Higher Mathematics I

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MCI

Introduction

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Introduction

- The goal of this lecture is to give you the foundations of mathematical methods you will employ during your M.Sc. studies.
- This lecture is a total of 3 SWS with a total of forty-five (45) UE.
  - With 43 UE devoted to teaching + tutorials and 2 UE for examination.
- To allow for a smoother flow of content, the lectures and tutorials will be done in a continuous form.
- There is a written exam at the end of the module worth two (2) UE.
- There is no assignment for this course:

Introduction

- Lecture materials and all possible supplements will be present in its Github page.
  - You can easily access the link to the web-page from here.

> Github is chosen for easy access to material management and CI/CD capabilities and allowing hosting websites.

- In the lecture some exercises are solved using SageMath and Python.

Introduction

- At the end of the lectures there will be a written examination which you will be asked from what was taught in the lectures.
- The duration of the exam is 2 UE (i.e., 90 mins)

Assessment Type	Overall Points	Breakdown	%
Written Exam	100		
		Question 1	25
		Question 2	25
		Question 3	25
		Question 4	25

Table 1: Assessment Grade breakdown for the lecture.

Introduction

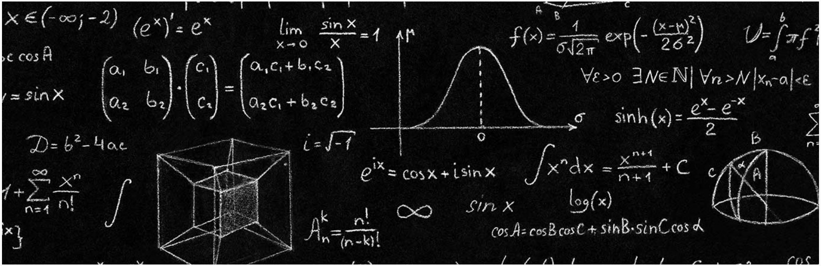
Covered Topic	Appointment
First-Order ODEs	1-2
Second-Order Linear ODEs	3-4
Higher-Order Linear ODEs	5-6
Systems of ODEs	7-8
Series Solutions of ODEs & Special Functions	8
Laplace Transforms	9-10
Linear Algebra I: Matrices and Vectors	11-12
Linear Algebra II: Eigenvalue Problems	13-14
Vector Calculus I: Grad, Div, Curl	14-15
Vector Calculus II: Curvilinear Coordinates	16-17
Vector Calculus III: Integral Theorems	17-18
Exam	19

Table 2: Distribution of materials across the semester.

Introduction

First-Order ODEs

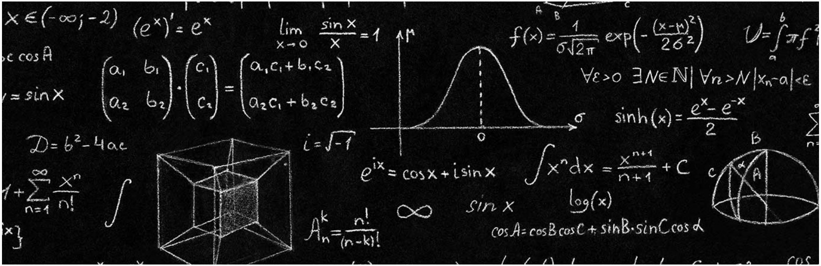
- Modelling Physical Systems,
- Initial Value Problems,
- Separable ODEs,
- Exact & Linear ODEs.



Introduction

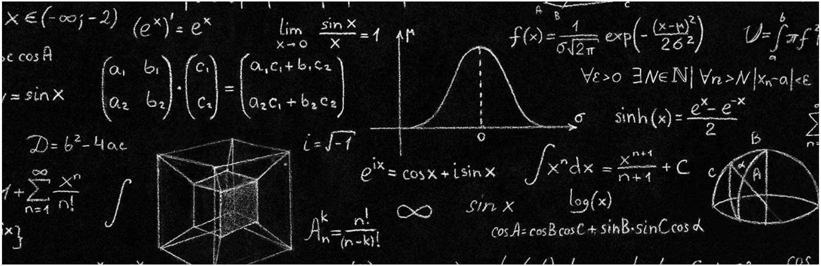
Second-Order ODEs

- 2^{nd} order Linear ODEs,
- Linear ODEs with Constant Coefficients,
- Modelling a Mass-Spring System,
- Euler–Cauchy Equations.



Higher-Order ODEs

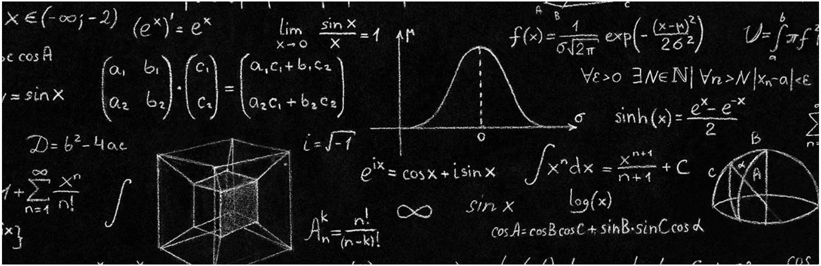
- Higher order Linear ODEs,
- Linear ODEs with Constant Coefficients,
- Non-homogeneous Linear ODEs,
- Application: Elastic Beams.



Introduction

Systems of ODEs

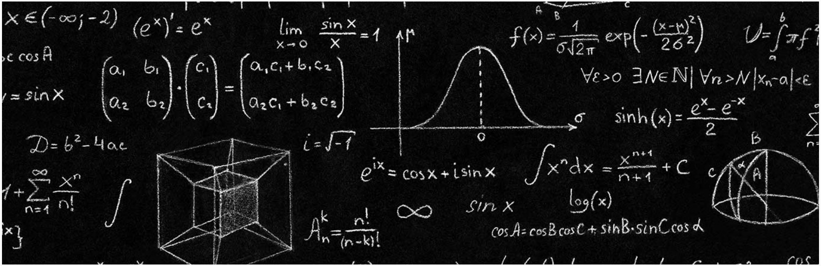
- Models in Engineering Applications,
- Linear ODEs with Constant Coefficients,
- Non-homogeneous Linear ODEs,
- Wronskian.



Introduction

Series Solutions of ODEs & Special Functions

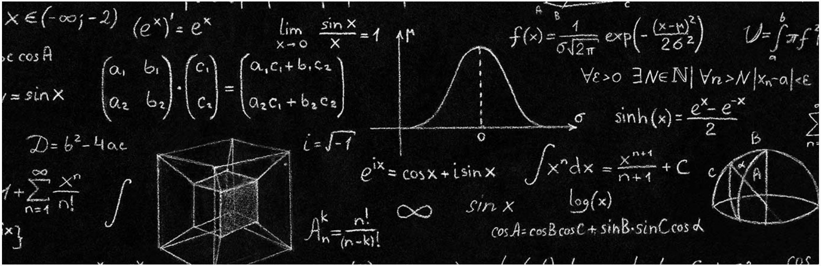
- Power Series,
- Legendre Polynomials,
- Bessel Function.



Introduction

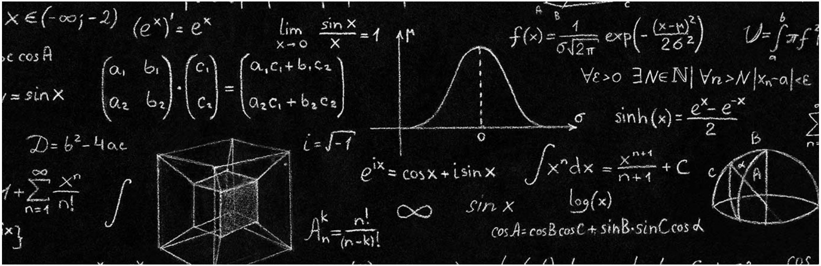
Laplace Transforms

- Unit Step Function,
- Convolution,
- Dirac Delta Function.



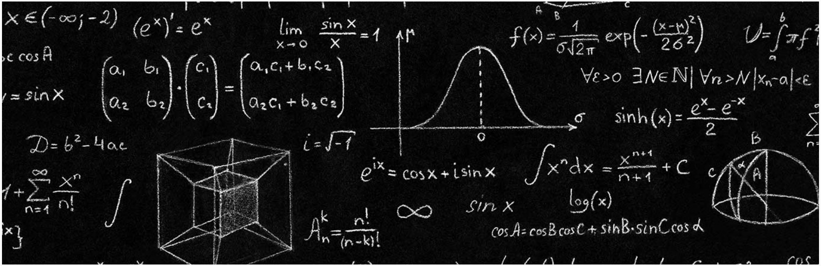
Linear Algebra I: Matrices and Vectors

- Gauss Elimination,
- Independence Rank, Vector Space,
- Determinants, Cramer's Rule,
- Gauss-Jordan Elimination .



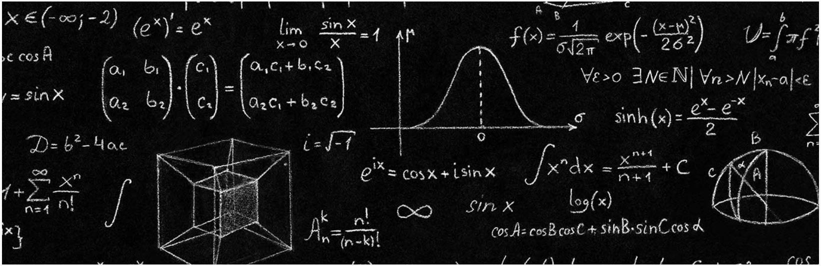
Linear Algebra II: Eigenvalue Problems

- Matrix Eigenvalue Problem,
- Applications of Eigenvalue Problems,
- Symmetric, Skew-Symmetric, and Orthogonal Matrices,
- Eigenbases, Diagonalisation, Quadratic Forms .



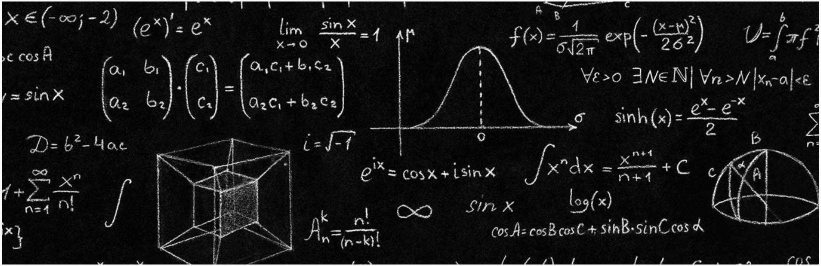
Vector Calculus I: Grad, Div & Curl

- Fields,
- Divergence,
- Curl.



Vector Calculus II: Curvilinear Coordinates

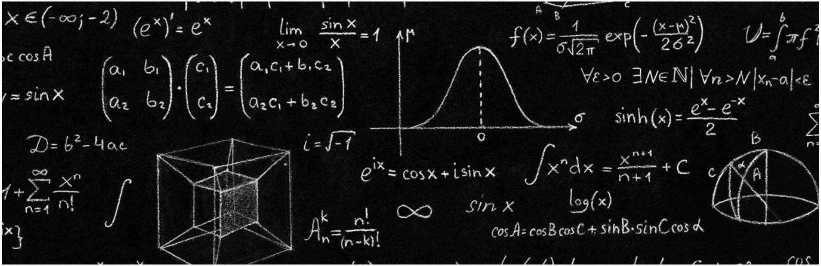
- Cartesian,
- Spherical,
- Cylindrical.



Introduction

Vector Calculus III: Integral Theorems

- Stokes Theorem,
- Triple Integrals, Divergence Theorem of Gauss,
- Green's Theorem in the Plane.



Introduction

Books

- Geore B. Thomas, et. al. "*Thomas Calculus (12th Edition)*" Pearson 2009.
- H. M. Schey "*Div, Grad, Curl, and All That: An Informal Text on Vector Calculus (4th Edition)*" W. W. Norton & Company 2004.
- E. Kreyszig "*Advanced Engineering Mathematics (10th Edition)*" Wiley 2011.
- D. Lay, et. al. "*Linear Algebra and Its Applications (5th Edition)*" Pearson 2015.
- K. F. Riley, et. al. "*Mathematical Methods for Physics and Engineering: A Comprehensive Guide (3rd Edition)*" Cambridge 2006.
- R. A. Adams "*Calculus: A Complete Course (5th Edition)*" Addison Wesley 2003.

Lecture Notes

- D. Tong, Vector Calculus "*University of Cambridge Part IA Mathematical Tripos*" University of Cambridge,

First-Order ODEs

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First-Order ODEs

- To solve an engineering problems of a physical nature, first formulate the problem as a mathematical expression in terms of:
  - variables,
  - functions,
  - equations.

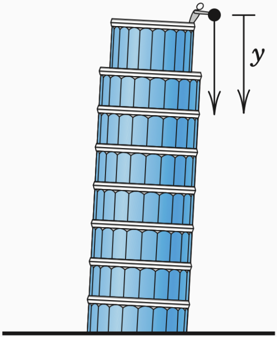
> Such an expression is known as a mathematical model.

The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modelling.

- Many physical concepts, such as velocity (v) and acceleration (a), are derivatives of a certain value (i.e., x).
- Therefore a model is an equation containing derivatives of an unknown function.
- Such a model is called a **differential equation**.

First-Order ODEs

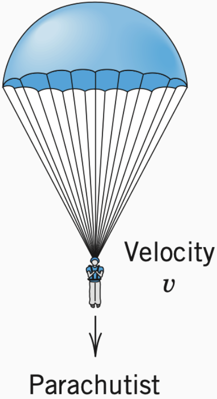
- Of course, we want to find in these differential equations:
  - a function that satisfies the equation,
  - explore its properties,
  - graph it
  - find values of it
  - interpret it in physical terms for the physical system under study.



Falling stone

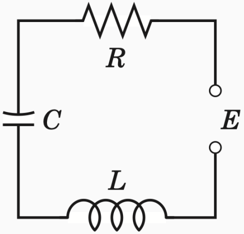
$$y'' = g = const.$$

Figure 1: The fall of the ball depends on the rate of change of the change of displacement.



$$mv' = mg - bv^2$$

Figure 2: A parachutist would experience acceleration and speed during descent.

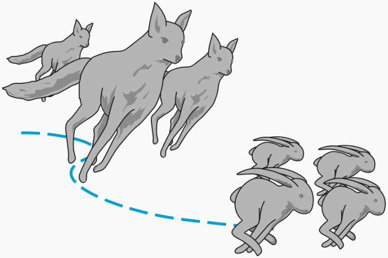


Current I in an
RLC circuit

$$LI'' + RI' + \frac{1}{C}I = E'$$

Figure 3: A simple RLC circuit can be modelled using differential equations.

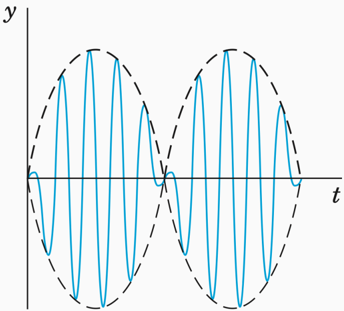
First-Order ODEs



Lotka–Volterra
predator–prey model

$$y_1' = ay_1 - by_1y_2$$
$$y_2' = ky_1y_2 - ly_2$$

Figure 4: The population of prey and predator oscillates with time.

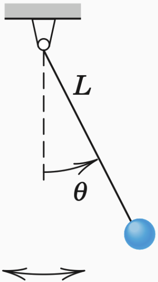


Beats of a vibrating system

$$y'' + \omega_0^2 y = \cos\ \omega t, \quad \omega_0 \approx \omega$$

Figure 5: The motion on a vibrating string

First-Order ODEs



Pendulum

$$L\theta'' + g \sin \theta = 0$$

Figure 6: The motion of a pendulum is a 2nd order differential equation with the variable being the angle of oscillation.

- An Ordinary Differential Equation (ODE) is an equation with one or several derivatives of an unknown function, usually called $y\,(x)$.
  - Sometimes $y\,(t)$ if the independent variable is time t.
- The equation may also contain y itself, known functions of x (or t), and constants (i.e., A, B, K).
- For example,

$$\begin{aligned} y' &= \cos x, \\ y'' + 9y &= e^{-2\,x}, \\ y'y''' - \frac{3}{2}\left(y'\right)^2 &= 0. \end{aligned}$$

where y' denotes dy/dt, y'' denotes d^2y/dt^2, ...

First-Order ODEs

- The term ordinary distinguishes them from Partial Differential Equation (PDE), which involve partial derivatives of an unknown function of two or more variables.
- i.e., a PDE with unknown function u of two variables x and y is:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDE play a vital role in engineering application but they are significantly harder to solve.

These will be the focus for **Higher Mathematics II**.

First-Order ODEs

- An ODE is said to be of order n if the n^{th} derivative of the unknown function y is the highest derivative of y in the equation.
- The concept of order gives a useful classification into ODEs of first order, second order, and so on.
- Therefore:

$$
\begin{aligned}
y' &= \cos x, && \text{First Order} \\
y'' + 9y &= e^{-2x}, && \text{Second Order} \\
y'y''' - \frac{3}{2}(y')^2 &= 0. && \text{Third Order}
\end{aligned}
$$

First-Order ODEs

- For now, we focus on the first order ODE.
- Such equations contain only the first derivative y' and may contain y and any given functions of x.
- These can write them as:

$$F(x, y, y') = 0 \tag{1}$$

- or presented as:

$$y' = f(x, y). \tag{2}$$

- The Eq. (1) is the implicit and Eq. (2) is the explicit.

First-Order ODEs

- The function:

$$y = h(x),$$

- is called a solution of a given ODE on some open interval:

$$a < x < b$$

- if h(x) is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively.
- The curve (the graph) of h is called a solution curve.

Example

Verify that:

$$y = \frac{c}{x}$$

is a solution to the ODE:

$$xy' = -y \qquad \text{for all } x \neq 0$$

Note: Here c is an arbitrary constant.

Example

Solve the following ODE:

$$y' = \frac{dy}{dx} = \cos x.$$

$$C + \sin(x)$$

Solution

```
# Define function
func = diff(y, x) - cos(x) == 0

# Solve Equation
desolve(func, y, ivar = x)
#+end_src

#+RESULTS: F-ODE-EX-1
```

```
#+NAME: F-ODE-A
```

First-Order ODEs

- ODE can have a solution containing an arbitrary constant c.
- Such a solution containing c is called a general solution of the ODE.

> c is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.

- Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c.
- Choosing a specific c (e.g., $c=6.45$ or 0 or $c=2.01$) we obtain the particular solution of the ODE.

> A particular solution does not contain any arbitrary constants.

- In most cases the unique (i.e., particular) solution, is obtained from the general solution by an initial condition:

$$y(x_0) = y_0.$$

with given values x_0 and y_0, used to determine a value of c.

> Geometrically the solution curve should pass through the point (x_0, y_0) in the xy-plane.

- An ODE, with initial condition, is called an **initial value problem**.
- Thus, if the ODE is explicit, the initial value problem is of the form:

$$y' = f(x, y), \qquad f(x_0) = y_0.$$

Example

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y \qquad y(0) = 5.7$$

$$\frac{57}{10}\, e^{(3\,x)}$$

Solution

```
# Define function
func = diff(y, x) - 3 * y == 0

# Solve Equation
desolve(func, y, ivar = x, ics=[0, 5.7])
#+end_src

#+RESULTS: F-ODE-EX-2
```

```
#+NAME: F-ODE-B
```

Example

Given an amount of a radioactive substance, say, 0.5 g, find the amount present at any later time.

As radioactive substance decomposes, i.e. decaying in time (y'), it is proportional (k) to the amount of substance present (y).

First-Order ODEs

- Many practically useful ODEs can be reduced to the form:

$$g(y)\, y' = f(x)$$

  using algebraic manipulations.
- Then we can integrate on both sides with respect to x, obtaining:

$$\int g(y)\, y' \,\mathrm{d}x = \int f(x)\, \mathrm{d}x + c.$$

- On the left we can switch to y as the variable of integration.
- By calculus, $y' = dy/dx \rightarrow y' \,\mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\, \mathrm{d}y = \int f(x)\, \mathrm{d}x + c.$$

- This method is called the method of separating variables.

Example

Solve the following ODE:

$$y' = 1 + y^2$$

$$y(x) = \tan(C + x)$$

Example

In September 1991 Ötzi, a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps in South Tyrol near the Austrian–Italian border, caused a scientific sensation.

When did Oetzi approximately live and die if the ratio of carbon-14 to carbon-12 in this mummy is 52.5% of that of a living organism?

Figure 7: Ötzi was found in the South Tyrolean Mountains.

5312.725

Solution

```
# Define variables and functions
k, t, _C = var('k, t, _C')
y = function('y')(t)

# Define function
func = diff(y, t) - k * y == 0

# Solve Equation
eq = desolve(func, y, ivar = t)

# Use the half-life value of t = 5715
sol = solve(eq == 0.5*_C, k)

_inter = sol[0].subs(t = 5715)

simplify(solve(e**(_inter.rhs()*t) == 0.525, t)[0].rhs().n())
```

Example

Solve the following ODE

$$y' = -2xy \qquad y(0) = 1.8$$

First-Order ODEs

- Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function.
- We discuss this technique for a class of ODEs of practical importance:

$$y' = f\left(\frac{y}{x}\right)$$

- Here, f is any (differentiable) function of y/x,
  - such as $\sin(y/x)$
- Such an ODE is sometimes called a homogeneous ODE,

- The form of such an ODE suggests that we set $y/x = u$:

$$y = ux \qquad \text{and} \qquad y' = u'x + u$$

- Substitution into $y' = f\left(y/x\right)$ gives:

$$u'x + u = f\left(u\right)$$

- If $f\left(u\right) - u \neq 0$, then:

$$\frac{\mathrm{d}u}{f\left(u\right) - u} = \frac{\mathrm{d}x}{x}.$$

Example

Solve the following ODE:

$$2xyy' = y^2 - x^2$$

$$y(x) = -\sqrt{-x^2 - \frac{x}{C}}$$

- If a function $u(x, y)$ has continuous partial derivatives, its differential is:

$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y.$$

- From this it follows:

$$\text{if} \quad u(x, y) = c = \text{const} \quad \text{then} \quad \mathrm{d}u = 0$$

- For example if $u = x + x^2y^3 = c$:

$$\mathrm{d}u = \left(1 + 2xy^3\right)\,\mathrm{d}x + 3x^2y^2\,\mathrm{d}y = 0$$
$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}.$$

First-Order ODEs

- A first order ODE in the form:

$$M(x,\, y) + N(x,\, y)\, y' = 0$$

can be re-written as:

$$M(x,\, y)\; \mathrm{d}x + N(x,\, y)\; \mathrm{d}y = 0$$

- To find whether this form is **exact**, it must conform to:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- and:

$$u = \int M\, \mathrm{d}x + k(y) \qquad \text{and} \qquad u = \int N\, \mathrm{d}y + l(x)$$

Example

Solve the following ODE:

$$\cos\left(x + y\right)\,\mathrm{d}x + \left(3y^2 + 2y + \cos\left(x + y\right)\right)\,\mathrm{d}y = 0$$

$$y\left(x\right)^3 + y\left(x\right)^2 + \sin\left(x + y\left(x\right)\right) = C$$

Solution

```
# Define function
func = diff(y, x) == - cos(x + y) / (3*y**2 + 2*y + cos(x + y))

# Solve Equation
eq = desolve(func, y, ivar = x)

print(eq)
#+end_src

#+RESULTS: F-ODE-B
```

Example

Solve the following ODE

$$(\cos y \sinh x + 1)\ \mathrm{d}x - \sin y \cosh x\, \mathrm{d}y = 0 \qquad \text{and} \qquad y(1) = 2.$$

Solution

```
# Define variables and functions
x = var('x')
y = function('y')(x)

# Define function
func = diff(y,x) - (cos(y)*sinh(x) + 1) / (sin(y) * cosh(x)) == 0

# Solve Equation
desolve(func, y, ics=[1,2])
```

Solution

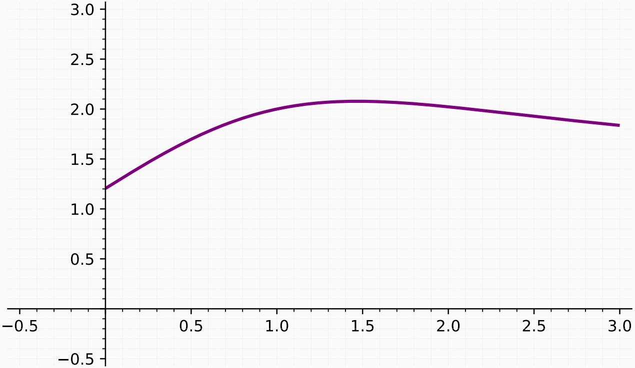


Figure 8: The solution curve to the equation with $c = 0.358$

First-Order ODEs

- Linear ODEs or ODEs are used in models of various phenomena in physics, biology, population dynamics, and ecology.
- A first-order ODE is said to be linear if it can be brought into the form

$$y' + p(x)\,y = r(x)\,,$$

- It is called **nonlinear** if it cannot be brought into this form.
- It is linear in both the y' and y, whereas p and r may be any given functions of x.

> In engineering, $x(x)$ is frequently called the input, and $y(x)$ is called the output or the response to the input

First-Order ODEs

- If $r(x) = 0$ within a given interval, it is **homogeneous**.

$$y' + p(x)\, y = 0. \tag{3}$$

- Solving is done by separation of variables:

$$\frac{\mathrm{d}y}{y} = -p(x)\ \mathrm{d}x, \quad \text{therefore} \quad \ln|y| = -\int p(x)\ \mathrm{d}x + c$$

- Simplifying this expression presents:

$$y(x) = ce^{-\int p(x)\,\mathrm{d}x}$$

- If $r(x)$ is non-zero, it is called non-homogeneous.
- To begin solving this form of equation, multiply both sides by $F(x)$.

$$Fy' + pFy = rF.$$

- LHS is the derivative $(Fy)' = F'y + Fy'$ if:

$$pFy = F'y \qquad \text{therefore} \qquad pF = F'$$

- By separating variables:

$$\frac{\mathrm{d}F}{F} = p\,\mathrm{d}x \qquad \text{and} \qquad h = \int p\,\mathrm{d}x$$

First-Order ODEs

- With F and $h' = p$, Eq. (3) becomes:

$$e^{h}y' + h'e^{h}y = e^{h}y' + \left(e^{h}\right)'y = \left(e^{h}y\right)' = re^{h}$$

- By integration:

$$e^{h} = \int e^{h}r\,\mathrm{d}x + c$$

- Dividing both sides by e^{h}, gives the solution:

$$y\left(x\right) = e^{-h}\left(\int e^{h}r\,\mathrm{d}x + c\right), \qquad \text{and} \qquad h = \int p\left(x\right)\,\mathrm{d}x \quad \blacksquare$$

Example

Solve the initial value problem:

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

$$y(x)^3 + y(x)^2 + \sin(x + y(x)) = C$$

Solution

```
func = diff(y, x) + y * tan(x) == sin(2*x)

desolve(func, y, ivar= x, ics = [0 , 1])
#+end_src

#+RESULTS: LINEAR-ODE-1
```

```
#+NAME: PLOT-ELECTRICAL-CIRCUIT
```

Example

Model the RL-circuit and solve the resulting ODE for the current $I(t)$ A (amperes), where t is time.

Assume the circuit contains a battery of $V = 48$ V (volts), which is constant, a resistor of $R = 11$ (ohms), and an inductor of $L = 0.1$ H (henrys), and that the current is initially zero.

Current causes a voltage drop of IR across the resistor and a voltage drop LI' across the inductor, and the sum of these two voltage drops equals the V.

$$\frac{48}{11}\left(e^{(110.0\,t)} - 1\right)e^{(-110.0\,t)}$$

Solution

```
func = diff(I, t) + R / L * I == E / L

_inter = desolve(func, I, ivar = t, ics = [0, 0])

eq = simplify(_inter.subs(E=48, R=11, L=0.1))

print(eq)
#+end_src

#+RESULTS: F-ODE-C
```

```
-cos(y(x))*cosh(x) - x == -cos(2)*cosh(1) - 1
```

Solution

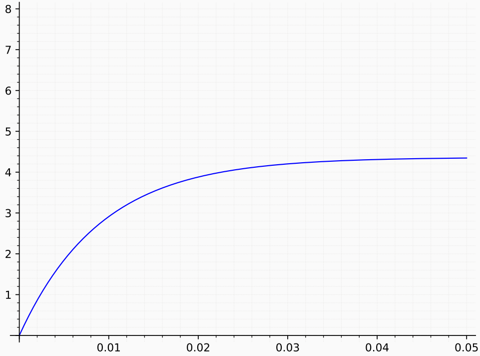


Figure 9: The solution curve to the RL circuit with $R/L = 100$ and $V = 48$ V.

Example

Assume the level of a certain hormone in the blood of a patient varies with time.

Suppose the time rate of change $(y'(t))$ is the difference between a sinusoidal input of a 24-hr period from the thyroid gland and a continuous removal rate proportional to the level present.

Set up a model for the hormone level in the blood and find its general solution.

Find the particular solution satisfying a suitable initial condition.

Solution: $-\dfrac{BK^2\cos(tw)e^{(Kt)}+BKwe^{(Kt)}\sin(tw)-AK^2e^{(Kt)}+(A-B)K^2-\left(Ae^{(Kt)}-A\right)w^2}{K^3e^{(Kt)}+Kw^2e^{(Kt)}}$

Solution

```
# Define the variables
A, B, K, w, t = var('A, B, K, w, t')
y = function('y')(t)

# Define the function
func = diff(y, t) + K * y - A + B * cos (w * t) == 0

# Solve the linear equation
hormone = desolve(func, y, ivar = t,  ics=[0,0])
```

Solution

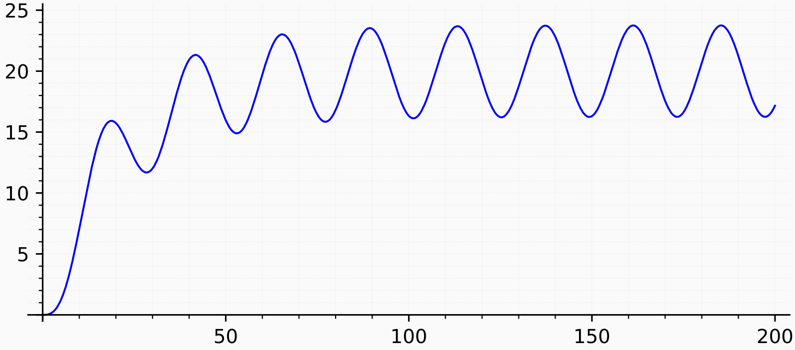


Figure 10: Graph of the solution ($A = B = 1,\ K = 0.05$).

- Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs.
- One of the most useful ones of these is the Bernoulli equation.

$$y' + p\,(y)\,y = g\,(x)\,y^{\,a}. \tag{4}$$

- If a is 0 or 1, Eq. (4) is linear,
- Otherwise it is non-linear.
- Differentiate Eq. (4) and do substitution of y in the form:

$$u\,(x) = [y\,(x)]^{1-a},$$
$$u' = (1-a)\,y^{-a}y' = (1-a)\,y^{-a}\,(gy^{\,a} - py)$$

- Simplifying this expression gives:

$$u' = (1 - a)\left(g - py^{1-a}\right)$$

where $y^{1-a} = u$ on the RHS, this turns our equation to a linear ODE.

$$u' = (1 - a)\, pu = (1 - a)\, g \quad \blacksquare$$

Example

Solve the following Bernoulli equation, known as the logistic equation (or Verhulst equation).

$$y' = Ay - By^2$$

Second Order ODEs

Table of Contents

- A second-order ODE is called linear if it can be written:

$$y'' + p(x)y' + q(x)y = r(x)$$

- and **non-linear** if it cannot be written in this form and are called coefficients.
- Here $p(x)$, $q(x)$, $r(x)$ can be a given function of x.
- If $r(x) = 0$ it is called **homogeneous**.
- If $r(x) \neq 0$ it is called **non-homogeneous**.
- A function of the form:

$$y = h(x)$$

is called a *solution* of a second-order ODE.

Second Order ODEs

- An example of **non-homogeneous** ODE:

$$y'' + 25y = e^{-x} \cos x.$$

- An example of **homogeneous** ODE:

$$xy'' + y' + xy = 0$$

- An example of **non-linear** ODE:

$$y''y + (y')^2 = 0$$

- Linear ODEs have a rich solution structure.
- For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle.

> we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

Example

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0,$$

for all x.

Both results **ARE** a solution to the ODE.

- From the previous example, we have obtained from $y_1 = \cos x$ and $y_2 = \sin x$ a function of the form:

$$y = c_1 y_1 + c_2 y_2$$

- This is called a **linear combination** of y_1 and y_2.

Fundamental Theorem for the Homogeneous Linear ODE
For a homogeneous linear ODE, any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

This theorem only works on homogeneous linear ODEs.

Example

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 0$$

Both results **ARE NOT** a solution to the ODE.

Second Order ODEs

- initial value problem consists of the ODE and one initial condition $x_0\,(y_0)$.
- The initial condition is used to determine the arbitrary constant c in the general solution of the ODE.
- This results in a unique solution, as we need it in most applications.
- That solution is called a particular solution of the ODE.

$$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

- These conditions prescribe given values K0 and K1 of the solution and its first derivative (the slope of its curve) at the same given x= x0 in the open interval considered.

Second Order ODEs

- The conditions (4) are used to determine the two arbitrary constants c1 and c2 in a general solution

$$y = c_1 y_1 + c_2 y_2$$

- of the ODE; here, y1 and y2 are suitable solutions of the ODE, with "suitable" to be explained after the next example. This results in a unique solution, passing through the point (x0, K0) with K1 as the tangent direction (the slope) at that point. That solution is called a particular solution of the ODE (2).

Example

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

$$y(x) = 3\cos(x) - \frac{1}{2}\sin(x)$$

Solution

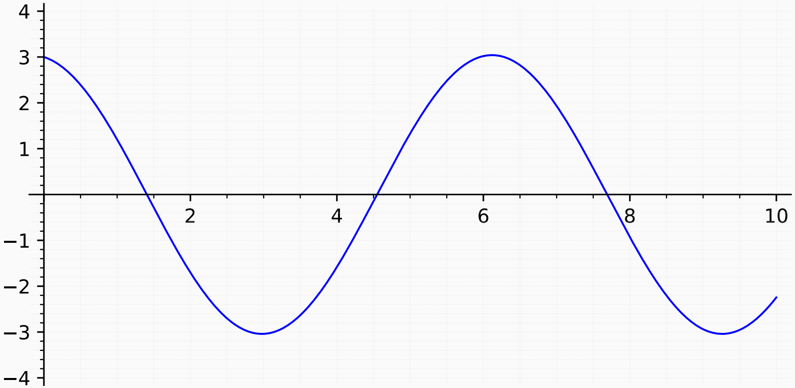


Figure 11: Particular solution of the initial value problem.

- Choice of y_1, y_2 was general enough to satisfy both conditions.
- Now let us take instead two proportional solutions $y_1 = \cos x$ and $y_2 = k\cos x$, so that $y_1/y_2 = 1/k$.
- Then we can write $y = c_1 y_1 + c_2 y_2$ in the form:

$$y = c_1 \cos x + c_2(k\cos x) = C\cos x \qquad \text{where} \qquad C = c_1 + c_2 k.$$

> We can't satisfy two (2) initial conditions with only one arbitrary constant C.

General Solution, Basis, Particular Solution

A general solution of an ODE (2) on an open interval I is a solution (5) in which y1 and y2 are solutions of (2) on I that are not proportional, and c1 and c2 are arbitrary constants. These y1, y2 are called a pair of linearly independent solutions of (2) on I. A particular solution of (2) on I is obtained if we assign specific values to c1 and c2 in (5).

Second Order ODEs

- Sometimes one solution can be found by inspection of the equation.
- Then a second linearly independent solution can be obtained by solving a first-order ODE.
- This is called the **method of reduction of order**.

- To see this method let's work through an example.

Example

Find a basis of solutions of the ODE:

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

$$y_1 = x \qquad y_2 = x \ln x + 1$$

Solution

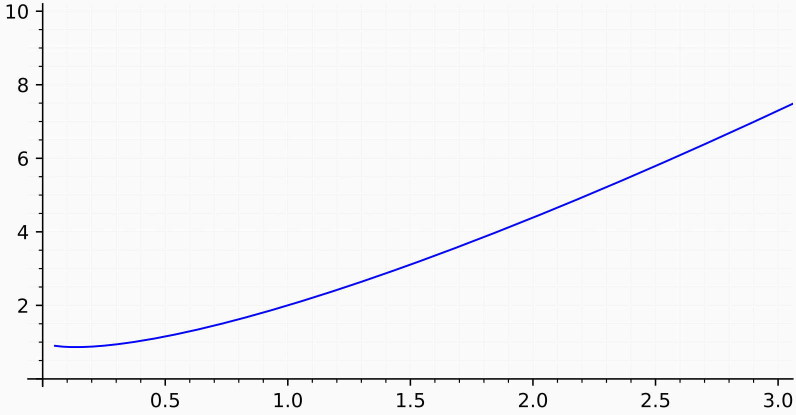


Figure 12

- We shall now consider second-order homogeneous linear ODEs whose coefficients a, b are **constant**,

$$y'' + ay' + by = 0 \tag{5}$$

- To solve recall the solution of the first-order linear ODE with constant coefficient k:

$$y' + ky = 0,$$

is an exponential function $y = ce^{-kx}$.

- This gives us the idea to try as a solution of Eq. (5) the function

$$y = e^{\lambda x} \tag{6}$$

- Substituting Eq. (6) and its derivatives

$$y' = \lambda e^{\lambda x}, \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}.$$

- into our equation Eq. (5), we obtain:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

- Hence if λ is a solution of the important **characteristic equation**[1]

$$\lambda^2 + a\lambda + b = 0 \tag{7}$$

[1] also known as the auxiliary equation.

Second Order ODEs

- The exponential in Eq. (6) is a solution of the ODE in Eq. (5).
- Now from algebra we recall the roots of this quadratic equation Eq. (92) are:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \quad (8)$$

- and will be basic because our derivation shows that the functions:

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

- From algebra we further know that the quadratic equation may have three (3) kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely:

Case	Description	Condition
I	Two real roots	$a^2 - 4b > 0$
II	A real double root	$a^2 - 4b = 0$
III	Complex conjugate roots	$a^2 - 4b < 0$

Table 3: Types of solutions.

Example

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

$$y\left(x\right) = 3\, e^{(-2\, x)} + e^{x}$$

Solution

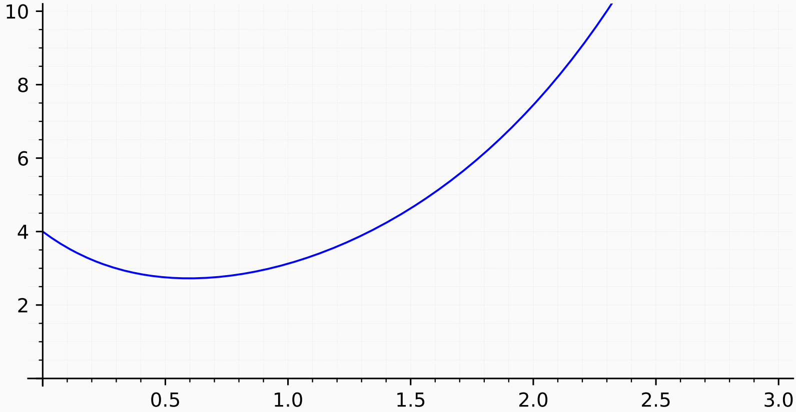


Figure 13: Solution to the case of distinct real roots.

Example

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \quad y(0) = -3.5.$$

$$y(x) = -(2x - 3)e^{\left(-\frac{1}{2}x\right)}$$

Solution

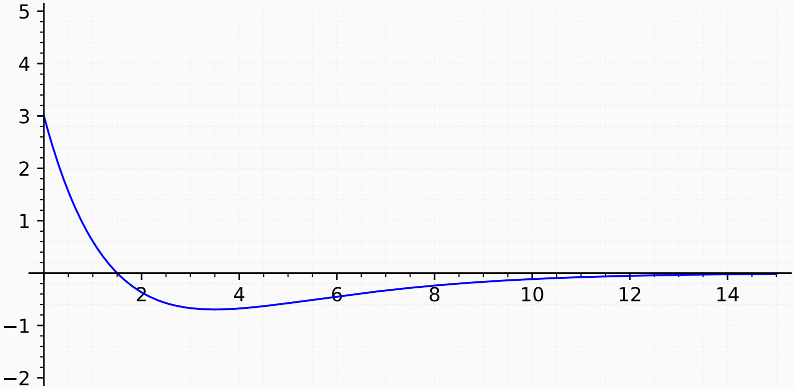


Figure 14: Solution to the case of a real double root.

Example

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

$$y\left(x\right) = e^{\left(-\frac{1}{5}x\right)} \sin\left(3x\right)$$

Solution

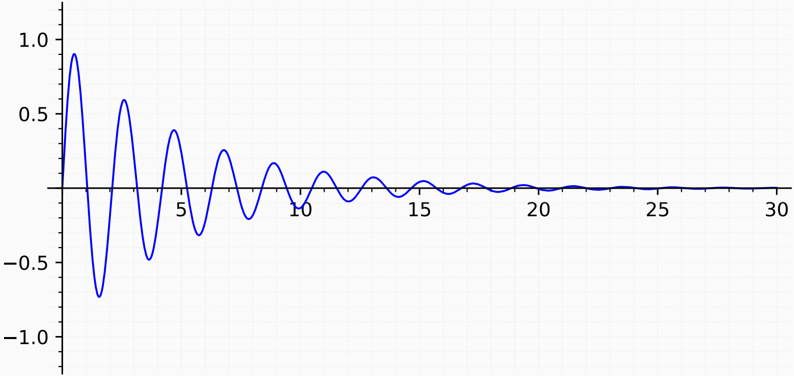


Figure 15: Solution to the case of a complex root.

- Linear ODEs with constant coefficients have important applications in mechanics,

Setting up the Model

- Take an ordinary coil spring that resists extension as well as compression.
- Suspend it vertically from a fixed support and attach a body at its lower end.
  - for instance, an iron ball.
- let $y = 0$ denote the position of the ball when the system is at rest.

Choose the downward direction as positive, regarding **downward forces as positive** and **upward forces as negative**.

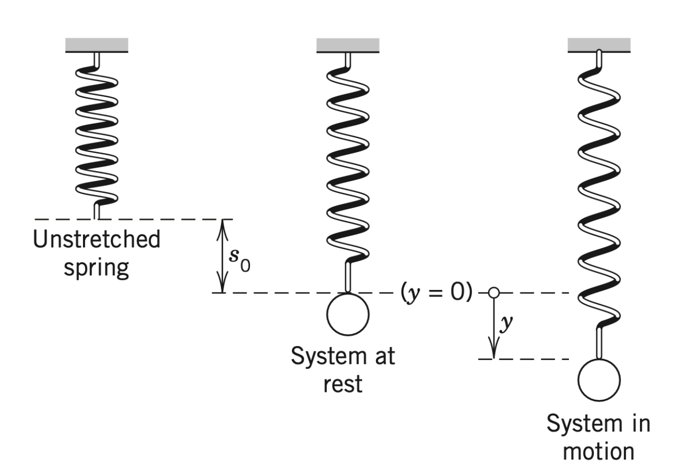


Figure 16: Mechanical mass–spring system

- We now let the ball move, as follows. We pull it down by an amount $y > 0$.
- This causes a spring force:

$$F_1 = -ky$$

where k is called the **spring constant**.

- This is known as Hooke's law.
- The motion of our mass–spring system is determined by Newton's second law.

$$\text{Mass} \times \text{Acceleration} = my'' = \text{Force}.$$

> Stiff springs have large k.

Second Order ODEs

Every system has damping. Otherwise it would keep moving forever. But if the damping is s mall and the motion of the system is considered over a relatively short time, we may disregard damping.

- If damping can be neglected, the model can be written as:

$$my'' + ky = 0.$$

- This is a homogeneous linear ODE with **constant coefficients** with the following general solution:

$$y(t) = Acos\omega_0 t + Bsin\omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

- This motion is called a harmonic oscillation.

Second Order ODEs

- Its frequency is $f = \omega_0 / 2\pi$ as cos and sin have the period $2\pi/\omega_0$.
- The frequency f is called the natural frequency of the system.
- An alternative representation which shows the physical characteristics of amplitude and phase shift of is

$$y(t) = C\cos(\omega_0 t - \delta)$$

- with $C = \sqrt{A^2 + B^2}$ and phase angle δ, where $\tan\delta = B/A$.

Second Order ODEs

- To our model we now add a damping force:

$$F_2 = -cy'$$

- Turning our equation into:

$$my'' + cy' + ky = 0$$

Second Order ODEs

- They are in the form:

$$x^2y'' + axy' + by = 0 \tag{9}$$

- with given constants a and b and unknown function $y(x)$.
- To solve, substitute:

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2},$$

into Eq. (9), which results in:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \tag{10}$$

> $y = x^m$ was a rather natural choice because as it has become the common factor

$y = x^{m}$ is a solution of Eq. (9) only if m is a root of Eq. (10).

- The roots of Eq. (10) are:

$$m_1 = \frac{1}{2}(1 - a) + \sqrt{\frac{1}{4}(1 - a)^2 - b},$$
$$m_2 = \frac{1}{2}(1 - a) - \sqrt{\frac{1}{4}(1 - a)^2 - b}.$$

- There are three (3) types of solutions.

Example

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of *Euler–Cauchy equation* $rv'' + 2v' = 0$.

$v(r) = \frac{1100}{r} - 110$

Example

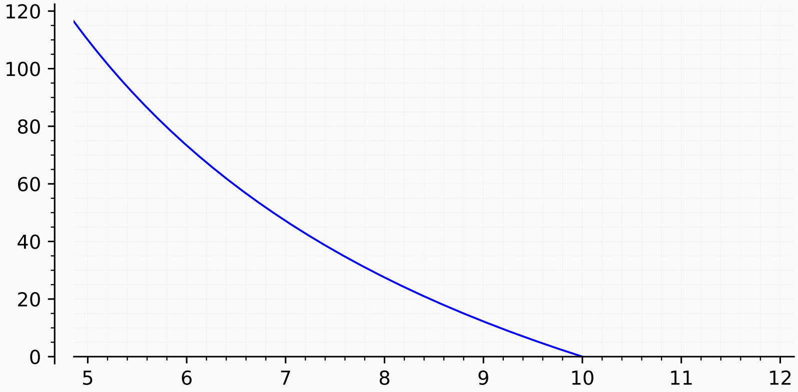


Figure 17: The potential of $v(r)$

Example

How does the motion of the damped system change if we change the damping constant c from one to another of the following three values, with:

- $c = 100\,\mathrm{kg}\cdot\mathrm{s}^{-1}$,
- $c = 60\,\mathrm{kg}\cdot\mathrm{s}^{-1}$,
- $c = 10\,\mathrm{kg}\cdot\mathrm{s}^{-1}$,

with $y(0) = 0.16$ and $y'(0) = 0$

$$y(x) = \frac{9}{50}\,e^{(-x)} - \frac{1}{50}\,e^{(-9\,x)}$$

$$y(x) = \frac{4}{25}\,(3\,x + 1)e^{(-3\,x)}$$

$$y(x) = \frac{4}{875}\left(\sqrt{35}\sin\left(\frac{1}{2}\sqrt{35}x\right) + 35\,\cos\left(\frac{1}{2}\sqrt{35}x\right)\right)e^{\left(-\frac{1}{2}\,x\right)}$$

Solution

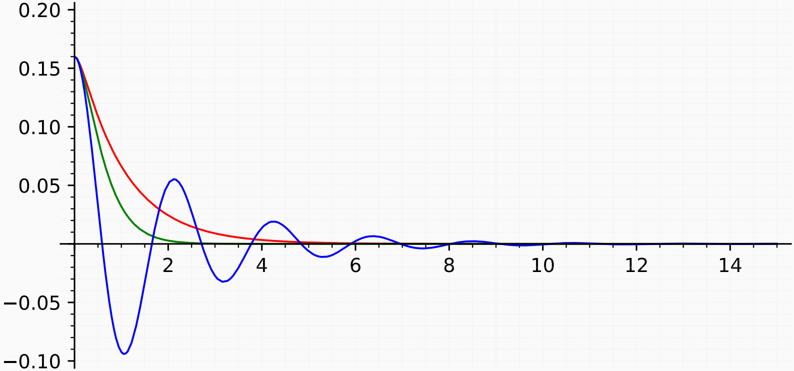


Figure 18: Three solutions for case I (red), case II (green) and case III (blue).

Second Order ODEs

- Let's discuss the general theory of homogeneous linear ODEs:

$$y'' + p(x)\,y' + q(x)\,y = 0, \tag{11}$$

with continuous, arbitrary, variable coefficients p and q.

Linear Independence of Solutions

- The solutions (y_1, y_2) are called **linearly independent** if:

$$k_1 y_1(x) + k_2 y_2(x) = 0 \qquad \text{on open interval I}$$

- The solutions (y_1, y_2) are called **linearly dependent** if:

$$y_1 = k y_2 \qquad \text{or} \qquad y_2 = l y_1 \qquad \text{on open interval I}$$

Linear Dependence and Independence of Solutions

Let the ODE in Eq. (11) have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two (2) solutions y_1 and y_2 of Eq. (11) on I are **linearly dependent** on I if and only if their *Wronskian*:

$$W(y_1, y_2) = y_1 y_2{}' - y_2 y_1{}'$$

is 0 at some x_0 in I.

Furthermore, if $W = 0$ at an $x = x_0$ in I, then $W = 0$ on I; hence, if there is an x_1 in I at which W is not 0, then y_1, y_2 are **linearly independent** on I.

Appendix

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Appendix

Slides were created using **GNU Emacs** version 29.1 with **AUCTeX** 14.0.7.

"Emacs, is a family of text editors that are characterised by their extensibility. The manual for the most widely used variant, GNU Emacs, describes it as "the extensible, customizable, self-documenting, real-time display editor."

"AUCTeX is a package for writing and formatting TeX files in GNU Emacs."

Beamer class was used as template with the **LuaTeX** engine.

"Beamer is a LaTeX class for generating slides."

"LuaTeX is a TeX-based computer typesetting system which started as a version of pdfTeX with a Lua scripting engine embedded."

All code presented in lectures are in **Python**, using version 3.9.13 and **SageMath** version 10.3.

"SageMath is a computer algebra system covering differentiable manifolds, numerical analysis, calculus and statistics and more..."

The lecture is based on the stellar book **Advanced Engineering Mathematics 10th Edition** by Erwin Kreyszig.

"Advanced Engineering Mathematics, 10th Edition is known for its comprehensive coverage, careful and correct mathematics, outstanding exercises, and self-contained subject matter parts for maximum flexibility..."

Significant portion of the Vector Calculus is based on the household book **Introduction to Electrodynamics 4th Edition** by David J. Griffiths.

"The Fourth Edition provides a rigorous, yet clear and accessible treatment of the fundamentals of electromagnetic theory and offers a sound platform for explorations of related applications (AC circuits, antennas, transmission lines, plasmas, optics and more)..."