Exam Electrodynamics Final

Neighbours

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SEMESTER: WS 2024
DATE: 24.01.2025
TIME: 10:45 - 12:15

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First and Last Name

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Student Registration Number

. . . .

Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

Result:

_ _ _/ max. 101 points

Grade:

Student Cohort	MA-MECH-24-VZ-ET
Study Programme	M.Sc Smart Technologies
Permitted Tools	Two hand-written sheets of paper are allowed.

Important Notes

Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

Writing materials

Do not use a pencil or red pen and write legibly.

Good Luck!

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Please read the following instructions carefully.

- You have **90 minutes** to complete this exam. This question booklet contains 4 question(s), 7 pages (including the cover) for the total of 101 points.
- Check to see if any pages are missing.
- All the questions are **compulsory** and all the notations used in the questions have their usual meaning taught at the lectures and done in practice.
- **Read the instructions for individual questions carefully** before answering the questions.

Question	Maximum Point	Result
Line Integrals	20	
Electric Fields	30	
Gaussian Sphere	20	
Magnetic Field of a Straight Wire	30	
Sum	101	

[Q1] Line Integrals — 20

Find the circulation of the field

$$\boldsymbol{F} = (x - y)\ \hat{x} + x\,\hat{y}$$

around the circle

$$\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$$

with a range of $0 \le t \le 2\pi$.

[Q2] Electric Fields — 30

One of these is an **impossible electrostatic field**. Please check each case.

a. $\boldsymbol{E}_1 = (y^2)\ \hat{x} + (2xy + z^2)\ \hat{y} + (2yz)\ \hat{z} \quad \mathrm{V\,m^{-1}}$, (15)

b. $\boldsymbol{E}_2 = (xy)\ \hat{x} + (2yz)\ \hat{y} + (3xz)\ \hat{z} \quad \mathrm{V\,m^{-1}}$. (15)

[Q3] Gaussian Sphere — 20

Find the field outside a uniformly charged solid sphere of radius R and total charge q. (20)

[Q4] Magnetic Field of a Straight Wire 30

Find the magnetic field a distance s from a long straight wire carrying a steady current I. (30)

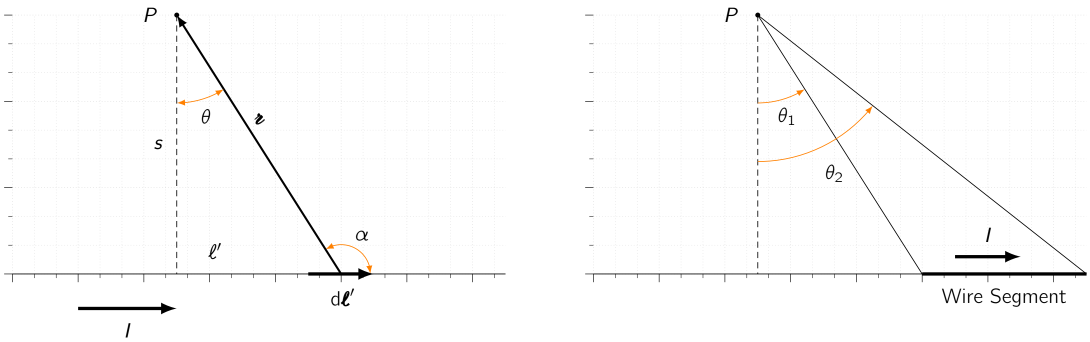

Figure 1: Diagrams to be use in question "Magnetic Field of a Straight Wire".

Formula Sheet

Vector Derivatives

Cartesian

Line Element	$d\boldsymbol{l}$	$= dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$
Volume Element	$d\tau$	$= dx\,dy\,dz$
Gradient	∇t	$= \frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z}$
Divergence	$\nabla \cdot v$	$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
Curl	$\nabla \times v$	$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$
Laplacian	∇t^2	$= \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical

Line Element	$d\boldsymbol{l}$	$= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$
Volume Element	$d\tau$	$= r^2\sin\theta\,dr\,d\theta\,d\phi$
Gradient	∇t	$= \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$
Divergence	$\nabla \cdot v$	$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$
Curl	$\nabla \times v$	$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$
Laplacian	∇t^2	$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical

Line Element	$d\boldsymbol{l}$	$= ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$
Volume Element	$d\tau$	$= s\,ds\,d\phi\,dz$
Gradient	∇t	$= \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$
Divergence	$\nabla \cdot v$	$= \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$
Curl	$\nabla \times v$	$= \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{z}$
Laplacian	∇t^2	$= \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Fundamental Constants

$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$	Permittivity of free space
$\mu_0 = 4\pi 1 \times 10^{-7}\,\mathrm{N\,A^{-2}}$	Permeability of free space
$c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$	Approximate speed of light
$e = 1.6 \times 10^{-19}\,\mathrm{C}$	Charge of electron
$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$	Mass of the electron

Vector Identities

Triple Products

(1) $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$

(2) $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$

(5) $\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

(7) $\nabla\times(f\boldsymbol{A}) = f(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times(\nabla f)$

(8) $\nabla\times\boldsymbol{A}\times\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$ (i.e., divergence of a curl is **always** zero.)

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$

Fundamental Theorems

Gradient Theorem:	$\int_a^b (\nabla f)\cdot d\boldsymbol{l} = f(b) - f(a)$
Divergence Theorem:	$\int (\nabla\cdot\boldsymbol{A})\,d\tau = \oint \boldsymbol{A}\cdot d\boldsymbol{a}$
Curl Theorem:	$\int (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{a} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}$

Coordinate Conversion

Spherical

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$$
$$z = r\cos\theta \qquad \phi = \tan^{-1}(y/x)$$
$$\hat{x} = \sin\theta\cos\phi\;\hat{r} + \cos\theta\cos\phi\;\hat{\theta} - \sin\phi\;\hat{\phi}$$
$$\hat{y} = \sin\theta\sin\phi\;\hat{r} + \cos\theta\sin\phi\;\hat{\theta} + \cos\phi\;\hat{\phi}$$
$$\hat{z} = \cos\theta\;\hat{r} - \sin\theta\;\hat{\theta}$$
$$\hat{r} = \sin\theta\cos\phi\;\hat{x} + \sin\theta\sin\theta\;\hat{y} + \cos\theta\;\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\;\hat{x} + \cos\theta\sin\phi\;\hat{y} - \sin\theta\;\hat{z}$$
$$\hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y}$$

Cylindrical

$$x = s\cos\phi \qquad s = \sqrt{x^2+y^2}$$
$$y = s\sin\phi \qquad \phi = \tan^{-1}(y/x)$$
$$z = z \qquad z = z$$
$$\hat{x} = \cos\phi\;\hat{s} - \sin\phi\;\hat{\phi}$$
$$\hat{y} = \sin\phi\;\hat{s} + \cos\phi\;\hat{\phi}$$
$$\hat{z} = \hat{z}$$
$$\hat{s} = \cos\phi\;\hat{x} + \sin\phi\;\hat{y}$$
$$\hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y}$$
$$\hat{z} = \hat{z}$$

Curl Operation

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Useful Physical Identities

Energy $U = \frac{1}{2}\int\left(\varepsilon_0|\boldsymbol{E}|^2 + \frac{1}{\mu_0}|\boldsymbol{B}|^2\right)\,d\tau$

Momentum $\boldsymbol{P} = \varepsilon_0\int(\boldsymbol{E}\times\boldsymbol{B})\,d\tau$

Poynting Vector $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$

Lorentz Force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

Potential $\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$

Coulombs Law $\boldsymbol{F} = \frac{qQ}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\imath}}}{\imath^2}$

Maxwell's Equations

Main Equations

In General:
$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

In Material:
$$\nabla\cdot\boldsymbol{D} = \rho_f$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t}$$

Auxiliary Fields

Definition:
$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$$
$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

Linear Media:
$$\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E},$$
$$\boldsymbol{M} = \chi_m\boldsymbol{H}$$
$$\boldsymbol{D} = \varepsilon\boldsymbol{E}$$
$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$$

[A1] Line Integrals 20

We first evaluate $\boldsymbol{F}$ on the curve:

$$\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z} = (\cos t)\ \hat{x} + (t)\ \hat{y} + (\sin t)\ \hat{z} \qquad \text{Substitute } \mathbf{x} = \cos\mathbf{t},\ \mathbf{z} = \mathbf{t},\ \mathbf{y} = \sin\mathbf{t}.$$

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\boldsymbol{F} \cdot (d\boldsymbol{\ell}/dt)$ from $t = 0$ to $t = \pi/2$:

$$\begin{aligned} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} &= (\cos t)\,(-\sin t) + (t)\,(\cos t) + (\sin t)\,(1)\,, \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

Which makes,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\,\mathrm{d}t = \int_0^{\pi/2} \left(-\sin t \cos t + t \cos t + \sin t\right)\, dt, \\ &= \left[\frac{\cos^2 t}{2} + t \sin t\right]\Bigg|_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare \end{aligned}$$

[A2] Electric Fields 30

To figure out the impossible we need to calculate the curl of $\boldsymbol{E}$.

(a) The first function:

$$\nabla \times \boldsymbol{E}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = (0 - 2y)\ \hat{x} + (0 - 3z)\ \hat{y} + (0 - x)\ \hat{z} \neq 0 \quad \mathrm{V\,m^{-1}} \quad \blacksquare$$

As the curl of this field is not equal to zero ($\nabla \times \boldsymbol{E} \neq 0$), this field cannot exist. (15)

(b) Second function:

$$\nabla \times \boldsymbol{E}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2y \end{vmatrix} = (2z - 2z)\ \hat{x} + (0 - 0)\ \hat{y} + (2y - 2y)\ \hat{z} = 0 \quad \mathrm{V\,m^{-1}} \quad \blacksquare$$

As the curl of this field is equal to zero ($\nabla \times \boldsymbol{E} = 0$), this field can exist. (15)

[A3] Gaussian Sphere 30

Imagine a spherical surface at radius $r > R$. this is called a Gaussian surface. Gausss law says that:

$$\oint_{\mathcal{S}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}},$$

For this case it is $Q_{\text{enc}} = q$. At first glance this doesnt seem to get us very far, as the quantity we want (E) is buried inside the surface integral. Luckily, symmetry allows us to extract E from under the integral sign: E certainly points radially outward,5 as does da, so we can drop the dot product,

$$\int_{\mathcal{S}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = \int_{\mathcal{S}} |\boldsymbol{E}| \; \mathrm{d}a$$

and the magnitude of E is constant over the Gaussian surface, so it comes outside the integral

$$\int_{\mathcal{S}} |\boldsymbol{E}| \; \mathrm{d}a = |\boldsymbol{E}| \int_{\mathcal{S}} \mathrm{d}a = |E| \, 4\pi r^2$$

Therefore:

$$\begin{aligned} |\boldsymbol{E}| \, 4\pi r^2 &= \frac{1}{\varepsilon_0} q \\ \boldsymbol{E} &= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \quad \blacksquare \end{aligned}$$

[A4] Magnetic Field of a Straight Wire 30

In the diagram it can be seen $(\mathrm{d}\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}})$ points out of the page and has the magnitude:

$$\mathrm{d}l' \sin\alpha = \mathrm{d}l' \cos\theta$$

Also from geometry we can observe $l' = s \tan\theta$, therefore:

$$\frac{1}{} = \frac{\cos^2\theta}{s^2}$$

Which gives:

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right) \left(\frac{s}{\cos^2\theta}\right) \cos\theta \, \mathrm{d}\theta \\ &= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, \mathrm{d}\theta \\ &= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1) \end{aligned}$$