

Topics on Fundamental Science

M.Sc

Electrodynamics

LectureSlide

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SS.2025





1. Vector Calculus
2. Theory of Vector Fields

Vector Calculus



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- Displacements (straight line segments going from one point to another) have **direction** as well as **magnitude**, and it is essential to take both into account when combining them.
- These objects in question are called **vectors**.
 - i.e., velocity (\mathbf{v}), acceleration (\mathbf{a}), force (\mathbf{F}), momentum (\mathbf{p}) ...
- If an object has magnitude but no direction, it is **scalar**.
 - i.e., mass (m), charge (q), density (d), temperature (T) ...
- In this lecture series boldface (i.e., \mathbf{A}) is used for *vectors* and normal type (A) is used for scalars.
- In diagrams, vectors are denoted by arrows (\rightarrow):
 - The **length** of the arrow is proportional to the **magnitude** of the vector, and the arrowhead indicates its direction.
- $-\mathbf{A}$ has the same magnitude of \mathbf{A} , but in **opposite** direction.



	Traits	Mathematical Example	Additional Notes
3*	Addition of multiple vectors	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ - This is known as <i>commutative</i> . $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ - This is known as <i>associative</i> . $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ - This is known as <i>distributive</i> .	
2*	Multiplication by a scalar	$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$ - If $a > 0$, direction remains. - If $a < 0$, direction is reversed.	
4*	Dot product of two vectors	$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta$ - θ : angle between two vectors. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ - $\mathbf{A} \cdot \mathbf{B}$ is a scalar value. $\mathbf{A} \cdot \mathbf{A} = A^2$ $\mathbf{A} \cdot \mathbf{B} = 0$ if $\mathbf{A} \perp \mathbf{B}$	
4*	Cross product of two vectors	$\mathbf{A} \times \mathbf{B} = AB \sin \theta \hat{n}$ - \hat{n} : unit vector. ¹ $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$ - This is known as <i>distributive</i> . $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$ - $\mathbf{A} \times \mathbf{B}$ is a vector. $\mathbf{A} \times \mathbf{A} = 0$	

The unit vector (\hat{n}) points perpendicular to the plane of \mathbf{A} and \mathbf{B} . Of course, there are two possible directions perpendicular to any plane: "*in*" and "*out*". This ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector-and curl around toward the second; then your thumb indicates the direction of (\hat{n}).



Example

Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$. Using this information, calculate the dot product of \mathbf{C} with itself. (i.e., $\mathbf{C} \cdot \mathbf{C}$)



Solution

As requested, let's do the dot product (\cdot) of \mathbf{C} with itself.

$$\begin{aligned}\mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}), \\ &= \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B}.\end{aligned}$$

This can also be represented using the **law of cosines**.



- It is often easier to set up Cartesian coordinates $x, y, z \dots$
 - ...and work with vector *components*.
- Let $\hat{x}, \hat{y}, \hat{z}$ be the unit vectors of x, y, z .
- A vector of \mathbf{A} can be therefore shown as:

$$\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z},$$

where A_x, A_y, A_z are the **components** of \mathbf{A} .

Geometrically, these are the projection of \mathbf{A} along the three coordinate axes.



Example

Find the angle between the face diagonals of a cube.

Tip: Use a cube of side 1.

**Solution**

We might as well use a cube of side 1, and place it as shown in question, with one corner at the origin. The face diagonals **A** and **B** are:

$$\mathbf{A} = 1 \hat{x} + 0 \hat{y} + 1 \hat{z} \quad \mathbf{B} = 0 \hat{x} + 1 \hat{y} + 1 \hat{z}$$

In component form,

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

In abstract form:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta = \sqrt{2}\sqrt{2} \cos \theta = 2 \cos \theta.$$

Therefore:



- As the **cross product** of two vectors is a **vector**, it can be dotted or crossed to form a **triple product**.
- **Scalar triple product:** This is simply $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.
 - This property also presents the following equivalence.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}).$$

- This behaviour could also be presented in its component form.
- The dot and cross can be interchanged.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}.$$



- This is shown as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.
- This can be simplified using the **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

This is a **different vector**. Cross product is **not commutative**.

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C}), \\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &\neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).\end{aligned}$$

- All higher vector products can be similarly reduced.

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \times \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}), \\ \mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D})) &= \mathbf{B}(\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}).\end{aligned}$$



- A point \mathbf{r} in 3D can be described using Cartesian (x, y, z) :

$$\mathbf{r} \equiv x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}.$$

where \mathbf{r} is the position of the **vector** with its magnitude (r):

$$r = \sqrt{x^2 + y^2 + z^2},$$

- This is the distance from the **origin**, and:

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}},$$

is the unit vector pointing radially outward.



- In electrodynamics, we often use two (2) points.
 - source point** (\mathbf{r}') where an electric charge is located,
 - field point** (\mathbf{r}) where the electric, magnetic field is calculated.
- For this lecture series, we shall adopt a short-hand notation for the **separation vector** from the source point to the field point.

$$\mathbf{z} \equiv \mathbf{r} - \mathbf{r}',$$

separation vector

$$z = |\mathbf{r} - \mathbf{r}'|,$$

magnitude

$$\hat{\mathbf{z}} = \frac{\mathbf{z}}{z} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

unit vector



■ in Cartesian coordinates:

$$\mathbf{z} = (x - x') \hat{\mathbf{x}} + (y - y') \hat{\mathbf{y}} + (z - z') \hat{\mathbf{z}},$$

$$z = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

$$\hat{\mathbf{z}} = \frac{(x - x') \hat{\mathbf{x}} + (y - y') \hat{\mathbf{y}} + (z - z') \hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}.$$



- Allows to change the position of an object in a coordinate frame
 - i.e., rotate, translate, flip.
- The \bar{x} , \bar{y} , \bar{z} system is rotated by angle ϕ relative to x , y , z about the common $x = \bar{x}$ axes.

$$A_y = A \cos \theta, \quad A_z = A \sin \theta,$$

- while ...

$$\begin{aligned}\bar{A}_y &= A \cos \bar{\theta} = A \cos (\theta - \phi), \\ &= A (\cos \theta \cos \phi + \sin \theta \sin \phi), \\ &= \cos \phi A_y + \sin \phi A_z.\end{aligned}$$

$$\begin{aligned}\bar{A}_z &= A \sin \bar{\theta} = A \sin (\theta - \phi), \\ &= A (\sin \theta \cos \phi - \cos \theta \sin \phi), \\ &= -\sin \phi A_y + \cos \phi A_z.\end{aligned}$$



- It is **simpler** to express this in a matrix:
- A more general approach would be:
- In a compact fashion

$$\bar{A}_i = \sum_{j=1}^3 R_{ij} A_j.$$



Assume we have a function that only has **one** variable. Call it $f(\cdot)$ x .
What is the point of calculating df/dx ?

It would tell us how rapidly the function $f(\cdot)$ x varies when we change the argument by an *infinitesimal* amount dx .

$$df = \left(\frac{df}{dx} \right) dx.$$

- If we change x by an amount dx , then f changes by an amount df ;
 - The derivative is the proportionality factor.
- Geometrically, df/dx is the slope of the graph of f vs. x



- A simple derivative can tell the **change of a variable**.
- For example dT/dx tells how T changes as we move along the x axis.
- However, last we checked we live in 3 dimensions so we need to model it in not only x , but y and z .
- Such as a temperature in a room...
you can move one place and it can rise it can stay or fall.
- We write it as following:

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz,$$

to simplify, we can write it as

$$dT = \overbrace{\left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right)}^{\nabla T} \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z}).$$



- Using nabla (∇) we can define **divergence**:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}), \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

- Divergence of a vector \mathbf{v} is itself a *scalar*.

You can't have the divergence of a scalar: **that's meaningless**.

- $\nabla \cdot$ is a measure of how much a vector **spreads**.

Positive Arrows point outward (i.e., source).

Negative Arrows point inward (i.e., sink).

Zero No change in magnitude.



- Has the following expression:

$$\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l} = \int_a^b \mathbf{v} \cdot d\mathbf{l}.$$

where \mathbf{v} is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path \mathcal{P} from point a to b .

- If the path is closed ($a = b$):

$$\oint \mathbf{v} \cdot d\mathbf{l}.$$

- At each point on the path, take the dot product of \mathbf{v} with the $d\mathbf{l}$ to the next point on the path.

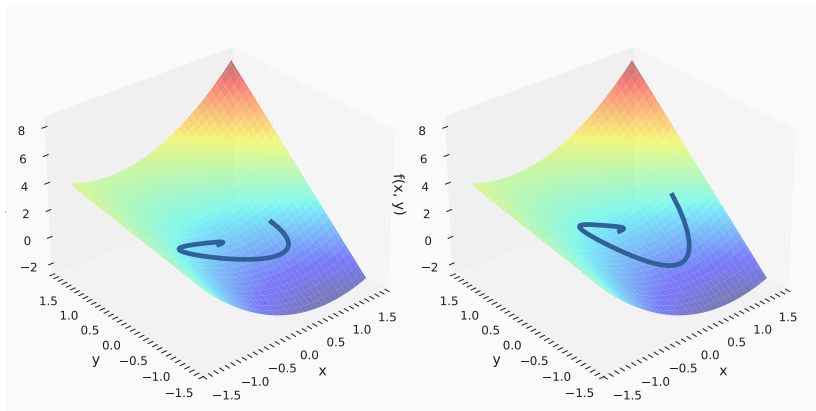


Figure 1:



- Has the following expression:

$$\int_S \mathbf{v} \cdot d\mathbf{a},$$

where \mathbf{v} is a vector function, $d\mathbf{a}$ is the infinitesimal area,

- with a direction pointing outward.

- There are two direction perpendicular to any surface so the sign is intrinsically ambiguous.
- If the area is closed (i.e., balloon):

$$\oint \mathbf{v} \cdot d\mathbf{a}.$$

- For analogy, if \mathbf{v} represents fluid flow, then $\int \mathbf{v} \cdot d\mathbf{a}$ represents the total mass per unit time passing through the surface (i.e., the flux).



- Has the following expression:

$$\int_V T \, d\tau,$$

where T is a scalar function and the $d\tau$ is an infinitesimal volume element.

- in Cartesian the volume element is:

$$d\tau = dx \, dy \, dz.$$

if T is the density of a substance (varying from point to point), the volume integral would give the total mass.



- Suppose $f(x)$ is a function of one variable.
- The **fundamental theorem of calculus** states.

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a),$$
$$\int_a^b F(x) dx = f(b) - f(a), \quad \text{where } df/dx = F(x).$$

- This gives us a relation between **differentiation** and **integration**.

For a function $f(\cdot)$, an anti-derivative may be obtained $F(x)$ as the integral of f over interval.

Implies the existence of anti-derivatives for continuous functions.



- Suppose $T(x, y, z)$ is a **scalar** function of three (3) variables.
 - These being x, y, z .
- The **fundamental theorem of gradients** states:

$$\int_a^b (\nabla T) \cdot d\mathbf{l} = T(b) - T(a).$$

This means, the integral of a gradient is given by the value of the function at the boundaries (i.e., a and b).

This also means the result is **independent** from the path taken.



- The fundamental theorem for divergence states:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) \, d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}.$$

- The integral of a derivative $(\nabla \cdot \mathbf{v})$ over a region (\mathcal{V}) is equal to the value of the function at boundary (surface that bounds the volume).
- Imagine \mathbf{v} as an incompressible fluid.
- Then $\mathbf{v} \cdot d\mathbf{a}$ would be the fluid passing through a surface.
- Divergence of \mathbf{v} would mean the spreading out of the fluid.
- To measure the amount of water in a region you could either:
 - (a) Count the water coming from a source (i.e., a faucet),
 - (b) Measure the flow coming from the region.



- The fundamental theorem for curl (i.e., Stokes' Theorem) states:

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}.$$

- The integral of the derivative $(\nabla \times \mathbf{v})$ over a region (S) is equal to the value of the function at boundary (the perimeter of the patch).

Remember, curl measures the **twist** of the vector \mathbf{v} .

- Now, the integral of the curl over some surface represents the **total amount of twist**.
- We can also determine this **twist** by finding how much flow is following the boundary.



- The technique known as **integration by parts** exploits the product rule for derivatives:

$$\frac{d}{dx} (fg) = f \left(\frac{dg}{dx} \right) + g \left(\frac{df}{dx} \right).$$

- where, f and g are *continuous* functions.
- Integrating both sides, and using the fundamental theorem:

$$\int_a^b \frac{d}{dx} (fg) \, dx = fg \Big|_a^b = \int_a^b f \left(\frac{dg}{dx} \right) \, dx + \int_a^b g \left(\frac{df}{dx} \right) \, dx.$$

When integrating the product of one function (f) and the derivative of another (g), you can transfer of the derivative from g to f , at the cost of a **minus sign and a boundary term**.



Example

Evaluate the following integral:

$$\int_0^{\infty} x \exp(-x) dx.$$

**Solution**

The exponential can be expressed as a derivative:

$$\exp(-x) = \frac{d}{dx} [-\exp(-x)].$$

Here $f(x) = x$, $g(x) = -\exp(-x)$, and $df/dx = 1$. Therefore:

$$\begin{aligned} \int_0^{\infty} x [-\exp(-x)] dx &= \{[-\exp(-x)] - x\} [-\exp(-x)] \Big|_0^{\infty} \\ &= -\exp(-x) \Big|_0^{\infty} = 1 \quad \blacksquare \end{aligned}$$



- The product rule is also applicable in **vector calculus**.
- Using the correct fundamental theorems, we can do calculus in parts.

$$\nabla \cdot (f \mathbf{A}) = f (\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f).$$

- Over a volume (\mathcal{V}), invoking the **divergence theorem**, presents:

$$\begin{aligned} \int \nabla \cdot (f \mathbf{A}) \, d\tau &= \int f (\nabla \cdot \mathbf{A}) \, d\tau + \int \mathbf{A} \cdot (\nabla f) \, d\tau = \oint f \mathbf{A} \cdot d\mathbf{a}, \\ \int_{\mathcal{V}} f (\nabla \cdot \mathbf{A}) \, d\tau &= - \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f) \, d\tau + \oint_S f \mathbf{A} \cdot d\mathbf{a}. \end{aligned}$$

- The integrand is $f (\nabla \cdot \mathbf{A})$.
- Using integration by parts to transfer the derivative from \mathbf{A} to f .
 - Where f becomes a gradient.
- We introduce a minus sign and a boundary term (a surface integral).



- The relation to Cartesian coordinates is:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

- The unit vectors are:

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}},$$

$$\hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}},$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}.$$

- The infinitesimal displacements are,

$$dl_r = dr, \quad dl_\theta = r d\theta, \quad dl_\phi = r \sin \theta d\phi.$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}.$$

- The range of r is $[0, \infty)$, θ is $[0, \pi]$, and ϕ is $[0, 2\pi)$.



- The relation to Cartesian coordinates is:

$$x = s \cos \phi, \quad y = s \sin \phi, \quad z = z.$$

- The unit vectors are:

$$\hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}},$$

$$\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}},$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}.$$

- The infinitesimal displacements are,

$$d\mathbf{l}_s = ds, \quad d\mathbf{l}_\phi = s d\phi, \quad d\mathbf{l}_z = dz,$$

$$d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\phi} + dz \hat{\mathbf{z}}.$$



- Consider the vector function,

$$\mathbf{v} = \frac{1}{r^2} \hat{\mathbf{r}}.$$

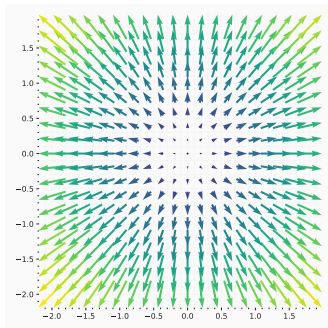


Figure 2: The vector field of the function.



- Here, \mathbf{v} is directed radially outward at every location.
- This function should have a **positive divergence**.

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0.$$

- It seems we have reached a paradox. Let's test this out with the **divergence theorem**.

$$\begin{aligned} \oint \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{R^2} \hat{\mathbf{r}} \right) \cdot (R \sin \theta d\theta d\phi \hat{\mathbf{r}}) \\ &= \left(\int_0^\pi \sin \theta d\theta \right) \left(\int_0^{2\pi} d\phi \right) = 4\pi \quad \blacksquare \end{aligned}$$



- Here, \mathbf{v} is directed radially outward at every location.
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- But the *volume* integral $(\int \nabla \cdot \mathbf{v} d\tau)$ is zero.
- Is the divergence theorem false?



- Problem lies at the point $r = 0$ where r approaches an incalculable value.
 - Where we, without meaning to, divide by zero.
- The divergence is 0 in every point in space, except at one point where it explodes to infinite.
- This bizarre behaviour can be remedied by defining a new mathematical function.

This new function is called Dirac delta function.



- The 1D delta function is an infinitely high, infinitesimally narrow **spike**, with an area of 1.

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

- The functions integration is equal to 1.

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1.$$

- Technically, $\delta(x)$ is not a function at all, since its value is *infinite* at $x = 0$.
- This is known as a **generalised function**¹, or distribution.

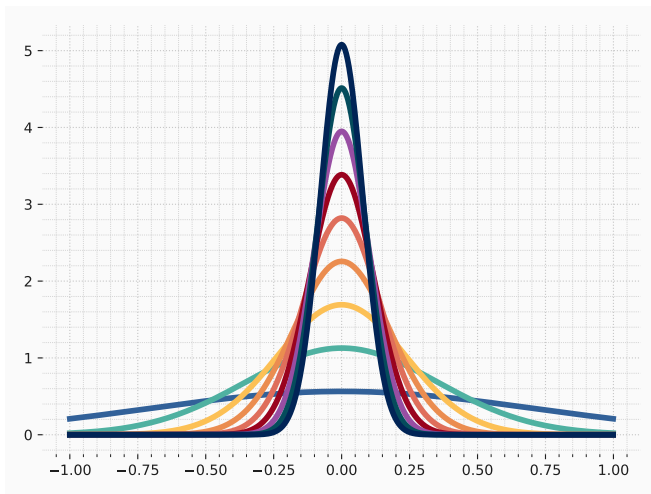


Figure 3: The Dirac delta as the limit as the limit reaches zero



- If $f(x)$ is some *ordinary continuous* function (i.e., not a delta function), the product $(f(x)\delta(x))$ is zero everywhere except at $x = 0$.

$$f(x)\delta(x) = f(0)\delta(x). \quad (1)$$

This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway except at $x = 0$, we may as well replace $f(x)$ by the value it assumes at the origin.

$$\int_{-\infty}^{+\infty} f(x)\delta(x) dx = f(0) \int_{-\infty}^{+\infty} \delta(x) dx = f(0). \quad (2)$$

- Under an integral, the delta function **picks out** $f(x)$ value at $x = 0$.

The 1D Delta (δ) Function

The integral need not run from $-\infty$ to $+\infty$;



- We can shift the spike from $x = 0$ to some arbitrary point, $x = a$.

$$\delta(x - a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x - a) dx = 1. \quad (3)$$

- Based on this Eq. (1) and Eq. (2) become:

$$f(x) \delta(x - a) = f(a) \delta(x - a), \quad (4)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a). \quad (5)$$



- Even though δ itself is not a legitimate function, integrals over δ are perfectly acceptable.
- Think of the delta function as something that is always intended for use under an integral sign.
- In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if:

$$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx. \quad (6)$$

The reason as to why we don't call Dirac delta function a valid one is no function can have a value at $x = 0$ and zero at any other point in space.

```
variables= "high" : 5, "low" : -5 , "vl" : 4 locals().update(variables)
import sympy as sy from sympy.abc import x
Eq = x ** 3 * sy.DiracDelta(x - vl) Sol = sy.integrate(Eq, (x, low, high))
print(locals())
```



Evaluate the following integral:

$$\int_{\text{«!low»}}^{\text{«!high»}} x^3 \delta(x - \text{«!vl»}) dx.$$

The delta function picks out the value of x^3 at the point $x = \text{«!vl»}$, so the integral is $2^3 = \text{«!Sol»}$.

Notice, if the upper limit had been 1 (instead of 3) the answer would be 0, because the spike would then be outside the domain of integration.



- From 1D, it is easy to create a 3D version of the delta function.

$$\delta^3(\mathbf{r}) = \delta(x) \delta(y) \delta(z),$$

- where $\mathbf{r} \equiv x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}$ is the position vector.
- This function is zero everywhere except at (0, 0, 0) where it is infinite.
- The volume integral is 1.

$$\int_V \delta^3(\mathbf{r}) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) dx dy dz = 1.$$

- Generalising Eq. (5) gives us the following:

$$\int_V f(\cdot) \delta^3(\mathbf{r} - \mathbf{a}) d\tau = f(\mathbf{a})$$

- We can now solve the paradox shown previously.

$$\delta^3(\mathbf{r}) = \delta(x) \delta(y) \delta(z)$$

Vector Theory



- As we are going to work on vectors *quite often*, it is worth covering some cases.
- For example,
 - say we define the scalar value of D as a divergence of \mathbf{F} and vector value \mathbf{C} as curl of \mathbf{F} :

$$\nabla \cdot \mathbf{F} = D, \quad \nabla \times \mathbf{F} = \mathbf{C}.$$

- Is it possible to determine the function \mathbf{F} ?



- Actually it is not possible ... yet.
- For example:
 - The following function is zero for both its divergence and its curl.

$$\mathbf{F} = yz \, \hat{\mathbf{x}} + xz \, \hat{\mathbf{y}} + xy \, \hat{\mathbf{z}}.$$

- To solve electrodynamics problems we need **boundary conditions**.
- Through this additional information **Helmholtz theorem** guarantees the field is uniquely determined by its divergence and curl.



- If the curl of a vector field (\mathbf{F}) is zero everywhere, then \mathbf{F} can be written as the gradient of a **scalar potential** (V):

$$\nabla \times \mathbf{F} = 0 \quad \rightarrow \quad \mathbf{F} = -\nabla V.$$

- The minus sign is there for convention.
- If the divergence of a vector field (\mathbf{F}) is zero everywhere, \mathbf{F} can be written as the curls of a **vector potential** (\mathbf{A}).

$$\nabla \cdot \mathbf{F} = 0 \quad \rightarrow \quad \mathbf{F} = -\nabla \times \mathbf{A}.$$

- For all cases a vector field (\mathbf{F}) can be written as sum of the scalar gradient and vector curl:

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}.$$



