Numerical Solution of a Variation Problem

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Contents

Introduction

This document was created in addition to 'Control Theory' subject third laboratory work 'Numerical Solution of a Variation Problem'. This document on GitHub is here. It contains performance of the task according to variant.

Problem Formulation

Variation Problem is optimization problem with infinite dimensions in set of continuous differentiated functions with fixed bounds.

$$J(x(\cdot)) = \int_{\alpha}^{\beta} I(t, x(t), \dot{x}(t))dt \to \min_{x(\cdot):x(\alpha)=a,x(\beta)=b}$$

Let approximate this problem next way:
Split interval $[\alpha, \beta]$ into N subintervals of same length by defining $N+1$ time points $t_i = \alpha + i \cdot \Delta t, i = 0 \cdots N$, where $\Delta t = \frac{\beta - \alpha}{N}$. Define $N-1$ variables as $x_i = x(t_i), i = 1 \cdots N-1$ and $x_0 = a, x_N = b$ are constant.

Then let approximate integral from Variation Problem by the Trapezoidal Formula

$$J(x(\cdot)) \approx \sum_{i=0}^{N-1} I(t_i, x_i, \dot{x}(t_i))\Delta t$$

And derivative $\dot{x}(t_i)$ can be also approximated with finite difference: $\dot{x}(t_i) \approx \frac{(x_{i+1} - x_i)}{\Delta t}$. And so let

$$J(x(\cdot)) \approx J_N(x_1, \cdots, x_{N-1}) = \sum_{i=0}^{N-1} I(t_i, x_i, \frac{(x_{i+1} - x_i)}{\Delta t})\Delta t \to \min_{x_1, \cdots, x_{N-1}}$$

Optimization problem in the infinite dimensions derived to optimization problem in finite dimensions. Also can be solved regularized problem:

$$J_N^{\lambda}(x_1, \cdots, x_{N-1}) = J_N(x_1, \cdots, x_{N-1}) + \lambda \sum_{i=0}^{N-1} (\frac{x_{i+1} - x_i}{\Delta t})^2 \Delta t$$

Where $\lambda \geq 0$ is regularization parameter.

The Task

1) Chosen example (variant 1 as 15-2*7) of variation problem to solve: $\int_0^1((x')^2 + x^2)dt \to \min, x_0 = x(0) = 0, x_N = x(1) = 1$.
2) To choose number of approximation points N.
3) To code function $J_N(x_1, \cdots, x_{N-1})$.
4) To choose optimization method (non-gradient or gradient) for minimization $J_N(x_1, \cdots, x_{N-1}$.
5) To apply chosen method to minimize $J_N(x_1, \cdots, x_{N-1}$ and build plot of function value dependence on iteration number.
6) To visualize got solution as part-line plot by points $(t_i, x_i), i = 0 \cdots N$.
7) To compare visually (by plots) got numeric solution and analytic one.
8) To research approximation accuracy dependently on number of approximation points N.
9) To prepare the report.

Provided Solution

To solve problem, it was written simple code using R language, that appears there on GitHub.

Solution provided with function

```
# Solves variation problem where
# I = function(t, x, dx) - underintegral function:
#        time t, value x (x=x(t)), and derivative dx (dx = dx/dt (t))
# t1, t2 - limits for integral
# x1, x2 - values for x at moments t1 and t2 (x1 = x(t1), x2 = x(t2))
# N - Number of intervals for splitting the time interval [t1, t2].
# lambda - regularization parameter. If not NULL, then problem solved
#          as regularized with value lambda >= 0
# method - optimization method passed to optim() function as cognominal argument
# ... - other arguments passed to optim() function,
#       for example, control=list(trace=3)
var.prb(I, t0, t1, x0, x1, N=50, lambda = NULL, ... , method = 'BFGS')
```

This function returns S3 class named `VariationProblemSolution` and so a generic methods developed for plotting and printing result.

So I chose `N=50` and set as default value into developed function. As method I chose the BFGS and set `method='BFGS'` as default. But both parameters can be changed. Method can be used according to available methods provided in `optim` function in R `stats` package.

Function $J_N(x_1, \cdots, x_{N-1}$ defined next way (inner code fragment):

```
J <- if(is.null(lambda)) function(X){
    X <- c(x0, X, x1)
    dX <- (X[(1:N)+1] - X[1:N])/dt
    I <- sapply(1:N, function(i) I(times[i], X[i], dX[i]))
    return(sum(I)*dt)
} else function(X){
    X <- c(x0, X, x1)
    dX <- (X[(1:N)+1] - X[1:N])/dt
```

```
    I <- sapply(1:N, function(i) I(times[i], X[i], dX[i]))
    return((sum(I) + lambda*sum(dX^2))*dt)
}
```

Result

Let's get solution for chosen variant via written code.

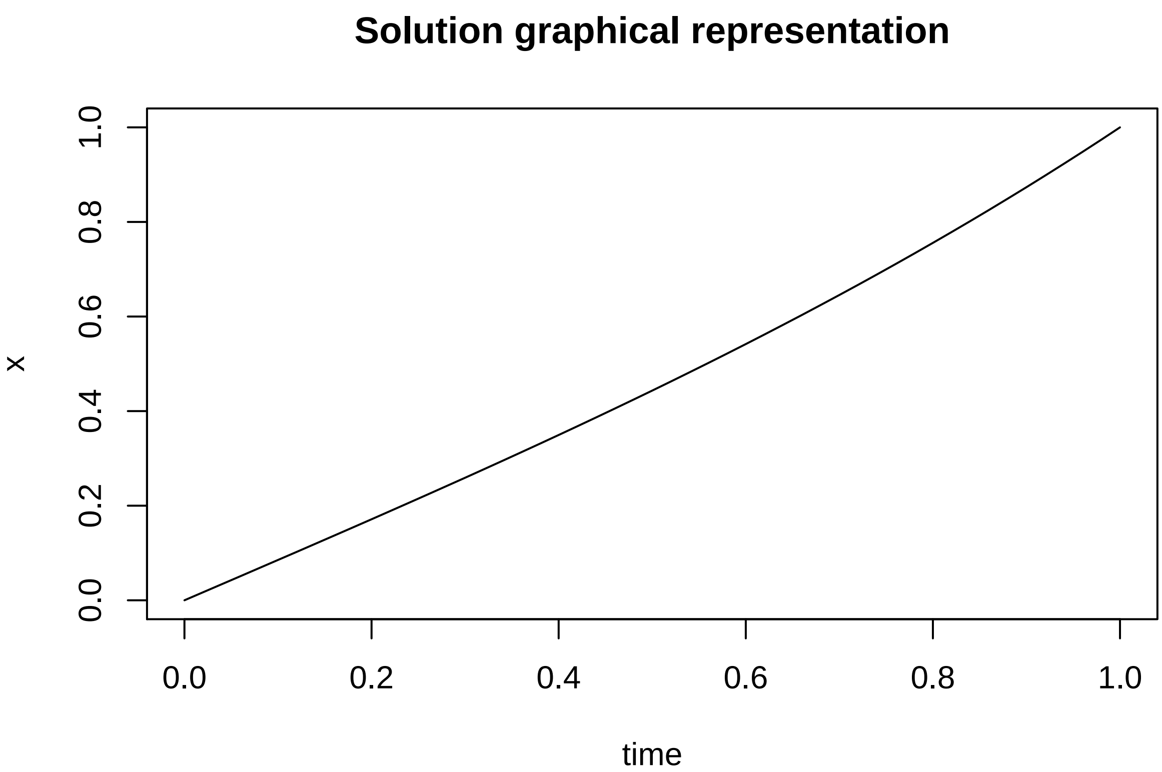
```
I <- function(t, x, dx) dx^2 + x^2
t0 <- 0; t1 <- 1
x0 <- 0; x1 <- 1

sol <- var.prb(I, t0, t1, x0, x1, control=list(trace=1))
```

```
## initial  value 1.323400
## iter  10 value 1.310372
## iter  20 value 1.304887
## iter  30 value 1.303386
## iter  40 value 1.303140
## iter  50 value 1.303122
## final  value 1.303113
## converged
```

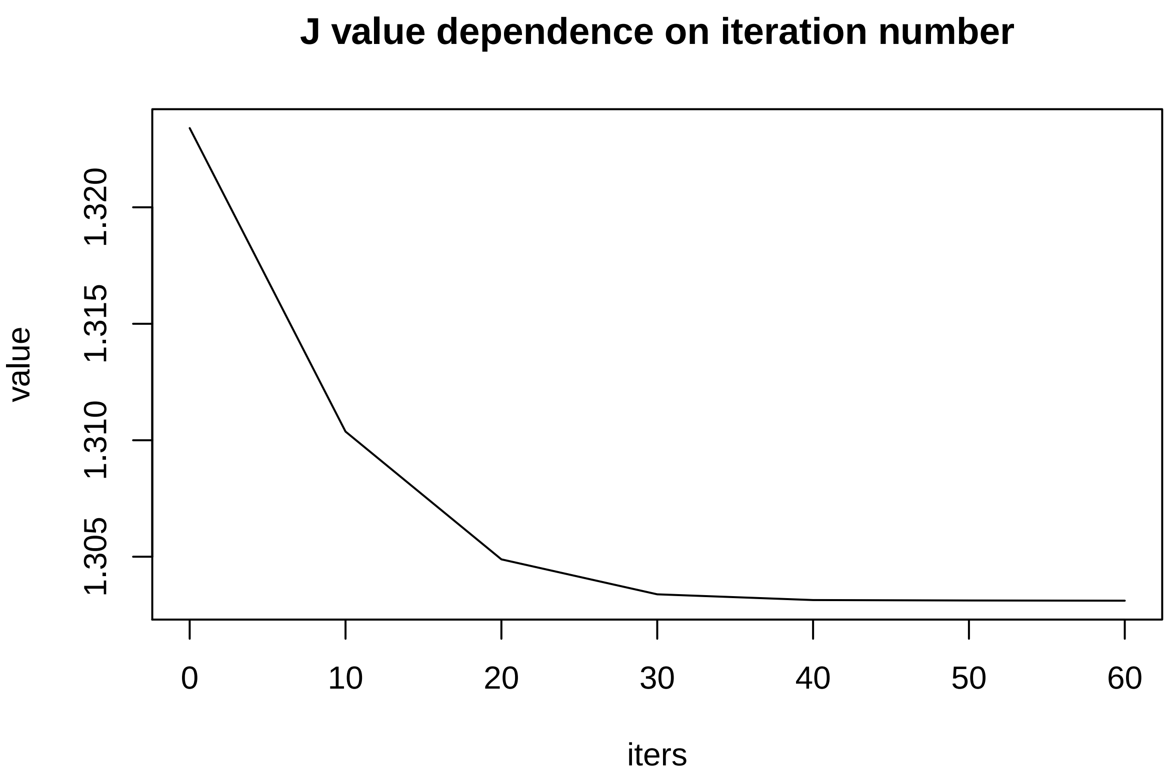
```
summary(sol)
```

```
##  Variatiion Problem solution
## Integrate I(t, x, dx) dt from 0 to 1
## Where I(t, x, dx) = dx^2 + x^2
## x = x(t): x(0) = 0, x(1) = 1
## Size of interval splitting N = 50
## J(...) depends on 49 variables (inner points)
## Regularization wasn't used
## Optimized via optim(par, fn, ...) with method='BFGS'
## Got J optimal value 1.303113
## While optimization evaluated J - 208 times and its gradient - 52 times.
```

```
plot(sol, main='Solution graphical representation')
```



It works easy and fine.
Plot for iterations and function J_N



Change of J function value is decreasing on increase of iterations number. It seems that after 30-th iteration we could stop having pretty accurate solution. Also, I noticed, that whole change of J is about 0.02 - it's little.

Analytical Solution

$$\frac{\partial I}{\partial x} = 2x$$

$$\frac{\partial I}{\partial \dot{x}} = 2\dot{x}$$

$$\frac{\partial^2 I}{\partial \dot{x} \partial t} = 2\ddot{x}$$

$$(\frac{\partial I}{\partial x} = \frac{\partial^2 I}{\partial \dot{x} \partial t}) \Rightarrow (2x = 2\ddot{x}) \Rightarrow (\ddot{x} - x = 0)$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$x(t) = C_1 e^t + C_2 e^{-t}, x(0) = 0, x(1) = 1$$

$$C_1 + C_2 = 0 \Rightarrow C_1 = -C_2 \quad and \quad C_1 e + \frac{C_2}{e} = 1$$

$$-C_2 e + \frac{C_2}{e} = 1 \Rightarrow C_2 = \frac{1}{e^{-1} - e}$$

$$C_1 = -C_2 = \frac{1}{e - e^{-1}}$$

$$C_1 = \frac{1}{e - e^{-1}}, \quad C_2 = \frac{1}{e^{-1} - e}$$

Finally

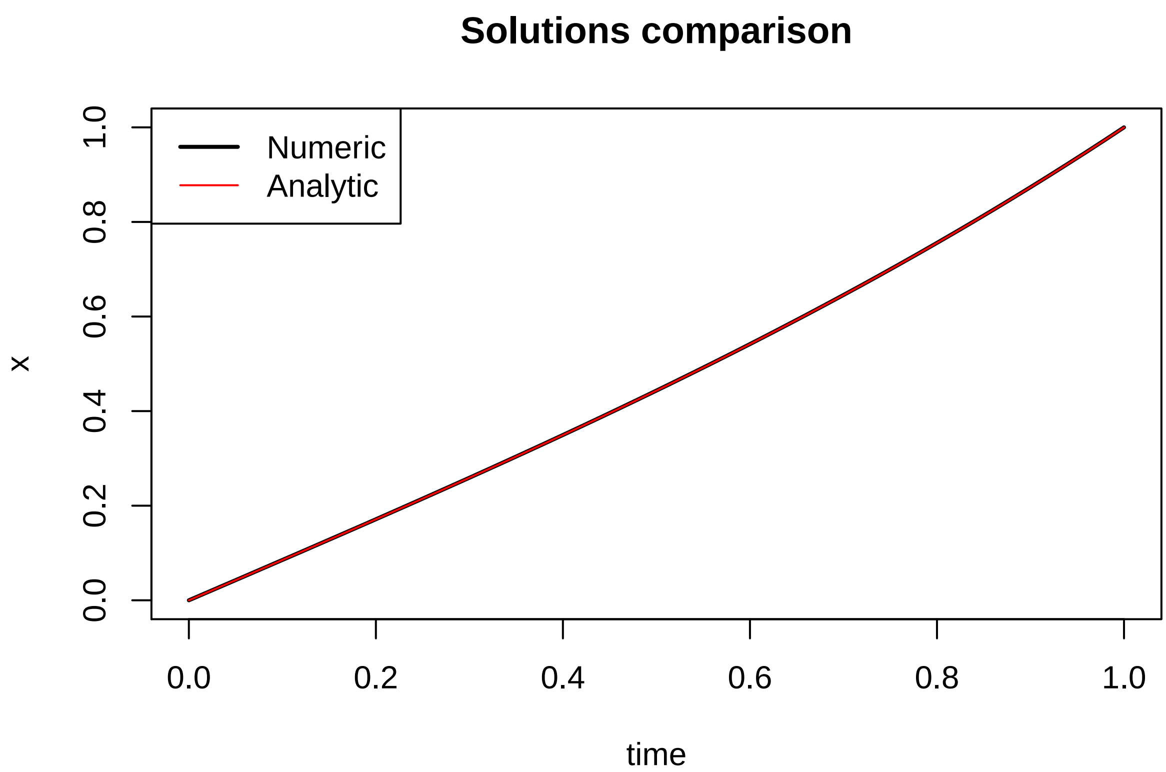
$$x(t) = \frac{e^t}{e - e^{-1}} + \frac{e^{-t}}{e^{-1} - e} = \frac{e^t - e^{-t}}{e - e^{-1}} = \frac{\sinh(t)}{\sinh(1)}$$

Comparisons

Set found $x(t)$ next way:

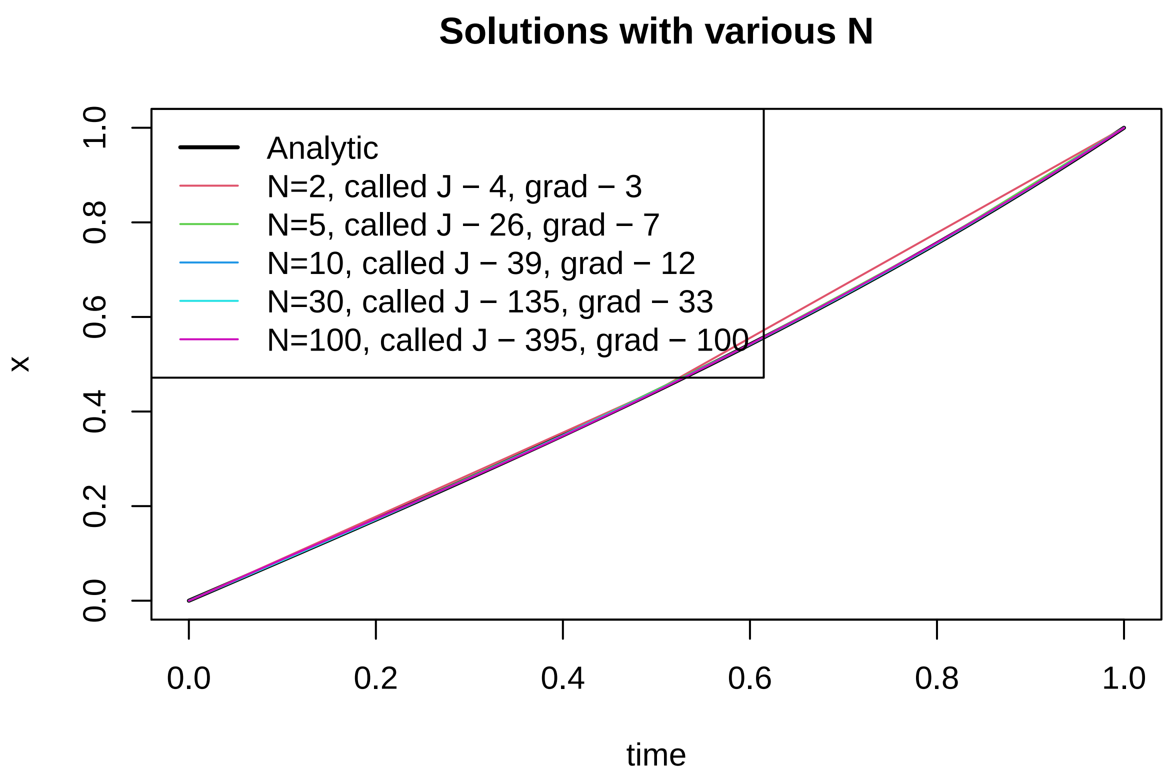
```
x <- function(t) sinh(t)/sinh(1)
time <- 0:100/100
x <- x(time)
```

Let's compare:



Lines are the same. So $N = 50$ is very enough to have correct solution.

Try different N values - $N \in \{2, 5, 10, 30, 100\}$



Just a mess. Probably this example is very simple to detect which N is nicer to use. I saw no valuable difference for all $N > 2$.