Integration

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Differentiation and integration are the two basic operations in integral calculus. While differentiation has easy rules by which the derivative of a complicated function can be found by differentiating its simpler component functions, integration does not. We shall discuss some of the methods used to obtain the antiderivatives. We shall omit the constant C that is required for any antiderivative.

♣ **Integration by Parts**. Consider the following integral:

$$I = \int u(x)v(x)\,dx.$$

By differentiating $u(x)$ and integrating $v(x)$ as many times as needed, we may find I by using the following chart:

$$\begin{array}{c} + \\ \\ - \\ \\ + \\ \\ \vdots \\ \\ (-1)^{k-1} \\ \\ (-1)^k \end{array} \left\{ \begin{array}{ccc} u_0(x) = u(x) & & v_0(x) = \int v(x)dx \\ & \searrow & \\ u_1(x) = u'(x) & & v_1(x) = \int v_1(x)dx \\ & \searrow & \\ u_2(x) = u''(x) & & v_2(x) = \int v_2(x)dx \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ u_{k-1}(x) = u^{(k-1)}(x) & \searrow & v_{k-1}(x) = \int v_{k-2}(x)dx \\ & & \\ u_k(x) = u^{(k)}(x) & \xrightarrow{\int} & v_k(x) = \int v_{k-1}(x)dx \end{array} \right.$$

Thus

$$I = \int u(x)v(x)\,dx = \sum_{i=0}^{k-1}(-1)^k u_i(x)v_{i+1}(x) + (-1)^k \int u_k(x)v_k(x)\,dx.$$

It is important to note that, we must choose the right u and v. For example in

$$\int x^n e^{ax}\,dx,$$

we must choose x^n as $u(x)$; since by differentiating it $n+1$ times, we end up with $u_n(x) = 0$. Choosing $v(x)$ as x^n leads us to a

$$u_1 = a\, e^a x \xrightarrow{\int} v_1 = \frac{x^{n+1}}{n+1}$$

which is a more difficult problem than the original. In the case of

$$J = \int e^{ax} \sin(b, x)\, dx$$

$u(x)$ may be chosen as either e^{ax} or $\sin(b, x)$.

♣ **Examples**

$$I = \int x^3 e^{2x}\, dx \qquad\qquad J = \int e^{2x} \sin\, x\, dx$$

$$\begin{array}{c} + \\ - \\ + \\ - \\ + \end{array} \begin{cases} u_0(x) = x^3 & & v_0(x) = e^{2x} \\ & \searrow & \\ u_1(x) = 3x^2 & & v_1(x) = \frac{1}{2}e^{2x} \\ & \searrow & \\ u_2(x) = 6x & & v_2(x) = \frac{1}{4}e^{2x} \\ & \searrow & \\ u_3(x) = 6 & & v_3(x) = \frac{1}{8}e^{2x} \\ & \searrow & \\ u_4(x) = 0 & \xrightarrow{\int} & v_4(x) = \frac{1}{16}e^{2x} \end{cases} \qquad \begin{array}{c} + \\ - \\ + \end{array} \begin{cases} u_0(x) = e^{2x} & & v_0(x) = \sin x \\ & \searrow & \\ u_1(x) = 2e^{2x} & & v_1(x) = -\cos x \\ & \searrow & \\ u_2(x) = 4e^{2x} & \xrightarrow{\int} & v_2(x) = -\sin x \end{cases}$$

$$J = e^{2x}[-\cos x + 2\sin x] - 4\, J$$

$$I = e^{2x}[\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}] + C \qquad\qquad J = \frac{1}{5}e^{2x}[2\sin x - \cos x] + C$$

♣ **Partial Fractions**. By using the following identity:

$$\frac{1}{(s+a)(s+b)} = \frac{1}{(b-a)} \left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

we may avoid some tedious calculations in finding partial fractions.

Note that although the above identity may be used instead of the traditional method for finding partial fractions, but it may not always be advantageous.

The following table shows how to use the above identity in various cases:

$$\frac{1}{(s^2+as+a^2)s}=\frac{(s-a)s^2}{(s^3-a^3)s^3}=\frac{1}{a^3}\left[\frac{s^2}{s^2+as+a^2}-\frac{s-a}{s}\right]=\frac{1}{a^2}\left[\frac{1}{s}-\frac{s-a}{s^2+as+a^2}\right]$$

$$\frac{1}{(s+a)(s^2+b)}=\frac{s-a}{(s^2-a)(s^2+b)}=\frac{1}{(b+a^2)}\left[\frac{1}{s+a}-\frac{s-a}{s^2+b}\right]$$

$$\frac{1}{(s+a)(s+b)^2}=\frac{1}{(b-a)}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]\frac{1}{(s+b)}=\frac{1}{(b-a)^2}\left[\frac{1}{s+a}-\frac{1}{s+b}-\frac{b-a}{(s+b)^2}\right]$$

$$\frac{1}{(s+a)(s+b)(s+c)}=\frac{1}{(b-a)}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]\frac{1}{(s+c)}$$

$$=\frac{1}{(b-a)(c-a)(c-b)}\left[\frac{c-b}{(s+a)}-\frac{c-a}{s+b}+\frac{b-a}{s+c}\right]$$

♣ Examples

(1)
$$\begin{aligned}\int\frac{d\,x}{(x-1)(x+3)}&=\int\frac{1}{3-(-1)}\left[\frac{1}{x+1}-\frac{1}{x+3}\right]\,d\,x\\&=\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+C\end{aligned}$$

(2)
$$\begin{aligned}\int\frac{d\,x}{(x+2)(x^2+1)}&=\int\frac{x-2}{(x^2-4)(x^2+1)}d\,x\\&=\int\frac{x-2}{(1-(-4)}\left[\frac{1}{x^2-4}-\frac{1}{x^2+1}\right]\,d\,x\\&=\frac{1}{5}\int\left[\frac{x-2}{x^2-4}-\frac{x-2}{x^2+1}\right]\,d\,x=\frac{1}{5}\int\left[\frac{1}{x+2}-\frac{x-2}{x^2+1}\right]\,d\,x\\&=\frac{1}{5}\int\left[\frac{1}{x+2}-\frac{x}{x^2+1}+\frac{2}{x^2+1}\right]\,d\,x\\&=\frac{1}{5}\ln|x+2|-\frac{1}{10}\ln|x^2+1|+\frac{1}{5}\tan^{-1}\,x+C\end{aligned}$$

(3)
$$\begin{aligned}\int\frac{d\,x}{(x+1)(x+3)^2}&=\frac{1}{(3-1)^2}\int\left[\frac{1}{x+1}-\frac{1}{x+3}-\frac{3-1}{(x+3)^2}\right]\,d\,x\\&=\frac{1}{4}\int\left[\frac{1}{x+1}-\frac{1}{x+3}-\frac{2}{(x+1)^2}\right]\,d\,x\\&=\frac{1}{4}\ln|x+1|-\frac{1}{4}\ln|x+3|+\frac{1}{2x+6}+C\end{aligned}$$

To integrate $\dfrac{x+a}{x+b}$, we change $\dfrac{x+a}{x+b}$ into $\dfrac{(x+b)+(a-b)}{(x+b}=1+\dfrac{a-b}{x+b}$, then we integrate. For example:

$$\int \frac{x+3}{x+1}\,d\,x=\int \frac{(x+1)+(3-1)}{x+1}\,dx=\int 1+\frac{3-1}{x+1}\,dx=x+2\ln|x+1|+C$$

$$\int \frac{x+3}{(x+1)(x+2)}\,d\,x=\int \frac{(x+1)+(3-1)}{(x+1)(x+2)}\,dx=\int \frac{1}{x+2}+\frac{2}{(x+1)(x+2)}\,dx$$
$$=\ln|\,x+2\,|+2\ln|\,x+1\,|-2\ln|\,x+2\,|+C=\ln\left|\,\frac{(x+1)^2}{C(x+2)}\,\right|$$

♣ **Integration by Trigonometric Substitution**. To integrate functions containing

$$a^2-x^2,\quad x^2-a^, \quad and \quad a^2+x^2,$$

sometimes we have to use the following trigonometric substitution, in order to be able to integrate.

$$\boxed{\begin{array}{llll} For & a^2-x^2, & we\ \ use & x=a\,\sin\theta \\ For & x^2-a^2, & we\ \ use & x=a\,\sec\theta \\ For & a^2+x^2, & we\ \ use & x=a\,\tan\theta \end{array}}$$

To evaluate

$$\int \frac{x}{\sqrt{(x^2+4)^3}}\,dx,$$

Instead of trigonometric substitution, we may simply substitute u for x^2+4 where $du = 2\,x\,dx$. To evaluate

$$\int \frac{1}{x^2-4}\,dx,$$

we just use the partial fractions method, which is an easier method than the trigonometric substitution one. Here is how to evaluate:

$$I=\int \frac{1}{\sqrt{x^2-2x+2}}\,d\,x=\int \frac{1}{\sqrt{(x-1)^2+1}}\,d\,x\,,$$

we use $(x-1)=\tan\,\theta$ with $dx=\sec^2\theta\,d\,\theta$ and $\theta=\tan^{-1}(x-1)$. Hence

$$I=\int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\,\theta=\int \sec\,\theta\,d\,\theta=\ln\left|\frac{\sec\,\theta+\tan\,\theta}{C}\right|.$$

We have $\tan(\tan^{-1}(x-1))=x-1$ and

$$\sec\,(\tan^{-1}(x-1))=\sqrt{1+\tan^2(\tan^{-1}(x-1))}=\sqrt{1+(x-1)^2}=\sqrt{x^2-2x+2}\,.$$

Thus

$$I=\ln\left|\,\frac{x-1+\sqrt{x^2-2x+2}}{C}\,\right|\,.$$

The integral of secant cubed is one of the more challenging integrals of elementary calculus:

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

This integral is used in evaluating any integral of the form

$$\int \sqrt{a^2 + x^2}\,dx$$

This antiderivative may be found by integration by parts with $u = \sec\, x$ and $dv = \sec^2 x\,dx$.

$$\begin{aligned}
I = \int \sec^3 x\,dx = \int u\,dv = uv - \int v\,du &= \sec x\tan x - \int \sec x\tan^2 x\,dx \\
&= \sec x\tan x - \int \sec x\,(\sec^2 x - 1)\,dx \\
&= \sec x\tan x - \left(\int \sec^3 x\,dx - \int \sec x\,dx.\right) \\
&= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
&= \sec x\tan x + \int \sec x\,dx + I.
\end{aligned}$$

Hence

$$2\,I = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C$$

$$\textit{or} \qquad I = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x| + C\right].$$

Basic Integrals	
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad n \neq 1$	$\int x^{-1}\,dx = \ln\lvert x\rvert$
$\int e^x\,dx = e^x$	$\int a^x\,dx = \dfrac{a^x}{\ln\, a}$
$\int \ln x\,dx = x\ln x - x$	$\int \log_a x\,dx = x\log_a x - \dfrac{x}{\ln a}$

Trigonometric functions	
$\int \sin\, x\,dx = -\cos\, x$	$\int \cos\, x\,dx = \sin\, x$
$\int \tan\, x\,dx = \ln\lvert\sec x\rvert$	$\int \cot\, x\,dx = \ln\lvert\sin x\rvert$
$\int \tan\, x\,dx = \ln\lvert\sec x\rvert$	$\int \cot\, x\,dx = \ln\lvert\sin x\rvert$
$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert$	$\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert$
$\int \sec^2 x\,dx = \tan x$	$\int \csc^2 x\,dx = -\cot x$
$\int \sec x\ \tan x\,dx = \sec x$	$\int \csc x\ \cot x\,dx = -\csc x$
$\int \sin^2 x\,dx = \dfrac{1}{2}(x - \sin x\cos x)$	$\int \cos^2 x\,dx = \dfrac{1}{2}(x + \sin x\cos x)$
$\int \cos^n x\,dx = -\dfrac{\cos^{n-1} x\sin x}{n} + \dfrac{n-1}{n}\int \sin^{n-2} x\,dx$	
$\int \sin^n x\,dx = -\dfrac{\sin^{n-1} x\cos x}{n} + \dfrac{n-1}{n}\int \cos^{n-2} x\,dx$	

To integrate inverse trigonometric functions, we use integration by parts with $v(x) = 1$.

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2}$	$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2}$
$\int \tan^{-1} x\,dx = x\tan^{-1} x - \dfrac{1}{2}\ln(1+x^2)$	$\int \cot^{-1} x\,dx = x\cot^{-1} x + \dfrac{1}{2}\ln(1+x^2)$
$\int \sec^{-1} x\,dx = x\sec^{-1} x - \ln\left\lvert x + \sqrt{x^2-1}\right\rvert$	$\int \csc^{-1} x\,dx = x\csc^{-1} x + \ln\lvert x + \sqrt{1-x^2}$

Hyperbolic Functions	Inverse Hyperbolic Functions
$\int \sinh x\, dx = \cosh x$	$\int \text{arsinh } x\, dx = x \text{ arsinh } x - \sqrt{x^2+1}$
$\int \cosh x\, dx = \sinh x$	$\int \text{arcosh } x\, dx = x \text{ arcosh } x - \sqrt{x+1}\,\sqrt{x-1}$
$\int \tanh x\, dx = \ln \lvert \cosh x \rvert$	$\int \text{artanh } x\, dx = x \text{ artanh } x + \dfrac{\ln(1-x^2)}{2}$
$\int \coth x\, dx = \ln \lvert \sinh x \rvert$	$\int \text{arcoth } x\, dx = x \text{ arcoth } x + \dfrac{\ln(1-x^2)}{2}$
$\int sech\, x\, dx = \arcsin(\tanh x)$	$\int \text{arsech } x\, dx = x \text{ arsech } x - 2 \arctan \sqrt{\dfrac{1-x}{1+x}}$
$\int csch\, x\, dx = \ln \left\lvert \tanh \dfrac{x}{2} \right\rvert$	$\int \text{arcsch } x\, dx = x \text{ arcsch } x + \text{artanh} \sqrt{\dfrac{1}{x^2}+1}$

There are some functions whose antiderivatives cannot be expressed in closed form. A few of them are given below.

$$\int \frac{\sin x}{x}\, dx\,, \quad \int \frac{\cos x}{x}\, dx\,, \quad \int \frac{e^x}{x}\, dx\,, \quad \int \sqrt{\frac{x}{e^x}}\, dx\,, \quad \int e^{-x^2}\, dx\,, \quad \int \frac{1}{\ln x}\, dx\,.$$

Wolfram Mathematica Online Integrator website is at:

http://integrals.wolfram.com/index.jsp