Tutorial Book

M.Sc Electrodynamics

D. T. McGuiness, Ph.D

Version: 2025.SS

This TutorialBook is a
complement to it's corresponding Lecture.
When in doubt regarding theory, please consult the LectureBook.

List of Questions

1

Vector Calculus

Q1 Simple Vectors

Find the components of the vector $\boldsymbol{v}$ with given initial point P and terminal point Q. Find $|\boldsymbol{v}|$ and unit vector $\hat{\boldsymbol{v}}$.

$$
\begin{array}{llll}
P\,(3,\ 2,\ 0)\,, & Q\,(5,\ -2,\ 2)\,, & P\,(1,\ 1,\ 1)\,, & Q\,(-4,\ -4,\ -4) \\
P\,\{\cdot\}\,1,\ 0,\ 1.2, & Q\,\{\cdot\}\,0,\ 0,\ 6.2, & P\,(2,\ -2,\ 0)\,, & Q\,(0,\ 4,\ 6) \\
P\,(4,\ 3,\ 2)\,, & Q\,(-4,\ -3,\ 2)\,, & P\,(0,\ 0,\ 0)\,, & Q\,\{\cdot\}\,6,\ 8,\ 10
\end{array}
$$

A1

The solution is as follows:

$$
\begin{aligned}
\boldsymbol{v} &= (5-3)\ \hat{x} + (-2-2)\ \hat{y} + (2-0)\ \hat{z} = (2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\ \hat{x} + \left(-\frac{2}{\sqrt{6}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{6}}\right)\ \hat{z} \quad \blacksquare \\
\boldsymbol{v} &= (-4-1)\ \hat{x} + (-4-1)\ \hat{y} + (-4-1)\ \hat{z} = (-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\ \hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{z} \quad \blacksquare \\
\boldsymbol{v} &= (0-1)\ \hat{x} + (0-0)\ \hat{y} + (6.2-1.2)\ \hat{z} = (-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\ \hat{x} + (0)\ \hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\ \hat{z} \quad \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{v} &= (0-2)\ \hat{x} + \big(4-(-2)\big)\ \hat{y} + (6-0)\ \hat{z} = (-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\ \hat{x} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{y} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{z} \quad \blacksquare \\
\boldsymbol{v} &= (-4-4)\ \hat{x} + (-3-3)\ \hat{y} + (2-2)\ \hat{z} = (-8)\ \hat{x} + (-6)\ \hat{y} + (0)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\ \hat{x} + (-8)\ \hat{y} + (0)\ \hat{z}}{10} = \left(-\frac{3}{5}\right)\ \hat{x} + \left(-\frac{4}{5}\right)\ \hat{y} + (0)\ \hat{z} \quad \blacksquare
\end{aligned}
$$

$$\boldsymbol{v} = (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},$$
$$|\boldsymbol{v}| = \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.$$
$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare$$

Q2 Finding the Points

Given the components of a vector $\boldsymbol{v} = \left[v_x,\ v_y,\ v_z\right]$ and a particular initial point P, find the corresponding terminal point Q and the length of $\boldsymbol{v}$ (i.e., $|\boldsymbol{v}|$).

$$\begin{aligned}
&\boldsymbol{v} = [3,\ -1,\ 0]\,; && P\,(4,\ 6,\ 0)\,, && \boldsymbol{v} = [8,\ 4,\ 2]\,; && P\,(-8,\ -4,\ -2)\,,\\
&\boldsymbol{v} = [0.25,\ 2,\ 0.75]\,; && P\,\{\cdot\}\ 0,\ -0.5,\ 0, && \boldsymbol{v} = [3,\ 2,\ 6]\,; && P\,(4,\ 6,\ 0)\,,\\
&\boldsymbol{v} = [4,\ 2,\ -2]\,; && P\,(4,\ 6,\ 0)\,, && \boldsymbol{v} = [3,\ -3,\ 3]\,; && P\,(4,\ 6,\ 0)\,,
\end{aligned}$$

A2

Previously we have defined $\boldsymbol{v} = Q - P$. Here we have $\boldsymbol{v}$ and P. To calculate Q we only need to add individual components of the vector with the initial point P.

$$\begin{aligned}
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare\\
Q &= \boldsymbol{v} + P = \big(8+(-8)\big)\ \hat{x} + \big(4+(-4)\big)\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare\\
Q &= \boldsymbol{v} + P = (0.25+0)\ \hat{x} + \big(2+(-0.5)\big)\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare\\
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare\\
Q &= \boldsymbol{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare\\
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare
\end{aligned}$$

Q3 Vector Addition

Let $\boldsymbol{a} = [2,\ 1,\ 0]$, $\boldsymbol{b} = [-4,\ 2,\ 5]$ and $\boldsymbol{c} = [0,\ 0,\ 3]$. Calculate the following vector operations:

$$\begin{aligned}
&2\boldsymbol{a}, && -\boldsymbol{a}, && -1/2\boldsymbol{a},\\
&5\,(\boldsymbol{a} - \boldsymbol{c})\,, && 5\boldsymbol{a} - 5\boldsymbol{c}, && (3\boldsymbol{a} - 5\boldsymbol{b}) + 2\boldsymbol{c},\\
&3\boldsymbol{a} + (-5\boldsymbol{b} + 2\boldsymbol{c})\,, && \boldsymbol{a} + 2\boldsymbol{b}, && 2\boldsymbol{b} + \boldsymbol{a}.
\end{aligned}$$

A3

The answers are the following:

$$
\begin{aligned}
2\boldsymbol{a} &= (4)\ \hat{x} + (2)\ \hat{y} + (0)\ \hat{z},\\
-\boldsymbol{a} &= (-2)\ \hat{x} + (-1)\ \hat{y} + (0)\ \hat{z},\\
-1/2\boldsymbol{a} &= (-1)\ \hat{x} + (-0.5)\ \hat{y} + (0)\ \hat{z},\\
5(\boldsymbol{a}-\boldsymbol{c}) &= (10)\ \hat{x} + (5)\ \hat{y} + (-15)\ \hat{z},\\
5\boldsymbol{a}-5\boldsymbol{c} &= (10)\ \hat{x} + (5)\ \hat{y} + (-15)\ \hat{z},\\
(3\boldsymbol{a}-5\boldsymbol{b})+2\boldsymbol{c} &= (26)\ \hat{x} + (-7)\ \hat{y} + (-19)\ \hat{z},\\
3\boldsymbol{a}+(-5\boldsymbol{b}+2\boldsymbol{c}) &= (26)\ \hat{x} + (-7)\ \hat{y} + (-19)\ \hat{z},\\
\boldsymbol{a}+2\boldsymbol{b} &= (-6)\ \hat{x} + (5)\ \hat{y} + (10)\ \hat{z},\\
2\boldsymbol{a}+\boldsymbol{b} &= (-6)\ \hat{x} + (5)\ \hat{y} + (10)\ \hat{z}.
\end{aligned}
$$

Q4 Finding the Dot Product

Find the dot product (i.e., $\boldsymbol{a}\cdot\boldsymbol{b}$) on the lengths of $\boldsymbol{a} = [1,\ 2,\ 0]$ and $\boldsymbol{b} = [3,\ -2,\ 1]$ as well as the angle (θ) between vectors.

A4

To find the angle between two vector we exploit the following identity:

$$
\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= |\boldsymbol{a}||\boldsymbol{b}|\cos\theta,\\
\theta &= \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right) = \cos^{-1}\left(\frac{3+(-4)+0}{\sqrt{(1)^2+(2)^2+(0)^2}\sqrt{(3)^2+(-2)^2+(1)^2}}\right),\\
&= \cos^{-1}\left(-\frac{1}{\sqrt{70}}\right) = 96.86\,\text{deg} \quad \blacksquare
\end{aligned}
$$

Q5 Finding the Vector Properties

Let $\boldsymbol{a} = [2,\ 1,\ 4]$, $\boldsymbol{b} = [-4,\ 0,\ 3]$ and $\boldsymbol{c} = [3,\ -2,\ 1]$. Find the following descriptions.

$$
\begin{aligned}
&|\boldsymbol{a}|\,,|\boldsymbol{b}|\,,|\boldsymbol{c}|\,, && \boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})\,,\ \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c},\\
&\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{c}\cdot\boldsymbol{a}, && 4\boldsymbol{a}\cdot 3\boldsymbol{c},\ 12\boldsymbol{a}\cdot\boldsymbol{c},\\
&|\boldsymbol{b}+\boldsymbol{c}|\,,|\boldsymbol{b}|+|\boldsymbol{c}|\,, && \boldsymbol{a}\cdot\boldsymbol{c},|\boldsymbol{a}||\boldsymbol{c}|\,.
\end{aligned}
$$

A5

The answers are the following:

$$
\begin{aligned}
|\boldsymbol{a}| = \sqrt{21}, \qquad |\boldsymbol{b}| &= 5, \qquad |\boldsymbol{c}| = \sqrt{14}.\\
\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) &= \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c} = 12,\\
\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{c}\cdot\boldsymbol{a} &= 3,\\
4\boldsymbol{a}\cdot 3\boldsymbol{c} &= 12\boldsymbol{a}\cdot\boldsymbol{c} = 96,
\end{aligned}
$$

$$|\boldsymbol{b}+\boldsymbol{c}| = \sqrt{21}, \qquad |\boldsymbol{b}| + |\boldsymbol{c}| = 5 + \sqrt{14},$$
$$\boldsymbol{a} \cdot \boldsymbol{c} = 8, \qquad |\boldsymbol{a}||\boldsymbol{c}| = 7\sqrt{6},$$

Q6 Find the Angle Between

Let $\boldsymbol{a} = [1,\ 1,\ 1]$, $\boldsymbol{b} = [2,\ 3,\ 1]$ and $\boldsymbol{c} = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(\boldsymbol{a}-\boldsymbol{c}) \text{ and } (\boldsymbol{b}-\boldsymbol{c}), \qquad (\boldsymbol{a}) \text{ and } (\boldsymbol{b}-\boldsymbol{c}).$$

A6

The answers are as follows:

$$33.21 \text{ deg}, \qquad 22.20 \text{ deg} \quad \blacksquare$$

Q7 Calculate the Vector Product

Find the vector product $\boldsymbol{a} \times \boldsymbol{b}$ of $\boldsymbol{a} = [1,\ 1,\ 0]$ and $\boldsymbol{b} = [3,\ 0,\ 0]$.

A7

Q8 Find the Cross Product

Let $\boldsymbol{a} = [1, 2, 0]$, $\boldsymbol{b} = [3,\ -4, 0]$, $\boldsymbol{c} = [3, 5, 2]$, $\boldsymbol{d} = [6, 2, 0]$. Calculate the cross product of:

$$\begin{aligned} &\boldsymbol{a}\times\boldsymbol{b}, \quad \boldsymbol{b}\times\boldsymbol{a}, && \boldsymbol{a}\times\boldsymbol{c}, \quad |\boldsymbol{a}\times\boldsymbol{c}|, \quad \boldsymbol{a}\cdot\boldsymbol{c}, \\ &(\boldsymbol{c}+\boldsymbol{d})\times\boldsymbol{d}, \quad \boldsymbol{c}\times\boldsymbol{d}, && \boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{b}, \\ &(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{b}+\boldsymbol{a}), && (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}, \quad \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}). \end{aligned}$$

A8

Q9 What is the Gradient

Find the gradient of $r = \sqrt{x^2+y^2+z^2}$.

A9

Q10 More of the Gradient

Find the gradient (∇f) of the following functions:

$$\begin{aligned} &f(x,\ y) = x^2 + \frac{1}{9}y^2, && f(x,\ y) = x^4 + y^4, \\ &f(x,\ y) = \ln\left(x^2+y^2\right), && f\{\cdot\}\, x,\ y,\ z = x^2 + 4y^2 + 9z^2, \\ &f(x,\ y) = \frac{y}{x^2+y^2}, && f(x,\ y) = x^2 y - \frac{1}{3}y^3. \end{aligned}$$

A10

Q11 Find the Divergence

Find the divergence ($\nabla \cdot$) to the following vector functions:

$$\boldsymbol{v} = \left(x^3 + y^3\right)\ \hat{x} + \left(3xy^2\right)\ \hat{y} + \left(3zy^2\right)\ \hat{z}, \qquad \boldsymbol{v} = \left(e^x\right)\ \hat{x} + \left(\ln xy\right)\ \hat{y} + \left(e^{xyz}\right)\ \hat{z},$$

$$\boldsymbol{v} = \left(x^2 + y^2\right)\ \hat{x} + \left(2xyz\right)\ \hat{y} + \left(z^2 + x^2\right)\ \hat{z}, \qquad \boldsymbol{v} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}.$$

A11

Q12 Cylindrical Divergence

Find the divergence of the function:

$$\boldsymbol{v} = \left(s\left(2 + \sin^2\phi\right)\right)\ \hat{s} + \left(s \sin\phi\cos\phi\right)\ \hat{\boldsymbol{\phi}} + (3z)\ \hat{z}.$$

A12

Q13 Curls of Functions

Find he curl ($\nabla \times$) of the following functions.

$$\boldsymbol{v} = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \qquad \boldsymbol{v} = \left(y^n\right)\ \hat{x} + \left(z^n\right)\ \hat{y} + \left(x^n\right)\ \hat{z},$$

$$\boldsymbol{v} = (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \qquad \boldsymbol{v} = \left(x^2 - z\right)\ \hat{x} + \left(xe^z\right)\ \hat{y} + (xy)\ \hat{z}.$$

A13

Q14 Checking the Product Rule

Check the product rule $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$ by calculating each term separately for the functions:

$$\boldsymbol{A} = (x)\ \hat{x} + (2y)\ \hat{y} + (3z)\ \hat{z}, \qquad \boldsymbol{B} = (3y)\ \hat{x} + (-2x)\ \hat{y} + (0)\ \hat{z}.$$

A14

Q15 Find the Laplacian

Calculate the Laplacian of the following functions:

$$T_a = x^2 + 3xy + 3z + 4, \qquad T_b = \sin x \sin y \sin z,$$

$$T_c = e^{-5x} \sin 4y \cos 3z, \qquad v = \left(x^2\right)\ \hat{x} + \left(3xz^2\right)\ \hat{y} + (-2xz)\ \hat{z}.$$

A15

Q16 Dirac Delta Function Integrals

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right) \delta\left(x - 3\right) \, dx, \qquad \int_0^5 \cos x \, \delta\left(x - \pi\right) \, dx,$$

$$\int_0^3 x^3 \, \delta\left(x + 1\right) \, dx, \qquad \int_{-\infty}^{+\infty} \ln\left(x + 3\right) \delta\left(x + 2\right) \, dx.$$

A16

Q17 Double Integrals

Find the following double integrals:

$$\int_0^1 \int_x^{2x} \left(x + y\right)^2 \, dy \, dx, \qquad \int_0^1 \int_y^{\sqrt{y}} \left(1 - 2xy\right) \, dx \, dy,$$

$$\int_0^3 \int_x^3 \cosh\left(x + y\right) \, dy \, dx, \qquad \int_0^1 \int_0^{y^3} \exp y^4 \, dx \, dy.$$

A17

The solution to integrations are as follows:

$$\begin{aligned}
\int_0^1 \int_x^{2x} \left(x + y\right)^2 \, dy \, dx &= \int_0^1 \int_x^{2x} x^2 + 2xy + y^2 \, dy \, dx, = \int_0^1 \left[yx^2 + xy^2 + \frac{y^3}{3} \right] \Bigg|_x^{2x} \, dx, \\
&= \int_0^1 \left(4x^3 + \frac{7x^3}{3} \right) \, dx, = \left[4x^3 + \frac{7x^4}{12} \right] \Bigg|_0^1 = \frac{19}{12} \quad \blacksquare \\
\int_0^1 \int_y^{\sqrt{y}} \left(1 - 2xy\right) \, dx \, dy &= \int_0^1 \left[x - x^2 y \right] \Bigg|_y^{\sqrt{y}} \, dy, \\
&= \int_0^1 \left[\left(\sqrt{y} - y^2 \right) - \left(y - y^3 \right) \right] \, dy = \int_0^1 \left[y^3 + \sqrt{y} - y^2 - y \right] \, dy, \\
&= \left[\frac{y^4}{4} + \frac{2}{3} y^{3/2} - \frac{y^3}{3} - \frac{y^2}{2} \right] \Bigg|_0^1, = \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} \right) - (0) = \frac{1}{12} \quad \blacksquare \\
\int_0^3 \int_x^3 \cosh\left(x + y\right) \, dy \, dx &= \int_0^1 \left[\sinh\left(x + y\right) \right] \Bigg|_x^3 \, dx = \int_0^1 \left[\sinh\left(3 + x\right) - \sinh\left(2x\right) \right] \, dx
\end{aligned}$$

Q18 Work in a Force Field

Find the work done by the force field $\boldsymbol{F} = \left(y - x^2\right) \hat{x} + \left(z - y^2\right) \hat{y} + \left(x - z^2\right) \hat{z}$ along the curve $\boldsymbol{r}\left(t\right) = \left(t\right) \hat{x} + \left(t^2\right) \hat{y} + \left(t^3\right) \hat{z}$, $0 \leq t \leq 1$ from 0 to 1.

A18

Q19 Fluid Flow

A fluid's velocity field is $\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z}$. Find the flow along the helix $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (t)\ \hat{z}$ with a range of $0 \leq t \leq \pi/2$.

A19

We first evaluate $\boldsymbol{F}$ on the curve:

$\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z} = (\cos t)\ \hat{x} + (t)\ \hat{y} + (\sin t)\ \hat{z}$ Substitute $\mathbf{x} = \cos \mathbf{t}$, $\mathbf{z} = \mathbf{t}$, $\mathbf{y} = \sin \mathbf{t}$.

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\boldsymbol{F} \cdot \left(d\boldsymbol{\ell}/dt\right)$ from $t = 0$ to $t = \pi/2$:

$$\begin{aligned}\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} &= (\cos t)\,(-\sin t) + (t)\,(\cos t) + (\sin t)\,(1)\,, \\ &= -\sin t \cos t + t \cos t + \sin t.\end{aligned}$$

Which makes,

$$\begin{aligned}\text{Flow} &= \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\, \mathrm{d}t = \int_0^{\pi/2} \left(-\sin t \cos t + t \cos t + \sin t\right)\, dt, \\ &= \left[\frac{\cos^2 t}{2} + t \sin t\right]\Bigg|_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare\end{aligned}$$

Q20 Field Circulation

Find the circulation of the field $\boldsymbol{F} = (x - y)\ \hat{x} + x\,\hat{y}$ around the circle $\boldsymbol{\ell}\,(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$ with a range of $0 \leq t \leq 2\pi$.

A20

On the circle, $\boldsymbol{F} = (x - y)\ \hat{x} + (x)\ \hat{y} + (0)\ \hat{z} = (\cos t - \sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}$ and

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1},$$

Gives.

$$\begin{aligned}\text{Circulation} &= \int_0^{2\pi} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\, dt = \int_0^{2\pi} \left(1 - \sin t \cos t\right)\, dt \\ &= \left[t - \frac{\sin^2 t}{2}\Bigg|_0^{2\pi}\right] = 2\pi \quad \blacksquare\end{aligned}$$

Q21 **Work Done by a Field**

Suppose the force field $\boldsymbol{F} = \nabla f$ is the gradient of the function:

$$f \{\cdot\} x, y, z = -\frac{1}{x^2 + y^2 + z^2}.$$

Find the work done by $\boldsymbol{F}$ in the moving an object along a smooth curve C joining (1,0,0) to (0,0,2) that does not pass through the origin.

A21

Q22 **Testing Stokes's Theorem**

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9$, $z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z}$. (**Tip:** Parametrisation of a circle is: $x = r\cos\theta$, $y = r\sin\theta$ and $da = \dfrac{3}{z} dA$)

A22

Q23 **Evaluating Divergence Theorem**

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}$ over the sphere $x^2 + y^2 + z^2 = a^2$

A23

Q24 **Measuring the Field Flux**

Find the flux of $\boldsymbol{F} = (xy)\ \hat{x} + (yz)\ \hat{y} + (xz)\ \hat{z}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

A24

Q25 **A Divergence or Not**

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2 \cos\theta\right)\ \hat{r} + \left(r^2 \cos\phi\right)\ \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right)\ \hat{\phi}.$$

using as your volume one octant of the sphere of radius R.

A25

Q26 **Calculating the Integral along a Path**

Compute the line integral of:

$$\boldsymbol{v} = (6)\ \hat{x} + \left(yz^2\right)\ \hat{y} + (3y + z)\ \hat{z},$$

along the triangular path shown below. Afterwards check your answer using Stokes' theorem.

A26

Q27 An Example of Divergence Theorem

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}$ over the sphere $x^2 + y^2 + z^2 = a^2$

A27

The outer unit normal to S, calculated from the gradient of $f\{\cdot\}\ x,\ y,\ z = x^2 + y^2 + z^2 - a^2$, is:

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{a}. \qquad \mathbf{x^2 + y^2 + z^2 = a^2} \text{ on } \mathbf{S}$$

Therefore:

$$(\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \frac{x^2 + y^2 + z^2}{a}\,\mathrm{d}a = \frac{a^2}{a}\ \mathrm{d}a = a\,\mathrm{d}a.$$

This in turn gives us:

$$\iint_S (\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \iint_S a\,\mathrm{d}a = a \iint_S \mathrm{d}a = a\left(4\pi a^2\right) = 4\pi a^3. \qquad \text{Area of } \mathbf{S} \text{ is } \mathbf{4ßa}^2$$

The divergence of $\boldsymbol{F}$ is:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

So,

$$\iiint_V (\nabla \cdot \boldsymbol{v})\ d\tau = \iiint_V 3\,d\tau = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3 \quad \blacksquare$$

Q28 Divergence Theorem of an Octant of a Sphere

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2 \cos\theta\right)\ \hat{r} + \left(r^2 \cos\phi\right)\ \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right)\ \hat{\phi}.$$

using as your volume one octant of the sphere of radius R.

A28

It is always useful to write the theorem we are going to work on:

$$\underbrace{\iiint_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\ dV}_{\text{Divergence integral}} = \underbrace{\iint_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{n}\,da.}_{\text{Outward flux}}$$

First solve the left hand side of the equation:

$$\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta r^2 \cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right), \\
&= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right), \\
&= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.
\end{aligned}$$

$$\int (\nabla \cdot \boldsymbol{v})\ d\tau = \int (4r\cos\theta)\ r^2 \sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta \int_0^{\pi/2} d\phi,$$

$$= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \blacksquare$$

Now it is time to solve the right hand side of the question. As we are aware from the shape, an octant of the sphere has 4 sides to it: the curved surface $xyz \to \mathbf{a}_1$, and $xz \to \mathbf{a}_2$, $yz \to \mathbf{a}_3$ and $xy \to \mathbf{a}_4$. These are

$$d\mathbf{a}_1 = \hat{\mathbf{r}}\, dl_\theta\, dl_\phi = \hat{\mathbf{r}}\, R^2 \sin\theta\, d\phi\, d\theta, \qquad d\mathbf{a}_2 = dl_r\, dl_\theta = -\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta,$$

$$d\mathbf{a}_3 = \hat{\boldsymbol{\phi}}\, dl_r\, dl_\theta = \hat{\boldsymbol{\phi}}\, r\, dr\, d\theta, \qquad d\mathbf{a}_4 = dl_r\, dl_\phi = \hat{\boldsymbol{\theta}}\, r\, dr\, d\theta. \quad (\theta = \pi/2)$$

$$\begin{aligned}
\iint_S \mathbf{v}\cdot d\mathbf{a} &= \iint_{S_1} \mathbf{v}\cdot d\mathbf{a} + \iint_{S_2} \mathbf{v}\cdot d\mathbf{a} + \iint_{S_3} \mathbf{v}\cdot d\mathbf{a} + \iint_{S_4} \mathbf{v}\cdot d\mathbf{a}, \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \left[r^2\cos\theta\ \hat{\mathbf{r}} + r^2\cos\phi\ \hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\ \hat{\boldsymbol{\phi}}\right]\Big|_{r=R} \cdot \left(\hat{\mathbf{r}}\, R^2\sin\theta\, d\phi\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{\mathbf{r}} + r^2\cos\phi\ \hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\ \hat{\boldsymbol{\phi}}\right]\Big|_{\phi=0} \cdot \left(-\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{\mathbf{r}} + r^2\cos\phi\ \hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\ \hat{\boldsymbol{\phi}}\right]\Big|_{\phi=\pi/2} \cdot \left(\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{\mathbf{r}} + r^2\cos\phi\ \hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\ \hat{\boldsymbol{\phi}}\right]\Big|_{\theta=\pi/2} \cdot \left(\hat{\boldsymbol{\theta}}\, r\, dr\, d\theta\right),
\end{aligned}$$

Time to do some integration.

$$\begin{aligned}
\iint_S \mathbf{v}\cdot d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2} \left[R^2\cos\theta\,\hat{\mathbf{r}} + R^2\cos\phi\,\hat{\boldsymbol{\theta}} - R^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}\right]\cdot\left(\hat{\mathbf{r}}\, R^2\sin\theta\, d\phi\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{\mathbf{r}} + r^2(1)\ \hat{\boldsymbol{\theta}} - (0)\sin\phi\ \hat{\boldsymbol{\phi}}\right]\cdot\left(-\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{\mathbf{r}} + (0)\phi\ \hat{\boldsymbol{\theta}} - r^2\cos\theta(1)\ \hat{\boldsymbol{\phi}}\right]\cdot\left(\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[(0)\ \hat{\mathbf{r}} + r^2\cos\phi\ \hat{\boldsymbol{\theta}} - (0)\ \hat{\boldsymbol{\phi}}\right]\cdot\left(\hat{\boldsymbol{\theta}}\, r\, dr\, d\theta\right).
\end{aligned}$$

Final touches and cleaning up,

$$\begin{aligned}
\iint_S \mathbf{v}\cdot d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2} R^4\sin\theta\cos\theta\, d\phi\, d\theta + 0 \overbrace{- \int_0^{\pi/2}\int_0^{R} r^3\cos\theta\, dr\, d\theta + \int_0^{\pi/2}\int_0^{R} r^3\cos\theta\, dr\, d\phi}^{=0}, \\
&= R^4\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right), \\
&= R^4\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right), \\
&= \frac{\pi R^4}{4} \quad \blacksquare
\end{aligned}$$

Q29 Surface Area of an Implicit Surface

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

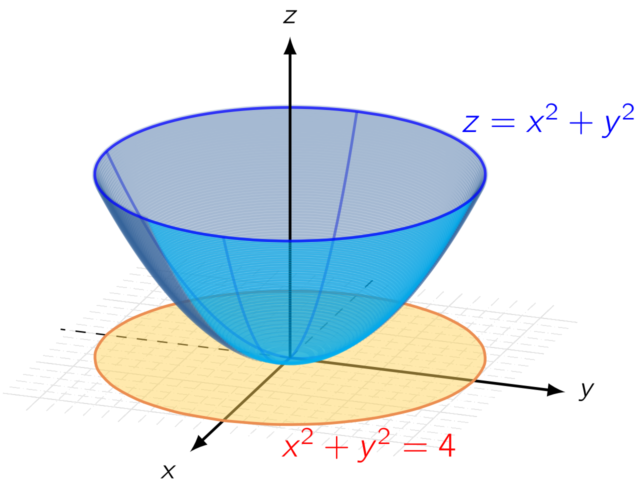

Figure 1.1: The paraboloid of "Surface Area of an Implicit Surface".

A29

We sketch the surface S and the region R below it in the xy-plane (**Fig.** 1.1). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, ans R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal (i.e., $\hat{n}$) to the plane R, we can take $\hat{n} = \hat{z}$. At any point (x, y, z) on the surface, we have:

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= (2x)\ \hat{x} + (2y)\ \hat{y} + (-1)\ \hat{z} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \hat{n}| &= |\nabla F \cdot \hat{z}| = |-1| = 1. \end{aligned}$$

In the region R, the area is defined to be $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y$. Therefore:

$$\begin{aligned} \text{Surface Area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{n}|}\, \mathrm{d}A \\ &= \iint\limits_{x^2+y^2 \leq 4} \sqrt{4x^2 + 4y^2 + 1}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(4r^2 + 1\right)^{3/2} \Big|_0^2\, \mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \frac{1}{12}\left(17^{3/2} - 1\right) \, \mathrm{d}\theta \\
&= \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \quad \blacksquare
\end{aligned}
$$

Q30 Stokes' Theorem Over a Hemisphere

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9$, $z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z}$.

Tip: Parametrisation of a circle is: $x = r\cos\theta$, $y = r\sin\theta$ and $da = \frac{3}{z} dA$)

A30

The start by calculating the counter-clockwise circulation around C using the following parametrisation:

$$
\begin{aligned}
&\boldsymbol{\ell}(\theta) = (3\cos\theta)\ \hat{x} + (3\sin\theta)\ \hat{y} + (0)\ \hat{z}, \\
&\text{where} \qquad 0 \leq \theta \leq 2\pi.
\end{aligned}
$$

Using this we can calculate the counter-clockwise circulation.

$$
\begin{aligned}
d\boldsymbol{\ell} &= (-3\sin\theta\, d\theta)\ \hat{x} + (3\cos\theta\, d\theta)\ \hat{y} + (0)\ \hat{z}, \\
\boldsymbol{F} &= (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z} \\
&= (3\sin\theta)\ \hat{x} + (-3\cos\theta)\ \hat{y} + (0)\ \hat{z}, \\
\boldsymbol{F}\cdot d\boldsymbol{\ell} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta, \\
\oint_C \boldsymbol{F}\cdot d\boldsymbol{\ell} &= \int_0^{2\pi} -9\, d\theta = -18\pi.
\end{aligned}
$$

For the curl of integral we have:

$$
\begin{aligned}
\nabla\times\boldsymbol{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (0-0)\ \hat{x} + (0-0)\ \hat{y} + (-1-1)\ \hat{z} = -2\,\hat{z} \\
\hat{n} &= \frac{\nabla S}{|\nabla S|} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{3} \qquad \textbf{Unit normal}
\end{aligned}
$$

Now it is time to define the area of integration (da):

$$
\begin{aligned}
da &= \frac{|\nabla S|}{|\nabla S\cdot\hat{z}|}\, dA \\
&= \frac{\left|(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}\right|}{2z} \\
&= \frac{2\overbrace{\sqrt{x^2+y^2+z^2}}^{a}}{2z}
\end{aligned}
$$

$$= \frac{3}{z} dA,$$
$$\nabla \times \boldsymbol{F} \cdot \boldsymbol{n}\, da = -\frac{2z}{3}\frac{3}{z}\, dA = -2\, dA$$

The cardinal direction $\hat{z}$ comes from being the direction **perpendicular** to the surface (S).

$$\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n}\, da = \iint_{x^2+y^2 \leq 9} -2\, dA = -18\pi$$

The circulation around the circle equals the integral of the curl over the hemisphere ■

Q31 The Volume of a Sphere

Find the volume of a sphere of radius R.

A31

The derivation is as follows:

$$\begin{aligned}
V &= \int \mathrm{d}\tau \\
&\int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi, \\
&= \left(\int_0^R r^2 \, \mathrm{d}r\right) \left(\int_0^\pi \sin\theta \, \mathrm{d}\theta\right) \left(\int_0^{2\pi} \mathrm{d}\phi\right) \\
&= \left(\frac{R^3}{3}\right)(2)(2\pi) = \frac{4}{3}\pi R^3 \quad ■
\end{aligned}$$

Q32 Finding Vector Components - I

Find the components of the vector $\boldsymbol{v}$ with given initial point P and terminal point Q. Find $|\boldsymbol{v}|$ and unit vector $\hat{\boldsymbol{v}}$.

$$\begin{aligned}
&P(3,\ 2,\ 0), && Q(5,\ -2,\ 2), && P(1,\ 1,\ 1), && Q(-4,\ -4,\ -4) \\
&P(1,\ 0,\ 1.2), && Q(0,\ 0,\ 6.2), && P(2,\ -2,\ 0), && Q(0,\ 4,\ 6) \\
&P(4,\ 3,\ 2), && Q(-4,\ -3,\ 2), && P(0,\ 0,\ 0), && Q(6,\ 8,\ 10)
\end{aligned}$$

A32

The solution is as follows:

$$\begin{aligned}
\boldsymbol{v} &= (5-3)\ \hat{x} + (-2-2)\ \hat{y} + (2-0)\ \hat{z} = (2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}. \\
\hat{\boldsymbol{v}} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\hat{x} + \left(-\frac{2}{\sqrt{6}}\right)\hat{y} + \left(\frac{1}{\sqrt{6}}\right)\hat{z} \quad ■ \\
\boldsymbol{v} &= (-4-1)\ \hat{x} + (-4-1)\ \hat{y} + (-4-1)\ \hat{z} = (-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.
\end{aligned}$$

$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\ \hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (0-1)\ \hat{x} + (0-0)\ \hat{y} + (6.2-1.2)\ \hat{z} = (-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.$$

$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\ \hat{x} + (0)\ \hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (0-2)\ \hat{x} + \big(4-(-2)\big)\ \hat{y} + (6-0)\ \hat{z} = (-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.$$

$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\ \hat{x} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{y} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (-4-4)\ \hat{x} + (-3-3)\ \hat{y} + (2-2)\ \hat{z} = (-8)\ \hat{x} + (-6)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.$$

$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\ \hat{x} + (-8)\ \hat{y} + (0)\ \hat{z}}{10} = \left(-\frac{3}{5}\right)\ \hat{x} + \left(-\frac{4}{5}\right)\ \hat{y} + (0)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.$$

$$\hat{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare$$

Q33 Finding Vector Components - II

Given the components of a vector $\boldsymbol{v} = \left[v_x, v_y, v_z\right]$ and a particular initial point P, find the corresponding terminal point Q and the length of $\boldsymbol{v}$ (i.e., $|\boldsymbol{v}|$).

$$\boldsymbol{v} = [3, -1, 0]\,; \quad P\,(4, 6, 0)\,, \qquad \boldsymbol{v} = [8, 4, 2]\,; \quad P\,(-8, -4, -2)\,,$$

$$\boldsymbol{v} = [0.25, 2, 0.75]\,; \quad P\,\{\cdot\}\,0, -0.5, 0, \qquad \boldsymbol{v} = [3, 2, 6]\,; \quad P\,(4, 6, 0)\,,$$

$$\boldsymbol{v} = [4, 2, -2]\,; \quad P\,(4, 6, 0)\,, \qquad \boldsymbol{v} = [3, -3, 3]\,; \quad P\,(4, 6, 0)\,,$$

A33

Previously we have defined $\boldsymbol{v} = Q - P$. Here we have $\boldsymbol{v}$ and P. To calculate Q we only need to add individual components of the vector with the initial point P.

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = \big(8+(-8)\big)\ \hat{x} + \big(4+(-4)\big)\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (0.25+0)\ \hat{x} + \big(2+(-0.5)\big)\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare$$
$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},$$
$$|\boldsymbol{v}| = \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare$$

Q34 Vector Addition and Scalar Multiplication

- Let $\boldsymbol{a} = [2,\ 1,\ 0]$, $\boldsymbol{b} = [-4,\ 2,\ 5]$ and $\boldsymbol{c} = [0,\ 0,\ 3]$. Calculate the following vector operations:

$$2\boldsymbol{a}, \qquad -\boldsymbol{a}, \qquad -1/2\boldsymbol{a},$$
$$5(\boldsymbol{a} - \boldsymbol{c}), \qquad 5\boldsymbol{a} - 5\boldsymbol{c}, \qquad (3\boldsymbol{a} - 5\boldsymbol{b}) + 2\boldsymbol{c},$$
$$3\boldsymbol{a} + (-5\boldsymbol{b} + 2\boldsymbol{c}), \qquad \boldsymbol{a} + 2\boldsymbol{b}, \qquad 2\boldsymbol{b} + \boldsymbol{a}.$$

reset

- Find the dot product (i.e., $\boldsymbol{a} \cdot \boldsymbol{b}$) on the lengths of $\boldsymbol{a} = [1,\ 2,\ 0]$ and $\boldsymbol{b} = [3,\ -2,\ 1]$ as well as the angle (θ) between vectors.

- Let $\boldsymbol{a} = [2,\ 1,\ 4]$, $\boldsymbol{b} = [-4,\ 0,\ 3]$ and $\boldsymbol{c} = [3,\ -2,\ 1]$. Find the following descriptions.

$$|\boldsymbol{a}|, |\boldsymbol{b}|, |\boldsymbol{c}|, \qquad \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}),\ \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c},$$
$$\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{c} \cdot \boldsymbol{a}, \qquad 4\boldsymbol{a} \cdot 3\boldsymbol{c}, 12\boldsymbol{a} \cdot \boldsymbol{c},$$
$$|\boldsymbol{b} + \boldsymbol{c}|, |\boldsymbol{b}| + |\boldsymbol{c}|, \qquad \boldsymbol{a} \cdot \boldsymbol{c}, |\boldsymbol{a}||\boldsymbol{c}|.$$

reset

- Let $\boldsymbol{a} = [1,\ 1,\ 1]$, $\boldsymbol{b} = [2,\ 3,\ 1]$ and $\boldsymbol{c} = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(\boldsymbol{a} - \boldsymbol{c}) \text{ and } (\boldsymbol{b} - \boldsymbol{c}), \qquad (\boldsymbol{a}) \text{ and } (\boldsymbol{b} - \boldsymbol{c}).$$

- Find the vector product $\boldsymbol{a} \times \boldsymbol{b}$ of $\boldsymbol{a} = [1,\ 1,\ 0]$ and $\boldsymbol{b} = [3,\ 0,\ 0]$.

- Let $\boldsymbol{a} = [1, 2, 0]$, $\boldsymbol{b} = [3,\ -4, 0]$, $\boldsymbol{c} = [3, 5, 2]$, $\boldsymbol{d} = [6, 2, 0]$. Calculate the cross product of:

$$\boldsymbol{a} \times \boldsymbol{b}, \quad \boldsymbol{b} \times \boldsymbol{a}, \qquad \boldsymbol{a} \times \boldsymbol{c}, \quad |\boldsymbol{a} \times \boldsymbol{c}|, \quad \boldsymbol{a} \cdot \boldsymbol{c},$$
$$(\boldsymbol{c} + \boldsymbol{d}) \times \boldsymbol{d}, \quad \boldsymbol{c} \times \boldsymbol{d}, \qquad \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{b},$$
$$(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{b} + \boldsymbol{a}), \qquad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}, \quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}).$$

A34

Q35 An Example of a Curl

Find he curl ($\nabla \times$) of the following functions.

$$\boldsymbol{v} = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \qquad \boldsymbol{v} = (y^n)\ \hat{x} + (z^n)\ \hat{y} + (x^n)\ \hat{z},$$
$$\boldsymbol{v} = (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \qquad \boldsymbol{v} = \left(x^2 - z\right)\ \hat{x} + (xe^z)\ \hat{y} + (xy)\ \hat{z}.$$

A35

The curl ($\nabla \times$) of the functions are as follows:

$$
\begin{aligned}
f(x, y, z) &= (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \\
\nabla \times f &= (0)\ \hat{x} + (0)\ \hat{y} + (-1 + 4x)\ \hat{z}. \\
f(x, y, z) &= \left(y^n\right)\ \hat{x} + \left(z^n\right)\ \hat{y} + \left(x^n\right)\ \hat{z}, \\
\nabla \times f &= \left(-nz^{n-1}\right)\ \hat{x} + \left(-nx^{n-1}\right)\ \hat{y} + \left(-ny^{n-1}\right)\ \hat{z}. \\
f(x, y, z) &= (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \\
\nabla \times f &= (\sin z)\ \hat{x} + \left(\sec^2 x\right)\ \hat{y} + (-\cos y)\ \hat{z}. \\
f(x, y, z) &= \left(x^2 - z\right)\ \hat{x} + \left(xe^{z}\right)\ \hat{y} + (xy)\ \hat{z}, \\
\nabla \times f &= \left(x - e^z x\right)\ \hat{x} + (-1 - y)\ \hat{y} + \left(e^z\right)\ \hat{z}.
\end{aligned}
$$

Q36 **The Laplacian of a Vector**

Calculate the Laplacian of the following functions:

$$
\begin{aligned}
&(i) \quad T_a = x_2 + 3xy + 3z + 4, && (ii) \quad T_b = \sin x \sin y \sin z, \\
&(iii) \quad T_c = e^{-5x} \sin 4y \cos 3z, && (iv) \quad \mathbf{v} = \left(x^2\right)\ \hat{x} + \left(3xz^2\right)\ \hat{y} + (-2xz)\ \hat{z}.
\end{aligned}
$$

A36

The solution to the Laplacian of the functions are as follows:

$$
\begin{aligned}
&\textbf{(i)} && \frac{\partial^2 T_a}{\partial x^2} = 2;\ \frac{\partial^2 T_a}{\partial y^2} = 0;\ \frac{\partial^2 T_a}{\partial z^2} = 0 && \rightarrow && \nabla^2 T_a = 2 \quad \blacksquare \\
&\textbf{(ii)} && \frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -3T_b && \rightarrow && \nabla^2 T_b = -3T_b = 3 \sin x \sin y \sin z \quad \blacksquare \\
&\textbf{(iii)} && \frac{\partial^2 T_c}{\partial x^2} = 25T_c; \\
& && \frac{\partial^2 T_c}{\partial y^2} = -16T_c;\quad \frac{\partial^2 T_c}{\partial z^2} = -9T_c && \rightarrow && \nabla^2 T_c = 0 \quad \blacksquare \\
&\textbf{(iii)} && \frac{\partial^2 v_x}{\partial x^2} = 2;\ \frac{\partial^2 v_x}{\partial y^2} = 0;\ \frac{\partial^2 v_x}{\partial z^2} = 0 && \rightarrow && \nabla^2 v_x = 0, \\
& && \frac{\partial^2 v_y}{\partial x^2} = 0;\ \frac{\partial^2 v_y}{\partial y^2} = 0;\ \frac{\partial^2 v_y}{\partial z^2} = 6 && \rightarrow && \nabla^2 v_y = 6x, \\
& && \frac{\partial^2 v_z}{\partial x^2} = 0;\ \frac{\partial^2 v_z}{\partial y^2} = 0;\ \frac{\partial^2 v_z}{\partial z^2} = 0 && \rightarrow && \nabla^2 v_z = 0, \\
& && \nabla^2 \mathbf{v} = 2\,\hat{x} + 6x\,\hat{y} \quad \blacksquare
\end{aligned}
$$

Q37 **A Simple Dirac Integral**

Evaluate the following integral:

$$
\int_0^3 x^3 \delta(x - 2)\ dx
$$

A37

The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Q38 **1D Dirac Delta Function**

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right) \delta(x-3)\, dx, \tag{i}$$

$$\int_0^5 \cos x\, \delta(x-\pi)\, dx, \tag{ii}$$

$$\int_0^3 x^3\, \delta(x+1)\, dx, \tag{iii}$$

$$\int_{-\infty}^{+\infty} \ln(x+3)\, \delta(x+2)\, dx. \tag{iv}$$

A38

The solution are as follows:

(a) $3\left(3^2\right) - 2(3) - 1 = 27 - 6 - 1 = 20$ ■

(b) $\cos\pi = -1$ ■

(c) 0 ■

(d) $\ln(-2+3) = \ln 1 = 0$ ■

2 Electrostatics

Q39 Electric Field at a Distance

Find the $\boldsymbol{E}$ field a distance z above the midpoint between two equal charges (q), a distance d apart:

A39

Let $\boldsymbol{E}_1$ be the field of the left charge, and $\boldsymbol{E}_2$ that of the right charge alone. Adding them, the horizontal components cancel and the vertical components add to:

$$E_z = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta.$$

Here $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$ and $\cos\theta = z/\mathscr{r}$, therefore:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\hat{z} \quad \blacksquare$$

Q40 A Gaussian Sphere

Find the field outside a uniformly charged solid sphere of radius R and total charge q.

A40

Imagine a spherical surface at radius $r > R$. this is called a Gaussian surface. Gauss's law says that:

$$\oint_{\mathcal{S}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{a} = \frac{1}{\varepsilon_0}Q_{\text{enc}},$$

For this case it is $Q_{\text{enc}} = q$. At first glance this doesn't seem to get us very far, as the quantity we want (E) is buried inside the surface integral. Luckily, symmetry allows us to extract E from under the integral sign: E certainly points radially outward,5 as does da, so we can drop the dot product,

$$\int_{\mathcal{S}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{a} = \int_{\mathcal{S}} |\boldsymbol{E}|\ \mathrm{d}a$$

and the magnitude of E is constant over the Gaussian surface, so it comes outside the integral

$$\int_{\mathcal{S}} |\boldsymbol{E}|\ \mathrm{d}a = |\boldsymbol{E}|\int_{\mathcal{S}} \mathrm{d}a = |E|\,4\pi r^2$$

Therefore:

$$|\boldsymbol{E}|\,4\pi r^2 = \frac{1}{\varepsilon_0}q \qquad \text{or} \qquad \boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \blacksquare$$

3

Advanced Mathematical Methods

Q41 2D Grounded Plates

Two infinitely long grounded metal plates (■) at $y = 0$ and $y = a$ are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0 (■) as shown. (a thin layer of insulation at each corner prevents the metal plates from shorting out just for pedantry). Find the potential inside the resulting rectangular pipe.

A41

As can be seen, the configuration is independent from z-direction. Therefore, our goal here is to solve a 2D Laplace's equation ($\nabla^2 f = 0$) is as follows:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

which are subject to the following boundary conditions:

$$\begin{array}{llllllll} \textbf{(i)} & V = 0 & \text{when} & y = 0, & \textbf{(ii)} & V = 0 & \text{when} & y = a, \\ \textbf{(iii)} & V = V_0 & \text{when} & x = b, & \textbf{(iv)} & V = V_0 & \text{when} & x = -b. \end{array}$$

To solve this problem we need to use Separation of Variables method which is as follows. Assume a solution with the following description,

$$V(x, y) = X(x)\,Y(y).$$

Putting this into the original Laplace equation gives us:

$$Y(y)\frac{d^2X}{dx^2} + X(x)\frac{d^2Y}{dy^2} = 0.$$

It is time to isolate the variables,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

First term depends on x, and the second term depends on y. From this we can write:

$$f(x) + g(y) = 0.$$

For this to be true both of these functions needs to be constants.

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y.$$

It is time to solve these equations, lets have a look at the first one:

$$\frac{d^2X}{dx^2} = k^2X.$$

In ODE this is classified as an homogeneous 2nd order linear differential equation with the generic form of

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y(x) = 0.$$

where a, b, c are arbitrary constants. The general solution for these kind of equation is,

$$y(x) = A\exp -r_1x + B\exp -r_2x \quad \text{where} \quad r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}, \quad r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

Based on this template, the solution to the first ODE is;

$$X(x) = A\exp kx + B\exp -kx,$$

If we were to apply the second ODE to the previously mentioned solution template we arrive at the following.

$$Y(y) = C\exp -jy + D\exp +jy \quad \text{or} \quad Y(y) = C\sin ky + D\cos ky.$$

While the imaginary solution is perfectly valid, we want to use the real only solution as the equation we are trying to solve is real as well. The solution to this problem is as follows the same path we have looked at the lecture up to the following part:

$$V(x,\,y) = (A\exp kx + B\exp -kx)\,(C\sin ky + D\cos ky)$$

Here, however, we cannot set $A = 0$ as the region in question does not extend to $x = \infty$, so $\exp kx$ is perfectly acceptable, whereas, the situation is symmetric with respect to x, so $V(-x,\,y) = V(x,y)$ and it follows $A = B$. Using the hyperbolic cosine function identity:

$$A\exp kx + A\exp -kx = 2A\cosh kx,$$

distributing A to C and D, we have;

$$V(x,\,y) = \cosh kx\,(2AC\sin ky + 2AD\cos ky)$$

Boundary conditions **(i)** and **(ii)** require D to be equal to zero and $k = n\pi/a$ so:

$$V(x,\,y) = 2AC\cosh\big(n\pi x/a\big)\sin\big(n\pi y/a\big)\,,$$

Because $V(x,\,y)$ is even in x, it will automatically meet condition **(iv)** if it fits **(iii)**. It remains, therefore, to construct the linear combination,

$$V(x,\,y) = \sum_{n=1}^{\infty} G_n\cosh\big(n\pi x/a\big)\sin\big(n\pi y/a\big)\,, \qquad \text{where} \quad G = 2AC.$$

and pick coefficients C_n such that it satisfies the condition on **(iii)**:

$$V(b,\,y) = \sum_{n=1}^{\infty} G_n\cosh\big(n\pi b/a\big)\sin\big(n\pi y/a\big) = V_0,$$

This is the same problem in *Fourier analysis* we worked in the exercise in part 3.
To solve this equation, we first need to multiply both sides with $\sin \frac{m\pi y}{a}$.

$$\sin \frac{m\pi y}{a} V_0(0) = \sum_1^\infty G_n \cosh\left(n\pi b/a\right) \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a},$$

Time to integrate both sides over y, from zero to a:

$$\int_0^a \sin \frac{m\pi y}{a} V_0(0)\ dy = \sum_1^\infty G_n \cosh\left(n\pi b/a\right) \int_0^a \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a}\, dy,$$

This is basically **Fourier Tansform** on both sides. It is time to focus on the integration on the right hand side.

$$\begin{aligned}\int_0^a \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a}\, dy &= \left[-\frac{a}{n\pi} \sin \frac{mxy}{a} \cos \frac{n\pi y}{a}\right]\Bigg|_0^a + \frac{m}{n} \int_0^a \cos \frac{m\pi y}{a} \cos \frac{n\pi y}{a}\, dy \\ &= \frac{m}{n}\left[\frac{a}{n\pi} \cos \frac{m\pi y}{a} \sin \frac{nxy}{a}\Bigg|_0^a + \frac{m}{n} \int_0^a \sin \frac{m\pi y}{a} \sin \frac{n\pi y}{a}\, dy\right],\end{aligned}$$

Simplification of this integral yields,

$$\left(1 - \frac{m^2}{n^2}\right) \int_0^a \sin \frac{m\pi y}{a} \sin \frac{n\pi y}{a}\, dy = 0.$$

There are two possibilities here: either $m = n$ or $m \neq n$. if the latter is the case, then:

$$\int_0^a \sin \frac{m\pi y}{a} \sin \frac{n\pi y}{a}\, dy = 0 \qquad (m \neq n)$$

If on the other hand $m = n$, then:

$$\begin{aligned}\int_0^a \sin^2 \frac{n\pi y}{a}\, dy &= -\frac{a}{n\pi} \sin \frac{n\pi y}{a} \cos \frac{n\pi y}{a}\Bigg|_0^a + \int_0^a \cos^2 \frac{n\pi y}{a}\, dy \\ 2\int_0^a \sin^2 \frac{n\pi y}{a}\, dy &= \int_0^a \sin^2 \frac{n\pi y}{a}\, dy + \int_0^a \cos^2 \frac{n\pi y}{a}\, dy = \int_0^a dy = a.\end{aligned}$$

Therefore:

$$\int_0^a \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a}\, dy = \begin{cases} 0, & \text{if } n \neq m \\ \frac{a}{2} & \text{if } n = m \end{cases}$$

Based on this, we only care about the solutions where $m = n$. If we take our focus to the rest of the equation.

$$\int_0^a \sin \frac{n\pi y}{a} V_0\, dy = \sum_1^\infty G_n \cosh\left(n\pi b/a\right) \overbrace{\int_0^a \sin \frac{n\pi y}{a} \sin \frac{n\pi y}{a}\, dy}^{=a/2},$$

If we isolate the G_n on one side:

$$G_n = \frac{2}{a} \frac{1}{\cosh(n\pi b/a)} \int_0^a \sin \frac{n\pi y}{a} V_0\, dy$$

Doing the integral on the right hand side yields the following:

$$G_n = \frac{2}{a}\frac{1}{\cosh(n\pi b/a)}\int_0^a \sin\frac{n\pi y}{a} V_0\, dy = \frac{1}{\cosh(n\pi b/a)}\frac{2V_0}{n\pi}(1-\cos n\pi).$$

The results are:

$$C_n \cosh\left(n\pi b/a\right) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

The potential in this case is given by:

$$V(x,\, y) = \frac{4V_0}{\pi}\sum_{n=1,\,3,\,5,\,\dots} \frac{1}{n}\frac{\cosh\left(n\pi x/a\right)}{\cosh\left(n\pi b/a\right)}\sin\left(n\pi y/a\right) \quad \text{V} \quad \blacksquare$$

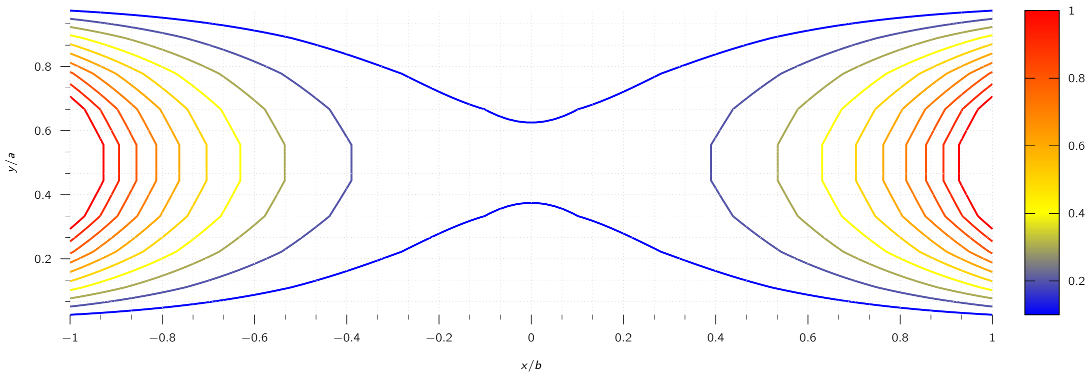

Figure 3.1

Q42 A Long Rectangular Tube

An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end at $x = 0$, is maintained at a specific potential $V_0\left(y,\, z\right)$ as indicated above. Find the potential inside the pipe.

A42

This is a 3D problem with the following **Laplace** equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

which are subject to the following boundary conditions given by the question either explicitly or implicitly:

(i)	$V = 0$	when	$y = 0,$	**(ii)**	$V = 0$	when	$y = a,$
(iii)	$V = 0$	when	$z = 0,$	**(iv)**	$V = 0$	when	$z = b,$
(v)	$V \to 0$	as	$x \to \infty,$	**(vi)**	$V = V_0(x,\, y)$	when	$x = 0.$

As usual, we are looking for solutions that are products using **separation of variables**:

$$V\left(x,\, y,\, z\right) = X\left(x\right) Y\left(y\right) Z\left(z\right)$$

Putting this into the Laplace equation presents to us:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0.$$

Where once we isolate the variables, we can get the following relation:

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_1,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2,\quad \frac{1}{Z}\frac{d^2Z}{dz^2}=C_3,\quad \text{with}\quad C_1+C_2+C_3=0.$$

To keep this solution relatively short, We will assume C_1 **must** be positive and C_2, C_3 are negative. Setting $C_2=-k^2$ and $C_3=-l^2$, we have $C_1=k^2+l^2$, and therefore:

$$\frac{d^2X}{dx^2}=\left(k^2+l^2\right)X,\qquad \frac{d^2Y}{dy^2}=-k^2Y,\qquad \frac{d^2Z}{dz^2}=-l^2Z$$

We have yet again turned a Partial Differential Equation (PDE) into a series of Ordinary Differential Equation (ODE)s. The solutions are:

$$\begin{aligned}
X(x)&=A\exp\left(\sqrt{k^2+l^2}x\right)+B\left(-\sqrt{k^2+l^2}x\right),\\
Y(y)&=C\sin ky+D\cos ky,\\
Z(z)&=E\sin lz+F\cos lz.
\end{aligned}$$

Boundary condition **(v)** implies $A=0$, **(i)** gives $D=0$ and **(iii)** yields $F=0$, whereas **(ii)** and **(iv)** require $k=n\pi/a$ and $l=m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are presented with:

$$V(x,\, y,\, z)=C\exp\left(-\pi\sqrt{\left(n/a\right)^2+\left(m/b\right)^2}x\right)\sin\left(n\pi y/a\right)\sin\left(m\pi z/b\right).$$

This solution meets all the boundary condition except **(vi)**. It contains two unspecified integers (m and n) and the most general linear combination is a double sum:

$$V(x,\, y,\, z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,\,m}\exp\left(-\pi\sqrt{\left(n/a\right)^2+\left(m/b\right)^2}x\right)\sin\left(n\pi y/a\right)\sin\left(m\pi z/b\right).$$

We hope to fit the remaining boundary condition:

$$V(0,\, y,\, z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,\,m}\sin\left(n\pi y/a\right)\sin\left(m\pi z/b\right)=V_0\left(y,\, z\right).$$

by choosing appropriate coefficients of $C_{n,\,m}$.
To determine these constants, we multiply by $\sin\left(n'\pi y/a\right)\sin\left(m'\pi z/b\right)$, where n' and m' are arbitrary positive integers, and integrate:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,\,m}\int_0^b\sin\left(n\pi y/a\right)\sin\left(n'\pi y/a\right)\,dy\int_0^b\sin\left(m\pi z/b\right)\sin\left(m'\pi z/b\right)\,dz\\
&\qquad=\int_0^a\int_0^b V_0\left(y,\, z\right)\sin\left(n'\pi z/b\right)\sin\left(m'\pi z/b\right)\,dy\,dz.
\end{aligned}$$

So:

$$C_{n,\,m}=\frac{4}{ab}\int_0^a\int_0^b V_0\left(y,\, z\right)\sin\left(n\pi z/b\right)\sin\left(m\pi z/b\right)\,dy\,dz.$$

We are almost at the end of our problem. For example, if the end of the tube is a conductor at constant period V_0:

$$C_{n,\,m} = \frac{4V_0}{ab} \int_0^a \sin\left(n\pi y/a\right)\, dy \int_0^b \sin\left(m\pi z/b\right)\, dz$$
$$= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$$

In this case:

$$V\left(x,\, y,\, z\right) = \frac{16V_0}{\pi^2} \sum_{n,\, m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \exp\left(-\pi\sqrt{\left(n/a\right)^2 + \left(m/b\right)^2} x\right) \sin\left(n\pi y/a\right) \sin\left(m\pi z/b\right) \qquad \blacksquare$$

As the successive terms decrease rapidly, a reasonable approximation would be obtained by keeping only the first few terms.

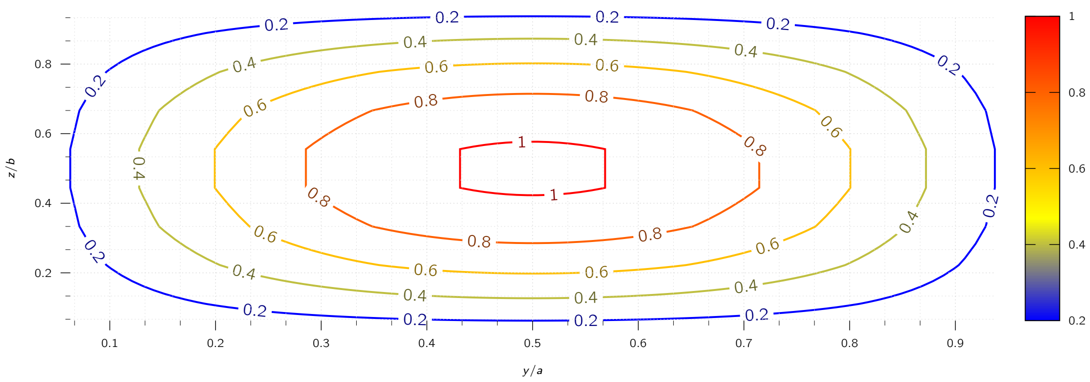

Figure 3.2

Q43 Charge Density

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

A43

You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma_0}{\lambda} da,$$

but separation of variables is often easier. For the interior region, we have

$$V\left(r,\theta\right) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad \left(r \leq R\right)$$

(no B_l terms–they blow up at the origin); in the exterior region

$$V\left(r,\theta\right) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad \left(r \geq R\right)$$

(no A_l terms–they don't go to zero at infinity). These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is *continuous* at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}[\infty] \frac{B_l}{R^{l+1}} P_l(\cos\theta).$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}.$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\Bigg|_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta).$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}[\infty] l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma_0(\theta),$$

or, using Eq. 3.81,

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0}\sigma_0(\theta).$$

From here, the coefficients can be determined using Fourier's trick:

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta d\theta.$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by Eqs. 3.81 and 3.84.
For instance, if

$$\sigma_0(\theta) = k\cos\theta = kP_1(\cos\theta),$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\varepsilon_0}\int_0^{\pi}\left[P_1(\cos\theta)\right]^2 \sin\theta \, d\theta = \frac{k}{3\varepsilon_0}.$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\varepsilon_0} r\cos\theta \qquad (r \leq R),$$

.
whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\varepsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \geq R).$$

In particular, if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{\boldsymbol{z}}$, so that $k = 3\epsilon_0 E_0$ (Eq. 3.77), then the potential inside is $E_0 r\cos\theta = E_0 z$, and the field is $-E_0\hat{\boldsymbol{z}}$—exactly right to cancel off the external field, as of course it should be. Outside the sphere the potential due to this surface charge is

$$E_0 \frac{R^3}{r^2}\cos\theta,$$

consistent with our conclusion in Ex. 3.8.

Q44 Hollow Sphere Spherical Coordinates

The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

A44

In this case, $B_l = 0$ for all lotherwise the potential would blow up at the origin. Thus.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{3.1}$$

[13]In rare cases where the z axis is excluded, these "other solutions" do have to be considered.
At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R,\theta) = \sum_{i=0}^{\infty} A_i R^i P_i(\cos\theta) = V_0(\theta). \tag{3.2}$$

Can this equation be satisfied, for an appropriate choice of coefficients A_1? Yes: The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \le x \le 1$ $(0 \le \theta \le \pi)$. How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are or- thogonal functions:¹⁴ $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta$ =

$$\begin{cases} 0, & \text{if } l' \ne l, \\ \frac{2}{2l+1}, & \text{if } l' = l. \end{cases} \tag{3.3}$$

Thus, multiplying Eq. 3.67 by $P_{V}(\cos\theta)\sin\theta$ and integrating, we have $A_{l'}R^{l'}\frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$, _{or}

$$A_l = \frac{2l+1}{2l}\int_l^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.4}$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69. It can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 "by eyeball."¹⁵ For instance, suppose we are told that the potential on the sphere is $V_0(\theta) = k\sin^2(\theta/2)$. (3.70) where k is a constant. Using the half-angle formula, we rewrite this as $V_0(\theta) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)]$.
Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other Al's vanish. Therefore,

$$V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right). \tag{3.5}$$

Q45 Uncharged Metal Sphere

An uncharged metal sphere of radius R is placed in an other- wise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the "northern" surface of the sphere, andsymmetricallynegative charge to the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.

A45

The sphere is an equipotentialwe may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does not go to zero at large z. In fact, far from the sphere the field is $E_0\hat{z}$, and hence $V \rightarrow -E_0 z + C$. $\overline{y}$
«FIGURE»
Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are (i) $V = 0$ when $r = R$,
 (ii) $V \rightarrow -E_0 r\cos\theta$ for $r \gg R$. (3.74) We must fit these boundary conditions with a function of the form 3.65. The first condition yields $A_l R^l + \frac{B_l}{R^{l+1}} = 0$, or

$$B_l = -A_l R^{2l+1}, \tag{3.6}$$

_{SO} $V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$. For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that $\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$. Evidently only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately $A_1 = -E_0$, all other A_l's zero. Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.7}$$

The first term $(-E_0 r\cos\theta)$ is due to the external field; the contribution attributable to the induced charge is $E_0\frac{R^3}{r^2}\cos\theta$. If you want to know the induced charge density, it can be calculated in the usual way: $\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R} = \epsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\Big|_{r=R} = 3\epsilon_0 E_0\cos\theta$. (3.77) As expected, it is positive in the "northern" hemisphere $(0 \leq \theta \leq \pi/2)$ and neg- ative in the "southern" $(\pi/2 \leq \theta \leq \pi)$.

Q46 Potential Outside the Sphere

The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential outside, assuming there is no charge there.

A46

In this case it's the A_i's that must be zero (or else V would not go to zero at ∞), so

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.8}$$

At the surface of the sphere, we require that $V(R,\theta) = \sum_{i=0}^{\infty} \frac{B_i}{R^{i+1}} P_i(\cos\theta) = V_0(\theta)$. Multiplying by $P_{V}(\cos\theta)\sin\theta$ and integratingexploiting, again, the orthogo- nality relation 3.68-we have $\frac{B_{l'}}{R^{l'+1}}\frac{2}{2l'+1} =$

$\int_0^\pi V_0(\theta) P_{l'}(\cos\theta)\sin\theta d\theta$, α

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.9}$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Q47 **Glued Charge**

A47