

# Exam Electrodynamics Final

Neighbours

**Lecturer:** Daniel T. McGuiness, Ph.D

**SEMESTER:** WS 2023

**DATE:** 01.12.2023

**TIME:** 10:45 - 12:15

**First and Last Name**

.....

**Student Registration Number**

.....

Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

**Result:**

\_\_\_ / max. 100 points

**Grade:**

**Student Cohort** MA-MECH-23-VZ

**Study Programme** M.Sc Smart Technologies

**Permitted Tools** Calculator and Exam Reference Sheet.

## Important Notes

### Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

### Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

### Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

### Writing materials

Do not use a pencil or red pen and write legibly.

**Good Luck!**

Please read the following instructions carefully.

- You have **90 minutes** to complete this exam. This question booklet contains 5 question(s), 8 pages (including the cover) for the total of 100 points.
- Check to see if any pages are missing.
- All the questions are **compulsory** and all the notations used in the questions have their usual meaning taught at the lectures and done in practice.
- **Read the instructions for individual questions carefully** before answering the questions.

Question	Maximum Point	Result
Electric Fields	30	
Field of Circulation	10	
Calculating the Volume Charge Density	20	
Self-Inductance of a Toroid	20	
The Wave Equation	20	
<b>Sum</b>	<b>100</b>	

**[Q1] Electric Fields** \_\_\_\_\_ 30

One of these is an **impossible electrostatic field**. Please check each case.

a.  $\vec{E}_1 = (y^2) \hat{x} + (2xy + z^2) \hat{y} + (2yz) \hat{z} \text{ V m}^{-1}$ , (15)

b.  $\vec{E}_2 = (xy) \hat{x} + (2yz) \hat{y} + (3xz) \hat{z} \text{ V m}^{-1}$ . (15)

**[Q2] Field of Circulation** \_\_\_\_\_ 10

Find the circulation of the field  $\vec{F} = (x - y) \hat{x} + x \hat{y}$  around the circle

$$\ell(t) = (\cos t) \hat{x} + (\sin t) \hat{y} + (0) \hat{z}$$

with a range of  $0 \leq t \leq 2\pi$ . (10)

**[Q3] Calculating the Volume Charge Density** \_\_\_\_\_ 20

If the electric field ( $\vec{E}$ ) in some region is given in spherical coordinates,  $(r, \theta, \phi)$  by the expression:

$$\vec{E}(\vec{r}) = \frac{1}{r} \left[ (3) \hat{r} + (2 \sin \theta \cos \theta \sin \phi) \hat{\theta} + (\sin \theta \cos \phi) \hat{\phi} \right] \text{ V m}^{-1},$$

what is the volume charge density ( $\rho$ ) in  $\text{C m}^{-3}$ ? (20)

**[Q4] Self-Inductance of a Toroid** \_\_\_\_\_ 20

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius  $a$ , outer radius  $b$ , height  $h$ ), which carries a total of  $N$  turns. (20)

(Tip: the  $\vec{B}$ -field inside the toroid is:  $B = \mu_0 N I / 2\pi s$  where  $I$  is the current and  $s$  is the distance from central axis to the toroid centre and the total flux is calculated as  $\Phi_T = N\Phi_B$ )

**[Q5] The Wave Equation** \_\_\_\_\_ 20

Check whether the following equations obey the wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

a.  $f_1(z, t) = A \exp[-b(z - vt)^2]$ , (10)

b.  $f_2(z, t) = A \sin(bz) \cos(bvt)^3$ . (10)

# Formula Sheet

## Vector Derivatives

### Cartesian

$$\begin{aligned}
 \text{Line Element} \quad d\vec{l} &= dx \hat{x} + dy \hat{y} + dz \hat{z} \\
 \text{Volume Element} \quad d\tau &= dx dy dz \\
 \text{Gradient} \quad \nabla t &= \frac{\partial t}{\partial x} \hat{x} + \frac{\partial t}{\partial y} \hat{y} + \frac{\partial t}{\partial z} \hat{z} \\
 \text{Divergence} \quad \nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
 \text{Curl} \quad \nabla \times \mathbf{v} &= \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} \\
 &\quad + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} \\
 &\quad + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} \\
 \text{Laplacian} \quad \nabla^2 t &= \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}
 \end{aligned}$$

### Spherical

$$\begin{aligned}
 \text{Line Element} \quad d\vec{l} &= dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \\
 \text{Volume Element} \quad d\tau &= r^2 \sin \theta dr d\theta d\phi \\
 \text{Gradient} \quad \nabla t &= \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\phi} \\
 \text{Divergence} \quad \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) \\
 &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) \\
 &\quad + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \\
 \text{Curl} \quad \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} \\
 &\quad + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} \\
 &\quad + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \\
 \text{Laplacian} \quad \nabla^2 t &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) \\
 &\quad + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) \\
 &\quad + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}
 \end{aligned}$$

### Cylindrical

$$\begin{aligned}
 \text{Line Element} \quad d\vec{l} &= ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z} \\
 \text{Volume Element} \quad d\tau &= s ds d\phi dz \\
 \text{Gradient} \quad \nabla t &= \frac{\partial t}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{z} \\
 \text{Divergence} \quad \nabla \cdot \mathbf{v} &= \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
 \text{Curl} \quad \nabla \times \mathbf{v} &= \left( \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{s} \\
 &\quad + \left( \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\phi} \\
 &\quad + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{z} \\
 \text{Laplacian} \quad \nabla^2 t &= \frac{\partial^2 t}{\partial s^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}
 \end{aligned}$$

## Fundamental Constants

$$\begin{aligned}
 \epsilon_0 &= 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2} & \text{Permittivity of free space} \\
 \mu_0 &= 4\pi \times 10^{-7} \text{ N A}^{-2} & \text{Permeability of free space} \\
 c &= 3 \times 10^8 \text{ m s}^{-1} & \text{Approximate speed of light} \\
 e &= 1.6 \times 10^{-19} \text{ C} & \text{Charge of electron} \\
 m_e &= 9.11 \times 10^{-31} \text{ kg} & \text{Mass of the electron}
 \end{aligned}$$

## Vector Identities

### Triple Products

$$\begin{aligned}
 (1) \quad \vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \\
 (2) \quad \vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})
 \end{aligned}$$

### Product Rules

$$\begin{aligned}
 (3) \quad \nabla (fg) &= f (\nabla g) + g (\nabla f) \\
 (4) \quad \nabla (\vec{A} \cdot \vec{B}) &= \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} \\
 (5) \quad \nabla \cdot (f \vec{A}) &= f (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f) \\
 (6) \quad \nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \\
 (7) \quad \nabla \times (f \vec{A}) &= f (\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \\
 (8) \quad \nabla \times \vec{A} \times \vec{B} &= (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})
 \end{aligned}$$

### Second Derivatives

$$\begin{aligned}
 (9) \quad \nabla \cdot (\nabla \times \vec{A}) &= 0 \quad (\text{i.e., divergence of a curl is always zero.}) \\
 (10) \quad \nabla \times (\nabla f) &= 0 \\
 (11) \quad \nabla \times (\nabla \times \vec{A}) &= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}
 \end{aligned}$$

## Fundamental Theorems

$$\begin{aligned}
 \text{Gradient Theorem:} \quad \int_a^b (\nabla f) \cdot d\vec{l} &= f(b) - f(a) \\
 \text{Divergence Theorem:} \quad \int (\nabla \cdot \vec{A}) d\tau &= \oint \vec{A} \cdot d\vec{a} \\
 \text{Curl Theorem:} \quad \int (\nabla \times \vec{A}) \cdot d\vec{a} &= \oint \vec{A} \cdot d\vec{l}
 \end{aligned}$$

## Coordinate Conversion

### Spherical

$$\begin{aligned}
 x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\
 y &= r \sin \theta \sin \phi & \theta &= \tan^{-1} \left( \sqrt{x^2 + y^2} / z \right) \\
 z &= r \cos \theta & \phi &= \tan^{-1} (y/x) \\
 \hat{x} &= \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\
 \hat{y} &= \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\
 \hat{z} &= \cos \theta \hat{r} - \sin \theta \hat{\theta} \\
 \hat{r} &= \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \\
 \hat{\theta} &= \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \\
 \hat{\phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y}
 \end{aligned}$$

## Cylindrical

$$\begin{aligned}x &= s \cos \phi & s &= \sqrt{x^2 + y^2} \\y &= s \sin \phi & \phi &= \tan^{-1}(y/x) \\z &= z & z &= z \\ \hat{x} &= \cos \phi \hat{s} - \sin \phi \hat{\phi} \\ \hat{y} &= \sin \phi \hat{s} + \cos \phi \hat{\phi} \\ \hat{z} &= \hat{z} \\ \hat{s} &= \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{\phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{z} &= \hat{z}\end{aligned}$$

## Curl Operation

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

## Useful Physical Identities

$$\begin{aligned}\text{Energy} \quad U &= \frac{1}{2} \int \left( \epsilon_0 \vec{p}_2 \vec{E} + \frac{1}{\mu_0} \vec{p}_2 \vec{B} \right) d\tau \\ \text{Momentum} \quad \vec{P} &= \epsilon_0 \int \left( \vec{E} \times \vec{B} \right) d\tau \\ \text{Poynting Vector} \quad \vec{S} &= \frac{1}{\mu_0} \left( \vec{E} \times \vec{B} \right) \\ \text{Lorentz Force} \quad \vec{F} &= q \left( \vec{E} + \vec{v} \times \vec{B} \right) \\ \text{Potential} \quad \vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \\ \text{Coulombs Law} \quad \vec{F} &= \frac{qQ}{4\pi\epsilon_0} \frac{\hat{z}}{z^2}\end{aligned}$$

## Maxwell's Equations

### Main Equations

$$\begin{aligned}\text{In General:} \quad \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\end{aligned}$$

$$\begin{aligned}\text{In Material:} \quad \nabla \cdot \vec{D} &= \rho_f \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t}\end{aligned}$$

### Auxiliary Fields

$$\begin{aligned}\text{Definition:} \quad \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ \vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M}\end{aligned}$$

$$\begin{aligned}\text{Linear Media:} \quad \vec{P} &= \epsilon_0 \chi_e \vec{E}, \\ \vec{M} &= \chi_m \vec{H} \\ \vec{D} &= \epsilon \vec{E} \\ \vec{H} &= \frac{1}{\mu} \vec{B}\end{aligned}$$

To figure out the impossible we need to calculate the curl of  $\vec{E}$ .

(a) The first function:

$$\nabla \times \vec{E}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = (0 - 2y) \hat{x} + (0 - 3z) \hat{y} + (0 - x) \hat{z} \neq 0 \quad \text{V m}^{-1} \quad \blacksquare$$

As the curl of this field is not equal to zero ( $\nabla \times \vec{E} \neq 0$ ), this field **cannot** exist.

(15)

(b) Second function:

$$\nabla \times \vec{E}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2y \end{vmatrix} = (2z - 2z) \hat{x} + (0 - 0) \hat{y} + (2y - 2y) \hat{z} = 0 \quad \text{V m}^{-1} \quad \blacksquare$$

As the curl of this field is equal to zero ( $\nabla \times \vec{E} = 0$ ), this field can exist.

(15)

## [A2] Field of Circulation

10

We first evaluate  $\vec{F}$  on the curve:

$$\vec{F} = (x) \hat{x} + (z) \hat{y} + (y) \hat{z} = (\cos t) \hat{x} + (t) \hat{y} + (\sin t) \hat{z} \quad \text{Substitute } x = \cos t, z = t, y = \sin t.$$

and then find  $d\ell/dt$ :

$$\frac{d\ell}{dt} = (-\sin t) \hat{x} + (\cos t) \hat{y} + (0) \hat{z}.$$

Then we integrate  $\vec{F} \cdot (d\ell/dt)$  from  $t = 0$  to  $t = \pi/2$ :

$$\begin{aligned} \vec{F} \cdot \frac{d\ell}{dt} &= (\cos t) (-\sin t) + (t) (\cos t) + (\sin t) (0), \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

So,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \vec{F} \cdot \frac{d\ell}{dt} dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t) dt, \\ &= \left[ \frac{\cos^2 t}{2} + t \sin t \right]_0^{\pi/2} = \left( 0 + \frac{\pi}{2} \right) - \left( \frac{1}{2} + 0 \right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare \end{aligned}$$

Which gives the solution to our problem.

(10)

**[A3] Calculating the Volume Charge Density**

20

First thing to do is to invoke Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0},$$
$$\rho = \epsilon_0 (\nabla \cdot \vec{E}).$$

$$\begin{aligned}\rho &= \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{3}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{2 \sin \theta \cos \theta \sin \phi}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\sin \theta \cos \phi}{r} \right) \right\} \text{ C m}^{-3}, \\&= \epsilon_0 \left[ \frac{1}{r^2} 3 + \frac{1}{r \sin \theta} \frac{2 \sin \phi (2 \sin \theta \cos^2 \theta - \sin^3 \theta)}{r} + \frac{1}{r \sin \theta} \frac{(-\sin \theta \sin \phi)}{r} \right] \text{ C m}^{-3}, \\&= \frac{\epsilon_0}{r^2} \left[ 3 + 2 \sin \phi (2 \cos^2 \theta - \sin^2 \theta) - \sin \phi \right] \text{ C m}^{-3}, \\&= \frac{\epsilon_0}{r^2} \left[ 3 + \sin \phi (4 \cos^2 \theta - 2 + 2 \cos^2 \theta - 1) \right] \text{ C m}^{-3}, \\&= 3 \frac{\epsilon_0}{r^2} \left[ 1 + \sin \phi (2 \cos^2 \theta - 1) \right] \text{ C m}^{-3}, \\&= 3 \frac{\epsilon_0}{r^2} (1 - \sin \phi \cos 2\theta) \text{ C m}^{-3} \quad \blacksquare\end{aligned}$$

Which gives the solution to our problem.

(20)

**[A4] Self-Inductance of a Toroid**

20

The magnetic field inside a toroid is:

$$B = \frac{\mu_0 N I}{2\pi s}$$

The flux through a single turn is:

$$\int \vec{B} \cdot d\vec{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s} ds = \frac{\mu_0 N I h}{2\pi} \ln \left( \frac{b}{a} \right)$$

The total flux is  $N$  times this, therefore the self inductance is:

$$= \frac{\mu_0 N^2 h}{2\pi} \ln \left( \frac{b}{a} \right) \quad \blacksquare$$

Which gives the solution to our problem.

(20)

Let us analyse the functions individually.

(i) Function 1:

(10)

$$\begin{aligned}\frac{\partial f_1}{\partial z} &= -2Ab(z - vt) \exp[-b(z - vt)^2], \\ \frac{\partial^2 f_1}{\partial z^2} &= 2Ab \{-v \exp[-b(z - vt)^2] + 2bv(z - vt)^2 \exp[-b(z - vt)^2]\}, \\ \frac{\partial f_1}{\partial t} &= -2Abv(z - vt) \exp[-b(z - vt)^2], \\ \frac{\partial^2 f_1}{\partial t^2} &= 2Abv^2 \{-v \exp[-b(z - vt)^2] + 2bv(z - vt)^2 \exp[-b(z - vt)^2]\}, \\ \frac{\partial^2 f_1}{\partial z^2} &= \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} \quad \blacksquare\end{aligned}$$

(iv) Function 2:

(10)

$$\begin{aligned}\frac{\partial f_2}{\partial z} &= Ab \cos(bz) \cos(bvt)^3, \\ \frac{\partial^2 f_2}{\partial z^2} &= -Ab^2 \sin(bz) \cos(bvt)^3, \\ \frac{\partial f_2}{\partial t} &= -3Ab^3 v^3 t^2 \sin(bz) \sin(bvt)^3, \\ \frac{\partial^2 f_2}{\partial t^2} &= -6Ab^3 v^3 t \sin(bz) \sin(bvt)^3 - 9A^6 v^6 t^4 \sin(bz) \cos(bvt)^4, \\ \frac{\partial^2 f_2}{\partial z^2} &\neq \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} \quad \blacksquare\end{aligned}$$