Exam Higher Mathematics I Final

Neighbours

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Lecturer: Daniel T. McGuiness, Ph.D

SEMESTER: WS 2024
DATE: 21.01.2025
TIME: 09:00 - 10:30

First and Last Name

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Student Registration Number

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Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

Result:

___/ max. 100 points

Grade:

Student Cohort	MA-MECH-24-VZ
Study Programme	M.Sc Smart Technologies
Permitted Tools	Non-programmable calculators are allowed.

Important Notes

Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

Writing materials

Do not use a pencil or red pen and write legibly.

Good Luck!

Please read the following instructions carefully.

- You have **90 minutes** to complete this exam. This question booklet contains 3 question(s), 9 pages (including the cover) for the total of 100 points.
- Check to see if any pages are missing.
- All the questions are **compulsory** and all the notations used in the questions have their usual meaning taught at the lectures and done in practice.
- **Read the instructions for individual questions carefully** before answering the questions.

Question	Maximum Point	Result
Ordinary Differential Equations	50	
Linear Algebra	20	
Vector Calculus	30	
Sum	100	

[Q1] Ordinary Differential Equations — 50

Please solve the following ODEs. You can use either standard methods or Laplace transform.

a. $y' = 1 + y^2$ (10)

b. $y' = -2xy, \qquad y(0) = 1.8$ (10)

c. $x^2y'' + 1.5xy' - 0.5y = 0$ (20)

d. $y'' + y' - 2y = 0$ (10)

[Q2] Linear Algebra — 20

Calculate the eigenvalues of the following matrix. (20)

$$\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

[Q3] Vector Calculus — 30

a. Please calculate the gradient of $r = \sqrt{x^2 + y^2 + z^2}$. (10)

b. Please calculate the divergence of $\boldsymbol{v} = \left(x^3 + y^3\right)\ \hat{x} + \left(3xy^2\right)\ \hat{y} + \left(3zy^2\right)\ \hat{z}$, (10)

c. Please calculate the curl of $\boldsymbol{v} = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}$ (10)

Ordinary Differential Equations

First-order ODEs

They are of the form:

$$F\left(x, y, y'\right) = 0 \quad \text{or in explicit form} \quad y' = f(x, y)$$

involving the derivative $y' = dy/dt$ of an unknown function y, given functions of x, and/or y itself. A first-order ODE usually has a **general solution**; a solution involving an arbitrary constant, which is denoted by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** of $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem** with the following mathematical expression:

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad (x_0, y_0 \text{given numbers})$$

and its solution is a particular solution of the ODE. A **separable ODE** is one that we can put into the form:

$$g(y)\ dy = f(x)\ dx$$

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides. An **exact ODE**, on the other hand, is of the form

$$M(x, y)\ dx + N(x, y)\ dy = 0$$

where $M\,dx + N\,dy$ is the **differential** for and,

$$du = u_x dx + u_y dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. The solution to u can be attained using two (2) ways:

$$u = \int M\,dx + k(y) \qquad \text{or} \qquad u = \int N\,dy + l(x)$$

Linear ODE's of the form:

$$y' + p(x)\,y = r(x)$$

are very important. Their solutions are given by the integral formula

$$y(x) = e^{-h}\left(\int e^{h} r\,dx + c\right), \quad h = \int p(x)\ dx$$

Certain nonlinear ODEs can be transformed to linear form in terms of new variables.

Second-order ODEs

A second-order ODE is called **linear** if it can be written as:

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

The above equation is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval, this is written $r(x) = 0$. Then, this equation becomes:

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

The above equation is called **non-homogeneous** if $r(x) \neq 0$ (meaning $r(x)$ is not zero for some x considered). For the homogeneous ODE we have the important **superposition principle** that a linear combination of two solutions is again a solution.

Two linearly independent solutions y_1, y_2 on an open interval I form a **basis** (or **fundamental system**) of solutions. and $y = c_1 y_1 + c_2 y_2$ with arbitrary constants c_1, c_2 a **general solution**. From it, we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two (2) **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \quad (x_0, K_0, K_1 \text{are given}).$$

This forms an **initial value problem**. For a nonhomogeneous ODE, a **general solution** is of the form

$$y = y_h + y_p$$

Here y_h is a **general** solution and y_p is a **particular** solution of. Such a y_p can be determined by a general method or in many practical cases by the *method of undetermined coefficients*. The latter applies when the above has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Table 3) Then we write (1) as

$$y'' + ay' + by = r(x)$$

Another large class of ODEs solvable "algebraically" consists of the **Euler-Cauchy equations**:

$$x^2 y'' + axy' + by = 0$$

These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

$$m^2 + (a - 1)m + b = 0.$$

Depending of the roots different solutions can be derived (Table 6). A homogeneous ODE of

$$y'' + ay' + by = 0$$

can be solved by deriving its characteristic equation in the form:

$$\lambda^2 + a\lambda + b = 0$$

Depending of its roots of this equation different solutions can be derived (Table 5)

Higher-order ODEs

A higher order system is of the form:

$$F\left(x,\ y,\ y',\ \cdots,\ y^{(n)}\right) = 0,$$

Which through substitutions (i.e., $y^{iv} \to \lambda = y''$) it can be reduced to lower order forms. From there on, all principles from 2nd order ODEs can be applied.

System of ODEs

A system of ODEs (linear) can be written in the form:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{g}, \qquad \text{where} \qquad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

If $\boldsymbol{g} = 0$, the system is called **homogeneous** and is of the form

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$$

If $a_{11}, \cdots, a_{22}$ are **constants**, it has solutions

$$\boldsymbol{y} = xe^{\lambda t},$$

where λ is a solution of the quadratic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

and $\boldsymbol{x} \neq 0$ has components x_1, x_2 determined up to a multiplicative constant by:

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0.$$

These λ's are called the **eigenvalues** and these vectors $\boldsymbol{x}$ **eigenvectors** of the matrix $\boldsymbol{A}$. To calculate the characteristic equation:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_2 - (a_{11} + a_{22})\,\lambda + \det(\boldsymbol{A}) = 0.$$

where:

$$p = a_{11} + a_{22}, \qquad q = \det\boldsymbol{A}, \qquad \Delta = p_2 - 4q.$$

or in another form:

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2$$

The stability criterion are given in Table 4.

Special Functions

The **power series method** gives solutions of linear ODEs:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

with variable coefficients p and q in the form of a power series (with any centre x_0, e.g., $x_0 = 0$)

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$
$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

Legendre's differential equation:

$$(1 - x^2)\,y'' - 2xy' + n(n + 1)\,y = 0$$

with its polynomial solution:

$$P_n(x) = \sum_{m-0}^{M} (-1)^m \frac{(2n - 2m)!}{2^n m!(n - m)!(n - 2m)!} x^{n-2m}$$
$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n - 2)!}{2^n 1!(n - 1)!(n - 2)!} x^{n-2} + - \cdots$$

Frobenius Method states: Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \qquad (1)$$

has **at least one solution** can represented in the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$
$$= x^r\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \quad (a_0 \neq 0)$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$). It's indicial equation is:

$$r(r - 1) + b_0 r + c_0 = 0$$

Depending on the equations roots, we have the following three (3) cases:

Case 1. Distinct Roots Not Differing by an Integer
A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

with coefficients obtained successively from Eq. (1) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.
A basis is

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)]$$

(of the same general form as before) and

$$y_2(x) = y_1(x)\ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0)$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(of the same general form as before) and

$$y_2(x) = ky_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero. **Bessel's Equation** is of the form:

$$x^2 y'' + xy' + (x^2 - \nu^2)\,y = 0$$

It solution of first kind is:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n + m)!} \quad (n \geq 0).$$

Laplace Transform

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)\, dt$$

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad \text{where a, b const.}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subdistiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0).$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the Table 1. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform**$y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem.

Linear Algebra

Fundamentals

A $m \times n$ **matrix** $\boldsymbol{A} = [a_{jk}]$ is a rectangular array of numbers or functions arranged in m horizontal **rows** and n vertical **columns**. If $m = n$, the matrix is called **square**. A $1 \times n$ matrix is call+ed a **row vector** and an $m \times 1$ matrix **column vector**.

The **sum** A + B of matrices of the same size (i.e., both $m \times n$) is obtained by adding corresponding entries. The **product** of A by a scalar c is obtained by multiplying each a_{jk} by c.

The **product** $\boldsymbol{C} = \boldsymbol{AB}$ of an $m \times n$ matrix $\boldsymbol{A}$ by an $r \times p$ matrix $\boldsymbol{B} = [b_{jk}]$ is defined only when $r = n$. It is associative, but is **not commutative**. If $\boldsymbol{AB}$ is defined, $\boldsymbol{BA}$ may not be defined, but even if $\boldsymbol{BA}$ is defined, $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general. Also $\boldsymbol{AB} = 0$ may not imply $\boldsymbol{A} = 0$ or $\boldsymbol{B} = 0$ or $\boldsymbol{BA} = 0$.

The **transpose** $\boldsymbol{A}^{\top}$ of a matrix $\boldsymbol{A} = [a_{jk}]$ is $\boldsymbol{A}^{\top} = [a_{kj}]$. rows become columns and conversely. Here, $\boldsymbol{A}$ need not be square. If it is and $\boldsymbol{A} = \boldsymbol{A}^{\top}$, then $\boldsymbol{A}$ is called **symmetric;** if $\boldsymbol{A} = -\boldsymbol{A}^{\top}$, it is called **symmetric**. For a product, $(\boldsymbol{AB})^{T} = \boldsymbol{B}^{\top}\boldsymbol{A}^{\top}$.

A main application of matrices concerns **linear systems of equations**

$$Ax = b$$

(m equations in n unknowns $x_1, \cdots, x_n$; $\boldsymbol{A}$ and $\boldsymbol{b}$ given). The most important method of solution is the **Gauss elimination**, which reduces the system to "triangular" form by *elementary row operations*, which leave the set of solutions unchanged.
The inverse $\boldsymbol{A}^{-1}$ of a square matrix satisfies $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. It exists if and only if det $\boldsymbol{A} \neq 0$. It can be computed by the *Gauss-Jordan elimination*.

The rank r of a matrix $\boldsymbol{A}$ is the maximum number of linearly independent rows or columns of $\boldsymbol{A}$ or, equivalently, the number of rows of the largest square sub-matrix of $\boldsymbol{A}$ with nonzero determinant.
The system of equations has solutions if and only if rank $\boldsymbol{A}$ = rank$[\boldsymbol{A} \quad \boldsymbol{b}]$, where $[\boldsymbol{A} \quad \boldsymbol{b}]$ is the **augmented matrix**.The **homogeneous system**

$$\boldsymbol{A}x = 0$$

has solutions $\boldsymbol{x} \neq 0$ ("nontrivial solutions") if and only if rank $\boldsymbol{A} < n$, in the case $m = n$ equivalently if and only if det $\boldsymbol{A} = 0$.

Eigenvalue Problems

The problems are defined by the vector equation

$$\boldsymbol{A}x = \lambda \boldsymbol{x}.$$

$\boldsymbol{A}$ is a given square matrix. λ is a scalar. To *solve* the problem means to determine values of λ, called **eigenvalues** (or **characteristic values**) of $\boldsymbol{A}$, such that, the above expression, has a nontrivial solution $\boldsymbol{x}$ (that is, $\boldsymbol{x} \neq 0$), called an **eigenvector** of $\boldsymbol{A}$ corresponding to that λ. A $n \times n$ matrix has at least one and at most n numerically different eigenvalues. These are the solutions of the **characteristic equation**

$$D(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})$$
$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$D(\lambda)$ is called the **characteristic determinant** of $\boldsymbol{A}$. By expanding it we get the **characteristic polynomial** of $\boldsymbol{A}$, which is of degree n in λ. Special matrices of importance are **symmetric** ($\boldsymbol{A}^{\top} = \boldsymbol{A}$), **skew-symmetric** ($\boldsymbol{A}^{\top} = -\boldsymbol{A}$), and **orthogonal matrices** ($\boldsymbol{A}^{\top} = \boldsymbol{A}^{-1}$). concerns the diagonalization of matrices and the transformation of quadratic forms to principal axes and its relation to eigenvalues.
If an $n \times n$ matrix $\boldsymbol{A}$ has a basis of eigenvectors, then

$$\boldsymbol{D} = \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X}$$

is **diagonal**, with the eigenvalues of $\boldsymbol{A}$ as the entries on the main diagonal.

Here $\boldsymbol{X}$ is the matrix with these eigenvectors as column vectors. Also,

$$\boldsymbol{D}^{m} = \boldsymbol{X}^{-1}\boldsymbol{A}^{m}\boldsymbol{X} \qquad (m = 2, 3, \cdots).$$

Vector Calculus

All vectors of the form:

$$\boldsymbol{a} = [a_1, a_2, a_3] = (a_1)\ \hat{x} + (a_2)\ \hat{y} + (a_3)\ \hat{z}$$

constitute the **real vector space** R^3 with componentwise vector addition

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and componentwise scalar multiplication (c a scalar)

$$c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3]$$

The **inner product** or **dot product** of two vectors is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. This gives for the **norm** or **length** $|\boldsymbol{a}|$ of $\boldsymbol{a}$:

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well as a formula for θ. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, we call $\boldsymbol{a}$ and $\boldsymbol{b}$ **orthogonal**. The **vector product** or **cross product** $\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b}$ is a vector of length

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$$

which can also be represented as a determinant.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This multiplication is anticommutative, and is *not* associative.

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Gradient is defined as:

$$\nabla f = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Divergence is defined as:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

Curl is defined as:

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_x & v_x \end{vmatrix}$$

Standard Integration Forms

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \int \frac{1}{x}dx = \ln|x|$$
$$\int u dv = uv - \int v du \qquad \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$$
$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$
$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x - a)}{(n+1)(n+2)}$$
$$\int \frac{1}{1+x^2}dx = \tan^{-1} x$$
$$\int \frac{1}{1+x^2}dx = \arctan x$$
$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$
$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$$
$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$$
$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$
$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$
$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, \ a \neq b$$
$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$$
$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$f(t)$	$\mathcal{L}(f(t)) = F(s)$	$f(t)$	$\mathcal{L}(f(t)) = F(s)$	$f(t)$	$\mathcal{L}(f(t)) = F(s)$
1	$\frac{1}{s}$	$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$	$\frac{e^{at} - e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$e^{at}f(t)$	$F(s-a)$	$\cosh kt$	$\frac{s}{s^2-k^2}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	e^{at}	$\frac{1}{s-a}$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$\sin kt$	$\frac{k}{s^2+k^2}$	$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\delta(t)$	1	$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$\delta(t-t_0)$	e^{-st_0}	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$	te^{at}	$\frac{1}{(s-a)^2}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	$e^{at} \sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at} \cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$	$t \sin kt$	$\frac{2ks}{(s^2+k^2)^2}$	$t \cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t \sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$	$t \cosh kt$	$\frac{s^2-k^2}{(s^2-k^2)^2}$	$\frac{\sin at}{t}$	$\arctan \frac{a}{s}$
$\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	$\text{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$

Table 1: Common Laplace transform operations.

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	**Comments**
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite signs
Centre	$p = 0$	$q > 0$		Pure imaginary
Spiral		$q \neq 0$	$\Delta < 0$	Complex (not pure imaginary)

Table 2: Criterion

Term in $r(x)$	**Choice for** $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \ldots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ or $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ or $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 3: Method of undetermined coefficients.

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$
Stable and attractive	$q < 0$	$q > 0$
Stable	$q \leq 0$	$q > 0$
Unstable	either $q \leq 0$	or $q > 0$

Table 4: Stability criterion for critical points.

Case	**Type**	**Roots**	**Basis**	**General Solution**
I	Distict real	(λ_1, λ_2)	$e^{\lambda_1 x}$ and $e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real Double Root	$(\lambda = -\frac{1}{2}a)$	$e^{-ax/2}$ and $xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex Conjugate	$\lambda_{1,2} = -1/2a \pm j\omega$	$e^{-ax/2} \cos \omega x$ and $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

Table 5: Possible roots of the characteristic equation based on the discriminant value.

Case	**Type**	**Roots**	**General Solution**
I	Distict real	(m_1, m_2)	$y = c_1 x^{m_1} + c_2 x^{m_2}$
II	Real Double Root	$m = \frac{1}{2}(1-a)$	$y = c_1 x^m \ln x + c_2 x^m$
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$	
		$\alpha = Re(m)$	
	$m_2 = \alpha - \beta j$	$\beta = Im(m)$	

Table 6: Cases for solving Euler-Cauchy.

a. The given ODE is separable because it can be written:

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x,$$
$$\int \frac{1}{1+y^2}\,\mathrm{d}y = \int \mathrm{d}x$$
$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x+c) \quad \blacksquare$$

where c is the integration constant (10)

b. By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \int \frac{1}{y}\,\mathrm{d}y = \int -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition:

$$y(0) = ce^0 = c = 1.8$$

Therefore the initial value problem has the solution:

$$y = 1.8e^{-x^2} \quad \blacksquare$$

This is a particular solution, representing a bell-shaped curve (10)

c. This equation can be classified as **Euler-Cauchy**. Doing the following substitution:

$$y = x^{\mathrm{m}} \qquad y' = mx^{\mathrm{m-1}} \qquad y'' = m(m-1)\,x^{\mathrm{m-2}}$$

which gives:

$$x^2 m(m-1)\,x^{\mathrm{m-2}} + 1.5xmx^{\mathrm{m-1}} - 0.5x^{\mathrm{m}} = 0$$
$$\left(m^2 - m\right)x^{\mathrm{m}} + 1.5mx^{\mathrm{m}} - 0.5x^{\mathrm{m}} = 0$$
$$m^2x^{\mathrm{m}} - mx^{\mathrm{m}} + 1.5mx^{\mathrm{m}} - 0.5x^{\mathrm{m}} = 0$$
$$x^2\left(m^2 + m(1.5-1) - 0.5\right) = 0$$

and has an auxiliary equation (10)

$$m^2 + 0.5m - 0.5 = 0$$

Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions is in the form:

$$y = c_1x^{m_1} + c_2x^{m_2}$$

for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution. (10)

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

d. Substituting:

$$y = e^{\lambda x}, \qquad y' = \lambda e^{\lambda x}, \qquad y'' = \lambda^2 e^{\lambda x}.$$

gives

$$\begin{aligned} \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} &= 0 \\ e^{\lambda x}\left(\lambda^2 + \lambda - 2\right) &= 0 \end{aligned}$$

Which gives the characteristic equation:

$$\lambda^2 + \lambda - 2 = 0$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution: (10)

$$y = c_1 e^{x} + c_2 e^{-2x} \quad \blacksquare$$

[A2] Linear Algebra 20

We start solving by first determining the eigenvalues:

$$\boldsymbol{A}x = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In component form:

$$\begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 \end{aligned}$$

Tidying up, we get the following:

$$\begin{aligned} (-5 - \lambda)\, x_1 + 2x_2 &= 0 \\ 2x_1 + (-2 - \lambda)\, x_2 &= 0. \end{aligned}$$

This can be written in matrix notation:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{x} = 0$$

By Cramer's theorem, it has a non-trivial solution $\boldsymbol{x} \neq 0$ (an eigenvector of $\boldsymbol{A}$ we are looking for) if and only if its **coefficient determinant is zero**, that is,

$$\begin{aligned} D(\lambda) = \det(\boldsymbol{A} - \lambda I) &= \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} \\ &= (-5 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0. \end{aligned}$$

We call $D(\lambda)$ the **characteristic determinant** or, if it is expanded, the **characteristic polynomial**, and $D(\lambda) = 0$ the **characteristic equation** of $\boldsymbol{A}$. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$.

These are the eigenvalues of $\boldsymbol{A}$.

Now we have calculated our eigenvalues, it is time for the eigenvectors. First, let's find the eigenvector of $\boldsymbol{A}$ corresponding to λ_1. This vector is obtained using $\lambda = \lambda_1 = -1$, that is,

$$-4x_1 + 2x_2 = 0$$
$$2x_1 - x_2 = 0.$$

A solution is $x_2 = 2x_1$, as we see from either of the two (2) equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We can do a check if our result is correct.

$$\boldsymbol{A}x_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1.$$

Time for the second equation. Eigenvector of $\boldsymbol{A}$ corresponding to λ_2. For $\lambda = \lambda_2 = -6$, it becomes

$$x_1 + 2x_2 = 0$$
$$2x_1 + 4x_2 = 0.$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Therefore, an eigenvector of $\mathbf{A}$ corresponding to $\lambda_2 = -6$ is:

$$\boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

To be complete, let's do the check for this as well:

$$\boldsymbol{A}x_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix}$$
$$= (-6)\,\boldsymbol{x}_2 = \lambda \boldsymbol{x}_2 \quad \blacksquare$$

[A3] Vector Calculus 30

a. The gradient is (10)

$$\nabla r = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

b. The divergence is (10)

$$
\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(x^3 + y^3\right) + \frac{\partial}{\partial y}\left(3xy^2\right) + \frac{\partial}{\partial z}\left(3zy_2\right) \\
&= 3x^2 + 6xy + 3y^2 \\
&= 3\left(x^2 + y^2\right) \quad \blacksquare
\end{aligned}
$$

c. The curl is: (10)

$$
\begin{aligned}
\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 2x^2 & 0 \end{vmatrix} \\
&= \frac{\partial}{\partial y}0\,\hat{\boldsymbol{x}} + \frac{\partial}{\partial z}y\,\hat{\boldsymbol{z}} + \frac{\partial}{\partial x}2x^2\,\hat{\boldsymbol{z}} - \left(\frac{\partial}{\partial y}y\,\hat{\boldsymbol{z}} + \frac{\partial}{\partial z}2x^2\,\hat{\boldsymbol{z}} + \frac{\partial}{\partial x}0\,\hat{\boldsymbol{y}}\right) \\
&= (0)\ \hat{\boldsymbol{x}} + (0)\ \hat{\boldsymbol{y}} + (4x-1)\ \hat{\boldsymbol{z}} \quad \blacksquare
\end{aligned}
$$