Chapter 1

First-Order Ordinary Differential Equations

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1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a *mathematical expression* in terms of: variables, functions, equations. Such an expression is known as a mathematical **model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**.

Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,

- graph our equation,
- find new values,
- interpret result in a physical terms.

This is all done to understand the behaviour of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter.

An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually $y(x)$. The equation may also contain y itself, known functions of x, and constants. For example all the equation shown below are classified as ODE.

$$y' = \cos x$$
$$y'' + 9y = e^{-2x}$$
$$y'y'' - \frac{3}{2}y = 0.$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from Partial Differential Equation (PDE)s, which involve **partial** derivatives of an unknown function of two or more variables[1]. For instance, a PDE with unknown function u of two (2) variables x and y is

[1] The topic of PDE will be the focus of **Higher Mathematics II**.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

For now, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F\left(x, y, y'\right) = 0 \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE:

$$x^{-3}y' - 4y^2 = 0 \qquad \text{where} \qquad x \neq 0$$

can be written explicitly as $y' = 4x^3y^2$.

1.1.1 Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y\left(x_0\right) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

Geometrically, this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE, together with an initial condition, is called an **initial value problem**.

Theory 1.0: Initial Value Problem

In multi-variable calculus, an Initial Value Problem (IVP) is an ODE together with an initial condition which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Exercise 1.1: Initial Value Problem - A

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

Solution

The general solution is:

$$y(x) = ce^{3x}$$

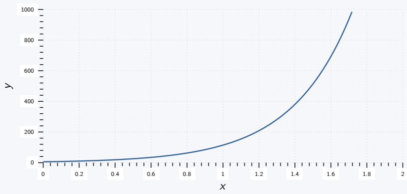
From this solution and the initial condition we obtain:

$$y(0) = ce^0 = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation ■



Exercise 1.2: Radioactive Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Note: The decay of Radium is measured to be $k = 1.4e - 11\,\mathrm{s}^{-1}$.

Solution

$y(t)$ is the amount of substance still present at t. By the physical law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives is the following:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is positive, so that, because of the minus, we get *decay*. The value of k is known from experiments for various radioactive substances which the question has given as $k = 1.4 \cdot 10^{-11} sec^{-1}$. Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

We have the initial condition $y(0) = 0.5$. This is the instant at which our observation of the process begins. In motivates the original condition which however, is also used when the independent variable is not time or when we choose a t other than $t = 0$.
Hence the mathematical model of the physical process is the initial value problem.

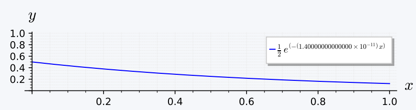
$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the ODE is an exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is:

$$y(t) = 0.5e^{-kt} \quad ■$$



1.2 Separable ODEs

Many practically useful ODEs can be reduced to the following form:

$$g(y)\,y' = f(x) \tag{1.4}$$

using *algebraic manipulations*. We can then integrate on both sides with respect to x, obtaining the following expression:

$$\int g(y)\,y'\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + c. \tag{1.5}$$

On the Left Hand Side (LHS) we can switch to y as the variable of integration. By calculus, we know the relation $y'\,\mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + c. \tag{1.6}$$

If f and g are **continuous functions**[2], the integrals in Eq. (1.6) exist, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

[2] a continuous function is a function such that a small variation of the argument induces a small variation of the value of the function.

Exercise 1.3: Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

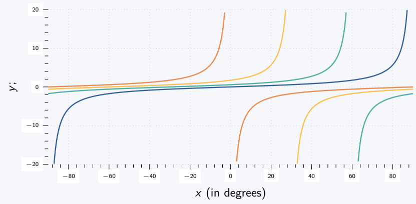
Solution

The given ODE is separable because it can be written:

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x \qquad \text{By integration} \quad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

Note: It is important to introduce the constant c when the integration is performed.



Exercise 1.4: A Bell Shaped Curve

Solve the following ODE:

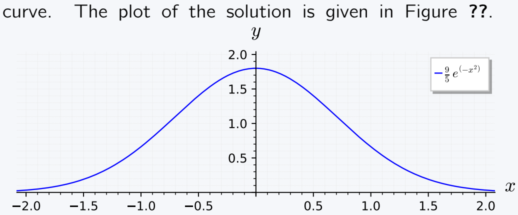
$$y' = -2xy, \qquad y(0) = 1.8$$

Solution

By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve. The plot of the solution is given in Figure **??**.



Exercise 1.5: Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the AustrianItalian border, caused a scientific sensation. When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

Note: The half-life of carbon is 5175 years.

Solution

Radioactive decay is governed by the ODE $y' = ky$ as we have developed previously. By separation and integration

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0 e^{kt} \qquad (y_0 = e^0).$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Ötzi died,

$$e^{k\mathcal{T}} = e^{-0.0001\,213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001213} = 5312. \qquad \blacksquare$$

Reduction to Separable Form

Certain non-separable ODEs can be made separable by transformations that introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right)$$

Here, f is any differentiable function of y x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests that we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f\left(y/x\right)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Exercise 1.6: Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

Solution

Reduction to Separable Form To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

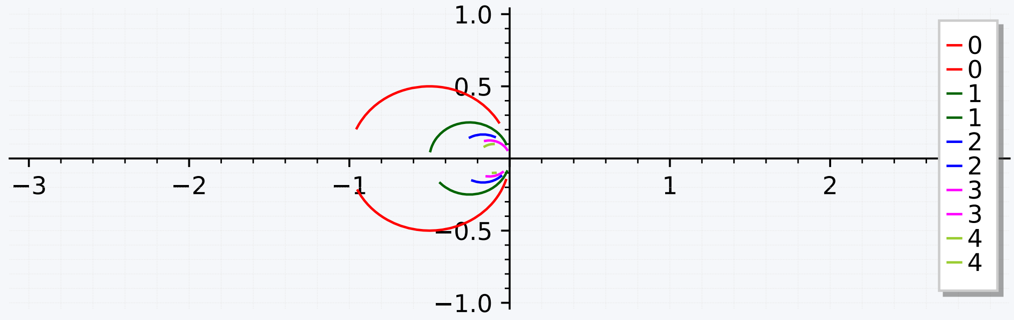
You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx. \quad \text{therefore} \quad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4} \quad \blacksquare$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen in Figure ??.



1.3 Exact ODEs

1.3.1 Integrating Factors

Recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (i.e., **total differential**) is:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

From this it follows that if $u(x, y) = c =$ is constant, then $du = 0$. As an example, let's have a look at the function $u = x + x^2 y^3 = c$. Finding its factors:

$$du = (1 + 2xy^3)dx + 3x^2 y^2 dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x, y) + N(x, y) y' = 0$, written as:

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1.7}$$

is called an **exact differential equation** if the differential form $M(x, y)\, dx + N(x, y)\, dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{1.8}$$

of some function $u(x, y)$. Then Eq. (1.7) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.7) in the form:

$$u(x, y) = c \tag{1.9}$$

Comparing Eq. (1.7) and Eq. (1.8), we see that Eq. (1.7) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N \tag{1.10}$$

From this we can derive a formula for checking whether Eq. (1.7) is exact or not, as follows.
Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.10),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x}, \tag{1.11}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{1.12}$$

By the assumption of continuity the two second partial derivatives are equal. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.13}$$

> This condition is not only necessary but also sufficient for Eq. (1.7) to be an exact differential equation.

If Eq. (1.7) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way.
From Eq. (1.11) we have by integration with respect to x:

$$u = \int M dx + k(y), \tag{1.14}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.14), use (4b) to get dk/dy, and integrate dk/dy to get k.
Formula Eq. (1.14) was obtained from Eq. (1.11).

> It is valid to use **either** of them and arrive at the same result.

Then, instead of Eq. (1.14), we first have by integration with respect to y

$$u = \int N\, \mathrm{d}y + l(x). \tag{1.15}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from , use Eq. (1.11) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Exercise 1.7: Initial Value Problem

Solve the initial value problem

$$(\cos y \sinh x + 1)dx - \sin y \cosh x dy = 0, \qquad y(1) = 2.$$

Solution

Verify that the given ODE is **exact**. We find u. For a change, let us use Eq. (1.16):

$$u = -\int \sin y \cosh x dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = u = \cos y \sinh x + 1$. Therefore $dl/dx = 1$ by integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$. Therefore the answer is $\cos y \cosh x + x = 0.358$.

Exercise 1.8: An Exact ODE

Solve the following ODE:

$$\cos(x + y)dx + (3y^2 + 2y + \cos(x + y))dy = 0. \qquad (1.16)$$

Step 1 - Test for exactness
First check if our equation is **exact**, try to convert the equation of the form Eq. (1.7):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

This proves our equation to be exact. **Step 2 - Implicit General Solution**

From Eq. (1.14), we obtain by integration:

$$u = \int M dx + k(y) = \int \cos(x + y)dx + k(y)$$
$$= \sin(x + y) + k(y) \qquad (1.17)$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (1.12), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Therefore $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into Eq. (1.17) and observing Eq. (1.9), we obtain:

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c \quad \blacksquare$$

Exercise 1.9: Breakdown of Exactness

Check the exactness of the following ODE:

$$-ydx + xdy = 0$$

Solution

Breakdown of Exactness The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\partial M/\partial y = -1 \qquad \partial N/\partial x = 1$$

Let us show that in such a case the present method does not work. From (6),

$$u = \int M dx + ky = -xy + ky, \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try ; it will also fail. Solve the equation by another method that we have discussed.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$).

1.4 Linear ODEs

1.4.1 Introduction

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A first-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)\,y = r(x) \tag{1.18}$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

Homogeneous Linear ODE

We want to solve in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) = 0$.) Then the ODE Eq. (1.18) becomes:

$$y' + p(x)\,y = 0 \tag{1.19}$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{therefore} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE Eq. (1.19),

$$y(x) = ce^{-\int p(x)\,\mathrm{d}x} \qquad \left(c = \pm e^{\,c^*} \qquad \text{when} \qquad y \neq 0\right) \tag{1.20}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Non-Homogeneous Linear ODE

We now solve Eq. (1.18) in the case that $r(x)$ in Eq. (1.18) is not everywhere zero in the interval J considered. Then the ODE Eq. (1.18) is called non-homogeneous. It turns out that in this case, Eq. (1.18) has a pleasant property.

Namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ as follows. We multiply Eq. (1.18) by $F(x)$, obtaining

$$Fy' + pFy = rF. \tag{1.21}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^{h}.$$

With this F and $h' = p$, Eq. (1.21) becomes

$$e^{h}y' + h'e^{h}y = e^{h}y' + (e^{h})'y = (e^{h}y)' = re^{h}.$$

By integration,

$$e^{h}y = \int e^{h}r\,\mathrm{d}x + c$$

Dividing by e^{h}, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^{h}r\,dx + c\right), \qquad h = \int p(x)\,dx. \tag{1.22}$$

This reduces solving Eq. (1.18) to the generally simpler task of evaluating integrals. For ODEs for which this is still difficult, you may have to use a numeric method for integrals or for the ODE itself.

h has nothing to do with $h(x)$ and that the constant of integration in h does not matter.

The structure of Eq. (1.22) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing Eq. (1.22) as a sum of two terms,

$$y(x) = e^{-h}\int e^{h}r\,dx + ce^{-h},$$

Exercise 1.10: First-Order ODE, General Solution Initial Value Problem

Solve the initial value problem

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.Here $p = \tan x$, $r = \sin 2x = 2\sin x\cos x$, and

$$h = \int p\,dx = \int \tan x\,dx = \ln|\sec x|.$$

From this we see that in Eq. (1.22),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h}r = (\sec x)(2\sin x\cos x) = 2\sin x,$$

and the general solution of our equation is

$$y(x) = \cos x\left(2\int \sin x\,dx + c\right) = c\cos x - 2\cos^2 x.$$

From this and the initial condition, $1 = c - 1 - 2\cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3\cos x - 2\cos^2 x$. Here $3\cos x$ is the response to the initial data, and $-2\cos^2 x$ is the response to the input $\sin 2x$ ■

Chapter 2

Second-Order Ordinary Differential Equations

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2.1 Introduction

A second-order ODE is called **linear**, if it can be written[1] as:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.1}$$

- when $r(x) = 0$ it is homogeneous,
- else it is **non-homogeneous**.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs.
An example of a **non-homogeneous linear** equation is:

$$y'' = 25y - e^{-x} \cos x$$

An example of a **homogeneous linear** equation is:

$$y'' + \frac{1}{x} y' + y = 0$$

[1] in its standard form

An example of **non-linear**ODE is:

$$y''y + \left(y''\right)^2 = 0$$

2.1.1 Superposition Principle

For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1 y_1 + c_2 y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or linearity principle

Theory 2.0: Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE of:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

any linear combination of two solutions on an open interval I is again a solution of on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Exercise 2.1: Homogeneous Linear ODEs: Superposition of Solutions

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0,$$

for all x.

Solution

Homogeneous Linear ODEs: Superposition of Solutions
Verify by differentiation and substitution. We obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)$$
$$= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare$$

Exercise 2.2: Example of a Non-homogeneous Linear ODE

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 0$$

Solution

Example of a Non-homogeneous Linear ODE Testing out the first function

$$(1+\cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1+\sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

For a second-order homogeneous linear ODE an initial value problem consists of two (2) initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1. \tag{2.2}$$

The conditions Eq. (2.2) are used to determine the two arbitrary constants c_1 and c_2 in a general solution

Exercise 2.3: Initial Value Problem

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution

Initial Value Problem
Step 1: General Solution
From Example 1, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

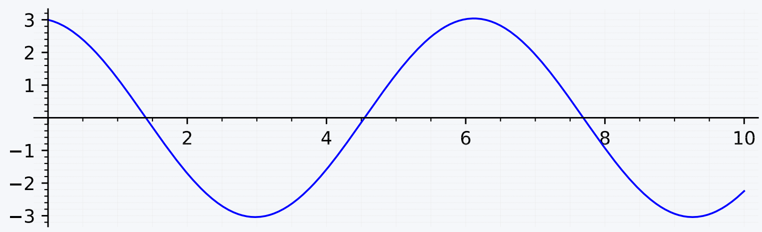
This will turn out to be a general solution as defined below.

Step 2: Particular Solution
We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$



2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.

2.2 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.3}$$

Solve by starting

$$y = e^{\lambda x} \tag{2.4}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.3) gives:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic** equation (or auxiliary equation),

$$\lambda^2 + a\lambda + b = 0 \tag{2.5}$$

then the exponential function Eq. (2.4) is a solution of the ODE given in Eq. (2.3). Now from algebra we recall the roots of the quadratic equation

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

From algebra we further know that the quadratic equation Eq. (2.5) may have three (3) kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

Case	Roots of	Basis	General Solution
I	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
	(λ_1, λ_2)	$e^{\lambda_2 x}$	
II	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
	$(\lambda = -1/2a)$	$xe^{-ax/2}$	
III	Complex Conjugate	$e^{-ax/2}\cos\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$
	$\lambda_1 = -1/2a + \mathbf{j}\omega$	$e^{-ax/2}\sin\omega x$	
	$\lambda_2 = -1/2a - \mathbf{j}\omega$		

Table 2.1.: Possible roots of the characteristic equation based on the discriminant value.

2.2.1 A Study of Damped System

To our previous undamped model $my'' = -ky$ we now add the damping force:

$$F_2 = -cy',$$

therefore, the ODE of the damped massspring system is:

$$my'' + cy' + ky = 0. \tag{2.6}$$

This can physically be done by connecting the ball to a dashpot. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

> This is generally a good approximation for small velocities.

The constant c is called the damping constant.

The damping force $F_2 = -cy'$ acts **against** the motion; hence for a downward motion we have $y\prime > 0$ which for positive c makes F negative (an upward force), as it should be.

Similarly, for an upward motion we have $y\prime < 0$ which, for $c > 0$ makes F_2 positive (a downward force).

The ODE Eq. (2.6) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}. \tag{2.7}$$

Depending on the amount of damping presentwhether a lot of damping, a medium amount of damping or little dampingthree types of motions occur, respectively:

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.2.: A Detailed look into the scientific method.

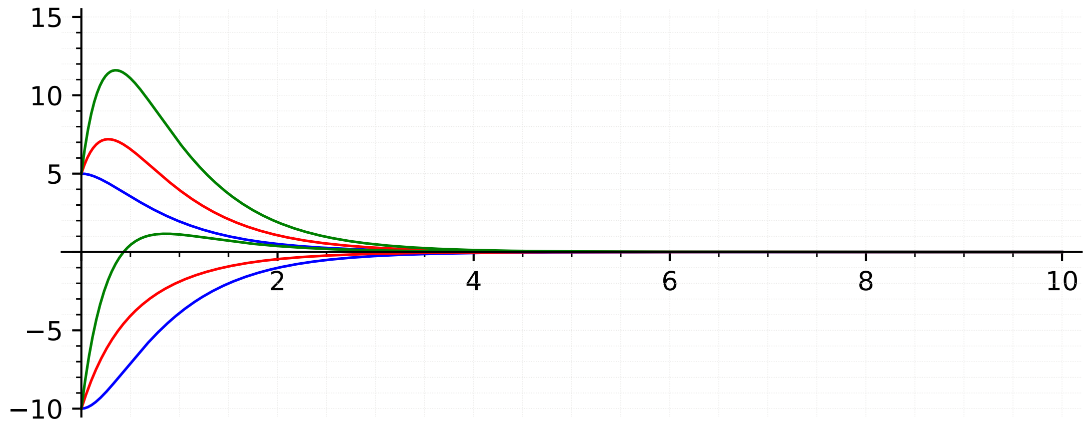


Figure 2.1.: The case of over damping.

Case I: Over-damping

If $c^2 > 4mk$, then λ_1 and λ_2 are **distinct real roots**. In this case the corresponding general solution is:

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}. \tag{2.8}$$

> In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.8) are negative because $\alpha > 0$ and $\beta > 0$ and:

$$\beta^2 = \alpha^2 - k/m < \alpha^2$$

Hence both terms in Eq. (2.8) approach zero as $t \to \infty$. Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Below are the results for some typical initial conditions.

Case II: Critical-Damping

Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III). Occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of Eq. (2.6) is:

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{2.9}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha\, t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.
Fig. 2.2 shows typical forms of Eq. (2.9).

> The graph above looks almost like those in the previous figure.

Case III: Under-Damping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 = 4mk$. Then β in Eq. (2.7) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (> 0).$$

> The asterisk (*) is used to differentiate from ω which is used predominantly in electrical engineering.

The roots of the characteristic equation are now complex conjugates,

$$y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t) = Ce^{-\alpha t}\cos(\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$ This represents **damped oscillations**. Their curve lies between the dashed curves: The roots of the characteristic equation are now complex conjugates. The frequency is $\omega^*/2\pi$ Hz (hertz, cycles/sec). From we see that the smaller $c > 0$ is, the larger is ω^* and the more rapid the oscillations become.

2.2.2 Euler-Cauchy Equations

Has the following form:

$$x^2y'' + axy' + by = 0 \tag{2.10}$$

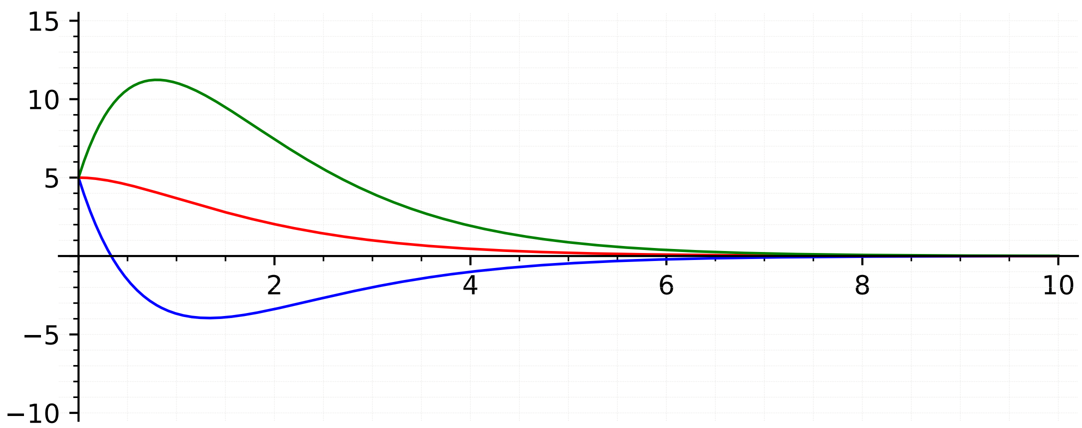


Figure 2.2.: The case of critical damping.

To solve do the following substitutions:

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Which gives:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

$y = x^m$ is a good choice as it produces a common factor x^m.

Simplifying the equation produces the **auxiliary** equation.

$$m^2 + (a-1)m + b = 0. \tag{2.11}$$

$y = x^m$ is a solution of Eq. (2.10) if and only if m is a root of Eq. (2.11).

The roots of Eq. (2.11) are:

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}.$$

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^m \ln x + c_2 x^m$
	(m)	
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta \mathrm{j}$	$\alpha = \mathrm{Re}(m)$
	$m_2 = \alpha - \beta \mathrm{j}$	$\beta = \mathrm{Im}(m)$

Table 2.3.: Possible solutions of the Euler-Cauchy based on the m value.

Complex conjugate roots are of minor practical importance for practical purposes.

Exercise 2.4: General Solution in the Case of Different Real Roots

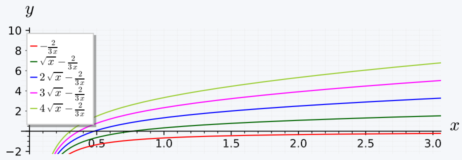
Solve the following ODE:

$$x^2y'' + 1.5xy' - 0.5y = 0$$

Solution

General Solution in the Case of Different Real Roots This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$



Exercise 2.5: General Solution in the Case of a Double Root

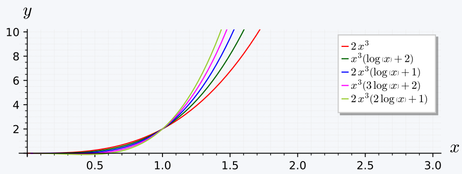
Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

Solution

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$



Exercise 2.6: BVP: Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of the *EulerCauchy equation* $rv'' + 2v' = 0$.

Solution

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

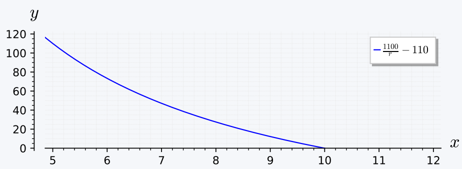
$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$



2.2.3 Non-homogeneous ODEs

They have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.12}$$

where $r(x) \neq 0$. a general solution of Eq. (2.12) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.13}$$

and a **particular solution** of Eq. (2.12). These two new terms **general solution** of Eq. (2.12) and **particular solution** of Eq. (2.12) are defined as follows:

Theory 2.6: General Solution and Particular Solution

A general solution of the nonhomogeneous ODE Eq. (2.12) on an open interval I is a solution of the form:

$$y(x) = y_h(x) + y_p(x) . \tag{2.14}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.13) on I and y_p is any solution of Eq. (2.12) on I containing **no arbitrary constants**. A particular solution of Eq. (2.12) on I is a solution obtained from Eq. (2.14) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.12) or an initial value problem for Eq. (2.12), we have to solve the homogeneous ODE Eq. (2.13) or an initial value problem for and find any solution y_p of Eq. (2.12), so that we obtain a general solution Eq. (2.14) of Eq. (2.12).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.15}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.
Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Theory 2.6: Choice Rules for the Method of Undetermined Coefficients

Basic Rule: If $r(x)$ in Eq. (2.15) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.15).

Modification Rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.15), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule: If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Table 2.4.: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.12) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.12) with $r = r_1 + r_2$. (Verify!)

> The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Exercise 2.7: Application of the Basic Rule (a)

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

Solution

Step 1: General Solution of the Homogeneous ODE

The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

Step 2: Solution of the non-Homogeneous ODE

First try $y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) **must be the same on both sides**. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

The looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0, \quad \text{Then} \quad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Hence:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$$

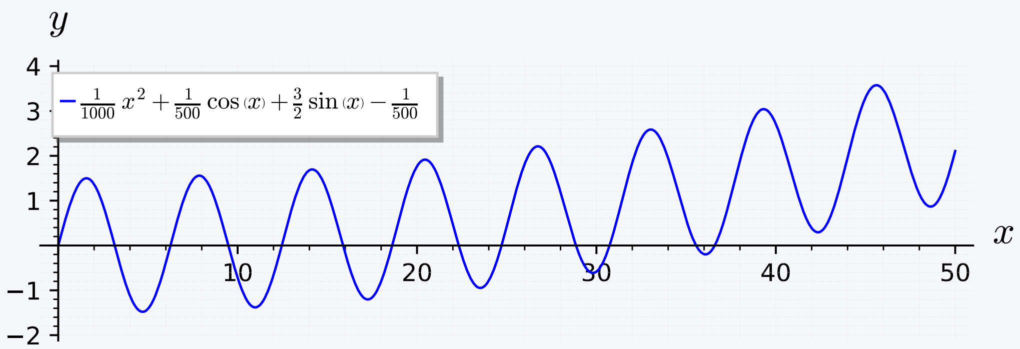
Step 3. Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \quad \text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002 \quad \blacksquare$$



Exercise 2.8: Application of the Modification Rule (b)

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution

Application of the Modification Rule (b)

Step 1.General solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\, e^{-1.5x}$$

Step 2.Solution y_p of the non-homogeneous ODE

The function $e^{-1.5x}$ on the RHS would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2 e^{-1.5x}, \quad \text{then} \quad y_p' = C(2x - 1.5x^2)e^{-1.5x}, y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5 e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2 e^{-1.5x}$$

Step 3. Solution of the initial value problem

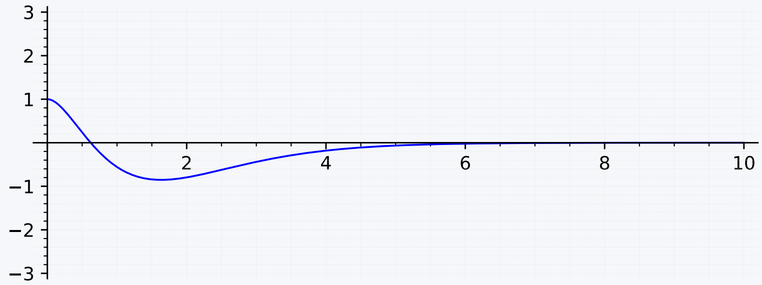
Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2 v)e^{-1.2x} - 10xe^{-1.2x} + 7.5x^2 e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2 e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.



Exercise 2.9: Application of the Sum Rule (c)

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

Solution

Step 1.The General Solution
The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}.$$

Step 2. The Particular Solution
We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1 x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE in (7) gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$
$$\text{therefore}$$
$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE in (7) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Step 3. Solution of the initial value problem

From y, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

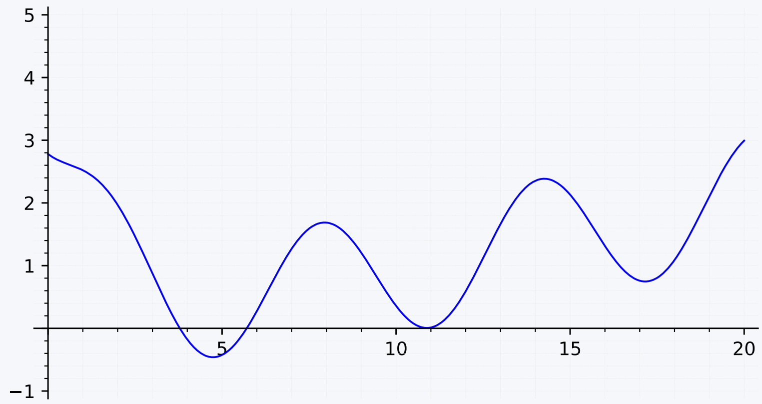


Figure 2.3.: Solution of Application of the Sum Rule (c)

2.2.4 A Study of Forced Oscillations and Resonance

Previously we considered vertical motions of a massspring system (vibration of a mass m on an elastic spring) and modeled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.16}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous massspring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia $my\prime\prime$, the damping force $cy\prime$ (if $c < 0$), and the spring force ky, a restoring force.

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.16) into:

$$my'' + cy' + ky = r(t). \tag{2.17}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.18}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.18) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.16) plus any solution y_p of Eq. (2.18). To find y_p, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.19}$$

By differentiating this function (remember the chain rule) we obtain:

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$
$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_p, $y_p\prime$, $y_p\prime\prime$, into Eq. (2.18) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2) equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.20}$$
$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.21}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $\left(km\omega^2\right)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.22}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.18) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.16) and y_p is given by Eq. (2.19) with coefficients Eq. (2.22).

2.2.5 Solving Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I \, \mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an integro-differential equation. To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_0 \omega cos \omega t. \tag{2.23}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.23) is the sum $I = I_\mathrm{h} + I_\mathrm{p}$, where I_h is a general solution of the homogeneous ODE corresponding to Eq. (2.23) and I_p is a particular solution of Eq. (2.23). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a \cos \omega t + b \sin \omega t, \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t), \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t). \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to E0v cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned} L\omega^2(-a) + R\omega b + a/C &= E_0 \omega \qquad \text{(Cosine terms)} \\ L\omega^2(-b) + R\omega(-a) + b/C &= 0 \qquad \text{(Sine terms)}. \end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.24}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$-Sa + Rb = E_0,$$
$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.25}$$

Equation (2) with coefficients a and b given by Eq. (2.25) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage. Using Eq. (2.25), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.26}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $\left(R^2 + S^2\right)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohms law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.26) and whose frequency is that of the input (i.e., voltage).

Exercise 2.10: Reduction of Order if a Solution Is Known

Find a basis of solutions of the ODE:

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

Solution

Inspection shows that $y_1 = x$ is a solution because $y_1{}' = 1$ and $y_1{}'' = 0$, so that the first term vanishes identically and the second and third terms cancel.
The idea of the method is to substitute

$$\begin{aligned} y &= uy_1 = ux, \\ y' &= u'x + ux' = u'x + u, && \text{(Chain Rule)} \\ y'' &= (u'x + u)\prime = u\prime\prime x + u\prime x\prime + u\prime = u\prime\prime x + 2u\prime. && \text{(Chain Rule)} \end{aligned}$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and xu cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u\prime$, namely:

$$\left(x^2 - x\right) v\prime + (x - 2)\, v = 0$$

Separation of variables and integration gives:

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}\,dx = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx, \qquad \ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}$$

We need no constant of integration because we want to obtain a particular solution.
Taking exponents and integrating again, we obtain:

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \left|\int v\,dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ \ln|x| + 1.\right|$$

Since $y_1 = x$ and $y_2 = x \ln \boldsymbol{p}x + 1$ are **linearly independent**.

> This means their quotient is not constant.

we have obtained a basis of solutions, valid for all positive x. ■

Exercise 2.11: IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution

IVP: Case of Distinct Real Roots **Step 1. General Solution**
The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

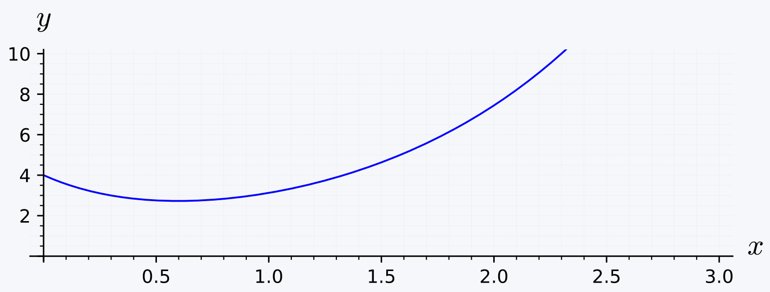
$$y = c_1e^x + c_2e^{-2x}.$$

Step 2. Particular Solution
As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$



Exercise 2.12: IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

Solution

IVP: Case of Real Double Roots The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = \left(\lambda + 2\right)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = \left(c_1 + c_2 x\right) e^{-0.5x}$$

We need its derivative:

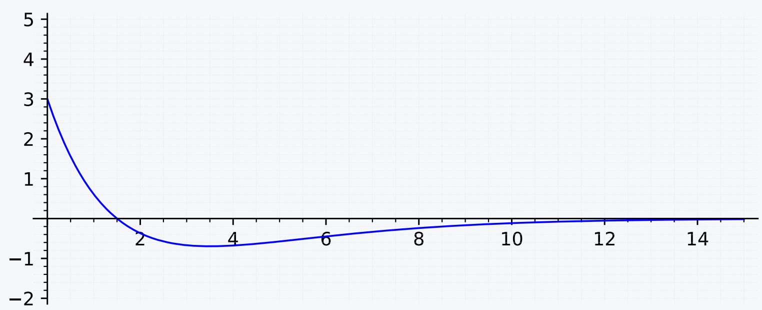
$$y\prime = c_2 e^{-0.5x} - 0.5\left(c_1 + c_2 x\right) e^{-0.5x}$$

From this and the initial conditions we obtain:

$$y(0) = c_1 = 3.0, \quad y\prime(0) = c_2 - 0.5c_1 = 3.5, \quad c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5x}$$



Exercise 2.13: IVP: Case of Complex Roots

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution

IVP: Case of Complex Roots **Step 1. General Solution**
The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3j$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A\cos 3x + B\sin 3x).$$

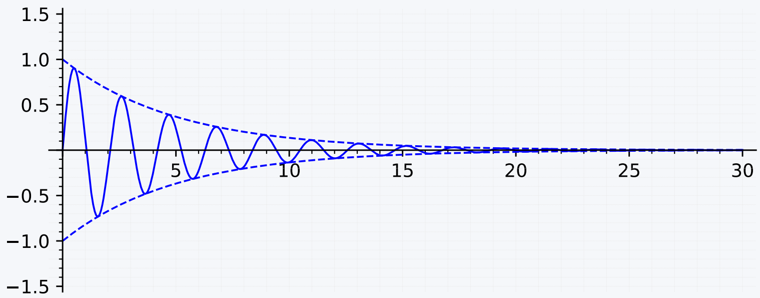
Step 2. Particular Solution
The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x}\sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such damped vibrations have important mechanical and electrical applications.



Exercise 2.14: Harmonic Oscillation of an Undamped MassSpring System

If a massspring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

Solution

Hookes law:

$$F_1 = -ky \tag{2.27}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0.48\,\text{Hz}$$

From Eq. (2.27) and the initial conditions, $y(0) = A = 0.16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$.
Hence the motion is

$$y(t) = 0.16\cos 3t\,\text{m} \quad \blacksquare$$

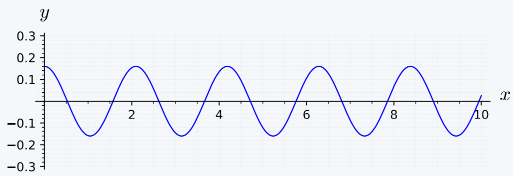


Figure 2.4.: The harmonic oscillation on a string.

Exercise 2.15: The Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped MassSpring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\text{kg}\cdot\text{s}^{-1}$

- $c = 60\,\text{kg} \cdot \text{s}^{-1}$
- $c = 10\,\text{kg} \cdot \text{s}^{-1}$

Solution

The Three Cases of Damped Motion It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I

With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped MassSpring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16\,\text{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9\,t} + 0.18e^{-\,t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II

The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3\,t}, \qquad \text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3\,t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y\prime(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III

The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96j$$

This gives the general solution:

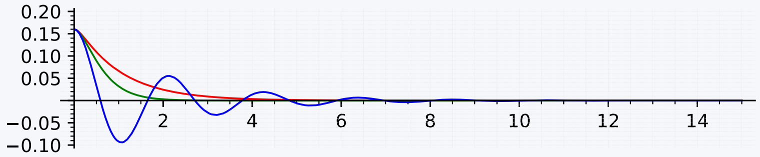
$$y = e^{-0.5\,t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$y\prime = e^{-0.5\,t}\left(-0.5A\cos 2.96t - 0.5B\sin 2.96t - 2.96A\sin 2.96t + 2.96B\cos 2.96t\right)$$

Hence $y\prime(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This give the solution:

$$y = e^{-0.5\,t}(0.16\cos 2.96t + 0.027\sin 2.96t) = 0.162e^{-0.5\,t}\cos(2.96t - 0.17) \quad \blacksquare$$



Exercise 2.16: Studying the RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11$ (Ohms), $L = 0.9$ H (Henry), $C = 0.01$ F (Farad), which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Assume that current and capacitor charge are 0 when $t = 0$.

Solution

Step 1. General solution of the homogeneous ODE

Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t.$$
$$0.1I'' + 11I' + 100I = 110 \cdot 377\cos 377t.$$

Hence the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1e^{-10t} + c_2e^{-100t}.$$

Step 2. Particular solution I_p

We calculate the reactance S = 37.7 - 0.3 = 37.4 and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1e^{-10t} + c_2e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

Step 3. Particular solution satisfying the initial conditions

How to use Q(0) 0? We finally determine c1 and c2 from the in initial conditions I(0) 0 and Q(0) 0. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to Q(0) 0. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

Chapter 3

Higher-Order Ordinary Differential Equations

Table of Contents

3.1 Homogeneous Linear ODEs

Recall from **First-Order ODEs** that an ODE is of n^{th} if the n^{th} derivative $y^{(\mathrm{n})} = d^{\mathrm{n}}y/dx^{n}$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F\left(x, y, y', \cdots, y^{(\mathrm{n})}\right) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(\mathrm{n})} + p_{\mathrm{n}-1}(x)y^{(\mathrm{n\text{-}1})} + \cdots + p_1(x)y' + p_0(x)y = r(x)\,. \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{\mathrm{n}-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

$y^{(\mathrm{n})}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_{\mathrm{n}}(x)y^{(\mathrm{n})}$, divide by $p_{\mathrm{n}}(x)$ to get this form.

An n^{th}-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.
A **solution** of an n$^{\text{th}}$-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to n$^{\text{th}}$-order homogeneous linear ODEs as following theorems.

> **Theory** 3.0: Fundamental Theorem for the Homogeneous Linear ODE (2)
>
> For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

> This does not hold for a nonhomogeneous or non-linear ODE.

> **Theory** 3.0: General Solution, Basis, Particular Solution
>
> A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:
>
> $$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \quad \text{arbitrary}) \tag{3.3}$$
>
> where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.
>
> That is, these solutions are linearly independent on I, as defined below.
> A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

> **Theory** 3.0: Linear Independence and Dependence
>
> Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:
>
> $$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on} I \tag{3.4}$$
>
> implies that all $k_1, \cdots, k_n$ are zero.
>
> These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \dots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n-1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (3.4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Exercise 3.1: Linear Dependence

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

Solution

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval
■

Exercise 3.2: General Solution

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{where} \qquad y^{\text{iv}} = \frac{d^4 y}{dx^4}$$

Solution

Similar to Chapter 2 we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Exercise 3.3: Initial Value Problem for a Third-Order EulerCauchy Equation

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \qquad y'(1) = 1, \qquad y''(1) = -4.$$

Solution

General solution

As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m-1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular solution

The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned} &\text{(a)} \quad y(1) = c_1 + c_2 + c_3 = 2 \\ &\text{(b)} \quad y'(1) = c_1 + 2c_2 + 3c_3 = 1 \\ &\text{(c)} \quad y'(1) = 2c_2 + 6c_3 = -4. \end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad ■$$

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

> Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n\text{-}1)} + \cdots + a_1y' + a_0y = 0$$

where $y^{(n)} = d^ny/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \ \cdots \ + a_1\lambda + a_0y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \qquad \cdots, \qquad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1e^{\lambda_1 x} + \ \cdots \ + c_ne^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Exercise 3.4: Distinct Real Roots

Solve the following ODE:

$$y''' - 2y'' - y' + 2y = 0$$

Solution

The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1e^{-x} + c_2e^{x} + c_3e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x}\cos\omega x, \qquad y_2 = e^{\gamma x}\sin\omega x.$$

Exercise 3.5: Simple Complex Roots

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0, \qquad y(0) = 4, \qquad y'(0) = 11, \qquad y''(0) = -299$$

Solution

The characteristic equation is:

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10j$.

Therefore, a general solution and its derivatives (obtained by differentiation) are:

$$y = c_1 e^x + A\cos 10x + B\sin 10x,$$
$$y' = c_1 e^x - 10A\sin 10x + 10B\cos 10x,$$
$$y'' = c_1 e^x - 100A\cos 10x - 100B\sin 10x.$$

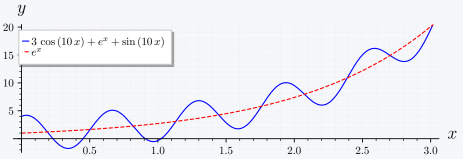
From this and the initial conditions we obtain, by setting $x = 0$,

$$\text{(a) } c_1 + A = 4, \qquad \text{(b) } c_1 + 108 = 11, \qquad \text{(c) } c_1 - 1004 = -299$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^x + 3\cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.



Multiple Real Roots

If a real double root occurs ($\lambda_1 = \lambda_2$) then $y_1 = y_2$ in Eq. (3.3), and we take y_1 and xy_1 as corresponding linearly independent solutions.
More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2 e^{\lambda x}, \quad \cdots, \quad x^{m-1} e^{\lambda x}$$

Exercise 3.6: Real Double and Triple Roots

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y^{\text{iv}} - y'' = 0$$

Solution

The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + \left(c_3 + c_4 x + c_5 x^2\right) e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda} = \gamma - i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x}\cos\omega x, \quad e^{\gamma x}\sin\omega x, \quad xe^{\gamma x}\cos\omega x, \quad xe^{\gamma x}\sin\omega x$$

The first two of these result from $e^{\lambda x}$ and $\overline{e}^{\lambda x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\overline{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}.$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\gamma x} \cos \omega x$, $x^2 e^{\gamma x} \sin \omega x$, and so on.

3.1.3 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of nth order. We write them in standard form:

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} +, \cdots, + p_1(x) y' + p_0(x) y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a **general solution** of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0 \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Thus Eq. (3.7) has no singular solutions. An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

The ideas of proof are the same as those for $n = 2$.

Exercise 3.7: IVP - Modification Rule

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \qquad y'(0) = -3, \qquad y''(0) = -47$$

Solution

Step 1

The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2) e^{-x} \end{aligned}$$

Step 2

If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and Cx^6e^{-x}. The Modification Rule calls for

$$y_p = Cx^3e^{-x}$$

Then

$$y_p{\prime} = C\left(3x^2 - x^3\right)e^{-x},$$
$$y_p{\prime\prime} = C\left(6x - 6x^2 + x^3\right)e^{-x},$$
$$y_p{\prime\prime\prime} = C\left(6 - 18x + 9x^2 - x^3\right)e^{-x}.$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C\left(6 - 18x + 9x^2 - x^3\right) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2e^{-x}$.

Step 3

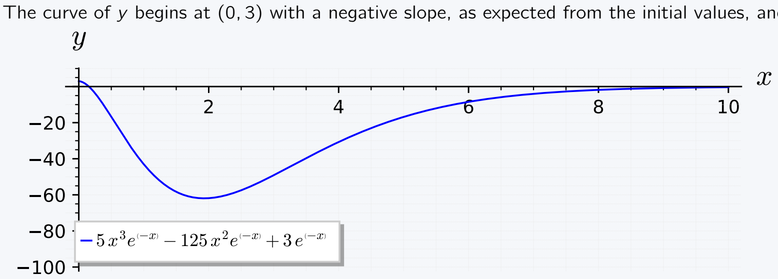
We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$\begin{aligned}
y = y_h + y_p &= (c_1 + c_2 + c_3x^2)e^{-x} + 5x^3e^{-x}, & y(0) &= c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, & y'(0) &= -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, & y''(0) &= 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2e^{-x}$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.



3.1.4 Application: Modelling an Elastic Beam

Whereas second-order ODEs have various applications, of which we have discussed some of the more important ones (i.e., RLC Circuit, Mass-Damper system), higher order ODEs have much fewer engineering applications.

An important fourth-order ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge.

A related application of vibration of beams does not fit in here since it leads to PDEs.

Problem Description

Consider a beam B of length L and constant (e.g., **rectangular**) cross section and homogeneous elastic material (e.g., **level**).
We assume under its own weight the beam is bent so little that it is certainly straight. If we apply a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.
Its axis is curved into the so-called **elastic curve** (or **deflection curve**).
It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ (determining the tangent direction of C) are small. Then, by calculus:

$$k = y''/(1 + y''x)^{1/2} \approx y''$$

Therefore:

$$M(x) = EIy''(x)$$

EI is the constant of proportionality. E Young's modulus of elasticity of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.
Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy^{\text{iv}} = f(x)$$

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows:
Simply supported

$$y = y'' = 0 \text{at} x = 0 \text{and} L$$

$$y = y' = 0 \text{at} x = 0 \text{and} L$$

$$\text{(C)Clamped at} x = 0, \text{free at} x = L$$

$$y(0) = y'(0) = 0, y''(L) = y'''(L) = 0.$$

The boundary condition $y = 0$ means no displacement at that point, $y'' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Solution Derivation

Let us apply this to the uniformly loaded simply supported beam. The load is $f(x) = f_0 = \text{const.}$ Then (8) is

$$y^{\text{iv}} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Two integrations give

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

$y''(0) = 0$ gives $c_2 = 0$. Then $y''(L) = L\left(\frac{1}{2}kL + c_1\right) = 0$, $c_1 = -kL/2$ (since $L \neq 0$). Hence

$$y'' = \frac{k}{2}(x^2 - Lx).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution

$$y = \frac{f_0}{24EI}(x^4 - 2Lx^3 + L^3 x).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L - x)$.

Verify this by setting $x = u + L/2$ and show that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0(x = L/2)$ is:

$$\frac{5f_0 L^4}{(16 \cdot 24EI)}$$

Recall that the positive direction points downward.

Part I.

Linear Algebra & Vector Calculus

Chapter 4

Vector Calculus

Table of Contents

4.1 Vector Algebra

Walking 5 kilometers north and then 12 kilometers east, you will have gone a total of 17 kilometers, but you're not 13 kilometers from where you set out, which is only 7. We need a set of mathematics principles to describe quantities like this, which evidently do not add in the ordinary way.

The reason they don't, is **displacements** have *direction* as well as *magnitude*, and it is essential to take both into account when you combine them. Such objects are called **vectors**.

> Examples include: velocity, acceleration, force, momentum ...

By contrast, quantities that have magnitude but no direction are called **scalars**.

> Examples include: mass, charge, density, temperature, ..

We shall use **boldface** ($\boldsymbol{A}$, $\boldsymbol{B}$, and so on) for vectors and ordinary type for scalars. The magnitude of a vector $\boldsymbol{A}$ is written $|\boldsymbol{A}|$ or, more simply, A. In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowed indicates its direction.

Minus $\boldsymbol{A}$ ($-\boldsymbol{A}$) is a vector with the same magnitude as $\boldsymbol{A}$ but of opposite direction.

> Vectors have magnitude and direction but *not location*

We define four (4) vector operations: addition and three kinds of multiplication.

(i) Addition of two vectors: Place the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$. The sum, $\boldsymbol{A}+\boldsymbol{B}$, is the vector from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$. Addition is *commutative*:

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A},$$

5 kilometers east followed by 12 kilometers north gets you to the same place as 12 kilometers north followed by 5 kilometers east. Addition is also *associative*:

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}).$$

To subtract a vector, add its opposite:

$$\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B}).$$

(ii) Multiplication by a scalar: Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged. This means if a is negative, the direction is reversed. Scalar multiplication is *distributive*:

$$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B}.$$

(iii) Dot product of two vectors: The dot product of two (2) vectors is defined by

$$\boldsymbol{A}\cdot\boldsymbol{B}\equiv AB\cos\theta \tag{4.1}$$

where θ is the angle they form when placed tail-to-tail.

$\boldsymbol{A}\cdot\boldsymbol{B}$ is itself a *scalar*[1] hence the alternative name **scalar product**

The dot product is *commutative*,

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A},$$

and *distributive*,

$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}, \tag{4.2}$$

Geometrically, $\boldsymbol{A}\cdot\boldsymbol{B}$ is the product of A times the projection of $\boldsymbol{B}$ along $\boldsymbol{A}$ (or the product of B times the projection of $\boldsymbol{A}$ along $\boldsymbol{B}$). If the two vectors are parallel, then $\boldsymbol{A}\cdot\boldsymbol{B}=AB$. In particular, for any vector $\boldsymbol{A}$,

$$\boldsymbol{A}\cdot\boldsymbol{A}=A^2 \tag{4.3}$$

Theory 4.0: Orthogonality Criterion

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

Exercise 4.1: Dot Product of Two Vectors

Find the inner product and the lengths of $\boldsymbol{a}=[1,2,0]$ and $\boldsymbol{b}=[3,-2,1]$ as well as the angle between these vectors;

Solution

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}&=1\cdot3+2\cdot(-2)+0\cdot1=-1\\ |\boldsymbol{a}|&=\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}=\sqrt{5}\\ |\boldsymbol{b}|&=\sqrt{\boldsymbol{b}\cdot\boldsymbol{b}}=\sqrt{14}\end{aligned}$$

$$\begin{aligned}\theta&=\arccos\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\\ &=\arccos(-0.11952)=1.69061\\ &=96.865^\circ \quad \blacksquare\end{aligned}$$

(iv) Cross product of two vectors: The cross product of two vectors is defined by

$$\boldsymbol{A} \times \boldsymbol{B} \equiv AB \sin\theta \, \hat{\boldsymbol{n}} \tag{4.4}$$

where $\hat{\boldsymbol{n}}$ is a **unit vector**[2] pointing perpendicular to the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$. Of course, there are *two* directions perpendicular to any plane: **in** and **out**.

[2] It is a vector of magnitude 1.

Exercise 4.2: Calculating Cross-Product

Find $\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b}$ with $\boldsymbol{a} = [1, 1, 0]$ and $\boldsymbol{b} = [3, 0, 0]$.

Solution

$$\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{x} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \hat{y} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \hat{z} = -3\boldsymbol{k} = [0, 0, -3] \quad \blacksquare$$

The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{\boldsymbol{n}}$.

$\boldsymbol{A} \times \boldsymbol{B}$ is itself a *vector* and it is also known as **vector product**.

The cross product is *distributive*,

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + \boldsymbol{A} \times \boldsymbol{C} \tag{4.5}$$

but **NOT** commutative:

$$(\boldsymbol{B} \times \boldsymbol{A}) = -(\boldsymbol{A} \times \boldsymbol{B}) \tag{4.6}$$

If two (2) vectors are parallel, their cross product is zero. In particular,

$$\boldsymbol{A} \times \boldsymbol{A} = 0,$$

for any vector $\boldsymbol{A}$.

4.1.1 Vector Component Forms

In the previous section, we defined the four (4) vector operations in abstract form, without reference to any particular coordinate system.

In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector **components**. Let $\hat{x}$, $\hat{y}$, $\hat{z}$ be unit vectors parallel to the x, y, and z axes, respectively.

An arbitrary vector $\boldsymbol{A}$ can be expanded in terms of these **basis vectors**:

$$\boldsymbol{A} = A_x \, \hat{x} + A_y \, \hat{y} + A_z \, \hat{z}$$

The symbols A_x, A_y, A_z, are the components of $\boldsymbol{A}$. In geometrical terms they are the **projections** of $\boldsymbol{A}$ along the three (3) coordinate axes (i.e., $A_x = \boldsymbol{A} \cdot \hat{x}$, $A_y = \boldsymbol{A} \cdot \hat{y}$, $A_z = \boldsymbol{A} \cdot \hat{z}$). We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned} \boldsymbol{A} + \boldsymbol{B} &= \left(A_x \, \hat{x} + A_y \, \hat{y} + A_z \, \hat{z}\right) + \left(B_x \, \hat{x} + B_y \, \hat{y} + B_z \, \hat{z}\right) \\ &= (A_x + B_x) \, \hat{x} + \left(A_y + B_y\right) \, \hat{y} + (A_z + B_z) \, \hat{z} \end{aligned}$$

The operation rules can be summarised as follows:

(i) To add vectors, add like components.

$$a\boldsymbol{A} = (aA_x)\ \hat{x} + \left(aA_y\right)\ \hat{y} + (aA_z)\ \hat{z}$$

(ii) To multiply by a scalar, multiply each component.

As $\hat{x}$, $\hat{y}$, $\hat{z}$ are mutually perpendicular unit vectors, the following properties are valid:

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1 \qquad \hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z} = 0$$

Accordingly,

$$\boldsymbol{A}\cdot\boldsymbol{B} = \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)\cdot\left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right)$$

(iii) To calculate the dot product, multiply like components, and add. In particular,

$$\boldsymbol{A}\cdot\boldsymbol{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Similarly,

$$\begin{aligned}\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0,\\ \hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z},\\ \hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x},\\ \hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}.\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)\times\left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right)\\ &= \left(A_yB_z - A_zB_y\right)\ \hat{x} + (A_zB_x - A_xB_z)\ \hat{y} + \left(A_xB_y - A_yB_x\right)\ \hat{z}.\end{aligned}$$

This expression can be written more neatly as a **determinant**:

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

(iv) To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\boldsymbol{A}$ (in component form), and whose third row is $\boldsymbol{B}$.

Exercise 4.3: Vector Addition. Multiplication by Scalars

With respect to a given coordinate system, let

$$\boldsymbol{a} = [4, 0, 1] \qquad \text{and} \qquad \boldsymbol{b} = [2, -5, \tfrac{1}{2}].$$

Calculate $2\boldsymbol{a} - 2\boldsymbol{b}$.

Solution

Then $-\boldsymbol{a} = [-4, 0, -1]$, $7\boldsymbol{a} = [28, 0, 7]$, $\boldsymbol{a} + \boldsymbol{b} = [6, -5, \frac{4}{2}]$, and

$$2\,(\boldsymbol{a} - \boldsymbol{b}) = 2[2, 5, \tfrac{2}{2}] = [4, 10, \tfrac{4}{2}] = 2\boldsymbol{a} - 2\boldsymbol{b}.$$

4.1.2 Triple Products

As the cross product of two (2) vectors is itself a vector, it can be dotted or crossed with a 3rd vector to form a *triple* product.

(i) Scalar triple product: Evidently,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}),$$

for they all correspond to the same value. Note that "alphabetical" order is preserved. The "nonalphabetical" triple products,

$$\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{C}) = \boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{A}),$$

have the opposite sign. In component form,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Note that the dot and cross can be interchanged:

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C},$$

however, the placement of the parentheses is critical:

> $(\boldsymbol{A}\cdot\boldsymbol{B})\times\boldsymbol{C}$ is a meaningless expression. You can't make a cross product from a scalar and a vector.

(ii) Vector triple product: $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$ The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}).$$

Notice that

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = -\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B}) = -\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C}) + \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})$$

is an entirely **different vector** (*cross-products are not associative*). All *higher* vector products can be similarly reduced, often by repeated application , so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) &= (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}),\\ \boldsymbol{A}\times\left[\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D})\right] &= \boldsymbol{B}\left[\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{D})\right] - (\boldsymbol{A}\cdot\boldsymbol{B})(\boldsymbol{C}\times\boldsymbol{D}).\end{aligned}$$

4.1.3 Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin $(\mathcal{O})$ is called the **position vector**:

$$\boldsymbol{r} \equiv x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$$

Throughout this course, $\boldsymbol{r}$ will be used to measure **distance**. Its magnitude:

$$r = \sqrt{x^2 + y^2 + z^2}$$

is the distance from the origin, and

$$\hat{r} = \frac{\boldsymbol{r}}{r} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$\mathrm{d}\boldsymbol{l} = (\mathrm{d}x)\ \hat{x} + (\mathrm{d}y)\ \hat{y} + (\mathrm{d}z)\ \hat{z}.$$

Exercise 4.4: Components and Length of a Vector

The vector $\boldsymbol{a}$ has the initial point P: (4, 0, 2) and terminal point Q: (6, −1, 2). Find its mangnitude:

Solution

$$a_1 = 6 - 4 = 2, \qquad a_2 = -1 - 0 = -1, \qquad a_1 = 2 - 2 = 0.$$

Hence $\boldsymbol{a} = [2, -1, 0]$. The length

$$|a| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

If we choose (−1, 5, 8) as the initial point of $\boldsymbol{a}$, the corresponding terminal point is (1, 4, 8).
If we choose the origin (0, 0, 0) as the initial point of $\boldsymbol{a}$, the corresponding terminal point is (2, −1, 0); its coordinates equal the components of $\boldsymbol{a}$. This suggests that we can determine each point in space with a vector. ■

4.2 Differential Calculus

4.2.1 Ordinary Derivatives

Assume a function of one variable: $f(x)$. Therefore, what does the derivative, df/dx, do. It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, $\mathrm{d}x$:

$$\mathrm{d}f = \left(\frac{df}{dt}\right) \mathrm{d}x$$

If we increment x by an infinitesimal amount dx, then f changes by an amount df.

the derivative is the proportionality factor

Geometrically, the derivative df/dt is the *slope* of the graph of f versus x.

4.2.2 Gradient

Assume a function of three (3) variables, for example, the temperature $T(x, y, z)$ in the lecture room. Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot. We want to generalise the notion of "derivative" to

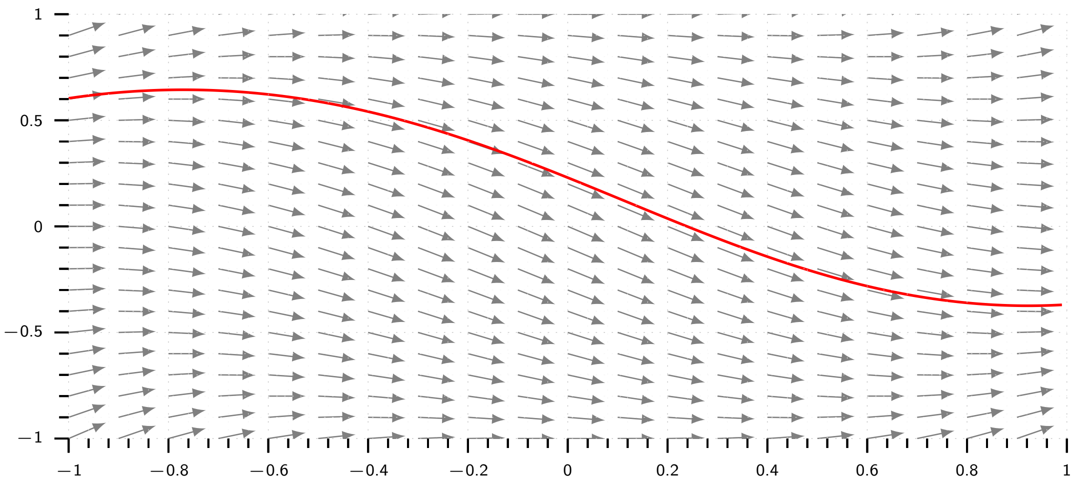


Figure 4.1.

functions like T, which depend not on *one* but on *three* variables.

A derivative tells us **how fast the function varies**, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move:

1. If we go straight up, then the temperature will probably increase fairly rapidly,

2. If we move horizontally, it may not change much at all.

In fact, the question "How fast does T vary?" has an infinite number of answers, one for each direction we might choose to explore.
Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states:

$$\mathrm{d}T = \left(\frac{dT}{dx}\right) \mathrm{d}x + \left(\frac{dT}{dy}\right) \mathrm{d}y + \left(\frac{dT}{dz}\right) \mathrm{d}z$$

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. We can write the aforementioned equation as a dot product:

$$\mathrm{d}T = \left(\frac{dT}{dx}\hat{\mathbf{x}} + \frac{dT}{dy}\hat{\mathbf{y}} + \frac{dT}{dz}\hat{\mathbf{z}}\right) \cdot \left((\mathrm{d}x)\ \hat{\mathbf{x}} + (\mathrm{d}y)\ \hat{\mathbf{y}} + (\mathrm{d}z)\ \hat{\mathbf{z}}\right) = (\nabla T) \cdot (\mathrm{d}\boldsymbol{l}),$$

where

$$\nabla T \equiv \frac{dT}{dx}\hat{\mathbf{x}} + \frac{dT}{dy}\hat{\mathbf{y}} + \frac{dT}{dz}\hat{\mathbf{z}}$$

is the **gradient** of T. Note that ∇T is a **vector quantity**, with three (3) components.

This is the generalized derivative we have been looking for.

Exercise 4.5: Finding Vector Components

Find the components of the vector $\boldsymbol{v}$ with given initial point P and terminal point Q. Find $|\boldsymbol{v}|$ and unit vector $\hat{v}$.

$$
\begin{aligned}
&P(3,\,2,\,0)\,, && Q(5,\,-2,\,2)\,, && P(1,\,1,\,1)\,, && Q(-4,\,-4,\,-4)\\
&P(1,\,0,\,1.2)\,, && Q(0,\,0,\,6.2)\,, && P(2,\,-2,\,0)\,, && Q(0,\,4,\,6)\\
&P(4,\,3,\,2)\,, && Q(-4,\,-3,\,2)\,, && P(0,\,0,\,0)\,, && Q(6,\,8,\,10)
\end{aligned}
$$

Given the components of a vector $\boldsymbol{v} = \left[v_x,\, v_y,\, v_z\right]$ and a particular initial point P, find the corresponding terminal point Q and the length of $\boldsymbol{v}$ (i.e., $|\boldsymbol{v}|$).

$$
\begin{aligned}
&\boldsymbol{v} = [3,\,-1,\,0]\,; && P(4,\,6,\,0)\,, && \boldsymbol{v} = [8,\,4,\,2]\,; && P(-8,\,-4,\,-2)\,,\\
&\boldsymbol{v} = [0.25,\,2,\,0.75]\,; && P\{\cdot\}\,0,\,-0.5,\,0, && \boldsymbol{v} = [3,\,2,\,6]\,; && P(4,\,6,\,0)\,,\\
&\boldsymbol{v} = [4,\,2,\,-2]\,; && P(4,\,6,\,0)\,, && \boldsymbol{v} = [3,\,-3,\,3]\,; && P(4,\,6,\,0)\,,
\end{aligned}
$$

Solution

The solution is as follows:

$$
\begin{aligned}
\boldsymbol{v} &= (5-3)\ \hat{x} + (-2-2)\ \hat{y} + (2-0)\ \hat{z} = (2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\ \hat{x} + \left(-\frac{2}{\sqrt{6}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{6}}\right)\ \hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (-4-1)\ \hat{x} + (-4-1)\ \hat{y} + (-4-1)\ \hat{z} = (-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\ \hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (0-1)\ \hat{x} + (0-0)\ \hat{y} + (6.2-1.2)\ \hat{z} = (-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\ \hat{x} + (0)\ \hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\ \hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (0-2)\ \hat{x} + \Big(4-(-2)\Big)\ \hat{y} + (6-0)\ \hat{z} = (-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\ \hat{x} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{y} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (-4-4)\ \hat{x} + (-3-3)\ \hat{y} + (2-2)\ \hat{z} = (-8)\ \hat{x} + (-6)\ \hat{y} + (0)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\ \hat{x} + (-8)\ \hat{y} + (0)\ \hat{z}}{10} = \left(-\frac{3}{5}\right)\ \hat{x} + \left(-\frac{4}{5}\right)\ \hat{y} + (0)\ \hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},\\
|\boldsymbol{v}| &= \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare
\end{aligned}
$$

Previously we have defined $\boldsymbol{v} = Q - P$. Here we have $\boldsymbol{v}$ and P. To calculate Q we only need to add individual components of the vector with the initial point P.

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = \Big(8 + (-8)\Big)\ \hat{x} + \Big(4 + (-4)\Big)\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (0.25+0)\ \hat{x} + \Big(2 + (-0.5)\Big)\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare$$

Exercise 4.6: Vector Addition and Scalar Multiplication

- Let $\boldsymbol{a} = [2,\ 1,\ 0]$, $\boldsymbol{b} = [-4,\ 2,\ 5]$ and $\boldsymbol{c} = [0,\ 0,\ 3]$. Calculate the following vector operations:

$$\begin{array}{lll} 2\boldsymbol{a}, & -\boldsymbol{a}, & -1/2\boldsymbol{a}, \\ 5\left(\boldsymbol{a}-\boldsymbol{c}\right), & 5\boldsymbol{a}-5\boldsymbol{c}, & (3\boldsymbol{a}-5\boldsymbol{b})+2\boldsymbol{c}, \\ 3\boldsymbol{a}+\left(-5\boldsymbol{b}+2\boldsymbol{c}\right), & \boldsymbol{a}+2\boldsymbol{b}, & 2\boldsymbol{b}+\boldsymbol{a}. \end{array}$$

  reset

- Find the dot product (i.e., $\boldsymbol{a}\cdot\boldsymbol{b}$) on the lengths of $\boldsymbol{a} = [1,\ 2,\ 0]$ and $\boldsymbol{b} = [3,\ -2,\ 1]$ as well as the angle (θ) between vectors.

- Let $\boldsymbol{a} = [2,\ 1,\ 4]$, $\boldsymbol{b} = [-4,\ 0,\ 3]$ and $\boldsymbol{c} = [3,\ -2,\ 1]$. Find the following descriptions.

$$\begin{array}{ll} |\boldsymbol{a}|\ ,|\boldsymbol{b}|\ ,|\boldsymbol{c}|\ , & \boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})\ ,\ \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}, \\ \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{c}\cdot\boldsymbol{a}, & 4\boldsymbol{a}\cdot 3\boldsymbol{c},\ 12\boldsymbol{a}\cdot\boldsymbol{c}, \\ |\boldsymbol{b}+\boldsymbol{c}|\ ,|\boldsymbol{b}|\ +|\boldsymbol{c}|\ , & \boldsymbol{a}\cdot\boldsymbol{c},|\boldsymbol{a}||\boldsymbol{c}|\ . \end{array}$$

  reset

- Let $\boldsymbol{a} = [1,\ 1,\ 1]$, $\boldsymbol{b} = [2,\ 3,\ 1]$ and $\boldsymbol{c} = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(\boldsymbol{a}-\boldsymbol{c})\ ,\ \text{and},\ (\boldsymbol{b}-\boldsymbol{c})\ , \qquad (\boldsymbol{a})\ ,\ \text{and},\ (\boldsymbol{b}-\boldsymbol{c})\ .$$

- Find the vector product $\boldsymbol{a}\times\boldsymbol{b}$ of $\boldsymbol{a} = [1,\ 1,\ 0]$ and $\boldsymbol{b} = [3,\ 0,\ 0]$.

- Let $\boldsymbol{a} = [1, 2, 0]$, $\boldsymbol{b} = [3,\ -4, 0]$, $\boldsymbol{c} = [3, 5, 2]$, $\boldsymbol{d} = [6, 2, 0]$. Calculate the cross product of:

$$\begin{array}{ll} \boldsymbol{a}\times\boldsymbol{b}, \quad \boldsymbol{b}\times\boldsymbol{a}, & \boldsymbol{a}\times\boldsymbol{c}, \quad |\boldsymbol{a}\times\boldsymbol{c}|\ , \quad \boldsymbol{a}\cdot\boldsymbol{c}, \\ (\boldsymbol{c}+\boldsymbol{d})\times\boldsymbol{d}, \quad \boldsymbol{c}\times\boldsymbol{d}, & \boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{b}, \\ (\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{b}+\boldsymbol{a})\ , & (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}, \quad \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}). \end{array}$$

4.2.3 The Del Operator

The gradient has the formal appearance of a vector, ∇, multiplying a scalar T:

$$\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) T$$

The term in parentheses is called **del** operator:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{y}\frac{\partial}{\partial z}$$

Del is **NOT** a vector, in the usual sense. It doesn't mean much until we provide it with a function to act upon. Furthermore, it does not "multiply" T; rather, it is an instruction to differentiate what follows. To be precise, then, we say that ∇ is a vector operator that acts upon T, not a vector that multiplies T.

With this qualification, though, $\boldsymbol{V}$ mimics the behaviour of an ordinary vector in virtually every way; almost anything that can be done with other vectors can also be done with ∇

Now, an ordinary vector $\boldsymbol{A}$ can multiply in three (3) ways:

1. By a scalar a: $\boldsymbol{A}a$;
2. By a vector $\boldsymbol{B}$, via the dot product: $\boldsymbol{A} \cdot \boldsymbol{B}$;
3. By a vector $\boldsymbol{B}$ via the cross product: $\boldsymbol{A} \times \boldsymbol{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\boldsymbol{v}$, via the dot product: $\nabla \cdot \boldsymbol{v}$ (divergence)
3. On a vector function $\boldsymbol{v}$, via the cross product: $\nabla \times \boldsymbol{v}$ (curl).

It is time to examine the other two vector derivatives: divergence and curl.

4.2.4 Divergence

From the definition of ∇ we construct the divergence:

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \left(\left(\frac{\partial}{\partial x} \right) \hat{x} + \left(\frac{\partial}{\partial y} \right) \hat{y} + \left(\frac{\partial}{\partial z} \right) \hat{z} \right) \cdot \left(\left(v_x \right) \hat{x} + \left(v_y \right) \hat{y} + \left(v_z \right) \hat{z} \right) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \end{aligned}$$

Observe that the divergence of a vector function $\boldsymbol{v}$ is itself a scalar $\nabla \cdot \boldsymbol{v}$.

4.2.5 Curl

From the definition of ∇ we construct the curl:

Exercise 4.7: Curl Example

Find the curl ($\nabla \times$) of the following functions.

$$\boldsymbol{v} = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \qquad \boldsymbol{v} = \left(y^n\right)\ \hat{x} + \left(z^n\right)\ \hat{y} + \left(x^n\right)\ \hat{z},$$

$$\boldsymbol{v} = (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \qquad \boldsymbol{v} = \left(x^2 - z\right)\ \hat{x} + \left(xe^z\right)\ \hat{y} + (xy)\ \hat{z}.$$

Solution

The curl ($\nabla \times$) of the functions are as follows:

$$\begin{aligned}
f(x, y, z) &= (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \\
\nabla \times f &= (0)\ \hat{x} + (0)\ \hat{y} + (-1 + 4x)\ \hat{z}. \\
f(x, y, z) &= \left(y^n\right)\ \hat{x} + \left(z^n\right)\ \hat{y} + \left(x^n\right)\ \hat{z}, \\
\nabla \times f &= \left(-nz^{n-1}\right)\ \hat{x} + \left(-nx^{n-1}\right)\ \hat{y} + \left(-ny^{n-1}\right)\ \hat{z}. \\
f(x, y, z) &= (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \\
\nabla \times f &= (\sin z)\ \hat{x} + \left(\sec^2 x\right)\ \hat{y} + (-\cos y)\ \hat{z}. \\
f(x, y, z) &= \left(x^2 - z\right)\ \hat{x} + \left(xe^z\right)\ \hat{y} + (xy)\ \hat{z}, \\
\nabla \times f &= \left(x - e^z x\right)\ \hat{x} + (-1 - y)\ \hat{y} + \left(e^z\right)\ \hat{z}.
\end{aligned}$$

4.2.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of rules, such as the sum rule:

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k\frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\nabla(f + g) = \nabla f + \nabla g, \qquad \nabla \cdot (\boldsymbol{A} + \boldsymbol{B}) = (\nabla \cdot \boldsymbol{A}) + (\nabla \cdot \boldsymbol{B}),$$

$$\nabla \times (\boldsymbol{A} + \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) + (\nabla \times \boldsymbol{B}),$$

and

$$\nabla(kf) = k\nabla f, \qquad \nabla \cdot (k\boldsymbol{A}) = k(\nabla \cdot \boldsymbol{A}), \qquad \nabla \times (k\boldsymbol{A}) = k(\nabla \times \boldsymbol{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

$$fg \quad \text{(product of two scalar functions)}, \boldsymbol{A} \cdot \boldsymbol{B} \quad \text{(dot product of two vector functions)},$$

and two ways to make a vector:

$$f\boldsymbol{A} \quad \text{(scalar times vector)},$$

$$\boldsymbol{A} \times \boldsymbol{B} \quad \text{(cross product of two vectors)}.$$

Accordingly, there are six product rules, two for gradients:
(i)

$$\boldsymbol{V}(fg) = fvVg + gvVf,$$

(ii) $\boldsymbol{V}(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\boldsymbol{V} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{V} \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{V})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{V})\boldsymbol{A}$, two for divergences:
(iii) $\boldsymbol{V} \cdot (f\boldsymbol{A}) = f(\boldsymbol{V} \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\boldsymbol{V}f)$,
(iv) $\boldsymbol{V} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{V} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{V} \times \boldsymbol{B})$,
and two for curls:
(v)

$$\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f),$$

(vi)

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}).$$

You will be using these product rules so frequently that I have put them inside the front cover for easy reference. The proofs come straight from the product rule for ordinary derivatives. For instance,

$$\nabla \cdot (f\boldsymbol{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right)$$

$$= (\nabla f) \cdot \boldsymbol{A} + f(\nabla \cdot \boldsymbol{A}).$$

It is also possible to formulate three quotient rules:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},$$

$$\nabla \cdot \left(\frac{\boldsymbol{A}}{g}\right) = \frac{g(\nabla \cdot \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla g)}{g^2},$$

$$\nabla \times \left(\frac{\boldsymbol{A}}{g}\right) = \frac{g(\nabla \times \boldsymbol{A}) + \boldsymbol{A} \times (\nabla g)}{g^2}.$$

However, since these can be obtained quickly from the corresponding product rules, there is no point in listing them separately.

4.2.7 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\nabla \cdot \boldsymbol{v}$ by applying ∇ twice, we can construct five (5) types of 2[nd] derivatives.

The gradient ∇T is a vector, so we can take the divergence and curl of it:

1. Divergence of gradient: $\nabla \cdot (\nabla T)$.

2. Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\boldsymbol{V} \cdot \boldsymbol{v}$ is a scalar, therefore all we can do is take its gradient:

3. Gradient of divergence: $\nabla(\nabla \cdot \boldsymbol{v})$.

The curl $\nabla \times \boldsymbol{v}$ is a vector, so we can take its divergence and curl:

4. Divergence of curl: $\nabla \cdot (\nabla \times \boldsymbol{v})$.

5. Curl of curl: $\nabla \times (\nabla \times \boldsymbol{v})$.

This exhausts the possibilities, and in fact not all of them give anything new. Let's consider them one at a time:

Divergence of a Gradient

This object, which we write as $\nabla^2 T$ for short, is called the **Laplacian** of T, which will be our focus later.

> The Laplacian of a scalar T is a scalar.

Occasionally, we will use the Laplacian of a vector, $\nabla^2 \boldsymbol{v}$. By this we mean a **vector** quantity whose x-component is the Laplacian of v_x, and so on.

$$\nabla^2 \boldsymbol{v} \equiv \left(\nabla^2 v_x\right) \hat{x} + \left(\nabla^2 v_y\right) \hat{y} + \left(\nabla^2 v_z\right) \hat{z}$$

This is nothing more than a convenient extension of the meaning of ∇^2.

Exercise 4.8: Laplacian of a Vector

Calculate the Laplacian of the following functions:

$$(i) \quad T_a = x_2 + 3xy + 3z + 4, \qquad (ii) \quad T_b = \sin x \sin y \sin z,$$
$$(iii) \quad T_c = e^{-5x} \sin 4y \cos 3z, \qquad (iv) \quad \boldsymbol{v} = \left(x^2\right) \hat{x} + \left(3xz^2\right) \hat{y} + (-2xz) \hat{z}.$$

Solution

The solution to the Laplacian of the functions are as follows:

$$\begin{aligned}
&\textbf{(i)} && \frac{\partial^2 T_a}{\partial x^2} = 2;\ \frac{\partial^2 T_a}{\partial y^2} = 0;\ \frac{\partial^2 T_a}{\partial z^2} = 0 && \to \quad \nabla^2 T_a = 2 \quad \blacksquare \\
&\textbf{(ii)} && \frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -3T_b && \to \quad \nabla^2 T_b = -3T_b = 3\sin x \sin y \sin z \quad \blacksquare \\
&\textbf{(iii)} && \frac{\partial^2 T_c}{\partial x^2} = 25T_c; && \\
& && \frac{\partial^2 T_c}{\partial y^2} = -16T_c;\quad \frac{\partial^2 T_c}{\partial z^2} = -9T_c && \to \quad \nabla^2 T_c = 0 \quad \blacksquare \\
&\textbf{(iii)} && \frac{\partial^2 v_x}{\partial x^2} = 2;\ \frac{\partial^2 v_x}{\partial y^2} = 0;\ \frac{\partial^2 v_x}{\partial z^2} = 0 && \to \quad \nabla^2 v_x = 0, \\
& && \frac{\partial^2 v_y}{\partial x^2} = 0;\ \frac{\partial^2 v_y}{\partial y^2} = 0;\ \frac{\partial^2 v_y}{\partial z^2} = 6 && \to \quad \nabla^2 v_y = 6x, \\
& && \frac{\partial^2 v_z}{\partial x^2} = 0;\ \frac{\partial^2 v_z}{\partial y^2} = 0;\ \frac{\partial^2 v_z}{\partial z^2} = 0 && \to \quad \nabla^2 v_z = 0, \\
& && \nabla^2 \boldsymbol{v} = 2\,\hat{x} + 6x\,\hat{y} && \blacksquare
\end{aligned}$$

Curl of a Gradient

The curl of a gradient is **always** zero:

$$\nabla \times (\nabla T)$$

This is an **important fact**, which will be used repeatedly. Without going into too much detail into the proof, it relies on the following relation:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$

If you think I'm being fussy, test your intuition on this one:

Gradient of Divergence

This operation rarely occurs in physical applications, and it has not been given any special name of its own.

Notice that $\nabla(\nabla \cdot \boldsymbol{v})$ is not the same as the Laplacian of a vector:

$$\nabla^2 = (\nabla \cdot \nabla) \neq \nabla(\nabla \cdot \boldsymbol{v})$$

Divergence of a Curl

Like the curl of a gradient, is always zero:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0.$$

Curl of a Curl

As you can check from the definition of ∇:

$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}.$$

So curl-of-curl gives nothing new; the first term is just number Divergence of a Curl, and the second is the Laplacian.

Really, then, there are just two kinds of second derivatives:

1. the Laplacian,
2. gradient-of-divergence

> It is possible to work out 3rd derivatives, but fortunately second derivatives suffice for practically all physical applications.

4.3 Integral Calculus

4.3.1 Line, Surface, and Volume Integrals

In electrodynamics, we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**, which will be the focus of this section.

a **Line Integrals** an expression of the form:

$$\int_a^b \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

where $\boldsymbol{v}$ is a vector function, $d\boldsymbol{l}$ is the infinitesimal displacement vector, and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\boldsymbol{a}$ to point $\boldsymbol{b}$. If the path forms a closed loop (i.e., if $\boldsymbol{b} = \boldsymbol{a}$), We put a circle on the integral sign:

$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

At each point on the path, we take the dot product of $\boldsymbol{v}$ (evaluated at that point) with the displacement $d\boldsymbol{l}$ to the next point on the path.

> A good example of a line integral is the work done by a force $\boldsymbol{F}$:
>
> $$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}$$

Ordinarily, the value of a line integral depends critically on the path taken from $\boldsymbol{a}$ to $\boldsymbol{b}$, but there is an important special class of vector functions for which the line integral is independent of path and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A force that has this property is called **conservative**.)

Exercise 4.9: Fluid Flow

A fluid's velocity field is $\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z}$.

Find the flow along the helix $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (t)\ \hat{z}$ with a range of $0 \leq t \leq \pi/2$.

Solution

We first evaluate $\boldsymbol{F}$ on the curve:

$$\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z} = (\cos t)\ \hat{x} + (t)\ \hat{y} + (\sin t)\ \hat{z} \qquad \text{Substitute } x = \cos t,\ z = t,\ y = \sin t.$$

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\boldsymbol{F} \cdot \left(d\boldsymbol{\ell}/dt\right)$ from $t = 0$ to $t = \pi/2$:

$$\begin{aligned} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} &= (\cos t)\,(-\sin t) + (t)\,(\cos t) + (\sin t)\,(1)\,, \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

Which makes,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{dy^{\boldsymbol{\ell}}}{dx}[t]\,\mathrm{d}t = \int_{0}^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\,dt, \\ &= \left[\frac{\cos^2 t}{2} + t \sin t\right]\Bigg|_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare \end{aligned}$$

Exercise 4.10: Circulation of a Field

Find the circulation of the field $\boldsymbol{F} = (x - y)\ \hat{x} + x\,\hat{y}$ around the circle $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$ with a range of $0 \leq t \leq 2\pi$.

Solution

On the circle, $\boldsymbol{F} = (x - y)\ \hat{x} + (x)\ \hat{y} + (0)\ \hat{z} = (\cos t - \sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}$ and

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1},$$

Gives.

$$\begin{aligned} \text{Circulation} &= \int_{0}^{2\pi} \boldsymbol{F} \cdot \frac{dy^{\boldsymbol{\ell}}}{dx}[t]dt = \int_{0}^{2\pi} (1 - \sin t \cos t)\ dt \\ &= \left[t - \frac{\sin^2 t}{2}\Bigg|_0^{2\pi}\right] = 2\pi \quad \blacksquare \end{aligned}$$

b. **Surface Integrals:** A surface integral is an expression of the form:

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$$

where $\boldsymbol{v}$ is a vector function, and the integral is over a specified surface $\mathcal{S}$. Here $\mathrm{d}\boldsymbol{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface. There are, two (2) directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous.
If the surface is closed (forming a "ballon"), we put a circle on the integral sign

$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$$

Tradition dictates that "outward" is positive, but for open surfaces it's arbitrary.

As an example, if $\boldsymbol{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$ represents the total mass per unit time passing through the surface.

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface and is determined entirely by the boundary line. An important task will be to characterize this special class of functions.

Exercise 4.11: Double Integrals

Find the following double integrals:

$$\int_0^1 \int_x^{2x} (x+y)^2 \, dy dx, \qquad \int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx dy,$$
$$\int_0^3 \int_x^3 \cosh(x+y) \, dy dx, \qquad \int_0^1 \int_0^{y^3} \exp y^4 dx dy.$$

Solution

The solution to integrations are as follows:

$$\begin{aligned}
\int_0^1 \int_x^{2x} (x+y)^2 \, dy dx &= \int_0^1 \int_x^{2x} x^2 + 2xy + y^2 dy dx, \\
&= \int_0^1 \left[yx^2 + xy^2 + \frac{y^3}{3} \right] \Bigg|_x^{2x} dx, \\
&= \int_0^1 \left(4x^3 + \frac{7x^3}{3} \right) dx, \\
&= \left[4x^3 + \frac{7x^4}{12} \right] \Bigg|_0^1 = \frac{19}{12} \quad \blacksquare \\
\int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx dy &= \int_0^1 \left[x - x^2 y \right] \Bigg|_y^{\sqrt{y}} dy, \\
&= \int_0^1 \left[\left(\sqrt{y} - y^2\right) - \left(y - y^3\right) \right] dy = \int_0^1 \left[y^3 + \sqrt{y} - y^2 - y \right] dy, \\
&= \left[\frac{y^4}{4} + \frac{2}{3} y^{3/2} - \frac{y^3}{3} - \frac{y^2}{2} \right] \Bigg|_0^1, \\
&= \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} \right) - (0) = \frac{1}{12} \quad \blacksquare \\
\int_0^3 \int_x^3 \cosh(x+y) \, dy dx &= \int_0^1 \left[\sinh(x+y) \right] \Bigg|_x^3 dx = \int_0^1 \left[\sinh(3+x) - \sinh(2x) \right] dx
\end{aligned}$$

c. **Volume Integrals** A volume integral is an expression of the form:

$$\int_{\mathcal{V}} T \, \mathrm{d}\tau$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$\mathrm{d}\tau = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

As an example, if T is the density of a substance (which might vary from point to point), then the volume integral would give the total mass.

Occasionally we shall encounter volume integrals of vector functions:

$$\int \boldsymbol{v}\, \mathrm{d}\tau = \int \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right) \mathrm{d}\tau = \hat{x} \int v_x \mathrm{d}\tau + \hat{y} \int v_y \mathrm{d}\tau + \hat{z} \int v_z \mathrm{d}\tau.$$

As the unit vectors ($\hat{x}$, $\hat{y}$, $\hat{z}$) are constants, they come outside the integral.

4.3.2 The Fundamental Theorem of Calculus

Assume $f(x)$ is a function of one (1) variable. The **fundamental theorem of calculus** says:

Theory 4.11: Calculus Theorem

the integral of a derivative over some region is given by the value of the function at the end points (boundaries)

$$\int_a^b \left(\frac{df}{dx} \right) \mathrm{d}x = f(x) - f(a) \qquad \text{or} \qquad \int_a^b F(x)\, \mathrm{d}x = f(x) - f(a)$$

In vector calculus there are three species of derivative (gradient, divergence, and curl,) and each has its own "fundamental theorem," with essentially the same format. I don't plan to prove these theorems here; rather, I will explain what they mean, and try to make them plausible. Proofs are given in Appendix A.

4.3.3 The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point **a**, move a small distance $\mathrm{d}\boldsymbol{l}_1$. The function T will change by an amount:

$$\mathrm{d}T = (\nabla T) \cdot \mathrm{d}\boldsymbol{l}_1$$

Now move an additional small displacement $\mathrm{d}\boldsymbol{l}_2$. The incremental change in T will be:

$$\mathrm{d}T = (\nabla T) \cdot \mathrm{d}\boldsymbol{l}_2$$

In this manner, proceeding by infinitesimal steps, we make the journey to point $\boldsymbol{b}$. At each step we compute the gradient of T (at that point) and dot it into the displacement $dv1_{...}$ this gives us the change in T.

Theory 4.11: Gradient Theorem

The total change in T in going from $\boldsymbol{a}$ to $\boldsymbol{b}$ (along the path selected) is:

$$\int_a^b (\nabla T) \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

Similar to "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (**a** and **b**).

Assumee you want to measure the height of GrossGlockner. You could climb the mountain from base, or take the high alpine road, or take a helicopter ride all the way up to top. Regardless or the options you take, you should get the same answer either way (that's the fundamental theorem).

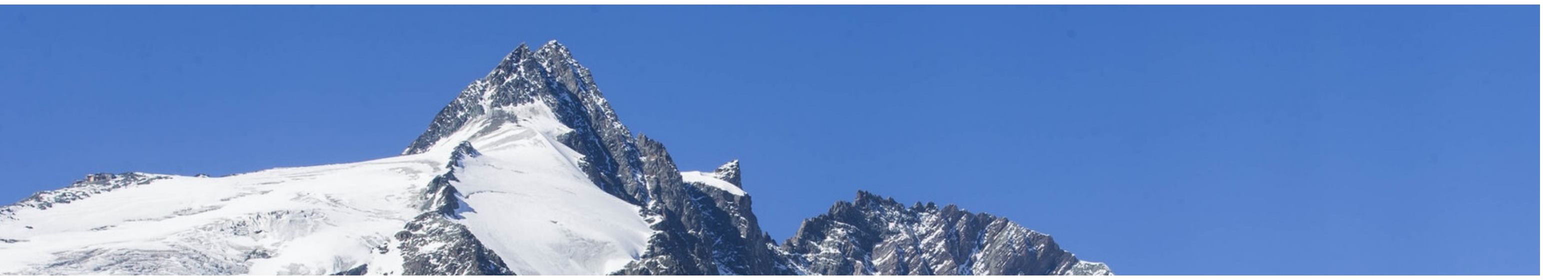


Figure 4.2.: To measure the height of a mountain, it doesn't matter what way you take, as long as you know the base and the top, you will know the height.

Theorem 1: Incliendent,
Incidentally, as we found in Ex. 1.6, line integrals ordinarily depend on the path taken from $\boldsymbol{a}$ to $\boldsymbol{b}$. But the right side of Eq. 1.55 makes no reference to the path–only to the end points. Evidently, gradients have the special property that their line integrals are path independent:
Corollary 1: $\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla T) \cdot \mathrm{d}\boldsymbol{l}$ is independent of the path taken from $\boldsymbol{a}$ to $\boldsymbol{b}$.
Corollary 2: $\oint (\nabla T) \cdot \mathrm{d}\boldsymbol{l} = 0$, since the beginning and end points are identical, and hence $T(\boldsymbol{b}) - T(\boldsymbol{a}) = 0$.

The Fundamental Theorem for Divergences

this theorem has at least three special names: **Gauss's theorem, Green's theorem,** or simply the **divergence theorem**, The fundamental theorem for divergences states that:

> **Theory** 4.11: Divergence Theorem
>
> the integral of a derivative (in this case the divergence) over a region (in this case the volume, $\mathcal{V}$) is equal to the value of the function at the boundary (in this case the surface$\mathcal{S}$ that bounds the volume).
>
> $$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\, \mathrm{d}\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}.$$

The boundary term is itself an integral, more specifically, a surface integral. This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

To create an analogy, if $\boldsymbol{v}$ represents the flow of an incompressible fluid, then the flux $\boldsymbol{v}$ is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the *spreading out* of the vectors from a point, a place of high divergence is like a tap, pouring out liquid. If we have a bunch of tap in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In fact, there are two (2) ways we could determine how much is being produced:

a. we could count up all the faucets, recording how much each puts out

b. we could go around the boundary, measuring the flow at each point, and add it all up

You get the same answer either way:

$$\int \text{(faucets within the volume)} = \oint \text{(flow out through the surface)}$$

Exercise 4.12: Divergence Theorem - I

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}$ over the sphere $x^2 + y^2 + z^2 = a^2$

Solution

The outer unit normal to S, calculated from the gradient of $f\{\cdot\}\ x,\ y,\ z = x^2 + y^2 + z^2 - a^2$, is:

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{a}. \qquad \text{x}^2 + \text{y}^2 + \text{z}^2 = \text{a}^2\text{, on, S}$$

Therefore:

$$(\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \frac{x^2 + y^2 + z^2}{a}\ \mathrm{d}a = \frac{a^2}{a}\ \mathrm{d}a = a\,\mathrm{d}a.$$

This in turn gives us:

$$\iint_S (\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \iint_S a\,\mathrm{d}a = a \iint_S \mathrm{d}a = a\left(4\pi a^2\right) = 4\pi a^3. \qquad \text{Area of, S, is, 4ßa}^2$$

The divergence of $\boldsymbol{F}$ is:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

So,

$$\iiint_V (\nabla \cdot \boldsymbol{v})\ d\tau = \iiint_V 3d\tau = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3 \quad \blacksquare$$

Exercise 4.13: Divergence Theorem - II

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2 \cos\theta\right)\ \hat{r} + \left(r^2 \cos\phi\right)\ \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right)\ \hat{\phi}.$$

using as your volume one octant of the sphere of radius R.

Solution

It is always useful to write the theorem we are going to work on:

$$\underbrace{\iiint_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\ dV}_{\text{Divergence integral}} = \underbrace{\iint_S \boldsymbol{v} \cdot \boldsymbol{n} da.}_{\text{Outward flux}}$$

First solve the left hand side of the equation:

$$\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta r^2 \cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right), \\
&= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right), \\
&= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\int (\nabla \cdot \boldsymbol{v})\ d\tau &= \int (4r\cos\theta)\ r^2 \sin\theta dr\, \mathrm{d}\theta\, \mathrm{d}\phi = 4\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\sin\theta\, \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\phi, \\
&= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \blacksquare
\end{aligned}$$

Now it is time to solve the right hand side of the question. As we are aware from the shape, an octant of the sphere has 4 sides to it: the curved surface $xyz \to \boldsymbol{a}_1$, and $xz \to \boldsymbol{a}_2$, $yz \to \boldsymbol{a}_3$ and $xy \to \boldsymbol{a}_4$. These are

$$d\boldsymbol{a}_1 = \hat{r} dl_\theta dl_\phi = \hat{r} R^2 \sin\theta \, d\phi \, d\theta, \qquad d\boldsymbol{a}_2 = dl_r dl_\theta = -\hat{\phi} r dr \, d\theta,$$

$$d\boldsymbol{a}_3 = \hat{\phi} dl_r dl_\theta = \hat{\phi} r dr \, d\theta, \qquad d\boldsymbol{a}_4 = dl_r dl_\phi = \hat{\theta} r dr d\theta. \quad \left(\theta = \pi/2\right)$$

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} &= \iint_{S_1} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_2} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_3} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_4} \boldsymbol{v} \cdot d\boldsymbol{a}, \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \left[r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi} \right] \Bigg|_{r=R} \cdot \left(\hat{r} R^2 \sin\theta d\phi d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi} \right] \Bigg|_{\phi=0} \cdot \left(-\hat{\phi} r dr d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi} \right] \Bigg|_{\phi=\pi/2} \cdot \left(\hat{\phi} r dr d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi} \right] \Bigg|_{\theta=\pi/2} \cdot \left(\hat{\theta} r dr \, d\theta \right),
\end{aligned}$$

Time to do some integration.

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} &= \int_0^{\pi/2} \int_0^{\pi/2} \left[R^2 \cos\theta \, \hat{r} + R^2 \cos\phi \, \hat{\theta} - R^2 \cos\theta \sin\phi \, \hat{\phi} \right] \cdot \left(\hat{r} R^2 \sin\theta d\phi d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta \, \hat{r} + r^2 (1) \, \hat{\theta} - (0) \sin\phi \, \hat{\phi} \right] \cdot \left(-\hat{\phi} r dr d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta \, \hat{r} + (0) \phi \, \hat{\theta} - r^2 \cos\theta (1) \, \hat{\phi} \right] \cdot \left(\hat{\phi} r dr d\theta \right) \\
&+ \int_0^{\pi/2} \int_0^{R} \left[(0) \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - (0) \, \hat{\phi} \right] \cdot \left(\hat{\theta} r dr d\theta \right).
\end{aligned}$$

Final touches and cleaning up,

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} &= \int_0^{\pi/2} \int_0^{\pi/2} R^4 \sin\theta \cos\theta \, d\phi \, d\theta + 0 \overbrace{- \int_0^{\pi/2} \int_0^{R} r^3 \cos\theta dr \, d\theta + \int_0^{\pi/2} \int_0^{R} r^3 \cos\theta dr d\phi}^{=0}, \\
&= R^4 \left(\int_0^{\pi/2} d\phi \right) \left(\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \right), \\
&= R^4 \left(\frac{\pi}{2} \right) \left(\frac{\pi}{2} \right), \\
&= \frac{\pi R^4}{4} \quad \blacksquare
\end{aligned}$$

4.3.4 The Fundamental Theorem for Curls

The fundamental theorem for curls, also known as **Stokes' theorem**, states:

Theory 4.13: Stokes' Theorem

the integral of a derivative over a region ($\mathcal{S}$) is equal to the value of the function at the boundary ($\mathcal{P}$).

$$\int_{\mathcal{S}} (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \oint_{\mathcal{P}} \boldsymbol{v} \cdot d\boldsymbol{l}.$$

Similar to the divergence theorem, the boundary term is itself an integral. Specifically, a *closed line integral*.

Remember the curl measures the *twist* of the vectors $\boldsymbol{v}$. Think of a region of high curl as a whirlpool, where if you put a wheel there, it will rotate. Now, the integral of the curl over some surface (or, more precisely, the *flux* of the curl through the surface) represents the *total amount of swirl*, and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

> $\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$ is sometimes called the **circulation** of $\boldsymbol{v}$.

There seems to be an ambiguity in Stokes' theorem: concerning the boundary line integral:

Which way are we supposed to go around (clockwise or counterclockwise)?

The answer is that it doesn't matter which way you go **as long as you are consistent**, for there is an additional sign ambiguity in the surface integral:

Which way does $\mathrm{d}\boldsymbol{a}$ point?

For a closed surface (i.e., the divergence theorem), $\mathrm{d}\boldsymbol{a}$ points in the direction of the outward normal. But for an open surface, which way would be defined as out? Consistency in Stokes' theorem is given by the right-hand rule. If your rings point in the direction of the line integral, then your thumb fixes the direction of $\mathrm{d}\boldsymbol{a}$.

Ordinary, a flux integral depends critically on what surface you integrate over, but this is **not** the case with curls. For Stokes' theorem says that $\int (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a}$ is equal to the line integral of $\boldsymbol{v}$ around the boundary, and the latter makes no reference to the specific surface you choose.

Preposition I $\int (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a}$ depends only on the boundary line, not on the particular surface used.

Preposition II $\oint (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

These corollaries are analogous to those for the gradient theorem.

Exercise 4.14: Surface Area of an Implicit Surface

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

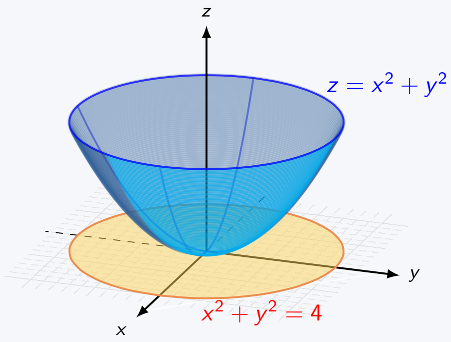


Figure 4.3.: We will calculate the area of the parabolic surface in Example 14.

Solution

We sketch the surface S and the region R below it in the xy-plane (**Fig. ??**). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, ans R is the disk $x^2 + y^2 \leq 4$ in the xy-plane.

To get a unit vector normal (i.e., $\hat{n}$) to the plane R, we can take $\hat{n} = \hat{z}$. At any point (x, y, z) on the surface, we have:

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= (2x)\ \hat{x} + (2y)\ \hat{y} + (-1)\ \hat{z} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \hat{n}| &= |\nabla F \cdot \hat{z}| = |-1| = 1. \end{aligned}$$

In the region R, the area is defined to be $dA = dx\,dy$. Therefore:

$$\begin{aligned} \text{Surface Area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{n}|}\, dA \\ &= \iint_{x^2+y^2\leq 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(4r^2 + 1\right)^{3/2} \Big|_0^2\, d\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(17^{3/2} - 1\right)\, d\theta \\ &= \frac{\pi}{6} \left(17\sqrt{17} - 1\right) \quad \blacksquare \end{aligned}$$

Exercise 4.15: Stokes Theorem Over a Hemisphere

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z}$.

Tip: Parametrisation of a circle is: $x = r\cos\theta$, $y = r\sin\theta$ and $da = \frac{3}{z} dA$)

Solution

The start by calculating the counter-clockwise circulation around C using the following parametrisation:

$$\begin{aligned} \boldsymbol{\ell}(\theta) &= (3\cos\theta)\ \hat{x} + (3\sin\theta)\ \hat{y} + (0)\ \hat{z}, \\ \text{where} \quad & 0 \leq \theta \leq 2\pi. \end{aligned}$$

Using this we can calculate the counter-clockwise circulation.

$$\begin{aligned} d\boldsymbol{\ell} &= (-3\sin\theta\, d\theta)\ \hat{x} + (3\cos\theta\, d\theta)\ \hat{y} + (0)\ \hat{z}, \\ \boldsymbol{F} &= (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z} \\ &= (3\sin\theta)\ \hat{x} + (-3\cos\theta)\ \hat{y} + (0)\ \hat{z}, \\ \boldsymbol{F} \cdot d\boldsymbol{\ell} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta, \\ \oint_C \boldsymbol{F} \cdot d\boldsymbol{\ell} &= \int_0^{2\pi} -9 d\theta = -18\pi. \end{aligned}$$

For the curl of integral we have:

$$
\begin{aligned}
\nabla \times \boldsymbol{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (0-0)\ \hat{x} + (0-0)\ \hat{y} + (-1-1)\ \hat{z} = -2\,\hat{z} \\
\hat{n} &= \frac{\nabla S}{|\nabla S|} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{3} \qquad \text{Unit normal}
\end{aligned}
$$

Now it is time to define the area of integration (da):

$$
\begin{aligned}
da &= \frac{|\nabla S|}{|\nabla S \cdot \hat{z}|} dA \\
&= \frac{\left|(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}\right|}{2z} \\
&= \frac{\overbrace{2\sqrt{x^2+y^2+z^2}}^{a}}{2z} \\
&= \frac{3}{z} dA, \\
\nabla \times \boldsymbol{F} \cdot \boldsymbol{n} da &= -\frac{2z}{3}\frac{3}{z} dA = -2dA
\end{aligned}
$$

The cardinal direction $\hat{z}$ comes from being the direction **perpendicular** to the surface (S).

$$
\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} da = \iint_{x^2+y^2 \leq 9} -2dA = -18\pi
$$

The circulation around the circle equals the integral of the curl over the hemisphere ■

4.4 Curvilinear Coordinates

4.4.1 Spherical Coordinate System

It is possible to label a point P in Cartesian coordinates (x, y, z), but sometimes it is more convenient to use **spherical** coordinates (r, θ, ϕ); r is the distance from the origin (the magnitude of the position vector r), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates can be read from Fig. **??**.

$$
x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta.
$$

Fig. **??** also shows three unit vectors, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, pointing in the direction of increase of the corresponding coordinates.

They constitute an **orthogonal** (mutually perpendicular) basis set, similar to $\hat{x}$, $\hat{y}$, $\hat{z}$), and any vector $\boldsymbol{A}$ can be expressed in terms of them, in the usual way:

$$
\boldsymbol{A} = (A_{\text{r}})\ \hat{r} + (A_{\text{ˋ}})\ \hat{\theta} + (A_{\text{Œ}})\ \hat{\phi}
$$

Here, A_{r}, $A_{\text{ˋ}}$, $A_{\text{Œ}}$ are the radial, polar, and azimuthal components of vector $\boldsymbol{A}$. In terms of the

Operator	Mathematical Definition
Gradient	$\nabla T = \frac{\partial T}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\,\hat{\phi}$
Divergence	$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$
Curl	$\nabla \times \boldsymbol{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta}$ $+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$
Laplacian	$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$

Table 4.1.: Defined mathematical operations in spherical coordinate system.

Cartesian unit vectors:

$$
\begin{aligned}
\hat{r} &= (\sin\theta\cos\phi)\ \hat{r} + (\sin\theta\sin\phi)\ \hat{\theta} + (\cos\theta)\ \hat{\phi},\\
\hat{\theta} &= (\cos\theta\cos\phi)\ \hat{r} + (\cos\theta\sin\phi)\ \hat{\theta} + (-\sin\theta)\ \hat{\phi},\\
\hat{\phi} &= (-\sin\phi)\ \hat{r} + (\cos\phi)\ \hat{\theta} + (0)\ \hat{\phi}.
\end{aligned}
$$

An infinitesimal displacement in the $\hat{r}$ direction is simply $\mathrm{d}r$, just as an infinitesimal element of length in the $\hat{x}$ direction is $\mathrm{d}x$:

$$\mathrm{d}l_{\mathrm{r}} = \mathrm{d}r$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (Fig. 1.38b) is not just $\mathrm{d}\theta$ rather,

$$\mathrm{d}l_{\theta} = r\,\mathrm{d}\theta$$

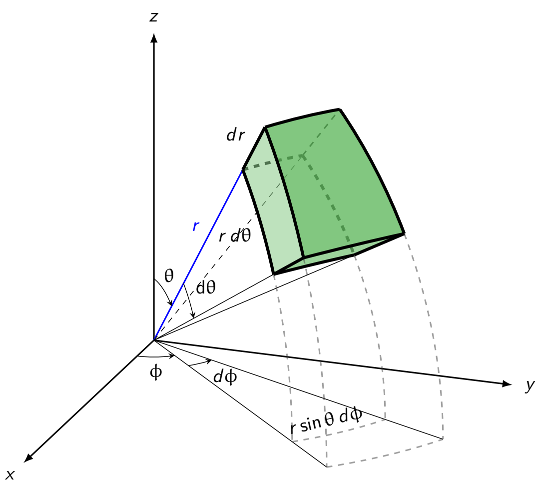


Figure 4.4.: The physics convention. Spherical coordinates (r,θ , ϕ) as commonly used: (ISO 80000-2:2019): radial distance r (slant distance to origin), polar angle (theta) (angle with respect to positive polar axis), and azimuthal angle (phi) (angle of rotation from the initial meridian plane)

Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (Fig. 1.38c) is

$$\mathrm{d}l_{Œ} = r \sin\theta \, \mathrm{d}\phi$$

Thus the general infinitesimal displacement $\mathrm{d}\boldsymbol{l}$ is:

$$\mathrm{d}\boldsymbol{l} = (\mathrm{d}r)\ \hat{r} + (r\,\mathrm{d}\theta)\ \hat{\theta} + (r \sin\theta)\ \hat{\phi}$$

This plays the role $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x)\ \hat{x} + (\mathrm{d}y)\ \hat{y} + (\mathrm{d}z)\ \hat{z}$ plays in Cartesian coordinates. The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three (3) infinitesimal displacements:

$$\mathrm{d}\tau = \mathrm{d}l_r\,\mathrm{d}l_{\grave{}}\,\mathrm{d}l_{Œ} = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

> It is not possible to give a general expression for surface elements $\mathrm{d}\boldsymbol{a}$, since these depend on the orientation of the surface. We simply have to analyze the geometry for any given case, which goes for Cartesian and curvilinear coordinates.

Integrating over the surface of a sphere, for instance, makes r constant, whereas θ and ϕ change:

$$\mathrm{d}\boldsymbol{a}_1 = \mathrm{d}l_{\grave{}}\,\mathrm{d}l_{Œ}\,\hat{r} = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{r}$$

On the other hand, if the surface lies in the xy plane, making θ is constant, while r and ϕ vary:

$$\mathrm{d}\boldsymbol{a}_2 = \mathrm{d}l_r\,\mathrm{d}l_{Œ}\,\hat{\theta} = r\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{\theta}$$

Finally: r ranges from 0 to ∞, ϕ from 0 to 2π, and θ from 0 to π.
Up to know, we only talked about the geometry of spherical coordinates. Now let's translate the vector derivatives (gradient, divergence, curl, and Laplacian) into r, θ, ϕ notation.
Here, then, are the vector derivatives in spherical coordinates:

Exercise 4.16: Volume of A Sphere

Find the volume of a sphere of radius R.

Solution

The derivation is as follows:

$$V = \int \mathrm{d}\tau = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi,$$
$$= \left(\int_0^R r^2\,\mathrm{d}r\right)\left(\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = \left(\frac{R^3}{3}\right)(2)\,(2\pi) = \frac{4}{3}\pi R^3 \quad \blacksquare$$

4.4.2 Cylindrical Coordinates

The cylindrical coordinates (s, ϕ, z) of a point P are defined in **Fig. ??** . Observe that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is:

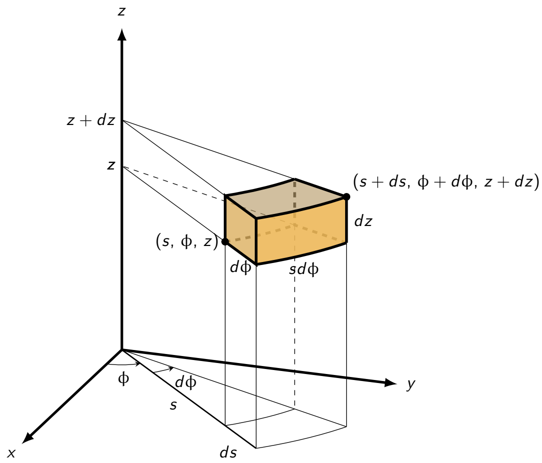


Figure 4.5.: A 3D representation of the cylindrical coordinate system.

$$x = s\cos\phi \qquad y = s\sin\phi \qquad z = z.$$

The unit vectors are:

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad \hat{z} = \hat{z}$$

The infinitesimal displacements are

$$\mathrm{d}l_{\mathrm{s}} = \mathrm{d}s \qquad \mathrm{d}l_{\text{Œ}} = s\,\mathrm{d}\phi, \qquad \mathrm{d}l_{\mathrm{z}} = \mathrm{d}z$$

which makes:

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{z}.$$

and the volume element is

$$\mathrm{d}\tau\,' = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$$

The range of s is $(0, \infty)$, ϕ is from 0 to 2π and z is from $-\infty$ to $+\infty$.

4.5 Dirac Delta Function

4.5.1 A Mathematical Anomaly

Consider the following vector function:

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

At every location, $\mathbf{v}$ is directed radially outward which can be seen in **Fig. ??** . Let's calculate its divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

This is interesting as this calculation gives us a unforeseen solution. Let's look at this closer. Suppose we integrate over a sphere of radius R, centered at the origin. The surface integral is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot \left(R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\hat{\mathbf{r}}\right) = \left(\int_0^{\pi} \sin\theta \,\mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = 4\pi$$

But the volume integral, $\int \nabla \cdot \mathbf{v} \,\mathrm{d}\tau$, is zero if we assume the aforementioned calculation to be true.

Does this mean that the divergence theorem is false? What's going on here?

> There seems to be a contradiction.

The source of the problem is the point $r = 0$, where $\mathbf{v}$ **blows up**. It is quite true that $\nabla \cdot \mathbf{v} = 0$ everywhere except the origin, but right at the origin is the situation is more complicated. Observe, the surface integral is independent of R. If the divergence theorem is right, we should expect $\int \nabla \cdot \mathbf{v} \,\mathrm{d}\tau = 4\pi$ for any non-zero vector and the origin.

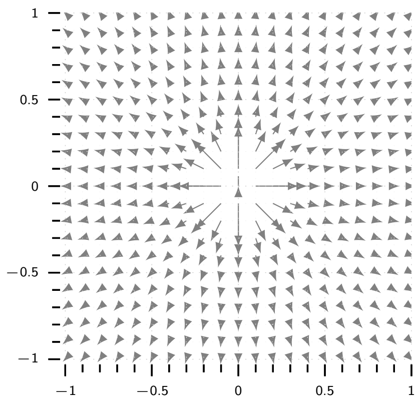


Figure 4.6.: Visually it is obvious there is positive divergence. Yet with our current definition of divergence it seems there is a contradiction.

This means the value of 4π must be coming from the point $r = 0$. Therefore, $\nabla \cdot \mathbf{v}$ has the unique property that it vanishes everywhere except at one point, and yet its integral is 4π.

No normal function behaves like that.

> To wrap our heads around this property think of **density**.
>
> The density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite–namely, the mass of the particle.)

What we have stumbled upon is called the **Dirac delta function**. It arises in numerous branches of theoretical physics and plays a central role in the theory of electrodynamics.

4.5.2 The 1D Dirac Delta Function

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow spike, with an area of one (1) . This approach to the infinitesimal width can be seen in **Fig. ??** .

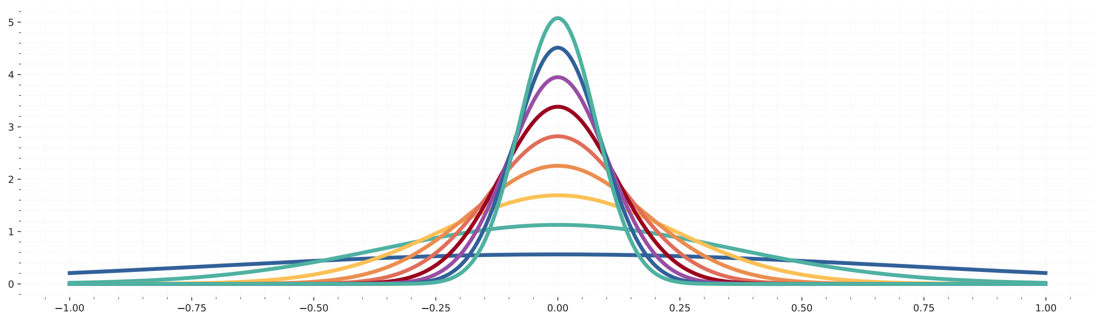


Figure 4.7.: A visual representation of a 1D Dirac Delta Function. Think of it as a distribution function being squeezed to an infinitely small width.

That is to say:

$$\delta(x) = \begin{cases} 0 & \text{if} \quad x \neq 0, \\ \infty & \text{if} \quad x = 0. \end{cases}$$

and in an integral form:

$$\int_{-\infty}^{\infty} \delta(x) \, \mathrm{d}x = 1 \tag{4.7}$$

In a strict sense of definition, $\delta(x)$ is not a function at all, as its value is not finite at $x = 0$. In literature it is known as a **generalized function**[3]

> If $f(x)$ is some ordinary function, then the product $f(x)\,\delta(x)$ is zero everywhere except at $x = 0$.

It follows that:

$$f(x)\,\delta(x) = f(0)\,\delta(x).$$

> The product is zero anyway except at $x = 0$. This allows us to replace $f(x)$ with the value it assumes at the origin.

In particular

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)\,.$$

Under an integral, the delta function picks out the value of $f(x)$ at $x = 0$ [4] . Of course, we can shift the spike from $x = 0$ to some other point, $x = a$:

$$\delta(x-a) = \begin{cases} 0 & \text{if} \quad x \neq a, \\ \infty & \text{if} \quad x = a. \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(x-a)\, dx = 1.$$

which becomes:

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a),$$

and finally generalises to:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)dx = f(a)\,. \tag{4.8}$$

Although δ itself is not a legitimate function, integrals over δ are perfectly acceptable. In fact, think of the delta function as something that is always intended for use under an integral sign. In particular, two (2) expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if:

$$\int_{-\infty}^{\infty} f(x)\,D_1(x)\, dx = \int_{-\infty}^{\infty} f(x)\,D_2(x)\, dx,$$

for all **ordinary** functions $f(x)$.

Exercise 4.17: A Simple Dirac Integral

Evaluate the following integral:

$$\int_0^3 x^3\delta(x-2)\, dx$$

Solution

The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$.

Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Exercise 4.18: 1D Dirac Delta

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right)\delta(x-3)\, dx, \tag{a}$$

$$\int_0^5 \cos x\,\delta(x-\pi)\, dx, \tag{b}$$

$$\int_0^3 x^3\delta(x+1)\, dx, \tag{c}$$

$$\int_{-\infty}^{+\infty} \ln(x+3)\,\delta(x+2)\, dx. \tag{d}$$

Solution

The solution are as follows:

- **(a)** $3\left(3^2\right) - 2(3) - 1 = 27 - 6 - 1 = 20$ ■
- **(b)** $\cos\pi = -1$ ■
- **(c)** 0 ■
- **(d)** $\ln(-2+3) = \ln 1 = 0$ ■

4.5.3 The 3D Dirac Delta Function

Once we have defined the 1D Dirac, it is trivial to generalise it to 3D:

$$\delta^3(\boldsymbol{r}) = \delta(x)\,\delta(y)\,\delta(z)\,,$$

and similar to 1D, 3D Dirac is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\boldsymbol{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

And, the general form is:

$$\int_{\text{all space}} f(\boldsymbol{r})\,\delta^3(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}\tau = f(\boldsymbol{a})\,. \tag{4.9}$$

As in the 1D case, integration with δ picks out the value of the function f at the location of the spike.

We can fix the paradox introduced in the beginning of Section **??**. Remember, the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, however, its integral over any volume containing the origin is a constant. These are precisely the defining conditions for the Dirac delta function; evidently

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r}) \quad \blacksquare \tag{4.10}$$

4.6 Vector Field Theory

4.6.1 Helmholtz Theorem

As an example, electricity and magnetism are generally expressed as electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ and like many physical laws, such as vortices in a fluid or the flow of a gas in an open environment, these are most compactly expressed as **differential equations**. As $\boldsymbol{E}$ and $\boldsymbol{B}$ are **vectors**, the differential equations naturally involve vector derivatives:

divergence and *curl*.

This formulation raises an interesting question:

To what extent is a vector function determined by its divergence and curl?

To study this case lets assume a vector of $\boldsymbol{F}$. if the divergence of $\boldsymbol{F}$ is a specified (scalar) function D,

$$\nabla \cdot \boldsymbol{F} = D,$$

and the curl of $\boldsymbol{F}$ is a specified (vector) function $\boldsymbol{C}$,

$$\nabla \times \boldsymbol{F} = \boldsymbol{C},$$

and for consistency, we assume $\boldsymbol{C}$ to have **NO** divergence,

$$\nabla \cdot \boldsymbol{C} = 0,$$

Remember, the divergence of a curl is **ALWAYS** zero.

Using this knowledge, is it possible to determine the function $\boldsymbol{F}$?

Without knowing more information, it is not really possible. There are many functions whose divergence and curl are both zero everywhere. Some examples are:

$$\boldsymbol{F} = 0,$$
$$\boldsymbol{F} = (y)\ \hat{x} + (zx)\ \hat{y} + (xy)\ \hat{z},$$
$$\boldsymbol{F} = (\sin x \cosh y)\ \hat{x} + (-\cos x \sinh y)\ \hat{y} + (.)\ \hat{z}$$

If you recall the beginning of **Higher Mathematics I**, to solve a differential equation with a particular solution, you must also be supplied with appropriate **boundary conditions**.

In electrodynamics, for example, we typically require that the fields go to zero at infinity[5]. With that extra information, the **Helmholtz theorem** guarantees the field is uniquely determined by its divergence and curl.

[5] To make calculations easier (a.k.a assume the cow is a sphere)

4.6.2 Potentials

If the curl of a vector field ($\boldsymbol{F}$) vanishes (everywhere), then $\boldsymbol{F}$ can be written as the **gradient of a scalar potential** (V):

$$\nabla \times \boldsymbol{F} = 0 \Longleftrightarrow \boldsymbol{F} = -\nabla V$$

The minus sign is purely conventional.

That's the essential burden of the following theorem:

Theory 4.18: Zero Curl Fields

The following conditions are equivalent.

(i) $\nabla \times \boldsymbol{F} = 0$ everywhere,

(ii) $\int_a^b \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}$ is independent of path, for any given end points,

(iii) $\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = 0$ for any closed loop,

(iv) $\boldsymbol{F}$ is the gradient of some scalar function: $\boldsymbol{F} = -\nabla V$.

The potential is **NOT** unique as any constant can be added to V, since this will not affect its gradient.

If the divergence of a vector field ($\boldsymbol{F}$) vanishes (everywhere), then $\boldsymbol{F}$ can be expressed as the curl of a **vector potential** ($\boldsymbol{A}$):

$$\nabla \cdot \boldsymbol{F} = 0 \Longleftrightarrow \boldsymbol{F} = \nabla \times \boldsymbol{A}$$

That's the main conclusion of the following theorem:

Theory 4.18: Zero Divergence Fields

The following conditions are equivalent:

(i) . $\nabla \cdot \boldsymbol{F} = 0$ everywhere.

(ii) $\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{a}$ is independent of surface, for any given boundary line.

(iii) $\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{a} = 0$ for any closed surface.

(iv) $\boldsymbol{F}$ is the curl of some vector function: $\boldsymbol{F} = \nabla \times \boldsymbol{A}$.

The vector potential is **NOT** unique as the gradient of any scalar function can be added to $\boldsymbol{A}$ without affecting the curl, given the curl of a gradient is zero.

Incidentally, in all cases, a vector field $\boldsymbol{F}$ can be written as the gradient of a scalar plus the curl of a vector.

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}$$