Numerical Solution of Ordinary Differential Equations

A first order differential equation may be expressed as follows:

$$\frac{dy}{dx} = f(x, y).$$

The problem

$$\begin{cases} \text{Solve:} & \frac{dy}{dx} = f(x,y) \\ \text{Subject to:} & y(x_0) = y_0 \end{cases} \tag{1}$$

is called an *initial-value problem.* The first equation gives the slope of the curve y at any point x, and the second equation specifies one particular value of the function $y(x)$.

In the numerical solution of differential equations, we rarely expect to obtain the solution directly as a *formula* giving $y(x)$ as a function of x. Instead, we usually construct a table of function values of the form

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \quad \ldots \quad \ldots \quad \ldots \quad (x_k, y_k), \quad \ldots \quad \ldots$$

Here, y_i is the computed approximation value of $y(x_i)$ at x_i. From those values, a spline function or other approximating functions can be constructed to represent our solution to the initial-value problem.

♣ **Taylor-Series Method**. For the Taylor-series method, it is necessary to assume that various partial derivatives of $f(x, y)$ exist. To illustrate the method we take a concrete example:

$$\begin{cases} y' = \cos x - \sin y + x^2 \\ \\ y(-1) = 3 \end{cases} \tag{2}$$

At the heart of the procedure is the Taylor series for y, which we write as

$$y(x+h) = x(h) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) + \cdots \tag{3}$$

The derivative appearing here can be obtained from the differential equation (5). They are

$$\begin{aligned} y'' &= -\sin x - y'\cos y + 2x \\ y''' &= -\cos x - y''\cos y + (y')^2 \sin y + 2 \\ y^{(4)} &= \sin x - y'''\cos y + 3y'y''\sin y + (y')^3 \cos y \end{aligned}$$

At this point, our patience wears thin and we decide to use only terms up to and including h^4 in the Formula (3). The term that we have not included start with a term in h^5, and they constitue collectively the *truncation error* inherent in our procedure. The resulting numerical method is said to be of *order 4*.

Here is an algorithm for this method and the problem (2).

Algorithm for Taylor-Series Method

INPUT : The initial $x_0 = -1$; the initial $y_0 = 3$; the step size h; integer n.

FOR $k = 1, 2, \ldots, n$ DO

$$y' := f(x, y) = \cos x - \sin y + x^2$$

$$y'' := df(x, y) = -\sin x - y' \cos y + 2x$$

$$y''' := d^2 f(x, y) = -\cos x - y'' \cos y + (y')^2 \sin y + 2$$

$$y^{(4)} := d^3 f(x, y) = \sin x - y''' \cos y + 3y'y'' \sin y + (y')^3 \cos y$$

$$y := y + h(y' + \frac{h}{2}(y'' + \frac{h}{3}(y''' + \frac{h}{4}y^{(4)})))$$

$$x := x + h$$

OUTPUT: k, x, y

END

♣ **Euler's Method**. The Taylor-series method with $n = 1$ is called *Euler's method*. It looks like this:

$$y(x + h) = y(x) + hf(x, y)$$

This formula has the obvious advantage of not requiring any differentiation of $f(x, y)$. This advantage is offset by the necessity of taking small values for h to gain acceptable precision. Still, the method serves as a useful example and is of great importance theoretically since existence theorems can be based on it.

Example. Apply the Euler's method to the initial-value problem

$$\begin{cases} y' = 2x + y \\ y(0) = 1, \end{cases} \tag{4}$$

where $h = 0.2$ and $n = 5$

Solution.

$$x_1 = x_0 + h = 0.2, f(x_0, y_0) = f(0, 1) = 1.000, \qquad (n = 1)$$

$$y_1 = y_0 + hf(x_0, y_0) = 1.000 + 0.2(1.000) = 1.200$$

$$x_2 = x_1 + h = 0.4, f(x_1, y_1) = f(0.2, 1.200) = 1.600, \qquad (n = 2)$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.200 + 0.2(1.600) = 1.520$$

$$x_3 = x_2 + h = 0.6, f(x_2, y_2) = f(0.4, 1.520) = 2.320, \qquad (n = 3)$$

$$y_3 = y_2 + hf(x_2, y_2) = 1.520 + 0.2(2.320) = 1.984$$

$$\begin{aligned}
x_4 =& x_3 + h = 0.8, f(x_3, y_3) = f(0.6, 1.984) = 3.184, && (n = 4)\\
y_4 =& y_3 + hf(x_3, y_3) = 1.984 + 0.2(3.184) = 2.621\\
x_5 =& x_4 + h = 1.0, f(x_4, y_4) = f(0.8, 2.621) = 4.221, && (n = 5)\\
y_5 =& y_4 + hf(x_4, y_4) = 2.621 + 0.2(4.221) = 3.465.
\end{aligned}$$

♣ **Runge-Kutta Methods**. The Taylor-series method has the drawback of requiring some analysis prior to programming it. We shall have to determine formulae for y'', y''', and $y^{(4)}$ by successive differentiation in (1). Then these functions will have to be programmed.

The Runge-Kutta methods avoid this difficulty although they do imitate the Taylor-series method by means of clever combinations of values of $f(x, y)$.

♡ **Second Order Runge-Kutta Method**. Let us begin with the Taylor series for $f(x, y)$:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y''(x) + \frac{h^4}{4!}y^{(4)}(x) + \cdots \tag{5}$$

From the differential equation, we have

$$\begin{aligned}
y'(x) =&\ f\\
y''(x) =&\ f_x + f_y y' = f_x + f_y f\\
y'''(x) =&\ f_{xx} + f_{xy} f + [f_x + f_y f] f_y + f[f_{xy} + f_{yy} f]\\
&\ etc
\end{aligned}$$

The first three terms in Equation (5) can be written now in the form

$$\begin{aligned}
y(x+h) =& y(x) + hf(x, y) + \frac{1}{2}h^2[f_x(x, y) + f(x, y)f_y(x, y)] + O(h^3)\\
=& y(x) + \frac{1}{2!}hf(x, y) + \frac{1}{2}h[f(x, y) + hf_x(x, y) + hf(x, y)f_y(x, y)] + O(h^3)
\end{aligned} \tag{6}$$

In order to eliminate the partial derivatives in Equation (6), we use the Taylor series in two variables for

$$f(x+h, y+hf(x, y)) = f(x, y) + hf_x(x, y) + hf(x, y)f_y(x, y) + O(h^2)$$

Thus Equation (6) becomes

$$y(x+h) = y(x) + \frac{1}{2}hf(x, y) + \frac{1}{2}hf(x+h, y+hf(x, y)) + O(h^3)$$

Hence, the formula for advancing the solution is

$$y(x+h) = y(x) + \frac{h}{2}f(x, y) + \frac{h}{2}f(x+h, y+hf(x, y))$$

or equivalently

$$y(x+h) = y(x) + \frac{1}{2}(F_1 + F_2) \tag{7}$$

where

$$\begin{cases} F_1 = & hf(x,y) \\ F_2 = & hf(x+h, y+F_1) \end{cases}$$

This formula can be used repeatedly to advance the solution one step at a time. It is called a *Second-Order Runge-Kutta Method.* It is also known as *Heun's Method.*

♡ **Fourth Order Runge-Kutta Method**. The higher-order Runge-Kutta formulae are very tedious to derive, and we shall not do so. The formulae are rather elegant, however, and are easily programmed once the have been derived. Here are the formulae for the *classical Fourth-Order Runge-Kutta Method:*

$$y(x+h) = y(x) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4) \tag{8}$$

where

$$\begin{cases} F_1 = & hf(x,y) \\ F_2 = & hf(x+\frac{1}{2}h, y+\frac{1}{2}F_1) \\ F_3 = & hf(x+\frac{1}{2}h, y+\frac{1}{2}F_2) \\ F_4 = & hf(x+h, y+F_3) \end{cases}$$

This is called a fourth-order method because it reproduces the terms in Taylor series up to and including the one involving h^4. The error is therefore $O(h^5)$.

Algorithm for Fourth-Order Runge-Kutta Method

INPUT : The function $f(x,y)$; The initial x_0; the initial y_0; the step size h; integer n.

FOR $k = 1, 2, \ldots, n$ DO

$F_1 := hf(x,y)$

$F_2 := hf(x+\frac{1}{2}h, y+\frac{1}{2}F_1)$

$F_3 := hf(x+\frac{1}{2}h, y+\frac{1}{2}F_2)$

$F_4 := hf(x+h, y+F_3)$

$y := y + \frac{1}{6}[F_1 + 2F_2 + 2F_3 + F_4]$

$x := x + h$

OUTPUT: k, x, y

END

Example. Apply the fourth-order Runge-Kutta method to the initial-value problem

$$\begin{cases} y' = 2x + y \\ \quad y(0) = 1, \end{cases} \tag{9}$$

where $h = 0.2$ and $n = 2$

Solution.

$$F_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2(1) = 0.2, \qquad (n = 1)$$
$$x_0 + \frac{h}{2} = 0 + \frac{1}{2}(0.2) = 0.1 \quad \text{and} \quad y_0 + \frac{1}{2}F_1 = 1 + \frac{1}{2}(0.2) = 1.1,$$
$$F_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{F_1}{2}) = 0.2f(0.1, 1.1) = 0.2(1.3) = 0.26$$
$$y_0 + \frac{1}{2}F_2 = 1 + \frac{1}{2}(0.26) = 1.13,$$
$$F_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{F_2}{2}) = 0.2f(0.1, 1.13) = 0.2(1.33) = 0.266$$
$$x_0 + h = 0 + 0.2 = 0.2 \quad \text{and} \quad y_0 + \frac{1}{2}F_3 = 1 + 0.266 = 1.266,$$
$$F_4 = hf(x_0 + h, y_0 + F_3) = 0.2f(0.2, 1.266) = 0.2(1.666) = 0.3332$$
$$y_1 = 1 + \frac{1}{6}[F_1 + 2F_2 + 2F_3 + F_4] = 1 + \frac{1}{6}(0.2 + 0.52 + 0.532 + 0.332) = 1.2642$$
$$F_1 = hf(x_1, y_1) = 0.2f(0.2, 1.2642) = 0.2(1.6642) = 0.33284, \qquad (n = 2)$$
$$x_1 + \frac{h}{2} = 0.2 + \frac{1}{2}(0.2) = 0.3 \quad \text{and} \quad y_1 + \frac{1}{2}F_1 = 1.2642 + \frac{1}{2}(0.33284) = 1.43062,$$
$$F_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{F_1}{2}) = 0.2f(0.3, 1.43062) = 0.2(2.03062) = 0.40612$$
$$y_1 + \frac{1}{2}F_2 = 1.2642 + \frac{1}{2}(0.40612) = 1.46726,$$
$$F_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{F_2}{2}) = 0.2f(0.3, 1.43062) = 0.2(2.06726) = 0.41345$$
$$x_1 + h = 0.2 + 0.2 = 0.4 \quad \text{and} \quad y_1 + \frac{1}{2}F_3 = 1.2642 + 0.41345 = 1.67765,$$
$$F_4 = hf(x_1 + h, y_1 + F_3) = 0.2f(0.4, 1.67765) = 0.2(2.47765) = 0.49553$$
$$y_2 = 1.2642 + \frac{1}{6}[F_1 + 2F_2 + 2F_3 + F_4] = 1.2642 + \frac{1}{6}(0.33284 + 0.81224 + 0.8269 + 0.49553 = 1.6754$$