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CLEAN TRIANGULATIONS

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A polyhedron on a surface is called a clean triangulation if each face is a triangle and each triangle is a face. Let S_p (resp. N_p) be the closed orientable (resp. nonorientable) surface of genus p. If $\tau(S)$ is the smallest possible number of triangles in a clean triangulation of S, the results are: $\tau(N_1)=20$, $\tau(S_1)=24$, $\lim \tau(S_p)p^{-1}=4$, $\lim \tau(N_p)p^{-1}=2$ for $p\to\infty$.

1. Introduction

A polyhedron on a closed surface S is called a *triangulation* if each face is a triangle with three distinct vertices, and the intersection of any two distinct triangles is either empty, a single vertex, or a single edge. In other words, a triangulation of S is a 2-cell embedding of a graph into S where every 2-cell is a triangle. A triangulation is called *minimal* if the number of triangles is minimal. We denote the number of triangles in a minimal triangulation of S by $\delta(S)$.

Let S_p (resp. N_q) denote the orientable (resp. nonorientable) surface of genus p (resp. q). Jungerman and Ringel [4] have shown that

$$\delta(S_p)=2\left\lceil\frac{7+\sqrt{1+48p}}{2}\right\rceil+4(p-1)\quad\text{if}\quad p\neq 2 \tag{1}$$

and

$$\delta(S_2)=24.$$

Ringel [6] has shown that the same formula holds for N_q if we replace p by $\frac{q}{2}$ in (1), except when q is equal to 2 or 3, and

$$\delta(N_2)=16,\quad \delta(N_3)=20.$$

The 1-skeleton of a triangulation is a graph, and if this graph has a cycle of length 3 which is not the boundary of a 2-cell in the embedding, we call this cycle an *extra triangle*. If a triangulation has no extra triangles, we call it a *clean triangulation*. In other words, in a clean triangulation, not only is every face a triangle, but also every triangle is a face. A clean triangulation of a surface S is called *minimal* if the number of triangles is minimal. We denote the number of triangles in a minimal clean

AMS subject classification (1980): 05 C 10 triangulation of S by $\tau(S)$. The tetrahedron, which is the minimal triangulation of the sphere, is also a clean triangulation, hence in this one instance $\delta(S_0) = \tau(S_0)$.

We find it very surprising that it seems to be impossible to derive a good lower bound for $\tau(S)$ using Euler's polyhedral formula.

We shall prove that

$$\tau(N_1) = 20, \quad \tau(S_1) = 24,$$

and we conjecture that $\tau(N_2) = 28$. If we compare these values with the δ values for the same surface, we find

$$\delta(N_1) = 10, \quad \delta(S_1) = 14, \quad \text{and} \quad \delta(N_2) = 16.$$

The values of τ seem to be significantly larger than the values of δ for the same surface. But we have another surprise. We shall prove that

$$\lim_{p\to\infty} \frac{\tau(S_p)}{p} = 4 \quad \text{and} \quad \lim_{q\to\infty} \frac{\tau(N_q)}{q} = 2.$$

Thus $\tau(S)$ exhibits the same asymptotic behavior as $\delta(S)$.

Tutte [7] considered clean triangulations also. He derived an asymptotic formula for the number of clean triangulations of the plane.

2. Low Order Cases

Theorem 1. $\tau(S_1) = 24$ *and Fig. 1 illustrates the only minimal clean triangulation of the torus.*

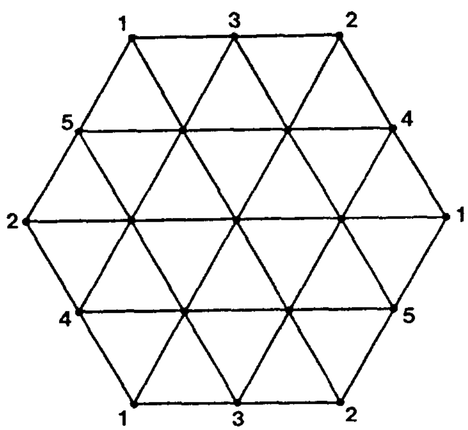

Fig. 1. Minimal clean triangulation of the torus

Proof. Suppose that we have a minimal clean triangulation of S_1 with V vertices, E edges, and F triangles. If we assume that the triangulation is regular of degree k, since F is equal to $2/3E$, by Euler's formula we find that k must equal 6.

Suppose that x is any vertex in the clean triangulation of S_1. Consider the triangles at distance at most 1 from x. (*See Fig. 1*, disregarding the numbers.)