Differentiation

Massoud Malek

CALIFORNIA STATE UNIVERSITY EAST BAY

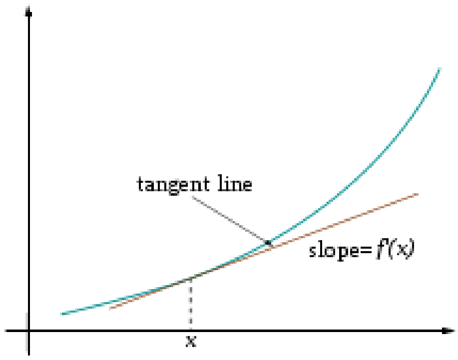
♣ Derivative

The concept of derivative is at the core of Calculus; It is a very powerful tool for understanding the behavior of mathematical functions. It allows us to optimize functions, which means to find their maximum or minimum values, as well as to determine other valuable qualities describing functions. Real-world applications are endless.

The geometrical definition of the derivative of a curve is the slope of the tangent lines at points on the curve. The physical definition is the instantaneous rate of change; for example, the derivative of the position of a moving object with respect to time is the object's instantaneous velocity.

♣ Differentiation

A method of computing the rate at which a dependent variable $y(x)$ changes with respect to the change in the independent variable x. This rate of change is called the derivative of y with respect to x and is denoted by $\dfrac{dy}{dx}$ or just $y'(x)$. If $y = f(x)$ is a function of x, then the derivative is denoted by $\dfrac{d\,f(x)}{dx}$ or just $f'(x)$. If the graph of y is plotted against x, the derivative measures the slope of the tangent line to this graph at each point.



Click here to see an animation of the derivative

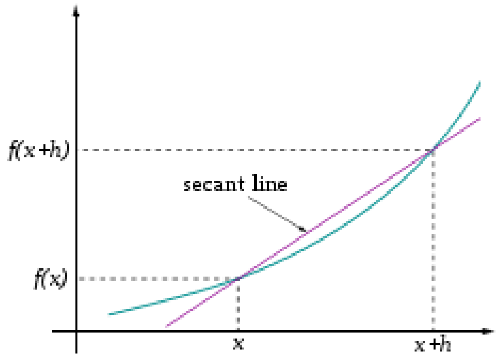
The derivative of $y = f(x)$ with respect to x at a is the slope of the tangent line to the graph of $f(x)$ at a. The slope of the tangent line is very close to the slope of the secant line through $(a,(a))$ and a nearby point on the graph, for example $(a+h,(a+h))$., h is very close to zero. The slope m of the secant line is the difference between the y values of these points divided by the difference between the x values, that is,

$$m = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

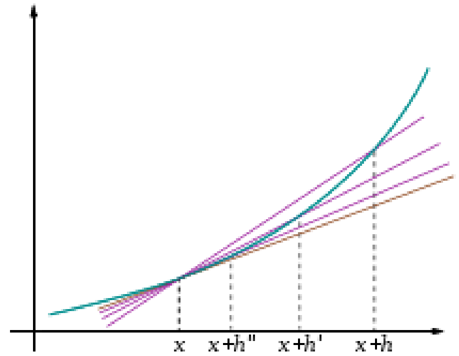
This expression is Newton's difference quotient. The derivative is the value of the difference quotient as the secant lines approach the tangent line. Formally, the derivative of the function $f(x)$ at a is the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

of the difference quotient as h approaches zero. If the limit exists, then $f(x)$ is differentiable at a.



The secant to curve $y = (x)$ passing through (x, f(x)) and (x+h, f(x+h))



The tangent line as limit of secants

$f'(a)$ may also be defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h} \qquad \text{or} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2\,h}$$

Hence

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2\,h}$$

Example

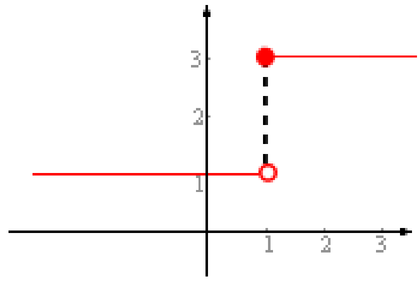
The function $f(x) = x^2$, representing a parabola, is differentiable at $x = 2$, with $f'(2) = 4$. This result is established by calculating the limit as h approaches zero of the difference quotient of $f(2)$:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} \\ &= \lim_{h \to 0} 4 + h = 4\,. \end{aligned}$$

♣ Continuity and differentiability

If $y = f(x)$ is differentiable at a, then $f(x)$ must also be continuous at a. Consider the non-continuous step function

$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ 3 & \text{if } x \geq 1. \end{cases}$$

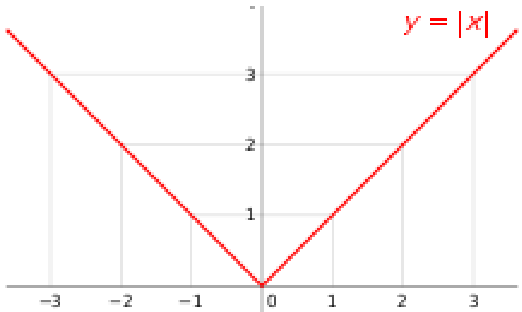


We have

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 0 \qquad \text{and} \qquad \lim_{h \to 0} \frac{f(1) - f(1-h)}{h} = \infty\,.$$

Therefore $f(x)$ is not differentiable at $x = 1$.

Even if a function is continuous at a point, it may not be differentiable there. For example, the absolute value function $f(x) = |x|$



is continuous at $x = 0$, but it is not differentiable there. Since

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 1 \qquad \text{and} \qquad \lim_{h \to 0} \frac{f(0) - f(0-h)}{h} = -1\,.$$

In summary: for a function to have a derivative it is necessary to be continuous, but continuity alone is not sufficient.

♣ Higher Derivatives

Let $f(x)$ be a differentiable function, and let $f'(x)$ be its derivative. The derivative of $f'(x)$ (if it has one) is written $f''(x)$ and is called the second derivative of $f(x)$. Similarly, the derivative of a second derivative, if it exists, is written $f'''(x)$ 0r $f^{(3)}(x)$ and is called the third derivative of $f(x)$. These repeated derivatives are called higher-order derivatives.

If a $f(x)$ has a derivative, it may not have a second or third derivative. For example, let

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x \leq 0. \end{cases}$$

Calculation shows that is a differentiable function whose derivative is

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x \leq 0. \end{cases}$$

$f'(x)$ is twice the absolute value function, and it does not have a derivative at zero. Similar examples show that a function can have k derivatives for any non-negative integer k, but no $(k+1)$ order derivative.

A function that has k successive derivatives is called k times differentiable. If in addition the $k-th$ derivative is continuous, then the function is said to be of differentiability class C^k. (This is a stronger condition than having k derivatives. A function that has infinitely many derivatives is called infinitely differentiable or smooth.

Every polynomial function is infinitely differentiable. By standard differentiation rules, if a polynomial of degree n is differentiated n times, then it becomes a constant function. All of its subsequent derivatives are identically zero. In particular, they exist, so polynomials are smooth functions.

The derivatives of a function $f(x)$ at a point k provide polynomial approximations to that function near k. For example, if $f(x)$ is twice differentiable, then

$$f(x+h) \approx f(x) + f'(x)h + \tfrac{1}{2}f''(x)h^2$$

in the sense that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h - \frac{1}{2}f''(x)h^2}{h^2} = 0.$$

♣ Critical Point

A critical point of a function is a point in the domain, where either the function is not differentiable or its derivative is 0. For example, the point $x = 0$ is a critical point of all three functions

$$f(x) = x^2, \quad g(x) = |x|, \quad \text{and} \quad h(x) = \sqrt{x}.$$

Because $f'(x) = 0$, so the tangent line to the curve of $f(x)$ at $x = 0$ is a horizontal line; $g(x) = |x|$ is not differentiable at $x = 0$; and finally, the tangent line to the curve of $h(x)$ at $x = 0$ is a vertical line. In the case, the derivative is zero; the point is called a stationary point of the function.

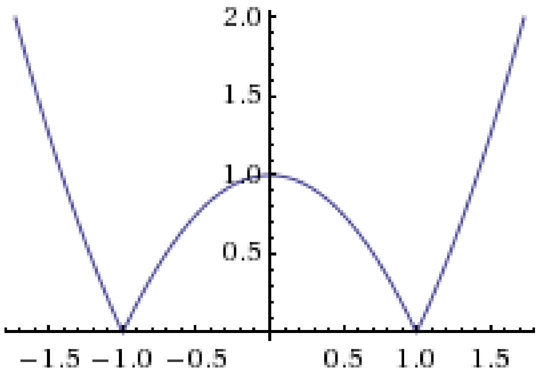
♣ Inflection point

An inflection point or a point of inflection, is a point on a curve at which the curvature changes sign. The curve changes from being concave upwards (positive curvature) to concave downwards (negative curvature), or vice versa. If one imagines driving a vehicle along a winding road, inflection point is the point at which the steering-wheel is momentarily "straight," when being turned from left to right or vice versa.
At an inflection point, the second derivative may be zero, as in the case of the inflection point $x = 0$ of the function $y = x^3$, or it may fail to exist, as in the case of the inflection point $x = 0$ of the function $y = x^{1/3}$.

♣ Local Maxima and Minima

By Fermat's theorem, local maxima and minima of a function can occur only at its critical points. However, not every stationary point is a maximum or a minimum of the function it may also correspond to an inflection point of the graph, as for $f(x) = x^3$ at $x = 0$, or the graph may oscillate in the neighborhood of the point, as in the case of the function

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$



This above graph represent the function $f(x) = |\,x^2 - 1\,|$ with three stationary points at $x = -1,\ 0, 1,$ where $(-1, 0)$ and $(1, 0)$ are local maxima and the point $(0, 1)$ is a local minimum and at the same time a point of inflection.

The derivative of a function can, in principle, be computed from the definition by considering the difference quotient, and computing its limit. In practice, once the derivatives of a few simple functions are known, the derivatives of other functions are more easily computed using rules for obtaining derivatives of more complicated functions from simpler ones. Most derivative computations eventually require taking the derivative of some common functions. The following incomplete list gives some of the most frequently used functions and their derivatives.

Basic Derivatives	
$y = k\,x^r, \qquad r \in \mathbb{R}$	$y' = r\,k\,x^{r-1}.$
$y = e^{c\,x}$	$y' = c\,e^{c\,x}.$
$y = a^{c\,x}$	$y' = c\,\ln\,a^{c\,x}.$
$y = \ln\,x, \qquad x > 0$	$y' = \dfrac{1}{x}$
$y = \log_a x, \qquad x > 0$	$y' = \dfrac{1}{x \ln\,a}$
$y = \sin\,x,$	$y' = \cos\,x$
$y = \cos\,x,$	$y' = -\,\sin\,x$
$y = \tan\,x,$	$y' = \sec^2\,x$
$y = \cot\,x,$	$y' = -\,\csc^2\,x$
$y = \sec\,x,$	$y' = \sec\,x \tan\,x$
$y = \csc\,x,$	$y' = -\,\csc\,x \cot\,x$
$y = \sinh\,x,$	$y' = \cosh\,x$
$y = \cosh\,x,$	$y' = \sinh\,x$
$y = \tanh\,x,$	$y' = \mathrm{sech}^2\,x$
$y = \coth x,$	$y' = -\,\mathrm{csch}^2\,x$
$y = \mathrm{sech}\,x,$	$y' = -\,\mathrm{sech}\,x \tanh\,x$
$y = \mathrm{csch}\,x,$	$y' = -\,\mathrm{csch}\,x \cot\,x$
$y = \arcsin\,x,$	$y' = \dfrac{1}{\sqrt{1-x^2}}$
$y = \arccos\,x,$	$y' = \dfrac{-1}{\sqrt{1-x^2}}$
$y = \mathrm{arcsec}\,x,$	$y' = \dfrac{1}{x\sqrt{1-x^2}}$
$y = \mathrm{arccsc}\,x,$	$y' = \dfrac{-1}{x\sqrt{1-x^2}}$
$y = \arctan\,x,$	$y' = \dfrac{1}{1+x^2}$
$y = \mathrm{arccot}\,x,$	$y' = \dfrac{-1}{1+x^2}$

Wolfram Mathematica Online Differentiation website