Ordinary Differential Equations

First-order ODEs

They are of the form:

$$F\left(x, y, y'\right) = 0 \quad \text{or in explicit form} \quad y' = f(x, y)$$

involving the derivative $y' = dy/dt$ of an unknown function y, given functions of x, and/or y itself. A first-order ODE usually has a **general solution**; a solution involving an arbitrary constant, which is denoted by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** of $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem** with the following mathematical expression:

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad (x_0, y_0 \text{given numbers})$$

and its solution is a particular solution of the ODE. A **separable ODE** is one that we can put into the form:

$$g(y)\ dy = f(x)\ dx$$

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides. An **exact ODE**, on the other hand, is of the form

$$M(x, y)\ dx + N(x, y)\ dy = 0$$

where $M\,dx + N\,dy$ is the **differential** for and,

$$du = u_x dx + u_y dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. The solution to u can be attained using two (2) ways:

$$u = \int M\,dx + k(y) \qquad \text{or} \qquad u = \int N\,dy + l(x)$$

Linear ODE's of the form:

$$y' + p(x)\,y = r(x)$$

are very important. Their solutions are given by the integral formula

$$y(x) = e^{-h}\left(\int e^{h} r\,dx + c\right), \quad h = \int p(x)\ dx$$

Certain nonlinear ODEs can be transformed to linear form in terms of new variables.

Second-order ODEs

A second-order ODE is called **linear** if it can be written as:

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

The above equation is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval, this is written $r(x) = 0$. Then, this equation becomes:

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

The above equation is called **non-homogeneous** if $r(x) \neq 0$ (meaning $r(x)$ is not zero for some x considered). For the homogeneous ODE we have the important **superposition principle** that a linear combination of two solutions is again a solution.

Two linearly independent solutions y_1, y_2 on an open interval I form a **basis** (or **fundamental system**) of solutions. and $y = c_1 y_1 + c_2 y_2$ with arbitrary constants c_1, c_2 a **general solution**. From it, we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two (2) **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \quad (x_0, K_0, K_1 \text{are given}).$$

This forms an **initial value problem**. For a nonhomogeneous ODE, a **general solution** is of the form

$$y = y_h + y_p$$

Here y_h is a **general** solution and y_p is a **particular** solution of. Such a y_p can be determined by a general method or in many practical cases by the *method of undetermined coefficients*. The latter applies when the above has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Table 3) Then we write (1) as

$$y'' + ay' + by = r(x)$$

Another large class of ODEs solvable "algebraically" consists of the **Euler-Cauchy equations**:

$$x^2 y'' + ay' + by = 0$$

These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

$$m^2 + (a-1)m + b = 0.$$

Depending of the roots different solutions can be derived (Table 6). A homogeneous ODE of

$$y'' + ay' + by = 0$$

can be solved by deriving its characteristic equation in the form:

$$\lambda^2 + a\lambda + b = 0$$

Depending of its roots of this equation different solutions can be derived (Table 5)

Higher-order ODEs

A higher order system is of the form:

$$F\left(x,\ y,\ y',\ \cdots,\ y^{(n)}\right) = 0,$$

Which through substitutions (i.e., $y^{iv} \to \lambda = y''$) it can be reduced to lower order forms. From there on, all principles from 2nd order ODEs can be applied.

System of ODEs

A system of ODEs (linear) can be written in the form:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{g}, \qquad \text{where} \qquad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

If $\boldsymbol{g} = 0$, the system is called **homogeneous** and is of the form

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$$

If $a_{11}, \cdots, a_{22}$ are **constants**, it has solutions

$$\boldsymbol{y} = xe^{\lambda t},$$

where λ is a solution of the quadratic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

and $\boldsymbol{x} \neq 0$ has components x_1, x_2 determined up to a multiplicative constant by:

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0.$$

These λ's are called the **eigenvalues** and these vectors $\boldsymbol{x}$ **eigenvectors** of the matrix $\boldsymbol{A}$. To calculate the characteristic equation:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_2 - (a_{11} + a_{22})\,\lambda + \det(\boldsymbol{A}) = 0.$$

where:

$$p = a_{11} + a_{22}, \qquad q = \det\boldsymbol{A}, \qquad \Delta = p_2 - 4q.$$

or in another form:

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2$$

The stability criterion are given in Table 4.

Special Functions

The **power series method** gives solutions of linear ODEs:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

with variable coefficients p and q in the form of a power series (with any centre x_0, e.g., $x_0 = 0$)

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m (x - x_0)^m \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \end{aligned}$$

Legendre's differential equation:

$$(1 - x^2)\,y'' - 2xy' + n(n+1)\,y = 0$$

with its polynomial solution:

$$\begin{aligned} P_n(x) &= \sum_{m-0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots \end{aligned}$$

Frobenius Method states: Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{1}$$

has **at least one solution** can represented in the form:

$$\begin{aligned} y(x) &= x^r \sum_{m=0}^{\infty} a_m x^m \\ &= x^r\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \quad (a_0 \neq 0) \end{aligned}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$). It's indicial equation is:

$$r(r-1) + b_0 r + c_0 = 0$$

Depending on the equations roots, we have the following three (3) cases:

Case 1. Distinct Roots Not Differing by an Integer
A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

with coefficients obtained successively from Eq. (1) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.
A basis is

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)]$$

(of the same general form as before) and

$$y_2(x) = y_1(x)\ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0)$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(of the same general form as before) and

$$y_2(x) = k y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero. **Bessel's Equation** is of the form:

$$x^2 y'' + xy' + (x^2 - \nu^2)\,y = 0$$

It solution of first kind is:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} \quad (n \geqq 0).$$

Laplace Transform

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\ dt$$

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad \text{where a, b const.}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subdistiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0).$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the Table 1. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform**$y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem.

Linear Algebra

Fundamentals

A $m \times n$ **matrix** $\boldsymbol{A} = [a_{jk}]$ is a rectangular array of numbers or functions arranged in m horizontal **rows** and n vertical **columns**. If $m = n$, the matrix is called **square**. A $1 \times n$ matrix is call+ed a **row vector** and an $m \times 1$ matrix **column vector**.

The **sum** A + B of matrices of the same size (i.e., both $m \times n$) is obtained by adding corresponding entries. The **product** of A by a scalar c is obtained by multiplying each a_{jk} by c.

The **product** $\boldsymbol{C} = \boldsymbol{AB}$ of an $m \times n$ matrix $\boldsymbol{A}$ by an $r \times p$ matrix $\boldsymbol{B} = [b_{jk}]$ is defined only when $r = n$. It is associative, but is **not commutative**. If $\boldsymbol{AB}$ is defined, $\boldsymbol{BA}$ may not be defined, but even if $\boldsymbol{BA}$ is defined, $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general. Also $\boldsymbol{AB} = 0$ may not imply $\boldsymbol{A} = 0$ or $\boldsymbol{B} = 0$ or $\boldsymbol{BA} = 0$.

The **transpose** $\boldsymbol{A}^\top$ of a matrix $\boldsymbol{A} = [a_{jk}]$ is $\boldsymbol{A}^\top = [a_{kj}]$. rows become columns and conversely. Here, $\boldsymbol{A}$ need not be square. If it is and $\boldsymbol{A} = \boldsymbol{A}^\top$, then $\boldsymbol{A}$ is called **symmetric;** if $\boldsymbol{A} = -\boldsymbol{A}^\top$, it is called **symmetric**. For a product, $(\boldsymbol{AB})^T = \boldsymbol{B}^\top \boldsymbol{A}^\top$.

A main application of matrices concerns **linear systems of equations**

$$\boldsymbol{A}x = \boldsymbol{b}$$

(m equations in n unknowns $x_1, \cdots, x_n$; $\boldsymbol{A}$ and $\boldsymbol{b}$ given). The most important method of solution is the **Gauss elimination**, which reduces the system to "triangular" form by *elementary row operations*, which leave the set of solutions unchanged.
The inverse $\boldsymbol{A}^{-1}$ of a square matrix satisfies $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. It exists if and only if $\det \boldsymbol{A} \neq 0$. It can be computed by the *Gauss-Jordan elimination.*

The rank r of a matrix $\boldsymbol{A}$ is the maximum number of linearly independent rows or columns of $\boldsymbol{A}$ or, equivalently, the number of rows of the largest square sub-matrix of $\boldsymbol{A}$ with nonzero determinant.
The system of equations has solutions if and only if $\operatorname{rank} \boldsymbol{A} = \operatorname{rank}[\boldsymbol{A} \quad \boldsymbol{b}]$, where $[\boldsymbol{A} \quad \boldsymbol{b}]$ is the **augmented matrix**.The **homogeneous system**

$$\boldsymbol{A}x = 0$$

has solutions $\boldsymbol{x} \neq 0$ ("nontrivial solutions") if and only if $\operatorname{rank} \boldsymbol{A} < n$, in the case $m = n$ equivalently if and only if $\det \boldsymbol{A} = 0$.

Eigenvalue Problems

The problems are defined by the vector equation

$$\boldsymbol{A}x = \lambda \boldsymbol{x}.$$

$\boldsymbol{A}$ is a given square matrix. λ is a scalar. To *solve* the problem means to determine values of λ, called **eigenvalues** (or **characteristic values**) of $\boldsymbol{A}$, such that, the above expression, has a nontrivial solution $\boldsymbol{x}$ (that is, $\boldsymbol{x} \neq 0$), called an **eigenvector** of $\boldsymbol{A}$ corresponding to that λ. A $n \times n$ matrix has at least one and at most n numerically different eigenvalues. These are the solutions of the **characteristic equation**

$$D(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})$$
$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$D(\lambda)$ is called the **characteristic determinant** of $\boldsymbol{A}$. By expanding it we get the **characteristic polynomial** of $\boldsymbol{A}$, which is of degree n in λ. Special matrices of importance are **symmetric** ($\boldsymbol{A}^\top = \boldsymbol{A}$), **skew-symmetric** ($\boldsymbol{A}^\top = -\boldsymbol{A}$), and **orthogonal matrices** ($\boldsymbol{A}^\top = \boldsymbol{A}^{-1}$). concerns the diagonalization of matrices and the transformation of quadratic forms to principal axes and its relation to eigenvalues.
If an $n \times n$ matrix $\boldsymbol{A}$ has a basis of eigenvectors, then

$$\boldsymbol{D} = \boldsymbol{X}^{-1}\boldsymbol{AX}$$

is **diagonal**, with the eigenvalues of $\boldsymbol{A}$ as the entries on the main diagonal.

Here $\boldsymbol{X}$ is the matrix with these eigenvectors as column vectors. Also,

$$\boldsymbol{D}^m = \boldsymbol{X}^{-1}\boldsymbol{A}^m\boldsymbol{X} \qquad (m = 2, 3, \cdots).$$

Vector Calculus

All vectors of the form:

$$\boldsymbol{a} = [a_1, a_2, a_3] = (a_1)\ \hat{x} + (a_2)\ \hat{y} + (a_3)\ \hat{z}$$

constitute the **real vector space** R^3 with componentwise vector addition

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and componentwise scalar multiplication (c a scalar)

$$c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3]$$

The **inner product** or **dot product** of two vectors is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. This gives for the **norm** or **length** $|\boldsymbol{a}|$ of $\boldsymbol{a}$:

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well as a formula for θ. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, we call $\boldsymbol{a}$ and $\boldsymbol{b}$ **orthogonal**. The **vector product** or **cross product** $\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b}$ is a vector of length

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$$

which can also be represented as a determinant.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This multiplication is anticommutative, and is *not* associative.

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Gradient is defined as:

$$\nabla f = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Divergence is defined as:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

Curl is defined as:

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_x & v_x \end{vmatrix}$$

Standard Integration Forms

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \int \frac{1}{x}dx = \ln|x|$$
$$\int u dv = uv - \int v du \qquad \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$$
$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$
$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$
$$\int \frac{1}{1+x^2}dx = \tan^{-1}x$$
$$\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x$$
$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$
$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$$
$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$$
$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$
$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$
$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b$$
$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$$
$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$f(t)$	$\mathcal{L}(f(t)) = F(s)$	$f(t)$	$\mathcal{L}(f(t)) = F(s)$	$f(t)$	$\mathcal{L}(f(t)) = F(s)$
1	$\frac{1}{s}$	$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$	$\frac{e^{at} - e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$e^{at}f(t)$	$F(s-a)$	$\cosh kt$	$\frac{s}{s^2-k^2}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	e^{at}	$\frac{1}{s-a}$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$\sin kt$	$\frac{k}{s^2+k^2}$	$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\delta(t)$	1	$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$\delta(t-t_0)$	e^{-st_0}	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$	te^{at}	$\frac{1}{(s-a)^2}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	$e^{at} \sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at} \cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$	$t \sin kt$	$\frac{2ks}{(s^2+k^2)^2}$	$t \cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t \sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$	$t \cosh kt$	$\frac{s^2-k^2}{(s^2-k^2)^2}$	$\frac{\sin at}{t}$	$\arctan \frac{a}{s}$
$\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$

Table 1: Common Laplace transform operations.

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	**Comments**
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite signs
Centre	$p = 0$	$q > 0$		Pure imaginary
Spiral		$q \neq 0$	$\Delta < 0$	Complex (not pure imaginary)

Table 2: Criterion

Term in $r(x)$	**Choice for** $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ or $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ or $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 3: Method of undetermined coefficients.

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$
Stable and attractive	$q < 0$	$q > 0$
Stable	$q \leq 0$	$q > 0$
Unstable	either $q \leq 0$	or $q > 0$

Table 4: Stability criterion for critical points.

Case	**Type**	**Roots**	**Basis**	**General Solution**
I	Distict real	(λ_1, λ_2)	$e^{\lambda_1 x}$ and $e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real Double Root	$(\lambda = -\frac{1}{2}a)$	$e^{-ax/2}$ and $xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex Conjugate	$\lambda_{1,2} = -1/2a \pm j\omega$	$e^{-ax/2} \cos \omega x$ and $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

Table 5: Possible roots of the characteristic equation based on the discriminant value.

Case	**Type**	**Roots**	**General Solution**
I	Distict real	(m_1, m_2)	$y = c_1 x^{m_1} + c_2 x^{m_2}$
II	Real Double Root	$m = \frac{1}{2}(1-a)$	$y = c_1 x^m \ln x + c_2 x^m$
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$	
		$\alpha = Re(m)$	
	$m_2 = \alpha - \beta j$	$\beta = Im(m)$	

Table 6: Cases for solving Euler-Cauchy.