## Series Solutions of Linear Differential Equations

In this chapter we shall solve some second-order linear differential equation about an initial point using The *Taylor series.*

In all that follows we assume that (E) is a second-order homogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \tag{1}$$

where the coefficients $a_0(x)$, $a_1(x)$, and $a_2(x)$ are polynomials. The *normalized form* of (E),

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

is obtained when the equation is divided by the polynomial $a_0(x)$.

A function $f(x)$ is said to be *analytic* at $x_0$ if its *Taylor series* about $x_0$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

exists and converges to $f(x)$ for all $x$ in some open interval containing $x_0$. Functions such as polynomials, $e^x$, $\sin x$, and $\cos x$ are analytic at all points. A rational function is analytic at all points except, when the denominator becomes zero.

If both $P(x)$ and $Q(x)$ are analytic at $x_0$, then $x_0$ is called an *ordinary* point of the equation. If either (or both) of these function is not analytic at $x_0$, then $x_0$ is called a *singular* point of the differential equation.

Let $x_0$ be a singular point of (E), then we define $deg_P(x_0)$ (rep $deg_Q(x_0)$ ) as the power of $(x - x_0)$ in the denominator of $P(x)$ (resp. $Q(x)$ ). If

$$0 \le deg_P(x_0) \le 1 \qquad 0 \le deg_Q(x_0) \le 2,$$

then $x_0$ is called a *regular singular point* (RSP), otherwise $x_0$ is said to be *irregular singular point* (ISP).

Consider the differential equation

$$x^3(x+1)^2(x-2)^3(x^2+1)y'' + x^2(x+1)(x^2-4)^2y' + xy = 0.$$

The normalized form of this equation is

$$y'' + \frac{(x+2)^2}{x(x+1)^2(x-2)(x^2+1)}y' + \frac{1}{x^2(x+1)^2(x-2)^3(x^2+1)}y = 0.$$

The point $0, -1, 2, -i,$ and $i$ are singular points. All singular points, except $2$ are regular.

♣ **Power Series Solutions About an Ordinary Point**. If the point $x_0$ is an ordinary point of the differential equation (E), then there is a power series solutions of the form

$$\sum_{k=0}^{\infty} C_k(x-x_0)^k \tag{3}$$

for the equation (E).

Now that we are asured that the equation about the regular point has a power series solution; we need to find the coefficients $C_k's$ in (3).

**Algorithm.** We assume that

$$y = \sum_{k=0}^{\infty} C_k(x-x_0)^k.$$

Hence

$$y' = \sum_{k=1}^{\infty} kC_k(x-x_0)^{k-1} \quad \textit{and} \quad y'' = \sum_{k=2}^{\infty} k(k-1)C_k(x-x_0)^{k-2}.$$

Next we put the power series corresponding to $y,\ y'$, and $y''$ in (2) (the non-normalized form of the equation (E) ), we obtain

$$a_0(x)\left[\sum_{k=2}^{\infty} k(k-1)C_k(x-x_0)^{k-2}\right] + a_1(x)\left[\sum_{k=1}^{\infty} kC_k(x-x_0)^{k-1}\right] + a_2(x)\left[\sum_{k=0}^{\infty} C_k(x-x_0)^k = 0\right].$$

We now need to set the coefficient of each $(x-x_0)$ term to zero. We illustrate the whole procedure with the following example.

**Example.** Consider the differential equation

$$y'' + xy' + (x^2+2)y = 0. \tag{4}$$

Notice that the point $x_0 = 0$ is an ordinary point, so we assume that

$$y = \sum_{k=0}^{\infty} C_k x^k$$

is a power series solution to the equation (4). Differentiating term by term we obtain

$$y' = \sum_{k=1}^{\infty} kC_k x^{k-1} \quad \textit{and} \quad y'' = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}.$$

Substituting the series for $y,\ y'$, and $y''$ in (4), we obtain

$$\left[\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}\right] + x\left[\sum_{k=1}^{\infty} kC_k x^{k-1}\right] + (x^2+2)\left[\sum_{k=0}^{\infty} C_k x^k = 0\right] = 0. \tag{5}$$

Multiplying $x$ by $y'$ and $(x^2+1)$ by $y$ in (5), we obtain

$$\left[\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}\right] + \left[\sum_{k=1}^{\infty} kC_k x^k\right] + \left[\sum_{k=0}^{\infty} C_k x^{k+2}\right] + 2\left[\sum_{k=0}^{\infty} C_k x^k\right] = 0. \tag{6}$$

In order to match the powers of $x$'s in (6), we need to start $k$ in each series from different values. Thus (6) becomes

$$\left[\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k\right] + \left[\sum_{k=1}^{\infty} kC_k x^k\right] + \left[\sum_{k=2}^{\infty} C_{k-2}x^k\right] + 2\left[\sum_{k=0}^{\infty} C_k x^k\right] = 0. \tag{7}$$

Now we start **k** from **2** in all four power series in (7); so we need to extract terms with indices less than two. Once this is done we combine all of the power series into a unique power serie. The equation (7) then becomes

$$[2\,C_2 + 6\,C_3 x] + [C_1 x] + [2\,C_0 + 2\,C_1 x] + \left[\sum_{k=0}^{\infty}\left[(k+2)\,(k+1)\,C_{k+2} + k\,C_k + C_{k-2} + C_k\right]x^k\right] = 0\,. \tag{8}$$

From (8) we obtain the system

$$\begin{cases} 2\,C_2 + 2\,C_0 = 0 \\ 6\,C_3 + 3\,C_1 = 0 \end{cases} \tag{9}$$

and a formula

$$[(k+2)(k+1)\,C_{k+2} + k\,C_k + C_{k-2} + C_k] = 0 \tag{10}$$

called the *recurrence formula.*

The system (9) gives us

$$C_2 = -\,C_0 \qquad \textit{and} \qquad C_3 = -\frac{1}{2}\,C_1. \tag{11}$$

The conditions (11) and the recurrence formula enable us to express each coefficient $C_{k+2}$ for $k \geq 2$ in terms of the previous coefficients $C_k$ and $C_{k-2}$, thus giving

$$C_{k+2} = -\,\frac{(k+1)\,C_k + C_{k-2}}{(k+1)\,(k+2)}, \qquad (k \geq 2) \tag{12}$$

♣ **Power Series Solutions About a Regular Singular Point**. We shall restrict our study to the interval $x > 0$ and if we then wish to find solutions for negative interval, by substituting $u = -\,x$ in (E), we may study the resulting equation for positive $u$. If the singular point is not zero, then by translating the origin to that point; this way, we obtain a new equation with a singular point at $x_0 = 0$. Since the equation (E) behave badly at an irregular singular point, we only study the case of regular singular points.

♡ **The Method of Frobenius**. If $x_0$ is a regular singular point of the differential equation (E), then for some real or complex constant (which may be determined), the power serie

$$\sum_{k=0}^{\infty} C_k x^{k+r}, \qquad (C_0 \neq 0) \tag{12}$$

is a solution to (E).

The fact that $x_0 = 0$ is a RSP implies that the Mac Laurin series of

$$xP(x) = p_0 + xp_1 + x^2 p_2 + \ldots = \sum_{k=0}^{\infty} p_k x^k \qquad \text{and}$$

$$x^2 Q(x) = q_0 + xq_1 + x^2 q_2 + \ldots = \sum_{k=0}^{\infty} q_k x^k$$

By putting the power series of $y$, $xP(x)$ and $x^2Q(x)$ into the differential equation (E), and equating the lowest term of this series to zero, we obtain the equation

$$r^2 + (p_0 - 1)r + q_0 = 0. \tag{13}$$

This equation is called the *indicial equation of (E).*

Let $r_1$ and $r_2$ be the roots of the indicial equation with $Re(r_1) \geq Re(r_2)$.

**Case 1.** If $r_1 - r_2$ is not an integer, then the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k$$

where $C_0 \neq 0$.

**Case 2.** If $r_1 - r_2$ is an integer, then the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0^1 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k + C y_1(x) \ln x$$

where $C_0^2 \neq 0$, and $C$ is a constant which may or may not be zero.

**Case 3.** If $r_1 = r_2$, the the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0^1 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k + y_1(x) \ln x$$

where $C_0^2 \neq 0$.

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|---|---|
| **Equation:** | $2x^2y'' + xy' + (x^2 - 3)y = 0$ |
| **Indicial Equation** | $2r^2 - r - 3 = 0$ |
| **Indicial Roots** | $r_1 = \frac{3}{2} \quad r_2 = -1$ |
| **Recurrence formulas** | $\begin{cases} r_1 = \frac{3}{2} & \{ C_1 = 0 \quad C_k = -\frac{C_{k-2}}{n(2n+5)} \quad n \geq 2, \\ r_2 = -1 & \{ C_1 = 0 \quad C_k = -\frac{C_{k-2}}{n(2n-5)} \quad n \geq 2. \end{cases}$ |

| | |
|---|---|
| **Equation:** | $x^2y'' - xy' - (x^2 - \frac{5}{4})y = 0$ |
| **Indicial Equation** | $r^2 - 2r - \frac{5}{4} = 0$ |
| **Indicial Roots** | $r_1 = \frac{5}{2} \quad r_2 = -\frac{1}{2}$ |
| **Recurrence formulas** | $\begin{cases} r_1 = \frac{5}{2} & \{ C_1 = 0 \quad C_k = \frac{C_{k-2}}{n(n+3)} \quad n \geq 2, \\ r_2 = -1 & \{ C_1 = 0 \quad C_k = \frac{C_{k-2}}{n(n-3)} \quad n \geq 2. \end{cases}$ |

♠ **Note**. If $r = \alpha \pm \beta i$ are the complex roots of the indicial equation, then the linearly independent solution to the differential equation (E) are

$$\begin{aligned} y(x) =& x^r \sum_{k=0}^{\infty} C_k x^k = x^{\alpha \pm \beta i} \sum_{k=0}^{\infty} C_k x^k = x^{\alpha} e^{\pm i\beta \ln x} \sum_{k=0}^{\infty} C_k x^k \\ =& x^{\alpha} \left[\cos(\beta \ln x) \pm \sin(\beta \ln x)\right] \sum_{k=0}^{\infty} C_k x^k \\ =& x^{\alpha} \cos(\beta \ln x) \sum_{k=0}^{\infty} C_k x^k \pm \sin(\beta \ln x) \sum_{k=0}^{\infty} C_k x^k. \end{aligned}$$