Margin Notes

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Topics on Fundamental Science

M.Sc
Electrodynamics
LectureSlide

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SS.2025

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Vector Calculus

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Vector Calculus

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Vector Calculus

- Displacements (straight line segments going from one point to another) have direction as well as magnitude, and it is essential to take both into account when combining them.
- These objects in question are called vectors.
  - i.e., velocity ($\boldsymbol{v}$), acceleration ($\boldsymbol{a}$), force ($\boldsymbol{F}$), momentum ($\boldsymbol{p}$) ...
- If an object has magnitude but no direction, it is scalar.
  - i.e., mass (m), charge (q), density (d), temperature (T) ...
- In this lecture series boldface (i.e., $\boldsymbol{A}$) is used for *vectors* and normal type (A) is used for scalars.
- In diagrams, vectors are denoted by arrows ($\rightarrow$):
  - The length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction.
- $-\boldsymbol{A}$ is a has the same magnitude of $\boldsymbol{A}$, but in opposite direction.

Vector Operations — Introduction

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Vector Calculus

@ r r@ | | @	Traits	Mathematical Example	Additional Notes
3*	Addition of multiple vectors	$\boldsymbol{A}+\boldsymbol{B} = \boldsymbol{B}+\boldsymbol{A}$	- This is known as *commutative*.
		$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C} = \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$	- This is known as *associative*.
		$\boldsymbol{A}-\boldsymbol{B} = \boldsymbol{A}+(-\boldsymbol{B})$	- This is known as *distributive*.
2*	Multiplication by a scalar	$a(\boldsymbol{A}+\boldsymbol{B}) = a\boldsymbol{A}+a\boldsymbol{B}$	- If $a>0$, direction remains.
			- If $a<0$, direction is reversed.
4*	Dot product of two vectors	$\boldsymbol{A}\cdot\boldsymbol{B} \equiv AB\cos\theta$	- θ: angle between two vectors.
		$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$	- $\boldsymbol{A}\cdot\boldsymbol{B}$ is a scalar value.
		$\boldsymbol{A}\cdot\boldsymbol{A} = A^2$	
		$\boldsymbol{A}\cdot\boldsymbol{B} = 0$ if $\boldsymbol{A}\perp\boldsymbol{B}$	
4*	Cross product of two vectors	$\boldsymbol{A}\times\boldsymbol{B} = AB\sin\theta\hat{\boldsymbol{n}}$	- $\hat{\boldsymbol{n}}$: unit vector.1
		$\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})+(\boldsymbol{A}\times\boldsymbol{C})$	- This is known as *distributive*.
		$\boldsymbol{B}\times\boldsymbol{A} = -(\boldsymbol{A}\times\boldsymbol{B})$	- $\boldsymbol{A}\times\boldsymbol{B}$ is a vector.
		$\boldsymbol{A}\times\boldsymbol{A} = 0$	

The unit vector ($\hat{\boldsymbol{n}}$) points perpendicular to the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$. Of course, there are two possible directions perpendicular to any plane: "*in*" and "*out*". This ambiguity is resolved by the right-hand rule: let your fingers point in the direction of the first vector-and curl around toward the second; then your thumb indicates the direction of ($\hat{\boldsymbol{n}}$).

Vector Operations — Introduction

Vector Calculus

Example

Let $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$. Using this information, calculate the dot product of $\boldsymbol{C}$ with itself. (i.e., $\boldsymbol{C}\cdot\boldsymbol{C}$)

Vector Operations — Introduction

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Vector Calculus

Solution

As requested, lets do the dot product ($\cdot$) of $\boldsymbol{C}$ with itself.

$$\begin{aligned}\boldsymbol{C}\cdot\boldsymbol{C} &= (\boldsymbol{A}-\boldsymbol{B})\cdot(\boldsymbol{A}-\boldsymbol{B}),\\ &= \boldsymbol{A}\cdot\boldsymbol{A}-\boldsymbol{A}\cdot\boldsymbol{B}-\boldsymbol{B}\cdot\boldsymbol{A}+\boldsymbol{B}\cdot\boldsymbol{B}.\end{aligned}$$

This can also be represented using the law of cosines.

Vector Operations — Introduction

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Vector Calculus

- It is often easier to set up Cartesian coordinates x, y, z ...
  - ...and work with vector *components*.
- Let $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ be the unit vectors of x, y, z.
- A vector of $\boldsymbol{A}$ can be therefore shown as:

$$\boldsymbol{A} = A_x \,\hat{\boldsymbol{x}} + A_y \,\hat{\boldsymbol{y}} + A_z \,\hat{\boldsymbol{z}},$$

where A_x, A_y, A_z are the components of $\boldsymbol{A}$.

> Geometrically, these are the projection of $\boldsymbol{A}$ along the three coordinate axes.

Component Form

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Vector Calculus

Example

Find the angle between the face diagonals of a cube.

> **Tip:** Use a cube of side 1.

Component Form

Vector Calculus

Solution

We might as well use a cube of side 1, and place it as shown in question, with one corner at the origin. The face diagonals $\boldsymbol{A}$ and $\boldsymbol{B}$ are:

$$\boldsymbol{A} = 1\,\hat{\boldsymbol{x}} + 0\,\hat{\boldsymbol{y}} + 1\,\hat{\boldsymbol{z}} \qquad \boldsymbol{B} = 0\,\hat{\boldsymbol{x}} + 1\,\hat{\boldsymbol{y}} + 1\,\hat{\boldsymbol{z}}$$

In component form,

$$\boldsymbol{A} \cdot \boldsymbol{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

In abstract form:

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.$$

Therefore:

$$\cos\theta = 0.5 \quad \text{or} \quad \theta = 60 \text{ deg} \quad \blacksquare$$

Component Form

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Vector Calculus

- As the cross product of two vectors is a **vector**, it can be dotted or crossed to form a triple product.
- **Scalar triple product:** This is simply $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$.
  - This property also presents the following equivalence.

$$\begin{aligned}\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) &= \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}),\\ \boldsymbol{A} \cdot (\boldsymbol{C} \times \boldsymbol{B}) &= \boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{C}) = \boldsymbol{C} \cdot (\boldsymbol{B} \times \boldsymbol{A}).\end{aligned}$$

- This behaviour could also be presented in its component form.
- The dot and cross can be interchanged.

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}.$$

Triple Products Examples

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Vector Calculus

- This is shown as $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$.
  - This can be simplified using the **BAC-CAB** rule:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A} \cdot \boldsymbol{B})\,.$$

> This is a **different vector**. Cross product is **not commutative**.
>
> $$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = -\boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = -\boldsymbol{A}\,(\boldsymbol{B} \cdot \boldsymbol{C}) + \boldsymbol{B}\,(\boldsymbol{A} \cdot \boldsymbol{C})\,,$$
> $$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} \neq \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})\,.$$

- All higher vector products can be similarly reduced.

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \times \boldsymbol{C})\,(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})\,(\boldsymbol{B} \cdot \boldsymbol{C})\,,$$
$$\boldsymbol{A} \times (\boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{D})) = \boldsymbol{B}\,(\boldsymbol{A} \cdot (\boldsymbol{C} \times \boldsymbol{D})) - (\boldsymbol{A} \cdot \boldsymbol{B})\,(\boldsymbol{C} \times \boldsymbol{D})\,.$$

Triple Products Vector Triple Products

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Vector Calculus

- A point $\boldsymbol{r}$ in 3D can be described using Cartesian $(x,\, y,\, z)$:

$$\boldsymbol{r} \equiv x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}.$$

where $\boldsymbol{r}$ is the position of the vector with its magnitude (r):

$$r = \sqrt{x^2 + y^2 + z^2},$$

- This is the distance from the origin, and:

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} = \frac{x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}}{\sqrt{x^2 + y^2 + z^2}},$$

is the unit vector pointing radially outward.

- The infinitesimal displacement vector $(d\boldsymbol{l})$:

Triple Products Vector Triple Products

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Vector Calculus

- In electrodynamics, we often use two (2) points.

source point $(\boldsymbol{r}')$ where an electric charge is located,
field point $(\boldsymbol{r})$ where the electric, magnetic field is calculated.

- For this lecture series, we shall adopt a short-hand notation for the separation vector from the source point to the field point.

$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}',$	separation vector
$\mathscr{r} = \lvert \boldsymbol{r} - \boldsymbol{r}' \rvert\,,$	magnitude
$\hat{\boldsymbol{\mathscr{r}}} = \dfrac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \dfrac{\boldsymbol{r} - \boldsymbol{r}'}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}.$	unit vector

Triple Products Short Hand Notation

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Vector Calculus

- in Cartesian coordinates:

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\boldsymbol{x}} + (y - y')\,\hat{\boldsymbol{y}} + (z - z')\,\hat{\boldsymbol{z}},$$
$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$
$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\,\hat{\boldsymbol{x}} + (y - y')\,\hat{\boldsymbol{y}} + (z - z')\,\hat{\boldsymbol{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}.$$

Triple Products Short Hand Notation

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Vector Calculus

- Allows to change the position of an object in a coordinate frame
  - i.e., rotate, translate, flip.
- The $\bar{x}$, $\bar{y}$, $\bar{z}$ system is rotated by angle ϕ relative to x, y, z about the common $x = \bar{x}$ axes.

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

- while ...

$$\begin{aligned}\bar{A}_y &= A\cos\bar{\theta} = A\cos(\theta - \phi), \\ &= A(\cos\theta\cos\phi + \sin\theta\sin\phi), \\ &= \cos\phi A_y + \sin\phi A_z. \\ \bar{A}_z &= A\sin\bar{\theta} = A\sin(\theta - \phi), \\ &= A(\sin\theta\cos\phi - \cos\theta\sin\phi), \\ &= -\sin\phi A_y + \cos\phi A_z.\end{aligned}$$

Triple Products — How Vectors Transform

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Vector Calculus

- It is simpler to express this in a matrix:
- A more general approach would be:
- In a compact fashion

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j.$$

Triple Products — How Vectors Transform

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Vector Calculus

Assume we have a function that only has one variable. Call it $f(\cdot)\,x$. What is the point of calculating df/dx?

It would tell us how rapidly the function $f(\cdot)\,x$ varies when we change the argument by an *infinitesimal* amount dx.

$$df = \left(\frac{df}{dx}\right) dx.$$

- If we change x by an amount dx, then f changes by an amount df;
  - The derivative is the proportionality factor.
- Geometrically, df/dx is the slope of the graph of f vs. x

Differentiation — Ordinary Derivatives

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Vector Calculus

- A simple derivative can tell the change of a variable.
- For example dT/dx tells how T changes as we move along the x axis.
- However, last we checked we live in 3 dimensions so we need to model it in not only x, but y an z.
- Such as a temperature in a room...
  you can move one place and it can rise it can stay or fall.
- We write it as following:

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz,$$

to simplify, we can write it as

$$dT = \overbrace{\left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)}^{\nabla T} \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}).$$

Differentiation — Gradient

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Vector Calculus

- Using nabla (∇) we can define divergence:

$$\nabla \cdot \boldsymbol{v} = \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{\boldsymbol{x}} + v_y \hat{\boldsymbol{y}} + v_z \hat{\boldsymbol{z}} \right),$$
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

- Divergence of a vector $\boldsymbol{v}$ is itself a *scalar*.

> You can't have the divergence of a scalar: **that's meaningless**.

- $\nabla \cdot$ is a measure of how much a vector spreads.

Positive Arrows point outward (i.e., source).
Negative Arrows point inward (i.e., sink).
Zero No change in magnitude.

Differentiation — Divergence

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Vector Calculus

- Has the following expression:

$$\int_{\mathcal{P}} \boldsymbol{v} \cdot d\boldsymbol{l} = \int_a^b \boldsymbol{v} \cdot d\boldsymbol{l}.$$

where $\boldsymbol{v}$ is a vector function, $d\boldsymbol{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point a to b.

- If the path is closed ($a = b$):

$$\oint \boldsymbol{v} \cdot d\boldsymbol{l}.$$

- At each point on the path, take the dot product of $\boldsymbol{v}$ with the $d\boldsymbol{l}$ to the next point on the path.

Integration — Line Integral

Vector Calculus

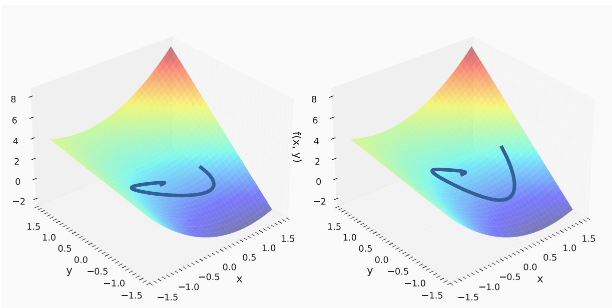

Figure 1:

Integration — Line Integral

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Vector Calculus

- Has the following expression:

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a},$$

where $\boldsymbol{v}$ is a vector function, $d\boldsymbol{a}$ is the infinitesimal area,
  - with a direction pointing outward.
- There are two direction perpendicular to any surface so the sign is intrinsically ambiguous.
- If the area is closed (i.e., balloon):

$$\oint \boldsymbol{v} \cdot d\boldsymbol{a}.$$

- For analogy, if $\boldsymbol{v}$ represents fluid flow, then $\int \boldsymbol{v} \cdot d\boldsymbol{a}$ represents the total mass per unit time passing through the surface (i.e., the flux).

Integration — Surface Integral

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Vector Calculus

- Has the following expression:

$$\int_{\mathcal{V}} T \, d\tau,$$

where T is a scalar function and the $d\tau$ is an infinitesimal volume element.
- in Cartesian the volume element is:

$$d\tau = dx \, dy \, dz.$$

> if T is the density of a substance (varying from point to point), the volume integral would give the total mass.

Integration — Volume Integral

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Vector Calculus

- Suppose $f(x)$ is a function of one variable.
- The fundamental theorem of calculus states.

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a),$$

$$\int_a^b F(x) \, dx = f(b) - f(a), \quad \text{where} \quad df/dx = F(x).$$

- This gives us a relation between differentiation and integration.

> For a function $f(\cdot)\, x$, an anti-derivative may be obtained $F(x)$ as the integral of f over interval.

> Implies the existence of anti-derivatives for continuous functions.

Integration — Volume Integral

Margin Notes

Vector Calculus

- Suppose $T(x, y, z)$ is a scalar function of three (3) variables.
  - These being x, y, z.
- The fundamental theorem of gradients states:

$$\int_a^b (\nabla T) \cdot d\boldsymbol{l} = T(b) - T(a).$$

> This means, the integral of a gradient is given by the value of the function at the boundaries (i.e., a and b).

> This also means the result is **independent** from the path taken.

Integration — Volume Integral

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Vector Calculus

- The fundamental theorem for divergence states:

$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{v}) \, d\tau = \oint_S \boldsymbol{v} \cdot d\boldsymbol{a}.$$

- The integral of a derivative $(\nabla \cdot \boldsymbol{v})$ over a region $(\mathcal{V})$ is equal to the value of the function at boundary (surface that bounds the volume).
- Imagine $\boldsymbol{v}$ as an incompressible fluid.
- Then $\boldsymbol{v} \cdot d\boldsymbol{a}$ would be the fluid passing through a surface.
- Divergence of $\boldsymbol{v}$ would mean the spreading out of the fluid.
- To measure the amount of water in a region you could either:
  (a) Count the water coming from a source (i.e., a faucet),
  (b) Measure the flow coming from the region.

Integration — Gauss's Theorem

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Vector Calculus

- The fundamental theorem for curl (i.e., Stokes' Theorem) states:

$\int_{\mathcal{S}} (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a} = \oint_{\mathcal{P}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}.$

- The integral of the derivative $(\nabla \times \boldsymbol{v})$ over a region $(\mathcal{S})$ is equal to the value of the function at boundary (the perimeter of the patch).

> Remember, curl measures the twist of the vector $\boldsymbol{v}$.

- Now, the integral of the curl over some surface represents the total amount of twist.
- We can also determine this twist by finding how much flow is following the boundary.

Integration | Gauss's Theorem

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Vector Calculus

- The technique known as integration by parts exploits the product rule for derivatives:

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right).$$

- where, f and g are *continuous* functions.
- Integrating both sides, and using the fundamental theorem:

$$\int_a^b \frac{d}{dx}(fg)\,\mathrm{d}x = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)\mathrm{d}x + \int_a^b g\left(\frac{df}{dx}\right)\mathrm{d}x.$$

> When integrating the product of one function (f) and the derivative of another (g), you can transfer of the derivative from g to f, at the cost of a minus sign and a boundary term.

Integration | Integration by Parts

Margin Notes

Vector Calculus

Example

Evaluate the following integral:

$$\int_0^{\infty} x \exp(-x)\,\mathrm{d}x.$$

Integration | Integration by Parts

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Vector Calculus

Solution

The exponential can be expressed as a derivative:

$$\exp(-x) = \frac{d}{dx}\left[-\exp(-x)\right].$$

Here $f(x) = x$, $g(x) = -\exp(-x)$, and $df/dx = 1$. Therefore:

$$\int_0^{\infty} x\left[-\exp(-x)\right]\mathrm{d}x = \{[-\exp(-x)] - x\}\left[-\exp(-x)\right]\Big|_0^{\infty}$$
$$= -\exp(-x)\Big|_0^{\infty} = 1 \quad \blacksquare$$

Integration | Integration by Parts

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Vector Calculus

- The product rule is also applicable in vector calculus.
- Using the correct fundamental theorems, we can do calculus in parts.

$$\nabla \cdot (f\boldsymbol{A}) = f\,(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla \cdot f)\,.$$

- Over a volume $(\mathcal{V})$, invoking the divergence theorem, presents:

$$\int \nabla \cdot (f\boldsymbol{A})\ \mathrm{d}\tau = \int f\,(\nabla \cdot \boldsymbol{A})\ \mathrm{d}\tau + \int \boldsymbol{A} \cdot (\nabla f)\ \mathrm{d}\tau = \oint f\boldsymbol{A} \cdot\ \mathrm{d}\boldsymbol{a},$$
$$\int_V f\,(\nabla \cdot \boldsymbol{A})\ \mathrm{d}\tau = -\int_V \boldsymbol{A} \cdot (\nabla \cdot f)\ \mathrm{d}\tau + \oint_S f\boldsymbol{A} \cdot\ \mathrm{d}\boldsymbol{a}.$$

- The integrand is $f\,(\nabla \cdot \boldsymbol{A})$.
- Using integration by parts to transfer the derivative from $\boldsymbol{A}$ to f.
  - Where f becomes a gradient.
- We introduce a minus sign and a boundary term (a surface integral).

Integration — Integration by Parts

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Vector Calculus

- The relation to Cartesian coordinates is:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

- The unit vectors are:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}},$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}},$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}.$$

- The infinitesimal displacements are,

$$\mathrm{d}l_r = \mathrm{d}r, \quad \mathrm{d}l_\theta = r\,\mathrm{d}\theta, \quad \mathrm{d}l_\phi = r\sin\theta\,\mathrm{d}\phi.$$
$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}.$$

- The range of r is $[0,\ \infty)$, θ is $[0,\ \pi]$, and ϕ is $[0, 2\pi)$.

Integration — Integration by Parts

Margin Notes

Vector Calculus

- The relation to Cartesian coordinates is:

$$x = s\cos\phi, \quad y = \sin\phi, \quad z = z.$$

- The unit vectors are:

$$\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}},$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}},$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}.$$

- The infinitesimal displacements are,

$$\mathrm{d}l_s = \mathrm{d}s, \quad \mathrm{d}l_\phi = s\,\mathrm{d}\phi, \quad \mathrm{d}l_z = \mathrm{d}z,$$
$$\mathrm{d}\boldsymbol{l} = \mathrm{d}s\,\hat{\boldsymbol{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}}.$$

- The range of s is $[0,\ \infty)$, ϕ is $[0,\ 2\pi]$, and z is $(-\infty, \infty)$.

Integration — Cylindrical Coordinates

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Vector Calculus

- Consider the vector function,

$$\boldsymbol{v} = \frac{1}{r^2}\,\hat{\boldsymbol{r}}.$$

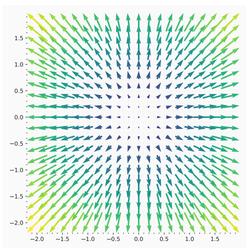

Figure 2: The vector field of the function.

Integration — Cylindrical Coordinates

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Vector Calculus

- Here, $\boldsymbol{v}$ is directed radially outward at every location.
- This function should have a positive divergence.

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

- It seems we have reached a paradox. Let's test this out with the divergence theorem.

$$\oint \boldsymbol{v} \cdot d\boldsymbol{a} = \int \left(\frac{1}{R^2}\hat{\boldsymbol{r}}\right) \cdot (R[][2] \sin\theta \, d\theta\phi \, \hat{\boldsymbol{r}})$$
$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi \quad \blacksquare$$

Integration — Cylindrical Coordinates

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Vector Calculus

- Here, $\boldsymbol{v}$ is directed radially outward at every location.
- This function should have a positive divergence.

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

- It seems we have reached a paradox. Let's test this out with the divergence theorem.

$$\oint \boldsymbol{v} \cdot d\boldsymbol{a} = \int \left(\frac{1}{R^2}\hat{\boldsymbol{r}}\right) \cdot (R[][2] \sin\theta \, d\theta\phi \, \hat{\boldsymbol{r}})$$
$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi \quad \blacksquare$$

- But the *volume* integral $\left(\int \nabla \cdot \boldsymbol{v} \, d\tau\right)$ is zero.
- Is the divergence theorem false?

Integration — Cylindrical Coordinates

Margin Notes

Vector Calculus

- Problem lies at the point $r = 0$ where $\boldsymbol{r}$ approaches an incalculable value.
  - Where we, without meaning to, divide by zero.
- The divergence is 0 in every point in space, except at one point where it explodes to infinite.
- This bizarre behaviour can be remedied by defining a new mathematical function.

> This new function is called Dirac delta function.

Integration — Cylindrical Coordinates

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Vector Calculus

- The 1D delta function is an infinitely high, infinitesimally narrow spike, with an area of 1.

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

- The functions integration is equal to 1.

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

- Technically, $\delta(x)$ is not a function at all, since its value is *infinite* at $x = 0$.
- This is known as a generalised function[1], or distribution.

The 1D Delta (δ) Function

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Vector Calculus

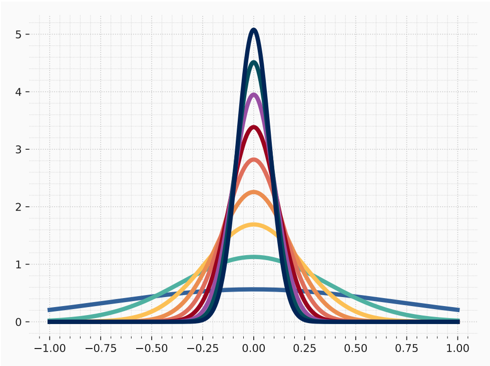

Figure 3: The Dirac delta as the limit as the limit reaches zero.

The 1D Delta (δ) Function

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Vector Calculus

- If $f(x)$ is some *ordinary continuous* function (i.e., not a delta function), the product $(f(x)\delta(x))$ is zero everywhere except at $x = 0$.

$$f(x)\delta(x) = f(0)\delta(x). \tag{1}$$

> This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway except at $x = 0$, we may as well replace $f(x)$ by the value it assumes at the origin.

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\, dx = f(0) \int_{-\infty}^{+\infty} \delta(x)\, dx = f(0). \tag{2}$$

- Under an integral, the delta function picks out $f(x)$ value at $x = 0$.
  - The integral need not run from ∞ to ∞;

The 1D Delta (δ) Function

Vector Calculus

- We can shift the spike from $x = 0$ to some arbitrary point, $x = a$.

$$\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x-a)\, dx = 1. \tag{3}$$

- Based on this Eq. (1) and Eq. (2) become:

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a), \tag{4}$$
$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a). \tag{5}$$

The 1D Delta (δ) Function

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Vector Calculus

- Even though δ itself is not a legitimate function, integrals over δ are perfectly acceptable.
- Think of the delta function as something that is always intended for use under an integral sign.
- In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if:

$$\int_{-\infty}^{\infty} f(x)\, D_1(x)\, dx = \int_{-\infty}^{\infty} f(x)\, D_2(x)\, dx. \tag{6}$$

> The reason as to why we don't call Dirac delta function a valid one is no function can have a value at $x = 0$ and zero at any other point in space.

The 1D Delta (δ) Function

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Margin Notes

Margin Notes

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variables= "high" : 5, "low" : -5 , "vl" : 4 locals().update(variables) import sympy as sy from sympy.abc import x

Eq = x ** 3 * sy.DiracDelta(x - vl) Sol = sy.integrate(Eq, (x, low, high))

print(locals())

The 1D Delta (δ) Function

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Vector Calculus

Evaluate the following integral:

$$\int_{«!low»}^{«!high»} x^3 \delta\left(x - «!vl»\right)\, dx.$$

The delta function picks out the value of x^3 at the point $x =$ «!vl », so the integral is $2^3 =$ «!Sol».
Notice, if the upper limit had been 1 (instead of 3) the answer would be 0, because the spike would then be outside the domain of integration.

The 1D Delta (δ) Function

Margin Notes

Vector Calculus

- From 1D, it is easy to create a 3D version of the delta function.

$$\delta^3(\boldsymbol{r}) = \delta(x)\,\delta(y)\,\delta(z),$$

- where $\boldsymbol{r} \equiv x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$ is the position vector.
- This function is zero everywhere except at (0, 0, 0) where it is infinite.
- The volume integral is 1.

$$\int_{\mathcal{V}} \delta^3(\boldsymbol{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1.$$

- Generalising Eq. (5) gives us the following:

$$\int_{\mathcal{V}} f\,(\cdot)\,\boldsymbol{r}\,\sigma^3\,(\boldsymbol{r} - \boldsymbol{a})\;d\tau = f\,(\boldsymbol{a})$$

- We can now solve the paradox shown previously.

The 3D Delta (δ) Function

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Vector Theory

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Vector Calculus

- As we are going to work on vectors *quite often*, it is worth covering some cases.
- For example,
  - say ve define the scalar value of D as a divergence of $\boldsymbol{F}$ and vector value $\boldsymbol{C}$ as curl of $\boldsymbol{F}$:

$$\nabla \cdot \boldsymbol{F} = D, \qquad \nabla \times \boldsymbol{F} = \boldsymbol{C}.$$

- Is it possible to determine the function $\boldsymbol{F}$?

The Helmholtz Theorem

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Vector Calculus

- Actually it is not possible ... yet.
- For example:
  - The following function is zero for both its divergence and its curl.

$$\boldsymbol{F} = yz\ \hat{\boldsymbol{x}} + xz\ \hat{\boldsymbol{y}} + xy\ \hat{\boldsymbol{z}}.$$

- To solve electrodynamics problems we need boundary conditions.
- Through this additional information Helmholtz theorem guarantees the field is uniquely determined by its divergence and curl.

The Helmholtz Theorem

Margin Notes

Vector Calculus

- If the curl of a vector field ($\boldsymbol{F}$) is zero everywhere, then $\boldsymbol{F}$ can be written as the gradient of a scalar potential (V):

$$\nabla \times \boldsymbol{F} \qquad \rightarrow \qquad \boldsymbol{F} = -\nabla V.$$

- The minus sign is there for convention.
- If the divergence of a vector field ($\boldsymbol{F}$) is zero everywhere, $\boldsymbol{F}$ can be written as the curls of a vector potential ($\boldsymbol{A}$).

$$\nabla \cdot \boldsymbol{F} \qquad \rightarrow \qquad \boldsymbol{F} = -\nabla \times \boldsymbol{A}.$$

- For all cases a vector field ($\boldsymbol{F}$) can be written as sum of the scalar gradient and vector curl:

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}.$$

Potentials

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List of Acronyms i

Potentials

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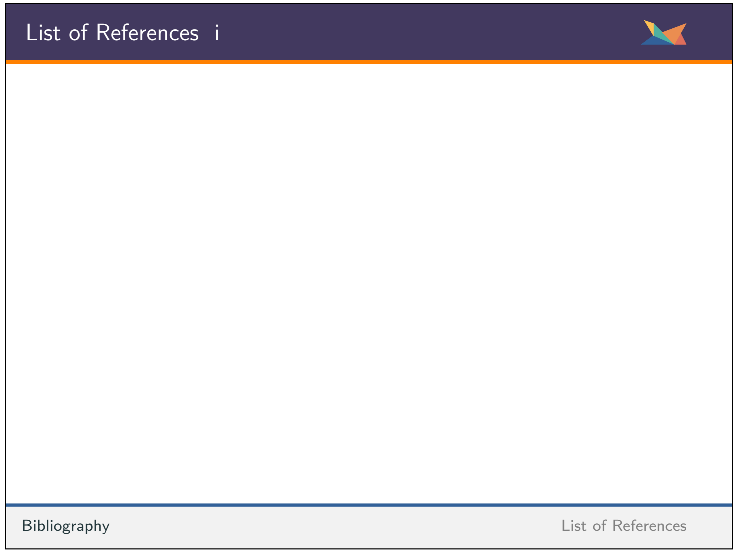

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