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Linear Differential Equation

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The set $\mathcal{F}$ of all complex-valued functions is known to be a vector space of infinite dimension. Solutions to any linear differential equations, form a subspace of $\mathcal{F}$ of dimension n. To define the subspace, we need to find a basis for the subspace; so finding n different solutions tis not enough, the must be linearly independent.

♣ **Linearly Independent functions**

A set $\{f_k(x)\} \quad (k = 1, 2, \dots, n)$ of functions is said to be *linearly independent*, if $f_k(x)$'s satisfy the following condition:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \textit{if and only if} \quad c_1 = c_2 = \ldots = c_n = 0.$$

Let $f_1(x), f_2(x), \dots, f_n(x)$ be $n-1$ times differentiable funtions. Then the function

$$W\left[f_1(x), f_2(x), \dots, f_n(x)\right] = \det \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f^{(n-1)}(x) \end{pmatrix}$$

is called the *Wronskian* of $f_k(x)$'s. To prove that $\{f_k(x)\} \quad (k = 1, 2, \dots, n)$ is a linearly independent set, it is usually simpler to show that

$$W\left(f_1(x), f_2(x), \dots, f_n(x)\right) \neq 0.$$

For example $\sin x$ and $\cos x$ are linearly independent since

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

Theorem 1. *Consider the n-th order homogeneous linear differential equation*

$$\left[a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)I\right] y = 0,$$

where the coefficients $a_i(x)$'s are continuous functions. Then the solutions to this equation form a vector space of dimension n. Thus if

$$f_1(x), \; f_2(x), \; \dots, \; f_n(x)$$

are solutions to the equation with $W\,[f_1(x), f_2(x), \ldots, f_n(x)] \neq 0$, *then*

$$\sum_{k=1}^{n} c_k f_k(x), \quad \textit{where} \quad c_k \in \mathbb{C}$$

is also a solution to the LDE.

Theorem 2. *Suppose the functions* $f(x)$ *and* $g(x)$ *are solutions to the homogeneous equations* $P(x,D)y = 0$ *and* $Q(x,D)y = 0$, *respectively. Then* $a\,f(x) + b\,g(x)$ *is a solution to the equation* $P(x,D)Q(x,D)\,y = 0$.
The general solution of the n-th order non-homogeneous linear differential equation

$$P(x,D)y = R(x)$$

is $y = y_c + y_p$, *where* y_c *is a solution to the homogeneous equation* $P(x,D)\,y = 0$, *called the complementary solution; and* y_p *is a particular solution to* $P(x,D)y = R(x)$. *If*

$$R(x) = R_1(x) + R_2(x) + \cdots + R_s(x)\,,$$

then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_s},$$

where each y_{p_k} *is a particular solution to the equation* $P(x,D)y = R_k(x)$, *is a solution to*

$$P(x,D)y = R(x)\,.$$

♣ Linear Differential Equations With Constant Coefficients
Recall that

$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$	$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

By using the fact that $y = e^{ax}$ is a solution to $(D - aI)y = 0$, we obtain:

Equation:	**Solution:**
$[(D - aI)^2 - b^2]y = 0$	$y = c_1 \cosh x + c_2 \sinh x$
$[(D - aI)^2 + b^2]y = 0$	$y = c_1 \cos x + c_2 \sin x$

If y is a solution of $[(D - aI)^2 \pm b^2]\,y = 0$, then

$$x^k y \qquad (k = 0, 1, \ldots, n-1)$$

are linearly independent solutions of $[(D - aI)^2 \pm b^2]^n y = 0$.

♠ Exponential Shift

Consider the linear differential equation of order n with constant coefficient

$$P(D)y = R(x).$$

If $P(D) = (D - aI)Q(D)$, then

$$e^{-ax}P(D)y = P(D + aI)(e^{-ax}y) = D\,Q(D + aI)(e^{-ax}y) = e^{-ax}R(x).$$

To solve this new equation, we set $z = e^{-ax}y$ and try to solve the linear differential equation of order $n-1$

$$Q(D + a)z = \int e^{-ax}R(x)\,dx.$$

The solution to the original equation will be

$$y = e^{ax}z = e^{ax}z_c + e^{ax}z_p.$$

♡ The Method of Undetermined Coefficients

let $P(D)y = R(x)$, where

$$P(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_nI.$$

To solve the equation, we proceed as follows

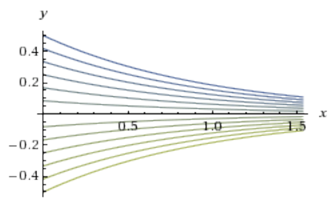
Step 1. Solve the homogeneous equation $P(D)y = 0$ and find y_c.

Step 2. Find an equation $F(D)y = 0$ which has y_p as a solution.

Step 3. Solve the homogeneous equation $P(D)F(D)y = 0$ and find the general solution $y = y_c + y_p$. Here y_p has undetermined coefficients.

Step 4. Find the undetermined coefficients of y_p by using the linear system $P(D)y_p = R(x)$.

Equation:	$[D^2 - 2D - 3I]y = 10\sin x$
Step 1.	$y_c = c_1e^{3x} + c_2e^{-x}$
Step 2.	$10\sin x \Rightarrow \quad F(D)y = [D^2 + 1]y = 0$
Step 3.	$y = c_1e^{3x} + c_2e^{-x} + A\cos x + B\sin x$
Step 4.	$y_p = \cos x - 2\sin x$
Sample Solutions:	



♣ Second Order Linear Equations

The exponential shift may be used to solve the equation

$$[a_0D^2 + a_1D + a_2I]y = R(x).$$

Algorithm. Any second order Linear differential equation with constant coefficients may be expressed as

$$[(D - aI)(D - bI)]y = R(x).$$

Step 1. Multiply both sides of the equation by e^{-ax}.

$$\boxed{\begin{array}{c} e^{-ax}[(D - aI)(D - bI)]y = e^{-ax}R(x) \\ D[D + (a - b)I)](e^{-ax}y) = e^{-ax}R(x) \end{array}}$$

Step 2. Set $z_1 = e^{-ax}$ and then integrate both sides of the equation.

$$\boxed{[D + (a - b)I]z_1 = \int e^{-ax}R(x) = R_1(x)}$$

Step 3. Use the first step, the new equation, and $e^{(a-b)x}$.

$$\boxed{\begin{array}{c} e^{(a-b)x}[D + (a - b)I]z_1 = e^{(a-b)x}R_1(x) \\ [D + (a - b)I)]\left(e^{(a-b)x}z_1\right) = e^{(a-b)x}R_1(x) \end{array}}$$

Step 4. To obtain the solution, set $z_2 = e^{(a-b)x}z_1 = e^{-bx}y$ and then integrate.

$$\boxed{z_2 = e^{-bx}y = \int e^{-ax}R_(x) = R_2(x)}$$

♠ **Note.** Clearly we can use this method for higher order equations, but it is not always possible to find the exact values of the roots of a polynomial of degree higher than 2.
Here are some examples.

$$\boxed{\text{If } [D^2 + I]y = R(x) \text{ then } y_p = (-\int R(x)\sin x dx)\cos x + (\int R(x)\cos x dx)\sin x}$$

If $[D^2 + a^2I]\, y = c\cos a\, x$, then $y_p = -\frac{cx}{2a}\cos a\, x$
If $[D^2 + a^2I]\, y = c\sin a\, x$ then $y_p = \frac{cx}{2a}\sin a\, x$

If $[D^2 \pm a^2I]\, y = c\cos b\, x$ then for $\lvert a\rvert \neq \lvert b\rvert$, we have $y_p = -\frac{c\cos b\, x}{\pm a^2 - b^2}$
If $[D^2 \pm a^2I]\, y = c\sin bx$then $y_p = \frac{c\sin bx}{\pm a^2 - b^2}$

♡ Reduction of Order

Consider the second order homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

If y_1 is one of the solutions of this equation, then we can always express the second solution as $y_2 = v(x)y_1$ for some function $v(x)$. So our task is to find the function $v(x)$.
In order to find $v(x)$, first we find

$$y_2' = y_1'v(x) + y_1v'(x) \qquad y_2'' = y_1''v(x) + 2y_1'v'(x) + y_1v''(x),$$

then we set y_2 in the equation and obtain

$$a_0(x)[y_1''v(x) + 2y_1'v'(x) + y_1v''(x)] + a_1(x)[y_1'v(x) + y_1v'(x)] + a_2(x)[y_1v(x)] = 0.$$

Next we set $w(x) = v'(x)$. Since y_1 is a solution to our homogeneous equation i.e.,

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0,$$

the above equation reduces to the following *first order linear differential equation*

$$a_0(x)[2y_1'w(x) + y_1w'(x)] + a_1(x)[y_1w(x)] = 0.$$

Once the solution $w(x)$ is found; $v(x)$ can be then obtained by integrating $w(x)$.

Equation:	$(x^2+1)y'' - 2xy' + 2y = 0$
Given	$y_1 = x$
Step 1.	$y_2 = xv(x) \quad y_2' = xv'(x) + v(x) \quad y_2'' = xv''(x) + 2v'(x)$
Step 2.	$w(x) = v'(x) \quad x(x^2+1)w'(x) + 2w(x) = 0$
Step 3.	$\int \frac{dw}{w} = -\int \frac{2dx}{x(x^2+1)} \Rightarrow \quad \ln w = \ln(\frac{x^2+1}{x^2})$
Step 4.	$w(x) = \frac{x^2+1}{x^2} \quad v(x) = \int (1 + \frac{1}{x^2})dx = x - \frac{1}{x} \quad y_2 = xv(x) = x^2 - 1$
Solution:	$y = C_1x + C_2(x^2-1)$

♡ Variation of Parameters

Consider the equation:

$$y'' + P(x)y' + Q(x)y = R(x).$$

Let $y_p = A(x)y_1 + B(x)y_2$, where y_1 and y_2 are the complementary solutions. The fact that there are infinitely many possibility to choose $A(x)$ and $B(x)$, we may assume that $A'(x)y_1 + B'(x)y_2 = 0$.
By solving the linear system:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

for $A'(x)$ and $B'(x)$; and then integrating $A'(x)$ and $B'(x)$, we will be able to find y_p.

Equation:	$y'' + y' = \tan x$
Step 1.	$y_1 = \cos x \quad y_2 = \sin x$
Step 2.	$y_1' = -\sin x \quad y_2' = \cos x$
Step 3.	$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$
Step 4.	$A'(x) = \cos x - \sec x \quad \text{and} \quad B'(x) = \sin x$
Step 5.	$A(x) = \sin x - \ln\lvert \sec x + \tan x\rvert \quad \text{and} \quad B(x) = -\cos x$
Solution:	$y_p = -\ln\lvert \sec x + \tan x\rvert \cos x$

♣ Linear Differential Equations With Non-Constant Coefficients

In the preceding section we showed how to obtain the general solution, using complementary solution. It is usually difficult to find complementary solution of a linear differential equation with non-constant coefficients even when the order is only 2. But sometimes one can use a transformation which changes the equation with non-constant coefficients into an equation with constant coefficients.

One special class of equations with non-constant coefficients, for which the complementary solution can be found is the so-called class of *Cauchy-Euler* equations.

♠ The Cauchy-Euler Equation

The equation

$$[a_0 x^n D_x^n + a_1 x^{n-1} D_x^{n-1} + \cdots + a_{n-1} x D_x + a_n I]y = R(x)$$

is known as the *Cauchy-Euler* equation. By setting $x = e^t$ in the equation, we obtain an equation with constant coefficient.

Theorem 3. *If $x = e^t$, then $t = \ln x$ and*

$$D_x^n y = \frac{1}{x^n}\left[\prod_{k=0}^{n-1}(D_t - kI)\right] y.$$

By using the above theorem, we change the Cauchy-Euler equation into

$$\left[a_0\left[\prod_{k=0}^{n-1}(D_t - kI)\right] + a_1\left[\prod_{k=0}^{n-2}(D_t - kI)\right] + \ldots + a_{n-1}D_t + a_n I\right] y = R(e^t).$$

This equation has only constant coefficients.