lambda > 1000$ km's

6-22 The Stokes drag D on a sphere can only depend on the velocity of the sphere u, its radius a, and the viscosity μ and the density of the fluid. Show by dimensional analysis that

$$\frac{D}{\rho u^2 a^2} = f\left(\frac{\rho u a}{\mu}\right) \qquad \textbf{(6-232)}$$

Where f is an arbitrary function. Because the equations of slow viscous flow are linear, D can only be directly proportional to u. Use this fact together with equation (6-232) to conclude that

$$D \propto \mu u a \qquad \textbf{(6-233)}$$

We have that

$$\begin{aligned}
[u] &= \left[\text{m s}^{-1}\right] \text{ - velocity} \\
[a] &= [\text{m}] \text{ - radius} \\
[\mu] &= [\text{Pa s}] = \left[\text{kg m}^{-1}\text{ s}^{-1}\right] \text{ - viscosity} \\
[\rho] &= \left[\text{kg m}^{-3}\right] \text{ - density} \\
[D] &= \left[\text{kg m s}^{-2}\right] \text{ - drag (a force)}
\end{aligned}$$

Now

$$\left[\frac{\rho u a}{\mu}\right] = \left[\frac{\text{kg m}^{-3}\cdot\text{m s}^{-1}\cdot\text{m}}{\text{kg m}^{-1}\text{ s}^{-1}}\right] = \text{ constant}$$

likewise

$$\left[\frac{D}{\rho u^2 a^2}\right] = \left[\frac{\text{kg m s}^{-2}}{\text{kg m}^{-3}\cdot\text{m}^2\text{ s}^{-2}\cdot\text{m}^2}\right] = \text{ constant}$$

Hence

$$\frac{D}{\rho u^2 a^2} = f\left(\frac{\rho U a}{\mu}\right)$$

The only combinations of the parameters above that has the same unit as the drag are

$$\mu u a \quad \text{and} \quad \rho u^2 a^2$$

which can easily be realised as we need s^2 in the denominator. Of these two combinations, only $\mu u a$ ia linear in u, hence

$$D \propto \mu u a$$

6-23 Consider a spherical bubble of a low-viscosity fluid with density ρ_b rising or falling through a much more viscous fluid with density ρ_f and viscosity μ_f because of buoyancy force. For this problem the appropriate boundary conditions at the surface of the sphere, $r = a$, are $u_r = 0$ and $\tau_{r\theta} = 0$. Using Equations (6-210), (6-211), and (6-220), show that

$$u_r = U\left(-1 + \frac{a}{r}\right)\cos\theta \tag{6-234}$$

$$u_\theta = U\left(1 - \frac{a}{2r}\right)\sin\theta \tag{6-235}$$

By integrating Equation (6-196), show that on $r = a$

$$p = \frac{\mu_f U}{a}\cos\theta \tag{6-236}$$

The drag force is obtained by carring out the integral

$$D = 2\pi a^2 \int_0^\pi \left(p - 2\mu_f \frac{\partial u_r}{\partial r}\right)_{r=a} \cos\theta \sin\theta d\theta \tag{6-237}$$

Show that

$$D = 4\pi\mu_f a U \tag{6-238}$$

and demonstrate that the terminal velocity of the bubble in the fluid is

$$U = \frac{a^2 g(\rho_f - \rho_b)}{3\mu_f} \tag{6-239}$$

Instead of considering the bubble moving through the surrounding fluid, an equivalent picture is the bubble being stationary and the surrounding fluid moving as depicted in figure 21.

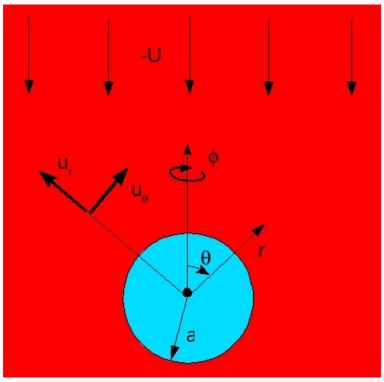

Figure 21

The velocity of the fluid relative to the bubble is then given by

$$u_r = \left(C_1 + \frac{C_2}{r^3} + \frac{C_3}{r} + C_4 r^2\right)\cos\theta \tag{6-210}$$

$$u_\theta = \left(-C_1 + \frac{C_2}{2r^3} - \frac{C_3}{2r} - 2C_4 r^2\right)\sin\theta \tag{6-211}$$

At a great distance from the sphere it must hold that

$$\begin{aligned} u_r &\to -U\cos\theta \\ u_\theta &\to U\sin\theta \end{aligned} \qquad \text{as} \qquad r \to \infty \tag{6-197}$$

As this is the velocity of the surrounding fluid. For this to hold true we find from Equations (6-210) and (6-211) that

$$C_1 = -U$$
$$C_4 = 0$$

At the surface of the sphere, the radial velocity is zero, i.e. at $r = a$, $u_r = 0$. plugging this into Equation (6-210) we find

$$\begin{aligned} &-U + \frac{C_2}{a^3} + \frac{C_3}{a} = 0 \\ \Rightarrow \quad & C_2 = a^2\left(aU - C_3\right) \end{aligned}$$

And so the velocity components are

$$u_r = \left(-U + \frac{a^2\left(aU - C_3\right)}{r^3} + \frac{C_3}{r}\right)\cos\theta \qquad \textbf{(6-210)}$$

$$u_\theta = \left(U + \frac{a^2\left(aU - C_3\right)}{2r^3} - \frac{C_3}{2r}\right)\sin\theta \qquad \textbf{(6-211)}$$

Now the shear stress in polar coordinates is given by

$$\tau_{r\theta} = \mu\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right\} \qquad \textbf{(6-220)}$$

And so using the free-slip (no-stick) condition

$$\tau_{r\theta}(r = a) = 0$$

we find that

$$\begin{aligned} \tau_{r\theta}(r = a) &= \mu\left\{a\sin\theta\left(-\frac{U}{a^2} - \frac{2\left(aU - C_3\right)}{a^3} + \frac{C_3}{a^3}\right) - \frac{\sin\theta}{a}\left(-U + \frac{\left(aU - C_3\right)}{a} + \frac{C_3}{a}\right)\right\} \\ &= \frac{\mu\sin\theta}{a^2}\left\{-Ua + 2\left(C_3 - aU\right) + C_3 + Ua + \left(C_3 - aU\right) - C_3\right\} \\ &= \left(C_3 - aU\right)\frac{\mu\sin\theta}{a^2} \end{aligned}$$

Since this should hold true for all angles θ we find that

$$C_3 = Ua$$

And hence the velocity of the surrounding fluid becomes

$$u_r = \left(-U + \frac{Ua}{r}\right)\cos\theta = U\left(-1 + \frac{a}{r}\right)\cos\theta \qquad \textbf{(6-234)}$$

$$u_\theta = \left(U - \frac{Ua}{2r}\right)\sin\theta = U\left(1 - \frac{a}{2r}\right)\sin\theta \qquad \textbf{(6-235)}$$

Assuming that the surrounding fluid is incompressible, we can use one of the equations of motion in polar coordinates to find an expression for the pressure in the fluid.

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_\theta}{\partial \theta}\right) + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2\sin^2\theta}\right\} \qquad \textbf{(6-196)}$$

plugging in our expressions for the velocity components, Equation (6-234) and (6-235) we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Ua}{2}\sin\theta\right) = 0$$

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right) = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(u\left[1-\frac{a}{2r}\right]\right)\sin\theta\cos\theta = \frac{U}{r^2}\left(1-\frac{a}{2r}\right)\frac{1-2\sin^2\theta}{\sin\theta}$$
$$\frac{2}{r^2}\frac{\partial u_r}{\partial\theta} = \frac{2U}{r^2}\left(1-\frac{a}{r}\right)\sin\theta$$
$$\frac{U_\theta}{r^2\sin^2\theta} = \frac{U}{r^2}\left(1-\frac{a}{2r}\right)\frac{1}{\sin\theta}$$

Hence summing all contributions together we find

$$\begin{aligned}\frac{\partial p}{\partial\theta} &= \mu\left\{\frac{U}{r}\left(1-\frac{a}{2r}\right)\left(\frac{1}{\sin\theta} - 2\sin\theta - \frac{1}{\sin\theta}\right) + \frac{2U}{r}\left(1-\frac{a}{r}\right)\sin\theta\right\} \\ &= -\frac{\mu a U}{r^2}\sin\theta\end{aligned}$$

And hence

$$dp = -\frac{\mu a U}{r^2}\sin\theta d\theta$$

Which upon integration yields

$$p = \frac{\mu a U}{r^2}\cos\theta$$

And so the pressure at the surface of the sphere is

$$p(r=a) = \frac{\mu U}{a}\cos\theta \tag{6-236}$$

The drag force on the sphere is then computed as

$$D = 2\pi a^2\int_0^\pi \left(p - 2\mu\frac{\partial u_r}{\partial r}\right)_{r=a}\cos\theta\sin\theta d\theta$$

Again using Equation (6-234) we find that

$$\begin{aligned}D &= 2\pi a^2\int_0^\pi\left(\frac{\mu U}{a}\cos\theta + \frac{2\mu U}{a}\cos\theta\right)\cos\theta\sin\theta d\theta \\ &= 6\pi\mu U a\int_0^\pi \cos^2\theta\sin\theta d\theta \\ &= 6\pi\mu U a\left[-\frac{cos^3\theta}{3}\right]_0^\pi \\ &= 6\pi\mu U a\left[\frac{2}{3}\right]\end{aligned}$$

Hence the drag on the sphere is

$$D = 4\pi\mu U a \tag{6-238}$$

Finally, according to Archimedes principle, the net upward buoyancy force on the bubble is

$$F = \Delta m g = V\Delta\rho g = \frac{4\pi a^3}{3}\left(\rho_f - \rho_b\right)g$$

equating this with the drag force then yields an expression for the terminal velocity

$$\begin{aligned}& 4\pi\mu U a = \frac{4\pi a^3}{3}\left(\rho_f - \rho_b\right)g \\ \Rightarrow\quad & U = \frac{a^2\left(\rho_f - \rho_b\right)g}{3\mu}\end{aligned} \tag{6-239}$$

6-24 Determine the radius of the plume conduit, the volume flux, the heat flux, the mean ascent velocity, and the plume head volume for the Azores plume. Assume that $T_p - T_1$ = 200 K, α_v = 3 x 10^{-5} K^{-1}, μ_p = 10^{19} Pa s, ρ_m = 3300 kg m^{-3}, μ_m = 10^{21} Pa s, and c_p = 1.25 kJ kg^{-1} K^{-1}

From Table 6-4 we find that for the Azores plume, the buoyancy flux is

$$B = 1.1 \times 10^3 \text{ kg s}^{-1}$$

The buoyancy flux is then equal to

$$B = \frac{\pi}{8} \frac{gR^4 \rho_m^2 \left(T_p - T_1\right)^2 \alpha_v^2}{\mu_p} \qquad \textbf{(6-245)}$$

From which we can find the radius R of the plume conduit as

$$R = \left[\frac{8\mu_p B}{\pi g \rho_m^2 \left(T_p - T_1\right)^2 \alpha_v^2}\right]^{1/4} = \left[\frac{8 \cdot 10^{19} \cdot 1.1 \times 10^3}{\pi \cdot 10 \cdot 3300^2 \cdot 200^2 \cdot (3 \times 10^{-5})^2}\right] \simeq 51700 \text{ m} = 51.7 \text{ km}$$

The volume flux Q_p is then given by

$$B = Q_p(\rho_m - \rho_p) \qquad \textbf{(6-244)}$$

Combined with

$$\rho_m - \rho_p = \rho_m \alpha_v (T_p - T_1) \qquad \textbf{(6-241)}$$

We find that

$$Q_p = \frac{B}{\rho_m \alpha_v (T_p - T_1)} = \frac{5.17 \times 10^3}{3300 \cdot 3 \times 10^{-5} \cdot 200} \simeq 55.6 \text{ m}^3 \text{ s}^{-1}$$

The heat flux Q_H can be found using equation (6-247)

$$Q_H = \frac{c_P B}{\alpha_v} = \frac{1.25 \times 10^3 \cdot 1.1 \times 10^3}{3 \times 10^{-5}} \simeq 4.58 \times 10^{10} \text{ W} = 45.8 \text{ GW}$$

The mean ascent velocity U is found from

$$Q_p = \pi R U \qquad \textbf{(6-248)}$$

$$\Rightarrow \qquad U = \frac{Q_p}{\pi R^2} = \frac{55.6}{\pi \cdot (5.71 \times 10^4)^2} \simeq 6.6 \times 10^{-9} \simeq 21 \text{ cm yr}^{-1}$$

Finally the volume V of the plume head can be found if we know the radius a of the plume head, which we can find from the Equation

$$U = \frac{a^2 g \rho_m \left(T_p - T_1\right) \alpha_v}{3\mu_m} \qquad \textbf{(6-242)}$$

$$\Rightarrow \qquad a = \left[\frac{3U\mu_m}{g\rho_m \left(T_p - T_1\right) \alpha_v}\right]^{1/2} = \left[\frac{3 \cdot 6.6 \times 10^{-9} \cdot 10^{19}}{10 \cdot 3300 \cdot 3 \times 10^{-5} \cdot 200}\right]^{1/2} \simeq 317000 \text{ m} = 317 \text{ km}$$

Hence the volume is

$$V = \frac{4\pi}{3} 317^3 \simeq 1.33 \times 10^8 \text{ km}^3$$

6-26 Consider the unidirectional flow driven by a constant horizontal pressure gradient through a channel with stationary plane parallel walls, as discussed in section 6-2. Determine the temperature distribution in the channel, the wall heat flux, the heat transfer coefficient, and the Nusselt number by assuming, as in the pipe flow problem above, that the temperature of both walls and the fluid varies linearly with distance x along the channel. You will need to form the temperature equation in two dimensions that balances horizontal heat advection against vertical heat conduction, as given in Equation (4-156)

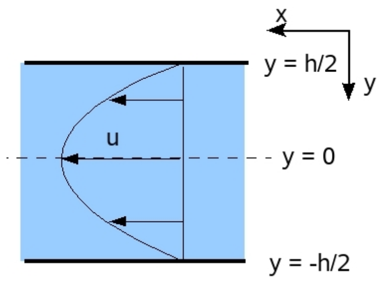

Figure 22

The velocity in the channel is given by (see figure 22)

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - \frac{h^2}{4}\right) \qquad \textbf{(6-16)}$$
$$v = 0$$

We are assuming the temperature field to be stationary, i.e.

$$\frac{\partial T}{\partial t} = 0$$

as well as linearly varying along the channel (i.e. in x)

$$\frac{\partial T}{\partial x} = C_1 \qquad \Rightarrow \qquad \frac{\partial^2 T}{\partial x^2} = 0$$

hence the thermal equation becomes a balance between advection along the channel and diffusion perpendicular to the channel, i.e. from

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\nabla^2 T = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

we find the governing equation to be

$$u\frac{\partial T}{\partial x} = uC_1 = \kappa\frac{\partial T^2}{\partial y^2} \qquad \textbf{(4-156)}$$

And so using equation (6-16) we find that governing equation to be

$$\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - \frac{h^2}{4}\right)C_1 = \kappa\frac{\partial T^2}{\partial y^2}$$

For this Equation we then seek a solution on the form (since T is linearly varying with x)

$$T = C_1 x + C_2 + \theta(y)$$

Plugging this into the heat equation yields

$$
\begin{aligned}
&\frac{\partial^2 T}{\partial y^2} = \frac{\partial \theta^2}{\partial y^2} = \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(y^2 - \frac{h^2}{4}\right) \\
\Rightarrow \quad &\frac{\partial \theta}{\partial y} = \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^3}{3} - \frac{yh^2}{4} + a\right) \\
\Rightarrow \quad &\theta = \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^4}{12} - \frac{y^2h^2}{8} + ay + b\right)
\end{aligned}
$$

By symmetry we find that at the center of the channel, i.e. at $y = 0$ the heat flux perpendicular to the channel must be zero, i.e.

$$q(0) = 0 \qquad \Rightarrow \qquad \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0$$

Hence we see immediately from the expression for $\frac{\partial \theta}{\partial y}$ above that

$$a = 0$$

Assuming that the temperature is equal on both walls, then

$$\theta(h/2) = \theta(-h/2) = 0$$

since any nonzero temperature can be incorporated into the constant C_2. Plugging this into the expression yields

$$
\begin{aligned}
&\theta = \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{h^4}{192} - \frac{h^4}{32} + b\right) = 0 \\
\Rightarrow \quad &b = \frac{h^4}{32} - \frac{h^4}{192} \\
\Rightarrow \quad &b = \frac{5h^4}{192}
\end{aligned}
$$

and so we find

$$T = C_1 x + C_2 + \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^4}{12} - \frac{y^2h^2}{8} + \frac{5h^4}{192}\right)$$

N.B. $T = C_1 x + C_2$ is the temperature of the channel walls, and we can not solve further for this due to lack of information.

Note that this expression was derrived using a coordinate system centered over the channel (i.e. y = 0 at the center of the channel). Had we instead used a coordinate with one side of the channel at $y = 0$, and the other at $y = h$, the velocity distibution in the channel would have been described by the equation

$$u = \frac{1}{2\mu}\frac{dp}{dy}\left(y^2 - hy\right) \qquad \textbf{(6-14)}$$

Which would have lead to the solution

$$T = C_1 x + C_2 + \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^4}{12} - \frac{hy^3}{6} + \frac{h^3 y}{12}\right)$$

The Wall heat flux is then

$$
\begin{aligned}
q &= -k \left.\frac{\partial T}{\partial y}\right|_{y=\pm h/2} \\
&= -k \left.\frac{\partial \theta}{\partial y}\right|_{y=\pm h/2} \\
&= -k \left[\frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^3}{3} - \frac{yh^2}{4}\right)\right]_{y=\pm h/2} \\
&= -\frac{kC_1}{2\mu\kappa}\frac{dp}{dx}\left(\pm\frac{h^3}{24} \mp \frac{h^3}{8}\right) \\
&= \pm\frac{kC_1 h^3}{24\mu\kappa}\frac{dp}{dx}
\end{aligned}
$$

I.e.

$$
q(h/2) = -q(-h/2) = -\frac{kC_1 h^3}{24\mu\kappa}\frac{dp}{dx}
$$

Hence if $C_1 > 0$, i.e. the temperature of the wall increases in the direction of the flow, heat will flow into the fluid through the walls (which can be understood as the fluid will be colder that the walls, comming from a colder region).

The heat transfer coefficient is defined as

$$
q_w = h\left(\bar{T} - T_w\right) = h\bar{\theta} \qquad \textbf{(6-264)}
$$

where q_w and T_w are heat flux and temperature at the wall, and $\bar{T}$ and $\bar{\theta}$ are flow weighted averages over the channel, i.e.

$$
\bar{\theta} = \frac{\int_{-h/2}^{h/2}\theta u dy}{\int_{-h/2}^{h/2} u dy}
$$

Starting with the lower integral, using Equation (6-16) we have

$$
\begin{aligned}
\int_{-h/2}^{h/2} u dy &= \frac{1}{2\mu}\frac{dp}{dx}\int_{-h/2}^{h/2}\left(y^2 - \frac{h^2}{4}\right) dy \\
&= \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{y^3}{3} - y\frac{h^2}{4}\right]_{-h/2}^{h/2} \\
&= \frac{h^3}{12\mu}\frac{dp}{dx}
\end{aligned}
$$

For the upper integral we find

$$
\begin{aligned}
\int_{-h/2}^{h/2} \theta u dy &= \int_{-h/2}^{h/2} \frac{C_1}{2\mu\kappa}\frac{dp}{dx}\left(\frac{y^4}{12}-\frac{y^2h^2}{8}+\frac{5h^4}{192}\right)\frac{1}{2\mu}\frac{dp}{dx}\left(y^2-\frac{h^2}{4}\right)dy \\
&= \frac{C_1}{4\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\int_{-h/2}^{h/2}\left(\frac{y^6}{12}-\frac{y^4h^2}{8}-\frac{y^4h^2}{48}+\frac{5h^4y^2}{192}+\frac{h^4y^2}{32}-\frac{5h^6}{768}\right)dy \\
&= \frac{2C_1}{4\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\int_{0}^{h/2}\left(\frac{y^6}{12}-\frac{7y^4h^2}{48}+\frac{11h^4y^2}{192}-\frac{5h^6}{768}\right)dy \\
&= \frac{C_1}{2\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\left[\frac{1}{2^7}\left(\frac{h^7}{12\cdot 7}-\frac{7h^7}{12\cdot 5}+\frac{11h^7}{12\cdot 3}-\frac{5h^7}{6}\right)\right] \\
&= \frac{h^7C_1}{2^9\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\left[\frac{1}{42}-\frac{7}{30}+\frac{11}{18}-\frac{5}{3}\right] \\
&= \frac{h^7C_1}{2^9\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\left[\frac{1}{3}\left(\frac{1}{14}-\frac{7}{10}+\frac{11}{6}-5\right)\right] \\
&= \frac{h^7C_1}{2^9\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\left[\frac{1}{3}\left(\frac{15}{210}-\frac{147}{210}+\frac{385}{210}-\frac{1050}{210}\right)\right] \\
&= \frac{h^7C_1}{2^9\mu^2\kappa}\left(\frac{dp}{dx}\right)^2\left[-\frac{797}{630}\right] \\
&= -\frac{797h^7C_1}{630\cdot 2^9\mu^2\kappa}\left(\frac{dp}{dx}\right)^2
\end{aligned}
$$

Hence after some more algebra we find that

$$\bar{\theta} = -\frac{797h^4C_1}{26880\mu\kappa}\frac{dp}{dx}$$

And so the heat transfer coefficient becomes

$$h = \frac{q_w}{\bar{\theta}} = \frac{k}{D}\frac{3360}{2391}$$

where we have denoted the width of the channel by D to separate it from the heat transfer coefficient. N.B. This value differs from the value given in Turcotte and Schubert (which is 70/17 at the end instead).

The Nusselt number is the defined as

$$\text{Nu} = \frac{hD}{k} \tag{6-267}$$

Hence

$$\text{Nu} = \frac{3360}{2391} \simeq 1.41$$

and according to Turcotte and Schubert Nu = 4.12

6-33 Consider convection in a fluid layer heated from below. The mean surface heat flux $\bar{q}$ is transferred through the cold thermal boundary layer by conduction. Therefore we can write

$$\bar{q} = \frac{k(T_c - T_o)}{\delta} \qquad \textbf{(6-386)}$$

where δ is a characteristic thermal boundary layer thickness. Show that

$$\frac{\delta}{b} = 1.7\ \mathrm{Ra}^{-1/3} \qquad \textbf{(6-387)}$$

Calculate δ for an upper mantle convection cell given the parameter values used in this section.

From equation (6-371) we have that the mean surface heat flow for the given problem is

$$\bar{q} = 0.294\mathrm{Ra}^{1/3}\frac{k\,(T_1 - T_0)}{b} \qquad \textbf{(6-371)}$$

Where the appropriate Rayleigh number is for heated-from-below convection. Combining Equation (6-371) with Equation (6-386) (given in the problem) we find

$$\begin{aligned} \frac{k\,(T_c - T_0)}{\delta} &= 0.294\mathrm{Ra}^{1/3}\frac{k\,(T_1 - T_0)}{b} \\ \Rightarrow \quad \frac{\delta}{d} &= \frac{(T_c - T_0)}{(T_1 - T_0)}\frac{1}{0.294}\mathrm{Ra}^{-1/3} \end{aligned}$$

Now for large Rayleigh number (and hence thermal convection) we have that

$$T_c - T_0 = \frac{1}{2}(T_1 - T_0) \qquad \textbf{(6-325)}$$

Hence

$$\frac{\delta}{d} = \frac{1}{2}\cdot\frac{1}{0.294}\mathrm{Ra}^{-1/3} \simeq 1.7\mathrm{Ra}^{-1/3}$$

For a fluid heated from below, the rayleigh number is given by

$$\mathrm{Ra} = \frac{\rho_0 g \alpha_v (T_1 - T_0) b^3}{\mu\kappa} \qquad \textbf{(6-316)}$$

Typical parameter values for upper mantle convection are

parameter	value
ρ_0	3700 kg m^3
g	10 m s^{-2}
α_v	3 x 10^{-5} K^{-1}
$T_1 - T_0$	1500 K
b	700 km
μ	10^{21} Pa s
κ	10^{-6} m^2 s^{-1}

Hence

$$\mathrm{Ra} = \frac{3700\cdot 10\cdot 3\times 10^{-5}\cdot 1500\cdot(700\times 10^3)^3}{10^{21}\cdot 10^{-6}} \simeq 5.71\times 10^5$$

Plugging this value into Equation (6-387) we find the boundary layer thickness to be

$$\delta = 1.7\mathrm{Ra}^{1/3}b = 1.7\cdot 11655^{-1/3}\cdot 700 \simeq 14.3\ \mathrm{km}$$

6-36 Apply the two-dimensional boundary-layer model for heated-from-below convection to the entire mantle. Calculate the mean surface heat flux, the mean horizontal velocity, and the mean surface thermal boundary layer thickness. Assume $T_1 - T_o$ = 3000 K, b = 2880 km, k = 4 W m^{-1} K^{-1}, κ = 1 mm^2 s^{-1}, α_v = 3 x 10^{-5} K^{-1}, g = 10 m s^{-2}, and ρ_o = 4000 kg m^{-3}

For the 2D boundary layer model for heated from below convection, the mean surface heat flux is given by

$$\bar{q} = 0.294 Ra^{1/3} \frac{k(T_1 - T_0)}{b} \qquad \textbf{(6-371)}$$

Using the expression for the Rayleigh number for a heated from below convection cell

$$Ra = \rho_o g \alpha_v (T_1 - T_0) \frac{b^3}{\mu\kappa} \qquad \textbf{(6-316)}$$

assuming a viscosity of the mantle of 10^{21} Pa s, we find

$$Ra = \frac{4000 \cdot 10 \cdot 3 \times 10^{-5} \cdot 3000 \cdot (2.88 \times 10^6)^3}{10^{-6} \cdot 10^{21}} \simeq 8.6 \times 10^7$$

And so

$$\bar{q} = 0.296 \cdot (8.6 \times 10^7)^{1/3} \cdot \frac{4 \cdot 3000}{2.88 \times 10^6} \simeq 0.54 \text{ w m}^2$$

The mean horizontal velocity U_o for the model is then given by

$$U_o = 0.271 \frac{\kappa}{b} Ra^{2/3} \qquad \textbf{(6-369)}$$

Hence

$$U_o = 0.271 \frac{10^{-6}}{2.88 \times 10^6} (8.6 \times 10^7)^{2/3} \simeq 57.6 \text{ cm yr}^{-1}$$

Finally the mean thermal boundary thickness δ is given by

$$\delta = 1.7 b Ra^{-1/3} \qquad \textbf{(6-387)}$$

Hence

$$\delta = 1.7 \cdot 2.88 \times 10^6 \cdot (8.6 \times 10^7)^{-1/3} \simeq 11091 \text{ m} = 11.1 \text{ km}$$

7-10 Show that the effective viscosity μ_{eff} for the channel flow of a power-law fluid is given by

$$\mu_{eff} = \frac{\tau}{du/dy} = \left(\frac{p_1 - p_o}{L}\right) \frac{h^2}{4(n+2)\bar{u}} \left(\frac{2y}{h}\right)^{1-n} \quad \textbf{(7-127)}$$

or

$$\frac{\mu_{eff}}{\mu_{eff,wall}} = \left(\frac{2y}{h}\right)^{1-n} \quad \textbf{(7-128)}$$

where $\mu_{eff,wall}$ is the value of μ_{eff} at $y = \pm h/2$. Plot $\mu_{eff}/\mu_{eff,wall}$ as a function of y/h for $n = 1, 3,$ and 5.

For shear stresses in 2D we have

$$\tau = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \textbf{(6-58)}$$

However, in channel flow (in the x.direction) $\frac{\partial v}{\partial x} = 0$ for laminar flow, hence

$$\tau = \mu \frac{\partial u}{\partial y}$$

And so we define the effective viscosity as

$$\mu_{eff} = \frac{\tau}{du/dy}$$

Now for channel flow the shear stress satisfies

$$\frac{d\tau}{dy} = -\frac{p_1 - p_0}{L} \quad \textbf{(7-120)}$$

$$\Rightarrow \quad \tau = -\frac{p_1 - p_0}{L} y$$

And for the channel flow of a power-law fluid it holds that

$$\frac{du}{dy} = -C_1 \left[\frac{p_1 - p_0}{L}\right]^n y^n \quad \textbf{(7-123)}$$

where the power n is an un-even integer, to allow for both positive and negative shear (had n been an even integer du/dy could only be negative which is unphysical). The mean velocity is then given by

$$\bar{u} = \frac{C_1}{n+2} \left[\frac{p_1 - p_0}{L}\right]^n \left(\frac{h}{2}\right)^{n+1} \quad \textbf{(7-125)}$$

Hence

$$C_1 = \bar{u}(n+2) \left[\frac{p_1 - p_0}{L}\right]^{-n} \left(\frac{2}{h}\right)^{n+1}$$

Combining these Equations we find

$$\begin{aligned} \mu_{eff} &= \frac{-\frac{p_1 - p_0}{L} y}{-\bar{u}(n+2) \left[\frac{p_1 - p_0}{L}\right]^{-n} \left(\frac{2}{h}\right)^{n+1} \left[\frac{p_1 - p_0}{L}\right]^n y^n} \\ &= \frac{p_1 - p_0}{L} \frac{1}{\bar{u}(n+2)} y^{1-n} \left(\frac{2}{h}\right)^{-1-n} \left(\frac{h}{2}\right)^2 \left(\frac{2}{h}\right)^2 \end{aligned}$$

Hence

$$\mu_{eff} = \frac{\tau}{du/dy} = \left(\frac{p_1 - p_o}{L}\right) \frac{h^2}{4(n+2)\bar{u}} \left(\frac{2y}{h}\right)^{1-n} \quad \textbf{(7-127)}$$

At the wall ($y = \pm h/2$) we have

$$\mu_{eff,wall} = \left(\frac{p_1 - p_o}{L}\right) \frac{h^2}{4(n+2)\bar{u}}$$

hence

$$\frac{\mu_{eff}}{\mu_{eff,wall}} = \left(\frac{2y}{h}\right)^{1-n} \qquad \textbf{(7-128)}$$

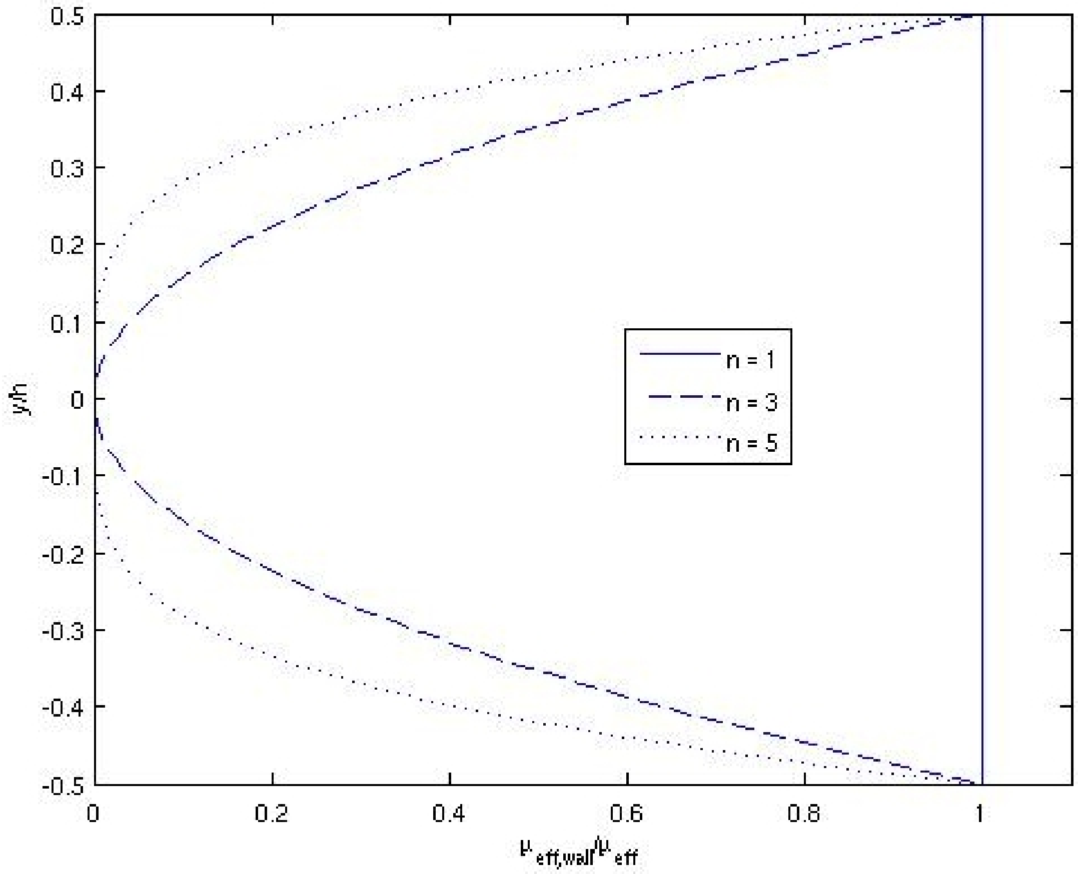

Figure 23: The viscosity of a powerlaw fluid for channel flow. Note that the inverse of eq. 7-128 has been plotted to avoid singularities at y/h = 0

7-12 Consider an ice sheet of thickness h lying on bedrock with a slope α, as shown in figure 7-16. The ice will creep slowly down hill under the force of its own weight. Determine the velocity profile $u(y)$ of the ice. The viscosity of ice has the temperature dependence given in Equation (7-130). Assume that the temperature profile in the ice is linear with the surface temperature T_o (at $y = 0$) and the bedrock-ice interface temperature T (at $y = h$). Assume that there is no melting at the base of the ice sheet so that the no-slip condition applies, that is $u = 0$ at $y = h$, and utilize the approximation given in Equation (7-133)

The geometry of the problem is displayed in figure 24

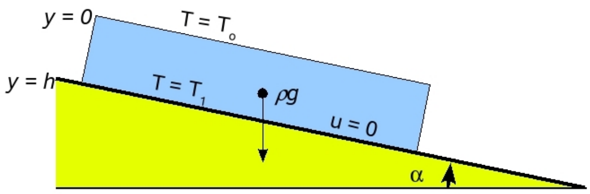

Figure 24

Consider the shear stress τ in the ice-block, choosing a coordinate system with the x-axis parallel to the surface of the bedrock we have

$$\tau = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \qquad \textbf{(6-58)}$$

Assuming the thickness of the ice to be constant, and the velocity to have reached steady state it holds that

$$\frac{\partial v}{\partial x} = 0$$

hence

$$\tau = \mu \frac{\partial u}{\partial y} = C e^{E_a/RT} \frac{du}{dy} \qquad \textbf{(7-131)}$$

where we have assumed a temperature dependence of the viscosity of

$$\mu = C e^{E_a/RT} \qquad \textbf{(7-130)}$$

Assuming a linear temperature profile

$$T = T_0 + (T_1 - T_0)\frac{y}{h} \qquad \textbf{(7-129)}$$

we find that

$$\frac{du}{dy} = \frac{\tau}{C} \exp\left[\frac{-E_a/R}{T_0 + (T_1 - T_0)\frac{y}{h}} \right]$$

Assuming $T_1 - T_0 \ll T_0$ we can use the linearly truncated Taylor expansion of the argument of the exponent

$$\frac{-E_a/R}{T_0 + (T_1 - T_0)\frac{y}{h}} \simeq -\frac{E_a}{RT_0}\left[1 - \frac{T_1 - T_0}{T_0}\frac{y}{h} \right] \qquad \textbf{(7-133)}$$

hence

$$\frac{du}{dy} = \frac{\tau}{C} \exp\left[\frac{-E_a}{RT_0} \right] \exp\left[\frac{E_a(T_1 - T_0)}{RT_0^2}\frac{y}{h} \right] \qquad \textbf{(7-134)}$$

For notational simplicity we then define the constants

$$\beta = \frac{-E_a}{RT_0}$$
$$\gamma = \frac{E_a(T_1 - T_0)}{hRT_0^2}$$

Hence we get

$$\frac{du}{dy} = \frac{\tau}{C} e^{\beta} e^{\gamma y}$$

The only force acting on the block of ice is the gravitational force, which needs to be balanced by the shear stress, hence (recall problem 2-11) we have that

$$\tau = -\rho g y \sin\alpha$$

And so we find that

$$du = -\frac{\rho g \sin\alpha}{C} e^{\beta} y e^{\gamma y} dy$$

Integration yields

$$\begin{aligned} u &= -\frac{\rho g \sin\alpha}{C} e^{\beta} \int y' e^{\gamma y'} dy' \\ &= -\frac{\rho g \sin\alpha}{C\gamma} e^{\beta} y e^{\gamma y} + \frac{\rho g \sin\alpha}{C\gamma} e^{\beta} \int e^{\gamma y'} dy' \\ &= \frac{\rho g \sin\alpha}{C\gamma} e^{\beta} e^{\gamma y} \left(\frac{1}{\gamma} - y \right) + D \end{aligned}$$

Utilizing the boundary condition $u = 0$ at $y = h$, we find

$$D = \frac{\rho g \sin\alpha}{C\gamma} e^{\beta} e^{\gamma h} \left(h - \frac{1}{\gamma} \right)$$

hence the velocity profile becomes

$$u = \frac{\rho g \sin\alpha}{C\gamma} e^{\beta} \left[\frac{1}{\gamma} \left(e^{\gamma y} - e^{\gamma h} \right) + h e^{\gamma h} - y e^{\gamma y} \right]$$

or equivalently

$$u = \frac{\rho g h R T_0^2 \sin\alpha}{C E_a (T_1 - T_0)} e^{\frac{-E_a}{RT_0}} \left[\frac{h R T_0^2}{E_a (T_1 - T_0)} \left(e^{\frac{E_a (T_1 - T_0)}{h R T_0^2} y} - e^{\frac{E_a (T_1 - T_0)}{R T_0^2}} \right) + h e^{\frac{E_a (T_1 - T_0)}{R T_0^2}} - y e^{\frac{E_a (T_1 - T_0)}{h R T_0^2} y} \right]$$

7-17 The way which subsolidus convection with temperature-dependent viscosity regulates the Earth's thermal history can be quantatively assessed using the following simple model. Assume that the Earth can be characterized by the mean temperature $\bar{T}$ and that Equation (7-200) gives the rate of cooling. Let the model Earth begin its thermal evolution at time $t = 0$ with a high temperature $\bar{T}(0)$ and cool there after. Disregard the heating due to the decay of radioactive isotopes and assume that the Earth cools by vigorous convection. Show that the mean surface heat flux $\bar{q}$ can be related to the mean temperature by

$$\bar{q} = 0.74k\left(\frac{\rho g \alpha_v}{\mu\kappa}\right)^{1/3}\left(\bar{T} - T_o\right)^{4/3} \quad \textbf{(7-213)}$$

Use Equations (6-316) and (6-337) and assume that the total temperature drop driving convection is twice the difference between the mean temperature $\bar{T}$ and the surface temperature T_o. Following Equation (7-100), assume that the viscosity is given by

$$\mu = C\bar{T}\exp\left(\frac{E_a}{R\bar{T}}\right) \quad \textbf{(7-214)}$$

and write the cooling formula as

$$\frac{d\bar{T}}{dt} = -\frac{2.2\kappa}{a}\left(\frac{\rho g \alpha_v}{C\kappa}\right)^{1/3}\bar{T}\exp\left(-\frac{E_a}{3R\bar{T}}\right) \quad \textbf{(7-215)}$$

Equation (7-215) was obtained assuming $(\bar{T} - T_o)^{4/3} \simeq \bar{T}^{4/3}$, a valid simplification since $T_o \ll \bar{T}$. Integrate the cooling formula and show that

$$\text{Ei}\left(\frac{E_a}{3R\bar{T}}\right) - \text{Ei}\left(\frac{E_a}{3R\bar{T}(0)}\right) = \frac{2.2\kappa}{a}\left(\frac{\rho g \alpha_v}{C\kappa}\right)^{1/3} t \quad \textbf{(7-216)}$$

Where Ei is the exponential integral. Calculate and plot $\bar{T}/\bar{T}(0)$ versus t for representative values of the parameters in Equation (7-216). Discuss the role of the temperature dependence in the cooling history

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Let us start by repeating the assumptions of the model

1. The Earth can be characterized by its mean temperature $\bar{T}$
2. The rate of cooling is given by (a is the mean Earth radius)

$$\frac{d\bar{T}}{dt} = -\frac{3\bar{q}}{a\bar{\rho}\bar{c}_p} \quad \textbf{(7-200)}$$

3. We can neglect internal heating, hence the Rayleigh number is given by (where b is the size of the convection cell)

$$\text{Ra} = \frac{\rho_o g \alpha_v (T_1 - T_0) b^3}{\mu\kappa} \quad \textbf{(6-136)}$$

   where we interpret the reference density ρ_o as the mean density $\bar{\rho}$

4. Convection within the Earth is vigorous, hence the mean heat flow is given by

$$\bar{q} = 0.120\frac{k(T_1 - T_0)}{b}\text{Ra}^{1/3} \quad \textbf{(6-337)}$$

5. The temperature drop driving convection is

$$(T_1 - T_0) = 2(\bar{T} - T_0)$$

6. The viscosity is given by

$$\mu = C\bar{T}\exp\left(\frac{E_a}{R\bar{T}}\right) \qquad \textbf{(7-214)}$$

7. $T_o \ll \bar{T}$

Combining assumptions 3, 4, and 5, we find that we can write the mean heat flow as

$$\begin{aligned}\bar{q} &= 0.120\frac{k(T_1-T_0)}{b}\left(\frac{\rho_o g\alpha_v(T_1-T_0)b^3}{\mu\kappa}\right)^{1/3}\\ &= 0.30k\left(\frac{\rho g\alpha_v}{\mu\kappa}\right)^{1/3}\left(\bar{T}-T_o\right)^{4/3}\end{aligned} \qquad \textbf{(7-213)}$$

including assumptions 6 and 7 into the expression for the mean heat flow we arrive at

$$\bar{q} = 0.30k\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}\bar{T}\exp\left(-\frac{E_a}{3R\bar{T}}\right)$$

using assumption 2 and remembering that $\kappa = \frac{k}{\rho c_p}$ we then find that we can express the temporal evolution of the mean temperature as

$$\frac{d\bar{T}}{dt} = -\frac{0.9\kappa}{a}\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}\bar{T}\exp\left(-\frac{E_a}{3R\bar{T}}\right) \qquad \textbf{(7-215)}$$

Which can be re-arranged to yield

$$-\frac{1}{\bar{T}}\exp\left(\frac{E_a}{3R\bar{T}}\right)d\bar{T} = \frac{0.9\kappa}{a}\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}dt$$

And so

$$-\int_{\bar{T}(0)}^{\bar{T}(t)}\frac{1}{\bar{T}'}\exp\left(\frac{E_a}{3R\bar{T}}\right)d\bar{T}' = \frac{0.9\kappa}{a}\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}\int_0^t dt = \frac{0.9\kappa}{a}\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}t$$

If we then make the variable substitution

$$\begin{aligned} & x = -\frac{E_a}{3R\bar{T}}\\ \Rightarrow\quad & \frac{1}{\bar{T}} = -\frac{3R}{E_a}x\\ \Rightarrow\quad & d\bar{T} = \frac{E_a}{3Rx^2}dx\end{aligned}$$

Hence we can rewrite the left hand side as

$$\begin{aligned} & -\int_{\bar{T}(0)}^{\bar{T}(t)}\frac{1}{\bar{T}'}\exp\left(\frac{E_a}{3R\bar{T}}\right)d\bar{T}'\\ &= \int_{-\frac{E_a}{3R\bar{T}(0)}}^{-\frac{E_a}{3R\bar{T}(t)}}\frac{1}{x}e^{-x}dx\\ &= \int_{-\frac{E_a}{3R\bar{T}(0)}}^{\infty}\frac{1}{x}e^{-x}dx + \int_{\infty}^{-\frac{E_a}{3R\bar{T}(t)}}\frac{1}{x}e^{-x}dx\\ &= \int_{-\frac{E_a}{3R\bar{T}(0)}}^{\infty}\frac{1}{x}e^{-x}dx - \int_{-\frac{E_a}{3R\bar{T}(t)}}^{\infty}\frac{1}{x}e^{-x}dx\\ &= -\mathrm{Ei}\left(\frac{E_a}{3R\bar{T}(0)}\right) + \mathrm{Ei}\left(\frac{E_a}{3R\bar{T}(t)}\right)\end{aligned}$$

And so we arrive at the final Equation

$$\mathrm{Ei}\left(\frac{E_a}{3R\bar{T}(t)}\right) - \mathrm{Ei}\left(\frac{E_a}{3R\bar{T}(0)}\right) = \frac{0.9\kappa}{a}\left(\frac{\rho g\alpha_v}{C\kappa}\right)^{1/3}t \qquad \textbf{(7-216)}$$

7-24 Consider a long circular cylinder of elastic-perfectly plastic material that is subjected to a torque at its outer surface $r = a$. The state of stress in the cylinder can be characterized by an azimuthal shear stress τ. Determine the torque for which an elastic core of radius c remains. Assume that the yield stress in shear is σ_o. In the elastic region the shear stress is proportional to the distance from the axis of the cylinder r. What is the torque for the onset of plastic yielding? What is the maximum torque that can be sustained by the cylinder?

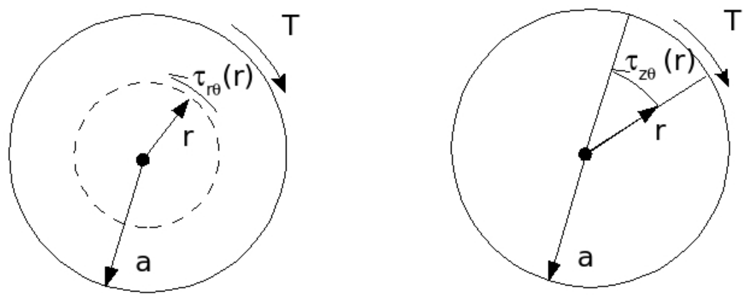

Figure 25

This problem is unfortunately a bit vague formulated as it stands, hence before we do anything we need to figure out what is red and what is green. To start with we can come to think of two types of azimuthal torques

- If the cylinder does not experience any torsion an azimuthal torque $\tau_{r\theta}(r)$ will act on any concentric plane around the center of the cylinder (see left panel of figure 25). However this leads to practical issues on how fixate the center of the cylinder, especially a long cylinder. But even more serious as the applied torque on every concentric plane needs to be balanced by the shear stress times the area the shear stress acts on, we find that the shear stress has to be $\propto 1/r$, hence it increases towards the center of the cylinder, thus any yielding will occur from the center and outwards, and not from the surface and inwards as asked for. Finally the shear stress is clearly not proportional to the distance from the center, but inversely proportional. Hence we conclude that this is not the situation considered
- In the second scenario we have torsion and the azimuthal torque $\tau_{z\theta}(r)$ acts on any cross sectional plane perpendicular to the cylinder axis (see right panel of figure 25). In this scenario we really do not have any problems fixating one end of the cylinder and apply a torque at the other, we can also make the assumption that the shear stress is proportional to the distance from the center of the cylinder. Hence this is what we want.

The torque acting on the cylinder can then be expressed as

$$T = 2\pi \int_0^a \tau(r) r dr = 2\pi \int_0^c \tau_{elastic} r dr + 2\pi \int_c^a \tau_{plastic} r dr$$

Since we know that the cylinder yields at a shear stress σ_o and that in the elastic region the shear stress is proportional to the distance from the center of the cylinder, assuming it to be zero at the center we can write the elastic shear stress as

$$\tau_{elastic} = \sigma_0 \frac{r}{c}$$

In the plastic region, since we are assuming perfect plasticity, the shear stress will be constant, i.e.

$$\tau_{plastic} = \sigma_0$$

And so we get

$$
\begin{aligned}
T &= 2\pi \int_0^c \sigma_0 \frac{r}{c} r dr + 2\pi \int_c^a \sigma_0 r dr \\
&= \frac{2\pi\sigma_o}{3} c^2 + \pi\sigma_o \left(a^2 - c^2\right) \\
&= \frac{\pi\sigma_o}{3} \left(3a^2 - c^2\right)
\end{aligned}
$$

Hence the torque at onset of yielding is, $c = a$

$$T = \frac{2\pi\sigma_o}{3} a^2$$

And the maximum torque that can be sustained by the the cylinder is, $c = 0$

$$T = \pi\sigma_o a^2$$

8-3 A number of criteria have been proposed to relate the brittle fracture of rock to the state of stress. One of these is the Coulomb-Navier criterion, which states that failure occurs on a plane when the shear stress τ attains the value

$$|\tau| = S + \mu\sigma_n \qquad \textbf{(8-37)}$$

where S is the inherent shear strength of the rock and μ is the coefficient of internal friction. Consider a two-dimensional state of stress with principal stresses σ_1 and σ_2 and show that $|\tau| - \mu\sigma_n$ has a maximum value for a plane whose normal makes an angle θ to the larger principal stress given by

$$\tan 2\theta = \frac{-1}{\mu} \qquad \textbf{(8-38)}$$

Show also that the quantity $|\tau| - \mu\sigma_n$ for this plane is

$$|\tau| - \mu\sigma_n = \frac{1}{2}(\sigma_1 - \sigma_2)(1+\mu^2)^{1/2} - \frac{1}{2}(\sigma_1+\sigma_2)\mu \qquad \textbf{(8-39)}$$

According to the Coulomb-Navier criterion, failure will occur if this quantity equals S, that is the failure criterion takes the form

$$\sigma_1\left\{[1+\mu^2] - \mu\right\} - \sigma_2\left\{[1+\mu^2]+\mu\right\} = 2S \qquad \textbf{(8-40)}$$

What is the compressive strength of the rock in terms of μ and S? From Equation (8-38) it is seen that θ must exceed 45^o, so that the direction of shear fracture makes an acute angle with σ_1. The Coulomb-Navier criterion is found to be reasonably valid for igneous rocks under compression.

For a 2D state of stress given σ_1 and σ_2 we have that

$$|\tau| = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta \qquad \textbf{(2-54,8-28)}$$

$$\sigma_n = \frac{1}{2}(\sigma_1+\sigma_2) + \frac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta \qquad \textbf{(2-53,8-27)}$$

And so we have that

$$|\tau| - \mu\sigma_n = \frac{1}{2}(\sigma_1-\sigma_2)(\sin 2\theta - \mu\cos 2\theta) - \frac{\mu}{2}(\sigma_1+\sigma_2)$$

To find the plane on which $|\tau| - \mu\sigma_n$ has a maximum value we then take the derivative with respect to θ and investigates its zero-points, i.e.

$$\frac{d(|\tau| - \mu\sigma_n)}{d\theta} = (\sigma_1+\sigma_2)(\cos 2\theta + \mu\sin 2\theta) = 0$$

$$\Rightarrow \quad \cos 2\theta + \mu \sin 2\theta = 0$$

$$\Rightarrow \quad \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = -\frac{1}{\mu} \qquad \textbf{(8-38)}$$

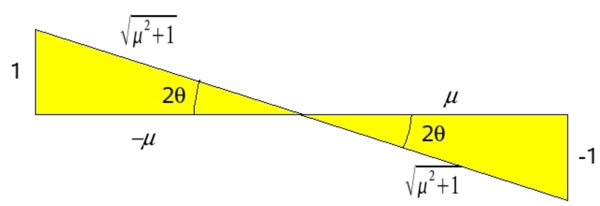

Figure 26

We then want to find the value of cosine and sine of 2θ to plug into our expression for $|\tau| - \mu\sigma_n$. Consider then the geometry displayed in figure 26, simple trigonometry gives that

$$\tan 2\theta = -\frac{1}{\mu}$$
$$\cos 2\theta = \mp\frac{\mu}{\sqrt{\mu^2+1}}$$
$$\sin 2\theta = \pm\frac{1}{\sqrt{\mu^2+1}}$$

Hence by choosing the left triangle (corresponding to the upper sign on the cosine and sine) we find that

$$\sin 2\theta - \mu\cos 2\theta = \frac{1}{\sqrt{\mu^2+1}} + \frac{\mu^2}{\sqrt{\mu^2+1}} = \frac{1+\mu^2}{\sqrt{\mu^2+1}} = \sqrt{\mu^2+1}$$

(N.B. choosing the right triangle would have resulted in a minimum of equal magnitude.) And so we find that

$$|\tau| - \mu\sigma_n = \frac{1}{2}\left(\sigma_1 - \sigma_2\right)\left(1+\mu^2\right)^{1/2} - \frac{\mu}{2}\left(\sigma_1 + \sigma_2\right) \qquad \textbf{(8-39)}$$

Equating Equation (8-39) with S and rearranging we find the compressive strength to be

$$\left(\sigma_1 - \sigma_2\right) = \frac{\mu\left(\sigma_1+\sigma_2\right) + 2S}{\sqrt{\mu^2+1}}$$

8-6 The spring force on the slider in Figure 8-18 at the time of slip initiation is $f_s F_n$. What is the spring force on the slider block at the end of slip?

The geometry of the problem is displayed in figure 27

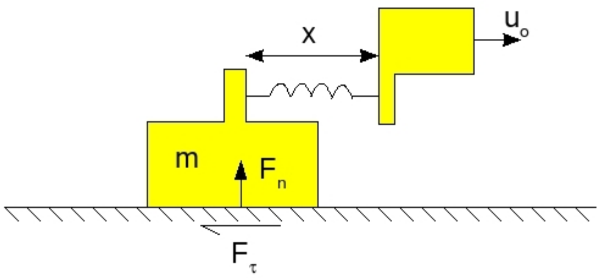

Figure 27: Equivalent to Figure 8-18 of Turcotte and Schubert

Once the block is sliding the Equation of motion for the block is

$$m\frac{d^2x}{dt^2} + kx = f_d F_n \tag{8-59}$$

which has the solution

$$x(t) = A\cos\left(\sqrt{k/m}t\right) + \frac{f_d F_n}{k}$$

Let the length of the spring at the time of slip initiation ($t = 0$) be $\tilde{x}$, then we have

$$\begin{aligned}
& k\tilde{x} = F = f_s Fn \\
\Rightarrow \quad & \tilde{x} = \frac{f_s Fn}{k} = x(0) = A + \frac{f_d F_n}{k} \\
\Rightarrow \quad & A = \frac{F_n}{k}\left(f_s - f_d\right) \\
\Rightarrow \quad & x(t) = \frac{F_n}{k}\left(f_s - f_d\right)\cos\left(\sqrt{k/m}t\right) + \frac{f_d F_n}{k}
\end{aligned}$$

When the slider stops $\frac{dx}{dt} = 0$ (i.e. no extension of the spring), hence

$$\begin{aligned}
& \frac{dx}{dt} = -\frac{F_n}{k}\left(f_s - f_d\right)\sqrt{k/m}\sin\left(\sqrt{k/m}t\right) = 0 \\
\Rightarrow \quad & \sin\left(\sqrt{k/m}t\right) = 0 \\
\Rightarrow \quad & \sqrt{k/m}t = n\pi; \qquad n \in \mathbb{Z}
\end{aligned}$$

Using n = 1 (i.e. first stop of the slider) we find

$$t = \pi\sqrt{m/k}$$

By that time the length of the spring is

$$\begin{aligned}
x(\pi\sqrt{m/k}) &= \frac{F_n}{k}\left(f_s - f_d\right)\cos\left(\pi\right) + \frac{f_d F_n}{k} \\
&= -\frac{F_n}{k}\left(f_s - f_d\right) + \frac{f_d F_n}{k} \\
&= \frac{F_n}{k}\left(2f_d - f_s\right)
\end{aligned}$$

Hence the spring force on the slider at the end of slip is

$$F_{spring} = kx = F_n \left(2f_d - f_s\right)$$

8-8 Compute the wave energy released in a magnitude 8.5 earthquake and compare it with the amount of heat lost through the surface of the Earth in an entire year.

Assuming thatthe magnitude is given as moment magnitude (which it probably is given the size), the energy-magnitude relation is

$$E_s = 10^{1.5m+4.8} \qquad \textbf{(8-75)}$$

Hence the released energy amounts to

$$E_s = 10^{1.5 \cdot 8.5+4.8} \simeq 3.55 \times 10^{17} \text{ J}$$

The heat flow through the surface of the Earth is estimated to about Q = 4.43 x 10^{13} W ($\sim$ 87 mW m^{-2}, Turcotte and Schubert p.136), hence

$$E_{heat,year} = 4.43 \times 10^{13} \cdot 365.25 \cdot 24 \cdot 3600 \simeq 1.4 \times 10^{21} \text{ J}$$

Thus

$$\frac{E_s}{E_{heat,year}} = \frac{3.55 \times 10^{17}}{1.4 \times 10^{21}} \simeq 2.54 \times 10^{-4}$$

I.e. the energy liberated by earthquakes is very small in comparison.