Generating Locally Cyclic Triangulations of Surfaces

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A locally cyclic graph is a connected graph such that for each vertex the induced subgraph on the set of its adjacent vertices is isomorphic to a cycle. These graphs correspond uniquely to locally cyclic triangulations of closed surfaces, i.e., triangulations where each cycle of length three in the underlying graph is facial. For each closed surface Σ, all locally cyclic triangulations of Σ can be obtained from a minimal basic set $\mathscr{B}(\Sigma)$ by applying the vertex-splitting operation. The main result proves that for an arbitrary closed orientable surface Σ, $\mathscr{B}(\Sigma)$ is finite. An application to the study of closed 2-cell embeddings of graphs in surfaces related to the double cycle cover conjecture is presented. © 1992 Academic Press, Inc.

1. INTRODUCTION

A connected graph G is *locally cyclic* if, for each vertex $v \in V(G)$, the induced subgraph $N(v, G)$ on the neighbours of v is a cycle. Note that a locally cyclic graph is not necessarily regular. By gluing a triangular face to each triangle (cycle of length three) of a locally cyclic graph G, a triangulated closed surface $\Sigma = \Sigma(G)$ is obtained. On the other hand, triangulations of closed surfaces where each cycle of length three (in the 1-skeleton of the triangulation) is facial correspond uniquely to locally cyclic graphs. We therefore call these triangulations *locally cyclic*. For any given closed surface Σ many such triangulation of Σ exist. For example, if K is a cell decomposition of Σ such that the closed faces are cells, then the barycentric subdivision of K is a locally cyclic triangulation of Σ. Triangulations of this kind were investigated by some authors in different contexts. For example, Hartsfield and Ringel [5] call them *clean*, and they were concerned with finding vertex-minimal such triangulations for each given closed surface; cf. also [1].

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For every given closed surface Σ, all locally cyclic graphs which triangulate Σ can be obtained from some minimal basic set of graphs by applying vertex-splitting operations (a vertex-splitting is inverse to an edge-contraction). The main result of this paper states that for each closed orientable surface its minimal basis is finite. It should be noted that this result can be applied in the study of the cycle double cover conjecture.

The paper is organized as follows. In Section 2 a quick overview of locally cyclic graphs is given. The main result, Theorem 2.3, is also included here, while its proof is deferred until Section 4. Section 3 contains some results about homotopy of curves on surfaces to be used in the proof of our main theorem. The last section contains an application of our main result to the double cycle cover conjecture. It provides a simple proof that the class of minor minimal closed 2-cell embeddings of graphs in any fixed closed orientable surface is finite.

2. LOCALLY CYCLIC TRIANGULATIONS

Let us review a few simple properties of locally cyclic graphs (triangulations). The proofs either are simple, or else can be found in [6]. See also [8]. First of all, it can be shown that non-trivial subgraphs of locally cyclic graphs cannot be locally cyclic. It is also easy to see that every locally cyclic graph, with the exception of K_4, is 4-connected. This implies, for example, that every planar locally cyclic triangulation is Hamiltonian, since by a well-known theorem of Tutte, each 4-connected planar graph admits a Hamiltonian cycle. We conjecture that this is always the case: *every locally cyclic graph is Hamiltonian.*

Let T be a locally cyclic triangulation of a closed surface Σ. The *vertex splitting* is an operation of replacing a vertex $v \in V(T)$ by an edge $e = v'v''$ and at the same time extending two edges in T containing v, to two new triangles as shown on Fig. 1. If T' is obtained from T by a *vertex splitting* which produces two vertices of degree $\geqslant 4$, then T' is also locally cyclic. The operation inverse to vertex-splitting is called *edge-contraction*. If T is obtained from T' by contracting e, we write $T = T'/e$.
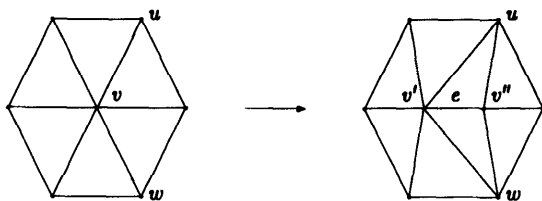

FIG. 1.   Vertex splitting.

An edge e of a locally cyclic triangulation T of Σ is called *contractible* if T/e is also locally cyclic. It may happen that no edge of T is contractible. In this case T is said to be an *irreducible* (or *contraction-minimal*) locally cyclic triangulation. Since every locally cyclic triangulation can be generated from irreducible ones by a sequence of vertex-splittings, the irreducible locally cyclic triangulations play a very important role. In [6] planar irreducible triangulations are determined:

THEOREM 2.1.   *The only irreducible locally cyclic triangulations of S^2 (the 2-sphere) are K_4 (the tetrahedron) and O_3 (the octahedron).*

Fisk *et al.* [3] determined all irreducible locally cyclic triangulations of the projective plane. It is easily seen that in a non-planar irreducible locally cyclic triangulation, every edge e lies on a 4-cycle whose underlying curve either is non-contractible (i.e., homotopically non-trivial), or else bounds a disc with at least one vertex in its interior. But a stronger result holds [6].

THEOREM 2.2.   *Let T be a locally cyclic triangulation of a non-planar surface Σ. Then T is irreducible if and only if each edge $e \in E(T)$ lies on a 4-cycle in T whose underlying curve belongs to a non-trivial homotopy class on Σ.*

The main result of this paper states that for each orientable surface Σ, the irreducible locally cyclic triangulations of Σ are bounded in size (see Section 4 for explicit bounds) which implies

THEOREM 2.3 (Main Theorem).   *For each orientable surface Σ there are only finitely many irreducible locally cyclic triangulations of Σ.*

3. CURVES ON SURFACES

Let Σ be a compact surface. A closed *curve* on Σ is a continuous mapping $\gamma : S^1 \to \Sigma$, where S^1 denotes the 1-dimensional sphere. The curve is *simple* if it is a 1–1 mapping. A curve γ will usually be identified with its image $\gamma(S^1)$ on Σ, particularly when considering topological properties: simple closed curves on Σ correspond to subsets of Σ, homeomorphic to the 1-sphere. If G is a graph embedded in Σ then any (graph) cycle may also be viewed as a simple closed curve on Σ.

Every simple closed curve $\gamma(S^1)$ in the interior of Σ has a *regular neighbourhood* which is either homeomorphic to a strip (cylinder), or a Möbius band. In the first case $\gamma(S^1)$ is *two-sided*, in the latter it is *one-sided*. Two-sided simple closed curves are either *bounding* (i.e., $\Sigma \backslash \gamma(S^1)$ has two

connected components) or *non-bounding* ($\Sigma \setminus \gamma(S^1)$ is connected). One-sided simple closed curves are always non-bounding.

In this section we establish some results about homotopy of simple closed curves on surfaces. These results will be used later in the proof of our Main Theorem. It will be assumed that the reader is familiar with elementary homotopy theory (cf., for example, [7]). Recall that closed curves γ_0, $\gamma_1 : S^1 \to \Sigma$ are *homotopic* if there is a continuous mapping $H : S^1 \times [0, 1] \to \Sigma$ such that $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ for each $s \in S^1$. The mapping H itself is called a *homotopy* between γ_0 and γ_1. If for some $s_0 \in S^1$, $\gamma_0(s_0) = \gamma_1(s_0) = v_0$, and there is a homotopy H between γ_0 and γ_1 such that $H(s_0, t) = v_0$ for all $t \in [0, 1]$ then the two curves are said to be *homotopic relative* to the point v_0. To distinguish these two types of homotopy we sometimes use the name *free homotopy* for the general case, and *homotopy in* $\pi_1(\Sigma, v_0)$ for the case of homotopy relative to v_0. Homotopy gives rise to the equivalence relation, also termed *homotopy*, and the corresponding equivalence classes are called *homotopy classes*. The trivial homotopy class, for instance, is the class of the constant mapping. Simple closed curves are *isotopic* if there is a homotopy H carrying one curve to another, such that for each t, $H(s, t)$ is a simple closed curve, i.e., an embedding of S^1. It is sometimes convenient, for technical reasons, to regard a closed curve $S^1 \to \Sigma$ as a closed path $[0, 1] \to \Sigma$. Homotopy of (closed) paths is defined analogously. Note that two closed curves are homotopic if and only if the corresponding closed paths are homotopic as loops; i.e., for each fixed t, $H(s, t)$ is a closed path.

PROPOSITION 3.1. *Closed curves γ_0 and γ_1 on Σ are free homotopic if and only if there is a path $\tau : [0, 1] \to \Sigma$ with $\tau(0) = v_0 \in \gamma_0(S^1)$ and $\tau(1) \in \gamma_1(S^1)$ such that the concatenated curve $\tau^- \gamma_1 \tau$ is homotopic to γ_0 in $\pi_1(\Sigma, v_0)$.*

Proof. ($\Rightarrow$) Let H be a homotopy between γ_0 and γ_1. Let $\tau(t) := H(s_0, t)$ where $s_0 \in S^1$ is chosen in such a way that $\gamma_0(s_0) = v_0$. By Theorem II.8.2 of [7], the curves γ_0 and $\tau^- \gamma_1 \tau$ are homotopic in $\pi_1(\Sigma, v_0)$.

($\Leftarrow$) First move γ_1 by a (free) homotopy to $\tau^- \gamma_1 \tau$, then use the relative homotopy between $\tau^- \gamma_1 \tau$ and γ_0. ∎

Next we introduce some terminology to be needed later on. Let $\gamma_1, \gamma_2 : S^1 \to \Sigma$ be distinct simple closed curves. Since their images are compact subsets of Σ, the inverse images of their intersection, i.e.,

$$\gamma_1^{-1}(\gamma_1(S^1) \cap \gamma_2(S^1)), \qquad \gamma_2^{-1}(\gamma_1(S^1) \cap \gamma_2(S^1)),$$

are compact, and therefore the connected components of these preimages are homeomorphic to intervals (possibly degenerated to points). We call them γ_1-*intervals of intersection with* γ_2 *and* γ_2-*intervals of intersection with*
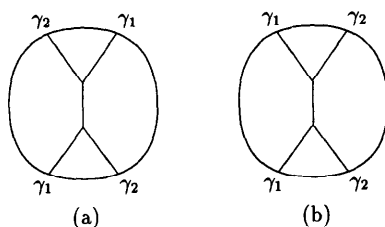
FIG. 2.   Crossing and touching.

γ_1, respectively. To each such interval there corresponds a connected component of the intersection $\gamma_1(S^1) \cap \gamma_2(S^1)$, each being a simple arc (possibly a point). Of course the two kinds of intervals are in a bijective correspondence, but usually there is no "natural" common enumeration of arcs with respect to S^1. The cardinal number of these arcs is termed the *intersection number*, and it may even be uncountable. In the sequel we assume that the intersection number is finite. By an application of Schoenflies' Theorem it can be shown that for two simple closed curves in the interior of Σ each common arc has a disc as its neighbourhood such that $\gamma_1(S^1)$ and $\gamma_2(S^1)$ look locally either as shown in Fig. 2a, or as shown in Fig. 2b. In the former case the curves *cross* along the common arc (at the common point), while in the latter they *touch*. The number of arcs along which the curves cross is called the *crossing number*, while the number of arcs along which the curves touch is the *touching number*. The proof of the following proposition is omitted.

PROPOSITION 3.2. *Let γ and γ' be homotopic simple closed curves on Σ. Then the crossing number is even or odd (respectively), depending on whether the curves are two-sided or one-sided (respectively).*

Let us continue with some homological properties of simple closed curves. The "if part" in the next proposition is obvious, and the "only if" statement can be found, for example, in [2].

PROPOSITION 3.3. *A simple closed curve γ on Σ is homotopic to a point (i.e., contractible) if and only if γ bounds a disc in Σ.*

We shall be mainly interested in non-contractible simple closed curves. Such curves are called *essential*. Now, from a well-known theorem in algebraic topology it follows that homotopic simple closed curves are also homologic, i.e., together they separate the surface. We need to be slightly more precise about this, and let us state two results taken from [2].

PROPOSITION 3.4. *Let γ_0 and γ_1 be simple closed curves on Σ with non-empty intersection. If the touching number of γ_0 and γ_1 is zero and γ_0, γ_1 are*
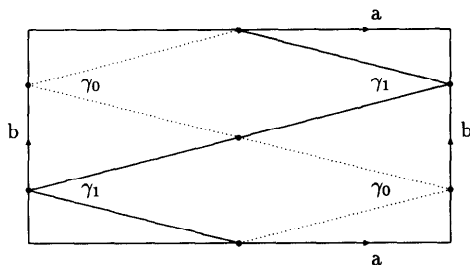
FIG. 3.   There is no disc.

free homotopic, then there is a disc D in Σ whose boundary consists of an arc
of γ_0 and an arc of γ_1, and no point of $\gamma_0(S^1) \cup \gamma_1(S^1)$ lies in the interior
of D.

The assumption that the touching number is 0 is essential. See Fig. 3 for
an example.

PROPOSITION 3.5.   *Let γ_0 and γ_1 be disjoint, free homotopic, two-sided*
simple closed curves on Σ. Then there is a cylinder $S^1 \times [0, 1]$ in Σ whose
boundary $S^1 \times 0$ and $S^1 \times 1$ consists of γ_0 and γ_1, respectively.

Finally, we present two results about families of pairwise non-homotopic
simple closed curves on closed surfaces.

PROPOSITION 3.6.   *Let Γ be a family of essential simple closed curves on*
a closed surface Σ. If $\gamma \cap \gamma' = \{v_0\}$ for all $\gamma, \gamma' \in \Gamma$, and the curves in Γ are
pairwise non-homotopic in $\pi_1(\Sigma, v_0)$, then

$$|\Gamma| \leqslant \begin{cases} 0 & \Sigma \text{ is the 2-sphere} \\ 1 & \Sigma \text{ is the projective plane} \\ 3(1 - \chi(\Sigma)) & \text{otherwise,} \end{cases}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Proof.   For the 2-sphere the statement is obvious since there are no
essential simple closed curves. Similarly for the projective plane: since its
fundamental group is isomorphic to Z_2, any pair of essential curves is
relatively homotopic. Assume now that $\chi(\Sigma) \leqslant 0$.

We bypass the proof that Γ should be finite by establishing the stated
upper bound for an arbitrary finite family Γ which satisfies the
requirements. Now, if Γ is finite, we may cut the surface along Γ to obtain
a family of compact surfaces with boundary, say $\Sigma_1, \Sigma_2, ..., \Sigma_k$. The
boundary $\partial \Sigma_i$ of Σ_i consists of "edges" that arise from curves of Γ.

Denote the number of these "edges" by $e(\Sigma_i)$. We have:

$$\sum_{i=1}^{k} e(\Sigma_i) = 2|\Gamma| \quad \text{and} \quad 3\chi(\Sigma_i) - e(\Sigma_i) \leqslant 0.$$

While the equality is obvious, the inequality is seen as follows. Since $e(\Sigma_i) \geqslant 1$, the inequality is trivial if $\chi(\Sigma_i) \leqslant 0$. Since Σ_i is a compact surface with boundary, the only remaining case to consider is $\chi(\Sigma_i) = 1$, i.e., a disc. Here $e(\Sigma_i) \neq 1$. Otherwise the curve $\gamma \in \Gamma$ corresponding to the boundary of the disc would be contractible. Suppose that $e(\Sigma_i) = 2$. Then the two "edges" cannot arise from two different curves of Γ since curves of Γ are pairwise nonhomotopic in $\pi(\Sigma, v_0)$. Now, if both "edges" arise from a single curve of Γ, then Σ is the projective plane, a case that we excluded already.

From the above relations it follows that

$$3 \sum_{i=1}^{k} \chi(\Sigma_i) - 2|\Gamma| = \sum_{i=1}^{k} (3\chi(\Sigma_i) - e(\Sigma_i)) \leqslant 0.$$

This inequality and the Euler–Poincaré formula

$$1 - |\Gamma| + \sum_{i=1}^{k} \chi(\Sigma_i) = \chi(\Sigma)$$

finally imply the bound of the proposition. ∎

PROPOSITION 3.7. *Let Γ be a family of pairwise non-homotopic and pairwise disjoint, essential simple closed curves on a closed surface Σ. Then*

$$|\Gamma| \leqslant \begin{cases} 0 & \Sigma \text{ is the 2-sphere} \\ 1 & \Sigma \text{ is the torus or the projective plane} \\ 3(g(\Sigma) - 1) & \text{otherwise,} \end{cases}$$

where $g(\Sigma)$ denotes the genus of Σ.

Proof. The inequality is easily seen to hold if Σ is the 2-sphere, the projective plane, or the torus.

Assume now that Σ is none of the above surfaces, and let Γ be a family of curves as in the proposition. Denote by Γ_1 and Γ_2 the subfamilies of all 1-sided and 2-sided curves of Γ, respectively. We may assume that $\Gamma_2 \neq \varnothing$, and that Γ_1 is maximal in the following sense: no 1-sided simple close curve on Σ is disjoint from Γ. Also, we may assume that $|\Gamma| > 1$.

As before we bypass the proof that Γ should be finite. This will follow indirectly. We first show that $|\Gamma_1| \leqslant g(\Sigma)$. This is clear if Σ is orientable.

But if Σ is non-orientable, let $\Gamma_1' \subseteq \Gamma_1$ be an arbitrary finite subfamily. Then we may cut the surface Σ along the curves in Γ_1'. The dissected surface is easily seen to be connected. Hence its characteristic is $2 - g(\Sigma)$, and since the dissected surface has exactly $|\Gamma_1'|$ boundary components, the characteristic is $\leqslant 2 - |\Gamma_1'|$. Therefore $|\Gamma_1'| \leqslant g(\Sigma)$ and hence $|\Gamma_1| \leqslant g(\Sigma)$.

We now prove the asserted bound on $|\Gamma|$ in case Γ_2 is finite (again, the obtained bound will show that this must indeed be the case). Now, cut Σ along the curves in Γ_2, and let $\Sigma_1, \Sigma_2, ..., \Sigma_k$ be the obtained surfaces with boundary. They are all compact and with non-empty boundary. Denote by $b(\partial \Sigma_i)$ the number of boundary components of Σ_i. Each curve of Γ_2 gives rise to two boundary components. Therefore

$$\sum_{i=1}^{k} b(\partial \Sigma_i) = 2 |\Gamma_2|.$$

Consider the closed surfaces $\Sigma_1^*, \Sigma_2^*, ..., \Sigma_k^*$ obtained by gluing discs to the boundary components of Σ_i. Then their Euler characteristics are related by $\chi(\Sigma_i) = \chi(\Sigma_i^*) - b(\partial \Sigma_i)$. Denote by n_s and n_p the number of 2-spheres and projective planes (respectively) among the obtained surfaces Σ_i^*, $1 \leqslant i \leqslant k$. Then

$$\sum_{i=1}^{k} \chi(\Sigma_i^*) \leqslant 2n_s + n_p.$$

Since we have cut the surface along disjoint simple closed curves, $\chi(\Sigma)$ is equal to the sum of all $\chi(\Sigma_i)$. Therefore

$$\chi(\Sigma) = \sum_{i=1}^{k} \chi(\Sigma_i) = \sum_{i=1}^{k} \chi(\Sigma_i^*) - \sum_{i=1}^{k} b(\partial \Sigma_i) \leqslant 2n_s + n_p - 2 |\Gamma_2|.$$

It remains to find upper bounds for n_s and n_p. Let us consider the number n_p first. Note that each curve from Γ_1 is completely contained in some Σ_i and hence in Σ_i^*. Since Γ_1 is maximal, each projective plane among Σ_i^*'s contains (exactly one) curve from Γ_1. Hence, $n_p \leqslant |\Gamma_1|$.

Consider now n_s. If Σ_i^* is 2-sphere we claim that $b(\Sigma_i) \geqslant 3$. Indeed, if $b(\Sigma_i) = 1$, then Σ_i is a disc, and the curve that gives rise to its boundary is contractible . If $b(\Sigma_i) = 2$, then Σ_i is a cylinder. If its boundary components come from two different curves of Γ_2 then these two curves are necessarily homotopic. Therefore the boundary components of the cylinder both come from the same curve of Γ_2. But then we know what the surface Σ itself should be: either the torus or the Klein bottle, and moreover, Γ consists of just one curve, which was excluded above. The claim is thus proved. By counting the number of boundary components of all the surfaces Σ_i we have $3n_s + n_p \leqslant 2 |\Gamma_2|$, or

$$n_s \leqslant \tfrac{2}{3} |\Gamma_2| - \tfrac{1}{3} n_p.$$

Finally, since $|\Gamma| = |\Gamma_1| + |\Gamma_2|$, we have the following bound for the Euler characteristic of Σ:

$$\chi(\Sigma) \leqslant 2(\tfrac{2}{3}|\Gamma_2| - \tfrac{1}{3}n_p) + n_p - 2|\Gamma_2| = -\tfrac{2}{3}|\Gamma_2| + \tfrac{1}{3}n_p \leqslant -\tfrac{2}{3}|\Gamma| + |\Gamma_1|.$$

Let Σ be orientable. Then $|\Gamma_1| = 0$ and $\chi(\Sigma) = 2 - 2g(\Sigma)$. Together with the above inequality this implies the desired result. Similarly in the non-orientable case. Here $|\Gamma_1| \leqslant g(\Sigma)$ and $\chi(\Sigma) = 2 - g(\Sigma)$. The proof of the proposition is now complete. ∎

4. Proof of the Main Theorem

Throughout this section, Σ will always be an orientable closed surface. Because of Theorem 2.1 we also assume that Σ is not the 2-sphere. Now the strategy in proving the theorem is the following. From Theorem 2.2 we know that each edge of an irreducible locally cyclic triangulation T lies on some essential 4-cycle. By counting pairwise disjoint and non-homotopic essential 4-cycles of T and using Proposition 3.7, an upper bound on $E(T)$ in terms of the genus $g(\Sigma)$ and the maximal vertex degree $\Delta(T)$ is found. An upper bound for $\Delta(T)$ is then obtained by considering essential 4-cycles passing through a vertex of large degree. The proof is split into a series of lemmas.

First, we introduce some terminology. Let T be a locally cyclic triangulation of Σ and Γ a class (usually a homotopy class) of closed curves on Σ. Denote by $Q(\Gamma)$ the set of all 4-cycles of T that belong to Γ. An edge $e \in E(T)$ is Γ-essential if it is contained in a 4-cycle from $Q(\Gamma)$. For instance, Theorem 2.2 says that in an irreducible triangulation T, each edge is Γ-essential for some non-trivial homotopy class Γ. For a fixed vertex $v \in V(T)$ let $Q(\Gamma, v) \subseteq Q(\Gamma)$ be the subset of those 4-cycles in $Q(\Gamma)$ which contain v. Note that for a free homotopy class Γ 4-cycles from $Q(\Gamma, v)$ need not be homotopic in $\pi_1(\Sigma, v)$.

LEMMA 4.1. *Let T be an irreducible locally cyclic triangulation of Σ and let Γ be a nontrivial free homotopy class. Then no five 4-cycles in $Q(\Gamma)$ can be pairwise disjoint.*

Proof. Suppose a family of five pairwise disjoint 4-cycles Q_1, Q_1, Q_3, Q_4, Q_5 from $Q(\Gamma)$ exists. According to Proposition 3.5, each pair Q_i, Q_j bounds a cylinder $A_{i,j}$ in Σ. We may assume that the cylinder $A_{1,5}$ contains Q_2, Q_3, Q_4 and that $A_{1,5} = A_{1,2} \cup A_{2,3} \cup A_{3,4} \cup A_{4,5}$, where $A_{i-1,i} \cap A_{i,i+1} = Q_i$. If necessary, the 4-cycles are re-enumerated. This is obvious for the torus. In all other cases, the cylinder of Proposition 3.5 is

unique. So we may choose the one containing the maximal number of cycles Q_i. Re-enumerate the cycles so that this maximal cylinder is $A_{1,5}$. Then this maximal cylinder contains all other 4-cycles, and the proof of the claim is straightforward.

Let $v \in V(T)$ be a vertex of Q_3. Note that any 4-cycle at v is completely contained in the cylinder $A_{1,5}$, and so, if it is essential, it belongs to $Q(\Gamma)$. Therefore, all essential 4-cycles at v belong to the same homotopy class. We now prove that this is impossible.

Choose a 4-cycle $Q = va_1 x a_2 \in Q(\Gamma, v)$ such that the distance between a_1 and a_2, measured on the cycle $N(v, T)$, is as small as possible. There is a vertex u_1 on $N(v, T)$ adjacent to a_1 which is closer to a_2 than a_1 is. Then $u_1 \neq a_2$. (Otherwise the triangle $a_1 a_2 x$ would not be facial.) Since T is irreducible, the edge $u_1 v$ belongs to some essential 4-cycle $Q' = u_1 v u_2 y \in Q(\Gamma, v)$. See Fig. 4. By minimality, the vertex u_2 is not on the arc $a_1 - u_1 - a_2$ of $N(v, T)$. Therefore, Q and Q' cross at v. By Proposition 3.2 $x = y$. But then the 4-cycle $vu_1 x a_2$ is essential (since $va_1 x a_2$ is essential and $va_1 x u_1$ is not), and this contradicts the minimality of $a_1 - a_2$. ∎

LEMMA 4.2. *Let T be an irreducible locally cyclic triangulation of Σ. If Δ is the maximal vertex degree of T, then*

$$|E(T)| \leqslant \begin{cases} 16\Delta^2 & g(\Sigma) = 1 \\ 48\Delta^2(g(\Sigma) - 1) & otherwise. \end{cases}$$

Proof. For each edge $e \in E(T)$ choose an essential 4-cycle Q_e such that $e \in Q_e$. Consider the set of pairs

$$\mathcal{Q} := \{(e, Q_e) \mid e \in E(T)\}.$$

Clearly, $|\mathcal{Q}| = |E(T)|$. Note that possibly $Q_e = Q_f$ for different edges e and f.

Let $\mathcal{Q}_v \subseteq \mathcal{Q}$ be the subset of pairs (e, Q_e), where Q_e contains a fixed vertex $v \in V(T)$. If (e, Q_e) is in this subset, then e is incident either with v, or e is
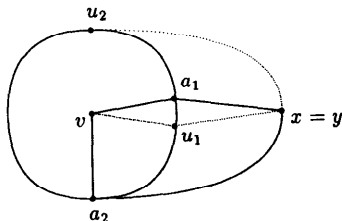

FIG. 4. A single homotopy class is impossible.

incident with a neighbour of v. There are at most $\deg(v) \leqslant \Delta$ elements of the first kind, and at most $\Delta(\Delta - 1)$ elements of the latter kind. Thus $|\mathcal{Q}_v| \leqslant \Delta + \Delta(\Delta - 1) = \Delta^2$.

Fix a pair (e, Q_e) and let $Q_e = v_1 v_2 v_3 v_4$. Then we have

$$|\{(f, Q_f) \mid Q_f \cap Q_e \neq \varnothing\}| \leqslant 1 + \sum_{i=1}^{4} (|\mathcal{Q}_{v_i}| - 1) < 4\Delta^2.$$

It follows that in T there are at least $|\mathcal{Q}|/(4\Delta^2) = |E(T)|/(4\Delta^2)$ pairwise disjoint essential 4-cycles. By Lemma 4.1, one can find at least $|E(T)|/(16\Delta^2)$ pairwise disjoint and pairwise non-homotopic essential 4-cycles of T. Finally, our lemma follows by using Proposition 3.7. ∎

In the next five lemmas we establish an upper bound for the maximal vertex degree Δ. Note that in the next two lemmas, T is not required to be irreducible.

LEMMA 4.3. *Let T be a locally cyclic triangulation of Σ and $\Gamma \in \pi(\Sigma, v)$, $v \in V(T)$, a non-trivial relative homotopy class. For an arbitrary pair of 4-cycles $Q_1, Q_2 \in Q(\Gamma, v)$ there exists a pair of 4-cycles $Q_1', Q_2' \in Q(\Gamma, v)$, on the same set of edges as Q_1 and Q_2, such that Q_1' and Q_2' do not cross.*

Proof. Figure 5 shows all possible cases for the intersection of two (free) homotopic essential 4-cycles. In the first three cases the intersection number is 1. By Proposition 3.2 the two curves must touch, so we may take
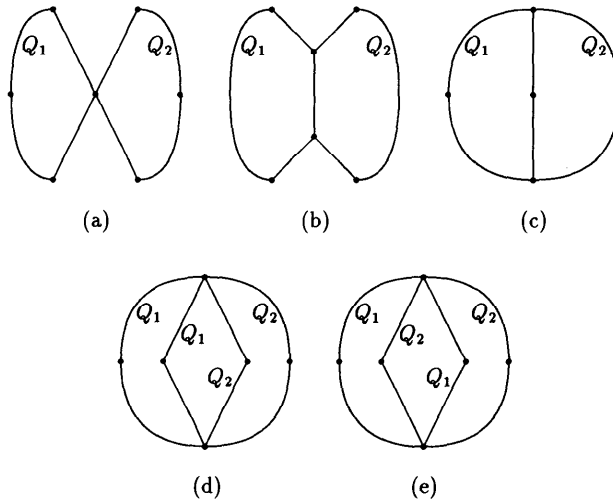

FIG. 5. Intersections of homotopic 4-cycles.

$Q'_1 := Q_1$ and $Q'_2 := Q_2$. In the last two cases the intersection number is 2. Here we have either two touchings or two crossings. In the latter case we make use of Proposition 3.4. There exists a disc D, whose boundary consists of an arc $\alpha \in Q_1$ and an arc $\beta \in Q_2$, such that no point of $Q_1 \cup Q_2$ lies in the interior of D. Now make a homotopic switch of these two arcs across the interior of D, keeping the end-points fixed. Since v belongs to the intersection, the relative homotopy is preserved. The required 4-cycles are $Q'_1 := Q_1 \backslash \alpha \cup \beta$ and $Q'_2 := Q_2 \backslash \beta \cup \alpha$. ∎

We now consider separation properties of a pair of homotopic 4-cycles at v. If they cross, there is at least one disc "in between." This is guaranteed by Proposition 3.4. But we need a slightly more specific result.

LEMMA 4.4. *Let T be a locally cyclic triangulation of Σ and $\Gamma \in \pi_1(\Sigma, v)$, $v \in V(T)$, a nontrivial relative homotopy class. If a pair of 4-cycles $Q_1, Q_2 \in Q(\Gamma, v)$ has crossing number 0, then either Q_1 and Q_2 bound an (open) disc (if the intersection number is 1), or Q_1 and Q_2 bound two discs (if the intersection number is 2). Each disc is bounded by an arc of Q_1 and an arc of Q_2, and none of the above disc(s) contains in its interior a point of either Q_1 or Q_2.*

Proof. We first consider the case $Q_1 \cap Q_2 = \{v\}$. Let $\gamma_i : [0, 1] \to Q_i \subset \Sigma$, $\gamma_i(0) = \gamma_i(1) = v$, $i = 1, 2$, be the corresponding relative homotopic closed paths. We claim that their orientation is as shown in Figure 6a. Suppose not. Construct disjoint simple closed paths γ'_1 and γ'_2 by pulling γ_1, γ_2 apart, using some free isotopy (as loops) in a neighbourhood of v. This is illustrated in Fig. 6b. By Proposition 3.5, γ'_1 and γ'_2 together bound a cylinder. There is a free isotopy along the cylinder carrying one boundary component of the cylinder, i.e., γ'_1, to the other boundary component γ'_2. Then γ'_1 and γ'_2 are coherently oriented if their orientation is compared by means of the isotopy along the cylinder. Since γ'_1 and γ'_2 are not coherently oriented if compared locally, there exists a 1-sided simple closed curve starting at v and going around the cylinder back to v, a contradiction since Σ is orientable. The claim is proved.
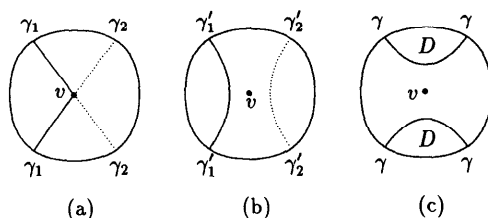

FIG. 6.   Orientation of 4-cycles homotopic in $\pi_1(\Sigma, v)$.

Consider now the concatenated path $\gamma_1 \cdot \gamma_2^-$ which clearly belongs to the trivial homotopy class. By the preceding discussion we may construct a simple closed path γ which is free isotopic (as loops) to $\gamma_1 \cdot \gamma_2^-$. See Fig. 6c. Now, since γ belongs to the trivial homotopy class, it bounds a disc. Figure 6c shows where the disc is. Suppose that the complementary part of the neighbourhood belongs to the disc. Then both, the original Q_1 and Q_2 belong to this disc, and are therefore not essential which is a contradiction.

The other cases are now at hand. In the second and the third case of Fig. 5 we simply perform contraction along the common arc. Since the homotopy is preserved, the two cases are reduced to the previous one. Similarly in the fourth case, i.e., when Q_1 and Q_2 touch at two different vertices. The two curves are first pulled apart (using some free isotopy), in the neighbourhood of the vertex, different from v. ∎

LEMMA 4.5. *Let T be an irreducible locally cyclic triangulation of Σ and $\Gamma \in \pi_1(\Sigma, v)$, $v \in V(T)$, some non-trivial relative homotopy class. Let $Q_1, Q_2 \in Q(\Gamma, v)$ with no crossings, and let D (D_1, D_2) be the disc(s) as in Lemma 4.4. Then each of the arcs in $N(v, T)$ lying in D (respectively, $D_1 \cup D_2$), contains at most five vertices, including those on Q_1 and Q_2.*

Proof. If Q_1 and Q_2 intersect twice then ∂D_1 and ∂D_2 are contractible 4-cycles. It can be shown easily (cf. [6]) that a contractible 4-cycle in an irreducible locally cyclic triangulation contains at most one vertex of T in its interior. In this case the corresponding arcs in the neighbourhood of v contain at most three vertices. Hence, assume that Q_1 and Q_2 touch once, in which case they bound a disc D. Moreover, we consider just the case when they touch at the vertex v. The other two possibilities (when Q_1 and Q_2 touch along a common edge or along two common edges) are similar.

Suppose $v_1, v_2, ..., v_r \in N(v, T)$, $r \geq 6$, in this order belong to an arc in D. See Fig. 7. Let $e_i := v_i v_{i+1}$ $(i = 1, 2, ..., r-1)$ and denote by x the vertex of
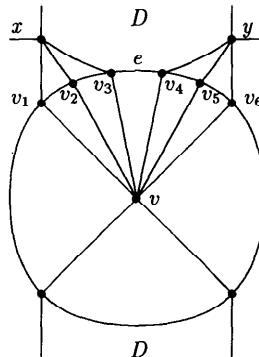

FIG. 7. At most 5 vertices on an arc in D.

Q_1 that is not adjacent to v, and let y be the vertex of Q_2 which is not adjacent to v. Consider the edge e_3. Since T is irreducible, e_3 is contained in an essential 4-cycle which clearly does not belong to Γ. This 4-cycle must contain x and y (since D is a disc and the induced subgraph $N(v, T)$ is a cycle), and the only possibility is that this cycle is $v_3 v_4 y x$. So, there exist edges $v_3 x$, $v_4 y \in E(T)$. Similarly for the edge e_{r-3}. So there exist edges $v_{r-3} x$, $v_{r-2} y \in E(T)$. It follows that $e_3 = e_{r-3}$, so $r = 6$. But we can show more. Since v_2 is the only possible vertex in the disc bounded by $v v_1 x v_3$, we have $v_2 x \in E(T)$, and similarly $v_5 y \in E(T)$. Consider the edge e_2. An essential 4-cycle containing e_2 must also contain e_1 and $v_3 y$. So $v_3 y \in E(T)$. Similarly we see that $v_4 x \in E(T)$, which is not possible since D is a disc. ∎

LEMMA 4.6.   *Let T be an irreducible locally cyclic triangulation of Σ and $\Gamma \in \pi_1(\Sigma, v)$, $v \in V(T)$, some non-trivial relative homotopy class. Then there exist vertices $v_1(\Gamma)$ and $v_2(\Gamma)$ adjacent to v, such that for each Γ-essential edge uv, the vertex u is at most two apart from $v_1(\Gamma)$, or from $v_2(\Gamma)$, where the distance is measured on the cycle $N(v, T)$. Moreover, each $Q \in Q(\Gamma, v)$ is of the form $u_1 v u_2 x$ where u_1 is at most two apart on $N(v, T)$ from $v_1(\Gamma)$, and similarly u_2 is at most two apart from $v_2(\Gamma)$.*

Proof. If $Q(\Gamma, v)$ contains at most one 4-cycle there is nothing to prove. Consider pairs of 4-cycles from $Q(\Gamma, v)$. By Lemma 4.3 we may consider just the pairs with crossing number 0. By Lemma 4.4 and Lemma 4.5 it is possible to choose a pair with the sum of the lengths of the arcs in the corresponding disc(s) as large as possible. Denote this maximal pair by Q_1, Q_2. Without loss of generality we may assume that Q_1 and Q_2 touch just at the vertex v, so that we really have one (open) disc D bounded by Q_1 and Q_2.

We claim that each Γ-essential edge at v is contained in D. Suppose not. Let the edge uv outside D belong to some $Q \in Q(\Gamma, v)$. Several possible cases must now be considered. If Q touches both Q_1 and Q_2 at v (possibly along an edge) we may w.l.o.g. assume the situation in Fig. 8a. Here the
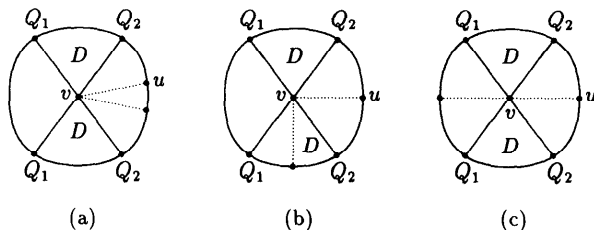

FIG. 8. Γ-essential edge belongs to D.

pair Q_1, Q contradicts the maximality of Q_1, Q_2. If Q touches with the one and crosses with the other cycle from the maximal pair, we may w.l.o.g. assume the situation in Fig. 8b. Construct a new pair Q_2', Q', using Lemma 4.3, where Q' is the 4-cycle containing uv. Then the pair Q_1, Q' contradicts the maximality of Q_1, Q_2. A similar procedure works in the remaining case when Q crosses both Q_1 and Q_2. See Fig. 8c.

We know now that any 4-cycle $Q \in Q(\Gamma, v)$ is completely contained in D. Since the homotopy class Γ is non-trivial, any such Q must have non-empty intersection with the two arcs of $N(v, T) \cap D$. Since these arcs contain at most five vertices each, the existence of vertices $v_1(\Gamma)$ and $v_2(\Gamma)$ is obvious. ∎

LEMMA 4.7.   *Let T be an irreducible locally cyclic triangulation of Σ and let Δ denote the maximal vertex degree of T. Then*

$$\Delta \leqslant 36(2g(\Sigma) - 1).$$

Proof. Choose an arbitrary vertex $v \in V(T)$. Inductively we shall construct at least $k = \lceil \frac{1}{12} \deg(v) \rceil$ 4-cycles Q_i that meet the following conditions:

- $Q_i = v_i v u_i w_i$, where v_i, u_i $(i = 1, 2, ..., k)$ are adjacent to v.
- $u_i = u_j$ if $w_i = w_j$.
- Q_i are essential and pairwise non-homotopic in $\pi_1(\Sigma, v)$.

At the beginning let E_0 contain all edges incident to v. Suppose $Q_1, Q_2, ..., Q_{i-1}$ have already been constructed. Then on the ith step:

- Choose an arbitrary edge $e_i = v_i v \in E_{i-1}$. If, for some $j < i$: $v_i w_j \in E(T)$, then set $w_i := w_j$, $u_i := u_j$, $Q_i := v_i v u_i w_i$. Otherwise choose an arbitrary essential 4-cycle $Q_i = v_i v u_i w_i$ containing e_i. Note that in this case $w_i \neq w_j$, $j < i$ (but it may happen that $u_i = u_j$). The relative homotopy class of Q_i is denoted by Γ_i.

- Define E_i as follows: delete all Γ_i-essential edges in E_{i-1}, plus edges, if any, that are at most two apart from $e_i' = v u_i$ (the distance is measured on the neighbourhood cycle $N(v, T)$ of v).

Let us prove that the above algorithm indeed produces 4-cycles with the required properties. The first and the second condition are satisfied by construction. It remains to prove that each homotopy class Γ_i is non-trivial, and that $\Gamma_i \neq \Gamma_j$ if $i \neq j$. That Γ_i is non-trivial needs verification only in the case when $w_i = w_j$, $u_i = u_j$, $j < i$. Now $v v_i$ is at least three apart from u_j, since at step j we have eliminated from E_j all edges at most two apart from u_j. Thus Q_i is not contractible, since contractible 4-cycles in an irreducible

locally cyclic triangulation contain at most one vertex in the disc they bound (cf. [6]). That no two of these cycles belong to the same homotopy class is also evident, since at each step of the algorithm, all essential edges for the current homotopy class are eliminated. Moreover, since at most 12 edges are eliminated at each step (cf. Lemma 4.6), the algorithm does not stop before $k = \lceil \frac{1}{12} \deg(v) \rceil$ steps.

Consider the subgraph $\mathcal{H}$ that consists of all the chosen Q_i. Because of the second properly, the subgraph which consists of all edges $v_i v$, $v u_i$, $u_i w_i$ is a spanning tree in $\mathcal{H}$. By contracting this spanning tree to v, the constructed 4-cycles give rise to k simple closed curves, pairwise non-homotopic in $\pi(\Sigma, v)$, and with v as the only common point. The proof of the lemma now follows by Proposition 3.6. ∎

Proof of Theorem 2.3.   The proof follows by Lemma 4.2 and Lemma 4.7. Since $|V(T)| = 2(1 - g(\Sigma)) + \frac{1}{3}|E(T)|$, it follows that $|V(T)|$ is bounded above by some constant depending just on the genus of the surface. ∎

Note.   We restrain ourselves from giving the explicit upper bound since we believe that the actual maximal number of vertices is a lot smaller than the one following from our proof. Indeed, we are able to slightly improve the bounds, but at the expense of the length and clarity of the paper. Moreover, in the non-orientable case the proof of Theorem 2.3 breaks down at (the corresponding) Lemma 4.3 and Lemma 4.4. Other analogous lemmas depend on the requirement that Σ be orientable only implicitly, or do not depend on this requirement at all.

5. AN APPLICATION

A graph H is a *minor* of G if it can be obtained from G by a sequence of edge-deletions, edge-contractions, and removals of isolated vertices. In other words, H is isomorphic to a contraction of a subgraph of G.

If ψ is an embedding of a graph G into a surface Σ, where Σ is not the 2-sphere, the *representativity* of ψ, denoted by $\rho(\psi)$, is the minimum number of intersections of a curve γ on Σ with (the image of) G on Σ, where γ is any non-contractible closed curve on Σ. This important invariant for graph embeddings was introduced by Robertson and Seymour [9] in their work on graph minors. Its importance is further investigated in [10, 11]. Embeddings with representativity $\geqslant 2$ are quite important. Such an embedding of a 2-connected graph G determines a cycle double cover of G. (By taking the boundaries of faces one gets a collection of cycles with the property that each edge belongs to exactly two of the cycles in the family.) There is a conjecture that every 2-connected graph has an embedding with representativity $\geqslant 2$ (the Strong Double Cycle Cover Conjecture). Minor

minimal embeddings with representativity ≥ 2 might give some insight into this problem (see Corollary 5.3). These are related to the irreducible locally cyclic triangulations as follows.

PROPOSITION 5.1. *Given a closed surface $\Sigma \neq S^2$, let ψ be a minor minimal embedding of G into Σ with $\rho(\psi) = 2$, i.e., every edge-deletion or edge-contraction in G gives rise to an embedding with representativity one. Then the barycentric subdivision of G in Σ is an irreducible locally cyclic triangulation of Σ.*

Proof. G is clearly connected. It must also be 2-connected. If not, let x be a vertex of G such that $G - x$ is disconnected. Let F be a face of ψ such that $F \cup x$ separates the graph on the surface. Then either there is a 1-representative curve on Σ using F and meeting G only at x (which contradicts $\rho(\psi) = 2$), or one of the components of $G - x$ is embedded in a planar way (which contradicts minor minimality).

Since $\rho(\psi) = 2$ and G is 2-connected, ψ is a closed-cell embedding [10]. Therefore the barycentric subdivision G' is a triangulation of Σ. But it is also locally cyclic. If v is its vertex corresponding to an edge of G, its neighbourhood is obviously an induced 4-cycle. If v corresponds to an original vertex v_0 of G, its neighbours in G' are edges and ψ-faces containing v_0. If there is a diagonal in $N(v_0, G')$ it must join and edge with a face. But every edge has only two edge-face joins which already appear as edges in $N(v_0, G')$. Similar (dual) is the proof that $N(v, G')$ is an induced cycle of G' if v corresponds to a ψ-face.

It remains to prove that every edge e' of G' lies on a non-contractible 4-cycle. Typical edges are $e' = ve$, $e' = vg$, or $e' = eg$, where v, e, g are vertices of G' corresponding to a vertex v, an edge e, and a ψ-face g of G, respectively. Since ψ is minimal, the contraction G/e produces an embedding with representativity one. Consequently, there is a ψ-face f which contains the end-points v, u, of e, but does not contain e, and the quadrilateral $veuf$ of G' is non-contractible. Similarly, the 4-cycle $vguf$ containing $e' = vg$ is essential. To prove that $e' = eg$ also belongs to such a 4-cycle, consider $G - e$. By the minimality of ψ, there is a 1-representative curve in the obtained embedding. If this curve intersects G at a vertex x, then it is obvious that the 4-cycle $egxh$ is essential and contains e'. Here h is the other ψ-face which contains e. ∎

THEOREM 5.2. *Let $\Sigma \neq S^2$ be a closed orientable surface. Then the set of graphs with a minor minimal 2-representative embedding into Σ is finite.*

Proof. By Proposition 5.1 every such minor minimal embedding determines an irreducible locally-cyclic triangulation of Σ. On the other hand, every such triangulation is the barycentric subdivision of at most six

different embeddings (usually only two!). Now we are done by our Main Theorem. ∎

It should be noted that the set of minor minimal embeddings with $\rho(\psi) \geqslant 2$ into the projective plane was determined by Vitray [12].

COROLLARY 5.3. *Let Σ be a closed orientable surface different from the 2-sphere. There are finitely many graphs $G_1, G_2, ..., G_N$ such that if a graph G has an embedding ψ into Σ with $\rho(\psi) \geqslant 2$ then G contains some G_i $(1 \leqslant i \leqslant N)$ as a minor.*

Proof. Having an embedding with representativity at least 2 we reach a minor minimal embedding with $\rho \geqslant 2$ by successive deletions and contractions of edges which do not cause the representativity to drop to 1. ∎

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