Vector Calculus

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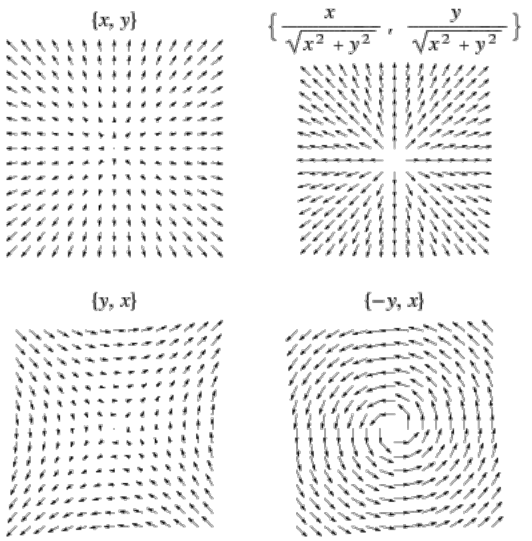
♣ **Vector Fields**. Let $\mathcal{D} = \{(x, y) : \ (x, y) \in \mathbb{R}^2\}$ and $\mathcal{S} = \{(x, y, z) : \ (x, y, z) \in \mathbb{R}^3\}$. Then a *vector field* on $\mathbb{R}^2$ or on $\mathbb{R}^3$, is a function

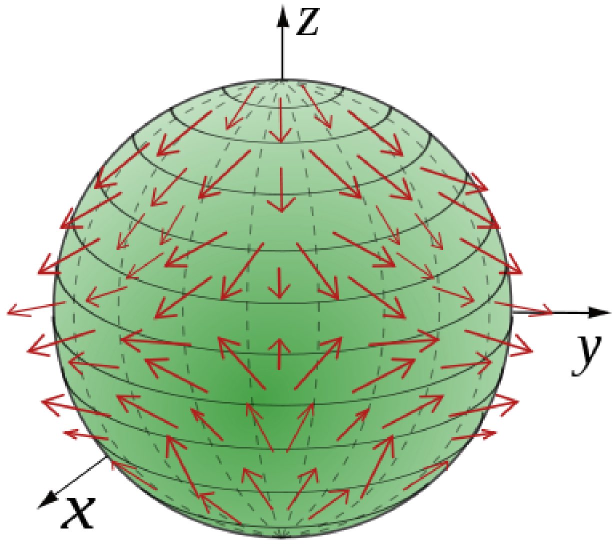
$$\overrightarrow{F} : \mathcal{D} \longrightarrow \mathbb{R}^2 \qquad \text{or} \qquad \overrightarrow{G} : \mathcal{S} \longrightarrow \mathbb{R}^3,$$

$$\text{with} \qquad \overrightarrow{F}(x, y) = P(x, y)\,\overrightarrow{i} + Q(x, y)\,\overrightarrow{j} = < P(x, y)\,,\, Q(x, y) > \qquad \text{and}$$

$$\overrightarrow{G}(x, y, z) = P(x, y, z)\,\overrightarrow{i} + Q(x, y, z)\,\overrightarrow{j} + R(x, y, z)\,\overrightarrow{k} = < P(x, y, z)\,,\, Q(x, y, z)\,,\, R(x, y, z) >.$$

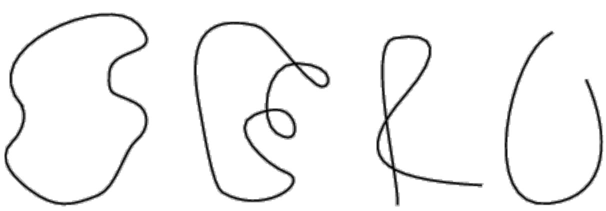


Four different vector fields on (x,y)-plane

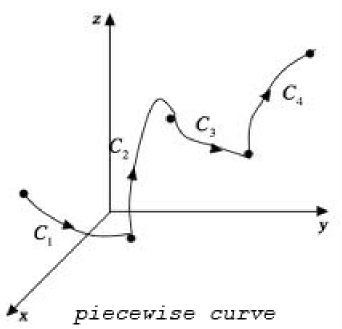


A vector field on a sphere

♣ **Curves**. A *smooth* curve is a continuous curve (no gaps or discontinuities) with no corners (no abrupt changes in slope at a point such as you would get from the intersection of two lines). A continuous and differentiable produces a smooth function.



closed curves   *open curves*



piecewise curve

A path C is called *closed* if its initial and final points are the same point. For example a circle is a closed path.

A path C is *simple*, if it doesn't cross itself. A circle is a simple curve while a curve in the shape of a figure 8 is not simple.

A curve C has a *positive orientation*, if it is traced out in a counter-clockwise direction.

A piecewise curve is a union of finite number of smooth curves $C_1, C_2, C_3, \cdots C_{n-1}, C_n$, where the initial point of C_{k+1} is the end point of C_k.

Here are some of the basic curves with their parametric equations:

Curve	**Parametric Equations**
$y = f(x)$	$x = t, \quad y = f(t)$
$x = g(y)$	$y = t, \quad x = g(t)$

The parametric equations of a line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1) are as follows:

$$x(t) = t\,x_0 + (1-t)\,x_1, \;\; y(t) = t\,y_0 + (1-t)\,y_1, \;\; z(t) = t\,z_0 + (1-t)\,z_1, \quad 0 \le t \le 1$$

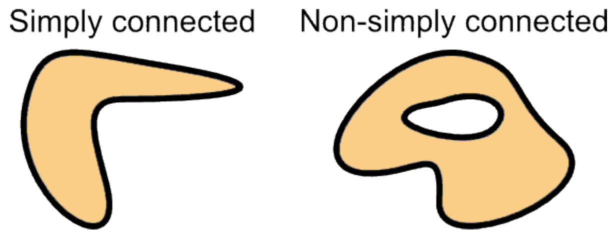
For a line in a two diminutional space, we just omit the third component.
The parametric equations of a circular helix may be as follows:

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t, \quad 0 \le t \le 2\,\pi\,.$$

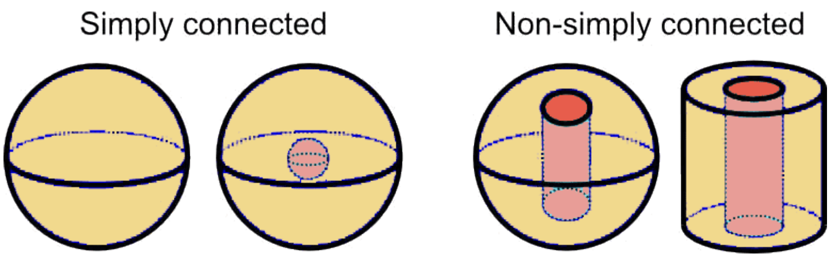
Curve	**Counter-Clockwise**	**Clockwise**
Circle $x^2 + y^2 = r^2$	$x = r\cos t$, $y = r\sin t$, $0 \le t \le 2\,\pi$	$x = r\cos t$, $y = -r\sin t$, $0 \le t \le 2\,\pi$
Elipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\,\pi$	$x = a\cos t$, $y = -b\sin t$, $0 \le t \le 2\,\pi$

Sphere	**Parametric Equations**	**Elipsoid**	**Parametric Equations**
$x^2 + y^2 + z^2 = \rho^2$	$x = \rho\cos u\sin v$, $y = \rho\sin u\sin v$, $z = \rho\cos v$, $0 \le u \le 2\,\pi,\; 0 \le v \le \pi$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	$x = a\cos u\sin v$, $y = b\sin u\sin v$, $z = c\cos v$, $0 \le u \le 2\,\pi,\; 0 \le v \le \pi$

A region $\mathcal{D}$ is *open* if it doesnt contain any of its boundary points.
A region $\mathcal{D}$ is *connected*, if we can connect any two points in the region with a path that lies completely in $\mathcal{D}$.
A *simply-connected* domain (region) is a connected domain, where one can continuously shrink any simple closed curve into a point while remaining in the domain.



Simply connected two-dimensional regions have no holes through them



For three-dimensional domains, the concept of simply connected is more subtle. A simply connected domain is one without holes going all the way through it. However, a domain with just a hole in the middle (like a ball whose center is hollow) is still simply connected, as we can continuously shrink any closed curve to a point by going around the hole and remaining in the domain. On the other hand, a ball with a hole drilled all the way through it, or a spool with a hollow central axis, is not simply connected. A closed curve that went around the hole could not be shrunk to a point while remaining in the domain. There is no way for the curve to bypass the the hole so it remains stuck around it.

♣ **Gradient**. Define the operator ∇ (pronounced 'del') on two dimensional or three dimensional vector fields:

$$\nabla = \frac{\partial}{\partial x}\overrightarrow{i} + \frac{\partial}{\partial y}\overrightarrow{j} \quad \text{or} \quad \nabla = \frac{\partial}{\partial x}\overrightarrow{i} + \frac{\partial}{\partial y}\overrightarrow{j} + \frac{\partial}{\partial z}\overrightarrow{k}\,.$$

Given a scalar field (also called scalar function) $f(x,y)$ or $g(x,y,z)$. Define

$$\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x}\overrightarrow{i} + \frac{\partial f(x,y)}{\partial y}\overrightarrow{j} = < \frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y} >$$

$$\nabla g\,(x,y,z) = \frac{\partial g\,(x,y,z)}{\partial x}\overrightarrow{i} + \frac{\partial g(x,y,z)}{\partial y}\overrightarrow{j} + \frac{\partial g(x,y,z)}{\partial z}\overrightarrow{k} = <\frac{\partial g(x,y,z)}{\partial x}, \frac{\partial g(x,y,z)}{\partial y}, \frac{\partial g(x,y,z)}{\partial z}>.$$

Notice that $\nabla f\,(x,y)$ and $\nabla g\,(x,y,z)$ are vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. They are called *gradient vector fields.* Vector field $\overrightarrow{F}(x,y)$ or $\overrightarrow{G}(x,y,z)$ is called *conservative vector field,* if $\overrightarrow{F}(x,y)$ or $\overrightarrow{G}(x,y,z)$ is the gradient field of a scalar field $f(x,y)$ or $g(x,y,z)$. The function $f(x,y)$ and $g(x,y,z)$ are called *potential functions* for $\overrightarrow{F}(x,y)$ and $\overrightarrow{G}(x,y,z)$, respectively.

Theorem 1. *Let $\overrightarrow{F}(x,y) = P(x,y)\overrightarrow{i} + Q(x,y)\overrightarrow{j}$ be a vector field, where P and Q have continuous first-order partial derivatives on a domain $\mathcal{D}$.*
(a) If $\overrightarrow{F}$ is conservative, then

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}\,.$$

(b) If $\mathcal{D}$ is an open simply-connected domain in $\mathbb{R}^2$ and

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

throughout $\mathcal{D}$, then $\overrightarrow{F}$ is conservative.

♡ **Examples**.
1. Let $f(x,y) = x^2 \sin y + 5\,y^3 \cos x$, then the vector field

$$\overrightarrow{F}(x,y) = < 2\,x \sin y - 5\,y^3 \sin x\,,\; x^2 \cos y + 15\,y^2 \cos x >)$$

is a gradient field for $f(x,y)$. This makes $\overrightarrow{F}(x,y)$ a conservative vector field, as $f(x,y)$ its potential function.
2. Let $g(x,y,z) = x^2 z^3 \sin y + 5\,y^3 \tan z \cos x$, then the vector field

$$\overrightarrow{F}(x,y) = < 2\,x\,z^3 \sin y - 5\,y^3 \tan z \sin x\,,\; x^2 z^3 \cos y + 15\,y^2 \tan z \cos x\,, 3\,x^2 z^2 \sin y + 5\,y^3 \sec^2 z \cos x >$$

is a gradient field for $g(x,y,z)$. This makes $\overrightarrow{G}(x,y,z)$ a conservative vector field, as $g(x,y,z)$ its potential function.
3. Determine whether or not the vector field $\overrightarrow{F}(x,y) = (x^2 - x\,y)\overrightarrow{i} + (y^2 - x\,y)\overrightarrow{j}$ is conservative.
Solution. We have

$$\frac{\partial\,(x^2 - x\,y)}{\partial y} = -x \neq -y = \frac{\partial\,(y^2 - x\,y)}{\partial x}\,.$$

So the vector field is not conservative.
4. Determine whether or not the vector field $\overrightarrow{F}(x,y) = (2\,x + x^2 y)\,e^{x\,y}\,\overrightarrow{i} + (x^3\,e^{x\,y} + 2\,y)\overrightarrow{j}$ is conservative.

Solution. We have

$$\frac{\partial\left[(2x+x^2y)\,e^{xy}\right]}{\partial y}=(3x^2+x^3y)e^{xy}=\frac{\partial\left[x^3\,e^{xy}+2y\right]}{\partial x}.$$

So the vector field is conservative.

5. If $\overrightarrow{F}(x,y)=(3+2xy)\overrightarrow{i}+(x^2-3y^2)\overrightarrow{j}$. Then find a potential function $f(x,y)$ for $\overrightarrow{F}(x,y)$.

Solution. If $f(x,y)$ is a potential function for $\overrightarrow{F}(x,y)$, then

$$f_x(x,y)=3+2xy \qquad \text{and} \qquad f_y(x,y)=x^2-3y^2.$$

By integrating $f_x(x,y)$ with respect to x, we obtain $f(x,y)=3x+x^2y+\phi(y)$, where $\phi(y)$ is a constant for x. Now, by differentiating this function with respect to y, we obtain:

$$x^2+\phi'(y)=x^2-3y^2 \implies \phi(y)=-y^3+K,$$

where K is a constant. Therefore $f(x,y)=3x+x^2y-y^3+K$ is the desired potential function for $\overrightarrow{F}(x,y)$.

♣ **Curl and Divergence**. In this section we use the cross product and the dot product of the operator

$$\nabla=\frac{\partial}{\partial x}\overrightarrow{i}+\frac{\partial}{\partial y}\overrightarrow{j}+\frac{\partial}{\partial z}\overrightarrow{k}$$

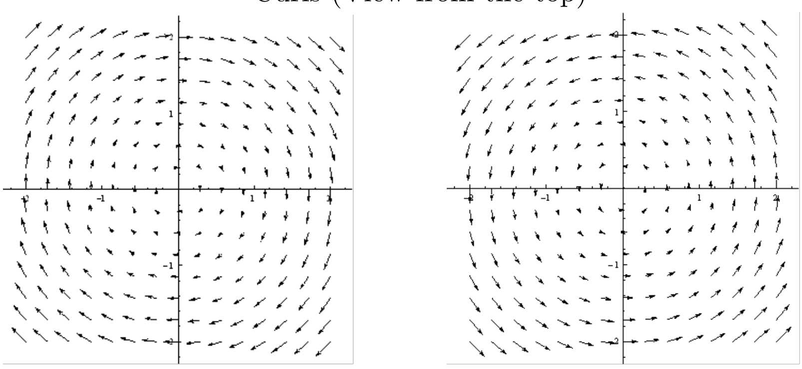
on a vector field $\overrightarrow{F}(x,y,z)=P(x,y,z)\overrightarrow{i}+Q(x,y,z)\overrightarrow{j}+R(x,y,z)\overrightarrow{k}$ in order to obtain a vector field called *curl* (sometimes called *circulation*) of $\overrightarrow{F}$ and a scalar field, called *divergence* of $\overrightarrow{F}$.

Clairaut's Theorem. *If $f:\mathbb{R}^3;\longrightarrow\mathbb{R}^m$ is a function whose second partial derivatives exist and are continuous on a set $\mathcal{S}\subset\mathbb{R}^n$, then $\frac{\partial f}{\partial x_i\partial x_j}=\frac{\partial f}{\partial x_j\partial x_i}$ on $\mathcal{S}$, where $1\le i,\,j\le n$.*

This theorem is commonly referred to as the equality of mixed partials. It is useful for proving basic properties of the interrelations of gradient, divergence, and curl.
For example, if $\overrightarrow{F}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ or $f:\mathbb{R}^3:\longrightarrow\mathbb{R}$, is a function satisfying the hypothesis, then $\nabla\cdot\left[\nabla\times\overrightarrow{F}\right]=0$ or $\nabla\cdot\nabla\times f\,[f]=0$.

♠ **Curl**. The curl is a vector operator that describes the rotation of a 3-dimensional vector field. The curl of a vector field captures the idea of how a fluid may rotate. Suppose that the vector field $\overrightarrow{F}$ is the velocity field of a flowing fluid. Then *curl* $\overrightarrow{F}$ represents the tendency of particles at the point (x,y,z) to rotate about the axis that points in the direction of *curl* $\overrightarrow{F}$. If *curl* $\overrightarrow{F}=0$ at a point P, then the fluid is free from rotations at P and $\overrightarrow{F}$ is called *irrotational* at P. In another words, there is no whirlpool at the point p.

Curls (View from the top)



Negative Curl (Clockwise) Positive Curl (Counter-clockwise)

$$curl\ \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right]\overrightarrow{i} + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right]\overrightarrow{j} + \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]\overrightarrow{k}.$$

In order to see whether or not a three dimensional vector field is conservative, we need the next theorem about curls.

Theorem 2. *If the scalar field $f(x, y, z)$ has continuous second-order partial derivatives, then*

$$curl\ (\nabla f(x, y, z)) = \nabla \times (\nabla f(x, y, z)) = 0\overrightarrow{i} + 0\overrightarrow{j} + 0\overrightarrow{k} = \theta.$$

Proof. Since $curl\ (\nabla f) = \nabla \times \nabla f$, then by Clairaut's theorem,

$$\begin{aligned} curl\ (\nabla f) &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right]\overrightarrow{i} + \left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right]\overrightarrow{j} + \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]\overrightarrow{k} \\ &= 0\overrightarrow{i} + 0\overrightarrow{j} + 0\overrightarrow{k} = \theta. \end{aligned}$$ □

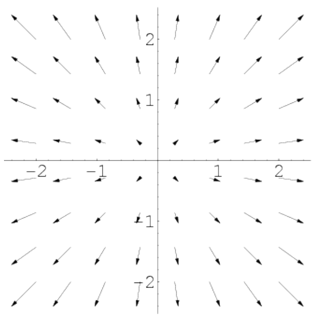
Since a conservative vector field $\overrightarrow{F}$ is one for which $\overrightarrow{F} = \nabla f$, for some scalar function f, this theorem may be rephrased as follows:

If $\overrightarrow{F}$ is conservative, then $\nabla \overrightarrow{F} = \theta$.

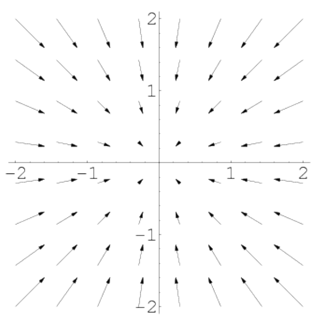
The converse of this theorem is not always true, but if the vector field $\overrightarrow{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous derivatives, then the converse is true. That is:

If $\nabla \overrightarrow{F} = \theta$, then $\overrightarrow{F}$ is conservative.

♠ **Divergence**. The divergence of a three dimensional vector field is the extent to which the vector field flow behaves as a source or sink (it explodes or implodes) at a given point. In another words, if a vector field $\overrightarrow{F}$ represents the flow of a fluid, then the divergence of $\overrightarrow{F}$ represents the rate of expansion or compression of the fluid. The divergence is defined for both two-dimensional vector fields $\overrightarrow{F}(x, y)$ and three-dimensional vector fields $\overrightarrow{F}(x, y, z)$.



Positive Divergence (Explosion)



Negative Divergence (Implosion)

$$div\ \overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \quad \text{or} \quad div\ \overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Theorem 3. *If the component of the vector field*

$$\overrightarrow{F}(x, y, z) = P(x, y, z)\overrightarrow{i} + Q(x, y, z)\overrightarrow{j} + R(x, y, z)\overrightarrow{k}$$

have continuous second-order partial derivatives, then

$$div\ \left[curl\ \overrightarrow{F}\right] = \nabla \cdot (\nabla \times \overrightarrow{F}) = 0.$$

Proof. The proof follows from Clairaut's theorem:

$$\nabla \cdot (\nabla \times \overrightarrow{F}) = \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right] + \left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right] + \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right] = 0 + 0 + 0 = 0\,.$$

♡ **Examples**.

1. Determine whether or not the vector field $\overrightarrow{F}(x,y,z) = x^2 y\,\overrightarrow{i} + x\,y\,z\,\overrightarrow{j}\,x^2y^2\,\overrightarrow{k}$ is conservative.

Solution. All we need to do is compute the curl and see if we get the zero vector θ or not.

$$curl\ \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & x\,y\,z & x^2y^2 \end{vmatrix} = -\left[2x^2y + x\,y\right]\overrightarrow{i} - 2x\,y^2\,\overrightarrow{j} + \left[y\,z - x^2\right]\overrightarrow{k} \neq \theta\,.$$

So, the curl isn't the zero vector; therefore this vector field is not conservative.
square

2. Show that the vector field $\overrightarrow{F}(x,y,z) = y^2z^3\,\overrightarrow{i} + 2\,x\,y\,z^3\,\overrightarrow{j} + 3\,x\,y^2z^2\,\overrightarrow{k}$ is conservative.

Solution.

$$curl\ \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2\,x\,y\,z^3 & 3\,x\,y^2z^2 \end{vmatrix} = \left[6\,x\,y\,z^2 - 6\,x\,y\,z^2\right]\overrightarrow{i} - \left[3\,y^2z^2 - 3\,y^2z^2\right]\overrightarrow{j} + \left[2\,y\,z^3 - 2\,y\,z^3\right]\overrightarrow{k} = \theta.$$

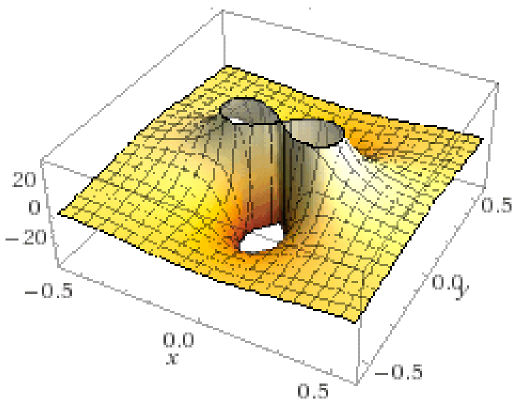
Since the curl is the zero vector; this vector field is conservative. □

3. Find the curl and divergence of the vector field

$$\overrightarrow{F}(x,y,z) = -\frac{y}{x^2+y^2}\,\overrightarrow{i} - \frac{x}{x^2+y^2}\,\overrightarrow{j} + z\,\overrightarrow{k}\,.$$

Solution.

$$curl\ \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & -\dfrac{x}{x^2+y^2} & z \end{vmatrix} = 0\,\overrightarrow{i} + 0\,\overrightarrow{j} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\,\overrightarrow{k}\,.$$



3D plot of $curl\,\overrightarrow{F}$

$$div\ \overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} + 1 = \frac{-4xy}{(x^2+y^2)^2} + 1\,.$$

4. First find the divergence of the vector field $\overrightarrow{F}(x,y,z) = x\,z\,\overrightarrow{i} + x\,y\,z\,\overrightarrow{j} - y^2\,\overrightarrow{k}$ and then prove that $\overrightarrow{F}(x,y,z)$ can't be written as the curl of another vector field $\overrightarrow{G}(x,y,z)$.

Solution. By the definition of divergence, we have

$$div\,\overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial\,(x\,z)}{\partial x} + \frac{\partial\,(x\,y\,z)}{\partial y} + \frac{\partial\,(-y^2)}{\partial z} = z + x\,z \neq 0\,.$$

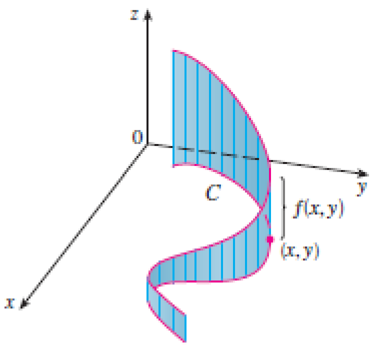
If $\overrightarrow{F}$ were *curl* $\overrightarrow{G}$ for some vector field $\overrightarrow{G}$, then *div* $\left[curl\,\overrightarrow{G}\right]$ would have been zero. □

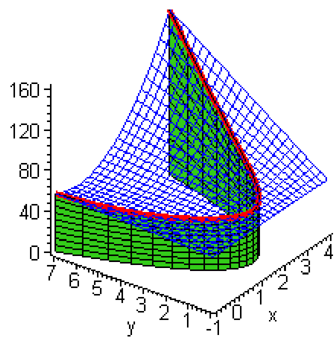
Click on the following link to evaluate the gradient, curl, divergence, etc...

`http://www.wolframalpha.com/examples/VectorAnalysis.html`

♣ **Line Integrals**. A *line integral* (sometimes called a *path integral*, or *curve integral*) is an integral where the function to be integrated is evaluated along a smooth (or piecewise smooth) curve) C instead of an interval $I = [a, b]$. The function to be integrated may be a scalar field or a vector field. In the case of a closed curve, the line integral is called a *contour integral*.

The line integral of a scalar field $f(x, y)$ along a curve C can be interpreted as the area created by $z = f(x, y)$ and a curve C in the x-y plane. The line integral of $f(x, y)$ would be the area of the "curtain" created when the points of the surface that are directly over C are carved out.





♠ **Line Integral of Scalar Fields with respect to the Arc Length**. Consider the smooth curve C given by $\overrightarrow{r}(t) = (x(t), y(t))$:

$$C:\ x = x(t)\,, \quad y = y(t)\,, \quad a \leq t \leq b\,.$$

The length of the curve C is obtained as follows:

$$L = \int_a^b ||\overrightarrow{r}'||\,d\,t = \int_a^b \sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2}\,d\,t\,.$$

The line integral of the scalar field $f(x, y)$ uses the length of the curve C from a to b. Here is how:

$$\int_C f(x, y)\,d\,s = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2}\,d\,t\,.$$

♠ **Line Integral of Scalar Fields in Space**. Consider the smooth space curve C given by $\overrightarrow{r}(t) = (x(t), y(t), z(t))$:

$$C:\ x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,, \quad a \leq t \leq b \qquad \text{scalar field: } g(x, y, z)\,.$$

The length of the space curve C is obtained as follows:

$$L = \int_a^b ||\overrightarrow{r}'||\,d\,t = \int_a^b \sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2 + \left(\frac{d\,z}{d\,t}\right)^2}\,d\,t\,.$$

The line integral of the scalar field $g(x,y,z)$ is:

$$\int_C g(x,y,z)\,d\,s = \int_a^b f(x(t),y(t))\,\sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2 + \left(\frac{d\,z}{d\,t}\right)^2}\;d\,t\,.$$

The line integrals of the scalar field $f(x,y)$ or $g(x,y,z)$ over a piecewise curve:
$C = C_1 + C_2 + C_3 + \cdots + C_{n-1} + C_n$ are:

$\int_C f(x,y)\,d\,s = \int_{C_1} f(x,y)\,d\,s + \int_{C_2} f(x,y)\,d\,s + \cdots + \int_{C_{n-1}} f(x,y)\,d\,s + \int_{C_n} f(x,y)\,d\,s$

$\int_C g(x,y,z)\,d\,s = \int_{C_1} g(x,y,z)\,d\,s + \int_{C_2} g(x,y,z)\,d\,s + \cdots + \int_{C_{n-1}} g(x,y,z)\,d\,s + \int_{C_n} g(x,y,z)\,d\,s\,.$

♡ **Examples**.

1. Consider the scalar field $f(x,y) = 2 + x^2 y$. If $C = C_1 \cup C_2$ is a piecewise smooth curve, where C_1 is the upper half of the unit circle $x^2 + y^2 = 1$ and C_2, the line segment from $[1,0]$ to $[4,4]$. Then evaluate

$$\int_C (2 + x^2 y)\,d\,s.$$

Solution. Since C is a piecewise curve, we have

$$\int_C (2 + x^2 y)\,d\,s = \int_{C_1} (2 + x^2 y)\,d\,s + \int_{C_2} (2 + x^2 y)\,d\,s\,.$$

To obtain the parametric equations of the half circle C_1, we use the clockwise option of the circle. Hence

$$x^2 + y^2 = 1 : \quad x(t) = -\cos t, \quad y(t) = \sin t, \quad 0 \le t \le \pi\,.$$

The line integral over the curve C_1 is:

$$\begin{aligned}
\int_{C_1} (2 + x^2 y)\,d\,s \quad &= \int_0^\pi (2 + \cos^2 t \sin t)\,\sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2}\;d\,t \\
&= \int_0^\pi (2 + \cos^2 t \sin t)\,\sqrt{\sin^2 t + \cos^2 t}\;d\,t \\
&= \int_0^\pi (2 + \cos^2 t \sin t)\,d\,t = \left[2\,t + \frac{\cos^3 t}{3}\right]_0^\pi \\
&= 2\,\pi - \tfrac{2}{3}\,.
\end{aligned}$$

The parametric equations of the line segment C_2 are:

$$x(t) = 1\,t + 4\,(1-t) = -3\,t + 4, \quad y(t) = 0\,t + 4\,(1-t) = -4\,t + 4\,, \quad 0 \le t \le 1\,.$$

The line integral over the curve C_2 is:

$$\begin{aligned}
\int_{C_2} (2 + x^2 y)\,d\,s &= \int_0^1 \big(2 + (-3\,t + 4)^2(-4\,t + 4)\big)\,\sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2}\;d\,t \\
&= \int_0^1 (-36\,t^3 + 132\,t^2 - 168\,t + 66)\,\sqrt{9 + 16}\;d\,t \\
&= \left[-45\,t^4 + 220\,t^3 - 420\,t^2 + 330\,t\right]_0^1 \\
&= 85\,.
\end{aligned}$$

Hence

$$\int_C (2 + x^2 y)\,d\,s = \int_{C_1} (2 + x^2 y)\,d\,s + \int_{C_2} (2 + x^2 y)\,d\,s = \left(2\,\pi + \frac{2}{3}\right) + 85 = 2\,\pi + \frac{257}{3}\,.$$

2. Evaluate $\displaystyle\int_C (y \sin z)\, d\,s$, where C is the circular helix given by the equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t, \quad 0 \leq t \leq 2\,\pi\,.$$

Solution.

$$\begin{aligned}
\int_C (y \sin z)\, d\,s &= \int_0^{2\,\pi} (\sin t\, \sin t) \sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2 + \left(\frac{d\,z}{d\,t}\right)^2}\; d\,t \\
&= \int_0^{2\,\pi} \sin^2 t\, \sqrt{\sin^2 t + \cos^2 t + 1}\; d\,t \\
&= \frac{\sqrt{2}}{2} \int_0^{2\,\pi} (1 - \cos 2\,t)\, d\,t = \frac{\sqrt{2}}{2} \left[t - \frac{\sin 2\,t}{2} \right]_0^{2\,pi} = \sqrt{2}\,\pi\,.
\end{aligned}$$

♠ **<u>Line Integrals of Vector Fields</u>**. Consider the vector field

$$\overrightarrow{F}(x, y, z) = P(x, y, z)\,\overrightarrow{i} + Q(x, y, z)\,\overrightarrow{j} + R(x, y, z)\,\overrightarrow{k}$$

and the three-dimensional, smooth curve C, given by

$$\overrightarrow{r}(t) = x(t)\,\overrightarrow{i} + y(t)\,\overrightarrow{j} + z(t)\,\overrightarrow{k}$$

The line integral of $\overrightarrow{F}$ along the curve C is

$$\int_C \overrightarrow{F} \cdot d\,\overrightarrow{r} = \int_C \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, d\,t = \int_C \overrightarrow{F}(x(t), y(t), z(t)) \cdot \overrightarrow{r}'(t)\, d\,t\,,$$

where $\cdot$ is the **dot product** of two vectors.

♡ **<u>Example</u>**. Evaluate the line integral of the vector field

$$\overrightarrow{F}(x, y, z) = 8\,x^2 y\, z\,\overrightarrow{i} + 5\,z\,\overrightarrow{j} - 4\,x\,y\,\overrightarrow{k}$$

along the smooth curve given by

$$\overrightarrow{r}(t) = t\,\overrightarrow{i} + t^2\,\overrightarrow{j} + t^3\,\overrightarrow{k}, \qquad 0 \leq t \leq 1\,.$$

Solution. We have

$$\overrightarrow{F}(\overrightarrow{r}(t)) = \left[8\,t^2 (t^2)(t^3)\,\overrightarrow{i} + 5\,t^3\,\overrightarrow{j} - 4\,t(t^2)\,\overrightarrow{k} \right] = 8\,t^7\,\overrightarrow{i} + 5\,t^3\,\overrightarrow{j} - 4\,t^3\,\overrightarrow{k}\,,$$

$$\overrightarrow{r}'(t) = \overrightarrow{i} + 2\,t\,\overrightarrow{j} + 3\,t^2\,\overrightarrow{k},$$

$$\overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) = 8\,t^7 + 10\,t^4 - 12\,t^5\,.$$

Hence

$$\int_C \overrightarrow{F} \cdot d\,\overrightarrow{r} = \int_0^1 \left[8\,t^7 + 10\,t^4 - 12\,t^5 \right]\, d\,t = \left[t^8 + 2\,t^4 - t^6 \right]_0^1 = 1\,.$$

Remark 1. *If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from the end point b to the initial point a), then we have*

$$\int_{-C} \overrightarrow{F}\, d\,s = -\int_C \overrightarrow{F}\, d\,s$$

♠ **<u>Line Integrals with respect to the Coordinates</u>**. Consider the curve

$$\overrightarrow{r}(t) = x(t)\,\overrightarrow{i} + y(t)\,\overrightarrow{j} + z(t)\,\overrightarrow{r}, \qquad a \leq t \leq b\,.$$

The line integral of the scalar field $f(x,y,z)$

Let C be a parametrized curve with respect to the parameter $t \in [\,a,b\,]$.

$$\int f(x,y,z)\,d\,s = \int_a^b f(x(t),y(t),z(t))\,\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,d\,t \quad (1)$$

with respect to x is :	$\int_C f(x,y,z)\,d\,x = \int_a^b f(\overrightarrow{r})\,x'(t)\,d\,t$	(2)
with respect to y is :	$\int_C f(x,y,z)\,d\,y = \int_a^b f(\overrightarrow{r})\,y'(t)\,d\,t$	(3)
with respect to z is :	$\int_C f(x,y,z)\,d\,z = \int_a^b f(\overrightarrow{r})\,z'(t)\,d\,t$	(4)

Let's compare the definitions of these four related, but distinct concepts.
Then:

In (1) we find the area of the "fence" built along the curve C whose height along any point (x,y) on C is given by $f(x,y,z)$. Alternatively, we are weighting the integrand $f(x(t),y(t),z(t))$ by the length of the velocity vector along the curve C.

In (2), we are weighting the integrand by only the x component of the velocity vector.

In (3), we are weighting the integrand by only the y component of the velocity vector.

In (4), we are weighting the integrand by only the z component of the velocity vector.

As a simple example, consider f(x,y)=1.

$\int 1\,d\,s{=}\int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2}\,d\,t$	length of C
$\int 1\,d\,x{=}\int_a^b x'(t) = x(b)-x(a)\,d\,t$	net displacement in x direction as C is traversed
$\int 1\,d\,x{=}\int_a^b x'(t) = x(b)-x(a)\,d\,t$	net displacement in y direction as C is traversed

If $\overrightarrow{F}(x,y,z) = <P(x,y,z),\ Q(x,y,z),\ R(x,y,z)>$, then we denote its line integral by:

$$\int_C P(x,y,z)\,d\,x + \int_C Q(x,y,z)\,d\,y + \int_C R(x,y,z)\,d\,z$$

♡ **Examples**.

1. Let $C : r(t) = <\sin(t),\, t^2>,\quad 0 \le t \le \pi$ and $f(x,y) = x^2 + y$. Evaluate the line integral of $f(x,y)$ over C with respect to the variable x.

Solution. We have $x'(t) = \cos t$ and

$$\int_C f(x,y)\,d\,s = \int_0^\pi (\sin^2 t + t^2)\cos t\,d\,t = \left[\frac{1}{3}\sin^3 t + (t^2-2)\sin t + 2\,t\,\cos t\right]_0^\pi = -\,2\,\pi\,.$$

2. Let $\overrightarrow{F}(x,y) = <y, x+2y>$. Then evaluate its line integral:

$$\int_C y\,d\,x + (x+2\,y)d\,y$$

from $(1,0)$ to $(0,1)$, where C is:
(a) the broken line from $(1,0)$ to $(1,1)$ to $(0,1)$;
(b) the arc of the circle $x=\cos t,\ y=\sin t$;
(c) the straight line segment $y = 1-x$.

Solution (a) Along the segment from $(1,0)$ to $(1,1)$ we have $x=1$ and $d\,x=0$; and along the segment from $(1,1)$ to $(0,1)$ we have $y=1$ and $d\,y=0$.
Since the complete line integral is the sum of the line integrals along each of the segments, we have:

$$\int_C y\,d\,x + (x+2\,y)d\,y = \int_0^1 (1+2\,y)\,d\,x + \int_1^0 d\,x = \left[y+y^2\Big|_0^1\right] + \left[x\Big|_1^0\right] = 1.$$

(b) Here we have $x = \cos t$ and $y = \sin t$ for $0 \le t \le \pi/2$, so $d\,x = -\sin t\,d\,t$ and $d\,y = \cos t\,d\,t$, and therefore

$$\begin{aligned}\int_C y\,d\,x + (x + 2\,y)\,d\,y &= \int_0^{\pi/2} -\sin^2\,t\,d\,t + (\cos t + 2\,\sin t)\cos t\,d\,t\\ &= \int_0^{\pi/2} (\cos^2\,t - \sin^2\,t + 2\,\sin t\cos t)d\,t\\ &= \int_0^{\pi/2} (\cos 2\,t + 2\,\sin t\cos t)d\,t\\ &= \left[\frac{1}{2}\sin 2\,t + \sin^2 t\Big|_0^{\pi/2}\right] = 1.\end{aligned}$$

(c) To integrate along the segment $y = 1 - x$ we can use x as the parameter, so that $d\,y = -\,d\,x$. Since x varies from 1 to 0 along this path, the integral is

$$\int_C y\,d\,x + (x + 2\,y)\,d\,y = \int_0^1 (1 - x)\,d\,x + [x + 2\,(1 - x)]\,(-dx) = \int_0^1 (-1)\,d\,x = 1.$$

In this example all three line integrals have the same value, and we might suspect that perhaps with this integrand we get the same value for any path from $(1, 0)$ to $(0, 1)$. This is indeed true because the vector field $\overrightarrow{F}(x, y) =< y, x + 2y >$ is the gradient field of the scalar field $f(x, y) = x\,y + y^2$.

Fundamental Theorem of Calculus for Line Integrals. *If a vector field $\overrightarrow{F}$ is the gradient of some scalar field f in a region R, so that $\overrightarrow{F} = \nabla f$ in R, and if C is any piecewise smooth curve in R with initial and final points A and B, then*

$$\int_C \overrightarrow{F}\,d\,R = f(B) - f(A).$$

Proof. Let $\overrightarrow{F}(x, y, z) = \nabla(f(x, y, z)$ and let $\overrightarrow{r}(t) =< x(t), y(t), z(t) >\quad a \le t \le b$ be the parametric equation of the curve C . We have:

$$\begin{aligned}\int_C \nabla f(x(t), y(t), z(t))\,\cdot d\,\overrightarrow{r}(t) &= \int_C \nabla f(\overrightarrow{r}(t)) \cdot < \frac{d\,x(t)}{d\,t}, \frac{d\,y(t)}{d\,t}, \frac{d\,z(t)}{d\,t} > d\,t\\ &= \int_a^b \left[\frac{\partial\,f}{\partial\,x}\,\frac{d\,x}{d\,t} + \frac{\partial\,f}{\partial\,y}\,\frac{d\,y}{d\,t} + \frac{\partial\,f}{\partial\,z}\,\frac{d\,z}{d\,t}\right] d\,t.\end{aligned}$$

Now we use the Chain Rule to change $\left[\frac{\partial\,f}{\partial\,x}\,\frac{d\,x}{d\,t} + \frac{\partial\,f}{\partial\,y}\,\frac{d\,y}{d\,t} + \frac{\partial\,f}{\partial\,z}\,\frac{d\,z}{d\,t}\right]$ into $\frac{d\,f(\overrightarrow{r}(t))}{d\,t}$. Thus

$$\int_C \nabla f(x(t), y(t), z(t))\,\cdot d\,\overrightarrow{r}(t) = \int_a^b \frac{d}{d\,t}\,[\,f(\overrightarrow{r}(t))\,]\,dt.$$

To conclude the proof, we just need to use the Fundamental Theorem of Calculus for single integrals to obtain

$$\int_C \nabla f(x(t), y(t), z(t))\,\cdot d\,\overrightarrow{r}(t) = f(\overrightarrow{b}) - f(\overrightarrow{a}).$$

□

According to this theorem, the line integral of a gradient field:
(a) is independent of path;
(b) around a closed path is zero.

♣ **Surface Integral.** The surface integrals are similar to line integrals. Like line integrals over curves, there are two types of surface integrals: surface integrals of scalar fields and surface integrals of vector fields.

♠ **Surface Integrals of Scalar Fields.** The surface integral of a scalar field is a generalization of a double integral. To calculate the mass of the surface given its density at each point x described by the scalar-valued field $f(x)$; we need to find the integral of the density over the surface (just like the mass of a wire is the integral of a density over the curve).

There are two ways to work with surface integrals of scalar fields; this depends on how the surface has been given. Either the equation of the surface is expressed in an implicit way, or for example, the function $z = g(x, y)$ represent the surface. First, we investigate the case of implicit equation, such as $x^2 + y^2 + z^2 = 1$.

To find an explicit formula for the surface integral of a scalar field over a surface $\mathcal{S}$, we need to parameterize the surface. Let such a parameterization be $r(u, v)$, where (u, v) varies in some region $\mathcal{D} \subset \mathbb{R}^2$. Then, the surface integral is given by

$$\int f\, d\, S = \iint_{\mathcal{D}} f[r(u, v)] \left\| \frac{\partial\, r}{\partial\, u} \times \frac{\partial\, r}{\partial\, v} \right\| d\, u\, d\, v\,.$$

where the expression between bars on the right-hand side is the magnitude (norm) of the cross product of the partial derivatives.

♡ **Example.** Evaluate $\displaystyle\iint_S x^2\, d\, S$, where $\mathcal{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. We parametrize the surface $\mathcal{S}$, using spherical coordinates; so

$$r(u, v) = \cos u\, \sin v\, \overrightarrow{i} + \sin u\, \sin v\, \overrightarrow{j} + \cos v\, \overrightarrow{k}\,, \quad \mathcal{D} = \{(u, v) : 0 \le u \le 2\,\pi, \quad 0 \le v \le \pi\}\,.$$

we need

$$\|r_u \times r_v\| = \| - \cos u\, \sin^2 v\, \overrightarrow{i} - \sin u\, \sin^2 v\, \overrightarrow{j} - \cos u\, \sin v\, \overrightarrow{k}\,\| = \sin v\,.$$

to evaluate

$$\begin{aligned}\iint_S x^2\, d\, S &= \iint_D \cos^2 u\, \sin^2 v\, \|r_u \times r_v\|\, d\, A = \int_0^{2\,\pi}\!\!\int_0^{\pi} \cos^2 u \sin^3 v\, d\, v\, du \\ &= \int_0^{2\,\pi} \cos^2 u \left[\int_0^{\pi} \sin^3 v\, d\, v\right] du = \frac{1}{2}\int_0^{2\,\pi} (1 + \cos 2\, u) \left[\int (1 - \cos^2 v)\, \sin v\, d\, v\right] du \\ &= \frac{1}{2}\left[u + \frac{1}{2}\sin 2\, u\right]_0^{2\,\pi} \left[- \cos v + \frac{1}{3}\sin^3\, v\right]_0^{\pi} = \frac{4\,\pi}{3}\,. \qquad \square\end{aligned}$$

To find the surface integral of $f(x, y, z)$ over a surface $\mathcal{S}$ with equation $z = g(x, y)$ in (x,y)-plane, we define

$$r(x, y, z) = r(x, y, g(x, y)) \quad \text{with} \quad \frac{\partial\, r}{\partial\, x} = \overrightarrow{i} + g_x(x, y)\, \overrightarrow{k} \quad \text{and} \quad \frac{\partial\, r}{\partial\, y} = \overrightarrow{j} + g_y(x, y)\, \overrightarrow{k}\,.$$

$$\begin{aligned}\text{So} \qquad \int f(x, y, g(x, y))\, d\, S &= \iint_{\mathcal{D}} f(x, y, g(x, y)) \left\| \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} \right\| d\, x\, d\, y \\ &= \iint_{\mathcal{D}} f(x, y, g(x, y)) \left\| -g_x\, \overrightarrow{i} - g_y\, \overrightarrow{j} + \overrightarrow{k} \right\| d\, x\, d\, y \\ &= \iint_{\mathcal{D}} f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1}\; d\, x\, d\, y.\end{aligned}$$

Note that because of the presence of the cross product, the above formulas only work for surfaces embedded in three-dimensional space. Note as well that there are similar formulas for surfaces given by

1. $y = g(x, z)$, when the surface is in the (x,z)-plane. In this case, we choose

$$r(x, y, z) = r(x, g(x, z), z) \quad \text{with} \quad \frac{\partial\, r}{\partial\, x} = \overrightarrow{i} + g_x(x, y)\, \overrightarrow{j} \quad \text{and} \quad \frac{\partial\, r}{\partial\, z} = g_z(x, z)\, \overrightarrow{j} + \overrightarrow{k}\,.$$

2. $x = g(y,z)$ when the surface is in the (y,z)-plane. In this case, we choose

$$r(x,y,z) = r(g(y,z),y,z) \quad \text{with} \quad \frac{\partial r}{\partial y} = g_y(y,z)\overrightarrow{i} + \overrightarrow{j} \quad \text{and} \quad \frac{\partial r}{\partial z} = g_y(y,z)\overrightarrow{i} + \overrightarrow{k}.$$

♡ **Example.** Evaluate $\int\int_S y\, dS$, where $\mathcal{S}$ is the surface

$$z = x + y^2, \quad \mathcal{D} = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}\ .$$

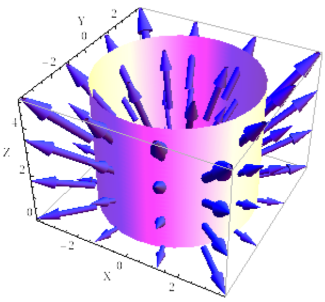
Solution. We have

$$\int\int_S y\, dS = \int_0^1\int_0^2 y\sqrt{z_x^2 + z_y^2 + 1}\, dy\, dx = \int_0^1\int_0^2 y\sqrt{4y^2+2}\, dy\, dx = \frac{13\sqrt{2}}{3}. \quad \square$$

♠ **Surface Integrals of Vector Fields.** If the vector field $\overrightarrow{F}$ represents the flow of a fluid, then the *surface integral* of $\overrightarrow{F}$ will represent the amount of fluid flowing through the surface per unit time. This amount is called the *flux* of fluid through the surface. For this reason, the surface integral of a vector field is often called the *flux integral.*
The dot product of the vector field $\overrightarrow{F}$ and the normal vector n of the surface $\mathcal{S}$ produces the scalar field $\overrightarrow{F} \cdot \overrightarrow{n}$. The total flux of fluid flow through the surface $\mathcal{S}$, denoted by $\int\int_S \overrightarrow{F} \cdot dS$ is defined as the surface integral of the scalar field $\overrightarrow{F} \cdot \overrightarrow{n}$ over the surface $\mathcal{S}$, parametrized by $r(t)$. So

$$\int\int_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS = \int\int \overrightarrow{F} \cdot \overrightarrow{n}\ du\, dv = \int\int \overrightarrow{F} \cdot \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\ du\, dv$$

♡ **Examples.**
1. Let $\mathcal{S}$ be the surface of the cylinder of radius 3 and height 5 given by $x^2 + y^2 = 3^2$ and $0 \le z \le 5$. Find the integral of the vector field $\overrightarrow{F}(x,y,z) = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ over $\mathcal{S}$.



The positive side is the outward pointing normal vector.

Solution. First we parameterize the cylinder by

$$r(\theta,t) = 3\cos\theta\ \overrightarrow{i} + 3\sin\theta\,\overrightarrow{j} + t\,\overrightarrow{k} \qquad 0 \le \theta \le 2\pi, \quad 0 \le t \le 5\,.$$

Next, we find the normal vector $\overrightarrow{n}$.

$$\overrightarrow{n} = \frac{\partial r(\theta,t)}{\partial\theta} \times \frac{\partial r(\theta t)}{\partial t} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\cos\theta\ \overrightarrow{i} + 3\sin\theta\,\overrightarrow{j} + 0\,\overrightarrow{k}\ .$$

We have

$$\begin{aligned}\iint_S \overrightarrow{F}(r(\theta,\, t)\cdot n\, dS &= \int_0^5\!\!\int_0^{2\,\pi} \overrightarrow{F}\cdot\overrightarrow{n}\, d\,\theta\, d\,t\\ &= \int_0^5\!\!\int_0^{2\,\pi} 9\left(\cos\theta\,\overrightarrow{i} + \sin\theta\,\overrightarrow{j} + t\,\overrightarrow{k}\right)\cdot\left(\cos\theta\,\overrightarrow{i} + \sin\theta\,\overrightarrow{j} + 0\,\overrightarrow{k}\right) d\,\theta\, d\,t\\ &= \int_0^5\!\!\int_0^{2\,\pi} 9\left(\cos^2\theta + \sin^2\theta\right) d\,\theta\, d\,t = 9\int_0^5\!\!\int_0^{2\,\pi} d\,\theta\, d\,t = 90\,\pi\,. \quad \square\end{aligned}$$

2. Find the flux of the vector field $\overrightarrow{F}(x,\, y,\, z) = z\,\overrightarrow{i} + y\,\overrightarrow{j} + x\,\overrightarrow{k}$ across the unit sphere $x^2+y^2+z^2 = 1$.
Solution. We parametrize the surface $\mathcal{S}$, using spherical coordinates; so

$$r(\theta,\varphi) = \cos\theta\,\sin\varphi\,\overrightarrow{i} + \sin\theta\,\sin\varphi\,\overrightarrow{j} + \cos\varphi\,\overrightarrow{k}, \quad \mathcal{D} = \{(\theta,\varphi) : 0 \le \theta \le 2\,\pi,\;\; 0 \le \varphi \le \pi\}\,.$$

Next, we find the normal vector $\overrightarrow{n}$.

$$\overrightarrow{n} = \frac{\partial\, r(\theta,\varphi)}{\partial\,\varphi} \times \frac{\partial\, r(\theta,\varphi)}{\partial\,\theta} = \cos\theta\,\sin^2\varphi\,\overrightarrow{i} + \sin\theta\,\sin^2\varphi\,\overrightarrow{j} + \cos\theta\,\sin\varphi\,\overrightarrow{k}\,.$$

$$\overrightarrow{F}(x,\, y,\, z) = \cos\varphi\,\overrightarrow{i} + \sin\theta\,\sin\varphi\,\overrightarrow{j} + \cos\theta\,\sin\varphi\,\overrightarrow{k}\,.$$

Now we can evaluate $\iint_S \overrightarrow{F}\, d\,S$:

$$\iint_S \overrightarrow{F}(r(\theta,\, \varphi)\cdot n\, dS = \int_0^{\pi}\!\!\int_0^{2\,\pi} \overrightarrow{F}\cdot\overrightarrow{n}\, d\,\theta\, d\,\varphi$$

$$= \int_0^{\pi}\!\!\int_0^{2\,\pi} < \cos\varphi,\, \sin\theta\,\sin\varphi,\, \cos\theta\,\sin\varphi > \cdot < \cos\theta\,\sin^2\varphi,\, \sin\theta\,\sin^2\varphi,\, \cos\theta\,\sin\varphi > d\,\theta\, d\,\varphi$$

$$= \int_0^{\pi}\!\!\int_0^{2\,\pi} \left(\cos\phi\cos\theta\sin\varphi + \sin^2\theta\sin^3\varphi + \cos^2\theta\sin^2\varphi\right) d\,\theta\, d\,\varphi = \int_0^{\pi}\!\!\int_0^{2\,\pi} \cdots\cdots$$

♣ **Fundamental Theorem of Vector Calculus.** The fundamental theorem of calculus states that the integral of a function $f(x)$ in $\mathbb{R}$ over the interval $I = [a, b]$ can be calculated by finding an antiderivative $F(x)$ of $f(x)$:

$$\int_a^b f(x)\, d\,x = F(b) - F(a)\,.$$

Similarly, the fundamental theorems of vector calculus state that an integral of some type of derivative over some object is equal to the values of the function along the boundary of that object.
We have fundamental theorems for one-dimensional (curves), two-dimensional (planar regions and surfaces), and three-dimensional (volumes) objects.
The fundamental theorems are:

I. Gradient theorem for line integrals relates a line integral to the values of a function at the "boundary" of the curve, i.e., its endpoints.

II. Green's theorem transforms the line integral around a simple closed curve C into a double integral over the plane region $\mathcal{D}$ bounded by C.

III. Stokes' theorem relates a line integral over a closed curve to a surface integral.

IV. Divergence theorem relates a surface integral to a triple integral.

Gradient Theorem for Line Integrals. *If a vector field $\overrightarrow{F}$ is a gradient field, meaning $\overrightarrow{F} = \nabla f$ for some scalar-valued field f, then the line integral of $\overrightarrow{F}$ along a curve C from some point a to some other point b, may be evaluated as*

$$\int_C \overrightarrow{F}\cdot d\,s = f(b) - f(a).$$

This integral does not depend on the entire curve C; it depends on only the endpoints a and b. If we replaced C by another curve with the same endpoints, the integral would be unchanged. Hence $\overrightarrow{F}$ is conservative (also called path-independent).

♡ **Example.** Consider the scalar function $f(x,y) = x\,y^2$ and define the gradient vector field $\overrightarrow{F}(x,y) = \nabla f(x,y) = y^2\,\overrightarrow{i} + 2\,y\,\overrightarrow{j}$. Let C be the path with

$$r(t) = t^2\,\overrightarrow{i} + 2\,(t-2)^3\overrightarrow{j} \quad \text{for} 1 \leq t \leq 3.$$

The starting point of the curve is $a = r(1) = (1,-2)$ and the ending point is $b = r(3) = (9,2)$ with $f(1,-2) = 4$ and $f(9,2) = 36$. Hence the integral must be

$$\int_C \overrightarrow{F} \cdot d\,s = f(b) - f(a) = f(9,2) - f(1,-2) = 36 - 4 = 32.$$

We could also compute $\int_C \overrightarrow{F} \cdot d\,s$ the direct way using the parametrization $r(t)$. The integral isn't difficult as it is just a polynomial, but it is messy, so we'll skip the details.

$$\int_C \overrightarrow{F} \cdot d\,s = \int_a^b \overrightarrow{F}(r(t)) \cdot r'(t)\,d\,t = \cdots\cdots = \cdots\cdots = 36 - 4 = 32,$$

which agrees with the first answer.

Green's Theorem. *Let C be a positively oriented, piecewise smooth, simple, closed curve in $\mathbb{R}^2$ and let $\mathcal{D}$ be the region enclosed by the curve. If $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives on $\mathcal{D}$, then*

$$\int_C P(x,y)\,d\,x + Q(x,y)\,d\,y = \int\int_{\mathcal{D}} \left(\frac{\partial\,Q(x,y)}{\partial x} - \frac{\partial\,P(x,y)}{\partial y} \right) d\,x\,d\,y$$

When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

$$\oint_C P(x,y)\,d\,x + Q(x,y)\,d\,y\,.$$

We can write Green's theorem as

$$\oint_C \overrightarrow{F} \cdot d\,s = \int\int (curl\,\overrightarrow{F}) \cdot \overrightarrow{k}\;d\,A\,,$$

where $\overrightarrow{k}$ is the unit vector in the z-direction and the component of the curl in the z-direction is given by the formula

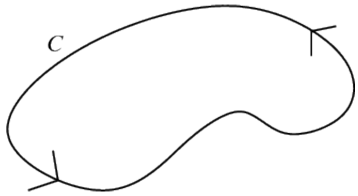
$$(curl\,\overrightarrow{F}) \cdot \overrightarrow{k} = \frac{\partial\,Q(x,y)}{\partial x} - \frac{\partial\,P(x,y)}{\partial y}$$

According to this theorem, the line integral of a conservative two-dimensional vector field along any simple, closed plane curve is zero.

♡ **Explanation**. When C is an oriented simple closed path that lives in the plane, the integral

$$\oint_C \overrightarrow{F}\,d\,s$$

represents the circulation of the vector field $\overrightarrow{F}$ around C. If $\overrightarrow{F}$ were the velocity field of water flow, for example, this integral would indicate how much the water tends to circulate around the path in the direction of its orientation.



One way to compute this circulation is, of course, to compute the line integral directly. But we can use Green's theorem as an alternative way to calculate the line integral.

♢ **Area Calculation.** Green's theorem can be used to compute area by line integral. Let C be a positively oriented, piecewise smooth, simple, closed curve in $\mathbb{R}^2$ and let $\mathcal{D}$ be the region enclosed by the curve. The area of $\mathcal{D}$ is given by

$$A = \int\int_C d\,A\,.$$

Now choose a vector field $\overrightarrow{F} = P(x,y)\overrightarrow{i} + Q(x,y)\overrightarrow{j}$ such that

$$\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = 1\,.$$

Then the area is given by:

$$A = \oint_C P(x,y)\,d\,x + Q(x,y)\,d\,y\,.$$

For example, if we choose $\overrightarrow{F} = -\frac{y}{2}\overrightarrow{i} + \frac{x}{2}\overrightarrow{j}$, then

$$A = \oint_C \left[\frac{\partial\left(\frac{x}{2}\right)}{\partial\,x} - \frac{\partial\left(\frac{-y}{2}\right)}{\partial y}\right] d\,x\,d\,y = \frac{1}{2}\oint_C [-y\,d\,x + x\,d\,y]\,.$$

Note that the vector field $\overrightarrow{F}(x,y) = (x^2y + 5\,y^2x)\,\overrightarrow{i} + (5\,x^2y + x^2 + x)\,\overrightarrow{j}$ also satisfies

$$\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = 1\,.$$

So, Green's theorem can be used for the area calculation for many vector fields.

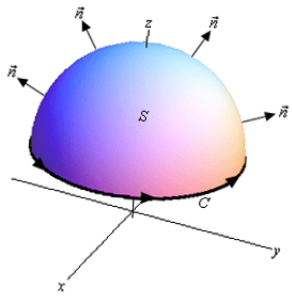
♡ **Example.** Calculate the area of the region $\mathcal{D}$ bounded by the curve C parametrized by

$$r(t) = \sin 2\,t\,\overrightarrow{i} + \sin t\,\overrightarrow{j} \quad \text{for} \quad 0 \le t \le \pi.$$

Solution. We'll use Green's theorem to calculate the area bounded by the curve. Since C is a counter-clockwise oriented boundary of $\mathcal{D}$, the area is just the line integral of the vector field $\overrightarrow{F} = -\frac{y}{2}\overrightarrow{i} + \frac{x}{2}\overrightarrow{j}$ around the curve C.

$$\begin{aligned} A &= \int_C \overrightarrow{F}\cdot d\,s = \int_0^\pi \overrightarrow{F}(r(t))\cdot r'(t)\,d\,t = \frac{1}{2}\int_0^\pi \left(-\sin t\,\overrightarrow{i} + \sin 2\,t\,\overrightarrow{j}\right)\cdot\left(2\,\cos 2\,t\,\overrightarrow{i} + \cos t\,\overrightarrow{j}\right) d\,t \\ &= \frac{1}{2}\int_0^\pi \left(-\sin t\,\overrightarrow{i} + 2\sin t\cos t\,\overrightarrow{j}\right)\cdot\left(2(\cos^2 t - \sin^2 t)\,\overrightarrow{i} + \cos t\,\overrightarrow{j}\right) d\,t = \int_0^\pi \sin^3 t\,d\,t \\ &= \int_0^\pi (1-\cos^2 t)\sin\,t\,d\,t = \left[-\cos t - \frac{1}{3}\cos^3 t\right]_0^\pi = 1 + \frac{1}{3} = \frac{4}{3}\,. \end{aligned}$$

♠ **Stokes' Theorem.** This theorem takes Green's theorem which relates a line integral to a double integral over some planar region to a higher dimension. Stokes' theorem relates the surface integral of the curl of a vector field over a surface S in $\mathbb{R}^3$ to the line integral of the vector field over its boundary.
Before stating the theorem we need to define the curve that is used in the line integral.



Around the edge of this surface we have a curve C. This curve is called the boundary curve. The orientation of the surface S will induce the positive orientation of C. You can see this using the right-hand rule. If you point the thumb of your right hand perpendicular to a surface, your fingers will curl in a direction corresponding to circulation parallel to the surface.

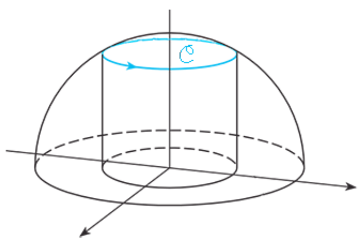
Stokes' Theorem. *Let $\mathcal{S}$ be an oriented smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. If $\overrightarrow{F}$ is a vector field defined over then,*

$$\oint_C \overrightarrow{F} \cdot d\,S = \iint (curl\,\overrightarrow{F}) \cdot \overrightarrow{n}\,d\,S\,.$$

Remark. *The surface integral of curl $\overrightarrow{F}$ over a surface $\mathcal{S}$ is the circulation of $\overrightarrow{F}$ around the boundary of the surface. So the surface $\mathcal{S}$ can actually be any surface as long as its boundary curve is given by C. This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.*

♣ **Stokes' Theorem versus Green's Theorem**. To go from Green's theorem to Stokes' theorem, we make the following changes.
1. We change the line integral in two dimensions (Green's theorem) to a line integral in three dimensions (Stokes' theorem).
2. We change the double integral of *curl* $\overrightarrow{F} \cdot \overrightarrow{k}$ over a region $\mathcal{D}$ in the plane (Green's theorem) to a surface integral of *curl* $\overrightarrow{F} \cdot n$ over a surface floating in space (Stokes' theorem).
3. We replace $\overrightarrow{k}$, the unit vector in the z-direction by $\overrightarrow{n}$, the unit normal vector to the surface.
4. The required relationship between the curve C and the surface $\mathcal{S}$ (Stokes' theorem) is identical to the relationship between the curve C and the region $\mathcal{D}$ (Green's theorem): the curve C must be the boundary of $\mathcal{D}$, the region or the boundary of $\mathcal{S}$, the surface.

♡ **Example.** Consider the vector field $\overrightarrow{F}(x,y,z) = y\,z\,\overrightarrow{i} + x\,z\,\overrightarrow{j} + x\,y\,\overrightarrow{k}$ and let $\mathcal{S}$ be part of the sphere $x^2+y^2+z^2 = 4$ that lies inside of the cylinder $x^2+y^2 = 1$ and above the (x,y)-plane. Then find $\iint (curl\,\overrightarrow{F}) \cdot \overrightarrow{n}\,d\,S\,.$



Sphere: $x^2+y^2+z^2 = 4$, Cylinder: $x^2+y^2 = 1$, $z \geq 0$

Solution. To find the boundary curve C, we solve the equations $x^2+y^2+z^2=4$ and $x^2+y^2=1$. Subtracting, we get $z^2=3$ and since the surface lies above the (x,y)-plane; we conclude that $z=\sqrt{3}$. Thus C is the circle given by the equations $x^2+y^2=1$ and $z=\sqrt{3}$. The vector equation of C is

$$r(t)=\cos t\,\overrightarrow{i}+\sin t\,\overrightarrow{j}+\sqrt{3}\,\overrightarrow{k} \qquad 0\le t\le 2\pi.$$

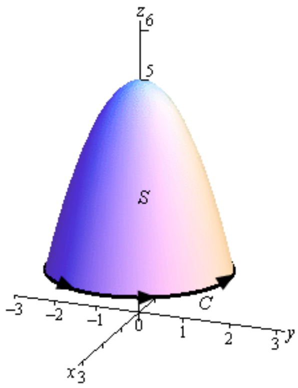
To integrate, we need

$$r'(t)=-\sin t\,\overrightarrow{i}+\cos t\,\overrightarrow{j}+0\,\overrightarrow{k} \quad \text{and} \quad \overrightarrow{F}(r(t))=\sqrt{3}\sin t\,\overrightarrow{i}+\sqrt{3}\cos t\,\overrightarrow{j}+\cos t\,\sin t\,\overrightarrow{k}.$$

By using the Stokes' Theorem, we obtain

$$\iint (curl\,\overrightarrow{F})\cdot\overrightarrow{n}\,dS=\oint_C \overrightarrow{F}(r(t))\cdot r'(t)dr=\sqrt{3}\int_0^{2\pi}\left[-\sin^2 t+\cos^2 t\right]dt=\sqrt{3}\int_0^{2\pi}\cos 2t=0.$$

Example. Use Stokes' Theorem to evaluate $\iint_S (curl\,\overrightarrow{F})\cdot\overrightarrow{n}\,dS$, where $\overrightarrow{F}(x,y,z)=<z^2,\,-3xy,\,x^3y^3>$ and S is the part of $z=5-x^2-y^2$ above the plane $z=1$. Assume that S is oriented upwards.



Solution. The boundary curve C is where the surface $z=5-x^2-y^2$ intersects the plane $z=1$. So the boundary curve of

$$z=5-x^2-y^2$$
$$z=1.$$

will be the circle of radius 2 that is in the plane $z=1$. Using cylindrical coordinates, the parameterization of this curve is:

$$\overrightarrow{r}(t)=<2\cos t,2\sin t,1> \quad \text{with} \quad \overrightarrow{r'}(t)=<-2\sin t,2\cos t,0>, \quad (\,0\le t\le\pi\,).$$

The first two components give the circle and the third component makes sure that it the curve is in the plane $z=1$. Using Stokes' Theorem we can write the surface integral as the following line integral:

$$\iint_S (curl\,\overrightarrow{F}(x,y,z))\cdot\overrightarrow{n}\,dS=\oint_C \overrightarrow{F}(t)\cdot\overrightarrow{r'}(t)\,dt.$$

We have

$$\overrightarrow{F}(t)\cdot\overrightarrow{r'}(t)=<1^2,-12\sin t,64\cos^3 t\sin^3 t>\cdot<-2\sin t,2\cos t,0>=(-2\sin t-\;24\cos^t\,\sin t).$$

Hence

$$\iint_S (curl\,\overrightarrow{F}(x,y,z))\cdot\overrightarrow{n}\,dS=\int_0^{2\pi}(-2\sin t\cos t-\;24\cos^t\,\sin t)\,dt=2\cos t-8\cos^3 t\Big|_0^{2\pi}=0.$$

♡ **The Divergence Theorem.** The theorem, more commonly known as *Gauss's theorem* and also as *Ostrogradsky's theorem*, is an important result for the mathematics of engineering, in particular in electrostatics and fluid dynamics. It is a result that relates the flow of a vector field through a surface to the behavior of the vector field inside the surface.
More precisely, the divergence theorem states that the outward flow of a vector field through a closed surface is equal to the volume integral of the divergence of the region inside the surface. Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region. The theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.
The following example illustrate the divergence theorem:
Let's say we have a rigid container filled with some gas. If the gas starts to expand but the container does not expand, what has to happen? Since we assume that the container does not expand (it is rigid) but that the gas is expanding, then gas has to somehow leak out of the container.
Recall that if a vector field $\overrightarrow{F}$ represents the flow of a fluid, then $\nabla \overrightarrow{F}$ represents the expansion or compression of the fluid.

The Divergence Theorem. *Let $\mathcal{V}$ be a region in space with boundary $\mathcal{S}$. Then the volume integral of the divergence of $\overrightarrow{F}$ over $\mathcal{V}$ and the surface integral of $\overrightarrow{F}$ over the boundary $\mathcal{S}$ of $\mathcal{V}$ are related by*

$$\iint_{\mathcal{S}} \overrightarrow{F}\, dS = \iiint_{\mathcal{V}} (\nabla \overrightarrow{F})\, dV\,.$$

Notice that the divergence theorem equates a surface integral with a triple integral over the volume inside the surface. In this way, it is analogous to Green's theorem, which equates a line integral with a double integral over the region inside the curve. Remember that Green's theorem applies only for closed curves. For the same reason, the divergence theorem applies only to closed surface.
In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions. In one dimension, it is equivalent to the fundamental theorem of calculus.

♡ **Examples.**

1. Compute $\displaystyle\iint_{\mathcal{S}} \overrightarrow{F}\, dS$ where

$$\overrightarrow{F} = \left(3\,x + z^{77}\right) \overrightarrow{i} + \left(y^2 - z\,\sin x^2\right) \overrightarrow{j} + \left(x\,z + y\,e^{x^5}\right) \overrightarrow{k}$$

and $\mathcal{S}$ is the boundary surface of the box $\mathcal{V} = \{(x, y, z)\,:\, 0 \le x \le 1, \quad 0 \le y \le 3, \quad 0 \le z \le 2\}$.

Solution. Given the ugly nature of the vector field, it would be hard to compute this integral directly. However,

$$\nabla \overrightarrow{F} = \frac{\partial\left(3\,x + z^{77}\right)}{\partial x} \overrightarrow{i} - \frac{\partial\left(y^2 - z\,\sin x^2\right)}{\partial y} \overrightarrow{i} + \frac{\partial\left(x\,z + y\,e^{x^5}\right)}{\partial z} \overrightarrow{k} = 3\,\overrightarrow{i} + 2\,y\,\overrightarrow{j} + x\,\overrightarrow{k}$$

is nice. We use the divergence theorem to convert the surface integral into a triple integral

$$\iint_{\mathcal{S}} \overrightarrow{F}\, dS = \iiint_{\mathcal{V}} (\nabla \overrightarrow{F})\, dV\,.$$

We compute the triple integral of $\nabla \overrightarrow{F} = 3 + 2\,y + x$ over $\mathcal{V}$:

$$\begin{aligned}\iint_{\mathcal{S}} \overrightarrow{F}\, dS &= \int_0^1\!\!\int_0^3\!\!\int_0^2 (3 + 2\,y + x)\, dz\, dy\, dx = \int_0^1\!\!\int_0^3 (6 + 4\,y + 2\,x)\, dy\, dx \\ &= \int_0^1 (18 + 18 + 6\,x)\, dx = 36 + 3 = 39.\end{aligned}$$

2. Compute $\displaystyle\iint_{\mathcal{S}} \overrightarrow{F}\, d\, S$ where $\overrightarrow{F} = x\, y^2\, \overrightarrow{i} + y\, z^2\, \overrightarrow{j} + x^2 z\, \overrightarrow{k}$ and $\mathcal{S}$ is the sphere of radius 3 centered at origin.

Solution. We convert the surface integral into a triple integral over the region inside the surface. Since $\nabla\, \overrightarrow{F} = y^2\, \overrightarrow{i} + z^2\, \overrightarrow{j} + x^2\, \overrightarrow{k}$, the surface integral is equal to the triple integral

$$\iiint_{\mathcal{V}} (\nabla\, \overrightarrow{F})\, d\, V = \iiint_{\mathcal{V}} y^2 + z^2 + x^2,$$

$\mathcal{V}$ is the ball of radius 3.

To evaluate the triple integral, it is easier to change variables to spherical coordinates. In spherical coordinates, the ball is

$$0 \leq \rho \leq 3, \quad 0 \leq \theta \leq 2\,\pi, \quad 0 \leq \varphi \leq \pi.$$

The integral is simply

$$\begin{aligned} \int_0^3\int_0^{2\,\pi}\int_0^{\pi} \rho^2 (\rho^2 \sin\varphi)\, d\,\varphi\, d\,\theta\, d\,\rho \ &= \int_0^3\int_0^{2\,\pi} \left[-\rho^4 \cos\varphi\right]_0^{\pi}\, d\,\theta\, d\,\rho = \int_0^3\int_0^{2\,\pi} 2\,\rho^4 \theta\, d\,\rho \\ &= \int_0^3 4\pi\rho^4 d\rho = \left[\frac{4}{5}\pi\,\rho^5\right]_0^3 = \frac{972\pi}{5}\,. \end{aligned}$$

Vincent Van Gogh transformed the chaos in his mind into beauty on the canvas. And, without any proper training in mathematics or physics, he explained how the turbulent flow works.

`http://www.greenvillegazette.com/p/billions-of-people-have-seen-this-painting-what-someone-j`

References :

1. *Mathematica* Search the site for the desired subject.
2. *WolframAlpha* Solve vector calculus problems and see examples.
3. *Wikipedia* Start here and then click on the different subjects.
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6. *PlanetMath.Org* Search the site for the desired subject.
7. *Proof Wiki* Search the site for the desired subject.
8. Several Calculus and Vector Calculus books.