Tutorial Book

# M.Sc Higher Mathematics I

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This TutorialBook is a
complement to it's corresponding Lecture.
When in doubt regarding theory, please consult the LectureBook.

# List of Questions

# 1

# First Order ODEs

**An Initial Value Problem**

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

**A1**

The general solution is:

$$y(x) = ce^{3x}$$

From the solution and the initial condition:

$$y(0) = ce^0 = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation. Visually the solution is plotted as follows ■
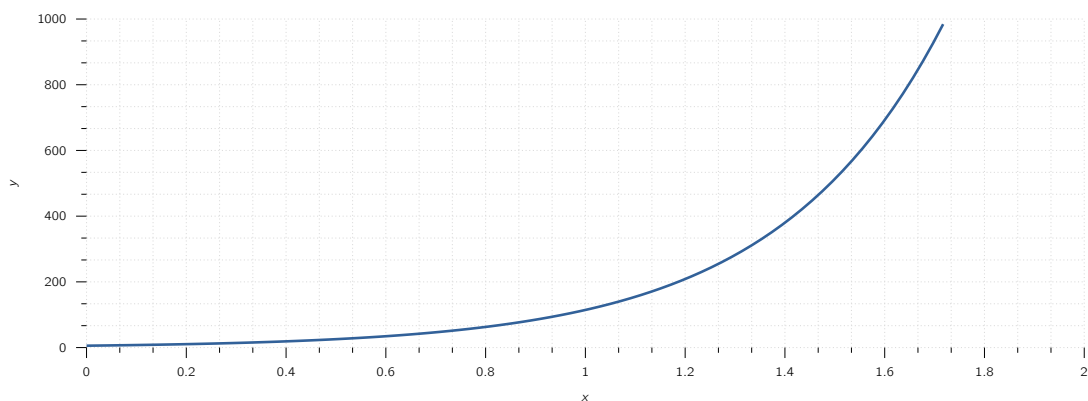


**Figure 1.1:** A Solution to the initial value problem.

**Q2** **Radioactive Decay**

Given $0{,}5\,\text{g}$ of a radioactive substance, find the amount present at any later time. The decay of Radium is measured to be $k = 1{,}4 \times 10^{-11}\,\text{s}^{-1}$.

**A2**

We know $y(t)$ is the substance amount still present at $t$. Using the law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives:

$$\frac{dy}{dt} = -ky \tag{1.1}$$

where the constant $k$ is **positive**, and due of the minus, we get *decay*. We know $k$ which the question has given as $k = 1,4 \times 10^{-11}\,\text{s}^{-1}$. Now the given initial amount is 0,5 g, and we can call the corresponding instant $t = 0$. We have the initial condition $y(0) = 0.5$, which is the instant the process begins. Therefore, the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the Ordinary Differential Equation (ODE) is an exponential decay and has the general solution:

$$y(0) = ce^{-kt}.$$

We now determine $c$ by using the initial condition which gives $y(0) = c = 0.5$. Therefore:

$$y(t) = 0.5e^{-kt} \quad \blacksquare$$
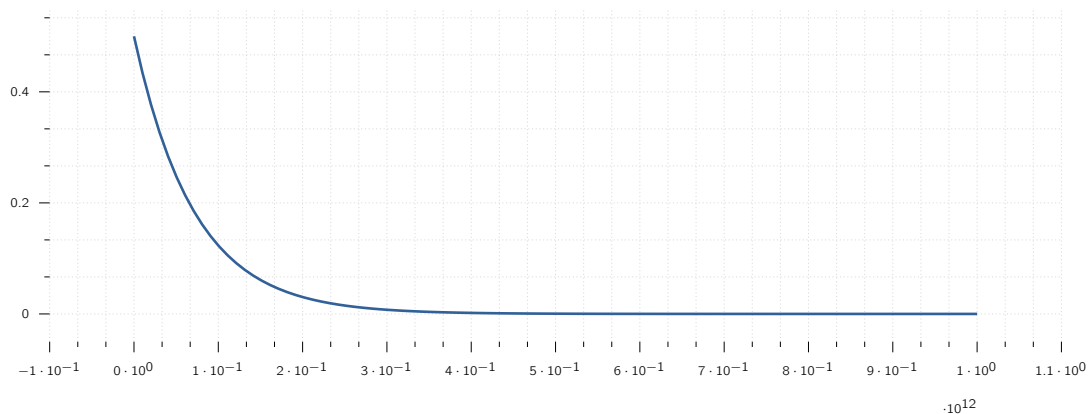


**Figure 1.2:** A Solution to the radioactive decay.

**Q3**  **Separable ODE**

Solve the following ODE:

$$y' = 1 + y^2$$

## A3

The given ODE is separable because it can be written:

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x \qquad \text{By integration} \qquad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

**Note**  It is important to introduce the constant $c$ when the integration is performed.
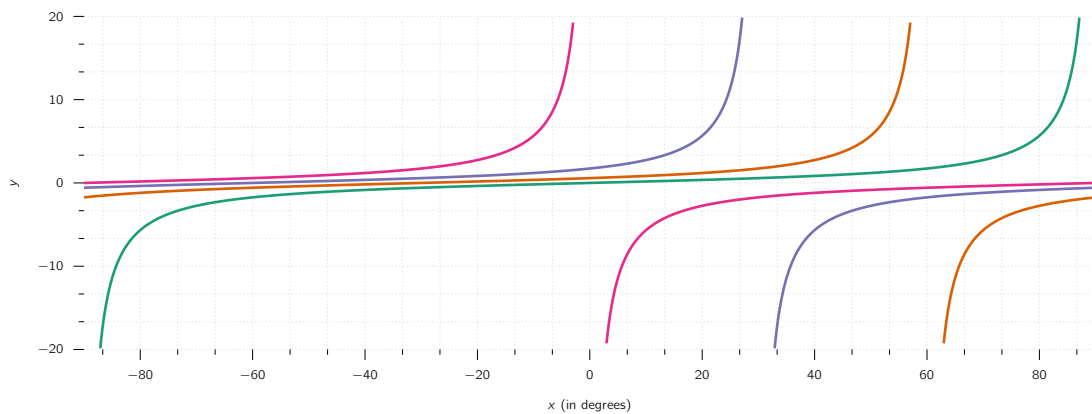


Figure 1.3: A Solution to the Separable ODE.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced $c$, we would have obtained $y = \tan x + c$, which is **NOT** a solution, when $c \neq 0$  ∎

## Q4  A Bell Shaped Curve

Solve the following ODE:
$$y' = -2xy, \qquad y(0) = 1.8$$

## A4

By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the Initial Value Problem (IVP) has the solution:

$$y = 1.8e^{-x^2}$$

This is a particular solution, representing a bell-shaped curve  ∎
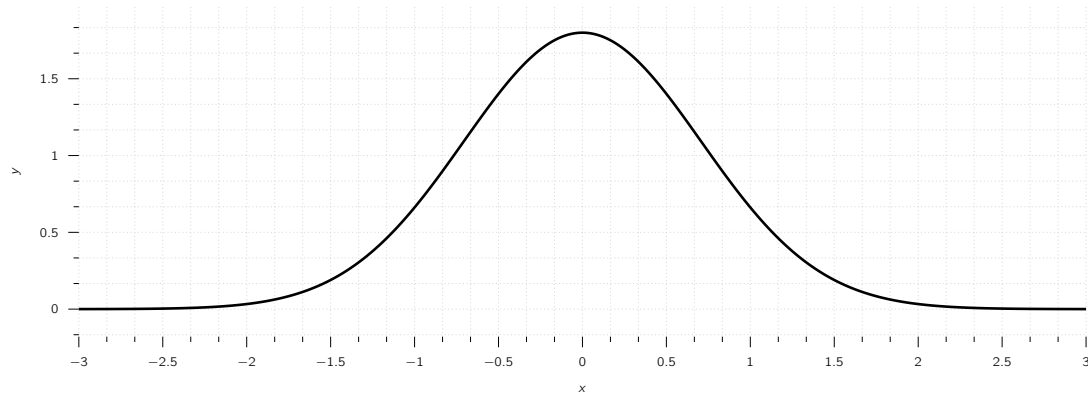
**Figure 1.4:** A Solution to the Separable ODE.

## Q5 Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian-Italian border, caused a scientific sensation.

When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

**NOTE** The half-life of carbon is 5175 years.

## A5

Radioactive decay is governed by the ODE $y' = ky$ as we have discussed previously. By separation and integration

$$\frac{dy}{y} = k \, dt, \ \ln|y| = kt + c, \ y = y_0 e^{kt}, \ y_0 = e^0.$$

Next we use the half-life $H = 5715$ to determine $k$. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \quad e^{kH} = 0.5,$$

$$k = \frac{\ln 0.5}{H} = \frac{0.693}{5715} = -0.0001213.$$

we then use the ratio 52.5% to determine the time:

$$e^{k \, t} = e^{-0.0001\,213t} = 0.525,$$

$$t = \frac{\ln 0.525}{-0.0001213} = 5312 \quad \blacksquare$$

## Q6 Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

## A6

To get the usual explicit form, we start by dividing the given equation by $2xy$ which gives,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute $y$ and $y'$ and then we simplify by subtracting $u$ on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

We see in the last equation that we can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x} \qquad \text{and by integration} \qquad \ln\left(1 + u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

We now take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx \qquad \text{therefore we can get} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centres on the $x$-axis, which can be seen below ∎
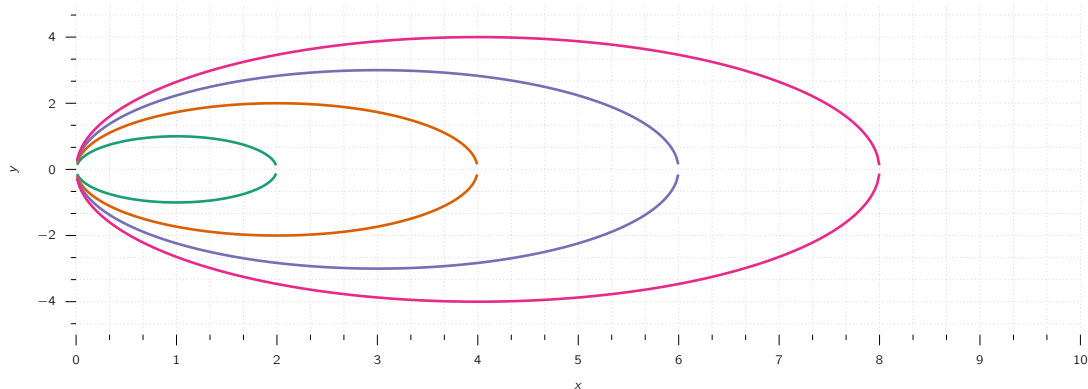


**Figure 1.5:** A Solution to the Separable ODE.

## Q7 Exact ODE - An Initial Value Problem

Solve the initial value problem:

$$(\cos y \sinh x + 1)\ dx - \sin y \cosh x\,dy = 0,$$

$$\text{with} \qquad y(1) = 2.$$

## A7

Let's begin by verifying the given equation is **exact**:

$$M(x, y) = (\cos y \sinh x + 1),$$
$$N(x, y) = -\sin y \cosh x.$$

We now apply our criteria:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin y \sinh x$$

This shows the given ODE is **exact**. We find $u$. For a change, let us use Eq. (**??**):

$$u = -\int \sin y \cosh x \, dy + I(x) = \cos y \cosh x + I(x).$$

From this:

$$\partial u / \partial x = \cos y \sinh x + \frac{dI}{dx} = u = \cos y \sinh x + 1$$

Therefore $dI/dx = 1$ and by integration,

$$I(x) = x + c^*.$$

This gives the general solution

$$u(x, y) = \cos y \cosh x + x = c.$$

From the initial condition:

$$\cos 2 \cosh 1 + 1 = 0.358 = c$$

Therefore the answer is:

$$\cos y \cosh x + x = 0.358 \quad \blacksquare$$

## Q8    An Exact ODE

Solve the following ODE:

$$\cos(x + y) \, dx + \left(3y^2 + 2y + \cos(x + y)\right) dy = 0.$$

## A8

The solution is as follows:

**Test for exactness**    First check if our equation is **exact**, try to convert the equation of the form Eq. (**??**):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

This proves our equation to be exact.

**Implicit General Solution**   From Eq. (**??**), we obtain by integration:

$$u = \int M \, \mathrm{d}x + k(y)$$
$$= \int \cos(x+y) \, \mathrm{d}x + k(y)$$
$$= \sin(x+y) + k(y) \tag{1.2}$$

To find $k(y)$, we differentiate this formula with respect to $y$ and use formula Eq. (**??**), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{\mathrm{d}k}{\mathrm{d}y} = N = 3y^2 + 2y + \cos(x+y)$$

Therefore $\dfrac{\mathrm{d}k}{\mathrm{d}y} = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$. Inserting this result into Eq. (**1.2**) and observing Eq. (**??**), we obtain:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c \quad \blacksquare$$

**Q9**  **The Breakdown of Exactness**

Check the exactness of the following ODE:

$$-y \, \mathrm{d}x + x \, \mathrm{d}y = 0$$

**A9**

The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

Let us show that in such a case the present method does **NOT** work.

$$u = \int M \, \mathrm{d}x + k(y) = -xy + k(y),$$
$$\frac{\partial u}{\partial y} = -x + \frac{\partial k}{\partial y}.$$

Now, $\partial u / \partial y$ should equal $N = x$, as required for this equation to be exact. However, this is impossible because $k(y)$ can depend only on $y$ \quad $\blacksquare$

**Q10**  **A Non Homogeneous Ordinary Differential Equation**

Solve the initial value problem of the following equation:

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

**A10**

Here we define the parameters as:

$$p = \tan x, \quad r = \sin 2x = 2 \sin x \cos x,$$

and

$$h = \int p \, dx = \int \tan x \, dx = \ln|\sec x|.$$

From this we see that in Eq. (**??**),

$$e^h = \sec x, \qquad e^{-h} = \cos x,$$

$$e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is:

$$y(x) = \cos x \left( 2 \int \sin x \, dx + c \right),$$

$$= c \cos x - 2 \cos^2 x.$$

From this and the initial condition

$$1 = c \cdot 1 - 2 \cdot 1^2, \qquad \text{therefore} \qquad c = 3,$$

and the solution of our initial value problem is:

$$y = 3 \cos x - 2 \cos^2 x$$

Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$ ∎

# 2

# Second Order ODEs

**Q11** **A Superposition of Solutions**

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0, \quad \text{for all } x.$$

**A11**

By differentiation and substitution, we obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:
$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:
$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)$$
$$= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare$$

**Q12** **Example of a Non-Homogeneous Linear ODE**

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.
$$y'' + y = 1$$

**A12**

Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$

$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1 + \sin x)'' = -\cos x$$

$$y'' + y = 1$$

$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

**Q13** **An Initial Value Problem**

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

**A13**

**General Solution**    From the previous examples, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

**Particular Solution**
We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

**Q14** **Reduction of Order**

Find a basis of solutions of the ODE

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

### A14

Inspection shows $y_1 = x$ is a solution as $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u,$$
$$y'' = u''x + 2u'$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

$ux$ and $-xu$ cancel and we are left with the following ODE, which we divide by $x$, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0,$$
$$(x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely,

$$(x^2 - x)w' + (x - 2)w = 0.$$

Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx,$$
$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We don't need constant of integration as we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain $n$

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \quad u = \int v\,dx = \ln|x| + \frac{1}{x},$$
$$\text{hence} \quad y_2 = ux = x\ln|x| + 1$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained as basic of solutions, valid for all positive $x$ ∎

### Q15 | A General Solution in the Case of Different Real Roots

Solve the following ODE:
$$x^2y'' + 1.5xy' - 0.5y = 0$$

### A15

This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and $-1$. Hence a basis of solutions for all positive $x$ is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad ■$$
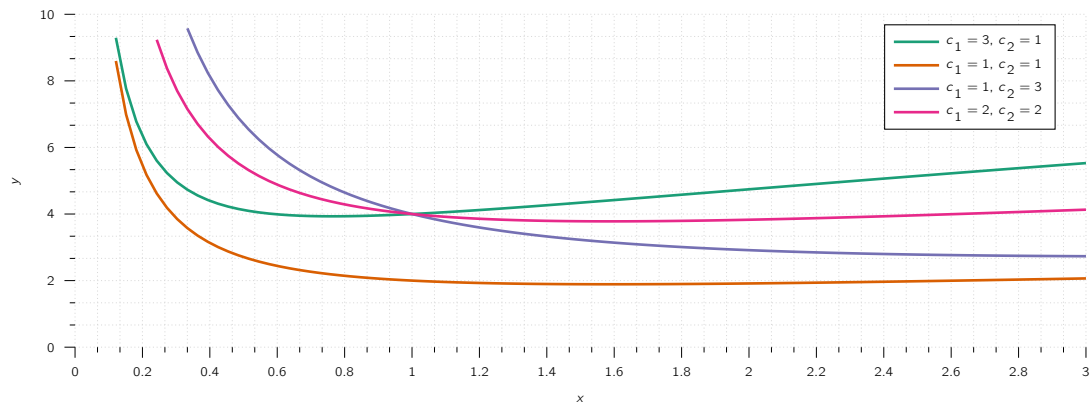
**Figure 2.1:** Solution to A General Solution in the Case of Different Real Roots with different constants.

## Q16    A General Solution in the Case of a Double Root

Solve the following ODE:

$$x^2 y'' - 5xy' + 9y = 0$$

## A16

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive $x$ is:

$$y = (c_1 + c_2 \ln x) x^2. \quad \blacksquare$$
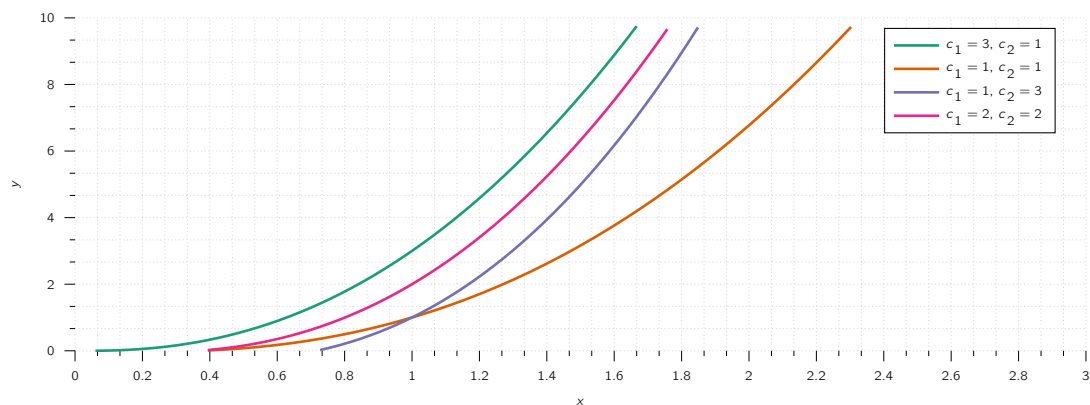


**Figure 2.2:** Solution to A General Solution in the Case of a Double Root with different constants.

## Q17    Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of the *Euler-Cauchy equation* $rv'' + 2v' = 0$.

**A17**

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$
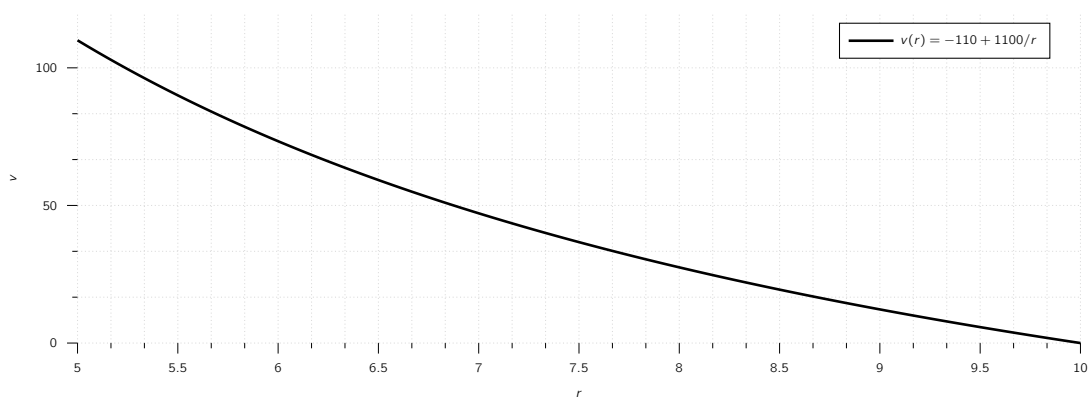


Figure 2.3: Solution to the Electric Potential Field Between Two Concentric Spheres.

**Q18** **Three Cases of Damped Motion**

How does the motion in *Harmonic Oscillation of an Undamped Mass-Spring System* change if we change the damping constant $c$ from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\text{kg} \cdot \text{s}^{-1}$

- $c = 60\,\text{kg} \cdot \text{s}^{-1}$

- $c = 10\,\text{kg} \cdot \text{s}^{-1}$

A18

It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

**Case I**   With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass-Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0{,}16 \, \text{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02 e^{-9\,t} + 0.18 e^{-\,t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

**Case II**   The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3\,t}, \qquad \text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3\,t}$$

The initial conditions give $y\left(0\right) = c_1 = 0.16$, $y\prime\left(0\right) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = \left(0.16 + 0.48t\right) e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

**Case III**   The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical $c$, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96\mathbf{j}$$

This gives the general solution:

$$y = e^{-0.5\,t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$y\prime = e^{-0.5\,t}(-0.5A\cos 2.96t - 0.5B\sin 2.96t - 2.96A\sin 2.96t + 2.96B\cos 2.96t)$$

Hence $y\prime(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution:

$$y = e^{-0.5\,t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) = 0.162e^{-0.5\,t}\cos\left(2.96t - 0.17\right) \quad \blacksquare$$
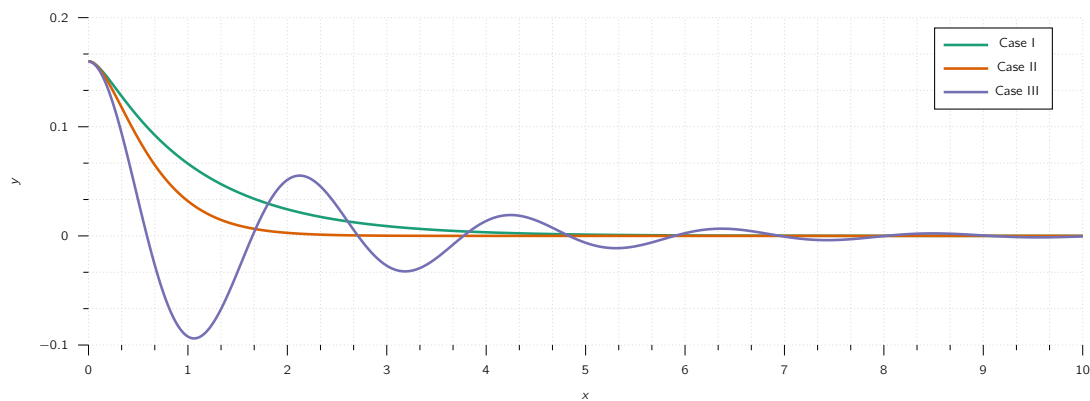


**Figure 2.4:** Three cases of damped motion.

## Q19     **Application of the Basic Rule A**

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

## A19

**General Solution of the Homogeneous ODE**    The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

**Solution of the non-Homogeneous ODE**    First we try $y_p = Kx^2$ and also $y_p'' = 2K$. By

substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all $x$, the coefficient of each power of $x$ ($x^2$ and $x^0$) must be the same on both sides. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \quad \text{a contradiction.}$$

Looking at the table suggests the choice:

$$y_p = K_2 x^2 + K_1 x + K_0,$$

$$y_p'' + y_p = 2K_2 + K_2 x^2 + K_1 x + K_0 = 0.001x^2.$$

Equating the coefficients of $x^2$, $x$, $x^0$ on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Therefore:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

**Solution of the initial value problem.**

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, therefore $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$
$$\text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$
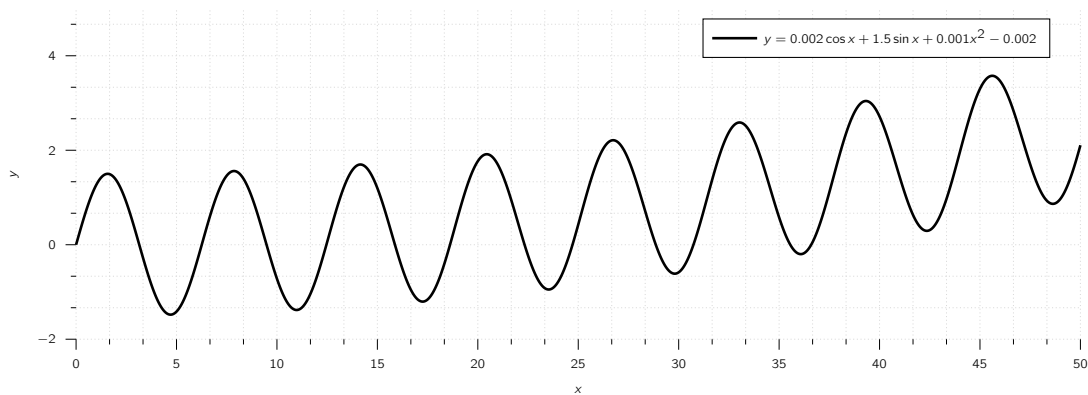


Figure 2.5: Solution to Application of the basic rule A.

## Q20  Application of the Basic Rule B

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x},$$
$$y(0) = 1, \qquad y'(0) = 0.$$

A20

**General solution of the homogeneous ODE**   The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$$

Therefore the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2y)\,e^{-1.5x}$$

**Solution $y_p$ of the non-homogeneous ODE**   The function $e^{-1.5x}$ on the Right Hand Side (RHS) would normally require the choice $Ce^{-1.5x}$. However, we see from $y_h$ that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Which means, according to the Modification Rule we have to multiply our choice function by $x^2$. That is, we choose:

$$y_p = Cx^2e^{-1.5x}, \quad \text{then}$$
$$y_p' = C(2x - 1.5x^2)e^{-1.5x},$$
$$y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of $x^2, x, x^0$ gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2e^{-1.5x}$$

**Step 3.  Solution of the initial value problem**   Setting $x = 0$ in $y$ and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of $y$ gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2v)e^{-1.2x}$$
$$-10xe^{-1.2x} + 7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$ and gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} =$$
$$(1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the $x$-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as $x$ increases.
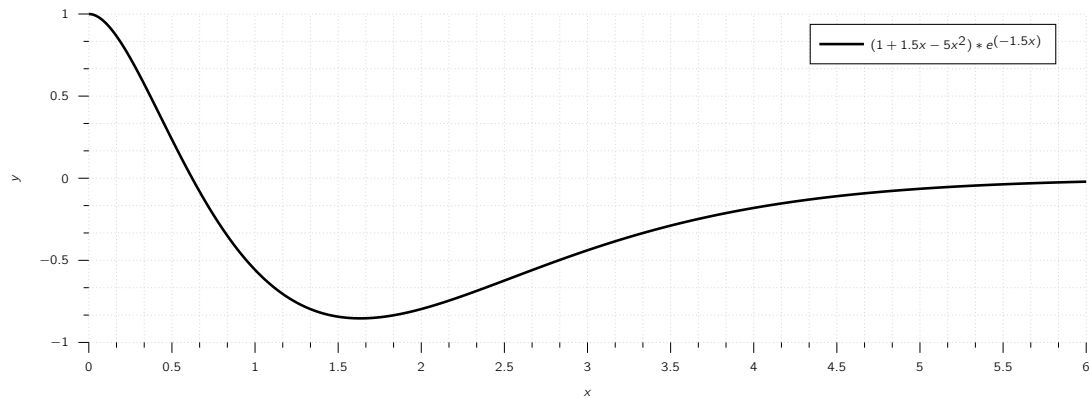
**Figure 2.6:** Solution to Application of the basic rule B.

## Q21 Application of the Basic Rule C

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

## A21

**The General Solution** The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}.$$

**The Particular Solution** We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1 x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of $y_{p1}$ into the ODE gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting $y_{p2}$ into the ODE in (7) and comparing the $x$ and $x^0$ terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

$$\text{therefore}$$

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

**Solution of the initial value problem**    From $y$, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$

$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$
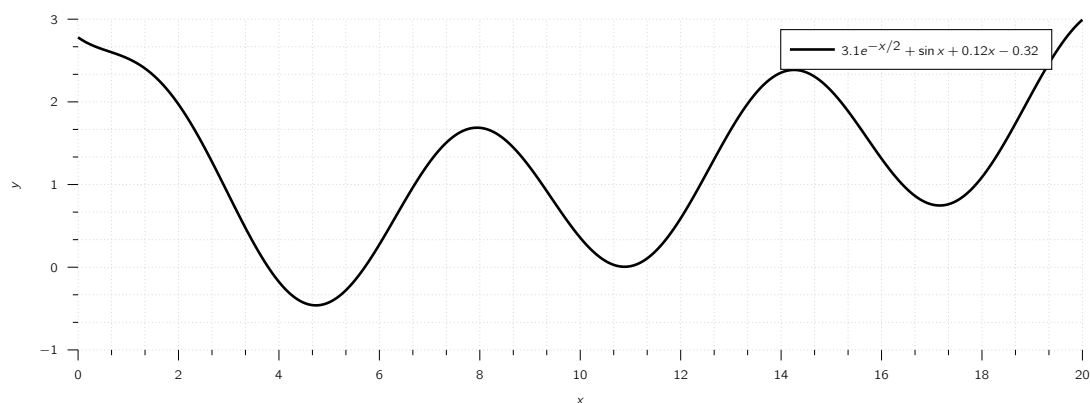


**Figure 2.7:** Solution to Application of the basic rule C.

Q22  **Studying a RLC Circuit**

Find the current $I(t)$ in an RLC-circuit with $R = 11\,\Omega$, $L = 0,9\,\text{H}$, $C = 0,01\,\text{F}$, which is connected to a source of $V(t) = 110\sin(120\pi t)$.

**Note**    Assume that current and capacitor charge are 0 when $t = 0$.

**A22**

**The General solution**    Substituting $R$, $L$, $C$ and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t.$$

$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Therefore the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

**The Particular solution**    We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from:

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Therefore in our present case, a general solution of the nonhomogeneous ODE is:

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t$$

**Particular solution satisfying the initial conditions**    How to use Q(0) 0? We finally determine $c_1$ and $c_2$ from the in initial conditions I(0) = 0 and Q(0) = 0. From the first condition and the general solution we have:

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824 sin(377t - 1.29) \quad \blacksquare$$
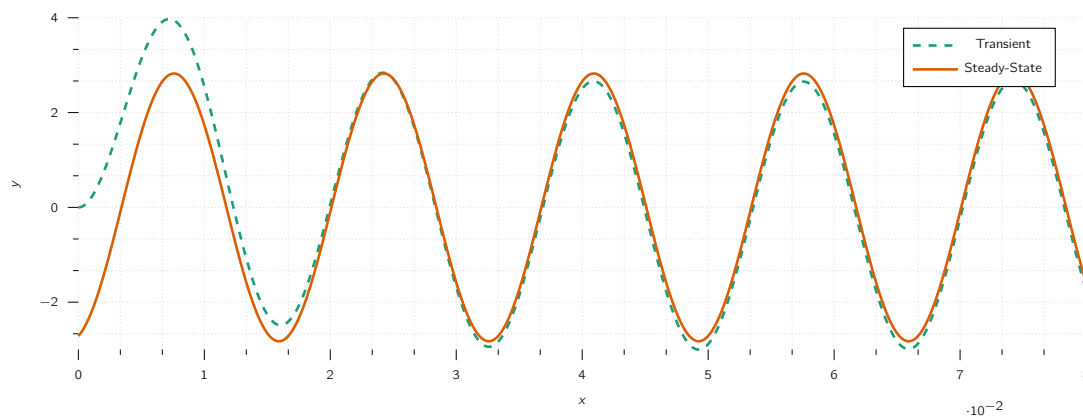


Figure 2.8: A comparison of the actual solution and the steady-state values.

## Q23    Harmonic Oscillation of an Undamped Mass-Spring System

If a mass-spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

## A23

Hooke's law:

$$F_1 = -ky \tag{2.1}$$

with $W$ as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass $(m)$ is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0{,}48\,\text{Hz}$$

From Eq. (2.1) and the initial conditions, $y(0) = A = 0{,}16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$.

Hence the motion is

$$y(t) = 0.16 \cos 3t \text{ m} \quad \blacksquare$$

**Q24**    **IVP: Case of Real Double Roots**

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

**A24**

The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = (\lambda + 2)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = (c_1 + c_2 x) e^{-0.5 x}$$

We need its derivative:

$$y' = c_2 e^{-0.5 x} - 0.5 (c_1 + c_2 x) e^{-0.5 x}$$

From this and the initial conditions we obtain:

$$y(0) = c_1 = 3.0,$$
$$y'(0) = c_2 - 0.5c_1 = 3.5,$$
$$c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x) e^{-0.5 x}$$
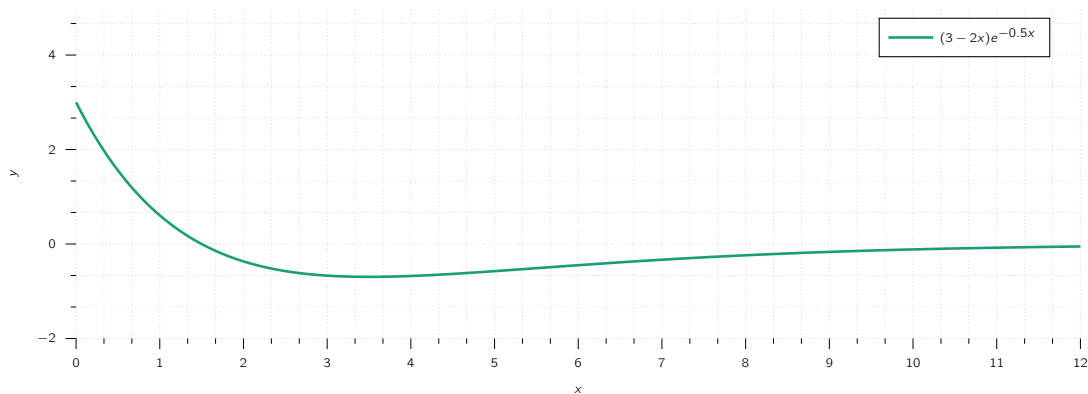
**Figure 2.9:** Solution to case of double roots.

**Q25**    **IVP: Case of Complex Roots**

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

**A25**

**General Solution**   The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathbf{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

**Particular Solution**   The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2\,x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows $y$ and $-e^{-0.2\,x}$ and $e^{-0.2\,x}$ (dashed), between which $y$ oscillates. Such "damped vibrations" have important mechanical and electrical applications.
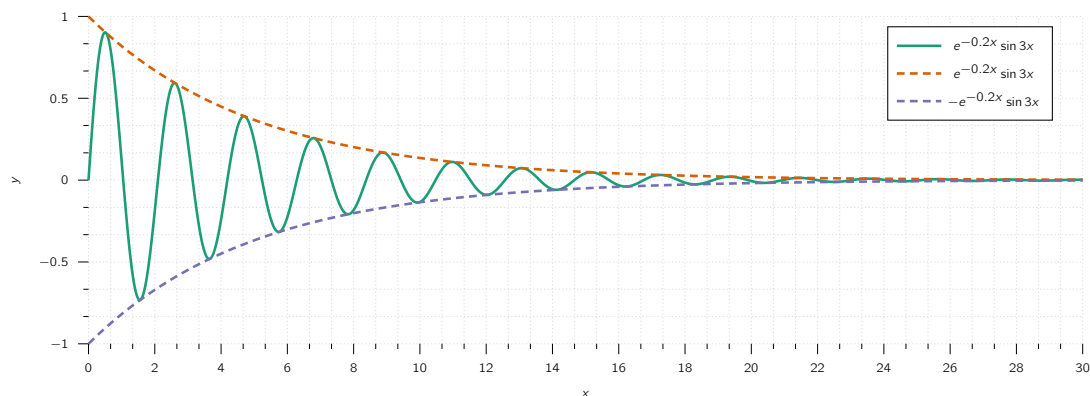


**Figure 2.10:** Solution to case of complex roots.

**Q26**   **IVP: Case of Distinct Real Roots**

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

A26

**General Solution**    The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1,$$

$$\text{and} \quad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

**Particular Solution**    As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$.  This gives the answer:

$$y = e^{\,x} + 3e^{-2\,x} \quad \blacksquare$$
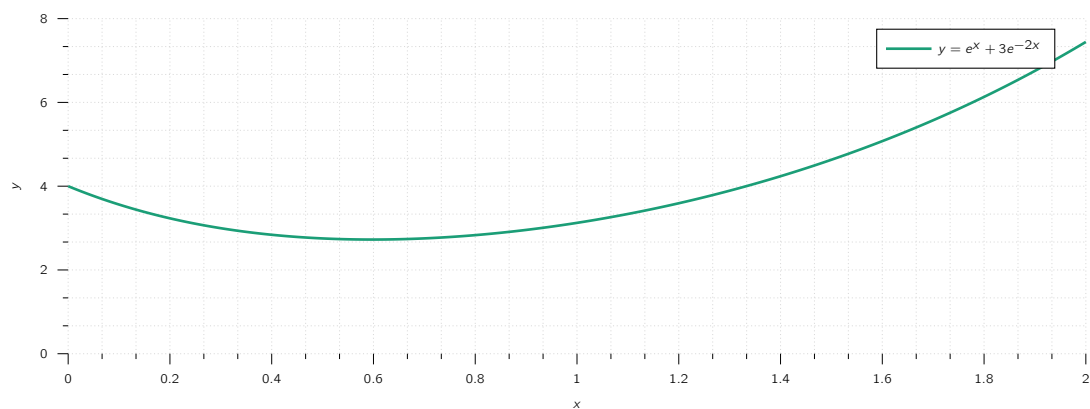


Figure 2.11: Solution to Case of distinct real roots.

# 3

# Higher Order ODEs

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**Q27** | **Linear Dependence** ─────────────────────────────

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

**A27** ─────────────────────────────────────────────

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ∎

**Q28** | **A General Solution** ─────────────────────────────

Solve the fourth-order ODE

$$y^{\,\text{iv}} - 5y'' + 4y = 0 \qquad \text{where} \qquad y^{\,\text{iv}} = \frac{d^4 y}{dx^4}$$

**A28** ─────────────────────────────────────────────

Similar to Chapter **??** we substitute $y = e^{4x}$. Omitting the common factor $e^{4x}$, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ∎

**Q29** | **Initial Value Problem for a Third-Order Euler-Cauchy Equation** ──────────────

Solve the following initial value problem on any open interval $I$ on the positive $x$-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4.$$

**A29**

**General Solution**   As in Chapter **??**, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping $x^m$ and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions $x, x^2, x^3$, which are linearly independent on $I$.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval $I$, even when it includes $x = 0$ where the coefficients of the ODE divided by $x^3$ (to have the standard form) we not continuous.

**Particular Solution**   The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and $y$ and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
\text{(a)} \quad y(1) = c_1 + && c_2 + c_3 = && 2 \\
\text{(b)} \quad y'(1) = c_1 + 2c_2 + && 3c_3 = 1 \\
\text{(c)} \quad y'(1) = 2c_2 + && 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

**Q30**  **Distinct Real Roots**

Solve the following ODE:
$$y''' - 2y'' - y' + 2y = 0$$

**A30**

The characteristic equation is:
$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots:
$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (**??**) is:
$$y = c_1 e^{-x} + c_2 e^x + c_3 e^{2x} \quad \blacksquare$$

**Q31** | **Real Double and Triple Roots**

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y^{\text{iv}} - y'' = 0$$

**A31**

The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + \left(c_3 + c_4 x + c_5 x^2\right) e^x \quad \blacksquare$$

**Q32** | **Simple Complex Roots**

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0,$$
$$y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299.$$

**A32**

The characteristic equation is:

$$\lambda_3 - \lambda_2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10\mathbf{j}$. Therefore, a general solution and its derivatives obtained by differentiation are:

$$y = c_1 e^x + A \cos 10x + B \sin 10x,$$
$$y' = c_1 e^x - 10A \sin 10x + 10B \cos 10x,$$
$$y'' = c_1 e^x - 100A \cos 10x - 100B \sin 10x.$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$(a) \quad c_1 + A = 4,$$
$$(b) \quad c_1 + 108 = 11,$$
$$(c) \quad c_1 - 1004 = -299.$$

We solve this system for the unknowns $A$, $B$, and $c_1$. Equation (a) minus Equation (c) gives $101A = 303$, therefore $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^x + 3 \cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about $e^x$.
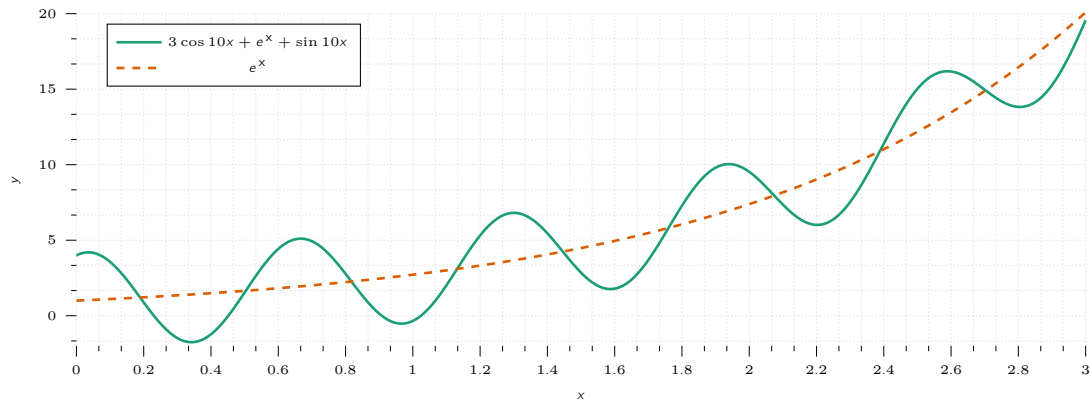
Figure 3.1: Solution to the question Simple Complex Roots

**Q33** **IVP: Modification Rule**

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47$$

**A33**

**Step 1** The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} = (c_1 + c_2 x + c_3 x^2)e^{-x}$$

**Step 2** If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try $Cxe^{-x}$ and $Cx^6 e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}$$

Then

$$y_p' = C\left(3x^2 - x^3\right)e^{-x},$$
$$y_p'' = C\left(6x - 6x^2 + x^3\right)e^{-x},$$
$$y_p''' = C\left(6 - 18x + 9x^2 - x^3\right)e^{-x}.$$

Substitution of these expressions into (6) and omission of the common factor $e^{-x}$ gives

$$C\left(6 - 18x + 9x^2 - x^3\right) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2 e^{-x}$.

**Step 3** We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find $c_1$ by the first initial condition. We insert the value, differentiate, and determine $c_2$ from the second

initial condition, insert the value, and finally determine $c_3$ from $y'(0)$ and the third initial condition:

$$y = y_h + y_p = (c_1 + c_2 + c_3 x^2)e^{-x} + 5x^3 e^{-x}, \qquad y(0) = c_1 = 3$$

$$y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, \qquad y'(0) = -3 + c_2 = -3, \quad c_2 = 0$$

$$y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, \qquad y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2 e^{-x} \qquad \blacksquare$$

The curve of $y$ begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.

# 4

# System of ODEs

**Mixing Problem Involving Two Tanks**
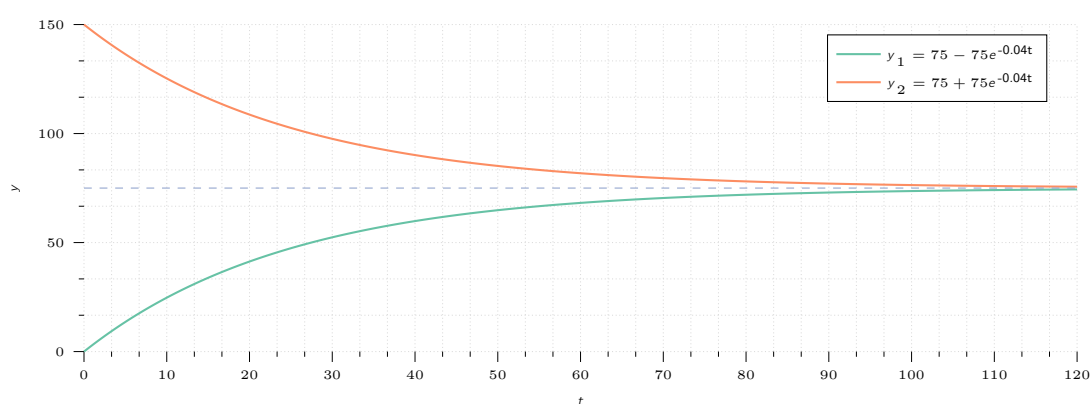
**An Electrical Network**

Find the currents $I_1(t)$ and $I_2(t)$ in the network.

**NOTE**   Assume all currents and charges zero at $t = 0$, the instant when the switch is **closed**.

The solution is as follows:

**Setting up the mathematical model**   The model of this network is obtained from Kirchhoff's Voltage Law.

> **Information**: Kirchhoff's Voltage Law
>
> The sum of the voltage differences around any closed loop in a circuit must be zero. A loop in a circuit is any path which ends at the same point at which it starts.

Let $I_1(t)$ and $I_2(t)$ be the currents in the left (L) and right (R) loops, respectively.

In (L), the voltage drops are:

$$L_1 I_1' = (1) I_1' \text{ V} \qquad \text{Over Inductor}$$
$$R_1 \left( I_1 - I_2 \right) = 4 \left( I_1 - I_2 \right) \text{ volt} \qquad \text{Over Resistor}$$

The difference is caused by $I_1$ and $I_2$ flowing through the resistor in **opposite** directions.

By Kirchhoff's Voltage Law the sum of these drops equals the voltage of the battery:

$$I_1' + 4 \left( I_1 - I_2 \right) = 12$$

Cleaning the aforementioned equation creates our first ODE:

$$I_1' = -4 I_1 + 4 I_2 + 12. \tag{4.1}$$

In (R), the voltage drops are:

$$R_2 I_2 = 6 I_2 \text{ V} \qquad \text{Over Resistor}$$
$$R_1 (I_2 - I_1) = 4 \left( I_2 - I_1 \right) \text{ V} \qquad \text{Over Resistor}$$
$$(1/C) \int I_2 \, dt = 4 \int I_2 \, dt \text{V} \qquad \text{Over Capacitor}$$

As there is no voltages sources in the (R) loop, the voltage sum **MUST** be zero.

$$6 I_2 + 4 \left( I_2 - I_1 \right) + 4 \int I_2 \, dt = 0 \qquad \text{or} \qquad 10 I_2 - 4 I_1 + 4 \int I_2 \, dt = 0.$$

Division by 10 and differentiation gives $I_2' - 0.4 I_1 [\prime] + 0.4 I_2 = 0$.

To simplify the solution process, we first get rid of $0.4 I_1'$, which by Eq. (**4.1**) equals $0.4 \left( -4 I_1 + 4 I_2 + 12 \right)$. Substitution into the present ODE gives

$$I_2' = 0.4 I_1 [\prime] - 0.4 I_2 = 0.4 \left( -4 I_1 + 4 I_2 + 12 \right) - 0.4 I_2$$

and by simplification

$$I_2' = -1.6 I_1 + 1.2 I_2 + 4.8. \tag{4.2}$$

In matrix form, Eq. (**4.1**) and Eq. (**4.2**) are (we write $J$ since $I$ is the unit matrix)

$$\boldsymbol{J}' = \boldsymbol{A}\,\boldsymbol{J} + \boldsymbol{g}, \qquad \text{where} \qquad \boldsymbol{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}. \tag{4.3}$$

**General Solution**

As we have a vector $\boldsymbol{g}$, this is a **non-homogeneous** system, and we try to proceed as for a single ODE, solving first the homogeneous system $\boldsymbol{J}' = \boldsymbol{A}\,\boldsymbol{J}$ (thus $\boldsymbol{J}' - \boldsymbol{A}\,\boldsymbol{J} = 0$) by substituting $\boldsymbol{J} = \boldsymbol{x} e^{\lambda t}$. This gives

$$\boldsymbol{J}' = \lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A}\boldsymbol{x} e^{\lambda t} \qquad \text{hence} \qquad \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Hence, to obtain a non-trivial solution, we again need the eigenvalues and eigenvectors. For the present matrix $\boldsymbol{A}$ they are:

$$\lambda_1 = -2, \qquad \boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

Hence a *general solution* of the homogeneous system is:

$$\boldsymbol{J}_{\text{h}} = c_1 \boldsymbol{x}^{(1)} e^{-2\,t} + c_2 \boldsymbol{x}^{(2)} e^{-0.8\,t}.$$

For a particular solution of the nonhomogeneous system Eq. (**4.3**), since $\boldsymbol{g}$ is constant, we try a constant column vector $\boldsymbol{J}_{\text{p}} = \boldsymbol{a}$ with components $a_1$, $a_2$. Then $\boldsymbol{J}'_{\text{p}} = 0$, and substitution into Eq. (**4.3**) gives $\boldsymbol{A}\,\boldsymbol{a} + \boldsymbol{g} = 0$, in components,

$$-4.0a_1 + 4.0a_2 + 12.0 = 0$$

$$-1.6a_1 + 1.2a_2 + 4.8 = 0.$$

The solution is $a_1 = 3$, $a_2 = 0$; thus $\boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

$$J = J_h + J_p = c_1 x^{(1)} e^{-2t} +$$

This is a parametric representation with time as the parameter $t$. It is often important to know in which sense such a curve is traced. This can be indicated by an arrow in the sense of increasing $t$. The $I_1$, $I_2$-plane is called the **phase plane** of our system Eq. (**4.3**), and the curve in **??** is called a trajectory.

In following chapters we will see that such *phase plane representations* are far more important than graphs because they will give a much better qualitative overall impression of the general behavior of whole families of solutions, not merely of one solution as in the present case. ∎



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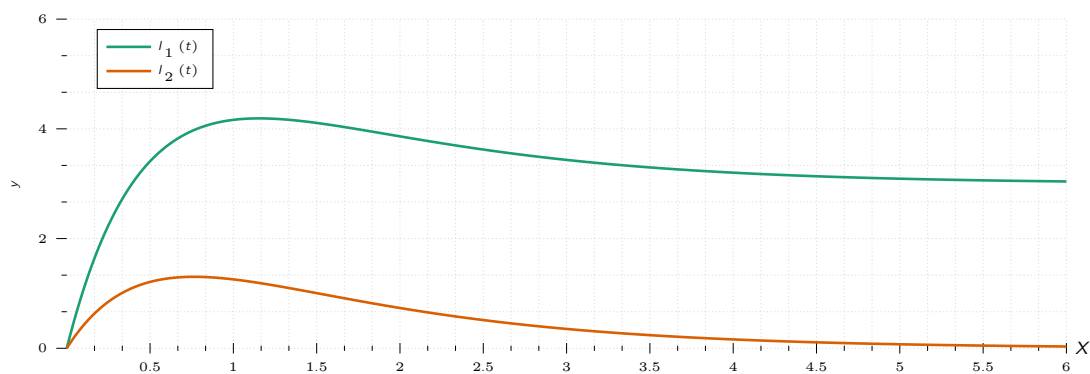
**Q36**  **Mass on a String**

To gain confidence in the conversion method, let us apply it to an old problem of ours:

modelling the free motions of a mass on a spring with value given as $m = 1$, $c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \qquad \text{or}$$

$$y'' = -\left(\frac{c}{m}\right)y' - \left(\frac{k}{m}\right)y.$$

**A36**

For this ODE given in the question can be written in the form ofEq. (**??**), making the system shown Eq. (**??**) as **linear** and **homogeneous**, applying to our system in question.

$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2.$$

Setting $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$y' = A y = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\det (A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Entering the values of $m = 1$, $c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0$$

This gives the eigenvalues:

$$\lambda_1 = -0.5 \qquad \text{and} \qquad \lambda_2 = -1.5$$

Eigenvectors follow from the first equation in $A - \lambda I = 0$, which is $-\lambda x_1 + x_2 = 0$. $\lambda_1 = 0.5$ produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$. $\lambda_2 = -1.5$ produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$. These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$x^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

Which gives:

$$y = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5 c_2 e^{-1.5t} \quad \blacksquare$$

**Q37** **Type I: Improper Node**

Find solutions of the following system:

$$y' = A y = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y, \quad \text{therefore} \quad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

**A37**

To see what is going on, let us find the solutions of the system. It is always a good idea to start with known solutions. Substituting $y = xe^{\lambda t}$ and $y' = \lambda xe^{\lambda t}$ and dropping the exponential function, as they exist both on the Left Hand Side (LHS) and RHS we can eliminate them, we get $Ax = \lambda x$. The characteristic equation is:

$$\det(A - \lambda I) = \begin{bmatrix} -3 - \lambda & 1 \\ 1 & -3 - \lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from:

$$(-3 - \lambda)x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $x^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $x^{(2)} = [1 \quad -1]^T$.

This gives the general solution:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 y^{(1)} + c_2 y^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad \blacksquare$$

**Q38**  **Type II: Proper Node**

A **proper node** is a critical point $P_0$ at which every trajectory has a definite limiting direction and for any given direction $d$ at $P_0$ there is a trajectory having $d$ as its limiting direction. Let's study the following system:

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y, \qquad \text{Therefore} \qquad y_1' = y_1 \qquad \text{and} \qquad y_2' = y_2$$

**A38**

The equation has a proper node at the origin with the matrix being the **identity matrix**. Its characteristic equation $(1 - \lambda)^2 = 0$ has the root $\lambda = 1$.

**Note:**  Any $x \neq 0$ is an eigenvector.

and we can take $[1 \quad 0]^T$ and $[0 \quad 1]^T$.

Hence, a general solution is:

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \qquad \text{or} \qquad \begin{matrix} y_1 = c_1 e^t, \\ y_2 = c_2 e^t. \end{matrix} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1 \quad \blacksquare$$

**Q39** | **Type III: Saddle Point**

A **saddle point** is a critical point $P_0$ at which there are two (2) incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of $P_0$ bypass $P_0$. Let's study the following system:

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} y, \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_1 \\ y_1' &= -y_2 \end{aligned}$$

**A39**

The equation has a saddle point at the origin and its characteristic equation

$$(1 - \lambda)(-1 - \lambda) = 0.$$

has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ in eigenvector $[1\ 0]^T$ is obtained from the second row of $(A - \lambda I)x = 0$, that is, $0x_1 + (-1 - 1)x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0\ 1]^T$. Hence a general solution is:

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned} \qquad \text{or} \qquad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas** ∎.

**Q40** | **Type IV: Centre Node**

A **centre** is a critical point that is enclosed by infinitely many closed trajectories. Let's study the following system:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y, \qquad \text{Therefore} \qquad y_1' = y_2 \qquad \text{and} \qquad y_2' = -4y_1 \qquad \textbf{(4.5)}$$

**A40**

The equation has a center at the origin.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathbf{j}$ and $-2\mathbf{j}$. For $2\mathbf{j}$, an eigenvector follows from the first equation $-2\mathbf{j}x_1 + x_2 = 0$ of $(A - \lambda I)x = 0$, which can be, $[1\ \ 2\mathbf{j}]^T$.

For $\lambda = -2\mathbf{j}$ that equation is $-(-2\mathbf{j})x_1 + x_2 = 0$ and gives, say, $[1\ \ -2\mathbf{j}]^T$. Hence a complex general solution is:

$$y = c_1 \begin{bmatrix} 1 \\ 2\mathbf{j} \end{bmatrix} e^{2\mathbf{j}t} + c_2 \begin{bmatrix} 1 \\ -2\mathbf{j} \end{bmatrix} e^{-2\mathbf{j}t}, \qquad \text{therefore} \qquad \begin{aligned} y_1 &= c_1 e^{2\mathbf{j}t} + c_2 e^{-2\mathbf{j}t}, \\ y_2 &= 2\mathbf{j}c_1 e^{2\mathbf{j}t} - 2\mathbf{j}c_2 e^{-2\mathbf{j}t}. \end{aligned} \qquad \textbf{(4.6)}$$

A real solution is obtained from Eq. (**4.6**) by the Euler formula or from Eq. (**4.5**).
Namely, we can create a relation of $-4y_1 y_1^2$.

$$-4y_1 y_1' = y_2 y_2' \qquad \text{By Integration} \qquad 2y_1^2 + \frac{1}{2}y_2^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. ∎

**Q41**    **Type V: Spiral Node**

A **spiral point** is a critical point $P_0$ about which the trajectories spiral, approaching $P_0$ as $t \to \infty$.

> or tracing these spirals in the opposite sense, away from $P_0$.

The system:

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y, \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.7}$$

**A41**

has a spiral point at the origin.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1 + \mathbf{j}$ and $-1 - \mathbf{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For $\lambda = -1 + \mathbf{j}$ this becomes $-\mathbf{j}x_1 + x_2 = 0$ and we can take $[1 \quad \mathbf{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathbf{j}$ is $[1 \quad -\mathbf{j}]^T$.

This gives the **complex** general solution:

$$y = c_1 \begin{bmatrix} 1 \\ \mathbf{j} \end{bmatrix} e^{(-1+\mathbf{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathbf{j} \end{bmatrix} e^{(-1-\mathbf{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (**4.7**) by $y_1$, the second by $y_2$- and add, obtaining:

$$y_1 y_1' + y_2 y_2' = -\left( y_1^2 + y_2^2 \right).$$

We now introduce polar coordinates $r$, $t$, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to $t$ gives:

$$2rr' = 2y_1 y_1' + 2y_2 y_2'$$

Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \quad dr/r = -dt, \qquad \ln |r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real $c$ this is a spiral.

**Q42**    **Linearisation of a Free Undamped Pendulum**

A pendulum consists of a body of mass $m$ (the bob) and a rod of length $L$. Determine the locations and type of the critical points.

**Warning**    Assume that the mass of the rod and an reference are negligible.

A42

Let us begin tackling the problem:

**Setting Up the Mathematical Model**    Let $\theta$ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is $mg$, where $g$ is the acceleration of gravity.

This causes a restoring force $mg \sin \theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2$^{nd}$ law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the **acceleration**.

Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg \sin \theta = 0.$$

Dividing this by $mL$, we have:

$$\theta'' + k \sin \theta = 0 \qquad \text{with} \qquad \left( k = \frac{g}{L} \right). \tag{4.8}$$

When $\theta$ is very small, we can approximate $\sin \theta$ rather accurately by $\theta$ and obtain as an approximate solution $A \cos \sqrt{k} + B \sin \sqrt{k}$, but the *exact* solution for any $\theta$ is not an **elementary function**.

**Critical Points and Linearisation**    To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from Eq. (4.8) we obtain a nonlinear system Eq. (??) of the form:

$$y_1' = f_1(y_1, y_2) = y_2,$$
$$y_2' = f_2(y_2, y_1) = -k \sin y_1.$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi, 0)$, where $n = 0, \pm 1, \pm 2, \cdots$.

We consider $(0, 0)$. Since the Maclaurin series is

$$\sin y_1 = y_1 - \tfrac{1}{6} y_1^2 + - \cdots \approx y_1,$$

the linearized system at $(0, 0)$ is:

$$y' = A y = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} y \qquad \text{Therefore} \qquad \begin{array}{l} y_1' = y_2 \\ y_2' = -k y_1. \end{array}$$

To apply our criteria in Sec. 4.4 we calculate:

$$p = a_{11} + a_{22} = 0,$$
$$q = \det(A) = k = g/L \qquad (> 0),$$

$$\Delta = p^2 - 4q = -4k.$$

From this and Table 4.1(c) in Sec. 4.4 we conclude that $(0,0)$ is a **centre**, which is **always stable**. Since $\sin \theta = \sin y_1$ is periodic with period of $2\pi$. **Warning**    This means the critical points $(n\pi, 0)$,

$n = \pm 2, \pm 4, \cdots$, are all centres. We now consider the critical point $(\pi, 0)$, setting:

$$y_1 = \theta - \pi$$
$$y_2 = (\theta - \pi)'$$

Then in Eq. (**4.8**), we can apply the MacLaurin series:

$$\sin \theta = \sin (y_1 + \pi) = -\sin y_1 = -y_1 + \frac{1}{2}y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$y' = Ay = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} y \qquad \text{Thus} \qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= ky_1. \end{aligned}$$

We see that:

$$p = 0,$$
$$q = -k \qquad (< 0),$$
$$\Delta = -4q = 4k.$$

> Hence, by Table 4.1(b), this gives a saddle point, which is always unstable.

Because of periodicity, the critical points $(n\pi, 0)$, $n = \pm 1, \pm 3, \cdots$, are all **saddle points**.

## Q43    Linearisation of a Damped Pendulum

To gain further experience in investigating critical points, as another practically Important, let us see how the previous example changes when we add a damping term $c\theta$, (damping proportional to the angular velocity) to equation Eq. (**4.8**), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

where $k > 0$ and $c \geq 0$ (which includes our previous case of no damping, $c = 0$).

## A43

First we start by setting $\theta = y_1$, $\theta'$ as before, we obtain the nonlinear system
(use $\theta'' = y_2$),

$$y_1' = y_2$$
$$y_2' = -k\sin y_1 - cy_2.$$

We see the critical points have the same locations as the example before, namely, $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, .... To analyse this system, we start with analysing $(0, 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0, 0)$.

$$y' = Ay = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} y \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2 \end{aligned}$$

This is identical with the system in previous example, except for the positive factor $m$ (and except for the physical meaning of $y_1$). Hence for $c = 0$ (no damping) we have a centre, for small damping we have a spiral point, and so on.

We now consider the critical point $(\pi, 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin(y_1 + \pi) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi, 0)$:

$$y' = Ay = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} y, \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2. \end{aligned}$$

For our criteria, we calculate:

$$p = a_{11} + a_{22} = -c$$
$$q = \det A = -k$$
$$\Delta = p^2 - 4q = c^2 + 4k$$

This gives the following results for the critical point $(\pi, 0)$.

**No Damping** $c > 0, p = 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 3b.

**Damping** $c > 0, p < 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 94.

As $\sin y_1$ is periodic with period of $2\pi$, the critical points $(\pm 2\pi, 0)$, $(\pm 4\pi, 0)$, ... are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished.   ■

# 5

# Special Functions for ODEs

**Power Series Solution**

Solve the following ODE:

$$y' - y = 0$$

**A44**

First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \tag{5.1}$$

We put these values into the ODE:

$$\left( a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \right) - \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right) = 0$$

Then we collect like powers of $x$, finding:

$$\left( a_1 - a_0 \right) + \left( 2a_2 - a_1 \right) x + \left( 3a_3 - a_2 \right) x^2 + \cdots = 0$$

Equating the coefficient of each power of $x$ to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express $a_1$, $a_2$, $\cdots$ in terms of $a_0$, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^x. \quad \blacksquare$$

**A Special Legendre Function**

Solve the following ODE:

$$\left( 1 - x^2 \right) y'' - 2xy' + 2y = 0$$

**A45**

Substitute Eq. (**??**), Eq. (**5.1**), and Eq. (**??**) into the ODE, $(1 - x^2) y''$ gives two (2) series: for $y''$, and for $-x^2 y''$. For the term $-2xy'$ use Eq. (**5.1**) and in $2y$ use Eq. (**??**). Write like powers of $x$ vertically aligned for easy viewing. This gives:

$$
\begin{aligned}
y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\
-xy'' &= \qquad\qquad\quad -2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots \\
-2xy' &= \qquad -2a_1 x - 4a_5 x^2 - 6a_9 x^3 - 8a_4 x^4 - \cdots \\
2y &= 2a_0 + 2a_1 x - 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots .
\end{aligned}
$$

Add terms of like powers of $x$. For each power $x^0$, $x$, $x^2$ equate the sum obtained to zero. Denote these sums by 0 (constant terms), 1 (first power of $x$), and so on and write it down to the following table:

| Sum | Power | Equation |
|---|---|---|
| 0 | $x^0$ | $a_2 = -a_0$ |
| 1 | $x$ | $a_3 = 0$ |
| 2 | $x^2$ | $14a_4 = 4a_2, \qquad a_4 = \frac{4}{12} a_2 = -\frac{1}{3} a_0$ |
| 3 | $x^3$ | $a_5 = 0 \qquad$ since $\qquad a_3 = 0$ |
| 4 | $x^4$ | $30a_6 = 18a_4, \qquad a_6 = \frac{18}{30} a_4 = \frac{18}{30}\left(-\frac{1}{3}\right) a_0 = -\frac{1}{5} a_0$ |

This gives the solution

$$
y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \cdots\right) \quad \blacksquare
$$

**Note:** $a_0$, $a_1$ remain **arbitrary**.

Therefore, this is a **general solution** consisting of two (2) solutions: $x$ and

$$
1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \cdots
$$

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(x)$ and *Legendre functions* $Q_1(x)$. Here we have

$$
x = P_1(x)
$$

and

$$
1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \cdots = -Q_1(x)
$$

**Note:** The minus is by convention. The index 1 is called the *order* of these functions and here the order is 1. $\quad \blacksquare$

**Q46** | **Frobenius Method of Euler-Cauchy**

Solve the following ODE:

$$x^2 y'' + b_0 x y' + c_0 y = 0 \quad \text{where} \quad b_0, \, c_0 \text{ are constant.}$$

As can be seen this equation obeys the form of Euler-Cauchy. Here we will solve the problem using the Frobenius method.

**A46**

Substitution of $y = x^r$ gives the auxiliary equation:

$$r(r-1) + b_0 r + c_0 = 0,$$

which is the **indicial equation**. For different roots $r_1$, $r_2$ we get a basis

$$y_1 = x^{r_1}, \, y_2 = x^{r_2},$$

and for a double root $r$ we get a basis $x^r$, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

**Q47** | **Frobenius Method - Double Root**

Solve the ODE
$$x(x-1) y'' + (3x-1) y'' + y = 0 \qquad (5.2)$$

**A47**

Writing Eq. (**5.2**) in the standard form Eq. (**??**):

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

we see it satisfies the assumptions in **Theorem 1** (i.e., analytic as $x = 0$). By inserting Eq. (**??**) and its derivatives Eq. (**??**) into Eq. (**5.2**) we obtain:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-1}$$
$$+ 3 \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \qquad (5.3)$$

The smallest power is $x^{r-1}$, occurring in the 2nd and the fourth series; by equating the sum of its coefficients to zero we have

$$\left[ -r(r-1) - r \right] a_0 = 0, \qquad \text{therefore} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.
**First Solution**

Insert this value $r = 0$ into Eq. (**5.3**) and equate the sum of the coefficients of the power $x^s$ to zero, obtaining:

$$s(s-1)a_s - (s+1)a_{s+1} + 3a_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

**Second Solution**

We get a 2$^{nd}$ independent solution $y_2$ by the method of reduction of order, substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9). Sec. 2.1, which we shall use in this example, instead of stating reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of $y'$ in (11) in standard form. By partial fractions,

$$-\left[\int p\,dx = -\left[\int \frac{3x-1}{3(x-1)}\,dx = -\left[\int \left(\frac{2}{x-1} + \frac{1}{x}\right)dx = -2\ln(x-1) - \ln x.\right.\right.\right.$$

Hence becomes

$$u' = U = y_1^{-2} e^{-\int y\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = w_1 = \frac{\ln x}{1-x}.$$

$y_1$ and $y_2$ are shown in Fig. 109. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$).

**Q48** | **Working with Bessel Functions**

Please calculate the bessel functions of $J_0(x)$ and $J_1(x)$.

**A48**

For $n = 0$ we obtain from Eq. (**??**) the *Bessel function* of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} - \cdots \qquad (5.4)$$

which looks similar to a cosine. For $n = 1$ we obtain in the *Bessel function* of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1}m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2} + \frac{x^5}{2^5 2!3} - \frac{x^7}{2^7 3!4} - \cdots \qquad (5.5)$$

which looks similar to a sine. But the zeros of these functions are not completely regularly spaced and the height of the "waves" decreases with increasing $z$. Heuristically, $n^2/x^2$ in Eq. (**??**) in standard form (i.e., Eq. (**??**) divided by $x^2$) is zero (if $n = 0$) or small in absolute value for large $x$, and so is $y'/x$, so that then Bessel's equation comes close to $y' + y = 0$, the equation of $\cos x$ and $\sin x$; also $y'/x$ acts as a "damping term," in part responsible for the decrease in height. One can show that for large $x$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.6}$$

where $\sim$ is read "asymptotically equal" and means that for fixed $n$ the quotient of the two sides approaches 1 as $x \to \infty$ $\frac{x^2}{2^2(1!)^2}$.



figures/Special-Functions-for-ODEs/figures.pdf

Figure 5.1

Formula Eq. (**5.6**) is surprisingly accurate even for smaller $x$ ($>0$). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of $f_0$, you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

# 6

# Laplace Transform

Let $f(t) = 1$ when $t \geq 0$ using this information, please find $F(s)$.

**A49** ──────────────────────────────────────────────────────

From Eq. (**??**) we obtain by integration:

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt \; = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad (s > 0)$$

Such an integral is called an <span style="color:crimson">improper integral</span> and, is evaluated according to the rule:

$$\int_0^\infty e^{-st}f(t)\; dt = \lim_{T\to\infty}\int_0^T e^{-st}f(t)\; dt.$$

Therefore our convention notation means:

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]\Big|_0^T = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}e^0\right] = \frac{1}{s}$$

$$\text{where} \qquad (s > 0) \qquad \blacksquare$$

**Q50**    **Laplace Transform of an Exponential Function** ─────────────────

Let $f(t) = e^{at}$ when $t \geq 0$, where $a$ is given as a constant. Using this information, please find $\mathcal{L}(f)$.

**A50** ──────────────────────────────────────────────────────

From Eq. (**??**) we obtain by integration:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}\, dt = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty$$

Therefore, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad \blacksquare.$$

**Q51**    **Hyperbolic Functions** ──────────────────────────────────

Find the transforms of the following hyperbolic functions:

$$\cosh at \qquad \text{and} \qquad \sinh at$$

**A51**

We know the following relations exist for the hyperbolic functions:

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at}) \qquad \text{and} \qquad \sinh at = \frac{1}{2}(e^{at} - e^{-at})$$

Using this, we can obtain the definitions of them using the exponential function definition from an earlier example.

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \quad \blacksquare$$

**Q52**    **Cosine and Sine**

Derive the following formulas:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \qquad \text{and} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

**A52**

We start by write $L_e = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit $\infty$, we obtain:

$$L_e = \int_0^\infty e^{-st} \cos \omega t \ dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \ dt = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \ dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \ dt = \frac{\omega}{s} L_c.$$

By substituting $L_s$ into the formula for $L_c$ on the right and then by substituting $L_c$ into the formula for $L_s$ on the right, we obtainOf course, this is not the only method as one can just use one of the operations <span style="color:purple">twice</span> over tho get the same result:

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2} \quad \blacksquare$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2} \quad \blacksquare$$

**Q53**    **Damped Vibrations**

Using the below definitions:

$$\mathcal{L}\{e^{at} \cos \omega t\} = \frac{s - a}{(s-a)^2 + \omega^2} \qquad \text{and} \qquad \mathcal{L}\{e^{at} \sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}.$$

Find the inverse of the transform:

$$\mathcal{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}.$$

Applying the inverse transform, and using its linearity, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t) \quad \blacksquare$$

**Q54** **Inverse Using Integrations**

Find the inverse of the following functions

$$\frac{1}{s(s^2+\omega^2)} \qquad \text{and} \qquad \frac{1}{s^2(s^2+\omega^2)}$$

**A54**

Using a standard Laplace Transform table we obtain the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^1 \frac{\sin\omega t}{\omega}\,d\tau = \frac{1}{\omega^2}(1-\cos\omega t).$$

The second one we obtain as the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^2}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}.$$

**Q55** **Inversion of Y**

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

**A55**

**Step 1:** Using a Standard Laplace Transform we get the subsidiary equation [ with $Y = \mathcal{L}(y)$]

$$s^2 Y - s y(0) - y'(0) - Y = 1/s^2,$$

therefore:

$$(s^2 - 1)Y = s + 1 + 1/s^2.$$

**Step 2:** The transfer function is $Q = 1/(s^2 - 1)$, and becomes

$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}.$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for $Y$ and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t \quad \blacksquare$$

**Q56**  **Comparison with Previous Methods**

Solve the IVP:
$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0$$

**A56**

We see that the subsidiary equation is:

$$s^2 Y - 0.16s + sY - 0.16 + 9Y = 0,$$

Therefore

$$\left(s^2 + s + 9\right)Y = 0.16(s+1)$$

The solution to the algebraic equation is:

$$Y = \frac{0.16(s+1)}{s^2+s+9} = \frac{0.16(s+\frac{1}{2}) + 0.08}{(s+\frac{1}{2})^2 + \frac{35}{4}}.$$

Therefore by the first shifting theorem and the formulas for cos and sin in from this Laplace Transform table we obtain:

$$y(t) = \mathcal{L}^{-1}(Y)$$

$$= e^{-t/2}\left(0.16\cos\sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\sqrt{\frac{35}{4}}t\right)$$

$$= e^{-0.5t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) \quad \blacksquare$$

**Q57**  **Shifted Data**

As a note for this question, shifted data means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$.

For such a problem, set $t = \widetilde{t} + t_0$, so that $t = t_0$ gives $\widetilde{t} = 0$ and the Laplace transform can be applied.

For instance, solve:

$$y'' + y = 2t, \qquad y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \qquad y'\left(\frac{1}{4}\pi\right) = 2 - \sqrt{2}.$$

**A57**

We have $t_0 = \frac{1}{4}\pi$ and we set $t = \widetilde{t} + \frac{1}{4}\pi$. Then the problem becomes:

$$\widetilde{y}'' + \widetilde{y} = 2\left(\widetilde{t} + \frac{1}{4}\pi\right)$$

$$\widetilde{y}(0) = \frac{1}{2}\pi, \qquad \widetilde{y}'(0) = 2 - \sqrt{2}.$$

where $\widetilde{y}\left(\widetilde{t}\right) = y(t)$. Using a standard Laplace Transform table and denoting the transform of $\widetilde{y}$ by $\widetilde{Y}$, we see that the subsidiary equation of the **shifted** initial value problem is:

$$s^2\widetilde{Y} - s\left(\frac{1}{2}\pi\right) - \left(2 - \sqrt{2}\right) + \widetilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\pi\right)}{s}$$

Therefore:

$$\left(s^2 + 1\right)\widetilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\right)}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this *algebraically* for $\widetilde{Y}$, we obtain

$$\widetilde{Y} = \frac{2}{\left(s^2 + 1\right)s^2} + \frac{\frac{1}{2}\pi}{\left(s^2 + 1\right)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}$$

The inverse of the first two terms can be seen from a previous example (with $\omega = 1$), and the last two terms give cos and sin,

$$\widetilde{y} = \mathcal{L}^{-1}\left(\widetilde{Y}\right)$$

$$= 2\left(\widetilde{t} - \sin\widetilde{t}\right) + \frac{1}{2}\pi\left(1 - \cos\widetilde{t}\right)$$

$$+ \frac{1}{2}\pi\cos\widetilde{t} + \left(2 - \sqrt{2}\right)\sin\widetilde{t}$$

$$= 2\widetilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\widetilde{t}.$$

Now $\widetilde{t} = t - 1/4\pi$, $\sin\widetilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is:

$$y = 2t - \sin t + \cos t \quad \blacksquare$$

**Q58**    **Use of Unit Step Function**

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } t > \frac{1}{2}\pi. \end{cases}$$

(Fig. 122)

**A58**

**Step 1** In terms of unit step functions,

$$f(t) = 2(1 - u(t - 1)) + \frac{1}{2}t^2(u(t - 1) - u(t - \frac{1}{2}\pi)) + (\cos f u(t - \frac{1}{2}\pi)).$$

Indeed, $2(1 - u(t - 1))$ gives $f(t)$ for $0 < t < 1$, and so on.

**Step 2** To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t - a)u(t - a)$. Thus, $2(1 - u(t - 1))$ remains as it is and gives the transform $2(1 - e^{-a})/s$. Then

$$\mathcal{R}\left\{\frac{1}{2}t^2 u(t - 1)\right\} = \mathcal{R}\left(\frac{1}{2}(t - 1)^2 + (t - 1) + \frac{1}{2}\right)u(t - 1)\right\} = \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-a}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2 u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t - \frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$= \left(\frac{1}{s^2} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{\cos\theta\, u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t - \frac{1}{2}\pi\right)\right\}$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2 + 1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t - a)$ is inconvenient, replace it by

$$\mathcal{L}\{f(t)u(t - a)\} = e^{-at}\mathcal{L}\{f(t + a)\}.$$

**Q59** **Application of Both Shifting Theorems**

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-2s}}{(s + 2)^2}.$$

**A59**

Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and $te^{-2t}$ as $1/s^{-2}$ has the inverse $t$, so that $1/(s+2)^2$ has the inverse $te^{-2t}$ by the *first shifting theorem*. Therefore by the *second shifting theorem* ($t$-shifting),

$$f(t) = \frac{1}{\pi} \sin(\pi(t-1)) u(t-1)$$
$$+ \frac{1}{\pi} \sin(\pi(t-2)) u(t-2)$$
$$+ (t-3) e^{-2(t-3)} u(t-3)$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so the first and second terms cancel each other when $t > 2$. Therefore we obtain:

$$
\begin{aligned}
f(t) = 0 &\quad \text{if} \quad 0 < t < 1, \\
-(\sin \pi t)/\pi &\quad \text{if} \quad 1 < t < 2, \\
0 &\quad \text{if} \quad 2 < t < 3, \\
(t-3) e^{-2(t-3)} &\quad \text{if} \quad t > 3 \quad \blacksquare
\end{aligned}
$$

**Q60**   **Response of a RC-Circuit to a Singular Rectangular Wave**

Find the current $i(t)$ in the $RC$-circuit if a single rectangular wave with voltage $V_0$ is applied. The circuit is assumed to be quiescentIn this state, it means it is in a state or period of inactivity or dormancy. before the wave is applied.

**A60**

The input is $V_0 (u(t-a) - u(t-b))$. Therefore the circuit is modeled by the integro-differential equation

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C} \int_0^t i(\tau) \, d\tau = v(t) = V_0 (u(t-a) - u(t-b))$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s} \left(e^{-as} - e^{-bs}\right)$$

Solving this equation algebraically for $I(s)$, we get:

$$I(s) = F(s) \left(e^{-as} - e^{-bs}\right) \quad \text{where} \quad F(s) = \frac{V_0/R}{s + 1/(RC)} \quad \text{and} \quad \mathcal{L}^{-1}(F) = \frac{V_0}{R} e^{-t(1/RC)}$$

**Q61**   **Mass-Spring System Under a Square Wave**

Determine the response of the damped mass-spring system under a square wave.

$$y'' + 3y' + 2y = u(t-1) - u(t-2),$$
$$y(0) = 0, \quad y'(0) = 0.$$

**A61**

nFrom (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2 Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}). \quad \text{Solution} \quad Y(s) = \frac{1}{s(s^2 + 3s + 2)}(e^{-s} - e^{-2s}).$$

We have seen that $Y(s)$ is bounded from $s$-direction.

Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2 + 3s + 2)} = \frac{1}{s(s + 1)(s + 2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s + 1} + \frac{\frac{1}{2}}{s + 2}.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = \mathcal{L}^{-1}(F) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

**Q62** | **Convolution - I**

Let $H(s) = 1/\big[(s - a)\, s\big]$. Find $h(t)$.

**A62**

$1/(s - a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{at}$ and $g(t - \tau) = 1$ we thus obtain from Eq. (**??**) the answer:

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau$$

$$= \frac{1}{a}\left(e^{at} - 1\right)$$

To check the above result we can calculate:

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s - a} - \frac{1}{s}\right)$$

$$= \frac{1}{s} \cdot \frac{a}{s^2 - as}$$

$$\frac{1}{s - a} \cdot \frac{1}{s}$$

$$\mathcal{L}\left(e^{at}\right)\mathcal{L}(1) \quad \blacksquare$$

# 7

# Vector Calculus

## Q63    Simple Vectors

Find the components of the vector $\boldsymbol{v}$ with given initial point $P$ and terminal point $Q$. Find $|\boldsymbol{v}|$ and unit vector $\hat{\boldsymbol{v}}$.

$$P\,(3,\,2,\,0)\,, \qquad Q\,(5,\,-2,\,2)\,, \qquad\qquad P\,(1,\,1,\,1)\,, \qquad Q\,(-4,\,-4,\,-4)$$
$$P\,\{\cdot\}\,1,\,0,\,1.2, \quad Q\,\{\cdot\}\,0,\,0,\,6.2, \qquad\qquad P\,(2,\,-2,\,0)\,, \qquad Q\,(0,\,4,\,6)$$
$$P\,(4,\,3,\,2)\,, \qquad Q\,(-4,\,-3,\,2)\,, \qquad\qquad P\,(0,\,0,\,0)\,, \qquad Q\,\{\cdot\}\,6,\,8,\,10$$

## A63

The solution is as follows:

$$\boldsymbol{v} = (5-3)\,\hat{\boldsymbol{x}} + (-2-2)\,\hat{\boldsymbol{y}} + (2-0)\,\hat{\boldsymbol{z}} = (2)\,\hat{\boldsymbol{x}} + (-4)\,\hat{\boldsymbol{y}} + (2)\,\hat{\boldsymbol{z}},$$
$$|\boldsymbol{v}| = \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}.$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\,\hat{\boldsymbol{x}} + (-4)\,\hat{\boldsymbol{y}} + (2)\,\hat{\boldsymbol{z}}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\hat{\boldsymbol{x}} + \left(-\frac{2}{\sqrt{6}}\right)\hat{\boldsymbol{y}} + \left(\frac{1}{\sqrt{6}}\right)\hat{\boldsymbol{z}} \quad \blacksquare$$
$$\boldsymbol{v} = (-4-1)\,\hat{\boldsymbol{x}} + (-4-1)\,\hat{\boldsymbol{y}} + (-4-1)\,\hat{\boldsymbol{z}} = (-5)\,\hat{\boldsymbol{x}} + (-5)\,\hat{\boldsymbol{y}} + (-5)\,\hat{\boldsymbol{z}},$$
$$|\boldsymbol{v}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\,\hat{\boldsymbol{x}} + (-5)\,\hat{\boldsymbol{y}} + (-5)\,\hat{\boldsymbol{z}}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\hat{\boldsymbol{x}} + \left(-\frac{1}{\sqrt{3}}\right)\hat{\boldsymbol{y}} + \left(-\frac{1}{\sqrt{3}}\right)\hat{\boldsymbol{z}} \quad \blacksquare$$
$$\boldsymbol{v} = (0-1)\,\hat{\boldsymbol{x}} + (0-0)\,\hat{\boldsymbol{y}} + (6.2-1.2)\,\hat{\boldsymbol{z}} = (-1)\,\hat{\boldsymbol{x}} + (0)\,\hat{\boldsymbol{y}} + (5)\,\hat{\boldsymbol{z}},$$
$$|\boldsymbol{v}| = \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\,\hat{\boldsymbol{x}} + (0)\,\hat{\boldsymbol{y}} + (5)\,\hat{\boldsymbol{z}}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\hat{\boldsymbol{x}} + (0)\,\hat{\boldsymbol{y}} + \left(-\frac{5}{\sqrt{26}}\right)\hat{\boldsymbol{z}} \quad \blacksquare$$

$$\boldsymbol{v} = (0-2)\,\hat{\boldsymbol{x}} + \left(4-(-2)\right)\hat{\boldsymbol{y}} + (6-0)\,\hat{\boldsymbol{z}} = (-2)\,\hat{\boldsymbol{x}} + (6)\,\hat{\boldsymbol{y}} + (6)\,\hat{\boldsymbol{z}},$$
$$|\boldsymbol{v}| = \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\,\hat{\boldsymbol{x}} + (6)\,\hat{\boldsymbol{y}} + (6)\,\hat{\boldsymbol{z}}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\hat{\boldsymbol{x}} + \left(\frac{3}{\sqrt{19}}\right)\hat{\boldsymbol{y}} + \left(\frac{3}{\sqrt{19}}\right)\hat{\boldsymbol{z}} \quad \blacksquare$$
$$\boldsymbol{v} = (-4-4)\,\hat{\boldsymbol{x}} + (-3-3)\,\hat{\boldsymbol{y}} + (2-2)\,\hat{\boldsymbol{z}} = (-8)\,\hat{\boldsymbol{x}} + (-6)\,\hat{\boldsymbol{y}} + (0)\,\hat{\boldsymbol{z}},$$
$$|\boldsymbol{v}| = \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\,\hat{\boldsymbol{x}} + (-8)\,\hat{\boldsymbol{y}} + (0)\,\hat{\boldsymbol{z}}}{10} = \left(-\frac{3}{5}\right)\hat{\boldsymbol{x}} + \left(-\frac{4}{5}\right)\hat{\boldsymbol{y}} + (0)\,\hat{\boldsymbol{z}} \quad \blacksquare$$

$$v = (6 - 0)\ \hat{x} + (8 - 0)\ \hat{y} + (10 - 0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},$$

$$|v| = \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.$$

$$\hat{v} = \frac{v}{|v|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare$$

---

**Q64**  **Finding the Points**

Given the components of a vector $v = \begin{bmatrix} v_x, v_y, v_z \end{bmatrix}$ and a particular initial point $P$, find the corresponding terminal point $Q$ and the length of $v$ (i.e., $|v|$).

$$v = [3, -1, 0]\,; \qquad P\,(4,\,6,\,0)\,, \qquad\qquad v = [8,\,4,\,2]\,; \qquad P\,(-8,\,-4,\,-2)\,,$$

$$v = [0.25,\,2,\,0.75]\,; \quad P\{\cdot\}\,0,\,-0.5,\,0, \qquad v = [3,\,2,\,6]\,; \qquad P\,(4,\,6,\,0)\,,$$

$$v = [4,\,2,\,-2]\,; \qquad P\,(4,\,6,\,0)\,, \qquad\qquad v = [3,\,-3,\,3]\,; \quad P\,(4,\,6,\,0)\,,$$

---

**A64**

Previously we have defined $v = Q - P$. Here we have $v$ and $P$. To calculate $Q$ we only need to add individual components of the vector with the initial point $P$.

$$Q = v + P = (3 + 4)\ \hat{x} + (-1 + 6)\ \hat{y} + (0 + 0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},$$

$$|v| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare$$

$$Q = v + P = \big(8 + (-8)\big)\ \hat{x} + \big(4 + (-4)\big)\ \hat{y} + (-2 + 2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},$$

$$|v| = \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare$$

$$Q = v + P = (0.25 + 0)\ \hat{x} + \big(2 + (-0.5)\big)\ \hat{y} + (0.75 + 0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},$$

$$|v| = \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare$$

$$Q = v + P = (3 + 4)\ \hat{x} + (2 + 6)\ \hat{y} + (6 + 0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},$$

$$|v| = \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare$$

$$Q = v + P = (4 + 4)\ \hat{x} + (2 + 6)\ \hat{y} + (-2 + 0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},$$

$$|v| = \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare$$

$$Q = v + P = (3 + 4)\ \hat{x} + (-3 + 6)\ \hat{y} + (3 + 0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},$$

$$|v| = \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare$$

---

**Q65**  **Vector Addition**

Let $a = [2,\,1,\,0]$, $b = [-4,\,2,\,5]$ and $c = [0,\,0,\,3]$. Calculate the following vector operations:

$$2a, \qquad\qquad -a, \qquad -1/2a,$$

$$5\,(a - c)\,, \qquad 5a - 5c, \qquad (3a - 5b) + 2c,$$

$$3a + (-5b + 2c)\,, \qquad a + 2b, \qquad 2b + a.$$

The answers are the following:

$$2\boldsymbol{a} = (4)\ \hat{x} + (2)\ \hat{y} + (0)\ \hat{z},$$
$$-\boldsymbol{a} = (-2)\ \hat{x} + (-1)\ \hat{y} + (0)\ \hat{z},$$
$$-1/2\boldsymbol{a} = (-1)\ \hat{x} + (-0.5)\ \hat{y} + (0)\ \hat{z},$$
$$5(\boldsymbol{a} - \boldsymbol{c}) = (10)\ \hat{x} + (5)\ \hat{y} + (-15)\ \hat{z},$$
$$5\boldsymbol{a} - 5\boldsymbol{c} = (10)\ \hat{x} + (5)\ \hat{y} + (-15)\ \hat{z},$$
$$(3\boldsymbol{a} - 5\boldsymbol{b}) + 2\boldsymbol{c} = (26)\ \hat{x} + (-7)\ \hat{y} + (-19)\ \hat{z},$$
$$3\boldsymbol{a} + (-5\boldsymbol{b} + 2\boldsymbol{c}) = (26)\ \hat{x} + (-7)\ \hat{y} + (-19)\ \hat{z},$$
$$\boldsymbol{a} + 2\boldsymbol{b} = (-6)\ \hat{x} + (5)\ \hat{y} + (10)\ \hat{z},$$
$$2\boldsymbol{a} + \boldsymbol{b} = (-6)\ \hat{x} + (5)\ \hat{y} + (10)\ \hat{z}.$$

## Q66    Finding the Dot Product

Find the dot product (i.e., $\boldsymbol{a} \cdot \boldsymbol{b}$) on the lengths of $\boldsymbol{a} = [1, 2, 0]$ and $\boldsymbol{b} = [3, -2, 1]$ as well as the angle ($\theta$) between vectors.

To find the angle between two vector we exploit the following identity:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta,$$

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right) = \cos^{-1}\left(\frac{3 + (-4) + 0}{\sqrt{(1)^2 + (2)^2 + (0)^2}\sqrt{(3)^2 + (-2)^2 + (1)^2}}\right),$$

$$= \cos^{-1}\left(-\frac{1}{\sqrt{70}}\right) = 96.86\,\text{deg} \quad \blacksquare$$

## Q67    Finding the Vector Properties

Let $\boldsymbol{a} = [2, 1, 4]$ , $\boldsymbol{b} = [-4, 0, 3]$ and $\boldsymbol{c} = [3, -2, 1]$. Find the following descriptions.

$$|\boldsymbol{a}|, |\boldsymbol{b}|, |\boldsymbol{c}|, \qquad\qquad \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}), \ \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c},$$
$$\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{c} \cdot \boldsymbol{a}, \qquad\qquad 4\boldsymbol{a} \cdot 3\boldsymbol{c}, \ 12\boldsymbol{a} \cdot \boldsymbol{c},$$
$$|\boldsymbol{b} + \boldsymbol{c}|, |\boldsymbol{b}| + |\boldsymbol{c}|, \qquad\qquad \boldsymbol{a} \cdot \boldsymbol{c}, |\boldsymbol{a}||\boldsymbol{c}|.$$

The answers are the following:

$$|\boldsymbol{a}| = \sqrt{21}, \qquad |\boldsymbol{b}| = 5, \qquad |\boldsymbol{c}| = \sqrt{14}.$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} = 12,$$
$$\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{c} \cdot \boldsymbol{a} = 3,$$
$$4\boldsymbol{a} \cdot 3\boldsymbol{c} = 12\boldsymbol{a} \cdot \boldsymbol{c} = 96,$$

$$|\boldsymbol{b} + \boldsymbol{c}| = \sqrt{21}, \qquad |\boldsymbol{b}| + |\boldsymbol{c}| = 5 + \sqrt{14},$$
$$\boldsymbol{a} \cdot \boldsymbol{c} = 8, \qquad |\boldsymbol{a}||\boldsymbol{c}| = 7\sqrt{6},$$

### Q68   Find the Angle Between

Let $\boldsymbol{a} = [1,\ 1,\ 1]$, $\boldsymbol{b} = [2,\ 3,\ 1]$ and $\boldsymbol{c} = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(\boldsymbol{a} - \boldsymbol{c}) \text{ and } (\boldsymbol{b} - \boldsymbol{c}), \qquad (\boldsymbol{a}) \text{ and } (\boldsymbol{b} - \boldsymbol{c}).$$

### A68

The answers are as follows:

$$33.21 \text{ deg}, \qquad 22.20 \text{ deg} \quad \blacksquare$$

### Q69   Calculate the Vector Product

Find the vector product $\boldsymbol{a} \times \boldsymbol{b}$ of $\boldsymbol{a} = [1,\ 1,\ 0]$ and $\boldsymbol{b} = [3,\ 0,\ 0]$.

### A69

### Q70   Find the Cross Product

Let $\boldsymbol{a} = [1, 2, 0]$, $\boldsymbol{b} = [3,\ -4, 0]$, $\boldsymbol{c} = [3, 5, 2]$, $\boldsymbol{d} = [6, 2, 0]$. Calculate the cross product of:

$$\boldsymbol{a} \times \boldsymbol{b}, \quad \boldsymbol{b} \times \boldsymbol{a}, \qquad\qquad \boldsymbol{a} \times \boldsymbol{c}, \quad |\boldsymbol{a} \times \boldsymbol{c}|, \quad \boldsymbol{a} \cdot \boldsymbol{c},$$
$$(\boldsymbol{c} + \boldsymbol{d}) \times \boldsymbol{d}, \quad \boldsymbol{c} \times \boldsymbol{d}, \qquad\qquad \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{b},$$
$$(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{b} + \boldsymbol{a}), \qquad\qquad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}, \quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}).$$

### A70

### Q71   What is the Gradient

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$.

### A71

### Q72   More of the Gradient

Find the gradient $(\nabla f)$ of the following functions:

$$f(x,\ y) = x^2 + \frac{1}{9}y^2, \qquad f(x,\ y) = x^4 + y^4,$$
$$f(x,\ y) = \ln\left(x^2 + y^2\right), \qquad f\{\cdot\}\ x,\ y,\ z = x^2 + 4y^2 + 9z^2,$$
$$f(x,\ y) = \frac{y}{x^2 + y^2}, \qquad f(x,\ y) = x^2 y - \frac{1}{3}y^3.$$

### A72

**Q73** **Find the Divergence**

Find the divergence $(\nabla \cdot)$ to the following vector functions:

$$\boldsymbol{v} = \left(x^3 + y^3\right) \hat{\boldsymbol{x}} + \left(3xy^2\right) \hat{\boldsymbol{y}} + \left(3zy^2\right) \hat{\boldsymbol{z}}, \qquad \boldsymbol{v} = (e^x) \hat{\boldsymbol{x}} + (\ln xy) \hat{\boldsymbol{y}} + (e^{xyz}) \hat{\boldsymbol{z}},$$

$$\boldsymbol{v} = \left(x^2 + y^2\right) \hat{\boldsymbol{x}} + (2xyz) \hat{\boldsymbol{y}} + \left(z^2 + x^2\right) \hat{\boldsymbol{z}}, \qquad \boldsymbol{v} = (x) \hat{\boldsymbol{x}} + (y) \hat{\boldsymbol{y}} + (z) \hat{\boldsymbol{z}}.$$

**A73**

**Q74** **Cylindrical Divergence**

Find the divergence of the function:

$$\boldsymbol{v} = \left(s \left(2 + \sin^2 \phi\right)\right) \hat{\boldsymbol{s}} + (s \sin \phi \cos \phi) \hat{\boldsymbol{\phi}} + (3z) \hat{\boldsymbol{z}}.$$

**A74**

**Q75** **Curls of Functions**

Find he curl $(\nabla \times)$ of the following functions.

$$\boldsymbol{v} = (y) \hat{\boldsymbol{x}} + \left(2x^2\right) \hat{\boldsymbol{y}} + (0) \hat{\boldsymbol{z}}, \qquad \boldsymbol{v} = (y^n) \hat{\boldsymbol{x}} + (z^n) \hat{\boldsymbol{y}} + (x^n) \hat{\boldsymbol{z}},$$

$$\boldsymbol{v} = (\sin y) \hat{\boldsymbol{x}} + (\cos z) \hat{\boldsymbol{y}} + (- \tan x) \hat{\boldsymbol{z}}, \qquad \boldsymbol{v} = \left(x^2 - z\right) \hat{\boldsymbol{x}} + (xe^z) \hat{\boldsymbol{y}} + (xy) \hat{\boldsymbol{z}}.$$

**A75**

**Q76** **Checking the Product Rule**

Check the product rule $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$ by calculating each term separately for the functions:

$$\boldsymbol{A} = (x) \hat{\boldsymbol{x}} + (2y) \hat{\boldsymbol{y}} + (3z) \hat{\boldsymbol{z}}, \qquad \boldsymbol{B} = (3y) \hat{\boldsymbol{x}} + (-2x) \hat{\boldsymbol{y}} + (0) \hat{\boldsymbol{z}}.$$

**A76**

**Q77** **Find the Laplacian**

Calculate the Laplacian of the following functions:

$$T_a = x^2 + 3xy + 3z + 4, \qquad T_b = \sin x \sin y \sin z,$$

$$T_c = e^{-5x} \sin 4y \cos 3z, \qquad \boldsymbol{v} = \left(x^2\right) \hat{\boldsymbol{x}} + \left(3xz^2\right) \hat{\boldsymbol{y}} + (-2xz) \hat{\boldsymbol{z}}.$$

**A77**

**Q78** | **Dirac Delta Function Integrals**

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right) \delta\left(x - 3\right) \, dx, \qquad \int_0^5 \cos x \, \delta\left(x - \pi\right) \, dx,$$

$$\int_0^3 x^3 \, \delta\left(x + 1\right) \, dx, \qquad \int_{-\infty}^{+\infty} \ln\left(x + 3\right) \delta\left(x + 2\right) \, dx.$$

**A78**

**Q79** | **Double Integrals**

Find the following double integrals:

$$\int_0^1 \int_x^{2x} (x + y)^2 \, dy \, dx, \qquad \int_0^1 \int_y^{\sqrt{y}} (1 - 2xy) \, dx \, dy,$$

$$\int_0^3 \int_x^3 \cosh\left(x + y\right) \, dy \, dx, \qquad \int_0^1 \int_0^{y^3} \exp y^4 \, dx \, dy.$$

**A79**

The solution to integrations are as follows:

$$\int_0^1 \int_x^{2x} (x + y)^2 \, dy \, dx \quad = \int_0^1 \int_x^{2x} x^2 + 2xy + y^2 \, dy \, dx, = \int_0^1 \left[yx^2 + xy^2 + \frac{y^3}{3}\right]\Bigg|_x^{2x} dx,$$

$$= \int_0^1 \left(4x^3 + \frac{7x^3}{3}\right) dx, = \left[4x^3 + \frac{7x^4}{12}\right]\Bigg|_0^1 = \frac{19}{12} \quad \blacksquare$$

$$\int_0^1 \int_y^{\sqrt{y}} (1 - 2xy) \, dx \, dy = \int_0^1 \left[x - x^2 y\right]\Bigg|_y^{\sqrt{y}} dy,$$

$$= \int_0^1 \left[\left(\sqrt{y} - y^2\right) - \left(y - y^3\right)\right] dy = \int_0^1 \left[y^3 + \sqrt{y} - y^2 - y\right] dy,$$

$$= \left[\frac{y^4}{4} + \frac{2}{3} y^{3/2} - \frac{y^3}{3} - \frac{y^2}{2}\right]\Bigg|_0^1, = \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2}\right) - (0) = \frac{1}{12} \quad \blacksquare$$

$$\int_0^3 \int_x^3 \cosh\left(x + y\right) \, dy \, dx = \int_0^1 \left[\sinh\left(x + y\right)\right]\Bigg|_x^3 dx = \int_0^1 \left[\sinh\left(3 + x\right) - \sinh\left(2x\right)\right] \, dx$$

**Q80** | **Work in a Force Field**

Find the work done by the force field $\mathbf{F} = \left(y - x^2\right) \hat{x} + \left(z - y^2\right) \hat{y} + \left(x - z^2\right) \hat{z}$ along the curve $\mathbf{r}\left(t\right) = \left(t\right) \hat{x} + \left(t^2\right) \hat{y} + \left(t^3\right) \hat{z}$, $0 \leq t \leq 1$ from 0 to 1.

**A80**

**Fluid Flow**

A fluid's velocity field is $\boldsymbol{F} = (x)\ \hat{\boldsymbol{x}} + (z)\ \hat{\boldsymbol{y}} + (y)\ \hat{\boldsymbol{z}}$. Find the flow along the helix $\boldsymbol{\ell}(t) = (\cos t)\ \hat{\boldsymbol{x}} + (\sin t)\ \hat{\boldsymbol{y}} + (t)\ \hat{\boldsymbol{z}}$ with a range of $0 \le t \le \pi/2$.

We first evaluate $\boldsymbol{F}$ on the curve:

$$\boldsymbol{F} = (x)\ \hat{\boldsymbol{x}} + (z)\ \hat{\boldsymbol{y}} + (y)\ \hat{\boldsymbol{z}} = (\cos t)\ \hat{\boldsymbol{x}} + (t)\ \hat{\boldsymbol{y}} + (\sin t)\ \hat{\boldsymbol{z}} \qquad \textcolor{blue}{\text{Substitute } \mathbf{x} = \cos\mathbf{t},\ \mathbf{z} = \mathbf{t},\ \mathbf{y} = \sin\mathbf{t}.}$$

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{\boldsymbol{x}} + (\cos t)\ \hat{\boldsymbol{y}} + (0)\ \hat{\boldsymbol{z}}.$$

Then we integrate $\boldsymbol{F} \cdot (d\boldsymbol{\ell}/dt)$ from $t = 0$ to $t = \pi/2$:

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = (\cos t)\ (-\sin t) + (t)\ (\cos t) + (\sin t)\ (1)\,,$$

$$= -\sin t \cos t + t \cos t + \sin t.$$

Which makes,

$$\text{Flow} = \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\ dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\ dt,$$

$$= \left[ \frac{\cos^2 t}{2} + t \sin t \right] \Bigg|_0^{\pi/2} = \left( 0 + \frac{\pi}{2} \right) - \left( \frac{1}{2} + 0 \right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare$$

**Field Circulation**

Find the circulation of the field $\boldsymbol{F} = (x - y)\ \hat{\boldsymbol{x}} + x\ \hat{\boldsymbol{y}}$ around the circle $\boldsymbol{\ell}(t) = (\cos t)\ \hat{\boldsymbol{x}} + (\sin t)\ \hat{\boldsymbol{y}} + (0)\ \hat{\boldsymbol{z}}$ with a range of $0 \le t \le 2\pi$.

On the circle, $\boldsymbol{F} = (x - y)\ \hat{\boldsymbol{x}} + (x)\ \hat{\boldsymbol{y}} + (0)\ \hat{\boldsymbol{z}} = (\cos t - \sin t)\ \hat{\boldsymbol{x}} + (\cos t)\ \hat{\boldsymbol{y}} + (0)\ \hat{\boldsymbol{z}}$ and

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{\boldsymbol{x}} + (\cos t)\ \hat{\boldsymbol{y}} + (0)\ \hat{\boldsymbol{z}}.$$

Then

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1},$$

Gives.

$$\text{Circulation} = \int_0^{2\pi} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\ dt = \int_0^{2\pi} (1 - \sin t \cos t)\ dt$$

$$= \left[ t - \frac{\sin^2 t}{2} \right] \Bigg|_0^{2\pi} = 2\pi \quad \blacksquare$$

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**Q83** | **Work Done by a Field**

Suppose the force field $\boldsymbol{F} = \nabla f$ is the gradient of the function:

$$f\{\cdot\}\, x, y, z = -\frac{1}{x^2 + y^2 + z^2}.$$

Find the work done by $\boldsymbol{F}$ in the moving an object along a smooth curve $C$ joining $(1,0,0)$ to $(0,0,2)$ that does not pass through the origin.

**A83** |

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**Q84** | **Testing Stokes's Theorem**

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y)\,\hat{\boldsymbol{x}} + (-x)\,\hat{\boldsymbol{y}} + (0)\,\hat{\boldsymbol{z}}$. (**Tip:** Parametrisation of a circle is: $x = r\cos\theta, y = r\sin\theta$ and $da = \frac{3}{z}dA$)

**A84** |

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**Q85** | **Evaluating Divergence Theorem**

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\,\hat{\boldsymbol{x}} + (y)\,\hat{\boldsymbol{y}} + (z)\,\hat{\boldsymbol{z}}$ over the sphere $x^2 + y^2 + z^2 = a^2$

**A85** |

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**Q86** | **Measuring the Field Flux**

Find the flux of $\boldsymbol{F} = (xy)\,\hat{\boldsymbol{x}} + (yz)\,\hat{\boldsymbol{y}} + (xz)\,\hat{\boldsymbol{z}}$ outward through the surface of the cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$.

**A86** |

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**Q87** | **A Divergence or Not**

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2\cos\theta\right)\,\hat{\boldsymbol{r}} + \left(r^2\cos\phi\right)\,\hat{\boldsymbol{\theta}} + \left(-r^2\cos\theta\sin\phi\right)\,\hat{\boldsymbol{\phi}}.$$

using as your volume one octant of the sphere of radius $R$.

**A87** |

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**Q88** | **Calculating the Integral along a Path**

Compute the line integral of:

$$\boldsymbol{v} = (6)\,\hat{\boldsymbol{x}} + \left(yz^2\right)\,\hat{\boldsymbol{y}} + (3y + z)\,\hat{\boldsymbol{z}},$$

along the triangular path shown below. Afterwards check your answer using Stokes' theorem.

A88

Q89 **An Example of Divergence Theorem**

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\ \hat{\boldsymbol{x}} + (y)\ \hat{\boldsymbol{y}} + (z)\ \hat{\boldsymbol{z}}$
over the sphere $x^2 + y^2 + z^2 = a^2$

A89

The outer unit normal to $S$, calculated from the gradient of $f\{\cdot\}\ x,\ y,\ z = x^2 + y^2 + z^2 - a^2$, is:

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(2x)\ \hat{\boldsymbol{x}} + (2y)\ \hat{\boldsymbol{y}} + (2z)\ \hat{\boldsymbol{z}}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x)\ \hat{\boldsymbol{x}} + (y)\ \hat{\boldsymbol{y}} + (z)\ \hat{\boldsymbol{z}}}{a}. \qquad \mathbf{x^2 + y^2 + z^2 = a^2}\ \text{on}\ \mathbf{S}$$

Therefore:

$$(\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \frac{x^2 + y^2 + z^2}{a}\ \mathrm{d}a = \frac{a^2}{a}\ \mathrm{d}a = a\,\mathrm{d}a.$$

This in turn gives us:

$$\iint_{\mathcal{S}} (\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \iint_{\mathcal{S}} a\,\mathrm{d}a = a \iint_{\mathcal{S}} \mathrm{d}a = a\left(4\pi a^2\right) = 4\pi a^3. \qquad \text{Area of } \mathbf{S} \text{ is } \mathbf{4\text{\ss} a^2}$$

The divergence of $\boldsymbol{F}$ is:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

So,

$$\iiint_V (\nabla \cdot \boldsymbol{v})\ \mathrm{d}\tau = \iiint_V 3\ \mathrm{d}\tau = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3 \quad \blacksquare$$

Q90 **Divergence Theorem of an Octant of a Sphere**

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2 \cos\theta\right)\ \hat{r} + \left(r^2 \cos\phi\right)\ \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right)\ \hat{\phi}.$$

using as your volume one octant of the sphere of radius $R$.

A90

It is always useful to write the theorem we are going to work on:

$$\underbrace{\iiint_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\ dV}_{\text{Divergence integral}} = \underbrace{\iint_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{n}\,da}_{\text{Outward flux}}.$$

First solve the left hand side of the equation:

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta r^2 \cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(-r^2 \cos\theta \sin\phi\right),$$

$$= \frac{1}{r^2}4r^3 \cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2 \cos\phi + \frac{1}{r\sin\theta}\left(-r^2 \cos\theta \cos\phi\right),$$

$$= \frac{r\cos\theta}{\sin\theta}[4\sin\theta + \cos\phi - \cos\phi] = 4r\cos\theta.$$

$$\int (\nabla \cdot \boldsymbol{v})\ d\tau = \int (4r\cos\theta)\ r^2 \sin\theta\ dr\ d\theta\ d\phi = 4\int_0^R r^3\ dr \int_0^{\pi/2} \cos\theta \sin\theta\ d\theta \int_0^{\pi/2} d\phi,$$

$$= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \blacksquare$$

Now it is time to solve the right hand side of the question. As we are aware from the shape, an octant of the sphere has 4 sides to it: the curved surface $xyz \to \boldsymbol{a}_1$, and $xz \to \boldsymbol{a}_2$, $yz \to \boldsymbol{a}_3$ and $xy \to \boldsymbol{a}_4$. These are

$$d\boldsymbol{a}_1 = \hat{\boldsymbol{r}}\, dl_\theta\, dl_\phi = \hat{\boldsymbol{r}}\, R^2 \sin\theta\, d\phi\, d\theta, \qquad d\boldsymbol{a}_2 = dl_r\, dl_\theta = -\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta,$$

$$d\boldsymbol{a}_3 = \hat{\boldsymbol{\phi}}\, dl_r\, dl_\theta = \hat{\boldsymbol{\phi}}\, r\, dr\, d\theta, \qquad d\boldsymbol{a}_4 = dl_r\, dl_\phi = \hat{\boldsymbol{\theta}}\, r\, dr\, d\theta. \quad (\theta = \pi/2)$$

$$\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} = \iint_{\mathcal{S}_1} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{\mathcal{S}_2} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{\mathcal{S}_3} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{\mathcal{S}_4} \boldsymbol{v} \cdot d\boldsymbol{a},$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}\right]\Big|_{r=R} \cdot \left(\hat{\boldsymbol{r}}\, R^2 \sin\theta\, d\phi\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}\right]\Big|_{\phi=0} \cdot \left(-\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}\right]\Big|_{\phi=\pi/2} \cdot \left(\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}\right]\Big|_{\theta=\pi/2} \cdot \left(\hat{\boldsymbol{\theta}}\, r\, dr\, d\theta\right),$$

Time to do some integration.

$$\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^{\pi/2} \int_0^{\pi/2} \left[R^2 \cos\theta\, \hat{\boldsymbol{r}} + R^2 \cos\phi\, \hat{\boldsymbol{\theta}} - R^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}\right] \cdot \left(\hat{\boldsymbol{r}}\, R^2 \sin\theta\, d\phi\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + r^2 (1)\, \hat{\boldsymbol{\theta}} - (0) \sin\phi\, \hat{\boldsymbol{\phi}}\right] \cdot \left(-\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[r^2 \cos\theta\, \hat{\boldsymbol{r}} + (0)\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta(1)\, \hat{\boldsymbol{\phi}}\right] \cdot \left(\hat{\boldsymbol{\phi}}\, r\, dr\, d\theta\right)$$

$$+ \int_0^{\pi/2} \int_0^{R} \left[(0)\, \hat{\boldsymbol{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - (0)\, \hat{\boldsymbol{\phi}}\right] \cdot \left(\hat{\boldsymbol{\theta}}\, r\, dr\, d\theta\right).$$

Final touches and cleaning up,

$$\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} = \int_0^{\pi/2} \int_0^{\pi/2} R^4 \sin\theta \cos\theta\, d\phi\, d\theta + 0 - \overbrace{\int_0^{\pi/2} \int_0^{R} r^3 \cos\theta\, dr\, d\theta + \int_0^{\pi/2} \int_0^{R} r^3 \cos\theta\, dr\, d\phi}^{=0},$$

$$= R^4 \left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2} \sin\theta \cos\theta\, d\theta\right),$$

$$= R^4 \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right),$$

$$= \frac{\pi R^4}{4} \quad \blacksquare$$

**Q91**    **Surface Area of an Implicit Surface**

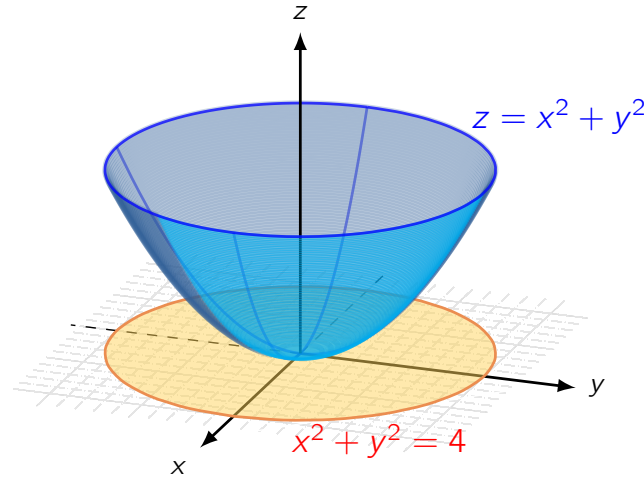Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.



Figure 7.1: The paraboloid of "Surface Area of an Implicit Surface".

**A91**

We sketch the surface $S$ and the region $R$ below it in the xy-plane (**Fig.** 7.1). The surface $S$ is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, ans $R$ is the disk $x^2 + y^2 \leq 4$ in the $xy$-plane. To get a unit vector normal (i.e., $\hat{n}$) to the plane $R$, we can take $\hat{n} = \hat{z}$. At any point $(x, y, z)$ on the surface, we have:

$$F(x, y, z) = x^2 + y^2 - z$$
$$\nabla F = (2x)\ \hat{x} + (2y)\ \hat{y} + (-1)\ \hat{z}$$
$$|\nabla F| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2}$$
$$= \sqrt{4x^2 + 4y^2 + 1}$$
$$|\nabla F \cdot \hat{n}| = |\nabla F \cdot \hat{z}| = |-1| = 1.$$

In the region $R$, the area is defined to be $dA = dx\,dy$. Therefore:

$$\text{Surface Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{n}|}\,dA$$
$$= \iint_{x^2+y^2\leq 4} \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$$
$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{1}{12}\left(4r^2 + 1\right)^{3/2}\Bigg|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} \left( 17^{3/2} - 1 \right) d\theta$$

$$= \frac{\pi}{6} \left( 17\sqrt{17} - 1 \right) \quad \blacksquare$$

**Q92**  **Stokes' Theorem Over a Hemisphere**

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y) \; \hat{\boldsymbol{x}} + (-x) \; \hat{\boldsymbol{y}} + (0) \; \hat{\boldsymbol{z}}$.

**Tip:** Parametrisation of a circle is: $x = r\cos\theta$, $y = r\sin\theta$ and $da = \frac{3}{z} dA$)

**A92**

The start by calculating the counter-clockwise circulation around $C$ using the following parametrisation:

$$\boldsymbol{l}(\theta) = (3\cos\theta) \; \hat{\boldsymbol{x}} + (3\sin\theta) \; \hat{\boldsymbol{y}} + (0) \; \hat{\boldsymbol{z}},$$

$$\text{where} \qquad 0 \leq \theta \leq 2\pi.$$

Using this we can calculate the counter-clockwise circulation.

$$d\boldsymbol{l} = (-3\sin\theta \, d\theta) \; \hat{\boldsymbol{x}} + (3\cos\theta \, d\theta) \; \hat{\boldsymbol{y}} + (0) \; \hat{\boldsymbol{z}},$$

$$\boldsymbol{F} = (y) \; \hat{\boldsymbol{x}} + (-x) \; \hat{\boldsymbol{y}} + (0) \; \hat{\boldsymbol{z}}$$

$$= (3\sin\theta) \; \hat{\boldsymbol{x}} + (-3\cos\theta) \; \hat{\boldsymbol{y}} + (0) \; \hat{\boldsymbol{z}},$$

$$\boldsymbol{F} \cdot d\boldsymbol{l} = -9\sin^2\theta \, d\theta - 9\cos^2\theta \, d\theta = -9 \, d\theta,$$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \int_0^{2\pi} -9 \, d\theta = -18\pi.$$

For the curl of integral we have:

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (0 - 0) \; \hat{\boldsymbol{x}} + (0 - 0) \; \hat{\boldsymbol{y}} + (-1 - 1) \; \hat{\boldsymbol{z}} = -2 \, \hat{\boldsymbol{z}}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(x) \; \hat{\boldsymbol{x}} + (y) \; \hat{\boldsymbol{y}} + (z) \; \hat{\boldsymbol{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{(x) \; \hat{\boldsymbol{x}} + (y) \; \hat{\boldsymbol{y}} + (z) \; \hat{\boldsymbol{z}}}{3} \qquad \textbf{\textcolor{blue}{Unit normal}}$$

Now it is time to define the area of integration ($da$):

$$da = \frac{|\nabla S|}{|\nabla S \cdot \hat{\boldsymbol{z}}|} dA$$

$$= \frac{|(2x) \; \hat{\boldsymbol{x}} + (2y) \; \hat{\boldsymbol{y}} + (2z) \; \hat{\boldsymbol{z}}|}{2z}$$

$$= \frac{2 \overbrace{\sqrt{x^2 + y^2 + z^2}}^{a}}{2z}$$

$$= \frac{3}{z} dA,$$

$$\nabla \times \boldsymbol{F} \cdot \boldsymbol{n} \, da = -\frac{2z}{3} \frac{3}{z} \, dA = -2 \, dA$$

The cardinal direction $\hat{z}$ comes from being the direction **perpendicular** to the surface $(S)$.

$$\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} \, da = \iint_{x^2+y^2 \leq 9} -2 \, dA = -18\pi$$

The circulation around the circle equals the integral of the curl over the hemisphere ∎

**Q93** **The Volume of a Sphere**

Find the volume of a sphere of radius $R$.

**A93**

The derivation is as follows:

$$V = \int d\tau$$

$$\int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

$$= \left( \int_0^R r^2 \, dr \right) \left( \int_0^\pi \sin\theta \, d\theta \right) \left( \int_0^{2\pi} d\phi \right)$$

$$= \left( \frac{R^3}{3} \right) (2)(2\pi) = \frac{4}{3}\pi R^3 \quad ∎$$

**Q94** **Finding Vector Components - I**

Find the components of the vector $\boldsymbol{v}$ with given initial point $P$ and terminal point $Q$. Find $|\boldsymbol{v}|$ and unit vector $\hat{\boldsymbol{v}}$.

$$P(3, 2, 0), \quad Q(5, -2, 2), \qquad P(1, 1, 1), \qquad Q(-4, -4, -4)$$
$$P(1, 0, 1.2), \quad Q(0, 0, 6.2), \qquad P(2, -2, 0), \qquad Q(0, 4, 6)$$
$$P(4, 3, 2), \quad Q(-4, -3, 2), \qquad P(0, 0, 0), \qquad Q(6, 8, 10)$$

**A94**

The solution is as follows:

$$\boldsymbol{v} = (5-3)\,\hat{x} + (-2-2)\,\hat{y} + (2-0)\,\hat{z} = (2)\,\hat{x} + (-4)\,\hat{y} + (2)\,\hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\,\hat{x} + (-4)\,\hat{y} + (2)\,\hat{z}}{2\sqrt{6}} = \left( \frac{1}{\sqrt{6}} \right)\hat{x} + \left( -\frac{2}{\sqrt{6}} \right)\hat{y} + \left( \frac{1}{\sqrt{6}} \right)\hat{z} \quad ∎$$

$$\boldsymbol{v} = (-4-1)\,\hat{x} + (-4-1)\,\hat{y} + (-4-1)\,\hat{z} = (-5)\,\hat{x} + (-5)\,\hat{y} + (-5)\,\hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\ \hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (0-1)\ \hat{x} + (0-0)\ \hat{y} + (6.2-1.2)\ \hat{z} = (-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\ \hat{x} + (0)\ \hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (0-2)\ \hat{x} + \left(4-(-2)\right)\ \hat{y} + (6-0)\ \hat{z} = (-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\ \hat{x} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{y} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (-4-4)\ \hat{x} + (-3-3)\ \hat{y} + (2-2)\ \hat{z} = (-8)\ \hat{x} + (-6)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\ \hat{x} + (-8)\ \hat{y} + (0)\ \hat{z}}{10} = \left(-\frac{3}{5}\right)\ \hat{x} + \left(-\frac{4}{5}\right)\ \hat{y} + (0)\ \hat{z} \quad \blacksquare$$

$$\boldsymbol{v} = (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.$$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare$$

## Q95 Finding Vector Components - II

Given the components of a vector $\boldsymbol{v} = \left[v_x,\ v_y,\ v_z\right]$ and a particular initial point $P$, find the corresponding terminal point $Q$ and the length of $\boldsymbol{v}$ (i.e., $|\boldsymbol{v}|$).

$$\boldsymbol{v} = [3,\ -1,\ 0]\ ; \qquad P\,(4,\ 6,\ 0)\,, \qquad \boldsymbol{v} = [8,\ 4,\ 2]\ ; \qquad P\,(-8,\ -4,\ -2)\,,$$

$$\boldsymbol{v} = [0.25,\ 2,\ 0.75]\ ; \quad P\,\{\cdot\}\,0,\ -0.5,\ 0, \qquad \boldsymbol{v} = [3,\ 2,\ 6]\ ; \qquad P\,(4,\ 6,\ 0)\,,$$

$$\boldsymbol{v} = [4,\ 2,\ -2]\ ; \qquad P\,(4,\ 6,\ 0)\,, \qquad \boldsymbol{v} = [3,\ -3,\ 3]\ ; \quad P\,(4,\ 6,\ 0)\,,$$

## A95

Previously we have defined $\boldsymbol{v} = Q - P$. Here we have $\boldsymbol{v}$ and $P$. To calculate $Q$ we only need to add individual components of the vector with the initial point $P$.

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (8+(-8))\ \hat{x} + (4+(-4))\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (0.25+0)\ \hat{x} + (2+(-0.5))\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},$$

$$|\boldsymbol{v}| = \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare$$

$$Q = \boldsymbol{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},$$

$$|v| = \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare$$

$$Q = v + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},$$

$$|v| = \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare$$

## Q96   Vector Addition and Scalar Multiplication

- Let $a = [2,\ 1,\ 0]$, $b = [-4,\ 2,\ 5]$ and $c = [0,\ 0,\ 3]$. Calculate the following vector operations:

$$2a, \qquad\qquad -a, \qquad -1/2a,$$
$$5(a-c), \qquad 5a-5c, \qquad (3a-5b)+2c,$$
$$3a+(-5b+2c), \qquad a+2b, \qquad 2b+a.$$

reset

- Find the dot product (i.e., $a \cdot b$) on the lengths of $a = [1,\ 2,\ 0]$ and $b = [3,\ -2,\ 1]$ as well as the angle ($\theta$) between vectors.

- Let $a = [2,\ 1,\ 4]$, $b = [-4,\ 0,\ 3]$ and $c = [3,\ -2,\ 1]$. Find the following descriptions.

$$|a|,|b|,|c|, \qquad\qquad a \cdot (b+c),\ a \cdot b + a \cdot c,$$
$$a \cdot b + b \cdot c + c \cdot a, \qquad 4a \cdot 3c, 12a \cdot c,$$
$$|b+c|,|b|+|c|, \qquad\qquad a \cdot c, |a||c|.$$

reset

- Let $a = [1,\ 1,\ 1]$, $b = [2,\ 3,\ 1]$ and $c = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(a-c)\ \text{and}\ (b-c), \qquad (a)\ \text{and}\ (b-c).$$

- Find the vector product $a \times b$ of $a = [1,\ 1,\ 0]$ and $b = [3,\ 0,\ 0]$.

- Let $a = [1,2,0]$, $b = [3,\ -4,0]$, $c = [3,5,2]$, $d = [6,2,0]$. Calculate the cross product of:

$$a \times b, \quad b \times a, \qquad\qquad a \times c, \quad |a \times c|, \quad a \cdot c,$$
$$(c+d) \times d, \quad c \times d, \qquad b \times c + c \times b,$$
$$(a+b) \times (b+a), \qquad (a \times b) \times c, \quad a \times (b \times c).$$

## A96

## Q97   An Example of a Curl

Find he curl ($\nabla \times$) of the following functions.

$$v = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \qquad\qquad v = (y^n)\ \hat{x} + (z^n)\ \hat{y} + (x^n)\ \hat{z},$$
$$v = (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \qquad v = \left(x^2 - z\right)\ \hat{x} + (xe^z)\ \hat{y} + (xy)\ \hat{z}.$$

## A97

The curl ($\nabla \times$) of the functions are as follows:

$$f(x, y, z) = (y) \, \hat{x} + \left(2x^2\right) \, \hat{y} + (0) \, \hat{z},$$

$$\nabla \times f = (0) \, \hat{x} + (0) \, \hat{y} + (-1 + 4x) \, \hat{z}.$$

$$f(x, y, z) = (y^n) \, \hat{x} + (z^n) \, \hat{y} + (x^n) \, \hat{z},$$

$$\nabla \times f = \left(-nz^{n-1}\right) \, \hat{x} + \left(-nx^{n-1}\right) \, \hat{y} + \left(-ny^{n-1}\right) \, \hat{z}.$$

$$f(x, y, z) = (\sin y) \, \hat{x} + (\cos z) \, \hat{y} + (-\tan x) \, \hat{z},$$

$$\nabla \times f = (\sin z) \, \hat{x} + \left(\sec^2 x\right) \, \hat{y} + (-\cos y) \, \hat{z}.$$

$$f(x, y, z) = \left(x^2 - z\right) \, \hat{x} + (xe^{\,z}) \, \hat{y} + (xy) \, \hat{z},$$

$$\nabla \times f = (x - e^z x) \, \hat{x} + (-1 - y) \, \hat{y} + (e^z) \, \hat{z}.$$

## Q98 The Laplacian of a Vector

Calculate the Laplacian of the following functions:

$$(i) \quad T_a = x_2 + 3xy + 3z + 4, \quad (ii) \quad T_b = \sin x \sin y \sin z,$$

$$(iii) \quad T_c = e^{-5x} \sin 4y \cos 3z, \quad (iv) \quad v = \left(x^2\right) \, \hat{x} + \left(3xz^2\right) \, \hat{y} + (-2xz) \, \hat{z}.$$

## A98

The solution to the Laplacian of the functions are as follows:

**(i)** $\dfrac{\partial^2 T_a}{\partial x^2} = 2; \; \dfrac{\partial^2 T_a}{\partial y^2} = 0; \; \dfrac{\partial^2 T_a}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 T_a = 2 \quad \blacksquare$

**(ii)** $\dfrac{\partial^2 T_b}{\partial x^2} = \dfrac{\partial^2 T_b}{\partial y^2} = \dfrac{\partial^2 T_b}{\partial z^2} = -3T_b \quad \rightarrow \quad \nabla^2 T_b = -3T_b = 3 \sin x \sin y \sin z \quad \blacksquare$

**(iii)** $\dfrac{\partial^2 T_c}{\partial x^2} = 25T_c;$

$\dfrac{\partial^2 T_c}{\partial y^2} = -16T_c; \quad \dfrac{\partial^2 T_c}{\partial z^2} = -9T_c \quad \rightarrow \quad \nabla^2 T_c = 0 \quad \blacksquare$

**(iii)** $\dfrac{\partial^2 v_x}{\partial x^2} = 2; \; \dfrac{\partial^2 v_x}{\partial y^2} = 0; \; \dfrac{\partial^2 v_x}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_x = 0,$

$\dfrac{\partial^2 v_y}{\partial x^2} = 0; \; \dfrac{\partial^2 v_y}{\partial y^2} = 0; \; \dfrac{\partial^2 v_y}{\partial z^2} = 6 \quad \rightarrow \quad \nabla^2 v_y = 6x,$

$\dfrac{\partial^2 v_z}{\partial x^2} = 0; \; \dfrac{\partial^2 v_z}{\partial y^2} = 0; \; \dfrac{\partial^2 v_z}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_z = 0,$

$\nabla^2 v = 2 \, \hat{x} + 6x \, \hat{y} \quad\quad\quad \blacksquare$

## Q99 A Simple Dirac Integral

Evaluate the following integral:

$$\int_0^3 x^3 \delta(x - 2) \, dx$$

The delta function picks out the value of $x^3$ at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

**Q100** **1D Dirac Delta Function**

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right) \delta(x - 3) \, dx, \tag{i}$$

$$\int_0^5 \cos x \, \delta(x - \pi) \, dx, \tag{ii}$$

$$\int_0^3 x^3 \, \delta(x + 1) \, dx, \tag{iii}$$

$$\int_{-\infty}^{+\infty} \ln(x + 3) \, \delta(x + 2) \, dx. \tag{iv}$$

The solution are as follows:

**(a)** $3\left(3^2\right) - 2\left(3\right) - 1 = 27 - 6 - 1 = 20$ ∎

**(b)** $\cos \pi = -1$ ∎

**(c)** $0$ ∎

**(d)** $\ln(-2 + 3) = \ln 1 = 0$ ∎

# List of Symbols

This document is incomplete. The external file associated with the glossary 'symbolslist' (which should be called `HigherMathematicsITutorialBook.syi`) hasn't been created.

This has probably happened because there are no entries defined in this glossary. Did you forget to use `type=symbolslist` when you defined your entries? If you tried to load entries into this glossary with `\loadglsentries` did you remember to use `[symbolslist]` as the optional argument? If you did, check that the definitions in the file you loaded all had the type set to `\glsdefaulttype`.

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# Acronyms

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Check the contents of the file `HigherMathematicsITutorialBook.acn`. If it's empty, that means you haven't indexed any of your entries in this glossary (using commands like `\gls` or `\glsadd`) so this list can't be generated. If the file isn't empty, the document build process hasn't been completed.

Try one of the following:

- Add `automake` to your package option list when you load `glossaries-extra.sty`. For example:

  ```
  \usepackage[automake]{glossaries-extra}
  ```

- Run the external (Lua) application:

  ```
  makeglossaries-lite.lua "HigherMathematicsITutorialBook"
  ```

- Run the external (Perl) application:

  ```
  makeglossaries "HigherMathematicsITutorialBook"
  ```

Then rerun LaTeX on this document.

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