Tutorial Book

M.Sc Higher Mathematics I

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This TutorialBook is a
complement to it's corresponding Lecture.
When in doubt regarding theory, please consult the LectureBook.

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1 First Order ODEs

Q1 An Initial Value Problem

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

A1

The general solution is:

$$y(x) = ce^{3x}$$

From the solution and the initial condition:

$$y(0) = ce^{0} = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation. Visually the solution is plotted as follows ■

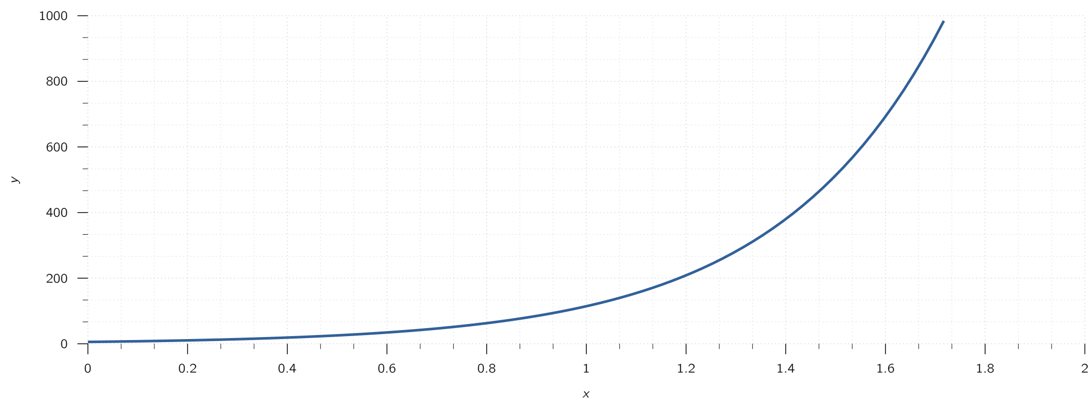


Figure 1.1: A Solution to the initial value problem.

Q2 Radioactive Decay

Given 0,5 g of a radioactive substance, find the amount present at any later time. The decay of Radium is measured to be $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$.

A2

We know $y(t)$ is the substance amount still present at t. Using the law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives:

$$\frac{dy}{dt} = -ky \tag{1.1}$$

where the constant k is **positive**, and due of the minus, we get *decay*. We know k which the question has given as $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$. Now the given initial amount is 0,5 g, and we can call the corresponding instant $t = 0$. We have the initial condition $y(0) = 0.5$, which is the instant the process begins. Therefore, the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the Ordinary Differential Equation (ODE) is an exponential decay and has the general solution:

$$y(0) = ce^{\text{-kt}}.$$

We now determine c by using the initial condition which gives $y(0) = c = 0.5$. Therefore:

$$y(t) = 0.5e^{\text{-kt}} \quad \blacksquare$$

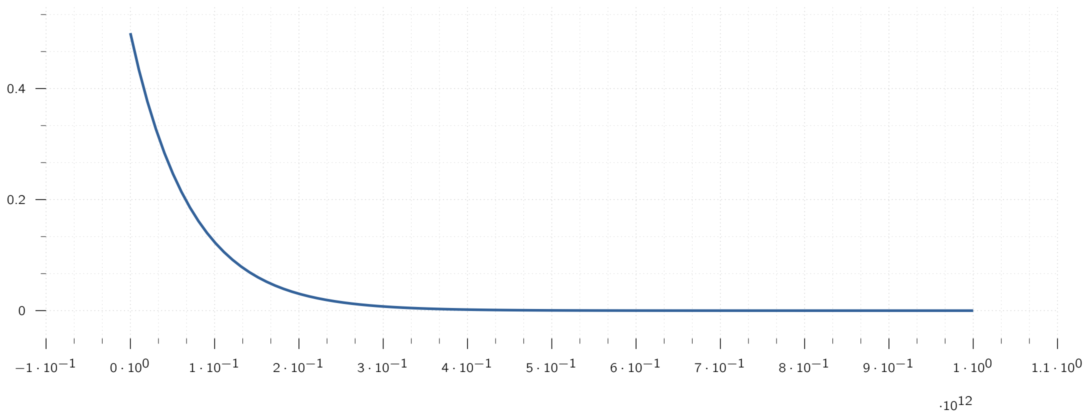


Figure 1.2: A Solution to the radioactive decay.

Q3 Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

A3

The given ODE is separable because it can be written:

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x \qquad \text{By integration} \qquad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x+c)$$

Note It is important to introduce the constant c when the integration is performed.

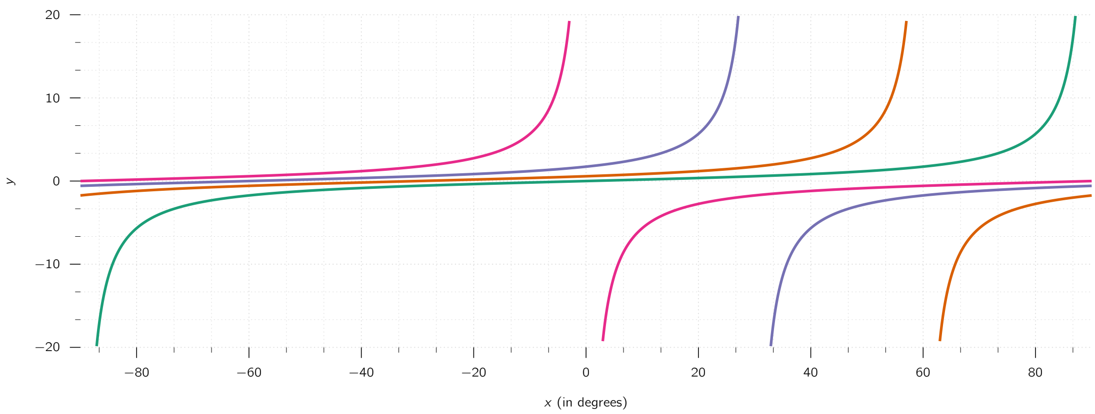


Figure 1.3: A Solution to the Separable ODE.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is **NOT** a solution, when $c \neq 0$ ■

Q4 A Bell Shaped Curve

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

A4

By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the Initial Value Problem (IVP) has the solution:

$$y = 1.8e^{-x^2}$$

This is a particular solution, representing a bell-shaped curve ■

Figure 1.4: A Solution to the Separable ODE.

Q5 Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian-Italian border, caused a scientific sensation.

When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

NOTE The half-life of carbon is 5175 years.

A5

Radioactive decay is governed by the ODE $y' = ky$ as we have discussed previously. By separation and integration

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t, \;\; \ln|y| = kt + c, \;\; y = y_0 e^{\mathrm{kt}}, \;\; y_0 = e^{0}.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{\mathrm{kH}} = 0.5 y_0, \quad e^{\mathrm{kH}} = 0.5,$$
$$k = \frac{\ln 0.5}{H} = \frac{0.693}{5715} = -0.0001213.$$

we then use the ratio 52.5% to determine the time:

$$e^{\mathrm{k}\;t} = e^{\text{-0.0001 213t}} = 0.525,$$
$$t = \frac{\ln 0.525}{-0.0001213} = 5312 \quad \blacksquare$$

Q6 Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

A6

To get the usual explicit form, we start by dividing the given equation by $2xy$ which gives,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then we simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

We see in the last equation that we can now separate the variables,

$$\frac{2u\,\mathrm{d}u}{1+u^2} = -\frac{\mathrm{d}x}{x} \qquad \text{and by integration} \qquad \ln\left(1+u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

We now take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx \qquad \text{therefore we can get} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen below ■

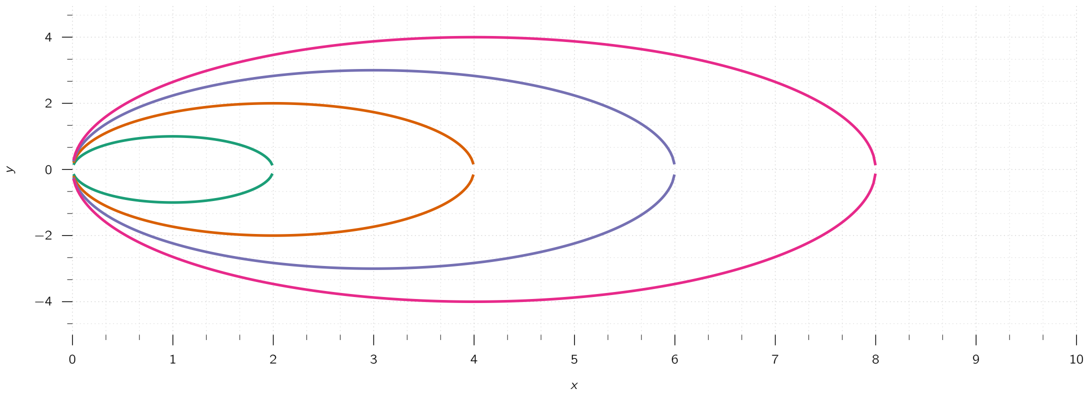


Figure 1.5: A Solution to the Separable ODE.

Q7 Exact ODE - An Initial Value Problem

Solve the initial value problem:

$$\left(\cos y \sinh x + 1\right)\,\mathrm{d}x - \sin y \cosh x\,\mathrm{d}y = 0,$$
$$\text{with} \qquad y(1) = 2.$$

A7

Let's begin by verifying the given equation is **exact**:

$$M(x,y) = (\cos y \sinh x + 1),$$
$$N(x,y) = -\sin y \cosh x.$$

We now apply our criteria:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin y \sinh x$$

This shows the given ODE is **exact**. We find u. For a change, let us use Eq. (**??**):

$$u = -\int \sin y \cosh x dy + I(x) = \cos y \cosh x + I(x).$$

From this:

$$\partial u/\partial x = \cos y \sinh x + \frac{dI}{dx} = u = \cos y \sinh x + 1$$

Therefore $dI/dx = 1$ and by integration,

$$I(x) = x + c^*.$$

This gives the general solution

$$u(x,y) = \cos y \cosh x + x = c.$$

From the initial condition:

$$\cos 2 \cosh 1 + 1 = 0.358 = c$$

Therefore the answer is:

$$\cos y \cosh x + x = 0.358 \quad \blacksquare$$

Q8 **An Exact ODE**

Solve the following ODE:

$$\cos(x+y)\ \mathrm{d}x + \left(3y^2 + 2y + \cos(x+y)\right)\ \mathrm{d}y = 0.$$

A8

The solution is as follows:

Test for exactness First check if our equation is **exact**, try to convert the equation of the form Eq. (**??**):

$$M = \cos(x+y),$$
$$N = 3y^2 + 2y + \cos(x+y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x+y),$$

$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

This proves our equation to be exact.

Implicit General Solution From Eq. (**??**), we obtain by integration:

$$\begin{aligned} u &= \int M \, dx + k(y) \\ &= \int \cos(x+y) \, dx + k(y) \\ &= \sin(x+y) + k(y) \end{aligned} \tag{1.2}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (**??**), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Therefore $\frac{dk}{dy} = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$. Inserting this result into Eq. (1.2) and observing Eq. (**??**), we obtain:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c \quad \blacksquare$$

Q9 The Breakdown of Exactness

Check the exactness of the following ODE:

$$-y \, dx + x \, dy = 0$$

A9

The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

Let us show that in such a case the present method does **NOT** work.

$$u = \int M \, dx + k(y) = -xy + k(y),$$
$$\frac{\partial u}{\partial y} = -x + \frac{\partial k}{\partial y}.$$

Now, $\partial u / \partial y$ should equal $N = x$, as required for this equation to be exact. However, this is impossible because $k(y)$ can depend only on y $\blacksquare$

Q10 A Non Homogeneous Ordinary Differential Equation

Solve the initial value problem of the following equation:

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

A10

Here we define the parameters as:

$$p = \tan x, \quad r = \sin 2x = 2 \sin x \cos x,$$

and

$$h = \int p \, \mathrm{d}x = \int \tan x \, \mathrm{d}x = \ln \left| \sec x \right| .$$

From this we see that in Eq. (**??**),

$$e^{\mathrm{h}} = \sec x, \qquad e^{\mathrm{-h}} = \cos x,$$

$$e^{\mathrm{h}} r = \left(\sec x\right) \left(2 \sin x \cos x\right) = 2 \sin x,$$

and the general solution of our equation is:

$$\begin{aligned} y\left(x\right) &= \cos x \left(2 \int \sin x \, \mathrm{d}x + c\right), \\ &= c \cos x - 2 \cos^2 x. \end{aligned}$$

From this and the initial condition

$$1 = c \cdot 1 - 2 \cdot 1^2, \qquad \text{therefore} \qquad c = 3,$$

and the solution of our initial value problem is:

$$y = 3 \cos x - 2 \cos^2 x$$

Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$ ■

2

Second Order ODEs

Q11 A Superposition of Solutions

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0, \quad \text{for all } x.$$

A11

By differentiation and substitution, we obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$\begin{aligned}&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\ &= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare\end{aligned}$$

Q12 Example of a Non-Homogeneous Linear ODE

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 1$$

A12

Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1 + \sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

Q13 An Initial Value Problem

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

A13

General Solution From the previous examples, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Particular Solution
We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as cos 0 = 1 and sin 0 = 0,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

Q14 Reduction of Order

Find a basis of solutions of the ODE

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

A14

Inspection shows $y_1 = x$ is a solution as $y_1{}' = 1$ and $y_1{}'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u,$$
$$y'' = u''x + 2u'$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0,$$
$$(x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely,

$$(x^2 - x)w' + (x - 2)w = 0.$$

Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx,$$
$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We don't need constant of integration as we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain n

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x},$$
$$\text{hence} \quad y_2 = ux = x\ln|x| + 1$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained as basic of solutions, valid for all positive x ■

Q15 **A General Solution in the Case of Different Real Roots**

Solve the following ODE:

$$x^2y'' + 1.5xy' - 0.5y = 0$$

A15

This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad ■$$

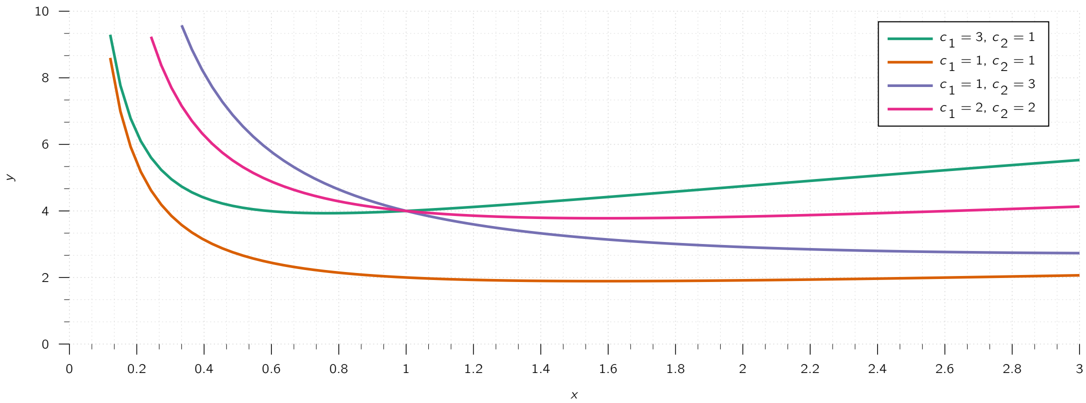


Figure 2.1: Solution to A General Solution in the Case of Different Real Roots with different constants.

Q16 A General Solution in the Case of a Double Root

Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

A16

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

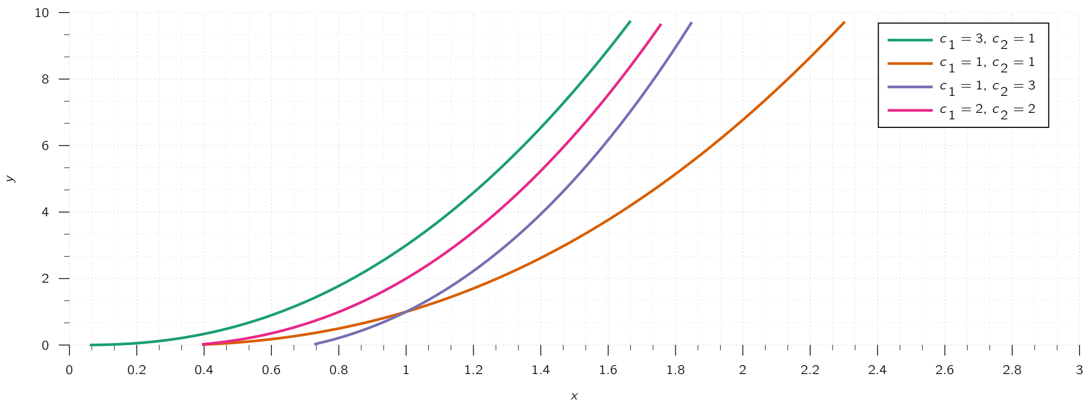


Figure 2.2: Solution to A General Solution in the Case of a Double Root with different constants.

Q17 Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii r_1 = 5 cm and r_2 = 10 cm kept at potentials v_1 = 110 V and v_2 = 0, respectively.

$v = v\,(r)$ is a solution of the *Euler-Cauchy equation* $rv'' + 2v' = 0$.

A17

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

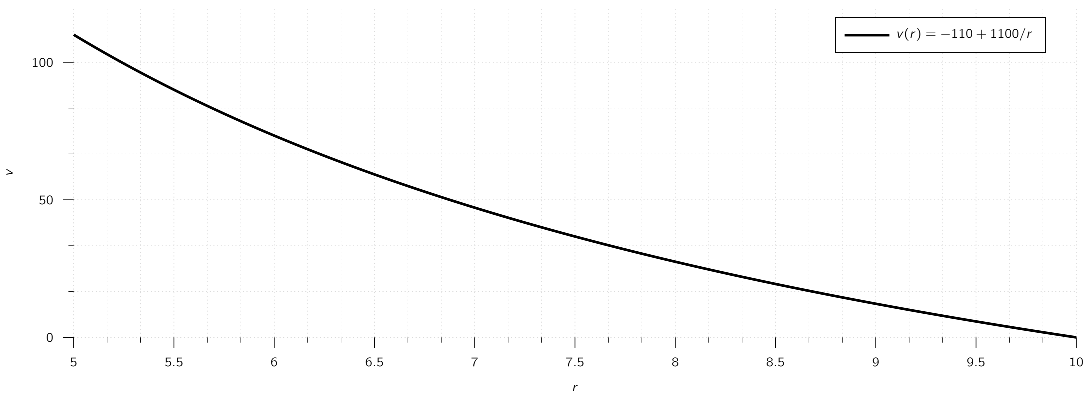


Figure 2.3: Solution to the Electric Potential Field Between Two Concentric Spheres.

Q18 Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped Mass-Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\text{kg} \cdot \text{s}^{-1}$
- $c = 60\,\text{kg} \cdot \text{s}^{-1}$
- $c = 10\,\text{kg} \cdot \text{s}^{-1}$

A18

It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass-Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0{,}16\,\text{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9t} + 0.18e^{-t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3t}, \qquad \text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y\prime(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96\mathbf{j}$$

This gives the general solution:

$$y = e^{-0.5\,t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$y\prime = e^{-0.5\,t}(-0.5A\cos 2.96t - 0.5B\sin 2.96t \;\; - 2.96A\sin 2.96t + 2.96B\cos 2.96t)$$

Hence $y\prime(0) = -0.5A + 2.96B = 0,\ B = 0.5A/2.96 = 0.027$. This gives the solution:

$$y = e^{-0.5\,t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) \;\; = 0.162e^{-0.5\,t}\cos\left(2.96t - 0.17\right) \quad \blacksquare$$

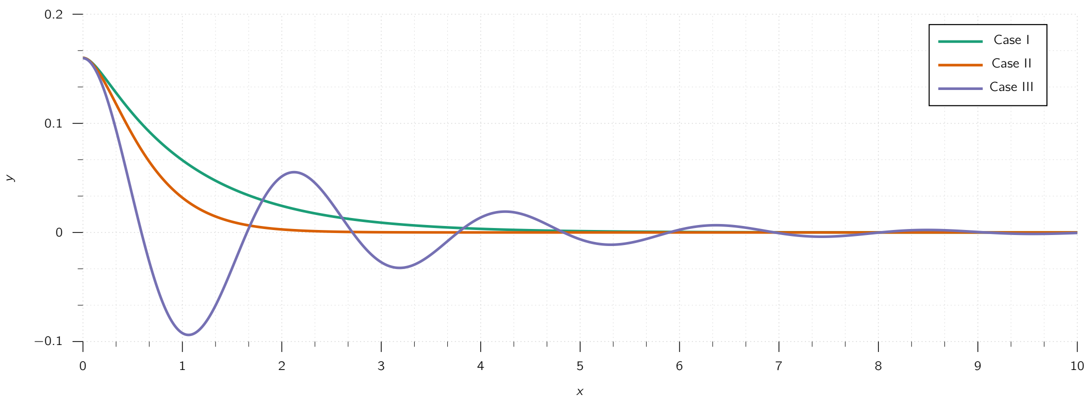


Figure 2.4: Three cases of damped motion.

Q19 Application of the Basic Rule A

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

A19

General Solution of the Homogeneous ODE The ODE $y'' + y = 0$ has the general solution

$$y_{\mathrm{h}} = A\cos x + B\sin x.$$

Solution of the non-Homogeneous ODE First we try $y_{\mathrm{p}} = Kx^2$ and also $y_{\mathrm{p}}\prime\prime = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

Looking at the table suggests the choice:

$$y_{\mathrm{p}} = K_2x^2 + K_1x + K_0,$$

$$y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Therefore:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Solution of the initial value problem.
Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, therefore $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$
$$\text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

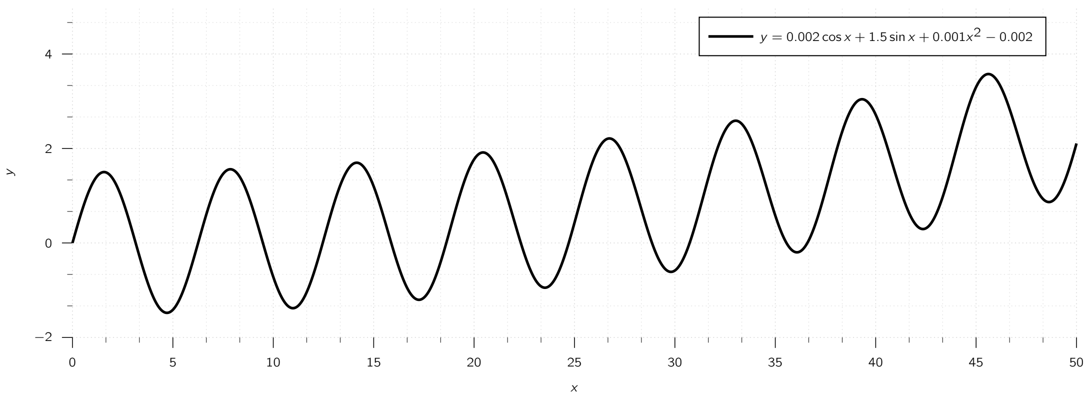


Figure 2.5: Solution to Application of the basic rule A.

Q20 Application of the Basic Rule B

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x},$$
$$y(0) = 1, \qquad y'(0) = 0.$$

A20

General solution of the homogeneous ODE The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$$

Therefore the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\, e^{-1.5x}$$

Solution y_p of the non-homogeneous ODE The function $e^{-1.5x}$ on the Right Hand Side (RHS) would normally require the choice $Ce^{-1.5x}$. However, we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Which means, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose:

$$\begin{aligned} y_p &= Cx^2 e^{-1.5x}, \quad \text{then} \\ y_p' &= C(2x - 1.5x^2)e^{-1.5x}, \\ y_p'' &= C(2 - 3x - 3x + 2.25x^2)e^{-1.5x} \end{aligned}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5 e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2 e^{-1.5x}$$

Step 3. Solution of the initial value problem Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$\begin{aligned} y' = (c_2 - 1.5c_1 - 1.5c_2 v)e^{-1.2x} \\ -10xe^{-1.2x} + 7.5x^2 e^{-1.2x} \end{aligned}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$ and gives the answer

$$\begin{aligned} y = (1 + 1.5x)e^{-1.5x} - 5x^2 e^{-1.5x} = \\ (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare \end{aligned}$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

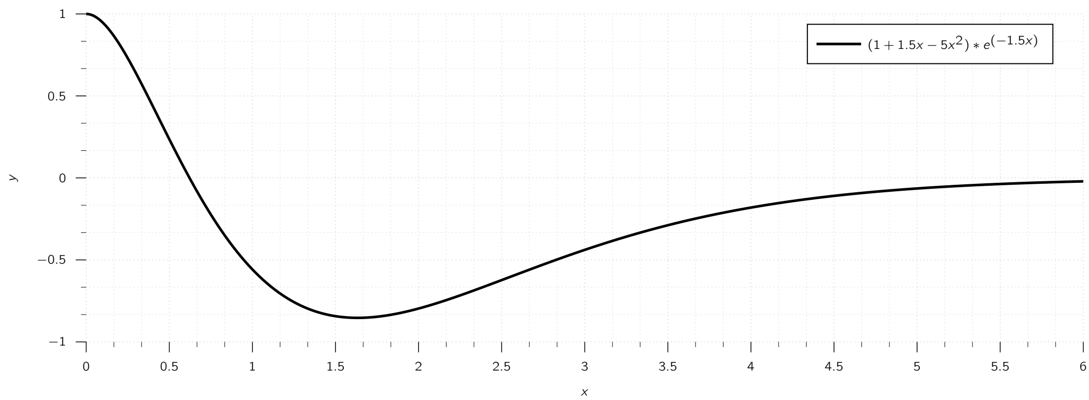


Figure 2.6: Solution to Application of the basic rule B.

Q21 Application of the Basic Rule C

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

A21

The General Solution The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}.$$

The Particular Solution We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1 x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

therefore

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Solution of the initial value problem From y, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

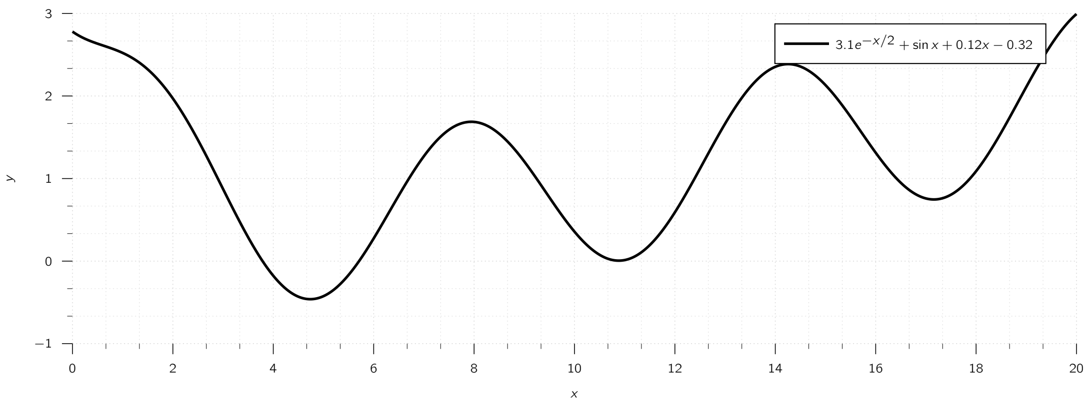


Figure 2.7: Solution to Application of the basic rule C.

Q22 Studying a RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\,\Omega$, $L = 0{,}9\,\mathrm{H}$, $C = 0{,}01\,\mathrm{F}$, which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Note Assume that current and capacitor charge are 0 when $t = 0$.

A22

The General solution Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t.$$
$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Therefore the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

The Particular solution We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from:

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Therefore in our present case, a general solution of the nonhomogeneous ODE is:

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t$$

Particular solution satisfying the initial conditions How to use Q(0) 0? We finally determine c_1 and c_2 from the in initial conditions I(0) = 0 and Q(0) = 0. From the first condition and the general solution we have:

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

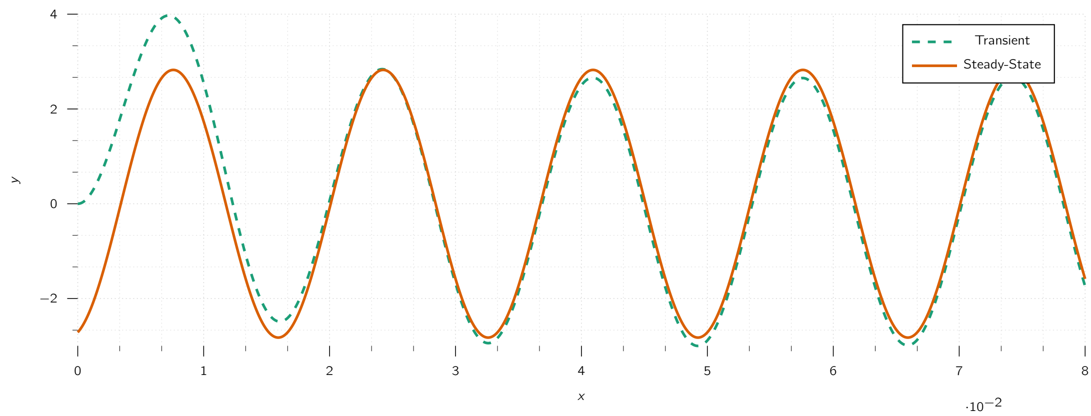


Figure 2.8: A comparison of the actual solution and the steady-state values.

Q23 Harmonic Oscillation of an Undamped Mass-Spring System

If a mass-spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

A23

Hooke's law:

$$F_1 = -ky \tag{2.1}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0{,}48\,\text{Hz}$$

From Eq. (2.1) and the initial conditions, $y(0) = A = 0{,}16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16\cos 3t\,\text{m} \quad \blacksquare$$

Q24 IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

A24

The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = (\lambda + 2)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = (c_1 + c_2 x)\, e^{-0.5\,x}$$

We need its derivative:

$$y\prime = c_2 e^{-0.5\,x} - 0.5\,(c_1 + c_2 x)\, e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$\begin{aligned} y(0) &= c_1 = 3.0, \\ y\prime(0) &= c_2 - 0.5c_1 = 3.5, \\ c_2 &= -2 \end{aligned}$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\,x}$$

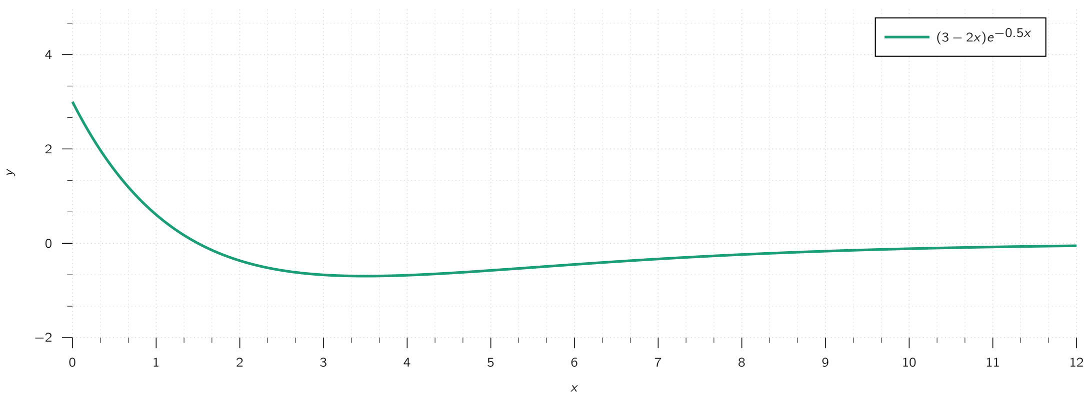


Figure 2.9: Solution to case of double roots.

Q25 IVP: Case of Complex Roots

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

A25

General Solution The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathbf{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Particular Solution The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

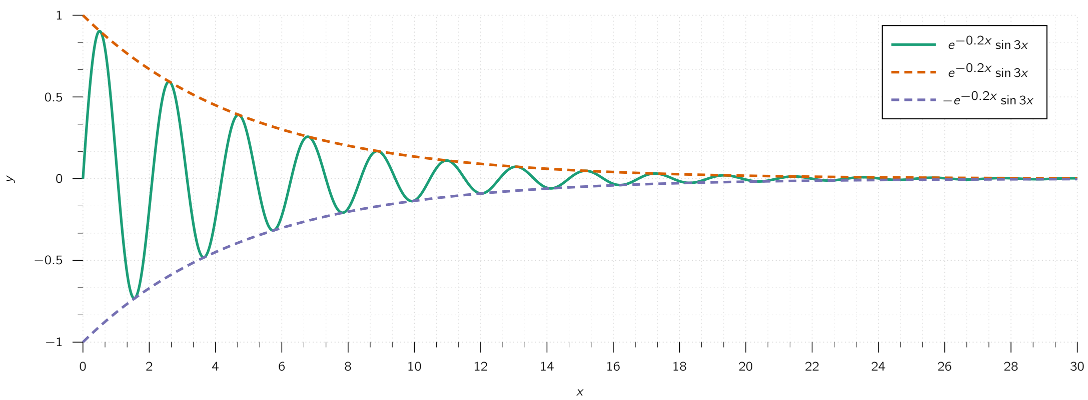


Figure 2.10: Solution to case of complex roots.

Q26 IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

A26

General Solution The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1,$$
$$\text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Particular Solution As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

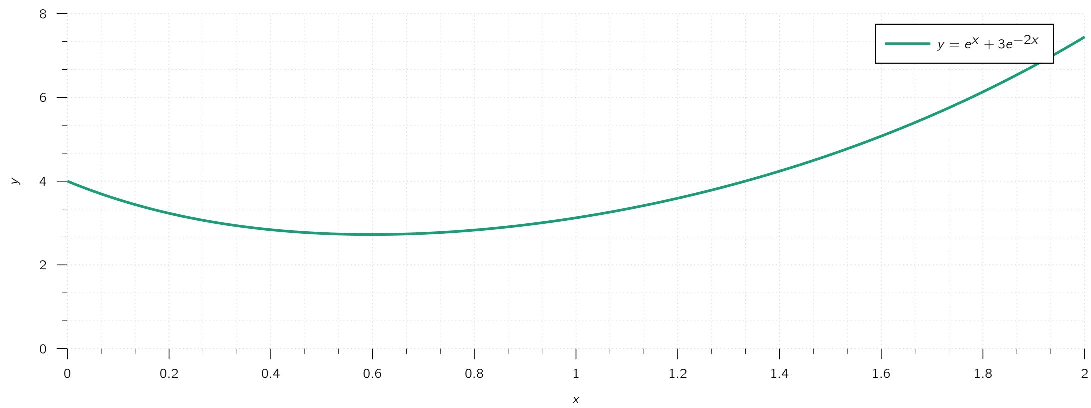


Figure 2.11: Solution to Case of distinct real roots.

3

Higher Order ODEs

Q27 Linear Dependence

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

A27

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Q28 A General Solution

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{where} \qquad y^{\text{iv}} = \frac{d^4y}{dx^4}$$

A28

Similar to Chapter **??** we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Q29 Initial Value Problem for a Third-Order Euler-Cauchy Equation

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4.$$

A29

General Solution As in Chapter **??**, try $y = x^{\mathrm{m}}$. By differentiation and substitution,

$$m(m-1)(m-2)x^{\mathrm{m}} - 3m(m-1)x^{\mathrm{m}} + 6mx^{m} - 6x^{m} = 0.$$

Dropping x^{m} and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular Solution The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
&(a) \quad y(1) = c_1 + c_2 + c_3 = 2 \\
&(b) \quad y'(1) = c_1 + 2c_2 + 3c_3 = 1 \\
&(c) \quad y''(1) = 2c_2 + 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

Q30 **Distinct Real Roots**

Solve the following ODE:

$$y_{'''} - 2y'' - y' + 2y = 0$$

A30

The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots:

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (**??**) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Q31 Real Double and Triple Roots

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y^{\text{iv}} - y'' = 0$$

A31

The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + \left(c_3 + c_4 x + c_5 x^2\right) e^x \quad \blacksquare$$

Q32 Simple Complex Roots

Solve the initial value problem:

$$\begin{aligned} y''' - y'' + 100y' - 100y = 0, \\ y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299. \end{aligned}$$

A32

The characteristic equation is:

$$\lambda_3 - \lambda_2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10\mathbf{j}$. Therefore, a general solution and its derivatives obtained by differentiation are:

$$\begin{aligned} y &= c_1 e^x + A\cos 10x + B\sin 10x, \\ y' &= c_1 e^x - 10A\sin 10x + 10B\cos 10x, \\ y'' &= c_1 e^x - 100A\cos 10x - 100B\sin 10x. \end{aligned}$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$\begin{aligned} \text{(a)} \quad c_1 + A = 4, \\ \text{(b)} \quad c_1 + 108 = 11, \\ \text{(c)} \quad c_1 - 1004 = -299. \end{aligned}$$

We solve this system for the unknowns A, B, and c_1. Equation (a) minus Equation (c) gives $101A = 303$, therefore $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^x + 3\cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.

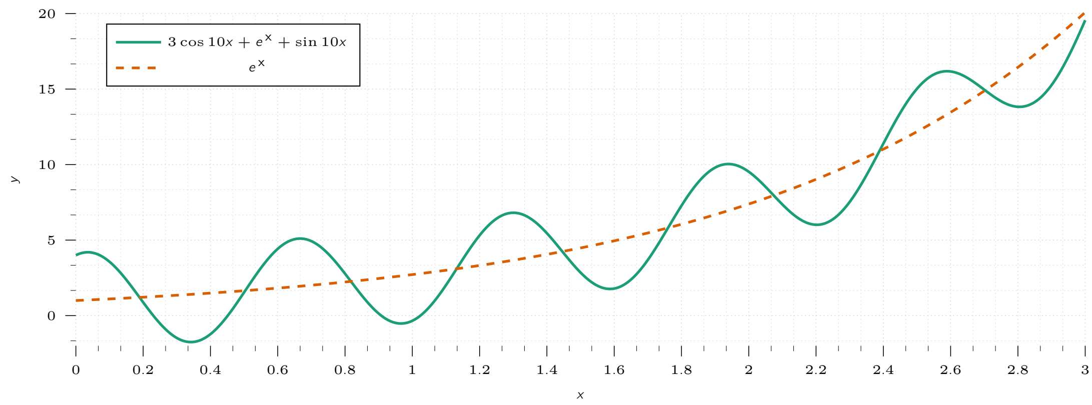


Figure 3.1: Solution to the question Simple Complex Roots

Q33 IVP: Modification Rule

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47$$

A33

Step 1 The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} = (c_1 + c_2 x + c_3 x^2) e^{-x}$$

Step 2 If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and $Cx^6 e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}$$

Then

$$\begin{aligned} y_p{}' &= C\left(3x^2 - x^3\right) e^{-x}, \\ y_p{}'' &= C\left(6x - 6x^2 + x^3\right) e^{-x}, \\ y_p{}''' &= C\left(6 - 18x + 9x^2 - x^3\right) e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C\left(6 - 18x + 9x^2 - x^3\right) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2 e^{-x}$.

Step 3 We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$
\begin{aligned}
y &= y_\text{h} + y_\text{p} = (c_1 + c_2 + c_3 x^2)e^{-x} + 5x^3 e^{-x}, \qquad && y(0) = c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, \qquad && y'(0) = -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, \qquad && y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}
$$

Hence the answer to our problem is:

$$
y = (3 - 25x^2)e^{-x} + 5x^2 e^{-x} \quad \blacksquare
$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.

4

System of ODEs

Q34 Mixing Problem Involving Two Tanks

A mixing problem involving a single tank is modeled by a single ODE which can be extended to two (2) sets of equations.

Assume two (2) Tanks T_1 and T_2 containing initially 100 L of water each, In T_1 the water is pure, whereas 150 kg of fertilizer are dissolved in T_2. By circulating liquid at rate of $2\,\mathrm{L}\cdot\mathrm{min}^{-1}$ and stirring the amount of fertiliser $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

How long should we let the liquid circulate so that T_1 will contain at least half as much fertiliser as there will be left in T_2? **Note** Assume the mixture is uniform.

A34

Setting Up The Model As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. Therefore:

$$
\begin{aligned}
y_1' &= \frac{2}{100}y_2 - \frac{2}{100}y_1 && \text{Tank 1,} \\
y_2' &= \frac{2}{100}y_1 - \frac{2}{100}y_2 && \text{Tank 2.}
\end{aligned}
$$

Therefore the mathematical model of our mixture problem is the system of first-order ODEs:

$$
\begin{aligned}
y_1' &= -0.02y_1 + 0.02y_2, \\
y_2' &= +0.02y_1 - 0.02y_2.
\end{aligned}
$$

As a vector equation with column vector:

$$
\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},
$$

and matrix $\boldsymbol{A}$ this becomes:

$$
\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \qquad \text{where} \qquad \boldsymbol{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.
$$

General Solution As for a single equation, we try an exponential function of t,

$$
\boldsymbol{y} = \boldsymbol{x}e^{\lambda t} \quad \text{and} \quad = \boldsymbol{y}' = \lambda\boldsymbol{x}e^{\lambda t} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}. \tag{4.1}
$$

Dividing the last equation $\lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A} \boldsymbol{x} e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\boldsymbol{A} \boldsymbol{x} = \lambda \boldsymbol{x}.$$

We need **nontrivial** solutions. Hence we have to look for eigenvalues and eigenvectors of $\boldsymbol{A}$. The eigenvalues are the solutions of the characteristic equation

$$\begin{aligned} \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) &= \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} \\ &= (-0.02 - \lambda)^2 - 0.02^2 \\ &= \lambda\,(\lambda + 0.04) = 0 \end{aligned} \tag{4.2}$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$. $\lambda = 0$ can very well happen but don't get mixed up. It is eigenvectors which must not be zero. Eigenvectors are obtained as $\lambda = 0$ and $\lambda = -0.04$. For our present $\boldsymbol{A}$ this gives:

$$\begin{aligned} -0.02x_1 + 0.02x_2 &= 0 \\ \text{and} \qquad (-0.02 + 0.04)\,x_1 + 0.02x_2 &= 0, \end{aligned}$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note This principle continues to hold for systems of homogeneous linear ODEs.

From Eq. (4.1) and the superposition principle, we thus obtain a solution:

$$\begin{aligned} \boldsymbol{y} &= c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned} \tag{4.3}$$

where c_1 and c_2 are arbitrary constants.

Use of Initial Conditions The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and Eq. (4.3) with $t = 0$ we obtain

$$\boldsymbol{y}\,(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer:

$$\begin{aligned} \boldsymbol{y} &= 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)} e^{-0.04t} \\ &= 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned}$$

In components,

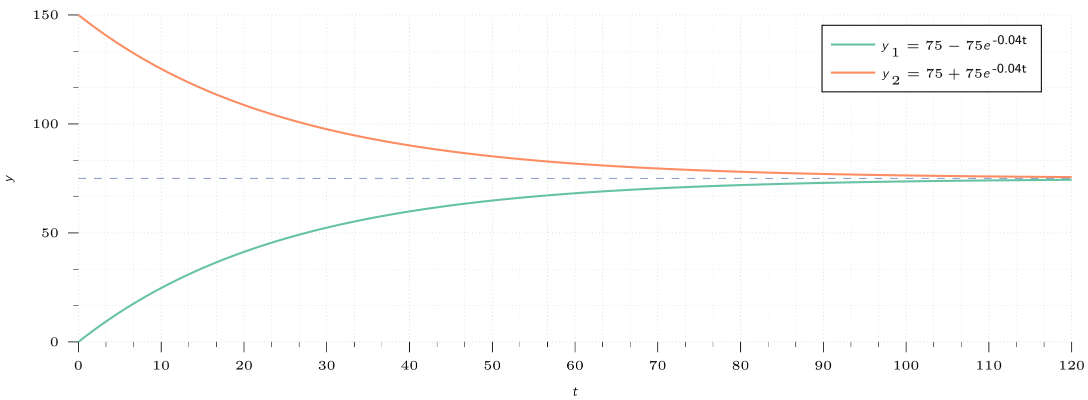
$$y_1 = 75 - 75e^{-0.04\,t} \qquad \text{Tank } T_1, \text{ lower curve,}$$
$$y_2 = 75 + 75e^{-0.04\,t} \qquad \text{Tank } T_2, \text{ upper curve.}$$

Figure below shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 kg.

Calculating the Answer T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 kg. Therefore:

$$y_1 = 75 - 75e^{-0.04t} = 50,$$
$$e^{-0.04t} = \frac{1}{3}, \qquad \text{and} \qquad t = (\ln 3)/0.04 = 27.5$$

Hence the fluid should circulate for roughly half an hour ■



Q35 An Electrical Network

Find the currents $I_1(t)$ and $I_2(t)$ in the network.

NOTE Assume all currents and charges zero at $t = 0$, the instant when the switch is **closed**.

A35

The solution is as follows:

Setting up the mathematical model The model of this network is obtained from Kirchhoff's Voltage Law.

Information: Kirchhoff's Voltage Law

The sum of the voltage differences around any closed loop in a circuit must be zero. A loop in a circuit is any path which ends at the same point at which it starts.

Let $I_1(t)$ and $I_2(t)$ be the currents in the left (L) and right (R) loops, respectively.

In (L), the voltage drops are:

$$\begin{aligned} L_1 I_1' &= (1) I_1' \text{V} \qquad \text{Over Inductor} \\ R_1 \left(I_1 - I_2 \right) &= 4 \left(I_1 - I_2 \right) \text{ volt} \qquad \text{Over Resistor} \end{aligned}$$

The difference is caused by I_1 and I_2 flowing through the resistor in **opposite** directions.

By Kirchhoff's Voltage Law the sum of these drops equals the voltage of the battery:

$$I_1' + 4 \left(I_1 - I_2 \right) = 12$$

Cleaning the aforementioned equation creates our first ODE:

$$I_1' = -4 I_1 + 4 I_2 + 12. \tag{4.4}$$

In (R), the voltage drops are:

$$\begin{aligned} R_2 I_2 &= 6 I_2 \text{V} \qquad \text{Over Resistor} \\ R_1 (I_2 - I_1) &= 4 \left(I_2 - I_1 \right) \text{ V} \qquad \text{Over Resistor} \\ (I/C) \int I_2 dt &= 4 \int I_2 dt \text{V} \qquad \text{Over Capacitor} \end{aligned}$$

As there is no voltages sources in the (R) loop, the voltage sum **MUST** be zero.

$$6 I_2 + 4 \left(I_2 - I_1 \right) + 4 \int I_2 dt = 0 \qquad \text{or} \qquad 10 I_2 - 4 I_1 + 4 \int I_2 dt = 0.$$

Division by 10 and differentiation gives $I_2' - 0.4 I_1[I] + 0.4 I_2 = 0$.

To simplify the solution process, we first get rid of $0.4 I_1'$, which by Eq. (4.4) equals $0.4 \left(-4 I_1 + 4 I_2 + 12 \right)$. Substitution into the present ODE gives

$$I_2' = 0.4 I_1[I] - 0.4 I_2 = 0.4 \left(-4 I_1 + 4 I_2 + 12 \right) - 0.4 I_2$$

and by simplification

$$I_2' = -1.6 I_1 + 1.2 I_2 + 4.8. \tag{4.5}$$

In matrix form, Eq. (4.4) and Eq. (4.5) are (we write $\boldsymbol{J}$ since $\boldsymbol{I}$ is the unit matrix)

$$\boldsymbol{J}' = \boldsymbol{A} \boldsymbol{J} + \boldsymbol{g}, \qquad \text{where} \qquad \boldsymbol{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}. \tag{4.6}$$

General Solution

As we have a vector $\boldsymbol{g}$, this is a **non-homogeneous** system, and we try to proceed as for a single ODE, solving first the homogeneous system $\boldsymbol{J}' = \boldsymbol{A} \boldsymbol{J}$ (thus $\boldsymbol{J}' - \boldsymbol{A} \boldsymbol{J} = 0$) by substituting $\boldsymbol{J} = \boldsymbol{x} e^{\lambda t}$. This gives

$$\boldsymbol{J}' = \lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A} \boldsymbol{x} e^{\lambda t} \qquad \text{hence} \qquad \boldsymbol{A} \boldsymbol{x} = \lambda \boldsymbol{x}$$

Hence, to obtain a non-trivial solution, we again need the eigenvalues and eigenvectors. For the present matrix $\boldsymbol{A}$ they are:

$$\lambda_1 = -2, \qquad \boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

Hence a *general solution* of the homogeneous system is:

$$\boldsymbol{J}_{\mathrm{h}} = c_1 \boldsymbol{x}^{(1)} e^{-2\,t} + c_2 \boldsymbol{x}^{(2)} e^{-0.8\,t}.$$

For a particular solution of the nonhomogeneous system Eq. (4.6), since $\boldsymbol{g}$ is constant, we try a constant column vector $\boldsymbol{J}_{\mathrm{p}} = \boldsymbol{a}$ with components a_1, a_2. Then $\boldsymbol{J}'_{\mathrm{p}} = 0$, and substitution into Eq. (4.6) gives $\boldsymbol{A}\,\boldsymbol{a} + \boldsymbol{g} = 0$, in components,

$$-4.0a_1 + 4.0a_2 + 12.0 = 0$$

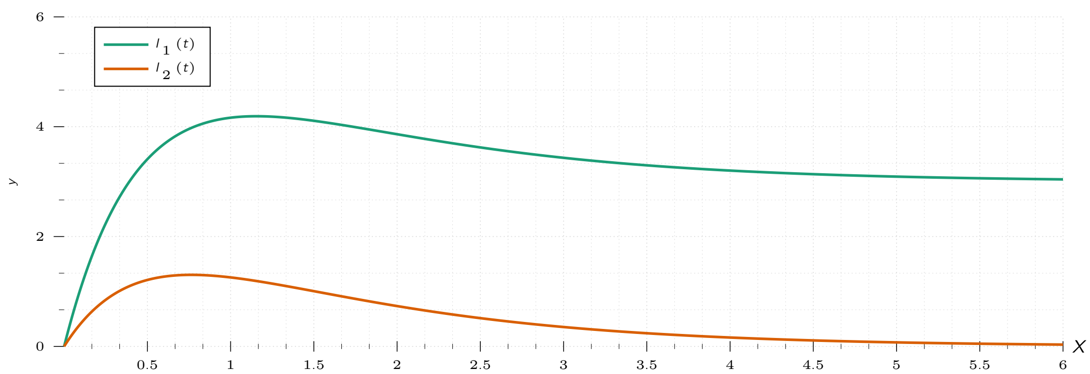
$$-1.6a_1 + 1.2a_2 + \ 4.8 = 0.$$

The solution is $a_1 = 3$, $a_2 = 0$; thus $\boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

$$J = J_h + J_p = c_1 x^{(1)} e^{-2t} +$$

This is a parametric representation with time as the parameter t. It is often important to know in which sense such a curve is traced. This can be indicated by an arrow in the sense of increasing t. The I_1, I_2-plane is called the **phase plane** of our system Eq. (4.6), and the curve in **??** is called a trajectory.

In following chapters we will see that such *phase plane representations* are far more important than graphs because they will give a much better qualitative overall impression of the general behavior of whole families of solutions, not merely of one solution as in the present case. ■



Q36 Mass on a String

To gain confidence in the conversion method, let us apply it to an old problem of ours:

modelling the free motions of a mass on a spring with value given as $m = 1$, $c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \qquad \text{or}$$

$$y'' = -\left(\frac{c}{m}\right) y' - \left(\frac{k}{m}\right) y.$$

A36

For this ODE given in the question can be written in the form ofEq. (**??**), making the system shown Eq. (**??**) as **linear** and **homogeneous**, applying to our system in question.

$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2.$$

Setting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I}\right) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Entering the values of $m = 1$, $c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)\,(\lambda + 1.5) = 0$$

This gives the eigenvalues:

$$\lambda_1 = -0.5 \qquad \text{and} \qquad \lambda_2 = -1.5$$

Eigenvectors follow from the first equation in $\boldsymbol{A} - \lambda\boldsymbol{I} = 0$, which is $-\lambda x_1 + x_2 = 0$. $\lambda_1 = 0.5$ produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$. $\lambda_2 = -1.5$ produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$. These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

Which gives:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t} \quad \blacksquare$$

Q37 **Type I: Improper Node**

Find solutions of the following system:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \boldsymbol{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

A37

To see what is going on, let us find the solutions of the system. It is always a good idea to start with known solutions. Substituting $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ and $\boldsymbol{y}' = \lambda \boldsymbol{x}e^{\lambda t}$ and dropping the exponential function, as they exist both on the Left Hand Side (LHS) and RHS we can eliminate them, we get $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$. The characteristic equation is:

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from:

$$(-3-\lambda)\, x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\boldsymbol{x}^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\boldsymbol{x}^{(2)} = [1 \quad -1]^{\mathsf{T}}$.

This gives the general solution:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \boldsymbol{y}^{(1)} + c_2 \boldsymbol{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad \blacksquare$$

Q38 **Type II: Proper Node**

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction $\boldsymbol{d}$ at P_0 there is a trajectory having $\boldsymbol{d}$ as its limiting direction. Let's study the following system:

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{y}, \qquad \text{Therefore} \qquad y_1' = y_1 \qquad \text{and} \qquad y_2' = y_2$$

A38

The equation has a proper node at the origin with the matrix being the **identity matrix**. Its characteristic equation $(1-\lambda)^2 = 0$ has the root $\lambda = 1$.

Note: Any $\boldsymbol{x} \neq 0$ is an eigenvector.

and we can take $[1 \quad 0]^{\mathsf{T}}$ and $[0 \quad 1]^{\mathsf{T}}$.

Hence, a general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t} \qquad \text{or} \qquad \begin{matrix} y_1 = c_1 e^t, \\ y_2 = c_2 e^t. \end{matrix} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1 \quad \blacksquare$$

Q39 Type III: Saddle Point

A **saddle point** is a critical point P_0 at which there are two (2) incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0. Let's study the following system:

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{y}, \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_1 \\ y_1' &= -y_2 \end{aligned}$$

A39

The equation has a saddle point at the origin and its characteristic equation

$$(1-\lambda)(-1-\lambda) = 0.$$

has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ in eigenvector $[1\ 0]^T$ is obtained from the second row of $(\boldsymbol{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $0x_1 + (-1-1)\,x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0 \quad 1]^T$. Hence a general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned} \qquad \text{or} \qquad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas** ■.

Q40 Type IV: Centre Node

A **centre** is a critical point that is enclosed by infinitely many closed trajectories. Let's study the following system:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y, \qquad \text{Therefore} \qquad y_1' = y_2 \qquad \text{and} \qquad y_2' = -4y_1 \tag{4.8}$$

A40

The equation has a center at the origin.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathbf{j}$ and $-2\mathbf{j}$. For $2\mathbf{j}$, an eigenvector follows from the first equation $-2\mathbf{j}\,x_1 + x_2 = 0$ of $(\boldsymbol{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$, which can be, $[1 \quad 2\mathbf{j}]^{\mathsf{T}}$.

For $\lambda = -2\mathbf{j}$ that equation is $-(-2\mathbf{j})x_1 + x_2 = 0$ and gives, say, $[1 \quad -2\mathbf{j}]^{\mathsf{T}}$. Hence a complex general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 2\mathbf{j} \end{bmatrix} e^{2\mathbf{j}\,t} + c_2 \begin{bmatrix} 1 \\ -2\mathbf{j} \end{bmatrix} e^{-2\mathbf{j}\,t}, \qquad \text{therefore} \qquad \begin{aligned} y_1 &= c_1 e^{2\mathbf{j}\,t} + c_2 e^{-2\mathbf{j}\,t}, \\ y_2 &= 2\mathbf{j}\, c_1 e^{2\mathbf{j}\,t} - 2\mathbf{j}\, c_2 e^{-2\mathbf{j}\,t}. \end{aligned} \tag{4.9}$$

A real solution is obtained from Eq. (4.9) by the Euler formula or from Eq. (4.8). Namely, we can create a relation of $-4y_1y_1^2$.

$$-4y_1y_1' = y_2y_2' \qquad \text{By Integration} \qquad 2y_1^2 + \frac{1}{2}y_2^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. ■

Q41 Type V: Spiral Node

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$.

> or tracing these spirals in the opposite sense, away from P_0.

The system:

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y, \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.10}$$

A41

has a spiral point at the origin.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1 + \mathbf{j}$ and $-1 - \mathbf{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For
$\lambda = -1 + \mathbf{j}$ this becomes $-\mathbf{j}x_1 + x_2 = 0$ and we can take $[1 \quad \mathbf{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathbf{j}$ is $[1 \quad -\mathbf{j}]^T$.

This gives the **complex** general solution:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ \mathbf{j} \end{bmatrix} e^{(-1+\mathbf{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathbf{j} \end{bmatrix} e^{(-1-\mathbf{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (4.10) by y_1, the second by y_2- and add, obtaining:

$$y_1 y_1' + y_2 y_2' = -\left(y_1^2 + y_2^2\right).$$

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to t gives:

$$2rr' = 2y_1 y_1' + 2y_2 y_2'$$

Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real c this is a spiral.

Q42 Linearisation of a Free Undamped Pendulum

A pendulum consists of a body of mass m (the bob) and a rod of length L. Determine the locations and type of the critical points.

Warning Assume that the mass of the rod and an reference are negligible.

A42

Let us begin tackling the problem:

Setting Up the Mathematical Model Let θ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is mg, where g is the acceleration of gravity.

This causes a restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2nd law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the**acceleration**.
Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg\sin\theta = 0.$$

Dividing this by mL, we have:

$$\theta'' + k\sin\theta = 0 \qquad \text{with} \qquad \left(k = \frac{g}{L}\right). \tag{4.11}$$

When θ is very small, we can approximate sin θ rather accurately by θ and obtain as an approximate solution $A \cos\sqrt{k} + B \sin\sqrt{k}$, but the *exact* solution for any θ is not an **elementary function**.

Critical Points and Linearisation To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from Eq. (4.11) we obtain a nonlinear system Eq. (**??**) of the form:

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = y_2, \\ y_2' &= f_2(y_2, y_1) = -k\sin y_1. \end{aligned}$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi,\, 0)$, where $n = 0,\, \pm1,\, \pm2,\, \cdots$.

We consider $(0,\, 0)$. Since the Maclaurin series is

$$\sin y_1 = y_1 - \tfrac{1}{6}y_1^2 + - \cdots \approx y_1,$$

the linearized system at $(0,\, 0)$ is:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \boldsymbol{y} \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1. \end{aligned}$$

To apply our criteria in Sec. 4.4 we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = 0, \\ q &= \det(\boldsymbol{A}) = k = g/L \qquad (> 0), \end{aligned}$$

$$\Delta = p^2 - 4q = -4k.$$

From this and Table 4.1(c) in Sec. 4.4 we conclude that $(0, 0)$ is a **centre**, which is **always stable**. Since $\sin\theta = \sin y_1$ is periodic with period of 2π. **Warning** This means the critical points $(n\pi,\ 0)$,

$n = \pm 2,\ \pm 4,\ \cdots$, are all centres. We now consider the critical point $(\pi, 0)$, setting:

$$\begin{aligned} y_1 &= \theta - \pi \\ y_2 &= (\theta - \pi)' \end{aligned}$$

Then in Eq. (4.11), we can apply the MacLaurin series:

$$\sin\ \theta = \sin\ (y_1 + \pi) = -\sin\ y_1 = -y_1 + \frac{1}{2}y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} \boldsymbol{y} \qquad \text{Thus} \qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= k y_1. \end{aligned}$$

We see that:

$$\begin{aligned} p &= 0, \\ q &= -k \qquad (< 0)\,, \\ \Delta &= -4q = 4k. \end{aligned}$$

Hence, by Table 4.1(b), this gives a saddle point, which is always unstable.

Because of periodicity, the critical points $(n\pi,\ 0)$, $n = \pm 1,\ \pm 3,\ \cdots$, are all **saddle points**.

Q43 Linearisation of a Damped Pendulum

To gain further experience in investigating critical points, as another practically Important, let us see how the previous example changes when we add a damping term $c\theta$, (damping proportional to the angular velocity) to equation Eq. (4.11), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

where $k > 0$ and $c \geq 0$ (which includes our previous case of no damping, $c = 0$).

A43

First we start by setting $\theta = y_1$, θ' as before, we obtain the nonlinear system
(use $\theta'' = y_2$),

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 - c y_2. \end{aligned}$$

We see the critical points have the same locations as the example before, namely, $(0,\ 0)\,,\ (\pm\pi,\ 0)\,,\ (\pm 2\pi,\ 0)\,,\ \ldots.$ To analyse this system, we start with analysing $(0,\ 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0,\ 0)$.

$$y' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \boldsymbol{y} \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2 \end{aligned}$$

This is identical with the system in previous example, except for the positive factor m (and except for the physical meaning of y_1). Hence for $c = 0$ (no damping) we have a centre, for small damping we have a spiral point, and so on.

We now consider the critical point $(\pi,\ 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin\left(y_1 + \pi\right) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi,\ 0)$:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \boldsymbol{y}, \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2. \end{aligned}$$

For our criteria, we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = -c \\ q &= \det \boldsymbol{A} = -k \\ \Delta &= p^2 - 4q = c^2 + 4k \end{aligned}$$

This gives the following results for the critical point $(\pi,\ 0)$.

No Damping $c > 0, p = 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 3b.

Damping $c > 0, p < 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 94.

As $\sin y_1$ is periodic with period of 2π, the critical points $(\pm 2\pi,\ 0)\,,\ (\pm 4\pi,\ 0)\,,\ \ldots$ are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished. ■

5

Special Functions for ODEs

Q44 Power Series Solution

Solve the following ODE:

$$y' - y = 0$$

A44

First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \tag{5.1}$$

We put these values into the ODE:

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

Then we collect like powers of x, finding:

$$(a_1 - a_0) + (2a_2 - a_1)\, x + (3a_3 - a_2)\, x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express a_1, a_2, $\cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^x. \quad \blacksquare$$

Q45 A Special Legendre Function

Solve the following ODE:

$$\left(1 - x^2\right) y'' - 2xy' + 2y = 0$$

Note: These equations usually occur in models with spherical symmetry.

A45

Substitute Eq. (**??**), Eq. (5.1), and Eq. (**??**) into the ODE, $\left(1-x^2\right)y''$ gives two (2) series: for y'', and for $-x^2y''$. For the term $-2xy'$ use Eq. (5.1) and in $2y$ use Eq. (**??**). Write like powers of x vertically aligned for easy viewing. This gives:

$$
\begin{aligned}
y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots \\
-xy'' &= \qquad\qquad\quad - 2a_2x^2 - 6a_3x^3 - 12a_4x^4 - \cdots \\
-2xy' &= \qquad - 2a_1x - 4a_5x^2 - 6a_9x^3 - 8a_4x^4 - \cdots \\
2y &= 2a_0 + 2a_1x - 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots .
\end{aligned}
$$

Add terms of like powers of x. For each power x^0, x, x^2 equate the sum obtained to zero. Denote these sums by 0 (constant terms), 1 (first power of x), and so on and write it down to the following table:

Sum	Power	Equation
0	x^0	$a_2 = -a_0$
1	x	$a_3 = 0$
2	x^2	$14a_4 = 4a_2, \qquad a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
3	x^3	$a_5 = 0 \qquad \text{since} \qquad a_3 = 0$
4	x^4	$30a_6 = 18a_4, \qquad a_6 = \frac{18}{30}a_4 = \frac{18}{30}\left(-\frac{1}{3}\right)a_0 = -\frac{1}{5}a_0$

This gives the solution

$$
y = a_1x + a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) \qquad \blacksquare
$$

Note: a_0, a_1 remain **arbitrary**.

Therefore, this is a **general solution** consisting of two (2) solutions: x and

$$
1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots
$$

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(x)$ and *Legendre functions* $Q_1(x)$. Here we have

$$
x = P_1(x)
$$

and

$$
1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(x)
$$

Note: The minus is by convention. The index 1 is called the *order* of these functions and here the order is 1. $\blacksquare$

Glossary

IVP Initial Value Problem. 3

LHS Left Hand Side. 36

ODE Ordinary Differential Equation. 2–7, 9, 11, 15, 17–20, 26, 27, 30, 31, 35, 39, 42, 43

RHS Right Hand Side. 17, 36