Exam Electrodynamics Final

Neighbours

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SEMESTER: WS 2023
DATE: 01.12.2023
TIME: 10:45 - 12:15

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First and Last Name

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Student Registration Number

. . . .

Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

Result:

___/ max. 101 points

Grade:

Student Cohort	MA-MECH-23-VZ
Study Programme	M.Sc Smart Technologies
Permitted Tools	Calculator and Exam Reference Sheet.

Important Notes

Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

Writing materials

Do not use a pencil or red pen and write legibly.

Good Luck!

Please read the following instructions carefully.

- You have **90 minutes** to complete this exam. This question booklet contains 5 question(s), 8 pages (including the cover) for the total of 101 points.
- Check to see if any pages are missing.
- All the questions are **compulsory** and all the notations used in the questions have their usual meaning taught at the lectures and done in practice.
- **Read the instructions for individual questions carefully** before answering the questions.

Maximum Point	Result
30	
10	
20	
20	
20	
101	

[Q1] Electric Fields — 30

One of these is an **impossible electrostatic field**. Please check each case.

a. $\boldsymbol{E}_1 = \left(y^2\right)\ \hat{x} + \left(2xy + z^2\right)\ \hat{y} + (2yz)\ \hat{z} \quad \mathrm{V\,m^{-1}}$, (15)

b. $\boldsymbol{E}_2 = (xy)\ \hat{x} + (2yz)\ \hat{y} + (3xz)\ \hat{z} \quad \mathrm{V\,m^{-1}}$. (15)

[Q2] Field of Circulation — 10

Find the circulation of the field $\boldsymbol{F} = (x - y)\ \hat{x} + x\,\hat{y}$ around the circle

$$\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$$

with a range of $0 \le t \le 2\pi$. (10)

[Q3] Calculating the Volume Charge Density — 20

If the electric field ($\boldsymbol{E}$) in some region is given in spherical coordinates, (r, θ, ϕ) by the expression:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{r}\left[(3)\ \hat{r} + (2\sin\theta\cos\theta\sin\phi)\ \hat{\theta} + (\sin\theta\cos\phi)\ \hat{\phi}\right] \quad \mathrm{V\,m^{-1}},$$

what is the volume charge density (ρ) in $\mathrm{C\,m^{-3}}$? (20)

[Q4] Self-Inductance of a Toroid 20

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns. (20)

(**Tip:** the $\boldsymbol{B}$-field inside the toroid is: $B = \mu_0 NI/2\pi s$ where I is the current and s is the distance from central axis to the toroid centre and the total flux is calculated as $\Phi_T = N\Phi_B$)

[Q5] The Wave Equation 20

Check whether the following equations obey the wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

a. $f_1(z, t) = A\exp\left[-b(z - vt)^2\right]$, (10)

b. $f_2(z, t) = A\sin(bz)\cos(bvt)^3$. (10)

Formula Sheet

Vector Derivatives

Cartesian

$$\begin{aligned}
&\text{Line Element} && d\boldsymbol{l} && = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \\
&\text{Volume Element} && d\tau && = dx\,dy\,dz \\
&\text{Gradient} && \nabla t && = \frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z} \\
&\text{Divergence} && \nabla \cdot v && = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&\text{Curl} && \nabla \times v && = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} \\
& && && + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} \\
& && && + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \\
&\text{Laplacian} && \nabla t^2 && = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}
\end{aligned}$$

Spherical

$$\begin{aligned}
&\text{Line Element} && d\boldsymbol{l} && = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \\
&\text{Volume Element} && d\tau && = r^2 \sin\theta\,dr\,d\theta\,d\phi \\
&\text{Gradient} && \nabla t && = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi} \\
&\text{Divergence} && \nabla \cdot v && = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) \\
& && && + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) \\
& && && + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
&\text{Curl} && \nabla \times v && = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} \\
& && && + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta} \\
& && && + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\
&\text{Laplacian} && \nabla t^2 && = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial r}\right) \\
& && && + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t}{\partial \theta}\right) \\
& && && + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}
\end{aligned}$$

Cylindrical

$$\begin{aligned}
&\text{Line Element} && d\boldsymbol{l} && = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z} \\
&\text{Volume Element} && d\tau && = s\,ds\,d\phi\,dz \\
&\text{Gradient} && \nabla t && = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z} \\
&\text{Divergence} && \nabla \cdot v && = \frac{1}{s}\frac{\partial}{\partial s}\left(s v_s\right) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
&\text{Curl} && \nabla \times v && = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} \\
& && && + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} \\
& && && + \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s v_\phi\right) - \frac{\partial v_s}{\partial \phi}\right]\hat{z} \\
&\text{Laplacian} && \nabla t^2 && = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}
\end{aligned}$$

Fundamental Constants

$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$	Permittivity of free space
$\mu_0 = 4\pi 1 \times 10^{-7}\,\mathrm{N\,A^{-2}}$	Permeability of free space
$c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$	Approximate speed of light
$e = 1.6 \times 10^{-19}\,\mathrm{C}$	Charge of electron
$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$	Mass of the electron

Vector Identities

Triple Products

(1) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$

(2) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A} \cdot \boldsymbol{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\,\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\,\boldsymbol{A}$

(5) $\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$

(7) $\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f)$

(8) $\nabla \times \boldsymbol{A} \times \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\,\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\,\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}\,(\nabla \cdot \boldsymbol{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ (i.e., divergence of a curl is **always** zero.)

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

Fundamental Theorems

Gradient Theorem: $\displaystyle\int_a^b (\nabla f) \cdot d\boldsymbol{l} = f(b) - f(a)$

Divergence Theorem: $\displaystyle\int (\nabla \cdot \boldsymbol{A})\, d\tau = \oint \boldsymbol{A} \cdot d\boldsymbol{a}$

Curl Theorem: $\displaystyle\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Coordinate Conversion

Spherical

$$\begin{aligned}
x &= r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}\\
y &= r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)\\
z &= r\cos\theta \qquad \phi = \tan^{-1}(y/x)\\
\hat{x} &= \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}\\
\hat{y} &= \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}\\
\hat{z} &= \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}\\
\hat{r} &= \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\theta\ \hat{y} + \cos\theta\ \hat{z}\\
\hat{\theta} &= \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}\\
\hat{\phi} &= -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}
\end{aligned}$$

Cylindrical

$$\begin{aligned}
x &= s\cos\phi \qquad s = \sqrt{x^2+y^2}\\
y &= s\sin\phi \qquad \phi = \tan^{-1}(y/x)\\
z &= z \qquad z = z\\
\hat{x} &= \cos\phi\ \hat{s} - \sin\phi\ \hat{\phi}\\
\hat{y} &= \sin\phi\ \hat{s} + \cos\phi\ \hat{\phi}\\
\hat{z} &= \hat{z}\\
\hat{s} &= \cos\phi\ \hat{x} + \sin\phi\ \hat{y}\\
\hat{\phi} &= -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}\\
\hat{z} &= \hat{z}
\end{aligned}$$

Curl Operation

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z \end{vmatrix}$$

Useful Physical Identities

Energy	$U = \frac{1}{2}\int\left(\varepsilon_0\lvert\boldsymbol{E}\rvert^2 + \frac{1}{\mu_0}\lvert\boldsymbol{B}\rvert^2\right)\, d\tau$
Momentum	$\boldsymbol{P} = \varepsilon_0\int(\boldsymbol{E}\times\boldsymbol{B})\, d\tau$
Poynting Vector	$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$
Lorentz Force	$\boldsymbol{F} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$
Potential	$\boldsymbol{E} = -\nabla V - \frac{\partial\boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$
Coulombs Law	$\boldsymbol{F} = \frac{qQ}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

Maxwell's Equations

Main Equations

In General:

$$\begin{aligned}
\nabla\cdot\boldsymbol{E} &= \frac{\rho}{\varepsilon_0}\\
\nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t}\\
\nabla\cdot\boldsymbol{B} &= 0\\
\nabla\times\boldsymbol{B} &= \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}
\end{aligned}$$

In Material:

$$\begin{aligned}
\nabla\cdot\boldsymbol{D} &= \rho_f\\
\nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t}\\
\nabla\cdot\boldsymbol{B} &= 0\\
\nabla\times\boldsymbol{H} &= \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t}
\end{aligned}$$

Auxiliary Fields

Definition:

$$\begin{aligned}
\boldsymbol{D} &= \varepsilon_0\boldsymbol{E} + \boldsymbol{P}\\
\boldsymbol{H} &= \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}
\end{aligned}$$

Linear Media:

$$\begin{aligned}
\boldsymbol{P} &= \varepsilon_0\chi_e\boldsymbol{E},\\
\boldsymbol{M} &= \chi_m\boldsymbol{H}\\
\boldsymbol{D} &= \varepsilon\boldsymbol{E}\\
\boldsymbol{H} &= \frac{1}{\mu}\boldsymbol{B}
\end{aligned}$$

[A1] Electric Fields 30

To figure out the impossible we need to calculate the curl of $\boldsymbol{E}$.

(a) The first function:

$$\nabla \times \boldsymbol{E}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = (0-2y)\ \hat{x} + (0-3z)\ \hat{y} + (0-x)\ \hat{z} \neq 0 \quad \mathrm{V\,m^{-1}} \quad \blacksquare$$

As the curl of this field is not equal to zero ($\nabla \times \boldsymbol{E} \neq 0$), this field cannot exist. (15)

(b) Second function:

$$\nabla\times\boldsymbol{E}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2y \end{vmatrix} = (2z-2z)\ \hat{x}+(0-0)\ \hat{y}+(2y-2y)\ \hat{z} = 0 \quad \mathrm{V\,m^{-1}} \quad \blacksquare$$

As the curl of this field is equal to zero ($\nabla \times \boldsymbol{E} = 0$), this field can exist. (15)

[A2] Field of Circulation 10

We first evaluate $\boldsymbol{F}$ on the curve:

$\boldsymbol{F} = (x)\ \hat{x}+(z)\ \hat{y}+(y)\ \hat{z} = (\cos t)\ \hat{x}+(t)\ \hat{y}+(\sin t)\ \hat{z}$ Substitute $\mathbf{x} = \cos\mathbf{t}$, $\mathbf{z} = \mathbf{t}$, $\mathbf{y} = \sin\mathbf{t}$.

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\boldsymbol{F}\cdot(d\boldsymbol{\ell}/dt)$ from $t=0$ to $t=\pi/2$:

$$\begin{aligned} \boldsymbol{F}\cdot\frac{d\boldsymbol{\ell}}{dt} &= (\cos t)\,(-\sin t) + (t)\,(\cos t) + (\sin t)\,(1)\,, \\ &= -\sin t\cos t + t\cos t + \sin t. \end{aligned}$$

So,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \boldsymbol{F}\cdot\frac{d\boldsymbol{\ell}}{dt}\,dt = \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\ dt, \\ &= \left[\frac{\cos^2 t}{2} + t\sin t\right]\Bigg|_0^{\pi/2} = \left(0+\frac{\pi}{2}\right) - \left(\frac{1}{2}+0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare \end{aligned}$$

Which gives the solution to our problem. (10)

[A3] Calculating the Volume Charge Density 20

First thing to do is to invoke Gauss's Law:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \qquad \rho = \varepsilon_0 (\nabla \cdot \boldsymbol{E}) .$$

$$\begin{aligned}
\rho &= \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{3}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{2\sin\theta\cos\theta\sin\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\sin\theta\cos\phi}{r}\right)\right\} \quad \mathrm{C\,m^{-3}}, \\
&= \varepsilon_0 \left[\frac{1}{r^2}3 + \frac{1}{r\sin\theta}\frac{2\sin\phi\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)}{r} + \frac{1}{r\sin\theta}\frac{(-\sin\theta\sin\phi)}{r}\right] \quad \mathrm{C\,m^{-3}}, \\
&= \frac{\varepsilon_0}{r^2}\left[3 + 2\sin\phi\left(2\cos^2\theta - \sin^2\theta\right) - \sin\phi\right] \quad \mathrm{C\,m^{-3}}, \\
&= \frac{\varepsilon_0}{r^2}\left[3 + \sin\phi\left(4\cos^2\theta - 2 + 2\cos^2\theta - 1\right)\right] \quad \mathrm{C\,m^{-3}}, \\
&= 3\frac{\varepsilon_0}{r^2}\left[1 + \sin\phi\left(2\cos^2\theta - 1\right)\right] \quad \mathrm{C\,m^{-3}}, \\
&= 3\frac{\varepsilon_0}{r^2}\left(1 - \sin\phi\cos 2\theta\right) \quad \mathrm{C\,m^{-3}} \quad \blacksquare
\end{aligned}$$

Which gives the solution to our problem. (20)

[A4] Self-Inductance of a Toroid 20

The magnetic field inside a toroid is:

$$B = \frac{\mu_0 N I}{2\pi s}$$

The flux through a single turn is:

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\,\mathrm{d}s = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

The total flux is N times this, therefore the self inductance is:

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \quad \blacksquare$$

Which gives the solution to our problem. (20)

[A5] The Wave Equation 20

Let us analyse the functions individually.

(i) Function 1: (10)

$$
\begin{aligned}
\frac{\partial f_1}{\partial z} &= -2Ab\,(z - vt)\exp\left[-b\,(z - vt)^2\right], \\
\frac{\partial^2 f_1}{\partial z^2} &= 2Ab\left\{-v\exp\left[-b\,(z - vt)^2\right] + 2bv\,(z - vt)^2\exp\left[-b\,(z - vt)^2\right]\right\}, \\
\frac{\partial f_1}{\partial t} &= -2Abv\,(z - vt)\exp\left[-b\,(z - vt)^2\right], \\
\frac{\partial^2 f_1}{\partial t^2} &= 2Abv^2\left\{-v\exp\left[-b\,(z - vt)^2\right] + 2bv\,(z - vt)^2\exp\left[-b\,(z - vt)^2\right]\right\}, \\
\frac{\partial^2 f_1}{\partial z^2} &= \frac{1}{v^2}\frac{\partial^2 f_1}{\partial t^2} \quad \blacksquare
\end{aligned}
$$

(iv) Function 2: (10)

$$
\begin{aligned}
\frac{\partial f_2}{\partial z} &= Ab\cos(bz)\cos(bvt)^3, \\
\frac{\partial^2 f_2}{\partial z^2} &= -Ab^2\sin(bz)\cos(bvt)^3, \\
\frac{\partial f_2}{\partial t} &= -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3, \\
\frac{\partial^2 f_2}{\partial t^2} &= -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9A^6v^6t^4\sin(bz)\cos(bvt)^4, \\
\frac{\partial^2 f_2}{\partial z^2} &\neq \frac{1}{v^2}\frac{\partial^2 f_2}{\partial t^2} \quad \blacksquare
\end{aligned}
$$