First Order Differential Equations

♣ Definitions and Notations

The k-th order derivative of the function $y(x)$ is denoted by $D^k_x y$ or simply $D^k y$. Thus

$$D^k y = D^k_x y = \frac{d^k y}{dx^k} \quad and \quad D^0_x y = Iy = y.$$

A general n-th order, ordinary differential equation is represented by

$$F(x, y, Dy, \cdots, D^n y) = 0;$$

so an ordinary differential equation is an equation (E) which contains terms such as $D^k y$. The highest power of D in (E) is called the order of the equation.
The equation $F(x, D)y = R(x)$, where

$$F(x, D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I,$$

is said to be *linear of order n.* When $R(x) = 0$, then the linear differential equation is called *homogeneous.*
If a solution of $F(x, y, D) = 0$ can be expressed as $y = f(x)$ (i.e., y is a function of x), then this solution is called an *explicit solution.* If we obtain $f(x, y) = 0$ as a solution of our differential equation, then we say that only an *implicit solution* has been found.

A first order differential equation may be expressed as follows:

$$\frac{dy}{dx} = f(x, y).$$

The problem

$$\begin{cases} \text{Solve:} & \dfrac{dy}{dx} = f(x, y) \\ \text{Subject to:} & y(x_0) = y_0 \end{cases} \tag{1}$$

is called an *initial-value problem (I.V.P.)* The first equation gives the slope of the curve y at any point x, and the second equation specifies one particular value of the function $y(x)$.

Existence. Will every initial-value problem have a solution? No, some assumptions must be made about $f(x,y)$, and even then we can only expect the solution to exist in a neighborhood of $x = x_0$. As an example of what could happen, consider

$$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(x_0) = 0 \end{cases} \tag{2}$$

The solution curve starts at $x = 0$ with slope one; that is, $y'(0) = 1$. Since the slope is positive, $y(x)$ is *increasing* near $x = 0$. Therefore, the expression $1 + y^2$ is also increasing. Hence, y' is increasing. Since y and y' are both increasing and are related by the equation $y' = 1 + x^2$, we can expect that at some finite value of x there will be no solution; that is, $y(x) = +\infty$. As a matter of fact, this occurs at $x = \pi/2$ because the *analytic solution* of the initial-value problem is $y(x) = \tan x$.

Theorem 1. *If $f(x,y)$ is continuous in a rectangle R centered at (x_0, y_0), say*

$$R = \{(x,y) : |x - x_0| \le \alpha, \quad |y - y_0| \le \beta\} \tag{3}$$

then the initial-value problem (1) *has a solution $y(x)$ for $|x - x_0| \le min\,(\alpha, \beta/M)$, where M is the maximum of $|f(x,y)|$ in the rectangle R.*

Uniqueness. It can happen, even if $F(x,y)$ is continuous, that the initial-value problem does not have a unique solution. A simple example of this phenomenon is given by the problem

$$\begin{cases} \dfrac{dy}{dx} = 1 + y^{2/3} \\ y(x_0) = 0 \end{cases} \tag{4}$$

It is obvious that the zero function, $y(x) \equiv 0$, is a solution of this problem. Another solution is the function

$$y(x) \equiv \frac{x^3}{27}\,.$$

To prove that the initial-value problem has a *unique solution* in a neighborhood of $x = x_0$, it is necessary to make some assumptions about $f(x,y)$. Here are the usual theorems used for the uniqueness of the solution of an initial-value problem.

Theorem 2. *If $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous in a rectangle R defined by* (2)*, then the initial-value problem (1) has a unique solution in the interval $|x - x_0| \le min\,(\alpha, \beta/M)$.*

Theorem 3. *If $f(x,y)$ is continuous in the strip $a \le x \le b$, where $-\infty < y < \infty$ and satisfies the inequality:*

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|, \tag{5}$$

then the initial-value problem (1) has a unique solution in the interval $[a,b]$.

The Inequality (5) is called a *Lipschitz condition* for the second variable y. This condition is *stronger* than the continuity.

Definition 1. Let $f(u)$ be a function where $u \in \mathbb{R}^n$ such that $f(\lambda u) = \lambda^k f(u)$, for a suitable real λ. Then $f(u)$ is called a *homogeneous function of degree k.*

♠ Methods of Solving First Order Differential Equations

There are several classes of differential equations of order one. We shall explain how to classify and solve some of these classes.

♣ Separation of Variables

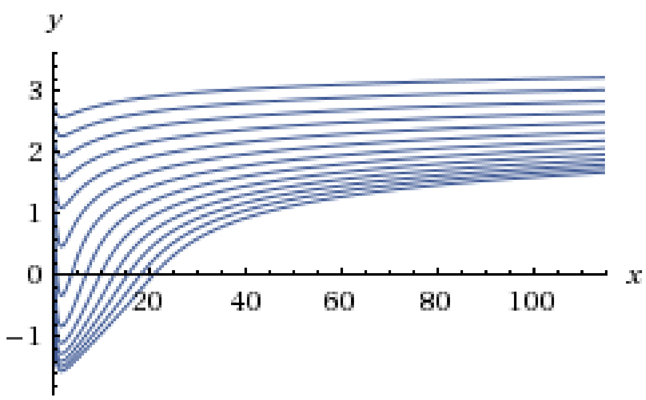
If the equation $M(x, y)\, dx + N(x, y)\, dy = 0$ can be changed into

$$\mathcal{M}\,(x)\, dx + \mathcal{N}\,(y) dy = 0,$$

then we may change it into $\mathcal{M}\,(x)\, dx = -\mathcal{N}\,(y)dy$. This way, terms with x are separated from terms with y. That is where the term "separation of variables" come from. By integrating both sides of the equality, we may obtain the solution to the equation.

Equation:	$(2x-4)\, d\,x - (x^2y^2 + x^2 + y^2 + 1)dy = 0$
Step 1.	$\dfrac{(2x-4)\, d\,x}{x^2+1} = (y^2+1)\, dy$
Step 2.	$\displaystyle\int \left[\frac{2x-4}{x^2+1}\right] dx = \int (y^2+1)\, dy$
Implicit Solution:	$\ln\,(x^2+1) - 4\,\arctan\, x = \dfrac{y^3}{3} + y + C$

Sample Solutions:



♣ Exact Equation

The equation

$$M(x, y)\, d\,x + N(x, y)\, d\,y = 0, \quad \text{where} \quad M_y(x, y) = N_x(x, y), \quad \text{is called exact,}$$

Given a smooth function $F(x, y)$, we have $F_{xy}(x, y) = F_{yx}(x, y)$. Thus we conclude that for our equation, there exists a constant function $C = F(x, y)$ such that

$$\frac{\partial F(x,y)}{\partial x} = F_x(x,y) = M(x,y) \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = F_y(x,y) = N(x,y),$$

with

$$0 = d\,C = d\,F(x,y) = M(x,y)\,d\,x + N(x,y)\,d\,y.$$

Algorithm. In an exact equation, we have:

$$C = F(x,y) = \int M(x,y)dx + \phi(y) \tag{6}$$

$$C = F(x,y) = \int N(x,y)dy + \psi(x). \tag{7}$$

Step 1. We must solve one of them. We should select the one with a simpler integral.
Step 2. If (6) is selected, then we need to find $\phi(y)$; for (7), $\psi(x)$ must be found.
Step 3. We evaluate

$$\frac{\partial F(x,y)}{\partial y} \quad \text{and equate with} \quad N(x,y)$$

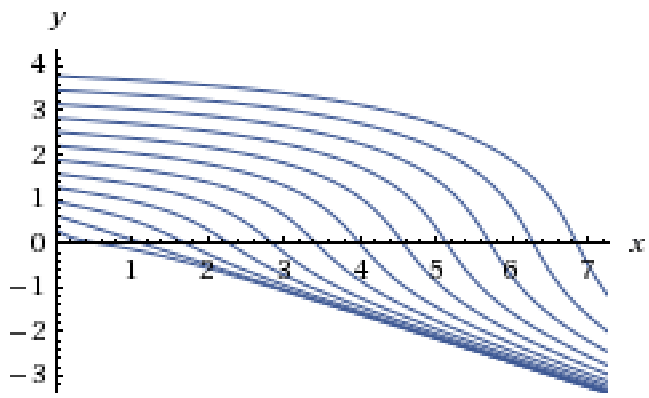
or

$$\frac{\partial F(x,y)}{\partial x} \quad \text{and equate with} \quad M(x,y).$$

This way, we obtain $\phi'(y)$ or $\psi'(x)$.
Step 4. To find $\phi(y)$ or $\psi(x)$, we just integrate $\phi'(y)$ or $\psi'(x)$. The solution of the equation is just

$$C = \int M(x,y)dx + \phi(y) \quad \text{or} \quad C = \int N(x,y)\,dy + \psi(x).$$

Equation:	$(x^2+2\,y)\,d\,x+(2\,x+6y^2)\,d\,y=0$
Step 1.	$M_y(x,y)=2=N_x(x,y)$
Step 2.	$C=F=\int M(x,y)\,dx=\int(x^2+2y)dx+\phi(y)\Rightarrow F=\dfrac{x^3}{3}+2\,x\,y+\phi(y)$
Step 3.	$0+2x+\phi'(y)=N(x,y)=2\,x+6y^2\ \Rightarrow\ \phi(y)=\int\phi'(y)\,dy=\int 6\,y^2\,dy=\dfrac{2y^3}{2}$
Implicit Solution:	$C=\dfrac{x^3}{3}+2\,x\,y+2\,y^3$ or $x^3+6\,x\,y+6\,y^3=C$
Sample Solutions:	



♡ **Integrating Factor.**

An integrating factor is a function which changes a non-exact equation (E) into an exact function, once it is multiplied by (E). Consider the differential equation:

$$M(x,y)\,d\,x+N(x,y)\,d\,y=0\,. \qquad (E)$$

1. If $\dfrac{M_y(x,y)-N_x(x,y)}{N(x,y)}$ is a function of x only, or $-\dfrac{[M_y(x,y)-N_x(x,y)]}{M(x,y)}$ is a function of y only, then the function :

$$u(x)=exp\left(\int\frac{M_y(x,y)-N_x(x,y)}{N(x,y)}\,dx\right) \quad \text{or} \quad v(y)=\exp\left(-\int\frac{M_y(x,y)-N_x(x,y)}{M(x,y)}\,dy\right).$$

is an integrating factor.

2. If the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree and $xM(x,y)+yN(x,y)\neq 0$, then

$$u(x,y)=\frac{1}{xM(x,y)+yN(x,y)}$$

is an integrating factor.

♣ **Differential Equation with Homogeneous Coefficients**

Suppose in the equation

$$M(x,y)\,d\,x + N(x,y)\,d\,y = 0\,, \qquad (E)$$

$M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree. Then by using a substitution we may solve the equation by the method of *separation of variables.*

♢ **Note.** Do not confound homogeneous equations with homogeneous functions. An equation with homogeneous coefficients $M(x,y)$ and $N(x,y)$ is not necessarily a homogeneous equation. All first order differential equations can be expressed as

$$M(x,y)\,d\,x + N(x,y)\,d\,y = 0.$$

The 0 on the right side of the equality does not make the equation homogeneous.

Algorithm. Suppose the functions $M(x,y)$ and $N(x,y)$ in (E) are both homogeneous of the same degree. Then by choosing

$$y = ux \;\text{ with }\; dy = xdu + udx \qquad \text{or} \qquad x = vy \;\text{ with }\; dx = ydv + vdy,$$

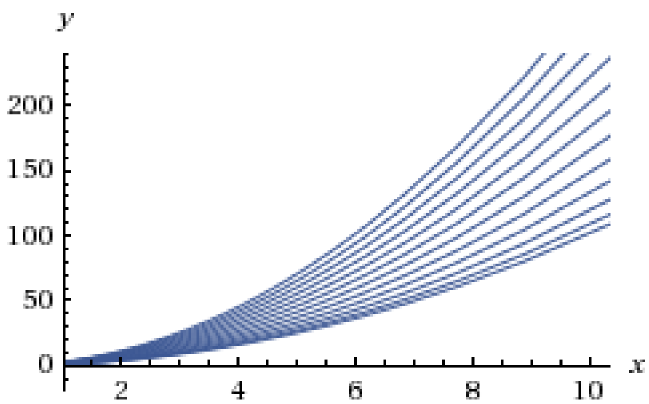
the equation (E) changes into

$$\widehat{M}(x,u)dx + \widehat{N}(x,u)du = 0 \quad \text{or} \quad \widehat{M}(v,y)dv + \widehat{N}(v,y)dy = 0.$$

The new equation can be solved by using the method of separation of variables.

Equation:	$(x^2 + 2y^2)\,d\,x - x\,y\,d\,y = 0$
Step 1.	$M(\lambda x, \lambda y) = \lambda^2 M(x,y)$ and $N(\lambda x, \lambda y) = \lambda^2 N(x,y)$ So $M(x,y)$ and $N(x,y)$ are both homogeneous of the same degree.
Step 2.	Let $y = xu$, then $dy = xdu + udx$
Step 3.	$(x^2 + 2\,x^2u^2)\,d\,x - x^2u(x\,d\,u + u\,d\,x) = 0$
Step 4.	$(x^2 + x^2u^2)\,dx = x^3u\,du \Rightarrow \quad (1+u^2)\,d\,x = xu\,d\,u$
Step 5.	$\displaystyle\int \left[\frac{1}{x}\right] dx = \int \left[\frac{u}{1+u^2}\right] d\,u \Rightarrow \quad \ln(Cx) = \frac{1}{2}\ln(1+u^2)$
Implicit Solution:	$C\,x^2 = 1 + \left[\dfrac{x}{y}\right]^2 \quad \Rightarrow \quad y^2 = Cx^4 - x^2$
Explicit Solution:	$y = \sqrt{Cx^4 - x^2}$ and $y = -\sqrt{Cx^4 - x^2}$

Sample Solutions:



Consider the equation:

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0. \qquad (E)$$

If the point (h,k) is a solution to the linear system:

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

Then by setting $x = u + h$ and $y = v + k$ in (E), we obtain the equation

$$(a_1u + b_v)du + (a_2u + b_2v)dv = 0$$

which has homogeneous coefficients of degree one.
If

$$(a_1x + b_1y + c_1) = s(a_2x + b_2y + c_2) + r,$$

then by substituting u for $(a_1x + b_1y + c_1)$ and eliminating x or y, we may solve the equation by seperation of variables.

♣ **Linear Differential Equation**
Consider the first order linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The function

$$u(x) = \exp\left(\int P(x)dx\right)$$

is an integrating factor. By multiplying the equation (E) by $u(x)$, we obtain

$$u(x)\,dy + y[P(x)u(x)\,dx] = u(x)Q(x)\,dx.$$

Notice that

$$P(x)u(x)\,dx = P(x)\exp\left(\int P(x)dx\right) = d[u(x)].$$

Thus the new equation is:

$$d[yu(x)] = u(x)Q(x)\,dx.$$

By integrating both sides of the equation, we obtain

$$yu(x) = \int u(x)Q(x)\,dx.$$

♡ **Bernouilli's Equation**
The following equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (n \neq 1)$$

is called a Bernouilli equation. For $n = 1$, the equation may be solved using the seperation of variables. By setting $z = y^{-n+1}$ in the equation, we obtain the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

♡ **Ricatti's Equation**
The nonlinear equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

which frequently occurs in physical applications is called Ricatti's equation. Its solution cannot be expressed in terms of elementary function. However when $R(x) = -1$, we can

change the equation into a second order linear differential equation by setting $y = \frac{z'}{z}$.
When $P(x) = 0$, then the equation becomes a Bernouilli equation:

$$\frac{dy}{dx} - Q(x)y = R(x)y^2.$$

♡ **Clairaut's Equation**
The nonlinear equation

$$y = xy' + f(y')$$

is called Clairaut's equation. By differentiating both sides of the equality with respect to x, we obtain the second order equation

$$[x + f'(y')]\, y'' = 0.$$

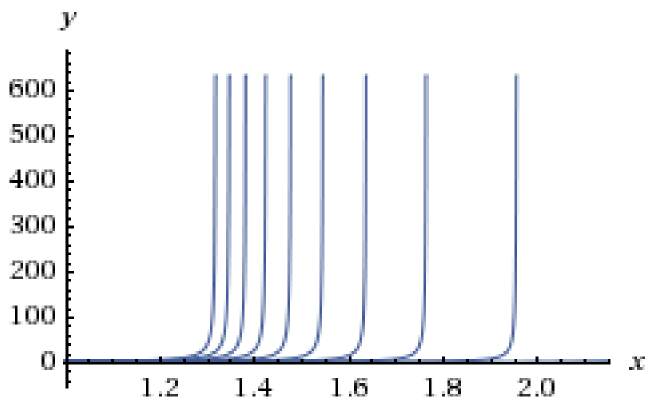
One set of solutions called *general solution,* is $y = c\,x + f(c)$ and is obtained from $y'' = 0$.

If $x + f'(y') = 0$, then we obtain the parametrized curve

$$x = -f'(t), \qquad y = f(t) - t f'(t).$$

This curve is also a solution, called the *singular solution.*

Equation:	$\dfrac{dy}{dx}+\dfrac{1}{x}\,y=xy^2$
Step 1.	$z=y^{-2+1}=y^{-1}$
Step 2.	$\dfrac{dz}{dx}-\dfrac{1}{x}z=-x$
Step 3.	$u(x)=e^{-\int\frac{1}{x}dx}=x^{-1}$
Step 4.	$x^{-1}z=\int -x^{-1}x\,d\,x=-\int 1\,d\,x \Rightarrow \quad x^{-1}y^{-1}=-x+C$
Explicit Solution:	$y=-\dfrac{1}{x(C-x)}$
Sample Solutions:	



We just solve an equation which is at the same time a Bernouilli and Ricatti equation.

♣ Solving an Equation by Inspection

The following identities may help you solve some differential equations.

$ydx+xdy=d(xy)$	$\dfrac{ydx-xdy}{y^2}=d\left(\dfrac{x}{y}\right)=-d\left(\dfrac{y}{x}\right)$
$mx^{m-1}y^ndx+nx^my^{n-1}dy=d(x^my^n)$	$\dfrac{mx^{m-1}y^ndx-nx^my^{n-1}dy}{y^{2n}}=d\left(\dfrac{x^m}{y^n}\right)$
$\dfrac{ydx+xdy}{xy}=d(\ln(xy))$	$\dfrac{ydx-xdy}{xy}=d\left[\ln\left(\dfrac{x}{y}\right)\right]=-d\left[\ln\left(\dfrac{y}{x}\right)\right]$
$\dfrac{ydx+xdy}{1+x^2y^2}=d\left[\arctan(xy)\right]$	$\dfrac{ydx-xdy}{x^2+y^2}=d\left[\arctan\left(\dfrac{x}{y}\right)\right]=-d\left[\arctan\left(\dfrac{y}{x}\right)\right]$

For example, to solve the equation:

$$(x^2y + y^3 - y - x^2y^3)\, d\,x + (x + x^3y^2 - x^3 - x\,y^2)\, d\,y = 0\,,$$

we factor out as follows:

$$y(x^2 + y^2)\, d\,x - y(1 + x^2y^2)\, d\,x + x(1 + x^2y^2)\, d\,y - x\,(x^2 + y^2)\, d\,y = 0 \qquad (8)$$

$$(x^2 + y^2)\,[\,y\,d\,x + x\,d\,y\,] = (1 + x^2y^2)\,[\,y\,d\,x - x\,d\,y\,] \qquad (9)$$

$$\frac{y\,d\,x + x\,d\,y}{1 + x^2y^2} = \frac{y\,d\,x - x\,d\,y}{x^2 + y^2} \quad \Longrightarrow \quad d\big[\,\arctan(xy)\big] = d\left[\arctan\left(\frac{y}{x}\right)\right] \qquad (10)$$

$$\arctan(xy) - \arctan\left(\frac{y}{x}\right) = C. \qquad (11)$$

♣ **Picard's Successive Approximations**

The method of successive approximations, or Picard iteration, provides a method that can, in principle, be used to solve any initial value problem.

Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0\,. \end{cases} \qquad \Longrightarrow \qquad \begin{cases} dy = f(x,y)\,d\,x \\ y(x_0) = y_0\,. \end{cases}$$

By integrating both sides of the differential equation form x_0 to x with respect to x and choosing y_0 as the constant C, we obtain the new equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt. \qquad (E_1)$$

Now, since $y(x_0) = y_0$, the new equation (E_1) is just an alternate way of writing the initial-value problem. Furthermore, if we differentiate both sides of (E_1), we obtain the differential equation

$$y'(x) = f(x, y(x)).$$

We now define a sequence of functions $\{y_n(x)\}_0^{\infty}$, called *Picard's iterations,* by successive formulae:

Picard's Successive Iteration Method

Equation:

$$\begin{cases} y' = x\,(y - x^2 + 2)\,; \\ y(0) = 1\,. \end{cases} \qquad \Longleftrightarrow \qquad y(x) = 1 + \int_0^x t(y(t) - t^2 + 2))dt.$$

The approximate solutions are:

$$y_0(x) = y(0) = 1$$

$$y_1(x) = 1 + \int_0^x t(3 - t^2))\,d\,t = 1 + \frac{3}{2}\,x^2 - \frac{1}{4}\,x^4$$

$$y_2(x) = 1 + \int_0^x t\left(3 + \frac{t^2}{2} - \frac{t^4}{4}\right)dt = 1 + \frac{3}{2}\,x^2 + \frac{1}{8}\,x^4 - \frac{1}{24}\,x^6$$

$$y_3(x) = 1 + \int_0^x t\left(3 + \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{24}\right)dt = 1 + \frac{3}{2}\,x^2 + \frac{1}{8}\,x^4 + \frac{1}{48}\,x^6 - \frac{1}{192}\,x^8$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_n(x) = x^2 + \left[1 + \frac{1}{1!}\frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(\frac{x^2}{2}\right)^4 + \cdots\ \cdots\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y(x) = x^2 + e^{\frac{x^2}{2}}$$

External Link: *http://www.wolframalpha.com/input/?*
To solve a differential equation, click on the link bellow:

Wolfram Mathematica Online website