Topics on Fundamental Science

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M.Sc
Higher Mathematics I
Lecture Book

WS.2025

WS.2025

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MCI

Current version is WS.2025.

This document includes the contents of Higher Mathematics I, official name being *Höhere Mathematik 1*, taught at MCI in the Mechatronik Smart Technologies. This document is the part of the module MECH-M-1-HMA-HMA-VO taught in the M.Sc degree.

All relevant code of the document is done using *SageMath* where stated and Python v3.13.7.

This document was compiled with LuaTEX v1.22.0, and all editing were done using GNU Emacs v30.1 using AUCTEX and org-mode package.

This document is based on the following books and resources shown in no particular order:

Thomas Calculus (12th Edition) by G. B. Thomas, Jr. et.al , Pearson (2010) *Probability: A Graduate Course* by A. Gut , Springer (2005) *Partial Differential Equations of Mathematical Physics* by S.L. Sobolev , Addison Wesley (2019) *Partial Differential Equations - An Introduction* by W. A. Strauss , Wiley (2008) *Probability and Statistics for Engineers & Scientists* by R. E. Walpole, et. al , Pearson (2012) *Mathematical Methods in the Physical Sciences (3rd Edition)* by M. L. Boas , Wiley (2006) *Mathematical Methods for Physics and Engineering (3rd Edition)* by K. F. Riley, et. al , Cambridge (2006) *Differential Equations with Applications and Historical Notes (3rd Edition)* by G. F. Simmons , CRC Press (2017) *Advanced Engineering Calculus (9th Edition)* by E. Kreyszig , Wiley (2011) *Applied Statistics and Probability for Engineers (3rd Edition)* by D. C. Montgomery , Wiley (2003) *A Students Guide to Fourier Transform* by J. F. James , Cambridge (2011) *Mathematics of Diffusion* by J. Crank , Oxford (1975) *Partial Differential Equations in Physics* by A. Sommerfeld , Academic Press (1949) *Probability and Stochastics* by E. Cinlar , Springer (2010) *Random Walks in Biology* by H. C. Berg , Princeton (1983)

The document is designed with no intention of publication and has only been designed for education purposes.

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Table of Contents

Supplemental List of Items

List of Figures

List of Tables

List of Examples

List of Theorems

Part I

Ordinary Differential Equations

Part Contents

Physics is written in this grand book . . . which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.

(Galilei, Galileo: Il Saggiatore, Chapter 6)

Chapter 1

First-Order Ordinary Differential Equations

Table of Contents

1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a mathematical expression in terms of its variables, functions, and equations. Such an expression is known as a **mathematical model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**. Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,

- find new values,
- interpret result in a physical terms.

This is all done to understand the behaviour of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout the chapter of our book.

An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually written as $y(x)$. The equation may also contain y itself, known functions of x, and constants.

For example all the equation shown below are classified as ODE:

$$y' = \sin x, \qquad y'' + 9y = e^{-3x}, \qquad y'y'' - \frac{5}{4}y = 0.$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from Partial Differential Equation (PDE)s, which involve **partial** derivatives of an unknown function of two or more variables[1]. For instance, a PDE with unknown function u of two (2) variables x and y is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

[1]The topic of PDE will be the focus of **Higher Mathematics II**.

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

For now, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F\left(x, y, y'\right) = 0, \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE:

$$x^{-4}y' - 3y^2 = 0 \qquad \text{where} \qquad x \neq 0$$

can be written explicitly as $y' = 3x^4y^2$.

1.1.1 The Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y\left(x_0\right) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

An ODE, together with an initial condition, is called an **initial value problem**.

Theory 1.1: Initial Value Problem

In multi-variable calculus, an Initial Value Problem (IVP) is an ODE together with an **initial condition** which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Exercise 1.1 An Initial Value Problem

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

SOLUTION The general solution is:

$$y(x) = ce^{3x}$$

From the solution and the initial condition:

$$y(0) = ce^{0} = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation. Visually the solution is plotted as follows ■

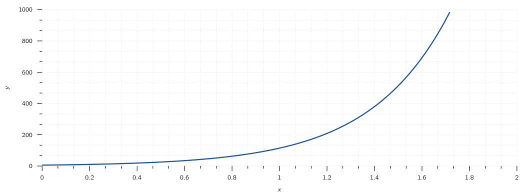


Figure 1.1: A Solution to the initial value problem.

Exercise 1.2 Radioactive Decay

Given 0,5 g of a radioactive substance, find the amount present at any later time. The decay of Radium is measured to be $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$.

SOLUTION We know $y(t)$ is the substance amount still present at t. Using the law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is **positive**, and due of the minus, we get *decay*. We know k which the question has given as $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$. Now the given initial amount is 0,5 g, and we can call the corresponding instant $t = 0$. We have the initial condition $y(0) = 0.5$, which is the instant the process begins. Therefore, the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the ODE is an exponential decay and has the general solution:

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition which gives $y(0) = c = 0.5$. Therefore:

$$y(t) = 0.5e^{-kt} \quad ■$$

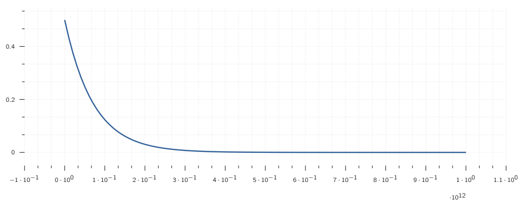


Figure 1.2: A Solution to the radioactive decay.

Information: Prey-Predator Model

The *Lotka - Volterra* equations, are a pair of 1st-order non-linear differential equations, used to describe the dynamics of biological systems in which two (**2**) species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = -\gamma y + \delta xy,$$

where:

- x, y is the population density of prey, predator,
- dy/dt growth rates of the two populations,
- t represents time;
- α, β are the maximum prey per capita growth rate, and the effect of of predators on the prey death rate.
- γ, δ are the predator's per capita death rate, and the effect of of prey on the predator's growth rate.

Figure 1.3: Bunny, natures fast food.

NOTE All parameters are positive and real.

The solution of the differential equations is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping.

1.2 Separable Ordinary Differential Equations

Many practically useful ODEs can be reduced to the following form:

$$g(y)\,y' = f(x) \tag{1.4}$$

using only algebraic manipulations. We can then do integration on both sides with respect to x, obtaining the following expression:

$$\int g(y)\,y'\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + c. \tag{1.5}$$

As a gentle reminder, c here is an integration constant. On the Left Hand Side (LHS) we can switch to y as the variable of integration.

By calculus, we know the relation $y'\,\mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + c. \tag{1.6}$$

If f and g are continuous functions,[2] the integrals in Eq. (1.6) **exist**, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

[2] a continuous function is a function such that a small variation of the argument induces a small variation of the value of the function.

Exercise 1.3 Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian-Italian border, caused a scientific sensation.

When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

NOTE The half-life of carbon is 5175 years.

SOLUTION Radioactive decay is governed by the ODE $y' = ky$ as we have discussed previously. By separation and integration

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t, \quad \ln|y| = kt + c, \quad y = y_0 e^{kt}, \quad y_0 = e^0.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5,$$
$$k = \frac{\ln 0.5}{H} = \frac{0.693}{5715} = -0.0001213.$$

we then use the ratio 52.5% to determine the time:

$$e^{kt} = e^{-0.0001\,213t} = 0.525,$$
$$t = \frac{\ln 0.525}{-0.0001213} = 5312 \quad \blacksquare$$

Exercise 1.4 A Bell Shaped Curve

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

SOLUTION By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution:

$$y = 1.8e^{-x^2}$$

This is a particular solution, representing a bell-shaped curve ■

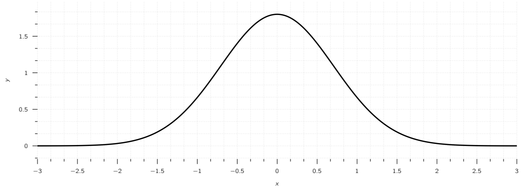


Figure 1.4: A Solution to the Separable ODE.

Exercise 1.5 Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

SOLUTION The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx \qquad \text{By integration} \qquad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

Note It is important to introduce the constant c when the integration is performed.

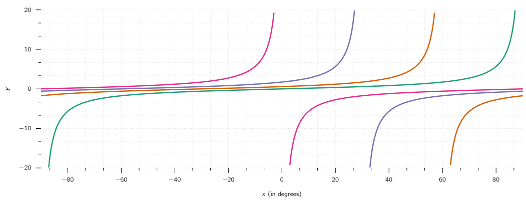


Figure 1.5: A Solution to the Separable ODE.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is **NOT** a solution, when $c \neq 0$ ■

1.2.1 Reduction to Separable Form

Certain **non-separable** ODEs can be made separable by transformations which introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right). \tag{1.7}$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Exercise 1.6 Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

SOLUTION To get the usual explicit form, we start by dividing the given equation by $2xy$ which gives,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then we simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

We see in the last equation that we can now separate the variables,

$$\frac{2u\,du}{1+u^2} = -\frac{dx}{x} \quad \text{and by integration} \quad \ln\left(1+u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

We now take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx \quad \text{therefore we can get} \quad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen below ■

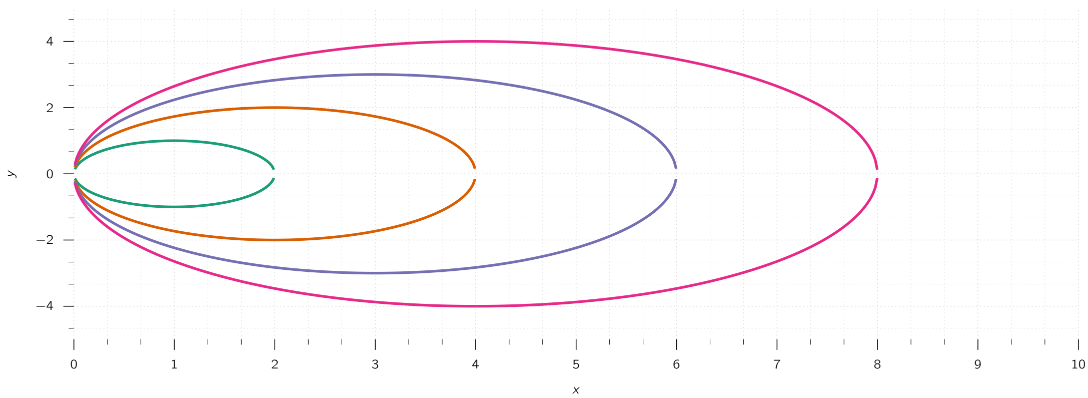


Figure 1.6: A Solution to the Separable ODE.

Information: Reason Behind the Leibniz Notation

To some of you, it might look archaic as to why the notation is written as d^2y/dx^2.

Purely symbolically, if we accept that $\mathrm{d}y = f'(x)\,\mathrm{d}x$, and treat $\mathrm{d}x$ as a constant, then:

$$\mathrm{d}^2 y = \mathrm{d}(\,\mathrm{d}y) = \mathrm{d}(f'(x)\,dx) = dx\,d(f'(x)) = dx\,f''(x)\,dx = f''(x)\,(dx)^2,$$

As to where this notation actually comes from, though: My guess is that it comes from a time when mathematicians primarily thought of and as "infinitesimal quantities."

It is just customary to write $\mathrm{d}x^2$to denote $(\,\mathrm{d}x)^2$ in all common theories of calculus,

1.3 Exact Ordinary Differential Equations

If we remember from calculus courses, if a function $u(x, y)$ has continuous partial derivatives, its differential[3] is:

$$\mathrm{d}u = \frac{\partial u}{\partial x} \, \mathrm{d}x + \frac{\partial u}{\partial y} \, \mathrm{d}y$$

From this it follows that if $u(x, y) = c =$ is constant, then $\mathrm{d}u = 0$.

As an example, let's have a look at the function:

$$u = x + x^2 y^3 = c$$

Finding its factors:

$$\mathrm{d}u = \left(1 + 2xy^3\right) \, \mathrm{d}x + 3x^2 y^2 \, \mathrm{d}y = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE in the form $M(x, y) + N(x, y) \, y' = 0$, written as:

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1.8}$$

is called an **exact differential equation** if the differential form $M(x, y) \, dx + N(x, y) \, dy$ is **exact**, that is, this form is the differential:

$$\mathrm{d}u = \frac{\partial u}{\partial x} \, \mathrm{d}x + \frac{\partial u}{\partial y} \, \mathrm{d}y \tag{1.9}$$

of some function $u(x, y)$. Then Eq. (1.8) can be written

$$\mathrm{d}u = 0$$

By integration we immediately obtain the general solution of Eq. (1.8) in the form:

$$u(x, y) = c \tag{1.10}$$

Comparing Eq. (1.8) and Eq. (1.9), we see that Eq. (1.8) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N. \tag{1.11}$$

From this we can derive a formula for checking whether Eq. (1.8) is exact or not, as follows.

[3] i.e., total differential.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.11),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \, \partial x}, \tag{1.12}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \, \partial y}. \tag{1.13}$$

By the assumption of continuity the two second partial derivatives are **equal**. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.14}$$

> This condition is not only necessary but also sufficient for Eq. (1.8) to be an exact differential equation.

If Eq. (1.8) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way. From Eq. (1.12) we have by integration with respect to x:

$$u = \int M \, \mathrm{d}x + k(y), \tag{1.15}$$

in this integration, y is to be regarded as a **constant**, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u / \partial y$ from Eq. (1.15), use Eq. (1.11) (a) to get dk/dy, and integrate dk/dy to get k.

Formula Eq. (1.15) was obtained from Eq. (1.12).

> It is valid to use **either** of them and arrive at the same result.

Then, instead of Eq. (1.15), we first have by integration with respect to y

$$u = \int N \, \mathrm{d}y + l(x). \tag{1.16}$$

To determine $l(x)$, we derive $\partial u / \partial x$ from , use Eq. (1.12) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Exercise 1.7 Exact ODE - An Initial Value Problem

Solve the initial value problem:

$$(\cos y \sinh x + 1) \, \mathrm{d}x - \sin y \cosh x \, \mathrm{d}y = 0, \quad \text{with} \quad y(1) = 2.$$

SOLUTION Let's begin by verifying the given equation is **exact**:

$$M(x, y) = (\cos y \sinh x + 1),$$

$$N(x,y) = -\sin y \cosh x.$$

We now apply our criteria:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin y \sinh x$$

This shows the given ODE is **exact**. We find u. For a change, let us use Eq. (1.16):

$$u = -\int \sin y \cosh x dy + l(x) = \cos y \cosh x + l(x).$$

From this:

$$\partial u/\partial x = \cos y \sinh x + \frac{dl}{dx} = u = \cos y \sinh x + 1$$

Therefore $dl/dx = 1$ and by integration,

$$l(x) = x + c^*.$$

This gives the general solution

$$u(x,y) = \cos y \cosh x + x = c.$$

From the initial condition:

$$\cos 2 \cosh 1 + 1 = 0.358 = c$$

Therefore the answer is:

$$\cos y \cosh x + x = 0.358 \quad \blacksquare$$

Exercise 1.8 An Exact ODE

Solve the following ODE:

$$\cos(x+y)\ dx + \left(3y^2 + 2y + \cos(x+y)\right)\ dy = 0.$$

SOLUTION

The solution is as follows:

Test for exactness First check if our equation is **exact**, try to convert the equation of the form Eq. (1.8):

$$M = \cos(x+y),$$
$$N = 3y^2 + 2y + \cos(x+y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x+y),$$
$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

This proves our equation to be exact.

Implicit General Solution From Eq. (1.15), we obtain by integration:

$$\begin{aligned} u &= \int M\,dx + k(y) \\ &= \int \cos(x+y)\ dx + k(y) \\ &= \sin(x+y) + k(y) \end{aligned} \tag{1.17}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (1.13), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Therefore $\frac{dk}{dy} = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$. Inserting this result into Eq. (1.17) and observing Eq. (1.10), we obtain:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c \quad \blacksquare$$

Exercise 1.9 The Breakdown of Exactness

Check the exactness of the following ODE:

$$-y\,dx + x\,dy = 0$$

SOLUTION The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

Let us show that in such a case the present method does **NOT** work.

$$u = \int M\,dx + k(y) = -xy + k(y),$$

$$\frac{\partial u}{\partial y} = -x + \frac{\partial k}{\partial y}.$$

Now, $\partial u/\partial y$ should equal $N = x$, as required for this equation to be exact. However, this is impossible because $k(y)$ can depend only on y $\blacksquare$

1.4 Linear Ordinary Differential Equations

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A 1st-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)\, y = r(x) \tag{1.18}$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

1.4.1 Homogeneous Linear Ordinary Differential Equations

We want to solve Eq. (1.18) in some interval $a < x < b$, let's call it J, and we begin with the simpler special case where $r(x)$ is zero for all x in J.[4] Then the ODE given in Eq. (1.18) becomes:

[4] This is sometimes written $r(x) = 0$.

$$y' + p(x)\, y = 0 \tag{1.19}$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\, dx, \qquad \text{therefore} \qquad \ln|y| = -\int p(x)\, dx + c^*.$$

Taking exponents on both sides, the general solution of the homogeneous ODE Eq. (1.19) is,

$$y(x) = ce^{-\int p(x)\, dx} \qquad \left(c = \pm e^{\,c^*} \qquad \text{when} \qquad y \neq 0\right) \tag{1.20}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

1.4.2 Non-Homogeneous Linear Ordinary Differential Equations

We now solve Eq. (1.18) in the case that $r(x)$ in Eq. (1.18) is **NOT** everywhere zero in the interval J considered. Then the ODE Eq. (1.18) is called non-homogeneous. It turns out that in this case, Eq. (1.18) has a useful property. Namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ as follows.

We multiply Eq. (1.18) by $F(x)$, obtaining:

$$Fy' + pFy = rF. \tag{1.21}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{therefore} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{therefore} \qquad F = e^{h}.$$

With this F and $h' = p$, Eq. (1.21) becomes:

$$e^{h}y' + h'e^{h}y = e^{h}y' + \left(e^{h}\right)'y = \left(e^{h}y\right)' = re^{h} \quad \text{by integration} \quad e^{h}y = \int e^{h}r\,dx + c$$

Dividing by e^{h}, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^{h}r\,dx + c\right), \qquad h = \int p(x)\,dx. \tag{1.22}$$

This reduces solving Eq. (1.18) to the generally simpler task of evaluating integrals.[5] The structure of Eq. (1.22) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing Eq. (1.22) as a sum of two terms,

[5] For ODEs for which this is still difficult, we may have to use a numeric method for integrals or for the ODE itself.

$$y(x) = e^{-h}\int e^{h}r\,dx + ce^{-h},$$

Exercise 1.10 A Non Homogeneous Ordinary Differential Equation

Solve the initial value problem of the following equation:

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

SOLUTION Here we define the parameters as:

$$p = \tan x, \quad r = \sin 2x = 2\sin x\cos x,$$

and

$$h = \int p\,dx = \int \tan x\,dx = \ln|\sec x|.$$

From this we see that in Eq. (1.22),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x,$$
$$e^{h}r = (\sec x)(2\sin x\cos x) = 2\sin x,$$

and the general solution of our equation is:

$$\begin{aligned} y(x) &= \cos x\left(2\int \sin x\,dx + c\right), \\ &= c\cos x - 2\cos^2 x. \end{aligned}$$

From this and the initial condition

$$1 = c\cdot 1 - 2\cdot 1^2, \qquad \text{therefore} \qquad c = 3,$$

and the solution of our initial value problem is:

$$y = 3\cos x - 2\cos^2 x$$

Here $3\cos x$ is the response to the initial data, and $-2\cos^2 x$ is the response to the input $\sin 2x$ ■

Chapter 2

Second-Order Ordinary Differential Equations

Table of Contents

2.1 Introduction

A second order ODE is a specific type of differential equation which consists of a derivative of a function of **order 2** and no other higher-order derivative of the function appears in the equation. These equations have significant engineering applications such as in the study of mechanical and electrical vibrations, wave motion, and heat conduction.

A second-order ODE is called **linear**, if it can be written[1] as:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.1}$$

Of course, we can extend most of what we learned in the study of first-order ODE and describe:

homogeneous if $r(x) = 0$,

non-homogeneous if otherwise.

[1] in its standard form.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs. For example:

$$
\begin{aligned}
y'' &= 25y - e^{-x}\cos x && \text{non-homogeneous linear}\\
y'' + \frac{1}{x}y' + y &= 0 && \text{homogeneous linear}\\
y''y + \left(y''\right)^2 &= 0 && \text{non-linear}
\end{aligned}
$$

2.1.1 The Principle of Superposition

For the homogeneous form the backbone of finding a useful solution is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1y_1 + c_2y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or **linearity principle**.

Theory 2.2: Fundamental Theorem for the Homogeneous Linear ODE

For a homogeneous linear ODE of the form:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

any linear combination of two (2) solutions on an open interval I is again a solution of on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions. This theorem is only applicable to **homogeneous** form.

While the iron is hot, lets do a couple of exercises to begin studying 2^{nd}-order ODEs:

Exercise 2.1 A Superposition of Solutions

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0, \quad \text{for all } x.$$

SOLUTION By differentiation and substitution, we obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$
\begin{aligned}
&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\
&= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare
\end{aligned}
$$

Exercise 2.2 **Example of a Non-Homogeneous Linear ODE**

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 1$$

SOLUTION Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1 + \sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

While the methodology is same as before, it is worth mentioning here the small difference. For a second-order homogeneous linear ODE, an initial value problem consists of two (2) initial conditions:[2]

[2]This makes sense as to properly evaluate a 2nd differential equation, we need two values.

$$y(x_0) = K_0 \qquad y'(x_0) = K_1. \tag{2.2}$$

The Eq. (2.2) are used to determine the two (2) arbitrary constants c_1 and c_2 in a general solution

Exercise 2.3 **An Initial Value Problem**

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

SOLUTION **General Solution** From the previous examples, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Particular Solution

We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.

Exercise 2.4 Reduction of Order

Find a basis of solutions of the ODE

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

SOLUTION Inspection shows $y_1 = x$ is a solution as $y_1{}' = 1$ and $y_1{}'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u, \qquad y'' = u''x + 2u'$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0,$$
$$(x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely,

$$(x^2 - x)w' + (x - 2)w = 0.$$

Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx,$$
$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We don't need constant of integration as we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain n

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x},$$
$$\text{hence} \qquad y_2 = ux = x\ln|x| + 1$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained as basic of solutions, valid for all positive x ■

Information: Differential Equations - Early Beginnings

Differential equations have been a major branch of pure and applied mathematics since mid 17^{th} century.

"Differential equations" began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's "fluxional equations" in the 1670s. Applications were made largely to geometry and mechanics with particular interest in optimisation. Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, which lead to partial differential equations. Generalisation of isoperimetrical problems led to the calculus of variations.

New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace. Development of the general theory of solutions included singular ones, functional solutions and those by infinite series. Many applications were made to mechanics, especially to astronomy and continuous media [1].

2.2 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.3}$$

We start to solve the above equation by starting:

$$y = e^{\lambda x} \tag{2.4}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.3) gives:

$$\left(\lambda^2 + a\lambda + b\right) e^{\lambda x} = 0.$$

Therefore if λ is a solution of the important **characteristic** equation,[3]

$$\lambda^2 + a\lambda + b = 0 \tag{2.5}$$

[3] This is also known as an auxiliary equation.

then the exponential function Eq. (2.4) is a solution of the ODE given in Eq. (2.3). Now from algebra we recall the roots of the quadratic equation:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \tag{2.6}$$

Using Eq. (2.5) and Eq. (2.6) we can see that

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

as solutions to Eq. (2.3). From algebra we further know that the quadratic equation Eq. (2.5) may have three (3) kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, which are shown in **Tbl.** 2.1.

Exercise 2.5 IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

SOLUTION **General Solution** The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1,$$

$$\text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Particular Solution As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

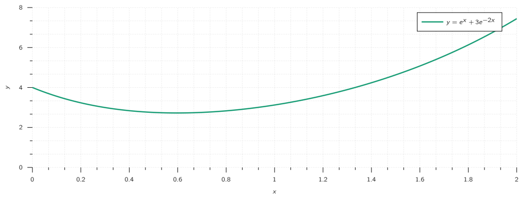


Figure 2.1: Solution to Case of distinct real roots.

Exercise 2.6 IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

SOLUTION The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = \left(\lambda + 2\right)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = \left(c_1 + c_2 x\right) e^{-0.5\,x}$$

We need its derivative:

$$y' = c_2 e^{-0.5\,x} - 0.5\left(c_1 + c_2 x\right) e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$y\,(0) = c_1 = 3.0,$$

$$y'(0) = c_2 - 0.5c_1 = 3.5,$$

$$c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\,x}$$

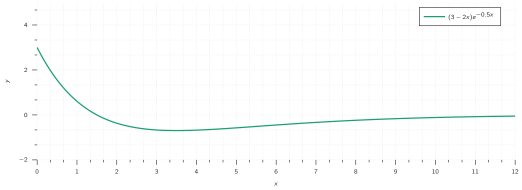


Figure 2.2: Solution to case of double roots.

Case	Condition	Roots of	Basis	General Solution
I	$a^2 - 4b > 0$	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
		$\left(\lambda_1, \lambda_2\right)$	$e^{\lambda_2 x}$	
II	$a^2 - 4b = 0$	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
		$\left(\lambda = -1/2a\right)$	$xe^{-ax/2}$	
III	$a^2 - 4b < 0$	Complex Conjugate	$e^{-ax/2}\cos\omega x$	$y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$
		$\lambda_1 = -1/2a + j\omega$	$e^{-ax/2}\sin\omega x$	
		$\lambda_2 = -1/2a - j\omega$		

Table 2.1: Possible roots of the characteristic equation based on the discriminant value.

Exercise 2.7 IVP: Case of Complex Roots

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

SOLUTION

General Solution The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathrm{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Particular Solution The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

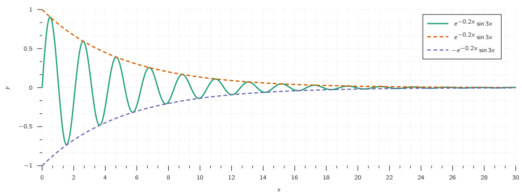


Figure 2.3: Solution to case of complex roots.

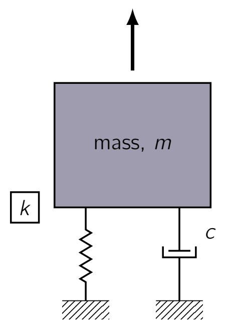
2.2.1 A Study of Damped System

Linear ODEs with constant coefficients have important applications in mechanics, and one of the important system to study is spring-mass-damper system[4], which has the following important component:

$$F_2 = -cy'.$$

Using this damping we can define the ODE of the damped mass-spring system:

$$my'' + cy' + ky = 0. \tag{2.7}$$



[4] A spring mass damper system.

This can physically be done by connecting the ball to a bowl containing a liquid. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

> This is generally a good approximation for small velocities.

The constant c is called the damping constant.

The damping force $F_2 = -cy'$ acts **against** the motion. Therefore for a downward motion we have $y' > 0$ which for positive c makes F negative,[5] as it should be.

[5] an upward force.

Similarly, for an upward motion we have $y' > 0$ which, for $c > 0$ makes F_2 positive.[6]

[6] a downward force.

The ODE Eq. (2.7) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \lambda_2 = -\alpha - \beta, \qquad \text{where} \qquad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}. \quad (2.8)$$

Depending on the amount of damping present-whether a lot of damping, a medium amount of damping or little damping-three types of motions occur, respectively. A summary of its behaviour is shown in **Tbl.** 2.2.

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.2: The three cases of behaviour depending on the condition.

A Deeper Look into the Three Cases

Case I: Over-damping If $c^2 > 4mk$, then λ_1 and λ_2 are said to be **distinct real roots**. In this case, the corresponding general solution becomes:[7]

$$y(t) = c_1 e^{-(\alpha - \beta)\, t} + c_2 e^{-(\alpha + \beta)\, t}. \tag{2.9}$$

[7]In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.9) are negative as $\alpha > 0$ and $\beta > 0$ with:

$$\beta^2 = \alpha^2 - k/m < \alpha^2$$

Therefore both terms in Eq. (2.9) approach zero as $t \to \infty$. Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position (i.e., $y = 0$). A graphical representation of this behaviour can be seen in **Fig. ??**.

Case II: Critical-Damping Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III) and occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$.

Then the corresponding general solution of Eq. (2.7) is:

$$y(t) = \left(c_1 + c_2 t\right) e^{-\alpha t}. \tag{2.10}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha\, t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. **Fig. ??** shows typical forms of **Fig. ??**.

The graph above looks almost like those in the previous figure.

Case III: Under-Damping This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in Eq. (2.8) is no longer real but pure **imaginary**, which we write as,

$$\beta = \mathbf{j}\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad \text{where} \qquad \beta > 0.$$

The asterisk $(*)$ is used to differentiate from ω which is used predominantly in electrical engineering do describe angular frequency.

The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + \mathbf{j}\omega^*, \qquad \lambda_2 = -\alpha - \mathbf{j}\omega^*.$$

with $\alpha = c/2m$. The corresponding general solution is:

$$y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right) \tag{2.11}$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$ This represents **damped oscillations**. Their curve lies between the two dashed curves:

$$y = Ce^{-\alpha t} \qquad \text{and} \qquad y = -Ce^{-\alpha t}$$

The frequency of the under-damping process is $\omega^*/2\pi$ Hz. Based on the equation, we see that the smaller c is,[8] the larger is ω^* and the more rapid the oscillations become.

[8] As long as it is bigger than 0.

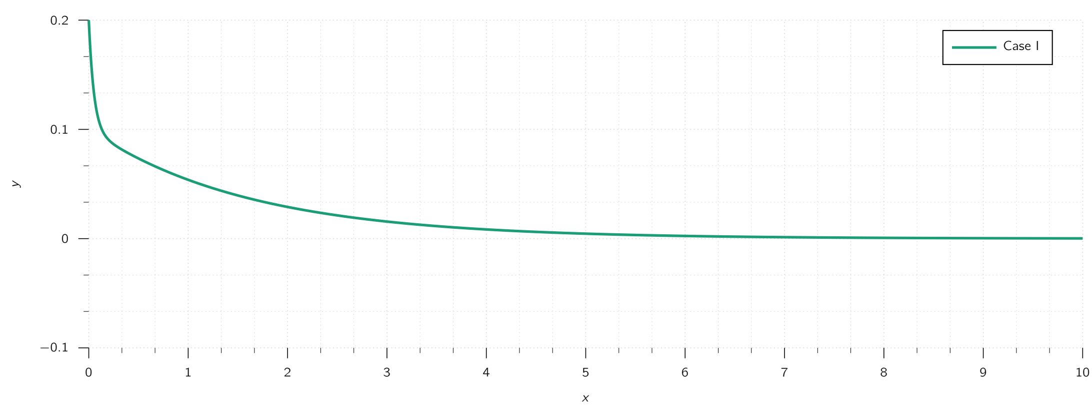


Figure 2.4: Standard behaviour of an over-damped system.

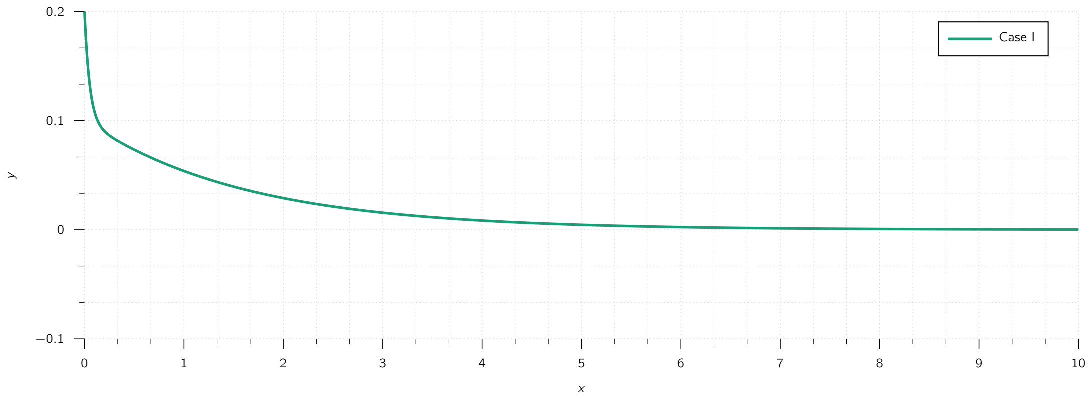


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Figure 2.5: Standard behaviour of an critical system.

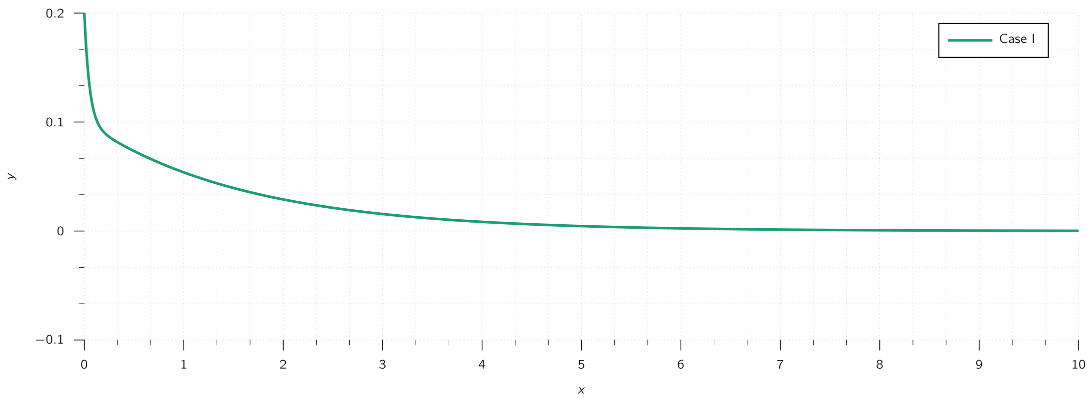


Figure 2.6

2.3 Euler-Cauchy Equations

Without much prior literature, let's get to the point. These class of equations have the form:[9]

$$x^2 y'' + axy' + by = 0 \tag{2.12}$$

To solve do the following substitutions:

$$y = x^{m}, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}.$$

Which gives:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^{m} = 0.$$

$y = x^{m}$ is a good choice as it produces a common factor x^{m}.

Simplifying the equation gives us the **auxiliary** equation.

$$m^2 + (a-1)m + b = 0. \tag{2.13}$$

$y = x^{m}$ is a solution of Eq. (2.12) if and only if m is a root of Eq. (2.13).

The roots of Eq. (2.13) are:

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}.$$

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^{m} \ln x + c_2 x^{m}$
	(m)	
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta j$ and $m_2 = \alpha - \beta j$	$\alpha = \mathrm{Re}(m)$ and $\beta = \mathrm{Im}(m)$

Table 2.3: Possible solutions of the Euler-Cauchy based on the m value.

Complex conjugate roots are of minor practical importance for practical purposes.

[9] **Augustin-Louis Cauchy** *(1789 - 1857)*

A French mathematician, engineer, and physicist. He was one of the first to rigorously state and prove the key theorems of calculus (thereby creating real analysis), pioneered the field complex analysis, and the study of permutation groups in abstract algebra. Cauchy also contributed to a number of topics in mathematical physics, notably continuum mechanics.

Exercise 2.8 A General Solution in the Case of Different Real Roots

Solve the following ODE:

$$x^2 y'' + 1.5xy' - 0.5y = 0$$

SOLUTION This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1 \sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

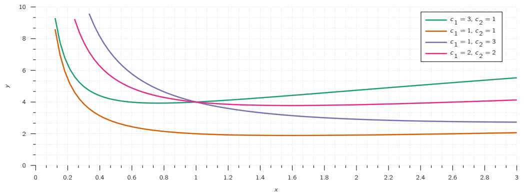


Figure 2.7: Solution to A General Solution in the Case of Different Real Roots with different constants.

Exercise 2.9 A General Solution in the Case of a Double Root

Solve the following ODE:

$$x^2 y'' - 5xy' + 9y = 0$$

SOLUTION Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

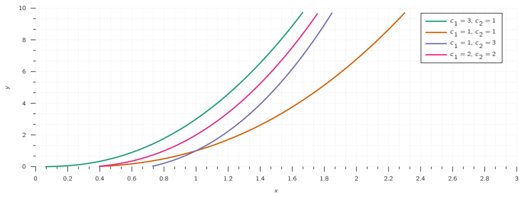


Figure 2.8: Solution to A General Solution in the Case of a Double Root with different constants.

Exercise 2.10 Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of the *Euler-Cauchy equation* $rv'' + 2v' = 0$.

SOLUTION The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

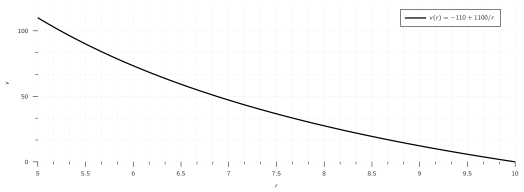


Figure 2.9: Solution to the Electric Potential Field Between Two Concentric Spheres.

2.4 Non-Homogeneous ODEs

To start, they have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.14}$$

where $r(x) \neq 0$. A general solution of Eq. (2.14) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.15}$$

and a **particular solution** of Eq. (2.14). These two (2) new terms **general solution** of Eq. (2.14) and **particular solution** of Eq. (2.14) are defined as follows:

Theory 2.3: General Solution and Particular Solution

A general solution of the non-homogeneous ODE Eq. (2.14) on an open interval I is a solution of the form:

$$y(x) = y_h(x) + y_p(x)\,. \tag{2.16}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.15) on I and y_p is any solution of Eq. (2.14) on I containing **no arbitrary constants**. A particular solution of Eq. (2.14) on I is a solution obtained from Eq. (2.16) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

2.4.1 Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.14) or an initial value problem for Eq. (2.14), we have to solve the homogeneous ODE Eq. (2.15) or an initial value problem for and find any solution y_p of Eq. (2.14), so that we obtain a general solution Eq. (2.16) of Eq. (2.14).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y\prime\prime + ay\prime + by = r(x) \tag{2.17}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.

Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Theory 2.4: Choice Rules for the Method of Undetermined Coefficients

Basic Rule

If $r(x)$ in Eq. (2.17) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.17).

Modification Rule

If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.17), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule

If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 2.4: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.14) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.14) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Exercise 2.11 Application of the Basic Rule A

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

SOLUTION **General Solution of the Homogeneous ODE** The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

Solution of the non-Homogeneous ODE First we try

$y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

Looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0,$$
$$y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Therefore:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, therefore $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$
$$\text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

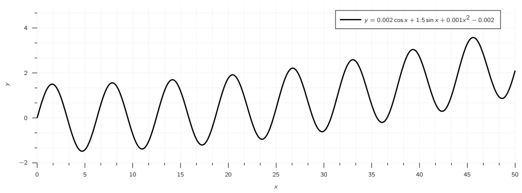


Figure 2.10: Solution to Application of the basic rule A.

Exercise 2.12 Application of the Basic Rule B

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x},$$
$$y(0) = 1, \qquad y'(0) = 0.$$

SOLUTION **General solution of the homogeneous ODE** The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$$

Therefore the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2y)\,e^{-1.5x}$$

Solution y_p of the non-homogeneous ODE The function $e^{-1.5x}$ on the Right Hand Side (RHS) would normally require the choice $Ce^{-1.5x}$. However, we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Which means, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose:

$$y_p = Cx^2e^{-1.5x}, \quad \text{then}$$
$$y_p' = C(2x - 1.5x^2)e^{-1.5x},$$
$$y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2e^{-1.5x}$$

Step 3. Solution of the initial value problem Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2v)e^{-1.2x}$$

$$-10xe^{-1.2x} + 7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$ and gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

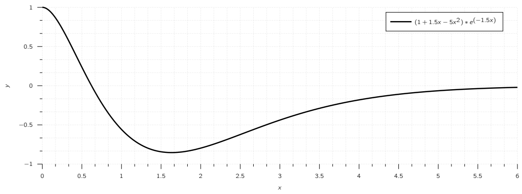


Figure 2.11: Solution to Application of the basic rule B.

Exercise 2.13 Application of the Basic Rule C

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

SOLUTION **The General Solution** The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1e^{-x/2} + c_2e^{-3x/2}.$$

The Particular Solution We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

therefore

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE is

$$y = c_1e^{-x/2} + c_2e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Solution of the initial value problem From y, y' and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

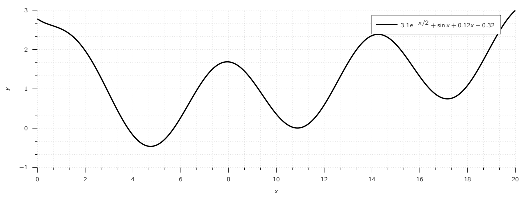


Figure 2.12: Solution to Application of the basic rule C.

2.5 A Study of Forced Oscillations and Resonance

Previously we considered vertical motions of a mass-spring system[10] and modelled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.18}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass-spring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia my'', the damping force cy' (if $c < 0$), and the spring force ky, a restoring force.

[10] as in the vibration of a mass m on an elastic spring

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.18) into:

$$my'' + cy' + ky = r(t). \tag{2.19}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.20}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.20) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.18) plus any solution y_p of Eq. (2.20). To find y_p, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.21}$$

By differentiating this function (remember the chain rule) we obtain:

$$\begin{aligned} y_p' &= -\omega a \sin \omega t + \omega b \cos \omega t, \\ y_p'' &= -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t. \end{aligned}$$

Substituting y_p, $y_p\prime$, $y_p\prime\prime$, into Eq. (2.20) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2)equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.22}$$

$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.23}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega\, c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $\left(km\omega^2\right)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.24}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.20) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.18) and y_p is given by Eq. (2.21) with coefficients Eq. (2.24).

2.5.1 Solving Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C} \int I \, dt = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t. \tag{2.25}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.25) is the sum $I = I_{\mathrm{h}} + I_{\mathrm{p}}$, where I_{h} is a general solution of the homogeneous ODE corresponding to Eq. (2.25) and I_{p} is a particular solution of Eq. (2.25). We first determine I_{p} by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a \cos \omega t + b \sin \omega t, \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t), \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t). \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to E0v cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned} L\omega^2(-a) + R\omega b + a/C = E_0\omega \qquad &\text{(Cosine terms)} \\ L\omega^2(-b) + R\omega(-a) + b/C = 0 \qquad &\text{(Sine terms)}. \end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.26}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$\begin{aligned} -Sa + Rb &= E_0, \\ -Ra - Sb &= 0. \end{aligned}$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.27}$$

Equation (2) with coefficients a and b given by Eq. (2.27) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage.

Using Eq. (2.27), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.28}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $(R^2 + S^2)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.28) and whose frequency is that of the input (i.e., voltage).

Exercise 2.14 Harmonic Oscillation of an Undamped Mass-Spring System

If a mass-spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

SOLUTION Hooke's law:

$$F_1 = -ky \tag{2.29}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0{,}48\,\text{Hz}$$

From Eq. (2.29) and the initial conditions, $y(0) = A = 0{,}16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$.

Hence the motion is

$$y(t) = 0.16 \cos 3t\,\text{m} \quad \blacksquare$$

Exercise 2.15 Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped Mass-Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 60\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 10\,\mathrm{kg}\cdot\mathrm{s}^{-1}$

SOLUTION It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass-Spring System*, the model is the initial value problem:

$$10y''+100y'+90y = 0, \qquad y(0) = 0{,}16\,\mathrm{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1e^{-9t}+c_2e^{-t} \quad \text{We also need} \quad y' = -9c_1e^{-9t}-c_2e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9\,t} + 0.18e^{-\,t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2t\right)e^{-3\,t},$$

$$\text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2t\right)e^{-3\,t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y\prime(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\,e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96j$$

This gives the general solution:

$$y = e^{-0.5\,t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$\begin{aligned} y\prime = e^{-0.5\,t}(&-0.5A\cos 2.96t - 0.5B\sin 2.96t \\ &- 2.96A\sin 2.96t + 2.96B\cos 2.96t) \end{aligned}$$

Hence $y\prime(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution:

$$\begin{aligned} y &= e^{-0.5\,t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) \\ &= 0.162e^{-0.5\,t}\cos\left(2.96t - 0.17\right) \quad \blacksquare \end{aligned}$$

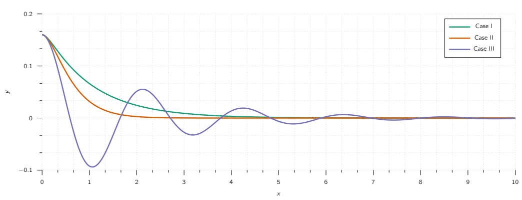


Figure 2.13: Three cases of damped motion.

Exercise 2.16 Studying a RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\,\Omega$, $L = 0{,}9\,\mathrm{H}$, $C = 0{,}01\,\mathrm{F}$, which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Note Assume that current and capacitor charge are 0 when $t = 0$.

SOLUTION **The General solution** Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t.$$

$$0.1I'' + 11I' + 100I = 110 \cdot 377\cos 377t.$$

Therefore the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1e^{-10t} + c_2e^{-100t}.$$

The Particular solution We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from:

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Therefore in our present case, a general solution of the nonhomogeneous ODE is:

$$I(t) = c_1e^{-10t} + c_2e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

Particular solution satisfying the initial conditions How to use Q(0) 0? We finally determine c_1 and c_2 from the in initial conditions I(0) = 0 and Q(0) = 0.

From the first condition and the general solution we have:

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 30$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

> You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

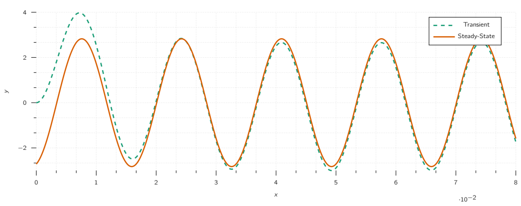


Figure 2.14: A comparison of the actual solution and the steady-state values.

Chapter 3

Higher-Order Ordinary Differential Equations

Table of Contents

3.1 Homogeneous Linear ODEs

Let's do a small revisit, and recall from **First-Order ODEs** that an ODE is of n^{th} if the n^{th} derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F\left(x, y, y', \cdots, y^{(n)}\right) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

> $y^{(n)}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.

An n$^{\text{th}}$-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.

A **solution** of an n$^{\text{th}}$-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to n$^{\text{th}}$-order homogeneous linear ODEs as following theorems.

Theory 3.5: Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

This does not hold for a nonhomogeneous or non-linear ODE.

Theory 3.6: General Solution, Basis, Particular Solution

A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \ldots, c_n \quad \text{arbitrary}) \tag{3.3}$$

where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.

That is, these solutions are linearly independent on I, as defined below.

A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

Theory 3.7: Linear Independence and Dependence

Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on} I \tag{3.4}$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \dots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n-1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (3.4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Exercise 3.1 Linear Dependence

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

SOLUTION By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Exercise 3.2 A General Solution

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{where} \qquad y^{\text{iv}} = \frac{d^4 y}{dx^4}$$

SOLUTION Similar to Chapter 2 we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Exercise 3.3 Initial Value Problem for a Third-Order Euler-Cauchy Equation

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0,$$
$$y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4.$$

SOLUTION **General Solution** As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular Solution The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
&\text{(a)} \quad y(1) = c_1 + c_2 + c_3 = 2 \\
&\text{(b)} \quad y'(1) = c_1 + 2c_2 + 3c_3 = 1 \\
&\text{(c)} \quad y'(1) = 2c_2 + 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W\left(y_1, \cdots, y_n\right) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(n)} = d^n y/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \ \cdots \ + a_1\lambda + a_0 y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \qquad \cdots, \qquad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Exercise 3.4 Distinct Real Roots

Solve the following ODE:

$$y_{'''} - 2y'' - y' + 2y = 0$$

SOLUTION The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots:

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

Exercise 3.5 Simple Complex Roots

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0,$$
$$y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299.$$

SOLUTION The characteristic equation is:

$$\lambda_3 - \lambda_2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10j$. Therefore, a general solution and its derivatives obtained by differentiation are:

$$y = c_1 e^{x} + A \cos 10x + B \sin 10x,$$
$$y' = c_1 e^{x} - 10A \sin 10x + 10B \cos 10x,$$
$$y'' = c_1 e^{x} - 100A \cos 10x - 100B \sin 10x.$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$\text{(a)} \quad c_1 + A = 4,$$
$$\text{(b)} \quad c_1 + 108 = 11,$$
$$\text{(c)} \quad c_1 - 1004 = -299.$$

We solve this system for the unknowns A, B, and c_1. Equation (a) minus Equation (c) gives $101A = 303$, therefore $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^{x} + 3 \cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.

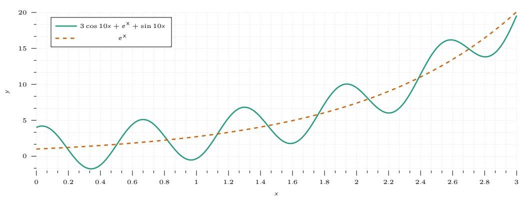


Figure 3.1: Solution to the question Simple Complex Roots

Multiple Real Roots

If a real double root occurs ($\lambda_1 = \lambda_2$) then $y_1 = y_2$ in Eq. (3.3), and we take y_1 and xy_1 as corresponding linearly independent solutions.

More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2 e^{\lambda x}, \quad \cdots, \quad x^{m-1} e^{\lambda x}$$

Exercise 3.6 Real Double and Triple Roots

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y^{\text{iv}} - y'' = 0$$

SOLUTION The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + \left(c_3 + c_4 x + c_5 x^2\right) e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda} = \gamma - i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x, \quad e^{\gamma x} \sin \omega x, \quad xe^{\gamma x} \cos \omega x, \quad xe^{\gamma x} \sin \omega x$$

The first two of these result from $e^{\lambda x}$ and $\overline{e}^{\lambda x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\overline{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}.$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\gamma x} \cos \omega x$, $x^2 e^{\gamma x} \sin \omega x$, and so on.

3.2 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of n^{th} order. As usual with other versions we looked at, we write them in standard form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} +, \cdots, +p_1(x)y' + p_0(x)y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a general solution of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Therefore Eq. (3.7) has no singular solutions.

An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

> The ideas of proof are the same as those for $n = 2$.

Exercise 3.7 IVP: Modification Rule

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x},$$
$$y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47$$

SOLUTION **Step 1** The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2) e^{-x} \end{aligned}$$

Step 2 If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and $Cx^6 e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}$$

Then

$$\begin{aligned} y_p' &= C\left(3x^2 - x^3\right) e^{-x}, \\ y_p'' &= C\left(6x - 6x^2 + x^3\right) e^{-x}, \\ y_p''' &= C\left(6 - 18x + 9x^2 - x^3\right) e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$\begin{aligned} C\left(6 - 18x + 9x^2 - x^3\right) &+ 3C(6x - 6x^2 + x^3) \\ &+ 3C(3x^2 - x^3) \\ &+ Cx^3 = 30. \end{aligned}$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2 e^{-x}$.

Step 3 We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$y = y_h + y_p = (c_1 + c_2 + c_3 x^2)e^{-x} + 5x^3 e^{-x}, \qquad y(0) = c_1 = 3.$$

$$y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, \qquad y'(0) = -3 + c_2 = -$$

$$y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, \qquad y''(0) = 3 + 2c_3 = -$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2 e^{-x} \quad \blacksquare$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as

3.2.1 Application: Modelling an Elastic Beam

While 2nd-order ODEs have numerous applications, of which we have discussed some of the more important ones,[1] higher order ODEs have much fewer engineering applications.

[1] i.e., RLC Circuit, mass-damper system, . . .

For electrical engineers, the order of a circuit can be increased by adding an inductor or a capacitor to the circuit. However, in those cases using computational resources to calculate the circuit behaviour (i.e., ngSpice, OpenModelica) would be an easier path.

For mechanical engineers, an important 4th ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge. This equation is also known as Euler-Bernoulli beam theory.[2]

[2] Daniel Bernoulli (1700 - 1782) was a Swiss mathematician and physicist and was one of the many prominent mathematicians in the Bernoulli family from Basel. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics.

A related application of vibration of beams does not fit in here since it leads to PDEs.

Describing the Problem

Let us consider a beam B of length L and constant, **rectangular**, cross section and homogeneous elastic material (e.g., **level**), which a representative diagram is shown in **Fig.** 3.2.

We assume, under its own weight the beam is bent so little that it is **straight**. Applying a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.

Its axis is curved into the so-called elastic curve[3].

[3] This is known as deflection curve

It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ are small.[4] Then, by the principles of calculus:

[4] The goal here is to determine the tangent direction of C

$$k = \frac{y''}{\sqrt{1 + y''x}} \approx y''$$

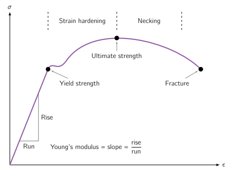
Therefore:

$$M(x) = EIy''(x)$$

where EI is the constant of proportionality. E Young's modulus of elasticity[5] of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.

Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy^{\text{iv}} = f(x) \tag{3.10}$$



[5] Young's modulus (or the Young modulus) is a mechanical property of solid materials that measures the tensile or compressive stiffness when the force is applied lengthwise. It is the elastic modulus for tension or axial compression. Young's modulus is defined as the ratio of the stress (force per unit area) applied to the object and the resulting axial strain (displacement or deformation) in the linear elastic region of the material.

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows:

Table 3.1: Different practical boundary conditions applicable to the problem.

Condition	Mathematical Definition
Simply supported	$y = y'' = 0$ at $x = 0$ and $x = L$
Clamped at both ends	$y = y' = 0$ at $x = 0$ and L
Clamped at $x = 0$, free at $x = L$	$y(0) = y'(0) = 0$ and $y''(L) = y'''(L) = 0$.

The boundary condition $y = 0$ means no displacement at that point, $y' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Information: Shear Force

In solid mechanics, shearing forces are unaligned forces acting on one part of a body in a specific direction, and another part of the body in the opposite direction. When the forces are collinear (aligned with each other), they are called tension forces or compression forces. Shear force can also be defined in terms of planes: "If a plane is passed through a body, a force acting along this plane is called a shear force or shearing force."

Deriving the Solution

Let us apply this to the uniformly loaded simply supported beam. The load is $f(x) = f_0 = \text{const}$. Then Eq. (3.10) is

$$y^{\text{iv}} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Integrating the above expression twice gives:

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

Time to apply the boundary conditions we have set. $y''(0) = 0$ gives $c_2 = 0$. Then

$$y''(L) = L\left(\frac{1}{2}kL + c_1\right) = 0, \quad c_1 = -k\frac{L}{2} \qquad (\text{since} \quad L \neq 0)$$

Therefore:

$$y'' = \frac{k}{2}\left(x^2 - Lx\right).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right),$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution:

$$y = \frac{f_0}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L-x)$. We can verify this by setting $x = u + L/2$ and see that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0$ (i.e., $x = L/2$) is:

$$\frac{5f_0L^4}{(16 \cdot 24EI)} \quad \blacksquare$$

Recall that the positive direction points downward.

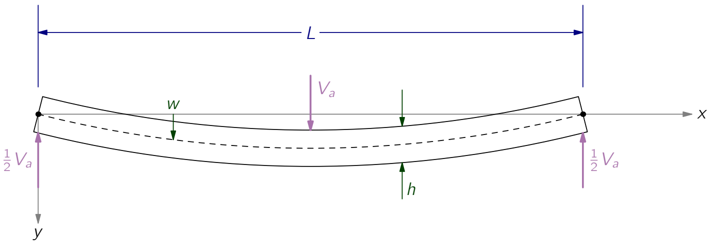


Figure 3.2: An Elastic beam under stress from load.

Chapter 4

Systems of ODEs

Table of Contents

4.1 Looking at Connected ODEs

We now introduce a different way of looking at systems of ODEs. The method consists of examining the general behaviour of families of solutions of ODEs in the phase plane, called the **phase plane** method.

Theory 4.8: Phase Plane

A visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two state variables, say (x, y), or (q, p) etc.

It gives information on the stability of solutions. This approach to systems of ODEs is a qualitative method as it depends only on the nature of the ODEs and does **NOT** require the actual solutions. This can be very useful because it is often difficult or sometimes even impossible to solve systems of ODEs. In contrast, the approach of actually solving a system is known as a **quantitative** method.

Theory 4.9: Qualitative Method

The qualitative analysis of ODEs is to be able to say something about the behavior of solutions of the equations, without solving them explicitly.

The phase plane method has many applications in control theory, circuit analysis theory, population dynamics and so on.

4.1.1 System of ODEs as Engineering Models

Time to see how systems of ODEs are of practical importance. We start by first illustrating how systems of ODEs can serve as models in different applications. Then we show how a higher order[1] ODE can be reduced to a first-order system. The following two (2) examples will look at a system of ODEs as a fluid mechanics problem and then an electrical engineering problem.

[1] with the highest derivative standing alone on one side.

Exercise 4.1 Mixing Problem Involving Two Tanks

A mixing problem involving a single tank is modeled by a single ODE which can be extended to two (2) sets of equations.

Assume two (2) Tanks T_1 and T_2 containing initially 100 L of water each, In T_1 the water is pure, whereas 150 kg of fertilizer are dissolved in T_2. By circulating liquid at rate of $2\,\text{L}\cdot\text{min}^{-1}$ and stirring the amount of fertiliser $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

How long should we let the liquid circulate so that T_1 will contain at least half as much fertiliser as there will be left in T_2? **Note** Assume the mixture is uniform.

SOLUTION **Setting Up The Model** As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. Therefore:

$$y_1' = \frac{2}{100}y_2 - \frac{2}{100}y_1 \qquad \text{Tank 1,}$$
$$y_2' = \frac{2}{100}y_1 - \frac{2}{100}y_2 \qquad \text{Tank 2.}$$

Therefore the mathematical model of our mixture problem is the system of first-order ODEs:

$$y_1' = -0.02y_1 + 0.02y_2,$$
$$y_2' = +0.02y_1 - 0.02y_2.$$

As a vector equation with column vector:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and matrix $\boldsymbol{A}$ this becomes:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \qquad \text{where} \qquad \boldsymbol{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

General Solution As for a single equation, we try an exponential function of t,

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{x}e^{\lambda t} \quad \text{and} \\ &= \boldsymbol{y}' = \lambda \boldsymbol{x}e^{\lambda t} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}. \end{aligned} \tag{4.1}$$

Dividing the last equation $\lambda \boldsymbol{x}e^{\lambda t} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}.$$

We need **nontrivial** solutions. Hence we have to look for eigenvalues and eigenvectors of $\boldsymbol{A}$. The eigenvalues are the solutions of the characteristic equation

$$\begin{aligned} \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) &= \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} \\ &= (-0.02-\lambda)^2 - 0.02^2 \\ &= \lambda(\lambda + 0.04) = 0 \end{aligned} \tag{4.2}$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$. $\lambda = 0$ can very well happen but don't get mixed up. It is eigenvectors which must not be zero. Eigenvectors are obtained as $\lambda = 0$ and $\lambda = -0.04$. For our present $\boldsymbol{A}$ this gives:

$$-0.02x_1 + 0.02x_2 = 0$$
$$\text{and} \qquad (-0.02 + 0.04)\,x_1 + 0.02x_2 = 0,$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathrm{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathrm{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note This principle continues to hold for systems of homogeneous linear ODEs.

From Eq. (4.1) and the superposition principle, we thus obtain a solution:

$$\begin{aligned} \boldsymbol{y} &= c_1 \mathrm{x}^{(1)} e^{\lambda_1 t} + c_2 \mathrm{x}^{(2)} e^{\lambda_2 t} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned} \tag{4.3}$$

where c_1 and c_2 are arbitrary constants.

Use of Initial Conditions The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and Eq. (4.3) with $t = 0$ we obtain

$$\boldsymbol{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer:

$$\begin{aligned} \boldsymbol{y} &= 75\mathrm{x}^{(1)} - 75\mathrm{x}^{(2)} e^{-0.04t} \\ &= 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned}$$

In components,

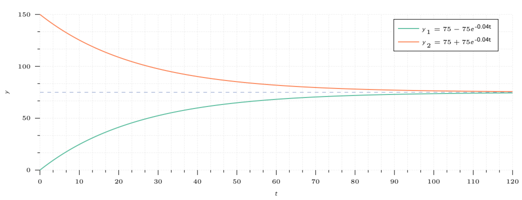
$$y_1 = 75 - 75e^{-0.04\,t} \qquad \text{Tank } T_1, \text{ lower curve,}$$
$$y_2 = 75 + 75e^{-0.04\,t} \qquad \text{Tank } T_2, \text{ upper curve.}$$

Figure below shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 kg.

Calculating the Answer T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 kg. Therefore:

$$y_1 = 75 - 75e^{-0.04t} = 50,$$
$$e^{-0.04t} = \tfrac{1}{3}, \qquad \text{and} \qquad t = (\ln 3)/0.04 = 27.5$$

Hence the fluid should circulate for roughly half an hour ■



4.1.2 Conversion of an n-th Order ODE to a System

An n$^{\text{th}}$-order ODE of the general form can be converted to a system of n first-order ODEs. This allows the study and solution of single ODEs by methods for systems, and opens a way of including the theory of higher order ODEs into that of first-order systems.

Theory 4.10: Conversion of an ODE

An nth-order ODE:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{4.4}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}. \tag{4.5}$$

This system is of the form

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned} \tag{4.6}$$

While the iron is hot, let's look at an example.

Exercise 4.2 **Mass on a String**

To gain confidence in the conversion method, let us apply it to an old problem of ours:

modelling the free motions of a mass on a spring with value given as $m = 1$, $c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \qquad \text{or}$$

$$y'' = -\left(\frac{c}{m}\right) y' - \left(\frac{k}{m}\right) y.$$

SOLUTION For this ODE given in the question can be written in the form ofEq. (4.4), making the system shown Eq. (4.5) as **linear** and **homogeneous**, applying to our system in question.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m} y_1 - \frac{c}{m} y_2. \end{aligned}$$

Setting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\begin{aligned} \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) &= \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} \\ &= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \end{aligned}$$

Entering the values of $m = 1$, $c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)\,(\lambda + 1.5) = 0$$

This gives the eigenvalues:

$$\lambda_1 = -0.5 \qquad \text{and} \qquad \lambda_2 = -1.5$$

Eigenvectors follow from the first equation in $\boldsymbol{A} - \lambda \boldsymbol{I} = 0$, which is $-\lambda x_1 + x_2 = 0$. $\lambda_1 = 0.5$ produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$. $\lambda_2 = -1.5$ produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$. These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

Which gives:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t} \quad \blacksquare$$

4.1.3 Linear Systems

Extending the notion of a linear ODE, we call a linear system if it is linear in $y_1, \ldots, y_n$. That is, if it can be written in the form:

$$
\begin{aligned}
y_1' &= a_{11}(t) y_1 + \cdots + a_{1n}(t) y_n + g_1(t) \\
&\vdots \\
y_n' &= a_{n1}(t) y_1 + \cdots + a_{nn}(t) y_n + g_n(t).
\end{aligned}
\tag{4.7}
$$

As a vector equation this becomes

$$
\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{g} \tag{4.8}
$$

where:

$$
\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ . & \cdots & . \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}
$$

This system is called **homogeneous** if $\boldsymbol{g} = \boldsymbol{0}$, so that it is:

$$
\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \tag{4.9}
$$

If $\boldsymbol{g} \neq \boldsymbol{0}$, then Eq. (4.9) is called **non-homogeneous**.

4.2 Constant Coefficient Systems

4.2.1 The Phase Plane Method

Continuing, we now assume our **homogeneous** linear system is of the form:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \tag{4.10}$$

under discussion has constant coefficients, so that the $n \times n$ matrix $\boldsymbol{A} = [a_{jk}]$ has entries **NOT** depending on t. We want to solve Eq. (4.10). Now a single ODE $y' = ky$ has the solution $y = Ce^{kt}$. So let us try:

$$\boldsymbol{y} = \boldsymbol{x}e^{\lambda t} \tag{4.11}$$

Substitution into Eq. (4.10) gives:

$$\boldsymbol{y}' = \lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}.$$

Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{4.12}$$

Therefore the nontrivial solutions of Eq. (4.10) are[2] of the form Eq. (4.11), where λ is an eigenvalue of λ and $\boldsymbol{x}$ is a corresponding eigenvector.

[2] i.e., non-zero vectors solutions.

We assume λ has a **linearly independent** set of n eigenvectors. This holds in most applications, particularly if $\boldsymbol{A}$ is symmetric ($a_{kj} = a_{jk}$) or skew-symmetric ($a_{kj} = -a_{jk}$) or has n different eigenvalues.

Let those eigenvectors be $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}$ and let them correspond to eigenvalues $\lambda_1, \ldots, \lambda_n$.[3] Then the corresponding solutions Eq. (4.11) are given as:

[3] which may be all different, or some, or even all, may be equal.

$$\boldsymbol{y}^{(1)} = \boldsymbol{x}^{(1)}e^{\lambda_1 t}, \quad \cdots, \quad \boldsymbol{y}^{(n)} = \boldsymbol{x}^{(n)}e^{\lambda_n t}. \tag{4.13}$$

Their Wronskian $W = W(y^{(1)}, \ldots, y^{(n)})$ is given by

$$W = \left(\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(n)}\right) = \begin{vmatrix} x_1^{(1)}e^{\lambda_1 t} & \cdots & x_1^{(n)}e^{\lambda_n t} \\ x_2^{(1)}e^{\lambda_1 t} & \cdots & x_2^{(n)}e^{\lambda_n t} \\ \cdot & \cdots & \cdot \\ x_n^{(1)}e^{\lambda_1 t} & \cdots & x_n^{(n)}e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t + \cdots + \lambda_n t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$

Theory 4.11: General Solution

If the constant matrix $\boldsymbol{A}$ in the system Eq. (4.10) has a linearly independent set of n eigenvectors, then the corresponding solutions $\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(n)}$ in Eq. (4.13) form a basis of solutions of Eq. (4.10), and the corresponding general solution is:

$$y = c_1 x^{(1)} e^{\lambda_1 t} + \cdots + c_n x^{(n)} e^{\lambda_n t} \tag{4.14}$$

On the right, the exponential function is never zero, and the determinant is not zero either because its columns are the n linearly independent eigenvectors. This proves the theorem, whose assumption is true if the matrix $\boldsymbol{A}$ is symmetric or skew-symmetric, or if the n eigenvalues of $\boldsymbol{A}$ are all different.

Exercise 4.3 Type I: Improper Node

Find solutions of the following system:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \boldsymbol{y}, \quad \text{therefore}$$

$$\begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

SOLUTION To see what is going on, let us find the solutions of the system. It is always a good idea to start with known solutions. Substituting $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ and $\boldsymbol{y}' = \lambda\boldsymbol{x}e^{\lambda t}$ and dropping the exponential function, as they exist both on the LHS and RHS we can eliminate them, we get $\boldsymbol{A}\,\boldsymbol{x} = \lambda\boldsymbol{x}$. The characteristic equation is:

$$\begin{aligned} \det\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right) &= \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8 = 0 \end{aligned}$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from:

$$(-3-\lambda)\,x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\boldsymbol{x}^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\boldsymbol{x}^{(2)} = [1 \quad -1]^T$.

This gives the general solution:

$$\begin{aligned} \boldsymbol{y} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad \blacksquare \end{aligned}$$

4.2.2 The Critical Points of a System

The point $\boldsymbol{y} = \boldsymbol{0}$ in the previous example seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from Eq. (4.10) we obtain:

$$\frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}. \tag{4.15}$$

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: $(0,\,0)$, where the right side of Eq. (4.15) becomes $0/0$.

> P_0, at which dy_2/dy_1 becomes **undetermined** is called a **critical point** of Eq. (4.15).

4.2.3 The Five Types of Critical Points

There are five (5) types of critical points depending on the geometric shape of the trajectories near them. These are:

1. improper nodes,
2. proper nodes,
3. saddle points,
4. centres, and

5. spiral points.

Let's look at them with examples.

Exercise 4.4 Type II: Proper Node

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction $\boldsymbol{d}$ at P_0 there is a trajectory having $\boldsymbol{d}$ as its limiting direction. Let's study the following system:

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{y}, \qquad \text{Therefore}$$

$$y_1' = y_1 \qquad \text{and} \qquad y_2' = y_2$$

SOLUTION The equation has a proper node at the origin with the matrix being the **identity matrix**. Its characteristic equation $(1-\lambda)^2 = 0$ has the root $\lambda = 1$.

Note: Any $\boldsymbol{x} \neq 0$ is an eigenvector.

and we can take $[1 \quad 0]^{\mathsf{T}}$ and $[0 \quad 1]^{\mathsf{T}}$.

Hence, a general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t}$$

$$\text{or} \quad \begin{aligned} y_1 &= c_1 e^{t}, \\ y_2 &= c_2 e^{t}. \end{aligned}$$

$$\text{or} \quad c_1 y_2 = c_2 y_1 \quad \blacksquare$$

Exercise 4.5 Type III: Saddle Point

A **saddle point** is a critical point P_0 at which there are two (2) incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0. Let's study the following system:

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{y}, \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_1 \\ y_1' &= -y_2 \end{aligned}$$

SOLUTION The equation has a saddle point at the origin and its characteristic equation

$$(1-\lambda)(-1-\lambda) = 0.$$

has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ in eigenvector $[1 \ 0]^T$ is obtained from the second row of $(\boldsymbol{A} - \lambda \mathrm{I})\mathrm{x} = 0$, that is, $0x_1 + (-1-1)\,x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0 \ \ 1]^T$. Hence a general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{aligned} y_1 &= c_1 e^{t} \\ y_2 &= c_2 e^{-t} \end{aligned} \quad \text{or} \quad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas** $\blacksquare$.

Exercise 4.6 Type IV: Centre Node

A **centre** is a critical point that is enclosed by infinitely many closed trajectories. Let's study the following system:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y, \qquad \text{Therefore} \qquad y_1' = y_2 \qquad \text{and} \tag{4.16}$$

SOLUTION The equation has a center at the origin.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathrm{j}$ and $-2\mathrm{j}$. For $2\mathrm{j}$, an eigenvector follows from the first equation $-2\mathrm{j}\,x_1 + x_2 = 0$ of $(\boldsymbol{A} - \lambda \mathrm{I})\mathrm{x} = 0$, which can be, $[1 \quad 2\mathrm{j}]^{\mathsf{T}}$.

For $\lambda = -2\mathrm{j}$ that equation is $-(-2\mathrm{j})x_1 + x_2 = 0$ and gives, say, $[1 \quad -2\mathrm{j}]^{\mathsf{T}}$. Hence a complex general solution is:

$$y_2' = -4y_1$$

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 2\mathrm{j} \end{bmatrix} e^{2\mathrm{j}\,t} + c_2 \begin{bmatrix} 1 \\ -2\mathrm{j} \end{bmatrix} e^{-2\mathrm{j}\,t}, \qquad \text{therefore} \qquad \begin{aligned} y_1 &= c_1 e^{2\mathrm{j}\,t} + c_2 e^{-2\mathrm{j}\,t}, \\ y_2 &= 2\mathrm{j}\, c_1 e^{2\mathrm{j}\,t} - 2\mathrm{j}\, c_2 e^{-2\mathrm{j}\,t}. \end{aligned} \tag{4.17}$$

A real solution is obtained from Eq. (4.17) by the Euler formula or from Eq. (4.16).

Namely, we can create a relation of $-4y_1 y_1^2$.

$$-4y_1 y_1' = y_2 y_2' \qquad \text{By Integration} \qquad 2y_1^2 + \frac{1}{2} y_2^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. $\blacksquare$

Exercise 4.7 Type V: Spiral Node

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$.

> or tracing these spirals in the opposite sense, away from P_0.

The system:

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y, \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.18}$$

SOLUTION has a spiral point at the origin.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1 + \mathrm{j}$ and $-1 - \mathrm{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For

$\lambda = -1 + \mathrm{j}$ this becomes $-\mathrm{j}x_1 + x_2 = 0$ and we can take $[1 \quad \mathrm{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathrm{j}$ is $[1 \quad -\mathrm{j}]^T$.

This gives the **complex** general solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ \mathrm{j} \end{bmatrix} e^{(-1+\mathrm{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathrm{j} \end{bmatrix} e^{(-1-\mathrm{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (4.18) by y_1, the second by y_2- and add, obtaining:

$$y_1 y_1' + y_2 y_2' = -\left(y_1^2 + y_2^2\right).$$

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to t gives:

$$2rr' = 2y_1 y_1' + 2y_2 y_2'$$

Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}$$

For each real c this is a spiral.

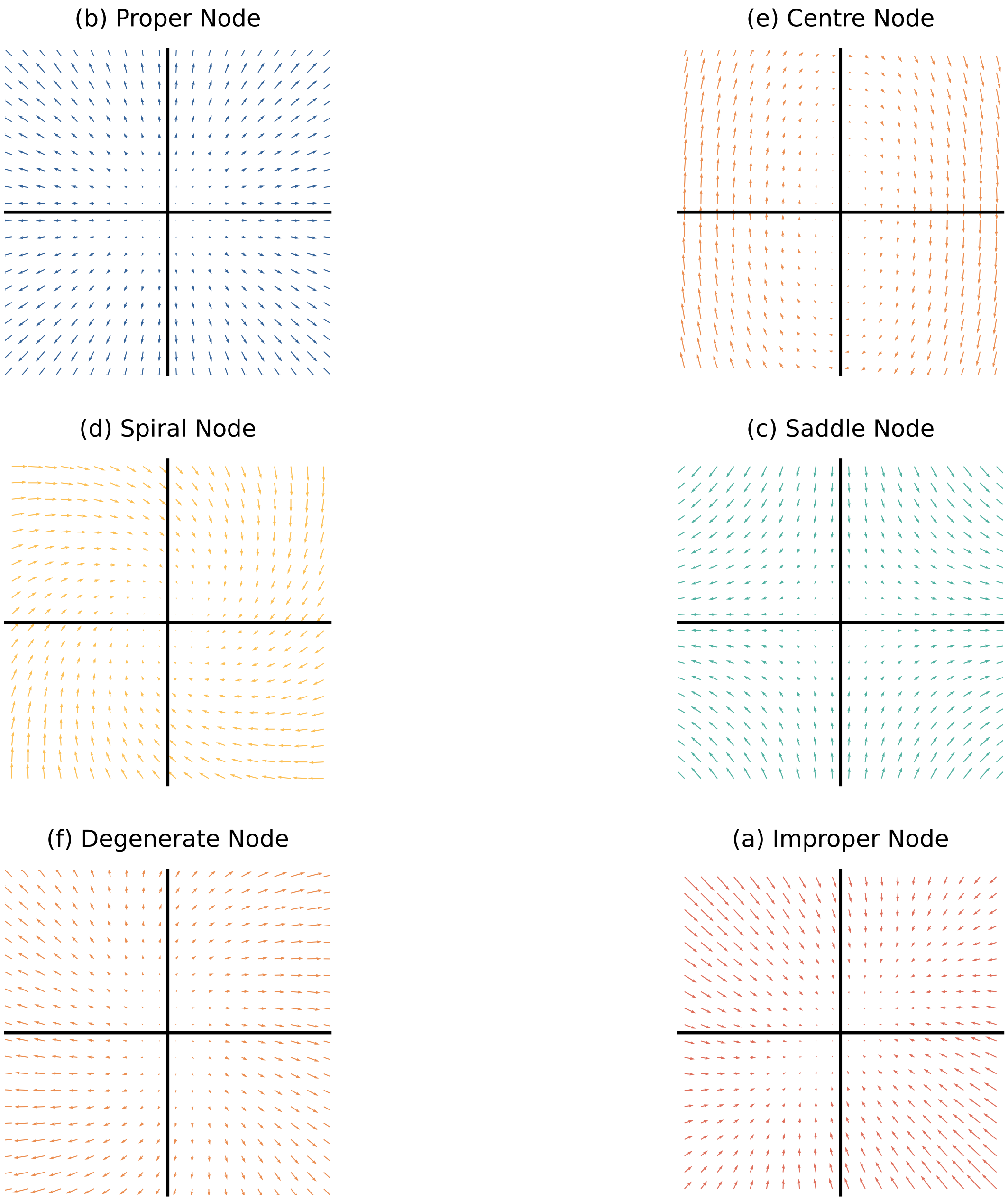


Figure 4.1: Types of possible systems encountered in the ODE System analysis.

4.3 Criteria for Critical Points and Stability

Continuing our discussion of homogeneous linear systems with **constant coefficients** given in Eq. (4.10), let us review where we are. From the previous section we have,

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \boldsymbol{y}, \qquad \text{in components,} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.19}$$

From the examples in the last section, we have seen, we can obtain an overview of families of solution curves if we represent them parametrically as $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T$ and graph them as curves in the y_1y_2-plane, called the **phase plane**.

Such a curve is called a **trajectory** of Eq. (4.10), and their totality is known as the **phase portrait** of Eq. (4.10). Now we have seen that solutions are of the form:

$$\boldsymbol{y}(t) = \mathbf{x}e^{\lambda t}. \qquad \text{Substitution into(1)gives} \qquad \boldsymbol{y}'(t) = \lambda\mathbf{x}e^{\lambda t} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\mathbf{x}e^{\lambda t}$$

Dropping the common factor $e^{\lambda t}$, we arrive at a similar equation.

$$\boldsymbol{A}\mathbf{x} = \lambda\mathbf{x} \tag{4.20}$$

> $\boldsymbol{y}(t)$ is a (nonzero) solution of Eq. (4.19) if λ is an eigenvalue of $\boldsymbol{A}$ and $\mathbf{x}$ a corresponding eigenvector.

Our examples in the last section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system Eq. (4.19) defined as a point at which $d\boldsymbol{y}^2/d\boldsymbol{y}^1$ becomes **undetermined** (i.e., 0/0).

$$\frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \tag{4.21}$$

Also recall from there are various types (5) of critical points.

What is new here, how these types of critical points are related to the eigenvalues. The latter are solutions $\lambda = \lambda_1$ and λ_2 of the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det A = 0. \tag{4.22}$$

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with coefficients p, q and discriminant Δ given by:

$$p = a_{11} + a_{22}, \qquad q = \det A = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta = p^2 - 4q. \tag{4.23}$$

From algebra we know that the solutions of this equation are

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}).$$

Furthermore, the product representation of the equation gives

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Hence p is the sum and q the product of the eigenvalues. Also $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ from (6). Together,

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2.$$

This gives the criteria in Table 4.1 for classifying critical points. A derivation will be indicated later in this section. Critical points may also be classified in terms of their **stability**. Stability concepts

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite signs
Centre	$p = 0$	$q > 0$		Pure imaginary
Spiral		$q \neq 0$	$\Delta < 0$	Complex (not pure imaginary)

Table 4.1: Eigenvalue Criteria for Critical Points.

are fundamental for engineering purposes where it means, a small change of a physical system at some instant changes the behavior of the system only slightly at all future times t.

Information: Stable Unstable Attractive

A critical point P_0 of Eq. (4.19) is called **stable** if, roughly, all trajectories of Eq. (4.19) that at some instant are close to P_0 remain close to P_0 at all future times, or in another way if for every disk D_ϵ of radius $\epsilon > 0$ with center P_0 there is a disk D_δ of radius $\delta > 0$ with center P_0 such that every trajectory of Eq. (4.19) that has a point P_1 in D_δ has all its points corresponding to $t \equiv t_1$ in D_ϵ.

P_0 is called **unstable** if P_0 is not stable.

P_0 is called **stable and attractive** if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \to \infty$.

In general term this can be written in a following table.

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$
Stable and attractive	$q < 0$	$q > 0$
Stable	$q \leq 0$	$q > 0$
Unstable	either $q \leq 0$	or $q > 0$

Table 4.2: Stability criteria for critical points.

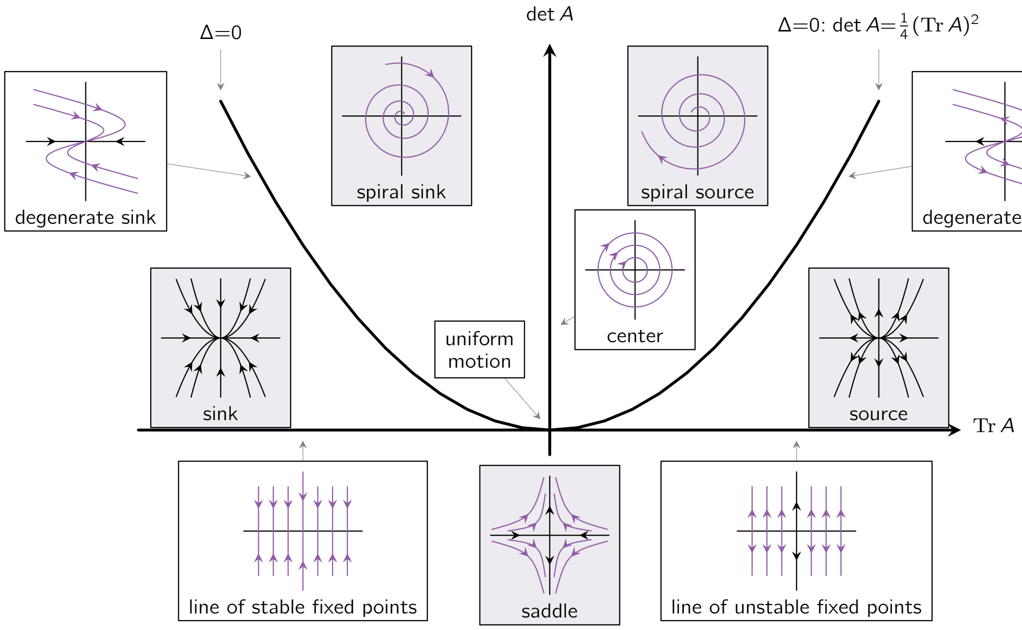


Figure 4.2: The Poincaré Phase-Plot diagram showcasing different behaviours. The shaded diagrams represent the region within the plot whereas the white boxes represent the behaviour when it is on the line.

Example Free Motions of a Mass-Spring System 8

What kind of critical point does the following equation have ?

$$my'' + c'y' + ky = 0$$

Solution

Free Motions of a Mass-Spring System

First, division by m gives:

$$y'' = -(k/m)y - (c/m)y'$$

To get a system, set $y_1 = y, y_2 = y'$. Then $y_2' = y'' = -(k/m)y_1 - (c/m)y_2$. Therefore:

$$y' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} y, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{bmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

We can see that:

$$p = -c/m, \qquad q = k/m, \qquad \Delta = (c/m)^2 - 4k/m$$

From this we obtain the following results.

> Note that in the last three cases the discriminant Δ plays an essential role.

No Damping $c = 0$, $p = 0$, $q > 0$ and has a centre.

Under damping $c^2 < 4mk$, $p < 0$, $q > 0$, $\Delta < 0$ and has a stable and attractive spiral point.

Critical Damping $c^2 = 4mk$, $p < 0$, $q > 0$, $\Delta = 0$ and has a stable and attractive node.

Overdamping $c^2 > 4mk$, $p < 0$, $q > 0$, $\Delta > 0$ and has a stable and attractive node.

4.4 Qualitative Methods for Non-Linear Systems

Qualitative methods are methods of obtaining qualitative information on solutions *without actually solving a system*. These methods are particularly valuable for systems whose solution by analytic methods is difficult or impossible.

This is the case for many practically important **non-linear systems**.

$$y' = f(y), \quad \text{therefore} \quad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned} \tag{4.24}$$

Here we will extend the previously discussed phase plane methods, from linear systems to nonlinear systems Eq. (4.24). We assume that Eq. (4.24) is autonomous, that is, the independent variable t does not occur explicitly.

> All examples in the last section are autonomous.

We shall, again exhibit entire families of solutions.

> This is an advantage over numeric methods, which give only one (approximate) solution at a time.

For this analysis we need to employ the previously defined concepts of **phase plane** (the y_1-y_2-plane), **trajectories** (solution curves of Eq. (4.24) in the phase), the **phase portrait** of Eq. (4.24) (the totality of these trajectories), and **critical points** of Eq. (4.24) points (y_1, y_2) at which both $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are zero).

Now Eq. (4.24) may have several critical points. Our approach shall be to discuss one critical point after another. If a critical point P_0 is not at the origin, then, for technical convenience, we shall move this point to the origin before analyzing the point.

More formally, if P_0: (a, b) is a critical point with (a, b) **NOT** at the origin (0, 0), then we apply the translation:

$$\overline{y_1} = y_1 - a, \qquad \overline{y_2} = y_2 - b,$$

which moves P_0 to $(0, 0)$ as desired. Thus we can assume P_0 to be the origin $(0, 0)$, and for simplicity we continue to write y_1, y_2 (instead of $\overline{y}_1, \overline{y}_2$). We also assume that P_0 is **isolated**, that is, it is the only critical point of Eq. (4.24) within a (sufficiently small) disk with center at the origin.

4.4.1 Linearisation of Non-Linear Systems

How to determine the kind and stability of a critical point P_0: $(0, 0)$ of Eq. (4.24)? In most cases this can be done by linearisation of Eq. (4.24) near P_0, writing Eq. (4.24) as $\boldsymbol{y}' = \mathbf{f}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y} + \mathbf{h}(\boldsymbol{y})$ and dropping $\mathbf{h}(\boldsymbol{y})$, as follows.

Since P_0 is critical, $f_1(0,0) = 0$, $f_2(0,0) = 0$, so that f_1 and f_2 have no constant terms and we can write

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \mathbf{h}(\boldsymbol{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2). \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2). \end{aligned} \tag{4.25}$$

A is constant as Eq. (4.24) is autonomous.

Theory 4.12: Linearisation

If f_1 and f_2 in Eq. (4.24) are continuous and have continuous partial derivatives in a neighborhood of the critical point P_0: $(0,0)$, and if $\det \boldsymbol{A} \neq 0$ in Eq. (4.25), then the kind and stability of the critical point of Eq. (4.24) are the same as those of the linearized system*

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.26}$$

Exceptions occur if $\boldsymbol{A}$ has equal or pure imaginary eigenvalues; then Eq. (4.24) may have the same kind of critical point as (3) or a spiral point.

Exercise 4.9 Linearisation of a Free Undamped Pendulum

A pendulum consists of a body of mass m (the bob) and a rod of length L. Determine the locations and type of the critical points.

Warning Assume that the mass of the rod and an reference are negligible.

SOLUTION Let us begin tackling the problem:

Setting Up the Mathematical Model Let θ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is mg, where g is the acceleration of gravity.

This causes a restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2nd law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the**acceleration**. Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg\sin\theta = 0.$$

Dividing this by mL, we have:

$$\theta'' + k\sin\theta = 0 \qquad \text{with} \qquad \left(k = \frac{g}{L}\right). \tag{4.27}$$

When θ is very small, we can approximate sin θ rather accurately by θ and obtain as an approximate solution $A \cos\sqrt{k} + B \sin\sqrt{k}$, but the *exact* solution for any θ is not an **elementary function**.

Critical Points and Linearisation To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from Eq. (4.27) we obtain a nonlinear system Eq. (4.24) of the form:

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = y_2, \\ y_2' &= f_2(y_2, y_1) = -k\sin y_1. \end{aligned}$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi,\ 0)$, where $n = 0, \pm1, \pm2, \cdots$.

We consider $(0,\ 0)$. Since the Maclaurin series is

$$\sin y_1 = y_1 - \tfrac{1}{6}y_1^2 + - \cdots \approx y_1,$$

the linearized system at $(0,\ 0)$ is:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\boldsymbol{y} \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1. \end{aligned}$$

To apply our criteria in Sec. 4.4 we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = 0, \\ q &= \det(\boldsymbol{A}) = k = g/L \qquad (> 0), \\ \Delta &= p^2 - 4q = -4k. \end{aligned}$$

From this and Table 4.1(c) in Sec. 4.4 we conclude that $(0, 0)$ is a **centre**, which is **always stable**. Since $\sin\theta = \sin y_1$ is periodic with period of 2π. **Warning** This means the critical points $(n\pi,\ 0)$, $n = \pm2, \pm4, \cdots$, are all centres.

We now consider the critical point $(\pi, 0)$, setting:

$$\begin{aligned} y_1 &= \theta - \pi \\ y_2 &= (\theta - \pi)' \end{aligned}$$

Then in Eq. (4.27), we can apply the MacLaurin series:

$$\sin\ \theta = \sin\ (y_1 + \pi) = -\sin\ y_1 = -y_1 + \frac{1}{2}y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\boldsymbol{y} \qquad \text{Thus} \qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= ky_1. \end{aligned}$$

We see that:

$$\begin{aligned} p &= 0, \\ q &= -k \qquad (< 0)\,, \\ \Delta &= -4q = 4k. \end{aligned}$$

Hence, by Table 4.1(b), this gives a saddle point, which is always unstable.

Because of periodicity, the critical points $(n\pi,\ 0)$, $n = \pm1, \pm3, \cdots$, are all **saddle points**.

Exercise 4.10 Linearisation of a Damped Pendulum

To gain further experience in investigating critical points, as another practically Important, let us see how the previous example changes when we add a damping term $c\theta$, (damping proportional to the angular velocity) to equation Eq. (4.27), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

where $k > 0$ and $c \geq 0$ (which includes our previous case of no damping, $c = 0$).

SOLUTION First we start by setting $\theta = y_1$, θ' as before, we obtain the nonlinear system (use $\theta'' = y_2$),

$$y_1' = y_2$$

$$y_2' = -k \sin y_1 - c y_2.$$

We see the critical points have the same locations as the example before, namely,
$(0,\ 0)$, $(\pm\pi,\ 0)$, $(\pm 2\pi,\ 0)$, ... To analyse this system, we start with analysing $(0,\ 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0,\ 0)$.

$$y' = Ay = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} y \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2 \end{aligned}$$

This is identical with the system in previous example, except for the positive factor m (and except for the physical meaning of y_1). Hence for $c = 0$ (no damping) we have a centre, for small damping we have a spiral point, and so on.

We now consider the critical point $(\pi,\ 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin\left(y_1 + \pi\right) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi,\ 0)$:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \boldsymbol{y}, \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2. \end{aligned}$$

For our criteria, we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = -c \\ q &= \det \boldsymbol{A} = -k \\ \Delta &= p^2 - 4q = c^2 + 4k \end{aligned}$$

This gives the following results for the critical point $(\pi,\ 0)$.

No Damping $c > 0, p = 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 3b.

Damping $c > 0, p < 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 94.

As $\sin y_1$ is periodic with period of 2π, the critical points $(\pm 2\pi,\ 0)$, $(\pm 4\pi,\ 0)$, ... are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished. ∎

A visual representation can be seen below:

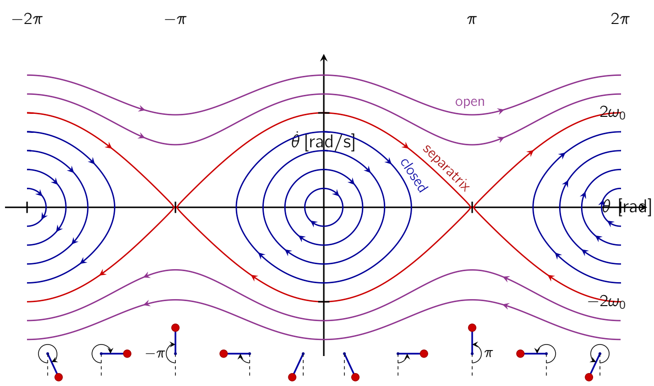


Figure 4.3

4.5 Self-Sustained Oscillations - Van der Pol Equation

There are physical systems such that for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system.

In other words, large oscillations will be damped, whereas for small oscillations there is *negative damping* (feeding of energy into the system). For physical reasons we expect such a system to approach a periodic behaviour, which will thus appear as a closed trajectory in the phase plane, called a **limit cycle**.

> An ODE describing such vibrations is the famous **van der Pol equation**.

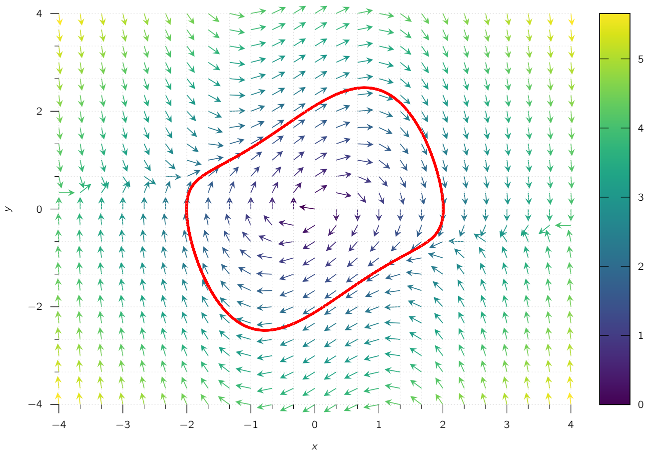


Figure 4.4

$$y'' - \mu\left(1 - y^2\right) y' + y = 0$$

It first occurred in the study of electrical circuits containing vacuum tubes.

Information: Vacuum Tube

A vacuum tube, electron tube, valve (British usage), or tube (North America) is a device that controls electric current flow in a high vacuum between electrodes to which an electric potential difference has been applied.

For $\mu = 0$ this equation becomes $y'' + y = 0$ and so with harmonic oscillations. If we define $\mu > 0$, then the damping term has the factor $-\mu\left(1 - y^2\right)$. This is a consequence for small oscillations, when $y^2 < 1$, so that we have negative damping, is zero for $y^2 = 1$ (no imaginary), and is positive if $y^2 > 1$ (positive damping. Loss of energy).

If μ is small, we expect a limit cycle almost a circle because then our equation differs but finite from $y'' + y =$. If μ is large, the limit cycle will probably look different.

Setting $y = y_1$, $y' = y_2$ and using $y'' = (dy_2/dy_1)y_2$ as in (8), we have from (10)

$$\frac{dy_2}{dy_1} y_2 - \mu(1 - y_1^2) y_2 + y_1 = 0.$$

The isoclines in the y_1y_2-plane (the phase plane) are the curves $dy_2/dy_1 = K =$ const, that is,

$$\frac{dy_2}{dy_1} = \mu(1 - y_1^2) - \frac{y_1}{y_2} = K.$$

Solving algebraically for y_2, we see that the icoclines are given by

$$y_2 = \frac{y_1}{\mu(1 - y_1^2) - K}$$

Chapter 5

Special Functions for ODEs

Table of Contents

5.1 Defining Special Functions

Linear ODEs with **constant coefficients** can be solved by algebraic methods, and their solutions are elementary functions known from calculus.

Theory 5.13: Elementary Functions

A function of a single variable, (real or complex) defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, and their inverses.

ODEs with **variable coefficients**, however, is more complicated, and their solutions may be non-elementary which means we can't write the solution explicitly.[1] We will look at the two (2) standard methods for solving ODEs:

Power Series Gives the solution in terms of a power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Frobenius Method Gives the solution in power series, multiplied by $\ln x$ or x^r.

[1] For engineering applications where explicit solutions are **NOT** possible, *Legendre's*, *Bessel's*, and the *hypergeometric* equations are important ODEs of this kind.

5.2 The Method of Power Series

The power series method is the primary method for solving linear ODEs with variable coefficients. It gives solutions in the form of a power series.[2] Remember, the **power series** (in powers of $x - x_0$) is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2 + \cdots . \quad (5.1)$$

[2] The power series method is used for computing values, graphing curves, proving formulas, and exploring properties of solutions.

Here, x is a variable and a_0, a_1, a_2, $\cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **centre** of the series. For $x_0 = 0$, we obtain a power series in powers of x:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \quad (5.2)$$

For the duration of the chapter we will assume all variables and constants are real.

The term **power series** usually refers to a series of the form Eq. (5.1), but does not include series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the *Legendre* and *Bessel equations* for integer values.

Exercise 5.1 Power Series Solution

Solve the following ODE:

$$y' - y = 0$$

SOLUTION First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \quad (5.3)$$

We put these values into the ODE:

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

Then we collect like powers of x, finding:

$$\left(a_1 - a_0\right) + \left(2a_2 - a_1\right) x + \left(3a_3 - a_2\right) x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express a_1, a_2, $\cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^{x}. \quad \blacksquare$$

We may now generalise this idea. For a given ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{5.4}$$

First represent $p(x)$, $q(x)$ by power series in powers of x.

> If $p(x)$, $q(x)$ are polynomials, and then nothing needs to be done in this first step.

Next we assume a solution in the form of a power series with unknown coefficients and insert it as well as Eq. (5.3) and:

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x_2 + \cdots = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} \tag{5.5}$$

into the ODE. Then we **collect same powers of** x and equate the sum of the coefficients of each occurring power of x to zero (0), starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of Eq. (5.3) successively.

Exercise 5.2 A Special Legendre Function

Solve the following ODE:

$$\left(1 - x^2\right) y'' - 2xy' + 2y = 0$$

Note: These equations usually occur in models with spherical symmetry.

SOLUTION Substitute Eq. (5.2), Eq. (5.3), and Eq. (5.5) into the ODE, $\left(1 - x^2\right) y''$ gives two (2) series: for y'', and for $-x^2 y''$. For the term $-2xy'$ use Eq. (5.3) and in $2y$ use Eq. (5.2). Write like powers of x vertically aligned for easy viewing. This gives:

$$\begin{aligned}
y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\
-xy'' &= \qquad\qquad\qquad - 2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots \\
-2xy' &= \qquad - 2a_1 x - 4a_5 x^2 - 6a_9 x^3 - 8a_4 x^4 - \cdots \\
2y &= 2a_0 + 2a_1 x - 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots .
\end{aligned}$$

Add terms of like powers of x. For each power x^0, x, x^2 equate the sum obtained to zero. Denote these sums by 0 (constant terms), 1 (first power of x), and so on and write it down to the following table:

Sum	Power	Equation
0	x^0	$a_2 = -a_0$
1	x	$a_3 = 0$
2	x^2	$14a_4 = 4a_2, \qquad a_4 = \frac{4}{12} a_2 = -\frac{1}{3} a_0$
3	x^3	$a_5 = 0$ since $a_3 = 0$
4	x^4	$30a_6 = 18a_4, \qquad a_6 = \frac{18}{30} a_4 = \frac{18}{30}\left(-\frac{1}{3}\right) a_0 = -\frac{1}{5} a_0$

This gives the solution

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) \quad \blacksquare$$

Note: a_0, a_1 remain **arbitrary**.

Therefore, this is a **general solution** consisting of two (2) solutions: x and

$$1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$$

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(x)$ and *Legendre functions* $Q_1(x)$. Here we have

$$x = P_1(x)$$

and

$$1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(x)$$

Note: The minus is by convention. The index 1 is called the *order* of these functions and here the order is 1. $\blacksquare$

5.3 Legendre's Equation

5.3.1 The Polynomials of Legendre

Legendre's[3] differential equation:

[3]Adrien-Marie Legendre (1752 - 1833) was a French mathematician who made numerous contributions to mathematics. Well-known and important concepts such as the Legendre polynomials and Legendre transformation are named after him.

$$\left(1-x^2\right) y'' - 2xy' + n\,(n+1)\,y = 0, \tag{5.6}$$

is an important ODE in physics. It arises in numerous problems, particularly in boundary value problems for spheres. The equation involves a parameter n, whose value depends on the physical or engineering problem. Therefore Eq. (5.6) is actually a whole family of ODEs. For $n = 1$ we solved it in the previous example.

Any solution of Eq. (5.6) is called a **Legendre function**.

The study of these and other higher functions **NOT** occurring in calculus is called the theory of special functions.

Dividing Eq. (5.6) by $1 - x^2$, we obtain the standard form:

$$y'' - \frac{2x}{\left(1-x^2\right)} y' + \frac{n\,(n+1)}{\left(1-x^2\right)} y = 0$$

We see that the coefficients $-2x/\left(1-x^2\right)$ and $n\,(n+1)\,/\left(1-x^2\right)$ of the new equation are analytic at $x = 0$, so the power series method is applicable for this equation. Substituting:

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{5.7}$$

and its derivatives into Eq. (5.6), and denoting the constant $n\,(n+1)$ simply as k, we obtain the following:

$$\left(1-x^2\right) \sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0.$$

By splitting the first expression as two (2) separate series we have the equation:

$$\sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} - \sum_{m=2}^{\infty} m\,(m-1)\,a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x_n in all four (4) series, set $m - 2 = s$ (therefore $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives:

$$\sum_{s=0}^{\infty} (s+2)\,(s+1)\,a_{s+2} x^s - \sum_{s=2}^{\infty} s\,(s-1)\,a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

Note the first series the summation begins with $s = 0$.

As this equation with the right side 0 must be an identity in x if Eq. (5.7) is to be a solution of Eq. (5.7), the sum of the coefficients of each power of x on the LHS must be zero.

Now x^0 occurs in the first and fourth series only, and gives:

remember $k = n(n+1)$

$$x^0 \qquad 2 \cdot 1 a_2 + n(n+1)\, a_0 = 0, \tag{5.8}$$

$$x^1 \qquad 3 \cdot 2 a_3 + \left[-2 + n(n+1)\right] a_1 = 0, \tag{5.9}$$

$$x^2,\, x_3,\, \cdots \qquad (s+2)(s+1)\, a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right] a_s = 0. \tag{5.10}$$

The expression in the brackets $[\cdots]$ can be simplified to $(n-s)(n+s+1)$.

Solving Eq. (5.8) for a_2 and Eq. (5.9) for a_3 as well as Eq. (5.10) for a_{s+2}, we obtain the **general formula**:

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad \text{for} \qquad s = 0,\, 1,\, 2,\, \cdots \tag{5.11}$$

This is called a **recurrence relation** or **recursion formula**. It gives each coefficient in terms of the 2nd one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively:

$$\begin{aligned}
a_2 &= -\frac{n(n+1)}{2 \cdot 1} a_0 \\
a_3 &= -\frac{(n-1)(n+2)}{3 \cdot 2} a_1 \\
a_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3} a_2 = \frac{(n-2)\, n(n+1)(n+3)}{4!} a_0 \\
a_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4} a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

By inserting these expressions for the coefficients into Eq. (5.7) we obtain:

$$y(x) = a_0 y_1(x) + a_1 y_2(x)\,. \tag{5.12}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2 \cdot 1} x^2 + \frac{(n-2)\, n(n+1)(n+3)}{4!} x^4 - \cdots \tag{5.13}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3 \cdot 2} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots \tag{5.14}$$

These series converge for $|x| < 1$. As Eq. (5.13) contains **even** powers of x only, while Eq. (5.14) contains **odd** powers of x only, the ratio y_1/y_2 is not a **constant**, This means y_1 and y_2 are not proportional and are thus linearly independent solutions.

Therefore Eq. (5.12) is a general solution of Eq. (5.6) on the interval $-1 < x < 1$.

$x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardised ODE are no longer analytic.

5.3.2 Polynomial Solutions

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions.

For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials. For *Legendre's equation* this happens when the parameter n is a non-negative integer because the RHS of Eq. (5.11) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0$, $\cdots$. Therefore if n is even, $y_1(x)$ reduces to a polynomial of degree n.

If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of such constants is done as follows.

We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad \text{where} \qquad n \text{ is a positive integer.} \tag{5.15}$$

and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from Eq. (5.11), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leq n-2) \tag{5.16}$$

The choice Eq. (5.15) makes $p_n(1) = 1$ for every n which makes our lives easier. From Eq. (5.16) with $s = n-2$ and Eq. (5.15) we obtain:

$$a_{n-2} = -\frac{n(n-1)}{2(n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2}$$

Using $(2n)! = 2n(2n-1)(2n-2)!$ in the numerator and $n! = n(n-1)!$ and $n! = n(n-1)(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n(n-1)\,2n(2n-1)(2n-2)!}{2(2n-1)\,2^n n(n-1)!\,n(n-1)(n-2)!}.$$

$n(n-1)\,2n(2n-1)$ cancels out, which we get:

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

Similarly,

$$\begin{aligned} a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} \\ &= \frac{(2n-4)!}{2^n 2!\ (n-2)!\ (n-4)!} \end{aligned}$$

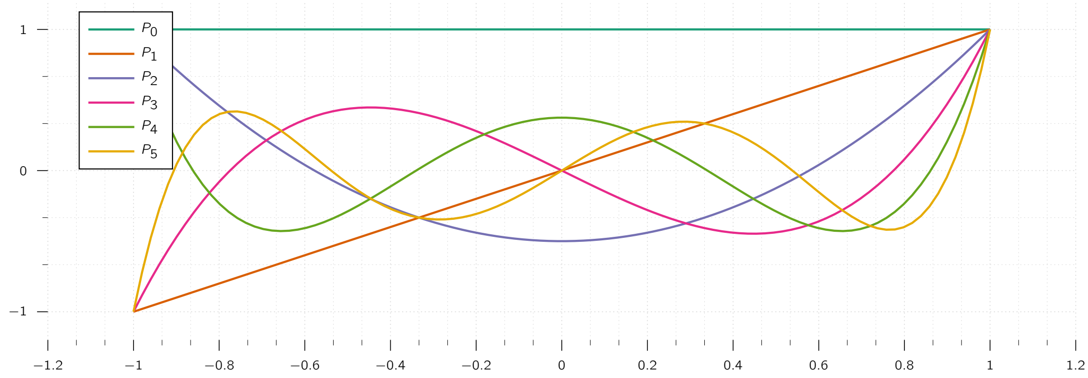


Figure 5.1: The first six Legendre polynomials.

and so on, and in general, when $n - 2m \geq 0$,

$$a_{\mathrm{n-2m}} = (-1)^m \frac{(2n-2m)!}{2^n m! \ (n-m)! \ (n-2m)!}. \tag{5.17}$$

The resulting solution of Legendre's differential equation Eq. (5.6) is called the *Legendre polynomial of degree n* and is denoted by $P_n(x)$.:

From Eq. (5.17) we obtain:

$$P_n(x) = \sum_{m-0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m! \, (n-m)! \, (n-2m)!} x^{n-2m} \tag{5.18}$$

$$= \frac{(2m)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1! \, (n-1)! \, (n-2)!} x^{n-2} + - \cdots \tag{5.19}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1), & P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

> The Legendre polynomials $P_n(x)$ are **orthogonal** on the interval $-1 \leqq x \leqq 1$, a basic property to be defined and used in making up "Fourier-Legendre series" which will be the focus for *Higher Mathematics II*.

5.4 Extending the Power Series using Frobenius Method

Several 2nd-order ODEs are important for engineering applications.

One of the famous ones **Bessel Equation** will be our focus in the continuing section.

Unfortunately, these practical 2nd-order ODEs have coefficients that are not analytic, but are possible to solve via series method.[4] The following theorem permits an extension of the power series method.

This method is called the Frobenius method.[5]

Theory 5.14: Frobenius Method

Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{5.20}$$

has **at least one solution** that can be represented in the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{5.21}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE Eq. (5.20) also has a 2nd solution (such that these two solutions are linearly independent) that may be similar to Eq. (5.21) (with a different r and different coefficients) or may contain a logarithmic term.

To see this theorem in action, let's look at the Bessel's equation.

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad \text{where } \nu \text{ is a parameter}$$

is of the form Eq. (5.20) with:

$$b(x) = 1 \qquad c(x) = x^2 - \nu^2 \qquad \text{analytic at } x = 0 \qquad .$$

This form allows us to use the Frobenius method.

This ODE could **NOT** be handled in full generality by the power series method as these functions are known as hyper-geometric differential equations. Therefore, this equation (also known as hypergeometric differential equation) requires the Frobenius method.

In Eq. (5.21) we have a power series times a single power of x whose exponent r is not restricted to be a non-negative integer.

[4] power series times a logarithm or times a fractional power of x, etc.

[5] Ferdinand Georg Frobenius (1849 - 1917) was a German mathematician, best known for his contributions to the theory of elliptic functions, differential equations, number theory, and to group theory. He is known for the famous determinantal identities, known as Frobenius-Stickelberger formulae, governing elliptic functions, and for developing the theory of biquadratic forms.

Regular and Singular Points

The following terms are practical and commonly used. A **regular point** of the ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a **regular point** of the ODE:

$$\widetilde{h}(x)y'' + \widetilde{p}(x)y'(x) + \widetilde{q}(x)y = 0,$$

is an x_0 at which $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$ are analytic and $\widetilde{h}(x_0) \neq 0$ (so what we can divide by $\widetilde{h}$ and get the previous standard form). Then the power series method can be applied. If x_0 is not a regular point, it is called a **singular point**.

5.4.1 Indicial Equation

Time to explain the Frobenius method for solving Eq. (5.20) which is the Bessel equation. Multiplication of Eq. (5.20) by x^2 gives the more convenient form which can be worked upon:

$$x^2 y'' + x b(x)\, y' + c(x)\, y = 0 \tag{5.22}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \qquad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

> If both $b(x)$ and $c(x)$ are polynomials, no actions are needed.

Then we differentiate Eq. (5.21) term by term, finding:

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \\ &= x^{r-1}[r a_0 + (r+1) a_1 x + \cdots] \\ y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} \\ &= x^{r-2}[r(r-1)a_0 + (r+1) r a_1 x + \cdots]. \end{aligned} \tag{5.23}$$

By inserting all these series into Eq. (5.22) we obtain:

$$x^r \left[r(r-1)a_0 + \cdots \right] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots) \tag{5.24}$$

$$+ (c_0 + c_1 x + \cdots) x^r (a_0 + a_1 x + \cdots) = 0. \tag{5.25}$$

We now equate the sum of the coefficients of each power x^r, x^{r+1}, x^{r+2}, ... to zero. This presents a system of equations involving the unknown coefficients a_m. The smallest power is x^r and the corresponding equation is:

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives:

$$r(r-1) + b_0 r + c_0 = 0 \tag{5.26}$$

This important quadratic equation is called the **indical equation** of the ODE Eq. (5.22). Its role is as follows.

The Frobenius method present a basis of solutions. One of the two solutions will always be of the form Eq. (5.23), where r is a root of Eq. (5.26). The other solution will be of a form indicated by the indicial equation.

There are three (3) cases:

Case 1 Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2 A double root.

Case 3 Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are related to the *Euler-Cauchy equation*, the simplest ODE of the form Eq. (5.20).

Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i\mathrm{Im}\, r_1$ is imaginary, so it cannot be a real integer.

Case 2 we must have a logarithm, whereas in Case 3 we may or may not.

Theory 5.15: Frobenius Method II - The Three Cases

Assume the ODE in Eq. (5.22) satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation Eq. (5.26).

Then we have the following three (3) cases:

Case 1. Distinct Roots Not Differing by an Integer

A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.27}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.28}$$

with coefficients obtained successively from Eq. (5.24) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.

A basis is

$$y_1(x) = x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{5.29}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^{r}(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0) \tag{5.30}$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.31}$$

(of the same general form as before) and

$$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.32}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

5.4.2 Typical Applications

Technically, the *Frobenius method* is similar to the power series method, once the roots of the indicial equation have been determined.

However, Eq. (5.27)-Eq. (5.32) merely indicate the general form of a basis, and a 2nd solution can often be obtained more rapidly by reduction of order.

Exercise 5.3: Euler-Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm

Solve the following ODE:

$$x^2 y'' + b_0 x y' + c_0 y = 0 \qquad (b_0,\ c_0 \text{ constant})$$

Solution

Substitution of $y = x^r$ gives the auxiliary equation:

$$r(r-1) + b_0 r + c_0 = 0,$$

which is the indicial equation. For different roots r_1, r_2 we get a basis $y_1 = x^{r_1}$, $y_2 = x^{r_2}$, and for a double root r we get a basis x^r, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

Exercise 5.4: Example of Case II - Double Root

Solve the ODE

$$x(x-1)y'' + (3x-1)y'' + y = 0 \tag{5.33}$$

Solution

Writing Eq. (5.33) in the standard form Eq. (5.22):

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$
$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

we see it satisfies the assumptions in **Theorem 1** (i.e., analytic as $x = 0$). By inserting Eq. (5.23) and its derivatives Eq. (5.23) into Eq. (5.33) we obtain:

$$\begin{aligned}&\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}\\&+3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r}=0\end{aligned} \tag{5.34}$$

The smallest power is x^{r-1}, occurring in the 2nd and the fourth series; by equating the sum of its coefficients to zero we have

$$\left[-r\,(r-1)-r\right]a_0=0, \qquad \text{therefore} \qquad r^2=0.$$

Hence this indicial equation has the double root $r=0$.

First Solution

Insert this value $r=0$ into Eq. (5.34) and equate the sum of the coefficients of the power x^s to zero, obtaining:

$$s(s-1)a_s-(s+1)a_{s+1}+3a_s-(s+1)a_{s+1}+a_s=0$$

thus $a_{s+1}=a_s$. Hence $a_0=a_1=a_2=\cdots$, and by choosing $a_0=1$ we obtain the solution

$$y_1(x)=\sum_{m=0}^{\infty}x^m=\frac{1}{1-x} \qquad (|x|<1).$$

Second Solution

We get a 2nd independent solution y_2 by the method of reduction of order, substituting $y_2=uy_1$ and its derivatives into the equation. This leads to (9). Sec. 2.1, which we shall use in this example, instead of stating reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p=(3x-1)/(x^2-x)$, the coefficient of y' in (11) in standard form. By partial fractions,

$$-\int p\,dx=-\int\frac{3x-1}{3(x-1)}\,dx=-\int\left(\frac{2}{x-1}+\frac{1}{x}\right)dx=-2\ln(x-1)-\ln x.$$

Hence becomes

$$u'=U=y_1^{-2}e^{-\int y\,dx}=\frac{(x-1)^2}{(x-1)^2x}=\frac{1}{x}, \qquad u=\ln x, \qquad y_2=w_1=\frac{\ln x}{1-x}.$$

y_1 and y_2 are shown in Fig. 109. These functions are linearly independent and thus form a basis on the interval $0<x<1$ (as well as on $1<x<\infty$).

The Frobenius method solves the **hypergeometric equation**, whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

5.5 Bessel's Function

One of the most important ODEs in applied mathematics in **Bessel's equation** which it's form is shown as:[6]

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \tag{5.35}$$

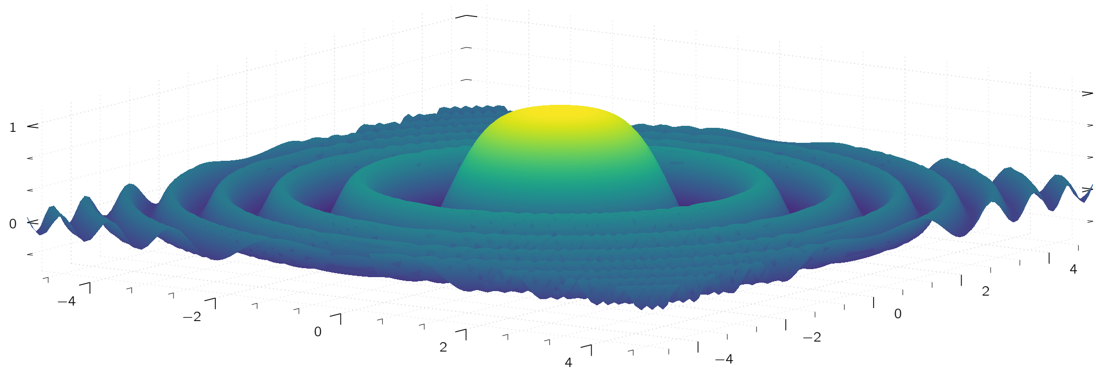


Figure 5.2: Bessel functions describe the radial part of vibrations of a circular membrane.

Converting this to the traditional Frobenius form:

$$y'' + \frac{1}{x} y' + \frac{1 - \nu^2}{x^2} y = 0 \qquad \text{where} \qquad b(x) = 1 \qquad c(x) = 1 - \nu^2 .$$

where the parameter ν is a given **real number** which is positive or zero.

> Bessel's equation often in problems showing cylindrical symmetry or membranes.

According to the Frobenius theory, it has a solution of the form:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad \left(a_0 \neq 0\right) \tag{5.36}$$

Substituting Eq. (5.36) and its 1st and 2nd derivatives into Bessel's equation, we obtain:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero.

> Note that this power x^{s+r} corresponds to $m = s$ in the first, 2nd, and fourth series, and to $m = s - 2$ in the third series.

[6]Friedrich Wilhelm Bessel (1784 - 1846) was a German astronomer, mathematician, physicist, and geodesist. He was the first astronomer who determined reliable values for the distance from the Sun to another star by the method of parallax. Certain important mathematical functions were first studied systematically by Bessel and were named Bessel functions in his honour.

Therefore, for $s = 0$ and $s = 1$, the third series does not contribute since $m \geq 0$. For $s = 2, 3, \ldots$ all four series contribute, so that we get a general formula for all these s. We find:

$$r(r-1)\,a_0 + ra_0 - v^2 a_0 = 0 \qquad (s = 0) \qquad (5.37)$$

$$(r+1)\,ra_1 + (r+1)\,a_1 - v^2 a_1 = 0 \qquad (s = 1) \qquad (5.38)$$

$$(s+r)\,(s+r-1)\,a_s + (s+r)\,a_s + a_{s-2} - v^2 a_s = 0 \qquad (s = 2, 3, \ldots) \qquad (5.39)$$

From Eq. (5.37) we obtain the **indicical equation** by dropping a_0.

$$(r+\nu)\,(r-\nu) = 0 \qquad (5.40)$$

The roots are $r_1 = \nu\,(\geq 0)$ and $r_2 = -\nu$.

Coefficient Recursion for $r = r_1 = \nu$ For $r = \nu$, Eq. (5.38) reduces to $(2\nu + 1)\,a_1 = 0$. Therefore $a_1 = 0$ as $\nu \geq 0$. Substituting $r = \nu$ in Eq. (5.39) and combining the three terms containing $a_s = 0$ gives simply:

$$(s + 2\nu)\,sa_s + a_{s-2} = 0 \qquad (5.41)$$

As $a_1 = 0$ and $v \equiv 0$, it follows from Eq. (5.41), $a_3 = 0, a_5 = 0, \cdots$. Hence we have to deal only with even-numbered coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (5.41) becomes:

$$(2m + 2\nu)\,2ma_{2m} + a_{2m-2} = 0$$

Solving for a_{2m} gives the recursion formula

$$a_{2m} = -\frac{1}{2^2 m\,(v+m)} a_{2m-2} \qquad m = 1, 2, \cdots \qquad (5.42)$$

From Eq. (5.42) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(v+1)}$$

$$a_4 = -\frac{a_2}{2^2 2(v+2)} = \frac{a_0}{2^4 2!\,(v+1)(v+2)}$$

and so on, and in general:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v+1)(v+2)\cdots(v+m)}, \qquad m = 1, 2, \cdots. \qquad (5.43)$$

5.5.1 Bessel Functions (J_n) for Integers

Integer values of ν are denoted by n, which is the standard mathematical notation.

For $\nu = n$ the relation Eq. (5.43) becomes:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n+1)(n+2)\cdots(n+m)}, \qquad m = 1, 2, \cdots \qquad (5.44)$$

a_0 is still arbitrary, so that the series Eq. (5.36) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function.

Accordingly, we have to make a choice.

The choice $a_0 = 1$ would be possible. A simpler series Eq. (5.36) could be obtained if we could absorb the growing product $(n+1)(n+2)\cdots(n+m)$ into a factorial function $(n+m)!$ What should be our choice? Our choice should be:

$$a_0 = \frac{1}{2^n n!} \tag{5.45}$$

because then $n!\,(n+1)\cdots(n+m) = (n+m)!$ in Eq. (5.44), so that Eq. (5.44) simply becomes:

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n+m)!}, \qquad \text{where} \qquad m = 1, 2, \ldots \tag{5.46}$$

By inserting these coefficients into Eq. (5.36) and remembering that $c_1 = 0$, $c_3 = 0$, $\cdots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} \qquad (n \geqq 0). \tag{5.47}$$

$J_n(x)$ is called the **Bessel function of the first kind** of order n. The series Eq. (5.47) converges for all x, as the ratio test shows.

$J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

Exercise 5.5: Bessel Function J_0 and J_1

Please calculate the bessel functions of $J_0(x)$ and $J_1(x)$.

Solution

For $n = 0$ we obtain from Eq. (5.47) the *Bessel function* of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} - \ldots \tag{5.48}$$

which looks similar to a cosine. For $n = 1$ we obtain in the *Bessel function* of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2} + \frac{x^5}{2^5 2!3} - \frac{x^7}{2^7 3!4} - \ldots \tag{5.49}$$

which looks similar to a sine (Fig. 110). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing z. Heuristically, n^2/x^2 in (1) in standard form (1) divided by x^2] is zero (if $n = 0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y' + y = 0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

where $\sim$ is read "asymptotically equal" and means that for fixed n the quotient of the two sides approaches 1 as $x \to \infty$ $\frac{x^2}{2^2(1!)^2}$.

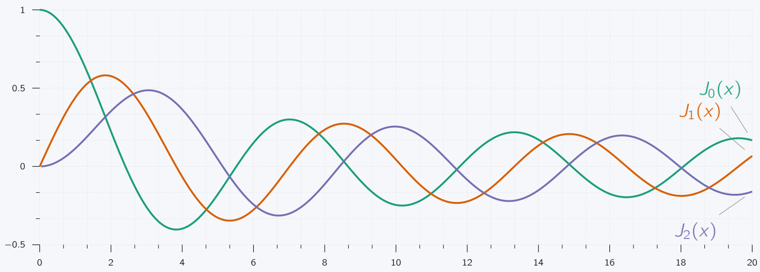


Figure 5.3

Formula (14) is surprisingly accurate even for smaller x (>0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of f_0, you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

Glossary

I

IVP	Initial Value Problem

L

LHS	Left Hand Side

O

ODE	Ordinary Differential Equation

P

PDE	Partial Differential Equation

R

RHS	Right Hand Side

Bibliography

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