Lecture Book

M.Sc Higher Mathematics I

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Topics on Fundamental Science

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Part I.

Ordinary Differential Equations

Chapter 1

First-Order Ordinary Differential Equations

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1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a *mathematical expression* in terms of: variables, functions, equations. Such an expression is known as a mathematical **model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**.

Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,
- find new values,
- interpret result in a physical terms.

This is all done to understand the behaviour of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter.

An ODE is an equation containing one or several derivatives of an unknown function, usually $y(x)$. The equation may also contain y itself, known functions of x, and constants. For example all the equation shown below are classified as ODE.

$$y' = \cos x$$
$$y'' + 9y = e^{-2x}$$
$$y'y'' - \frac{3}{2}y = 0.$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from Partial Differential Equation (PDE)s, which involve **partial** derivatives of an unknown function of two or more variables[1]. For instance, a PDE with unknown function u of two (2) variables x and y is

[1] The topic of PDE will be the focus of **Higher Mathematics II**.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

For now, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F\left(x, y, y'\right) = 0 \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE:

$$x^{-3}y' - 4y^2 = 0 \qquad \text{where} \qquad x \neq 0$$

can be written explicitly as $y' = 4x^3y^2$.

1.1.1 Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

> Geometrically, this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE, together with an initial condition, is called an **initial value problem**.

> **Theory 1.0: Initial Value Problem**
>
> In multi-variable calculus, an Initial Value Problem (IVP) is an ODE together with an initial condition which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Exercise 1.1: Initial Value Problem - A

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

Solution

The general solution is:

$$y(x) = ce^{3x}$$

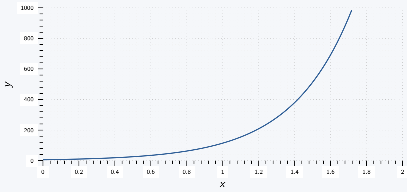
From this solution and the initial condition we obtain:

$$y(0) = ce^{o} = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation ■



Exercise 1.2: Radioactive Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Note: The decay of Radium is measured to be $k = 1.4e - 11\,\mathrm{s}^{-1}$.

Solution

$y(t)$ is the amount of substance still present at t. By the physical law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives is the following:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is positive, so that, because of the minus, we get *decay*. The value of k is known from experiments for various radioactive substances which the question has given as $k = 1.4 \cdot 10^{-11} sec^{-1}$. Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

We have the initial condition $y(0) = 0.5$. This is the instant at which our observation of the process begins. In motivates the original condition which however, is also used when the independent variable is not time or when we choose a t other than $t = 0$.

Hence the mathematical model of the physical process is the initial value problem.

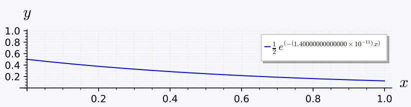
$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the ODE is an exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is:

$$y(t) = 0.5e^{-kt} \quad ■$$



1.2 Separable ODEs

Many practically useful ODEs can be reduced to the following form:

$$g(y)\, y' = f(x) \tag{1.4}$$

using *algebraic manipulations*. We can then integrate on both sides with respect to x, obtaining the following expression:

$$\int g(y)\, y' \, \mathrm{d}x = \int f(x)\, \mathrm{d}x + c. \tag{1.5}$$

On the Left Hand Side (LHS) we can switch to y as the variable of integration. By calculus, we know the relation $y' \mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\, \mathrm{d}y = \int f(x)\, \mathrm{d}x + c. \tag{1.6}$$

[2] a continuous function is a function such that a small variation of the argument induces a small variation of the value of the function.

If f and g are **continuous functions**[2], the integrals in Eq. (1.6) exist, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

Exercise 1.3: Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

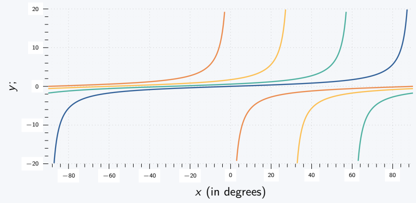
Solution

The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx \qquad \text{By integration} \quad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

Note: It is important to introduce the constant c when the integration is performed.



Exercise 1.4: A Bell Shaped Curve

Solve the following ODE:

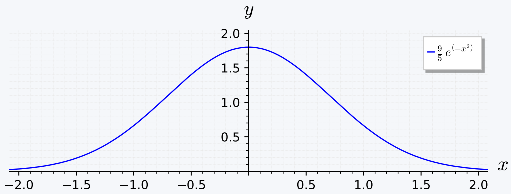
$$y' = -2xy, \qquad y(0) = 1.8$$

Solution

By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve. The plot of the solution is given in Figure **??**.



Exercise 1.5: Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian–Italian border, caused a scientific sensation. When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

Note: The half-life of carbon is 5175 years.

Solution

Radioactive decay is governed by the ODE $y' = ky$ as we have developed previously. By separation and integration

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0 e^{kt} \qquad (y_0 = e^{o}).$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Ötzi died,

$$e^{kT} = e^{-0.0001\,213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001213} = 5312. \qquad \blacksquare$$

Reduction to Separable Form

Certain non-separable ODEs can be made separable by transformations that introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right)$$

Here, f is any differentiable function of y x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests that we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x+u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Exercise 1.6: Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

Solution

Reduction to Separable Form To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

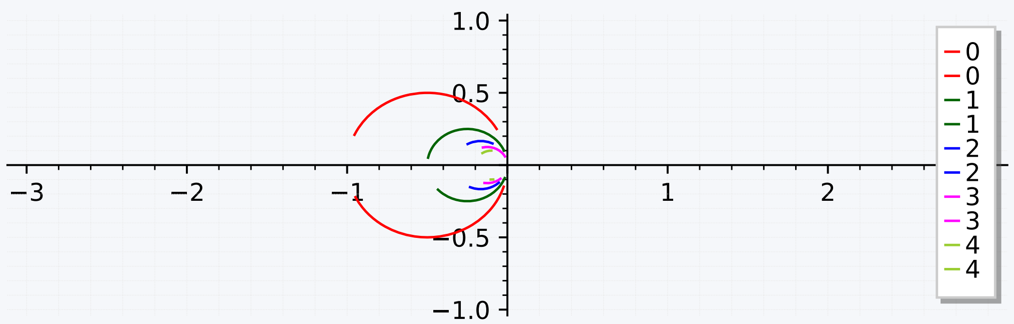
You see that in the last equation you can now separate the variables,

$$\frac{2udu}{1+u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1+u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx. \qquad \text{therefore} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4} \quad \blacksquare$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen in Figure **??**.



1.3 Exact ODEs

1.3.1 Integrating Factors

Recall from calculus that if a function $u(x,y)$ has continuous partial derivatives, its **differential** (i.e., **total differential**) is:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

From this it follows that if $u(x,y) = c =$ is constant, then $du = 0$. As an example, let's have a look at the function $u = x + x^2 y^3 = c$. Finding its factors:

$$du = (1 + 2xy^3) dx + 3x^2 y^2 dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x,y) + N(x,y)\, y' = 0$, written as:

$$M(x,y)\, dx + N(x,y)\, dy = 0 \tag{1.7}$$

is called an **exact differential equation** if the differential form $M(x,y)\, dx + N(x,y)\, dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{1.8}$$

of some function $u(x,y)$. Then Eq. (1.7) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.7) in the form:

$$u(x,y) = c \tag{1.9}$$

Comparing Eq. (1.7) and Eq. (1.8), we see that Eq. (1.7) is an exact differential equation if there is some function $u(x,y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N \tag{1.10}$$

From this we can derive a formula for checking whether Eq. (1.7) is exact or not, as follows.
Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.10),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x}, \tag{1.11}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{1.12}$$

By the assumption of continuity the two second partial derivatives are equal. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.13}$$

This condition is not only necessary but also sufficient for Eq. (1.7) to be an exact differential equation.

If Eq. (1.7) is proved to be **exact**, the function $u(x,y)$ can be found by inspection or in the following systematic way.
From Eq. (1.11) we have by integration with respect to x:

$$u = \int M dx + k(y), \tag{1.14}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.14), use (4b) to get dk/dy, and integrate dk/dy to get k.
Formula Eq. (1.14) was obtained from Eq. (1.11).

It is valid to use **either** of them and arrive at the same result.

Then, instead of Eq. (1.14), we first have by integration with respect to y

$$u = \int N \, dy + l(x). \tag{1.15}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from , use Eq. (1.11) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Exercise 1.7: Initial Value Problem

Solve the initial value problem

$$(\cos y \sinh x + 1)dx - \sin y \cosh x dy = 0, \qquad y(1) = 2.$$

Solution

Verify that the given ODE is **exact**. We find u. For a change, let us use Eq. (1.16):

$$u = -\int \sin y \cosh x dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = u = \cos y \sinh x + 1$. Therefore $dl/dx = 1$ by integration, $l(x) = x + c^*$. This gives the general solution $u(x,y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$.
Therefore the answer is $\cos y \cosh x + x = 0.358$.

Exercise 1.8: An Exact ODE

Solve the following ODE:

$$\cos(x+y)dx + (3y^2 + 2y + \cos(x+y))dy = 0. \tag{1.16}$$

Step 1 - Test for exactness
First check if our equation is **exact**, try to convert the equation of the form Eq. (1.7):

$$M = \cos(x+y),$$
$$N = 3y^2 + 2y + \cos(x+y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x+y),$$
$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

This proves our equation to be exact. **Step 2 - Implicit General Solution**
From Eq. (1.14), we obtain by integration:

$$u = \int M dx + k(y) = \int \cos(x+y) dx + k(y)$$
$$= \sin(x+y) + k(y) \tag{1.17}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (1.12), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Therefore $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$.
Inserting this result into Eq. (1.17) and observing Eq. (1.9), we obtain:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c \quad \blacksquare$$

Exercise 1.9: Breakdown of Exactness

Check the exactness of the following ODE:

$$-ydx + xdy = 0$$

Solution

Breakdown of Exactness The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\partial M/\partial y = -1 \qquad \partial N/\partial x = 1$$

Let us show that in such a case the present method does not work. From (6),

$$u = \int M dx + ky = -xy + ky, \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try ; it will also fail. Solve the equation by another method that we have discussed.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$).

1.4 Linear ODEs

1.4.1 Introduction

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A first-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)\,y = r(x) \tag{1.18}$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

Homogeneous Linear ODE

We want to solve in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) = 0$.) Then the ODE Eq. (1.18) becomes:

$$y' + p(x)\,y = 0 \tag{1.19}$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{therefore} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE Eq. (1.19),

$$y(x) = ce^{-\int p(x)\,\mathrm{d}x} \qquad \left(c = \pm e^{c*} \quad \text{when} \quad y \neq 0\right) \tag{1.20}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Non-Homogeneous Linear ODE

We now solve Eq. (1.18) in the case that $r(x)$ in Eq. (1.18) is not everywhere zero in the interval J considered. Then the ODE Eq. (1.18) is called non-homogeneous. It turns out that in this case, Eq. (1.18) has a pleasant property.

Namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ as follows. We multiply Eq. (1.18) by $F(x)$, obtaining

$$Fy' + pFy = rF. \tag{1.21}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\ dx$. By integration, writing $h = \int p\ dx$,

$$\ln|F| = h = \int p\ dx, \qquad \text{thus} \qquad F = e^{\mathrm{h}}.$$

With this F and $h' = p$, Eq. (1.21) becomes

$$e^{\mathrm{h}}y' + h'e^{\mathrm{h}}y = e^{\mathrm{h}}y' + (e^{\mathrm{h}})'y = (e^{\mathrm{h}}y)' = re^{h}.$$

By integration,

$$e^{\mathrm{h}}y = \int e^{\mathrm{h}}r\,\mathrm{d}x + c$$

Dividing by e^{h}, we obtain the desired solution formula

$$y(x) = e^{-\mathrm{h}}\left(\int e^{\mathrm{h}}r dx + c\right), \qquad h = \int p(x)\,dx. \tag{1.22}$$

This reduces solving Eq. (1.18) to the generally simpler task of evaluating integrals. For ODEs for which this is still difficult, you may have to use a numeric method for integrals or for the ODE itself.

> h has nothing to do with $h(x)$ and that the constant of integration in h does not matter.

The structure of Eq. (1.22) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing Eq. (1.22) as a sum of two terms,

$$y(x) = e^{-\mathrm{h}}\int e^{\mathrm{h}}r dx + ce^{-h},$$

Exercise 1.10: First-Order ODE, General Solution Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$h = \int p dx = \int \tan x dx = \ln|\sec x|.$$

From this we see that in Eq. (1.22),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h} r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x dx + c \right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c - 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$ ■

Chapter 2

Second-Order Ordinary Differential Equations

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2.1 Introduction

A second-order ODE is called **linear**, if it can be written[1] as:

[1] in its standard form

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.1}$$

- when $r(x) = 0$ it is homogeneous,
- else it is **non-homogeneous**.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs.
An example of a **non-homogeneous linear** equation is:

$$y'' = 25y - e^{-x} \cos x$$

An example of a **homogeneous linear** equation is:

$$y'' + \frac{1}{x} y\prime + y = 0$$

An example of **non-linear**ODE is:

$$y''y + \left(y''\right)^2 = 0$$

2.1.1 Superposition Principle

For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1 y_1 + c_2 y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or linearity principle

Theory 2.0: Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE of:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

any linear combination of two solutions on an open interval I is again a solution of on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Exercise 2.1: Homogeneous Linear ODEs: Superposition of Solutions

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0,$$

for all x.

Solution

Homogeneous Linear ODEs: Superposition of Solutions
Verify by differentiation and substitution. We obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$\begin{aligned}&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\ &= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare\end{aligned}$$

Exercise 2.2: Example of a Non-homogeneous Linear ODE

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 0$$

Solution

Example of a Non-homogeneous Linear ODE Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$\begin{aligned}y'' + y &= 1\\ -\sin x + 1 + \cos x &\neq 1 \quad \blacksquare\end{aligned}$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$\begin{aligned} (1+\sin x)'' &= -\cos x \\ y'' + y &= 1 \\ -\cos x + 1 + \sin x &\neq 1 \quad \blacksquare \end{aligned}$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

For a second-order homogeneous linear ODE an initial value problem consists of two (2) initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1. \tag{2.2}$$

The conditions Eq. (2.2) are used to determine the two arbitrary constants c_1 and c_2 in a general solution

Exercise 2.3: Initial Value Problem

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution

Initial Value Problem

Step 1: General Solution

From Example 1, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

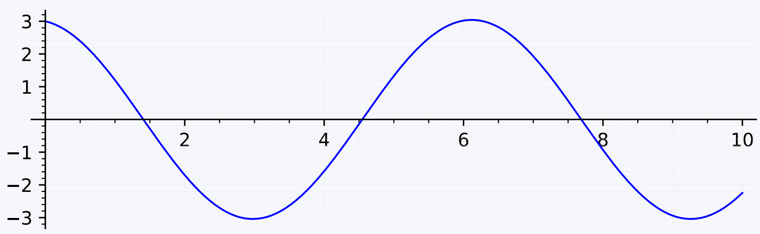
Step 2: Particular Solution

We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as cos 0 = 1 and sin 0 = 0,

$$y(0) = c_1 = 3 \quad \text{and} \quad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$



2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.

2.2 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.3}$$

Solve by starting

$$y = e^{\lambda x} \tag{2.4}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.3) gives:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic** equation (or auxiliary equation),

$$\lambda^2 + a\lambda + b = 0 \tag{2.5}$$

then the exponential function Eq. (2.4) is a solution of the ODE given in Eq. (2.3). Now from algebra we recall the roots of the quadratic equation

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

From algebra we further know that the quadratic equation Eq. (2.5) may have three (3) kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

Case	Roots of	Basis	General Solution
I	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
	(λ_1, λ_2)	$e^{\lambda_2 x}$	
II	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
	$(\lambda = -1/2a)$	$xe^{-ax/2}$	
III	Complex Conjugate	$e^{-ax/2}\cos\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$
	$\lambda_1 = -1/2a + \mathrm{j}\omega$	$e^{-ax/2}\sin\omega x$	
	$\lambda_2 = -1/2a - \mathrm{j}\omega$		

Table 2.1.: Possible roots of the characteristic equation based on the discriminant value.

2.2.1 A Study of Damped System

To our previous undamped model $my'' = -ky$ we now add the damping force:

$$F_2 = -cy',$$

therefore, the ODE of the damped mass–spring system is:

$$my'' + cy' + ky = 0. \tag{2.6}$$

This can physically be done by connecting the ball to a dashpot. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

> This is generally a good approximation for small velocities.

The constant c is called the damping constant.

The damping force $F_2 = -cy'$ acts **against** the motion; hence for a downward motion we have $y\prime > 0$ which for positive c makes F negative (an upward force), as it should be.

Similarly, for an upward motion we have $y\prime < 0$ which, for $c > 0$ makes F_2 positive (a downward force).

The ODE Eq. (2.6) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}. \tag{2.7}$$

Depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.2.: A Detailed look into the scientific method.

Case I: Over-damping

If $c^2 > 4mk$, then λ_1 and λ_2 are **distinct real roots**. In this case the corresponding general solution is:

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}. \tag{2.8}$$

> In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.8) are negative because $\alpha > 0$ and $\beta > 0$ and:

$$\beta^2 = \alpha^2 - k/m < \alpha^2$$

Hence both terms in Eq. (2.8) approach zero as $t \to \infty$. Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Below are the results for some typical initial conditions.

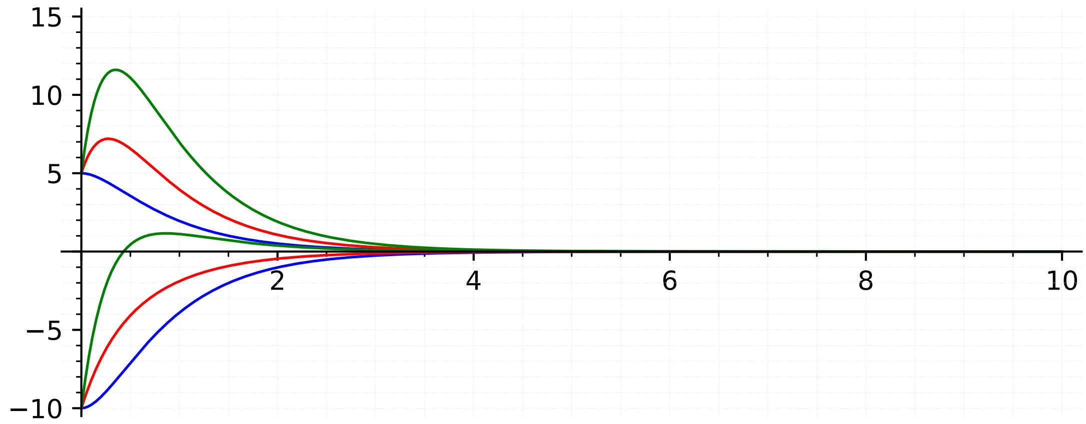


Figure 2.1.: The case of over damping.

Case II: Critical-Damping

Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III). Occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0,\ \lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of Eq. (2.6) is:

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{2.9}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.
If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.
Fig. 2.2 shows typical forms of Eq. (2.9).

> The graph above looks almost like those in the previous figure.

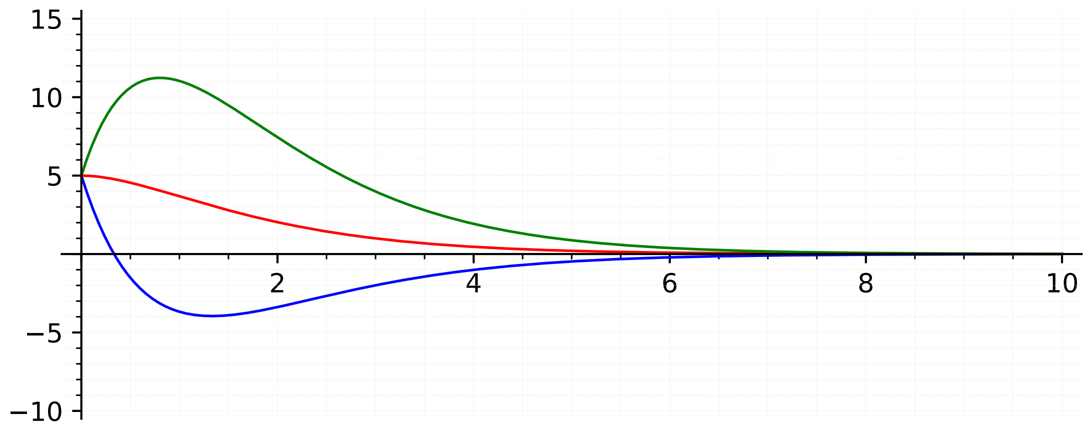


Figure 2.2.: The case of critical damping.

Case III: Under-Damping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 = 4mk$. Then β in Eq. (2.7) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (> 0).$$

The asterisk (*) is used to differentiate from ω which is used predominantly in electrical engineering.

The roots of the characteristic equation are now complex conjugates,

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos(\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$ This represents **damped oscillations**. Their curve lies between the dashed curves: The roots of the characteristic equation are now complex conjugates.
The frequency is $\omega^*/2\pi$ Hz (hertz, cycles/sec). From we see that the smaller $c > 0$ is, the larger is ω^* and the more rapid the oscillations become.

2.2.2 Euler-Cauchy Equations

Has the following form:

$$x^2 y'' + axy' + by = 0 \tag{2.10}$$

To solve do the following substitutions:

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Which gives:

$$x^2 m(m - 1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

$y = x^m$ is a good choice as it produces a common factor x^m.

Simplifying the equation produces the **auxiliary** equation.

$$m^2 + (a - 1)m + b = 0. \tag{2.11}$$

$y = x^m$ is a solution of Eq. (2.10) if and only if m is a root of Eq. (2.11).

The roots of Eq. (2.11) are:

$$m_1 = \frac{1}{2}(1 - a) + \sqrt{\frac{1}{4}(1 - a)^2 - b}, \qquad m_2 = \frac{1}{2}(1 - a) - \sqrt{\frac{1}{4}(1 - a)^2 - b}.$$

Complex conjugate roots are of minor practical importance for practical purposes.

Case	Roots of	General Solution
I	Distict real (m_1, m_2)	$y = c_1 x^{m_1} + c_2 x^{m_2}$
II	Real Double Root (m)	$y = c_1 x^{m} \ln x + c_2 x^{m}$
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta \mathbf{j}$	$\alpha = \mathrm{Re}(m)$
	$m_2 = \alpha - \beta \mathbf{j}$	$\beta = \mathrm{Im}(m)$

Table 2.3.: Possible solutions of the Euler-Cauchy based on the m value.

Exercise 2.4: General Solution in the Case of Different Real Roots

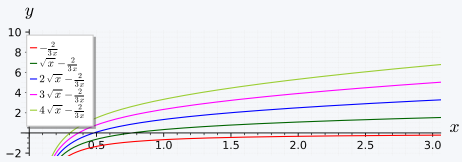
Solve the following ODE:

$$x^2 y'' + 1.5 x y' - 0.5 y = 0$$

Solution

General Solution in the Case of Different Real Roots This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1 \sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$



Exercise 2.5: General Solution in the Case of a Double Root

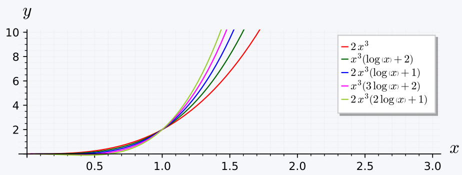
Solve the following ODE:

$$x^2 y'' - 5 x y' + 9 y = 0$$

Solution

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$



Exercise 2.6: BVP: Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii r_1 = 5 cm and r_2 = 10 cm kept at potentials v_1 = 110 V and v_2 = 0, respectively.

$v = v(r)$ is a solution of the *Euler–Cauchy equation* $rv'' + 2v' = 0$.

Solution

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

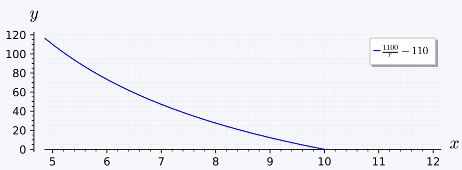
$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$



2.2.3 Non-homogeneous ODEs

They have the form:

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2.12}$$

where $r(x) \neq 0$. a general solution of Eq. (2.12) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2.13}$$

and a **particular solution** of Eq. (2.12). These two new terms **general solution** of Eq. (2.12) and **particular solution** of Eq. (2.12) are defined as follows:

> **Theory** 2.6: General Solution and Particular Solution
>
> A general solution of the nonhomogeneous ODE Eq. (2.12) on an open interval I is a solution of the form:
>
> $$y(x) = y_h(x) + y_p(x)\,. \tag{2.14}$$
>
> here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.13) on I and y_p is any solution of Eq. (2.12) on I containing **no arbitrary constants**. A particular solution of Eq. (2.12) on I is a solution obtained from Eq. (2.14) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.12) or an initial value problem for Eq. (2.12), we have to solve the homogeneous ODE Eq. (2.13) or an initial value problem for and find any solution y_p of Eq. (2.12), so that we obtain a general solution Eq. (2.14) of Eq. (2.12).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.15}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

> These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.
Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Theory 2.6: Choice Rules for the Method of Undetermined Coefficients

Basic Rule: If $r(x)$ in Eq. (2.15) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.15).

Modification Rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.15), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule: If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 2.4.: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.12) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.12) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Exercise 2.7: Application of the Basic Rule (a)

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

Solution

Step 1: General Solution of the Homogeneous ODE

The ODE $y'' + y = 0$ has the general solution

$$y_h = A \cos x + B \sin x.$$

Step 2: Solution of the non-Homogeneous ODE

First try $y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) **must be the same on both sides**. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

The looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0, \quad \text{Then} \quad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Hence:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$$

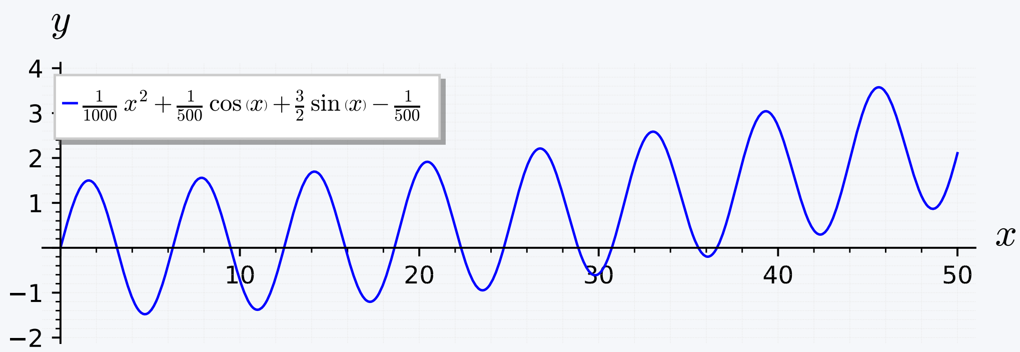
Step 3. Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x \quad \text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$



Exercise 2.8: Application of the Modification Rule (b)

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution

Application of the Modification Rule (b)

Step 1.General solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\, e^{-1.5x}$$

Step 2.Solution y_p of the non-homogeneous ODE

The function $e^{-1.5x}$ on the RHS would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2e^{-1.5x}, \quad \text{then} \quad y_p' = C(2x - 1.5x^2)e^{-1.5x}, y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5 e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2 e^{-1.5x}$$

Step 3. Solution of the initial value problem

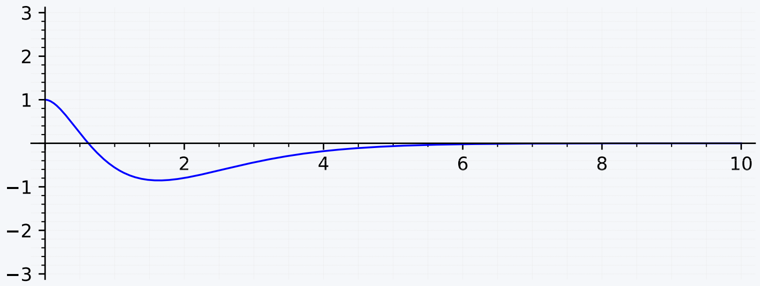
Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2 v)e^{-1.2x} - 10xe^{-1.2x} + 7.5x^2 e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2 e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.



Exercise 2.9: Application of the Sum Rule (c)

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

Solution

Step 1.The General Solution
The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}.$$

Step 2. The Particular Solution
We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1 x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE in (7) gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

therefore

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE in (7) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Step 3. Solution of the initial value problem

From y, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

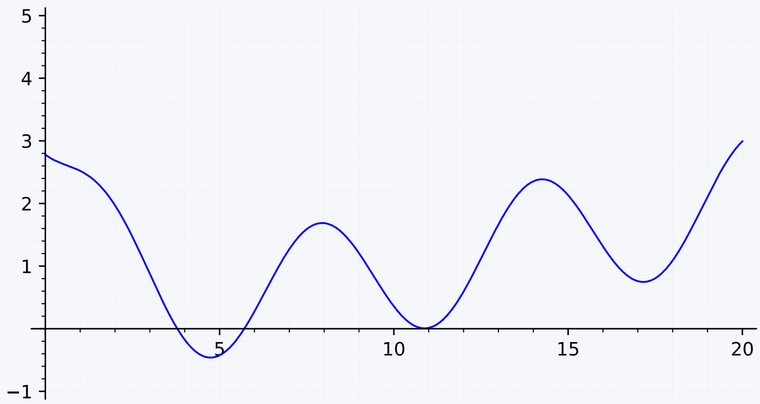


Figure 2.3.: Solution of Application of the Sum Rule (c)

2.2.4 A Study of Forced Oscillations and Resonance

Previously we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring) and modeled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.16}$$

Here $y\left(t\right)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass–spring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia $my\prime\prime$, the damping force $cy\prime$ (if $c < 0$), and the spring force ky, a restoring force.

Now extend our model by including an additional force, that is, the external force $r\left(t\right)$, on the RHS. This turns Eq. (2.16) into:

$$my'' + cy' + ky = r(t). \tag{2.17}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.18}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.18) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.16) plus any solution y_p of Eq. (2.18). To find y_p, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.19}$$

By differentiating this function (remember the chain rule) we obtain:

$$\begin{aligned} y_\mathrm{p}' &= -\omega a \sin \omega t + \omega b \cos \omega t, \\ y_\mathrm{p}'' &= -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t. \end{aligned}$$

Substituting y_p, $y_\mathrm{p}\prime$, $y_\mathrm{p}\prime\prime$, into Eq. (2.18) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2) equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.20}$$

$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.21}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega\, c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $\left(km\omega^2\right)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.22}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.18) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.16) and y_p is given by Eq. (2.19) with coefficients Eq. (2.22).

2.2.5 Solving Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I\,\mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t. \tag{2.23}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.23) is the sum $I = I_\mathrm{h} + I_\mathrm{p}$, where I_h is a general solution of the homogeneous ODE corresponding to Eq. (2.23) and I_p is a particular solution of Eq. (2.23). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a\cos \omega t + b \sin \omega t, \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t), \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t). \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to E0v cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$L\omega^2(-a) + R\omega b + a/C = E_0\omega \qquad \text{(Cosine terms)}$$
$$L\omega^2(-b) + R\omega(-a) + b/C = 0 \qquad \text{(Sine terms)}.$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.24}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$-Sa + Rb = E_0,$$
$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.25}$$

Equation (2) with coefficients a and b given by Eq. (2.25) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage.

Using Eq. (2.25), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.26}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $(R^2 + S^2)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.26) and whose frequency is that of the input (i.e., voltage).

Exercise 2.10: Reduction of Order if a Solution Is Known

Find a basis of solutions of the ODE:

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

Solution

Inspection shows that $y_1 = x$ is a solution because $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel.
The idea of the method is to substitute

$$\begin{aligned} y &= uy_1 = ux, \\ y' &= u'x + ux' = u'x + u, \qquad \text{(Chain Rule)} \\ y'' &= (u'x + u)' = u''x + u'x' + u' = u''x + 2u'. \qquad \text{(Chain Rule)} \end{aligned}$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and xu cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely:

$$\left(x^2 - x\right) v' + (x - 2)\, v = 0$$

Separation of variables and integration gives:

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}\, dx = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx, \qquad \ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}$$

We need no constant of integration because we want to obtain a particular solution.
Taking exponents and integrating again, we obtain:

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \left|\int v\, dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ln|x| + 1.\right|$$

Since $y_1 = x$ and $y_2 = x \ln \mathbf{p} x + 1$ are **linearly independent**.

> This means their quotient is not constant.

we have obtained a basis of solutions, valid for all positive x. ■

Exercise 2.11: IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution

IVP: Case of Distinct Real Roots **Step 1. General Solution**
The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

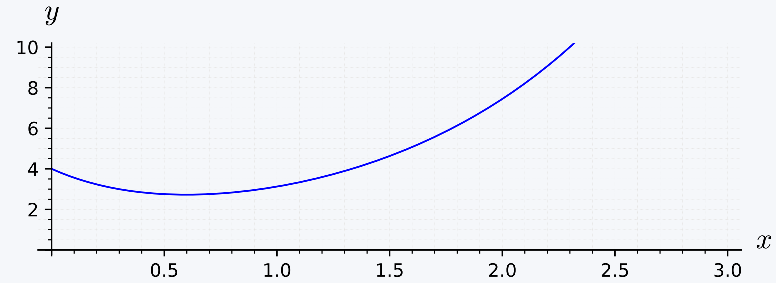
$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular Solution
As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^x + 3e^{-2x} \quad ■$$



Exercise 2.12: IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

Solution

IVP: Case of Real Double Roots The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = (\lambda + 2)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = (c_1 + c_2 x)\, e^{-0.5\,x}$$

We need its derivative:

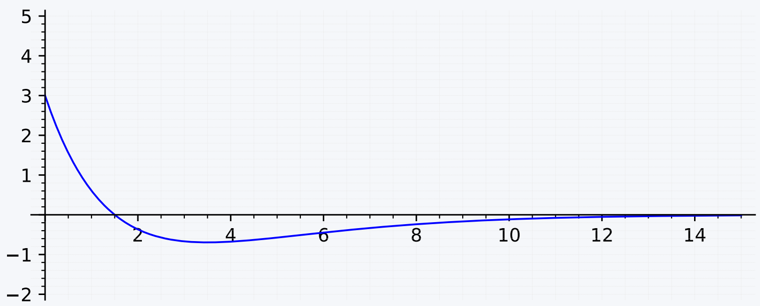
$$y\prime = c_2 e^{-0.5\,x} - 0.5\,(c_1 + c_2 x)\, e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$y\,(0) = c_1 = 3.0, \quad y\prime\,(0) = c_2 - 0.5c_1 = 3.5, \quad c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\,x}$$



Exercise 2.13: IVP: Case of Complex Roots

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution

IVP: Case of Complex Roots **Step 1. General Solution**
The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathrm{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A\cos 3x + B\sin 3x).$$

Step 2. Particular Solution

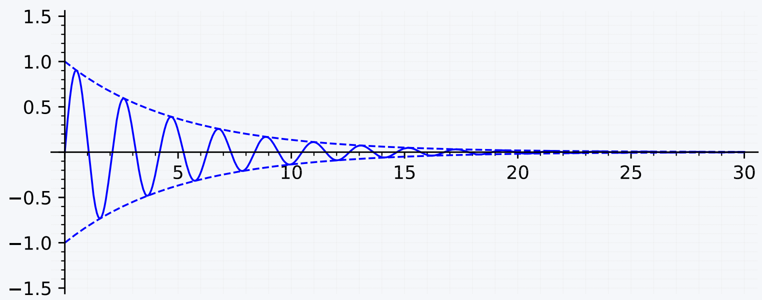
The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x}\sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.



Exercise 2.14: Harmonic Oscillation of an Undamped Mass–Spring System

If a mass–spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

Solution

Hooke's law:

$$F_1 = -ky \tag{2.27}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\mathrm{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0.48\,\mathrm{Hz}$$

From Eq. (2.27) and the initial conditions, $y(0) = A = 0.16\,\mathrm{m}$ and $y\prime(0) = \omega_0 B = 0$.
Hence the motion is

$$y(t) = 0.16\cos 3t\,\mathrm{m} \quad \blacksquare$$

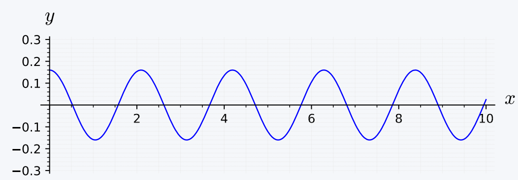


Figure 2.4.: The harmonic oscillation on a string.

Exercise 2.15: The Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped Mass–Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 60\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 10\,\mathrm{kg}\cdot\mathrm{s}^{-1}$

Solution

The Three Cases of Damped Motion It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I

With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass–Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16\,\mathrm{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02,\ c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9t} + 0.18e^{-t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II

The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3t}, \qquad \text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3t}$$

The initial conditions give $y\left(0\right) = c_1 = 0.16,\ y\prime\left(0\right) = c_2 - 3c_1 = 0,\ c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III

The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96j$$

This gives the general solution:

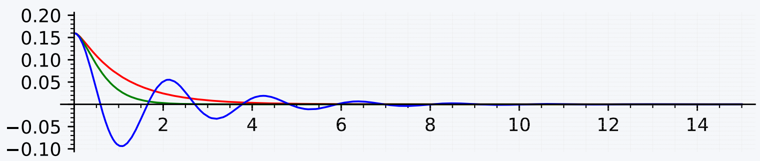
$$y = e^{-0.5t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y\left(0\right) = A = 0.16$. We also need the derivative

$$y\prime = e^{-0.5t}\left(-0.5A\cos 2.96t - 0.5B\sin 2.96t - 2.96A\sin 2.96t + 2.96B\cos 2.96t\right)$$

Hence $y\prime\left(0\right) = -0.5A + 2.96B = 0,\ B = 0.5A/2.96 = 0.027$. This give the solution:

$$y = e^{-0.5t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) = 0.162e^{-0.5t}\cos\left(2.96t - 0.17\right) \quad \blacksquare$$



Exercise 2.16: Studying the RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11$ (Ohms), $L = 0.9$ H (Henry), $C = 0.01$ F (Farad), which is connected to a source of $V(t) = 110 \sin(120\pi t)$.

Assume that current and capacitor charge are 0 when $t = 0$.

Solution

Step 1. General solution of the homogeneous ODE

Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t.$$

$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Hence the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

Step 2. Particular solution I_p

We calculate the reactance S = 37.7 - 0.3 = 37.4 and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t$$

Step 3. Particular solution satisfying the initial conditions

How to use Q(0) 0? We finally determine c1 and c2 from the in initial conditions I(0) 0 and Q(0) 0. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to Q(0) 0. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

Chapter 3

Higher-Order Ordinary Differential Equations

Table of Contents

3.1 Homogeneous Linear ODEs

Recall from **First-Order ODEs** that an ODE is of n^{th} if the n^{th} derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F\left(x, y, y', \cdots, y^{(n}\right)) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

> $y^{(n)}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.

> An n^{th}-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.
A **solution** of an n^{th}-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to n^{th}-order homogeneous linear ODEs as following theorems.

Theory 3.0: Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

This does not hold for a nonhomogeneous or non-linear ODE.

Theory 3.0: General Solution, Basis, Particular Solution

A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \quad \text{arbitrary}) \tag{3.3}$$

where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.

That is, these solutions are linearly independent on I, as defined below.
A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

Theory 3.0: Linear Independence and Dependence

Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on} I \tag{3.4}$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \ldots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n-1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (3.4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Exercise 3.1: Linear Dependence

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

Solution

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Exercise 3.2: General Solution

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \quad \text{where} \quad y^{\text{iv}} = \frac{d^4y}{dx^4}$$

Solution

Similar to Chapter 2 we substitute $y = e^{\lambda x}$. Omitting the common factor $e^{\lambda x}$, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Exercise 3.3: Initial Value Problem for a Third-Order Euler–Cauchy Equation

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \quad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4.$$

Solution

General solution

As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular solution

The derivatives are $y' = c_1 + 2c_2x + 3c_3x^2$ and $y'' = 2c_2 + 6c_3x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
\text{(a)} \quad y(1) &= c_1 + c_2 + c_3 = 2 \\
\text{(b)} \quad y'(1) &= c_1 + 2c_2 + 3c_3 = 1 \\
\text{(c)} \quad y''(1) &= 2c_2 + 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3$$ ■

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

> Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(n)} = d^n y/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \qquad \cdots, \qquad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Exercise 3.4: Distinct Real Roots

Solve the following ODE:

$$y''' - 2y'' - y' + 2y = 0$$

Solution

The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$.

> If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x}\cos\omega x, \qquad y_2 = e^{\gamma x}\sin\omega x.$$

Exercise 3.5: Simple Complex Roots

Solve the initial value problem:

$$y'''-y''+100y'-100y=0, \qquad y(0)=4, \qquad y'(0)=11, \qquad y''(0)=-299$$

Solution

The characteristic equation is:

$$\lambda^3-\lambda^2+100\lambda-100=0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda-1$ shows that the other roots are $\pm 10j$.

Therefore, a general solution and its derivatives (obtained by differentiation) are:

$$\begin{aligned} y &= c_1e^x + A\cos 10x + B\sin 10x, \\ y' &= c_1e^x - 10A\sin 10x + 10B\cos 10x, \\ y'' &= c_1e^x - 100A\cos 10x - 100B\sin 10x. \end{aligned}$$

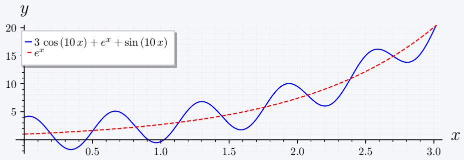
From this and the initial conditions we obtain, by setting $x=0$,

$$\text{(a) } c_1+A=4, \qquad \text{(b) } c_1+108=11, \qquad \text{(c) } c_1-1004=-299$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A=303$, $A=3$. Then $c_1=1$ from (a) and $B=1$ from (b). The solution is:

$$y=e^x+3\cos 10x+\sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.



Multiple Real Roots

If a real double root occurs ($\lambda_1=\lambda_2$) then $y_1=y_2$ in Eq. (3.3), and we take y_1 and xy_1 as corresponding linearly independent solutions.

More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2e^{\lambda x}, \quad \cdots, \quad x^{m-1}e^{\lambda x}$$

Exercise 3.6: Real Double and Triple Roots

Solve the following ODE:

$$y^{v}-3y^{iv}+3y^{iv}-y''=0$$

Solution

The characteristic equation is:

$$\lambda^5-3\lambda^4+3\lambda^3-\lambda^2=0$$

and has the roots $\lambda_1=\lambda_2=0$, and $\lambda_3=\lambda_4=\lambda_0=1$, and the answer is

$$y=c_1+c_2x+\left(c_3+c_4x+c_5x^2\right)e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda=\gamma+i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda}=\gamma-i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x}\cos\omega x, \quad e^{\gamma x}\sin\omega x, \quad xe^{\gamma x}\cos\omega x, \quad xe^{\gamma x}\sin\omega x$$

The first two of these result from $e^{\lambda x}$ and $e^{\overline{\lambda}x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\overline{\lambda}x}$ in the same fashion. Obviously, the corresponding general solution is

$$y=e^{\gamma x}.$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2e^{\gamma x}\cos\omega x$, $x^2e^{\gamma x}\sin\omega x$, and so on.

3.1.3 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of nth order. We write them in standard form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)}+, \cdots, +p_1(x)y' + p_0(x)y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a **general solution** of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Thus Eq. (3.7) has no singular solutions. An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

The ideas of proof are the same as those for $n = 2$.

Exercise 3.7: IVP - Modification Rule

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \qquad y'(0) = -3, \qquad y''(0) = -47$$

Solution

Step 1

The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2) e^{-x} \end{aligned}$$

Step 2

If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and $Cx^6 e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}$$

Then

$$\begin{aligned} y_p' &= C\left(3x^2 - x^3\right) e^{-x}, \\ y_p'' &= C\left(6x - 6x^2 + x^3\right) e^{-x}, \\ y_p''' &= C\left(6 - 18x + 9x^2 - x^3\right) e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C\left(6-18x+9x^2-x^3\right)+3C(6x-6x^2+x^3)+3C(3x^2-x^3)+Cx^3=30.$$

The linear, quadratic, and cubic terms drop out, and $6C=30$. Hence $C=5$, giving $y_p=5x^2e^{-x}$.

Step 3

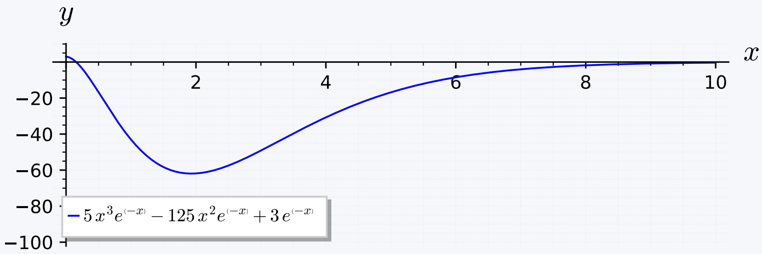
We now write down $y=y_h+y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$\begin{aligned}
y &= y_h+y_p=(c_1+c_2+c_3x^2)e^{-x}+5x^3e^{-x}, & y(0)&=c_1=3\\
y' &= [-3+c_2+(-c_2+2c_3)x+(15-c_3)x^2-5x^3]e^{-x}, & y'(0)&=-3+c_2=-3, \quad c_2=0\\
y'' &= [3+2c_3+(30-4c_3)x+(-30+c_3)x^2+5x^3]e^{-x}, & y''(0)&=3+2c_3=-47, \quad c_3=-25.
\end{aligned}$$

Hence the answer to our problem is:

$$y=(3-25x^2)e^{-x}+5x^2e^{-x}$$

The curve of y begins at $(0,3)$ with a negative slope, as expected from the initial values, and approaches zero as $x\to\infty$.



3.1.4 Application: Modelling an Elastic Beam

Whereas second-order ODEs have various applications, of which we have discussed some of the more important ones (i.e., RLC Circuit, Mass-Damper system), higher order ODEs have much fewer engineering applications.

An important fourth-order ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge.

A related application of vibration of beams does not fit in here since it leads to PDEs.

Problem Description

Consider a beam B of length L and constant (e.g., **rectangular**) cross section and homogeneous elastic material (e.g., **level**).
We assume under its own weight the beam is bent so little that it is certainly straight. If we apply a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.
Its axis is curved into the so-called **elastic curve** (or **deflection curve**).
It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ (determining the tangent direction of C) are small. Then, by calculus:

$$k=y''/(1+y''x)^{1/2}\approx y''$$

Therefore:

$$M(x) = EIy''(x)$$

EI is the constant of proportionality. E Young's modulus of elasticity of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.
Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy^{\text{iv}} = f(x)$$

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows:
Simply supported

$$y = y'' = 0 \text{at} x = 0 \text{and} L$$

$$y = y' = 0 \text{at} x = 0 \text{and} L$$

$$\text{(C)Clamped at} x = 0, \text{free at} x = L$$

$$y(0) = y'(0) = 0, y''(L) = y'''(L) = 0.$$

The boundary condition $y = 0$ means no displacement at that point, $y'' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Solution Derivation

Let us apply this to the uniformly loaded simply supported beam. The load is $f(x) = f_0 = \text{const}$. Then (8) is

$$y^{\text{iv}} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Two integrations give

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

$y''(0) = 0$ gives $c_2 = 0$. Then $y''(L) = L\left(\frac{1}{2}kL + c_1\right) = 0$, $c_1 = -kL/2$ (since $L \neq 0$). Hence

$$y'' = \frac{k}{2}(x^2 - Lx).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution

$$y = \frac{f_0}{24EI}(x^4 - 2Lx^3 + L^3 x).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L - x)$.

Verify this by setting $x = u + L/2$ and show that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0 (x = L/2)$ is:

$$\frac{5f_0L^4}{(16 \cdot 24EI)}$$

> Recall that the positive direction points downward.

Chapter 4

Systems of ODEs

4.1 Introduction

Table of Contents

In this chapter of our book, we introduce a different way of looking at systems of ODEs. The method consists of examining the general behaviour of whole families of solutions of ODEs in the phase plane, called the **phase plane** method.

> **Theory** 4.0: Phase Plane
>
> A visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two state variables, say (x, y), or (q, p) etc.

It gives information on the stability of solutions. This approach to systems of ODEs is a qualitative method because it depends only on the nature of the ODEs and does not require the actual solutions. This can be very useful because it is often difficult or even impossible to solve systems of ODEs. In contrast, the approach of actually solving a system is known as a **quantitative** method.

> **Theory** 4.0: Qualitative Method
>
> The qualitative analysis of ODEs is to be able to say something about the behavior of solutions of the equations, without solving them explicitly.

> The phase plane method has many applications in control theory, circuit theory, population dynamics and so on.

4.1.1 System of ODEs as Models in Engineering

Time to see how systems of ODEs are of practical importance. We first illustrate how systems of ODEs can serve as models in various applications. Then we show how a higher order ODE (with the highest derivative standing alone on one side) can be reduced to a first-order system.

Exercise 4.1: Mixing Problem Involving Two Tanks

A mixing problem involving a single tank is modeled by a single ODE which can be extended to two sets of equations.
Tanks T_1 and T_2 contain initially 100 L of water each, In T_1 the water is pure, whereas 150 kg of fertilizer are dissolved in T_2. By circulating liquid at rate of 2 l/min and stirring the amount of fertiliser $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain at least half as much fertiliser as there will be left in T_2? **Note:**Assume the mixture is uniform.

Solution

Setting Up the Model

As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. Therefore:

$$y_1' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}\,y_2 - \frac{2}{100}\,y_1 \qquad (Tank\,T_1),$$
$$y_2' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}\,y_1 - \frac{2}{100}\,y_2 \qquad (Tank\,T_2).$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs:

$$y_1{\prime} = -0.02y_1 + 0.02y_2,$$
$$y_2{\prime\prime} = 0.02y_1 - 0.02y_2.$$

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix $\mathbf{A}$ this becomes:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y}\,, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

General Solution

As for a single equation, we try an exponential function of t,

$$\mathbf{y} = \mathbf{x}e^{\lambda t}. \qquad \text{Then} \qquad \mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{x}e^{\lambda t}. \tag{1}$$

Dividing the last equation $\lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x}.$$

We need nontrivial solutions (solutions that are not identically zero). Hence we have to look for eigenvalues and eigenvectors of $\mathbf{A}$. The eigenvalues are the solutions of the characteristic equation

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} = (-0.02-\lambda)^2 - 0.02^2 = \lambda(\lambda+0.04) = 0 \tag{4.1}$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$.

> $\lambda = 0$ can very well happen but don't get mixed up. It is eigenvectors that must not be zero.

Eigenvectors are obtained as $\lambda = 0$ and $\lambda = -0.04$. For our present $\mathbf{A}$ this gives:

$$-0.02x_1 + 0.02x_2 = 0 \qquad \text{and} \qquad (-0.02+0.04)x_1 + 0.02x_2 = 0,$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From Eq. (1) and the superposition principle, we thus obtain a solution:

> This principle continues to hold for systems of homogeneous linear ODEs

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.04t} \tag{4.2}$$

where c_1 and c_2 are arbitrary constants.

Use of initial conditions

The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and Eq. (4.2) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

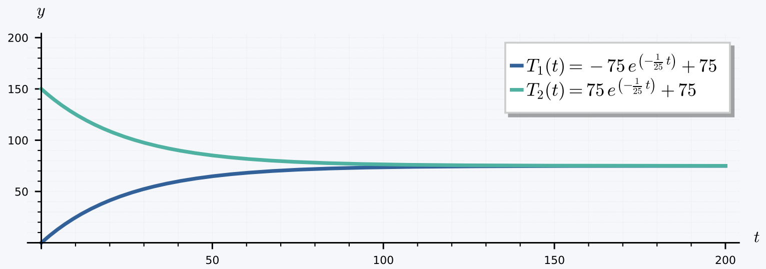
In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer:

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)} e^{-0.04t} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

In components,

$$y_1 = 75 - 75e^{-0.04\,t} \qquad \text{Tank } T_1\text{, lower curve,}$$
$$y_2 = 75 + 75e^{-0.04\,t} \qquad \text{Tank } T_2\text{, upper curve.}$$

Figure **??** shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 kg.



Answer

T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 kg. Therefore:

$$y_1 = 75 - 75e^{-0.04t} = 50, \qquad e^{-0.04t} = \tfrac{1}{3}, \qquad t = (\ln 3)/0.04 = 27.5$$

Hence the fluid should circulate for roughly half an hour ■

Exercise 4.2: Electrical Network

Find the currents $I_1(t)$ and $I_2(t)$ in the network.

Assume all currents and charges zero at $t = 0$, the instant when the switch is **closed**.

Solution

Setting up the mathematical model

The model of this network is obtained from Kirchhoff's Voltage Law.

Kirchhoff's Voltage Law

The sum of the voltage differences around any closed loop in a circuit must be zero. A loop in a circuit is any path which ends at the same point at which it starts.

Let $I_1(t)$ and $I_2(t)$ be the currents in the left (L) and right (R) loops, respectively.
In (L), the voltage drops are:

$$L_1' I_1' = (1) I_1' \; V \qquad \text{Over Inductor}$$
$$R_1(I_1 - I_2) = 4\,(I_1 - I_2) \;\; V \qquad \text{Over Resistor}$$

The difference is caused by I_1 and I_2 flowing through the resistor in **opposite** directions.

By Kirchhoff's Voltage Law the sum of these drops equals the voltage of the battery:

$$I_1' + 4(I_1 - I_2) = 12$$

Cleaning the aforementioned equation creates our first ODE:

$$I_1' = -4I_1 + 4I_2 + 12. \tag{4.3}$$

In (R), the voltage drops are:

$$R_2 I_2 = 6I_2 \text{ [V]}$$
$$R_1(I_2 - I_1) = 4(I_2 - I_1) \text{ [V]}$$
$$(I/C)\int I_2\,dt = 4\int I_2\,dt \text{ [V]}\,.$$

As there is no voltages sources in the (R) loop, the voltage sum **MUST** be zero.

$$6I_2 + 4(I_2 - I_1) + 4\int I_2\,dt = 0 \qquad \text{or} \qquad 10I_2 - 4I_1 + 4\int I_2\,dt = 0.$$

Division by 10 and differentiation gives $I_2' - 0.4I_1' + 0.4I_2 = 0$.

To simplify the solution process, we first get rid of $0.4I_1'$, which by Eq. (4.3) equals $0.4(-4I_1 + 4I_2 + 12)$. Substitution into the present ODE gives

$$I_2' = 0.4I_1' - 0.4I_2 = 0.4(-4I_1 + 4I_2 + 12) - 0.4I_2$$

and by simplification

$$I_2' = -1.6I_1 + 1.2I_2 + 4.8. \tag{4.4}$$

In matrix form, Eq. (4.3) and Eq. (4.4) are (we write $\mathbf{J}$ since $\mathbf{I}$ is the unit matrix)

$$\mathbf{J}' = \mathbf{A}\,\mathbf{J} + \mathbf{g}, \qquad \text{where} \qquad \mathbf{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}. \tag{6}$$

General Solution

As we have a vector $\mathbf{g}$, this is a **non-homogeneous** system, and we try to proceed as for a single ODE, solving first the homogeneous system $\mathbf{J}' = \mathbf{A}\,\mathbf{J}$ (thus $\mathbf{J}' - \mathbf{A}\,\mathbf{J} = v0$) by substituting $\mathbf{J} = \mathbf{x}e^{\lambda t}$. This gives

$$\mathbf{J}' = \lambda \mathbf{x} e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t} \qquad \text{hence} \qquad \mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

Hence, to obtain a non-trivial solution, we again need the eigenvalues and eigenvectors. For the present matrix $\mathbf{A}$ they are:

$$\lambda_1 = -2, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

Hence a *general solution* of the homogeneous system is:

$$\mathbf{J}_h = c_1 \mathbf{x}^{(1)} e^{-2\,t} + c_2 \mathbf{x}^{(2)} e^{-0.8\,t}.$$

For a particular solution of the nonhomogeneous system Eq. (6), since $\mathbf{g}$ is constant, we try a constant column vector $\mathbf{J}_p = \mathbf{a}$ with components a_1, a_2. Then $\mathbf{J}_p' = 0$, and substitution into Eq. (6) gives $\mathbf{A}\,\mathbf{a} + \mathbf{g} = 0$, in components,

$$-4.0a_1 + 4.0a_2 + 12.0 = 0$$

$$-1.6a_1 + 1.2a_2 + 4.8 = 0.$$

The solution is $a_1 = 3$, $a_2 = 0$; thus $\mathbf{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

$$J = J_h + J_p = c_1 x^{(1)} e^{-2t} + c_2 x^{(2)} e^{-0.8t} + a; \tag{4.5}$$

in components,

$$I_1 = 2c_1 e^{-2t} + c_2 e^{-0.8t} + 3$$
$$I_2 = c_1 e^{-2t} + 0.8c_2 e^{-0.8t}$$

The initial conditions give

$$I_1(0) = 2c_1 + c_2 + 3 = 0$$
$$I_2(0) = c_1 + 0.8c_2 = 0.$$

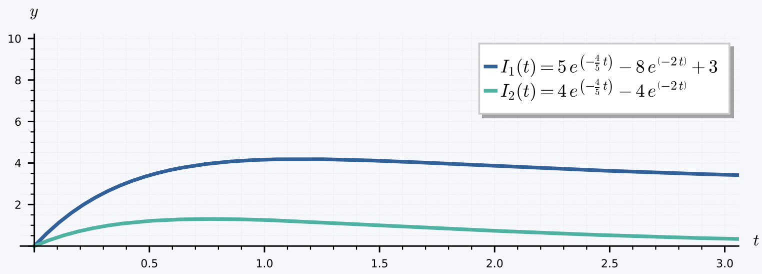
Hence $c_1 = -4$ and $c_2 = 5$. As the solution of our problem we thus obtain

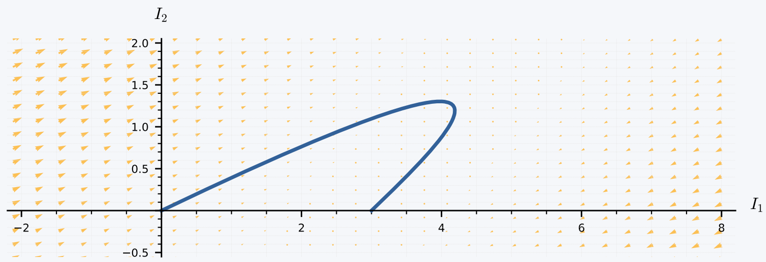
$$J = -4x^{(1)} e^{-2t} + 5x^{(2)} e^{-0.0t} + a \tag{4.6}$$

In components:

$$I_1 = -8e^{-2t} + 5e^{-0.8t} + 3$$
$$I_2 = -4e^{-2t} + 4e^{-0.8t}.$$

Now comes an important idea, on which we shall elaborate further. Figure **??** shows $I_1(t)$ and $I_2(t)$ as two (2) separate curve. Figure **??** shows those two currents as a **single curve** $[I_1(t), I_2(t)]$ in the $I_1 I_2$-plane.
This is a parametric representation with time as the parameter t. It is often important to know in which sense such a curve is traced. This can be indicated by an arrow in the sense of increasing t. The I_1, I_2-plane is called the **phase plane** of our system Eq. (6), and the curve in **??** is called a trajectory.
In following chapters we will see that such *phase plane representations* are far more important than graphs because they will give a much better qualitative overall impression of the general behavior of whole families of solutions, not merely of one solution as in the present case. ■





4.1.2 Conversion of an n-th Order ODE to a System

An nth-order ODE of the general form can be converted to a system of n first-order ODEs. This permits the study and solution of single ODEs by methods for systems, and opens a way of including the theory of higher order ODEs into that of first-order systems.

Theory 4.2: Conversion of an ODE

An nth-order ODE:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{4.7}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}. \tag{4.8}$$

This system is of the form

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned} \tag{4.9}$$

While the iron is hot, let's look at an example.

Exercise 4.3: Mass on a Spring

To gain confidence in the conversion method, let us apply it to an old friend of ours, modelling the free motions of a mass on a spring with value given as $m = 1,\ c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \quad \text{or} \quad y'' = -\left(\frac{c}{m}\right)y, -\left(\frac{k}{m}\right)y.$$

Solution

For this ODE given in the question can be written in the form ofEq. (4.7), making the system shown Eq. (4.8) as **linear** and **homogeneous**, applying to our system in question.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m}y_1 - \frac{c}{m}y_2. \end{aligned}$$

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Entering the values of $m = 1$, $c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0$$

This gives the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$.
Eigenvectors follow from the first equation in $A - \lambda I = 0$, which is $-\lambda x_1 + x_2 = 0$.

$\lambda_1 = 0.5$ Produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$.

$\lambda_2 = -1.5$ Produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$.

These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$x^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \quad \text{give} \quad y = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t} \quad \blacksquare$$

4.1.3 Linear Systems

Extending the notion of a linear ODE, we call a linear system if it is linear in $y_1, \cdots, y_n$; that is, if it can be written

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t). \end{aligned} \tag{4.10}$$

As a vector equation this becomes

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} + \mathbf{g} \tag{4.11}$$

where:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ . & \cdots & . \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

This system is called **homogeneous** if $\mathbf{g} = 0$, so that it is:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} \tag{4.12}$$

If $\mathbf{g} \neq 0$, then Eq. (4.12) is called **non-homogeneous**.

4.2 Constant-Coefficient Systems

4.2.1 Phase Plane Method

Continuing, we now assume that our **homogeneous** linear system:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{4.13}$$

under discussion has **constant coefficients**, so that the $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ has entries not depending on t. We want to solve Eq. (4.13). Now a single ODE $y' = ky$ has the solution $y = Ce^{kt}$. So let us try:

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \tag{4.14}$$

Substitution into Eq. (4.13) gives:

$$\mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{y} = \mathbf{A}\,\mathbf{x}e^{\lambda t}.$$

Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**:

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x} \tag{4.15}$$

Thus the nontrivial solutions of Eq. (4.13) (i.e., non-zero vectors solutions) are of the form Eq. (4.14), where λ is an eigenvalue of λ and $\mathbf{x}$ is a corresponding eigenvector.

We assume that λ has a **linearly independent** set of n eigenvectors. This holds in most applications, in particular if $\mathbf{A}$ is symmetric ($a_{\text{kj}} = a_{\text{jk}}$) or skew-symmetric ($a_{\text{kj}} = -a_{\text{jk}}$) or has n different eigenvalues.

Let those eigenvectors be $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ and let them correspond to eigenvalues $\lambda_1, \cdots, \lambda_n$ (which may be all different, or some–or even all–may be equal). Then the corresponding solutions Eq. (4.14) are

$$y^{(1)} = x^{(1)} e^{\lambda_1 t}, \quad \cdots, \quad y^{(n)} = x^{(n)} e^{\lambda_n t}. \tag{4.16}$$

Their Wronskian $W = W(y^{(1)}, \cdots, y^{(n)})$ is given by

$$W = (y^{(1)}, \cdots, y^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ x_2^{(1)} e^{\lambda_1 t} & \cdots & x_2^{(n)} e^{\lambda_n t} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} e^{\lambda_1 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t + \cdots + \lambda_n t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$

On the right, the exponential function is never zero, and the determinant is not zero either because its columns are the n linearly independent eigenvectors. This proves the following theorem, whose assumption is true if the matrix $\mathbf{A}$ is symmetric or skew-symmetric, or if the n eigenvalues of $\mathbf{A}$ are all different.

Theory 4.3: Theorem: General Solution

If the constant matrix $\mathbf{A}$ in the system Eq. (4.13) has a linearly independent set of n eigenvectors, then the corresponding solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ in Eq. (4.16) form a basis of solutions of Eq. (4.13), and the corresponding general solution is:

$$y = c_1 x^{(1)} e^{\lambda_1 t} + \cdots + c_n x^{(n)} e^{\lambda_n t} \tag{4.17}$$

Exercise 4.4: Type I: Improper Node -Trajectories in the Phase Plane

Find and graph solutions of the system.

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

Solution

To see what is going on, let us find and graph solutions of the system. It is always a good idea to start with known solutions. Substituting $\mathbf{y} = \mathbf{x}e^{\lambda t}$ and $\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t}$ and dropping the exponential function (as they exist both on the LHS and RHS we can eliminate them) we get $\mathbf{A}\,\mathbf{x} = \lambda \mathbf{x}$.

The characteristic equation is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$.
Eigenvectors are then obtained from:

$$(-3-\lambda)x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\mathbf{x}^{(1)} = [1 \quad 1]^{\mathsf{T}}$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\mathbf{x}^{(2)} = [1 \quad -1]^{\mathsf{T}}$.

This gives the general solution:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}.$$

Figure below shows a phase portrait of some of the trajectories (to which more trajectories could be added if so desired).

The two straight trajectories correspond to $c_1 = 0$ and c_2 and the others to other choices of c_1, c_2.

4.2.2 Critical Points of the System

The point $\mathbf{y} = 0$ in Figure seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from Eq. (4.13) we obtain:

$$\frac{dy}{dt} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}. \tag{4.18}$$

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: (0, 0), where the right side of Eq. (4.18) becomes 0/0.

> This point P_0, at which dy_2/dy_1 becomes **undetermined** and called a **critical point** of Eq. (4.18).

Five Types of Critical Points

There are five types of critical points depending on the geometric shape of the trajectories near them. These are: (1) improper nodes, (2) proper nodes, (3) saddle points, (4) centres, and (5) spiral points.

Let's look at them with examples.

Exercise 4.5: Type II: Proper Node

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction **d** at P_0 there is a trajectory having **d** as its limiting direction.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{y}, \quad \text{Therefore} \quad y_1' = y_1 \quad \text{and} \quad y_2' = y_2$$

has a proper node at the origin with the matrix being the **identity matrix**. Its characteristic equation $(1-\lambda)^2 = 0$ has the root $\lambda = 1$.

> Any $\mathbf{x} \neq 0$ is an eigenvector.

and we can take $[1 \quad 0]^{\mathsf{T}}$ and $[0 \quad 1]^{\mathsf{T}}$.

Hence, a general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \quad \text{or} \quad \begin{matrix} y_1 = c_1 e^t, \\ y_2 = c_2 e^t. \end{matrix} \quad \text{or} \quad c_1 y_2 = c_2 y_1 \quad \blacksquare$$

Exercise 4.6: Type III: Saddle Node

A **saddle point** is a critical point P_0 at which there are two (2) incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}, \quad \text{Therefore} \quad \begin{matrix} y_1' = y_1 \\ y_1' = -y_2 \end{matrix}$$

has a saddle point at the origin.

Its characteristic equation $(1-\lambda)(-1-\lambda) = 0$ has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$.

For $\lambda = 1$ in eigenvector $[1\ 0]^T$ is obtained from the second row of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, that is, $0x_1 + (-1-1)\,x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0 \quad 1]^T$. Hence a general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{matrix} y_1 = c_1 e^t \\ y_2 = c_2 e^{-t} \end{matrix} \quad \text{or} \quad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas** ■.

Exercise 4.7: Type IV: Centre Node

A **centre** is a critical point that is enclosed by infinitely many closed trajectories.

The system:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y, \quad \text{Therefore} \quad y_1' = y_2 \quad \text{and} \quad y_2' = -4y_1 \tag{4.19}$$

has a center at the origin.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathrm{j}$ and $-2\mathrm{j}$. For $2\mathrm{j}$, an eigenvector follows from the first equation $-2\mathrm{j}\, x_1 + x_2 = 0$ of $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$, which can be, $[1 \quad 2\mathrm{j}]^{\mathsf{T}}$.

For $\lambda = -2\mathrm{j}$ that equation is $-(-2\mathrm{j})x_1 + x_2 = 0$ and gives, say, $[1 \quad -2\mathrm{j}]^{\mathsf{T}}$. Hence a complex general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2\mathrm{j} \end{bmatrix} e^{2\mathrm{j}\,t} + c_2 \begin{bmatrix} 1 \\ -2\mathrm{j} \end{bmatrix} e^{-2\mathrm{j}\,t}, \quad \text{therefore} \quad \begin{aligned} y_1 &= c_1 e^{2\mathrm{j}\,t} + c_2 e^{-2\mathrm{j}\,t}, \\ y_2 &= 2\mathrm{j}\, c_1 e^{2\mathrm{j}\,t} - 2\mathrm{j}\, c_2 e^{-2\mathrm{j}\,t}. \end{aligned} \tag{4.20}$$

A real solution is obtained from Eq. (4.20) by the Euler formula or from Eq. (4.19).
Namely, we can create a relation of $-4y_1 y_1^2$.

$$-4y_1 y_1' = y_2 y_2' \quad \text{By Integration} \quad 2y_1^2 + \frac{1}{2} y_2^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. ■

Exercise 4.8: Type V: Spiral Point

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$.

> or tracing these spirals in the opposite sense, away from P_0.

The system:

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y, \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.21}$$

has a spiral point at the origin.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1 + \mathrm{j}$ and $-1 - \mathrm{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For
$\lambda = -1 + \mathrm{j}$ this becomes $-\mathrm{j}\, x_1 + x_2 = 0$ and we can take $[1 \quad \mathrm{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathrm{j}$ is $[1 \quad -\mathrm{j}]^T$.

This gives the **complex** general solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ \mathrm{j} \end{bmatrix} e^{(-1+\mathrm{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathrm{j} \end{bmatrix} e^{(-1-\mathrm{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (4.21) by y_1, the second by y_2- and add, obtaining:

$$y_1 y_1' + y_2 y_2' = -\left(y_1^2 + y_2^2\right).$$

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to t gives:

$$2rr' = 2y_1 y_1' + 2y_2 y_2'$$

Hence the previous equation can be written

$$rr' = -r^2, \quad \text{Thus,} \quad r' = -r, \quad dr/r = -dt, \quad \ln|r| = -t + c^*, \quad r = ce^{-t}.$$

For each real c this is a spiral.

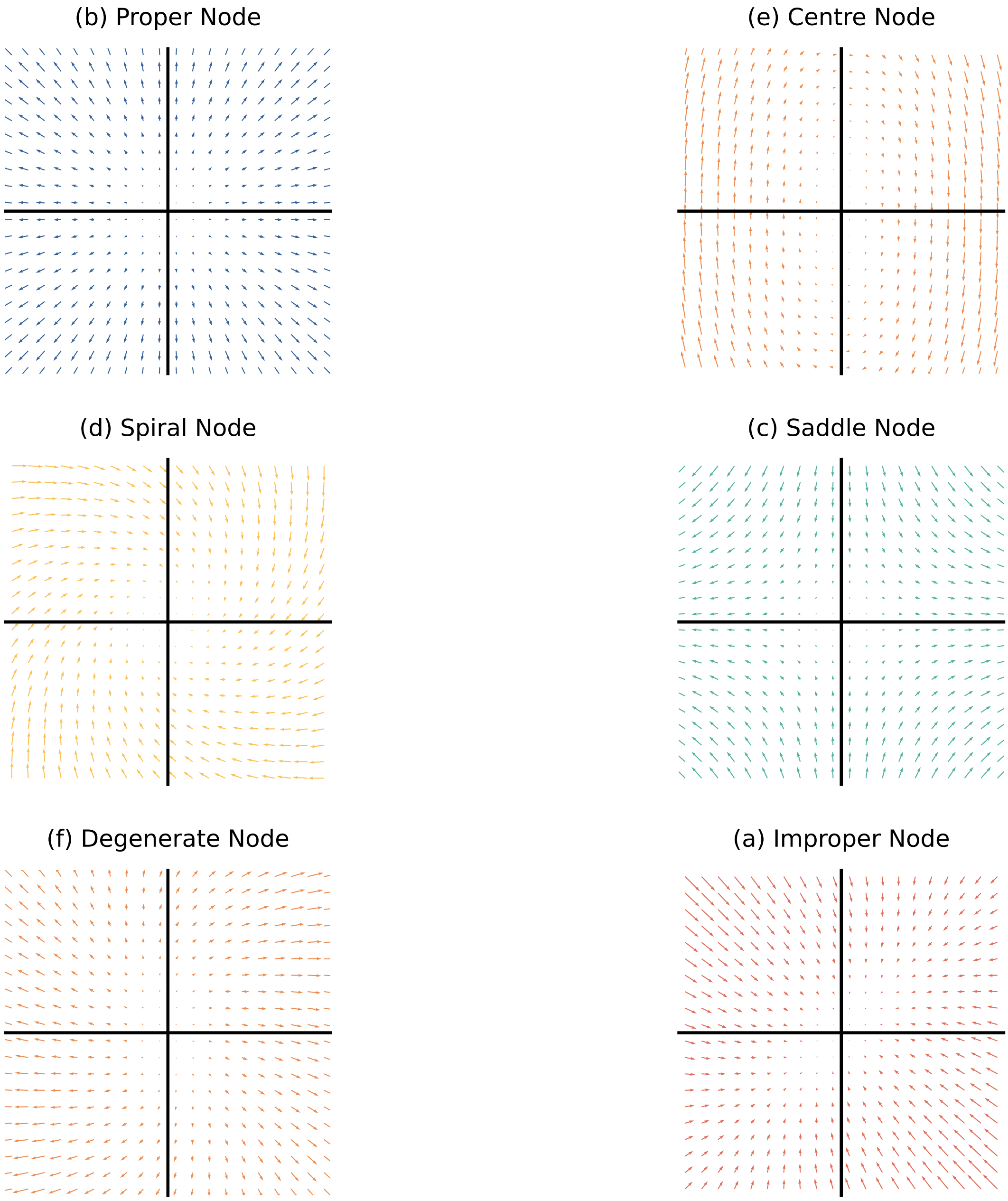


Figure 4.1.: Types of possible systems encountered in the ODE System analysis.

4.3 Criteria for Critical Points & Stability

Continuing our discussion of homogeneous linear systems with **constant coefficients** Eq. (4.13). Let us review where we are. From the previous section we have,

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}, \qquad \text{in components, } y_1' = a_{11}y_1 + a_{12}y_2$$

$$y_2' = a_{21}y_1 + a_{22}y_2. \tag{4.22}$$

From the examples in the last section, we have seen that we can obtain an overview of families of solution curves if we represent them parametrically as $\mathbf{y}(t) = [y_1(t) \quad y_2(t)]^\mathsf{T}$ and graph them as curves in the $y_1 y_2$-plane, called the **phase plane**.
Such a curve is called a **trajectory** of Eq. (4.13), and their totality is known as the **phase portrait** of Eq. (4.13).

Now we have seen that solutions are of the form:

$$\mathbf{y}(t) = \mathbf{x}e^{\lambda t}. \qquad \text{Substitution into(1)gives} \qquad \mathbf{y}'(t) = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{y} = \mathbf{A}\,\mathbf{x}e^{\lambda t}$$

Dropping the common factor $e^{\lambda t}$, we arrive at a similar equation.

$$\mathbf{A}\,\mathbf{x} = \lambda \mathbf{x} \tag{4.23}$$

> $\mathbf{y}(t)$ is a (nonzero) solution of Eq. (4.3) if λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ a corresponding eigenvector.

Our examples in the last section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system Eq. (4.3) defined as a point at which $d\mathbf{y}^2/d\mathbf{y}^1$ becomes **undetermined** (i.e., 0/0).

$$\frac{dy_2}{dy_1} = \frac{y_2' \, dt}{y_1' \, dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \tag{4.24}$$

Also recall from there are various types (5) of critical points.

What is new here, how these types of critical points are related to the eigenvalues. The latter are solutions $\lambda = \lambda_1$ and λ_2 of the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det A = 0. \tag{4.25}$$

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with coefficients p, q and discriminant Δ given by:

$$p = a_{11} + a_{22}, \qquad q = \det A = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta = p^2 - 4q. \tag{4.26}$$

From algebra we know that the solutions of this equation are

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}).$$

Furthermore, the product representation of the equation gives

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Hence p is the sum and q the product of the eigenvalues. Also $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ from (6). Together,

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2.$$

This gives the criteria in Table 4.1 for classifying critical points. A derivation will be indicated later in this section. Critical points may also be classified in terms of their **stability**. Stability concepts are fundamental for engineering purposes where it means, a small change of a physical system at some instant changes the behavior of the system only slightly at all future times t.

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments
Node		$q>0$	$\Delta\geq 0$	Real, same sign
Saddle Point		$q<0$		Real, opposite signs
Centre	$p=0$	$q>0$		Pure imaginary
Spiral		$q\neq 0$	$\Delta<0$	Complex (not pure imaginary)

Table 4.1.: Eigenvalue Criteria for Critical Points.

Stable Unstable Attractive

A critical point P_0 of Eq. (4.3) is called **stable** if, roughly, all trajectories of Eq. (4.3) that at some instant are close to P_0 remain close to P_0 at all future times, or in another way if for every disk D_ϵ of radius $\epsilon>0$ with center P_0 there is a disk D_δ of radius $\delta>0$ with center P_0 such that every trajectory of Eq. (4.3) that has a point P_1 in D_δ has all its points corresponding to $t\equiv t_1$ in D_ϵ.
P_0 is called **unstable** if P_0 is not stable.
P_0 is called **stable and attractive** if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t\to\infty$.

In general term this can be written in a following table.

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
Stable and attractive	$q<0$	$q>0$
Stable	$q\leq 0$	$q>0$
Unstable	either $q\leq 0$	or $q>0$

Table 4.2.: Stability criteria for critical points.

Free Motions of a Mass-Spring System 9 **Example**

What kind of critical point does the following equation have ?

$$my''+c'y'+ky=0$$

Solution

Free Motions of a Mass-Spring System
First, division by m gives:

$$y''=-(k/m)y-(c/m)y'$$

To get a system, set $y_1=y, y_2=y'$. Then $y_2'=y''=-(k/m)y_1-(c/m)y_2$. Therefore:

$$y'=\begin{bmatrix}0 & 1\\ -k/m & -c/m\end{bmatrix}y,\qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{bmatrix}-\lambda & 1\\ -k/m & -c/m-\lambda\end{bmatrix}=\lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0$$

We can see that:

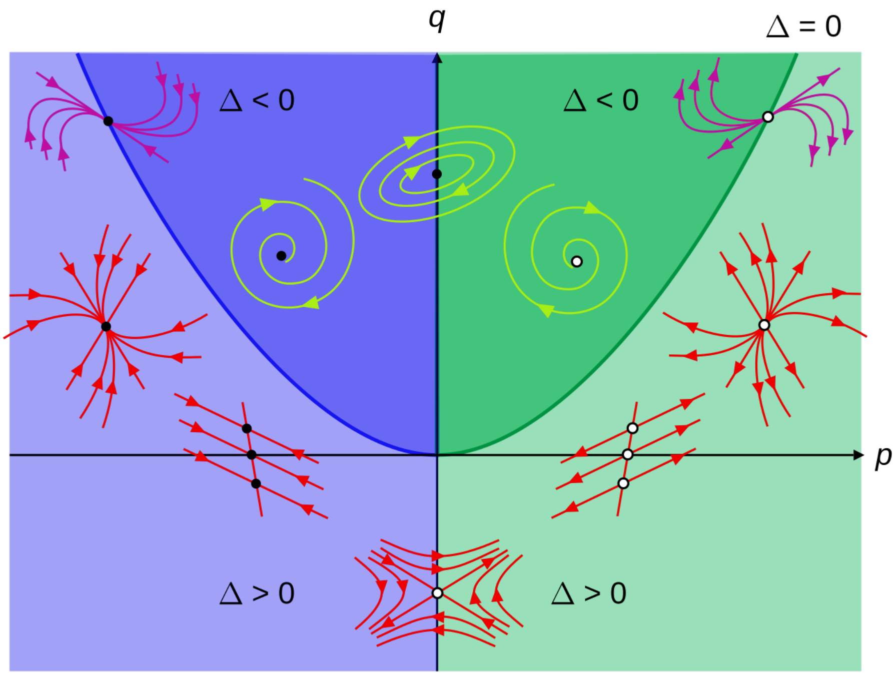
$$p=-c/m,\qquad q=k/m,\qquad \Delta=(c/m)^2-4k/m$$

From this we obtain the following results.

Note that in the last three cases the discriminant Δ plays an essential role.

No Damping $c=0,\ p=0,\ q>0$ and has a centre.

Under damping $c^2<4mk,\ p<0,\ q>0,\ \Delta<0$ and has a stable and attractive spiral point.



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$q = AD - BC$$
$$\frac{dy}{dt} = Cx + Dy \qquad \Delta = p^2 - 4q$$

Figure 4.2.: A diagram showing the stability criteria.

Critical Damping $c^2 = 4mk$, $p < 0$, $q > 0$, $\Delta = 0$ and has a stable and attractive node.

Overdamping $c^2 > 4mk$, $p < 0$, $q > 0$, $\Delta > 0$ and has a stable and attractive node.

4.4 Qualitative Methods for Non-Linear Systems

Qualitative methods are methods of obtaining qualitative information on solutions *without actually solving a system*. These methods are particularly valuable for systems whose solution by analytic methods is difficult or impossible.
This is the case for many practically important **non-linear systems**.

$$y' = f(y), \quad \text{therefore} \quad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned} \tag{4.27}$$

Here we will extend the previously discussed phase plane methods, from linear systems to nonlinear systems Eq. (4.27). We assume that Eq. (4.27) is autonomous, that is, the independent variable t does not occur explicitly.

> All examples in the last section are autonomous.

We shall, again exhibit entire families of solutions.

> This is an advantage over numeric methods, which give only one (approximate) solution at a time.

For this analysis we need to employ the previously defined concepts of **phase plane** (the y_1-y_2-plane), **trajectories** (solution curves of Eq. (4.27) in the phase), the **phase portrait** of Eq. (4.27) (the totality of these trajectories), and **critical points** of Eq. (4.27) points (y_1, y_2) at which both $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are zero).
Now Eq. (4.27) may have several critical points. Our approach shall be to discuss one critical point after another. If a critical point P_0 is not at the origin, then, for technical convenience, we shall move this point to the origin before analyzing the point.
More formally, if P_0: (a, b) is a critical point with (a, b) **NOT** at the origin (0, 0), then we apply the translation:

$$\overline{y_1} = y_1 - a, \qquad \overline{y_2} = y_2 - b,$$

which moves P_0 to $(0, 0)$ as desired. Thus we can assume P_0 to be the origin $(0, 0)$, and for simplicity we continue to write y_1, y_2 (instead of $\overline{y}_1, \overline{y}_2$). We also assume that P_0 is **isolated**, that is, it is the only critical point of Eq. (4.27) within a (sufficiently small) disk with center at the origin.

4.4.1 Linearisation of Non-Linear Systems

How to determine the kind and stability of a critical point P_0: $(0, 0)$ of Eq. (4.27)?

In most cases this can be done by linearisation of Eq. (4.27) near P_0, writing Eq. (4.27) as $\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \mathbf{A}\,\mathbf{y} + \mathbf{h}(\mathbf{y})$ and dropping $\mathbf{h}(\mathbf{y})$, as follows.
Since P_0 is critical, $f_1(0, 0) = 0$, $f_2(0, 0) = 0$, so that f_1 and f_2 have no constant terms and we can write

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} + \mathbf{h}(\mathbf{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2). \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2). \end{aligned} \tag{4.28}$$

A is constant as Eq. (4.27) is autonomous.

> **Theory** 4.9: Linearisation
>
> If f_1 and f_2 in Eq. (4.27) are continuous and have continuous partial derivatives in a neighborhood of the critical point P_0: $(0, 0)$, and if $\det \mathbf{A} \neq 0$

in Eq. (4.28), then the kind and stability of the critical point of Eq. (4.27) are the same as those of the linearized system*

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y}, \quad \text{thus} \quad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.29}$$

Exceptions occur if $\mathbf{A}$ has equal or pure imaginary eigenvalues; then Eq. (4.27) may have the same kind of critical point as (3) or a spiral point.

Exercise 4.10: Linearisation of a Free Un-damped Pendulum

A pendulum consists of a body of mass m (the bob) and a rod of length L. Determine the locations and type of the critical points.

Assume that the mass of the rod and an reference are negligible.

Solution

Setting Up the Mathematical Model

Let θ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is mg, where g is the acceleration of gravity.

This causes a restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2nd law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the**acceleration**.
Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg\sin\theta = 0.$$

Dividing this by mL, we have:

$$\theta'' + k\sin\theta = 0 \quad \text{with} \quad \left(k = \frac{g}{L}\right). \tag{4.30}$$

When θ is very small, we can approximate $\sin\,\theta$ rather accurately by θ and obtain as an approximate solution $A\cos\sqrt{k} + B\sin\sqrt{k}$, but the *exact* solution for any θ is not an **elementary function**.

Critical Points $(\pm 2\pi n,\ 0)$ and Linearisation

To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from Eq. (4.30) we obtain a nonlinear system Eq. (4.27) of the form:

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = y_2, \\ y_2' &= f_2(y_2, y_1) = -k\sin y_1. \end{aligned}$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi,\ 0)$, where $n = 0, \pm1, \pm2, \cdots$.

We consider $(0,\ 0)$. Since the Maclaurin series is

$$\sin y_1 = y_1 - \tfrac{1}{6}y_1^2 + - \cdots \approx y_1,$$

Theory 4.10: Maclaurin Series

A Maclaurin series is a Taylor series expansion of a function about 0,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

Maclaurin series are named after the Scottish mathematician *Colin Maclaurin*.

the linearized system at $(0,\ 0)$ is:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\mathbf{y} \quad \text{Therefore} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1. \end{aligned}$$

To apply our criteria in Sec. 4.4 we calculate:

$$\begin{aligned}p &= a_{11} + a_{22} = 0,\\ q &= \det(\mathbf{A}) = k = g/L \qquad (> 0),\\ \Delta &= p^2 - 4q = -4k.\end{aligned}$$

From this and Table 4.1(c) in Sec. 4.4 we conclude that $(0, 0)$ is a **centre**, which is **always stable**. Since $\sin\theta = \sin y_1$ is periodic with period of 2π.

This means the critical points $(n\pi,\ 0)$, $n = \pm 2,\ \pm 4,\ \cdots$, are all centres.

Critical Points $(\pm(2n-1)\pi,\ 0)$ and Linearisation

We now consider the critical point $(\pi, 0)$, setting:

$$\begin{aligned}y_1 &= \theta - \pi\\ y_2 &= (\theta - \pi)'\end{aligned}$$

Then in Eq. (4.30), we can apply the MacLaurin series:

$$\sin\ \theta = \sin(y_1 + \pi) = -\sin\ y_1 = -y_1 + \frac{1}{2}y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix}0 & 1\\ k & 0\end{bmatrix}\mathbf{y} \qquad \text{Thus} \qquad \begin{aligned}y_1' &= y_2,\\ y_2' &= ky_1.\end{aligned}$$

We see that:

$$\begin{aligned}p &= 0,\\ q &= -k \qquad (< 0)\,,\\ \Delta &= -4q = 4k.\end{aligned}$$

Hence, by Table 4.1(b), this gives a saddle point, which is always unstable.

Because of periodicity, the critical points $(n\pi,\ 0)$, $n = \pm 1,\ \pm 3,\ \cdots$, are all **saddle points**.

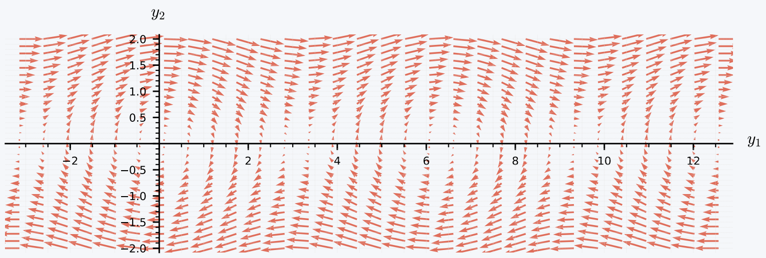


Figure 4.3.

Exercise 4.11: Linearisation of a Damped Pendulum

To gain further experience in investigating critical points, as another practically Important, let us see how the previous example changes when we add a damping term $c\theta$, (damping proportional to the angular velocity) to equation Eq. (4.30), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

where $k > 0$ and $c \geq 0$ (which includes our previous case of no damping, $c = 0$).

Solution

First we start by setting $\theta = y_1,\ \theta'$ as before, we obtain the nonlinear system (use $\theta'' = y_2$),

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k \sin y_1 - c y_2. \end{aligned}$$

We see the critical points have the same locations as the example before, namely, $(0,\ 0)\ ,\ (\pm\pi,\ 0)\ ,\ (\pm 2\pi,\ 0)\ ,\ \dots$. To analyse this system, we start with analysing $(0,\ 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0,\ 0)$.

$$y' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \mathbf{y} \quad \text{therefore} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2 \end{aligned}$$

This is identical with the system in previous example, except for the positive factor m (and except for the physical meaning of y_1). Hence for $c = 0$ (no damping) we have a centre, for small damping we have a spiral point, and so on.

We now consider the critical point $(\pi,\ 0)$. We set $\theta - \pi = y_1,\ (\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin\left(y_1 + \pi\right) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi,\ 0)$:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \mathbf{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2. \end{aligned}$$

For our criteria, we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = -c \\ q &= \det \mathbf{A} = -k \\ \Delta &= p^2 - 4q = c^2 + 4k \end{aligned}$$

This gives the following results for the critical point $(\pi,\ 0)$.

No Damping $c > 0$, $p = 0$, $q < 0$, $\Delta > 0$, a saddle point, Sec. Fig. 3b.

Damping $c > 0$, $p < 0$, $q < 0$, $\Delta > 0$, a saddle point, Sec. Fig. 94.

As $\sin y_1$ is periodic with period of 2π, the critical points $(\pm 2\pi,\ 0)\ ,\ (\pm 4\pi,\ 0)\ ,\ \dots$ are of the same type as $(0, 0)$, and the critical points $(-\pi, 0)$, $(\pm 3\pi, 0)$, $\cdots$ are of the same type as $(\pi, 0)$, so that our task is finished. ■

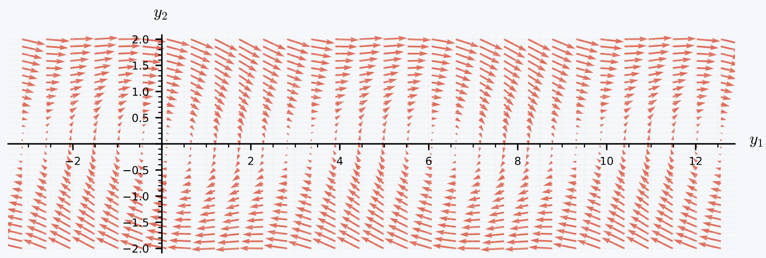


Figure 4.4.: The vector field of a damped medium with $k = 0.5,\ c = 0.2$.

Model Self-Sustained Oscillations - Van der Pol Equation

There are physical systems such that for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system.

In other words, large oscillations will be damped, whereas for small oscillations there is *negative damping* (feeding of energy into the system). For physical reasons we expect such a system to approach a periodic behaviour, which will thus appear as a closed trajectory in the phase plane, called a **limit cycle**.

> An ODE describing such vibrations is the famous **van der Pol equation**.

$$y'' - \mu\left(1 - y^2\right) y' + y = 0$$

It first occurred in the study of electrical circuits containing vacuum tubes.

Vacuum Tube

A vacuum tube, electron tube, valve (British usage), or tube (North America) is a device that controls electric current flow in a high vacuum between electrodes to which an electric potential difference has been applied.

For $\mu = 0$ this equation becomes $y'' + y = 0$ and so with harmonic oscillations. If we define $\mu > 0$, then the damping term has the factor $-\mu\left(1 - y^2\right)$. This is a consequence for small oscillations, when $y^2 < 1$, so that we have negative damping, is zero for $y^2 = 1$ (no imaginary), and is positive if $y^2 > 1$ (positive damping. Loss of energy).

If μ is small, we expect a limit cycle almost a circle because then our equation differs but finite from $y'' + y =$. If μ is large, the limit cycle will probably look different.
Setting $y = y_1$, $y' = y_2$ and using $y'' = (dy_2/dy_1)y_2$ as in (8), we have from (10)

$$\frac{dy_2}{dy_1} y_2 - \mu(1 - y_1^2)y_2 + y_1 = 0.$$

The isoclines in the y_1y_2-plane (the phase plane) are the curves $dy_2/dy_1 = K =$ const, that is,

$$\frac{dy_2}{dy_1} = \mu(1 - y_1^2) - \frac{y_1}{y_2} = K.$$

Solving algebraically for y_2, we see that the icoclines are given by

$$y_2 = \frac{y_1}{\mu(1 - y_1^2) - K}$$

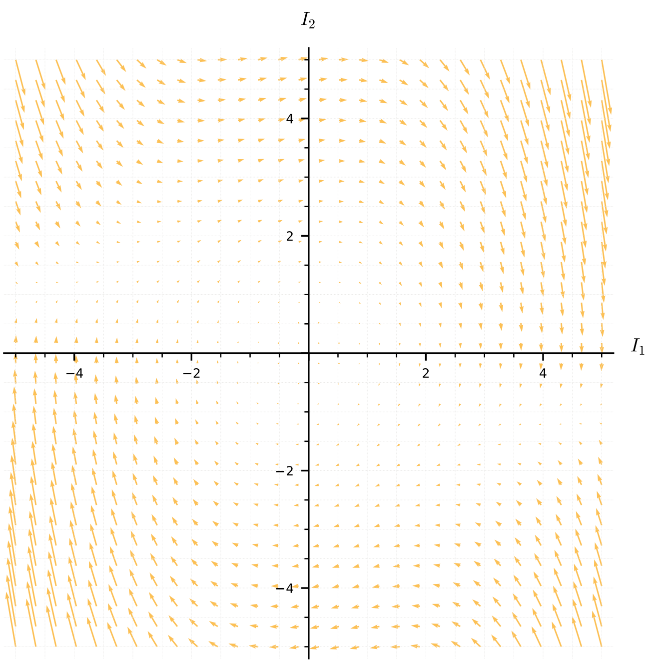


Figure 4.5.: Vector plot of the van der pol equation.

Chapter 5

Special Functions for ODEs

5.1 Introduction

Table of Contents

Linear ODEs with **constant coefficients** can be solved by *algebraic* methods, and their solutions are elementary functions known from calculus.

> **Theory** 5.0: Elementary Functions
>
> A function of a single variable, (real or complex) defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, and their inverses (e.g., $\arcsin$, $\log$, $x^{1/n}$).

ODEs with **variable coefficients**, however, it is more complicated, and their solutions may be non-elementary functions which means we cant write the solution with explicit functions.
For engineering applications where explicit solutions are not possible, *Legendre's*, *Bessel's*, and the *hypergeometric* equations are important ODEs of this kind.
We will look at the two (2) standard methods for solving ODEs:

Power Series Gives the solution in terms of a power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Frobenius Method Gives the solution in power series (similar to the power series, multiplied by a logarithmic term $\ln x$ or a fractional power x^r.

5.2 Power Series Method

The power series method is the standard method for solving linear ODEs with *variable* coefficients. It gives solutions in the form of **power series**.

> The power series method is used for computing values, graphing curves, proving formulas, and exploring properties of solutions

Remember, the **power series** (in powers of $x - x_0$) is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots . \tag{5.1}$$

Here, x is a variable and $a_0, a_1, a_2, \cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **centre** of the series. For $x_0 = 0$, we obtain a power series in powers of x:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \tag{5.2}$$

For the duration of the chapter we will assume all variables and constants are real.

The term **power series** usually refers to a series of the form Eq. (5.1), but does not include series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the *Legendre* and *Bessel equations* for integer values.

Exercise 5.1: Power Series Solution

Solve the following ODE:

$$y' - y = 0$$

Solution

First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \tag{5.3}$$

We put these values into the ODE:

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

Then we collect like powers of x, finding:

$$(a_1 - a_0) + (2a_2 - a_1)\, x + (3a_3 - a_2)\, x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express $a_1, a_2, \cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^x. \quad \blacksquare$$

We may now generalise this idea. For a given ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{5.4}$$

First represent $p(x)$, $q(x)$ by power series in powers of x.

If $p(x)$, $q(x)$ are polynomials, and then nothing needs to be done in this first step.

Next we assume a solution in the form of a power series with unknown coefficients and insert it as well as Eq. (5.3) and:

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x_2 + \cdots = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} \tag{5.5}$$

into the ODE. Then we **collect same powers of** x and equate the sum of the coefficients of each occurring power of x to zero (0), starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of Eq. (5.3) successively.

Exercise 5.2: A Special Legendre Function

Solve the following ODE:

$$\left(1 - x^2\right) y'' - 2xy' + 2y = 0$$

These equations usually occur in models with spherical symmetry.

Solution

Substitute Eq. (5.2), Eq. (5.3), and Eq. (5.5) into the ODE, $\left(1 - x^2\right) y''$ gives two (2) series:

- For y'',
- For $-x^2 y''$.

For the term $-2xy'$ use Eq. (5.3) and in $2y$ use Eq. (5.2). Write like powers of x vertically aligned for easy viewing. This gives:

$$\begin{aligned}
y'' &= 2a_2 + 6a_3 x + \quad 12a_4 x^2 + \quad 20a_5 x^3 + 30a_6 x^4 + \cdots \\
-xy'' &= \qquad\qquad\qquad - 2a_2 x^2 - 6a_3 x^3 - \ 12a_4 x^4 - \cdots \\
-2xy' &= \qquad - 2a_1 x - 4a_5 x^2 - \quad 6a_9 x^3 - \ 8a_4 x^4 - \ \cdots \\
2y &= 2a_0 + 2a_1 x + \quad 2a_2 x^2 + \quad 2a_3 x^3 + \ 2a_4 x^4 + \ \cdots .
\end{aligned}$$

Add terms of like powers of x. For each power x^0, x, x^2 equate the sum obtained to zero. Denote these sums by [0] (constant terms), [1] (first power of x), and so on:

Sum	Power	Equation
[0]	x^0	$a_2 = -a_0$
[1]	x	$a_3 = 0$
[2]	x^2	$14a_4 = 4a_2, \qquad a_4 = \frac{4}{12} a_2 = -\frac{1}{3} a_0$
[3]	x^3	$a_5 = 0 \quad \text{since} \quad a_3 = 0$
[4]	x^4	$30a_6 = 18a_4, \qquad a_6 = \frac{18}{30} a_4 = \frac{18}{30}\left(-\frac{1}{3}\right) a_0 = -\frac{1}{5} a_0$

Table 5.1.: Coefficient table for the example "*A Special Legendre Function*".

This gives the solution

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) \quad \blacksquare$$

a_0, a_1 remain **arbitrary**.

Therefore, this is a **general solution** consisting of two (2) solutions: x and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$.

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(\cdot)\, x$ and *Legendre functions* $Q_1(\cdot)\, x$.

Here we have $x = P_1(\cdot)\, x$ and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(\cdot)\, x$

The minus is by convention.

The index 1 is called the *order* of these functions and here the order is 1.

5.3 Legendre's Equation

5.3.1 Legendre Polynomials ($P_n(x)$)

Legendre's differential equation:

$$\left(1-x^2\right) y'' - 2xy' + n\left(n+1\right) y = 0 \tag{5.6}$$

is an important ODE in physics. It arises in numerous problems, particularly in boundary value problems for spheres.

The equation involves a **parameter** n, whose value depends on the physical or engineering problem. Therefore Eq. (5.6) is actually a whole family of ODEs. For $n = 1$ we solved it in the previous example.

> Any solution of Eq. (5.6) is called a **Legendre function**.

> The study of these and other higher functions not occurring in calculus is called the theory of special functions.

Dividing Eq. (5.6) by $1 - x^2$, we obtain the standard form:

$$y'' - \frac{2x}{\left(1-x^2\right)} y' + \frac{n\left(n+1\right)}{\left(1-x^2\right)} y = 0$$

We see that the coefficients $-2x/\left(1-x^2\right)$ and $n\left(n+1\right)/\left(1-x^2\right)$ of the new equation are analytic at $x = 0$, so the power series method is applicable for this equation.

Substituting:

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{5.7}$$

and its derivatives into Eq. (5.6), and denoting the constant $n\left(n+1\right)$ simply as k, we obtain the following:

$$\left(1-x^2\right) \sum_{m=2}^{\infty} m\left(m-1\right) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0.$$

By splitting the first expression as two (2) separate series we have the equation:

$$\sum_{m=2}^{\infty} m\left(m-1\right) a_m x^{m-2} - \sum_{m=2}^{\infty} m\left(m-1\right) a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x_n in all four (4) series, set $m - 2 = s$ (therefore $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives:

$$\sum_{s=0}^{\infty} \left(s+2\right)\left(s+1\right) a_{s+2} x^s - \sum_{s=2}^{\infty} s\left(s-1\right) a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

> Note the first series the summation begins with $s = 0$.

As this equation with the right side 0 must be an identity in x if Eq. (5.7) is to be a solution of Eq. (5.7), the sum of the coefficients of each power of x on the LHS must be zero.

Now x^0 occurs in the first and fourth series only, and gives:

remember $k = n(n+1)$

$$x^0 \qquad 2\cdot 1 a_2 + n(n+1)\, a_0 = 0, \tag{5.8}$$

$$x^1 \qquad 3\cdot 2 a_3 + \left[-2 + n(n+1)\right] a_1 = 0, \tag{5.9}$$

$$x^2,\ x_3,\ \cdots \qquad (s+2)(s+1)\, a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right] a_s = 0. \tag{5.10}$$

The expression in the brackets $[\cdots]$ can be simplified to $(n-s)(n+s+1)$.
Solving Eq. (5.8) for a_2 and Eq. (5.9) for a_3 as well as Eq. (5.10) for a_{s+2}, we obtain the **general formula**:

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad \text{for} \qquad s = 0,\ 1,\ 2,\ \cdots \tag{5.11}$$

This is called a **recurrence relation** or **recursion formula**. It gives each coefficient in terms of the 2nd one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively:

$$\begin{aligned}
a_2 &= -\frac{n(n+1)}{2\cdot 1} a_0 \\
a_3 &= -\frac{(n-1)(n+2)}{3\cdot 2} a_1 \\
a_4 &= -\frac{(n-2)(n+3)}{4\cdot 3} a_2 = \frac{(n-2)\, n(n+1)(n+3)}{4!} a_0 \\
a_5 &= -\frac{(n-3)(n+4)}{5\cdot 4} a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

By inserting these expressions for the coefficients into Eq. (5.7) we obtain:

$$y(x) = a_0 y_1(x) + a_1 y_2(x). \tag{5.12}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2\cdot 1} x^2 + \frac{(n-2)\, n(n+1)(n+3)}{4!} x^4 - \cdots \tag{5.13}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3\cdot 2} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots \tag{5.14}$$

These series converge for $|x| < 1$. As Eq. (5.13) contains **even** powers of x only, while Eq. (5.14) contains **odd** powers of x only, the ratio y_1/y_2 is not a **constant**, This means y_1 and y_2 are not proportional and are thus linearly independent solutions.
Therefore Eq. (5.12) is a general solution of Eq. (5.6) on the interval $-1 < x < 1$.

$x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardised ODE are no longer analytic.

Polynomial Solutions

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions.

For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials. For *Legendre's equation* this happens when the parameter n is a non-negative integer because the Right Hand Side (RHS) of Eq. (5.11) is zero for $s = n$, so that $a_{n+2} = 0,\ a_{n+4} = 0,\ a_{n+6} = 0, \cdots$. Therefore if n is even, $y_1(x)$ reduces to a polynomial of degree n.

If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of such constants is done as follows.

We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad \text{where} \qquad n \text{ is a positive integer.} \tag{5.15}$$

and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from Eq. (5.11), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leq n-2) \tag{5.16}$$

The choice Eq. (5.15) makes $p_n(1) = 1$ for every n which makes our lives easier. From Eq. (5.16) with $s = n-2$ and Eq. (5.15) we obtain:

$$a_{n-2} = -\frac{n(n-1)}{2(n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2}$$

Using $(2n)! = 2n(2n-1)(2n-2)!$ in the numerator and $n! = n(n-1)!$ and $n! = n(n-1)(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n(n-1)\,2n(2n-1)(2n-2)!}{2(2n-1)\,2^n n(n-1)!\,n(n-1)(n-2)!}.$$

$n(n-1)\,2n(2n-1)$ cancels out, which we get:

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

Similarly,

$$\begin{aligned} a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} \\ &= \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!} \end{aligned}$$

and so on, and in general, when $n - 2m \geq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}. \tag{5.17}$$

The resulting solution of Legendre's differential equation Eq. (5.6) is called the *Legendre polynomial of degree n* and is denoted by $P_n(x)$.:
From Eq. (5.17) we obtain:

$$P_n(x) = \sum_{m-0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \tag{5.18}$$

$$= \frac{(2m)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots \tag{5.19}$$

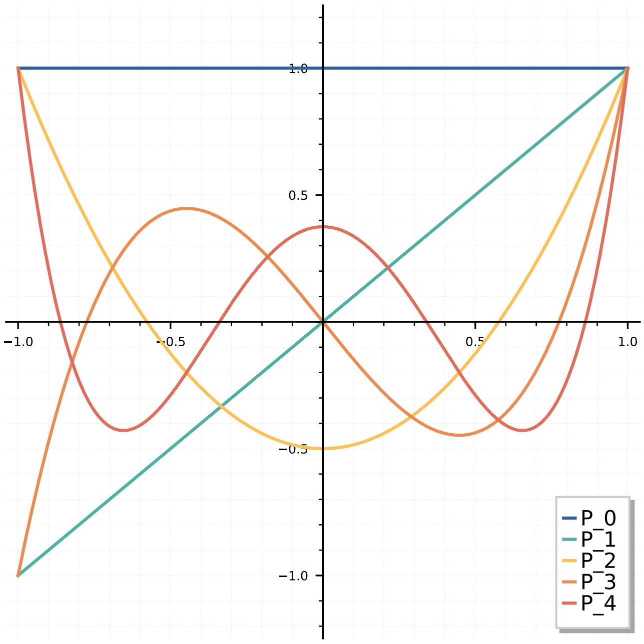


Figure 5.1.

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1), & P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

The Legendre polynomials $P_n(x)$ are o**rthogonal** on the interval $-1 \leqq x \leqq 1$, a basic property to be defined and used in making up "Fourier-Legendre series" which will be the focus for *Higher Mathematics II*.

5.4 Extended Power Series: Frobenius Method

Several 2nd-order ODEs are important for engineering applications.

One of the famous ones **Bessel Equation** will be our focus in the continuing section.

Unfortunately, these practical 2nd-order ODEs have coefficients that are not analytic, but are possible to solve via series method (power series times a logarithm or times a fractional power of x, etc.).

The following theorem permits an extension of the power series method.

The new method is called the **Frobenius method**.

Theory 5.2: Frobenius Method

Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{5.20}$$

has **at least one solution** that can be represented in the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{5.21}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE Eq. (5.20) also has a 2nd solution (such that these two solutions are linearly independent) that may be similar to Eq. (5.21) (with a different r and different coefficients) or may contain a logarithmic term.

To see this theorem in action, let's look at the Bessel's equation.

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad \text{where } \nu \text{ is a parameter}$$

is of the form Eq. (5.20) with:

$$b(x) = 1 \qquad c(x) = x^2 - \nu^2 \qquad \text{analytic at } x = 0 \qquad .$$

This form allows us to use the Frobenius method.

This ODE could **NOT** be handled in full generality by the power series method as these functions are known as hyper-geometric differential equations. Therefore, this equation (also known as hypergeometric differential equation) requires the Frobenius method.

In Eq. (5.21) we have a power series times a single power of x whose exponent r is not restricted to be a non-negative integer.

Regular and Singular Points

The following terms are practical and commonly used. A **regular point** of the ODE:

$$y'' + p(x) y' + q(x) y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a **regular point** of the ODE:

$$\widetilde{h}(x)y'' + \widetilde{p}(x)y'(x) + \widetilde{q}(x)y = 0,$$

is an x_0 at which $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$ are analytic and $\widetilde{h}(x_0) \neq 0$ (so what we can divide by $\widetilde{h}$ and get the previous standard form). Then the power series method can be applied. If x_0 is not a regular point, it is called a **singular point**.

5.4.1 Indicial Equation

Time to explain the *Frobenius method* for solving Eq. (5.20) which is the Bessel equation. Multiplication of Eq. (5.20) by x^2 gives the more convenient form which can be worked upon:

$$x^2 y'' + x b(x) y' + c(x) y = 0 \tag{5.22}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \qquad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

> If both $b(x)$ and $c(x)$ are polynomials, no actions are needed.

Then we differentiate Eq. (5.21) term by term, finding:

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \\ &= x^{r-1}[r a_0 + (r+1) a_1 x + \cdots] \\ y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} \\ &= x^{r-2}[r(r-1) a_0 + (r+1) r a_1 x + \cdots]. \end{aligned} \tag{5.23}$$

By inserting all these series into Eq. (5.22) we obtain:

$$x^r \left[r(r-1) a_0 + \cdots \right] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots) \tag{5.24}$$

$$+ (c_0 + c_1 x + \cdots) x^r (a_0 + a_1 x + \cdots) = 0. \tag{5.25}$$

We now equate the sum of the coefficients of each power x^r, x^{r+1}, x^{r+2}, ... to zero. This presents a system of equations involving the unknown coefficients a_m. The smallest power is x^r and the corresponding equation is:

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives:

$$r(r-1) + b_0 r + c_0 = 0 \tag{5.26}$$

This important quadratic equation is called the **indical equation** of the ODE Eq. (5.22).

Its role is as follows.

The Frobenius method present a basis of solutions. One of the two solutions will always be of the form Eq. (5.23), where r is a root of Eq. (5.26). The other solution will be of a form indicated by the indicial equation.

There are three (3) cases:

Case 1 Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2 A double root.

Case 3 Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are related to the *Euler-Cauchy equation*, the simplest ODE of the form Eq. (5.20).

Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i\mathrm{Im}\, r_1$ is imaginary, so it cannot be a real integer.

Case 2 we must have a logarithm, whereas in Case 3 we may or may not.

Theory 5.2: Frobenius Method II - The Three Cases

Assume the ODE in Eq. (5.22) satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation Eq. (5.26).

Then we have the following three (3) cases:

Case 1. Distinct Roots Not Differing by an Integer
A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.27}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.28}$$

with coefficients obtained successively from Eq. (5.24) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.
A basis is

$$y_1(x) = x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{5.29}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^{r}(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0) \tag{5.30}$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.31}$$

(of the same general form as before) and

$$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.32}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

5.4.2 Typical Applications

Technically, the *Frobenius method* is similar to the power series method, once the roots of the indicial equation have been determined.

However, Eq. (5.27)-Eq. (5.32) merely indicate the general form of a basis, and a 2^{nd} solution can often be obtained more rapidly by reduction of order.

Exercise 5.3: Euler-Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm

Solve the following ODE:

$$x^2 y'' + b_0 x y' + c_0 y = 0 \qquad (b_0,\ c_0 \text{ constant})$$

Solution

Substitution of $y = x^r$ gives the auxiliary equation:

$$r(r-1) + b_0 r + c_0 = 0,$$

which is the indicial equation. For different roots $r_1,\ r_2$ we get a basis $y_1 = x^{r_1},\ y_2 = x^{r_2}$, and for a double root r we get a basis $x^r,\ x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

Exercise 5.4: Example of Case II - Double Root

Solve the ODE

$$x(x-1)\,y'' + (3x-1)\,y'' + y = 0 \tag{5.33}$$

Solution

Writing Eq. (5.33) in the standard form Eq. (5.22):

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$
$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

we see it satisfies the assumptions in **Theorem 1** (i.e., analytic as $x = 0$). By inserting Eq. (5.23) and its derivatives Eq. (5.23) into Eq. (5.33) we obtain:

$$\begin{aligned} &\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} \\ &+3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \end{aligned} \tag{5.34}$$

The smallest power is x^{r-1}, occurring in the 2^{nd} and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]\,a_0 = 0, \qquad \text{therefore} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.

First Solution

Insert this value $r = 0$ into Eq. (5.34) and equate the sum of the coefficients of the power x^s to zero, obtaining:

$$s(s-1)a_s - (s+1)a_{s+1} + 3a_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

Second Solution

We get a 2^{nd} independent solution y_2 by the method of reduction of order, substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9). Sec. 2.1, which we shall use in this example, instead of stating reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of y' in (11) in standard form. By partial fractions,

$$-\int p\,dx = -\int \frac{3x-1}{3(x-1)}\,dx = -\int \left(\frac{2}{x-1} + \frac{1}{x}\right) dx = -2\ln(x-1) - \ln x.$$

Hence becomes

$$u' = U = y_1^{-2} e^{-\int y\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = w_1 = \frac{\ln x}{1-x}.$$

y_1 and y_2 are shown in Fig. 109. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$).

The Frobenius method solves the **hypergeometric equation**, whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

5.5 Bessel's Function

One of the most important ODEs in applied mathematics in **Bessel's equation** which it's form is shown as:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \tag{5.35}$$

Converting this to the traditional *Frobenius* form:

$$y'' + \frac{1}{x} y' + \frac{1 - \nu^2}{x^2} y = 0$$
$$b(x) = 1 \qquad c(x) = 1 - \nu^2.$$

where the parameter ν is a given **real number** which is positive or zero.

> Bessel's equation often in problems showing cylindrical symmetry or membranes.

According to the *Frobenius theory*, it has a solution of the form:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{5.36}$$

Substituting Eq. (5.36) and its 1st and 2nd derivatives into Bessel's equation, we obtain:

$$\sum_{m=0}^{\infty} (m + r)(m + r - 1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m + r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero.

> Note that this power x^{s+r} corresponds to $m = s$ in the first, 2nd, and fourth series, and to $m = s - 2$ in the third series.

Therefore, for $s = 0$ and $s = 1$, the third series does not contribute since $m \geq 0$. For $s = 2, 3, \cdots$ all four series contribute, so that we get a general formula for all these s. We find:

$$r(r - 1)a_0 + r a_0 - \nu^2 a_0 = 0 (s = 0) \tag{5.37}$$
$$(r + 1)r a_1 + (r + 1)a_1 - \nu^2 a_1 = 0 (s = 1) \tag{5.38}$$
$$(s + r)(s + r - 1)a_s + (s + r)a_s + a_{s-2} - \nu^2 a_s = 0 (s = 2, 3, \cdots) \tag{5.39}$$

From Eq. (5.37) we obtain the **indicical equation** by dropping a_0.

$$(r + \nu)(r - \nu) = 0 \tag{5.40}$$

The roots are $r_1 = \nu (\geq 0)$ and $r_2 = -\nu$.

Coefficient Recursion for $r = r_1 = \nu$

For $r = \nu$, Eq. (5.38) reduces to $(2\nu + 1)\, a_1 = 0$. Therefore $a_1 = 0$ as $\nu \geq 0$. Substituting $r = \nu$ in Eq. (5.39) and combining the three terms containing $a_s = 0$ gives simply:

$$(s + 2\nu)\, s a_s + a_{s-2} = 0 \tag{5.41}$$

As $a_1 = 0$ and $v \equiv 0$, it follows from Eq. (5.41), $a_3 = 0$, $a_5 = 0, \cdots$. Hence we have to deal only with even-numbered coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (5.41) becomes:

$$(2m + 2\nu)\, 2m a_{2m} + a_{2m-2} = 0$$

Solving for a_{2m} gives the recursion formula

$$a_{2m} = -\frac{1}{2^2 m\,(v + m)} a_{2m-2} \qquad m = 1,\ 2,\ \cdots \tag{5.42}$$

From Eq. (5.42) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(v + 1)}$$

$$a_4 = -\frac{a_2}{2^2 2(v + 2)} = \frac{a_0}{2^4 2!\,(v + 1)(v + 2)}$$

and so on, and in general:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v + 1)(v + 2)\cdots(v + m)}, \qquad m = 1, 2, \cdots. \tag{5.43}$$

5.5.1 Bessel Functions (J_n) for Integers

> Integer values of ν are denoted by n, which is the standard mathematical notation.

For $\nu = n$ the relation Eq. (5.43) becomes:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n + 1)(n + 2)\cdots(n + m)}, \qquad m = 1,\ 2, \cdots \tag{5.44}$$

a_0 is still arbitrary, so that the series Eq. (5.36) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function.

Accordingly, we have to make a choice.

The choice $a_0 = 1$ would be possible. A simpler series Eq. (5.36) could be obtained if we could absorb the growing product $(n+1)(n+2)\cdots(n+m)$ into a factorial function $(n+m)!$ What should be our choice? Our choice should be:

$$a_0 = \frac{1}{2^n n!} \tag{5.45}$$

because then $n!\,(n + 1)\cdots(n + m) = (n + m)!$ in Eq. (5.44), so that Eq. (5.44) simply becomes:

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n + m)!}, \; m = 1,\ 2, \cdots \tag{5.46}$$

By inserting these coefficients into Eq. (5.36) and remembering that $c_1 = 0$, $c_3 = 0, \cdots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n + m)!} \qquad (n \geqq 0). \tag{5.47}$$

$J_n(x)$ is called the **Bessel function of the first kind** of order n. The series Eq. (5.47) converges for all x, as the ratio test shows.

> $J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

Exercise 5.5: Bessel Function J_0 and J_1

Please calculate the bessel functions of $J_0(x)$ and $J_1(x)$.

Solution

For $n = 0$ we obtain from Eq. (5.47) the *Bessel function* of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} - \dots \tag{5.48}$$

which looks similar to a cosine. For $n = 1$ we obtain in the *Bessel function* of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2} + \frac{x^5}{2^5 2! 3} - \frac{x^7}{2^7 3! 4} - \dots \tag{5.49}$$

which looks similar to a sine (Fig. 110). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing z. Heuristically, n^2/x^2 in (1) in standard form (1) divided by x^2] is zero (if $n = 0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y' + y = 0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

where $\sim$ is read "asymptotically equal" and means that for fixed n the quotient of the two sides approaches 1 as $x \to \infty$ $\frac{x^2}{2^2(1!)^2}$.

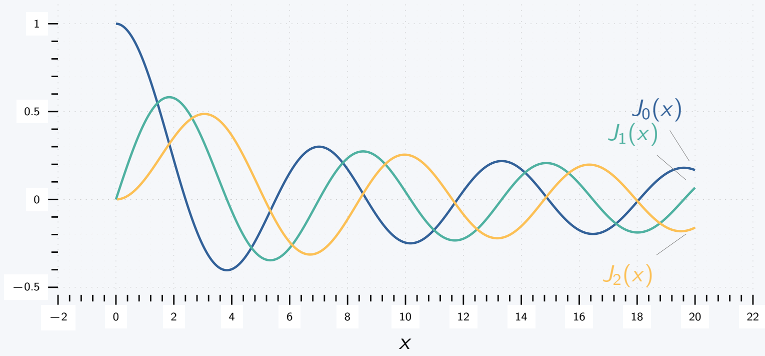


Figure 5.2.

Formula (14) is surprisingly accurate even for smaller x (>0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of f_0, you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

Chapter 6

Laplace Transform

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6.1 Introduction

Laplace transform are important for any engineer as it makes solving **linear** ODEs and related initial value problems, as well as systems of linear ODEs, much easier. There are numerous applications which can be significantly simplified such as:

- electrical networks,
- springs,
- mixing problems,
- signal processing,

and other areas of engineering and physics. The process of solving an ODE using Laplace transform consists of three (3) steps:

Part 1 The given ODE is transformed into an algebraic equation, called the **subsidiary equation**.

Part 2 The subsidiary equation is solved by purely algebraic manipulations.

Part 3 The solution in Step 2 is transformed back, resulting in the solution of the given problem.

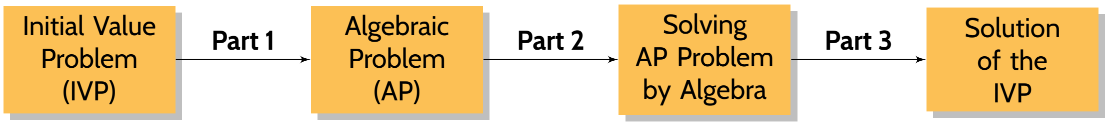

Figure 6.1.

The diagram explaining the thought process can be seen in **Fig.** 6.1 .

> The idea of Laplace transforms converting an ODE to an algebraic problem.

Laplace Transform has two (2) major advantages over the previous methods.

1. **Problems are solved more directly:** Initial value problems are solved without first determining a general solution. Non-homogeneous ODEs are solved without first solving the corresponding homogeneous ODE.

2. **Solving Discontinuities:** More importantly, the use of the unit step function (**Heaviside** function) and Dirac's **delta** [1] make the method particularly powerful for problems with inputs with discontinuities or represent short impulses or complicated periodic functions.

[1] Both these functions play pivotal roles in solving engineering problems, particularly control theory, signal processing, and electrodynamics.

6.2 First Shifting Theorem (s-Shifting)

Laplace transform, when applied to a function, **changes the function into a new function** by using a process involving integration.

If $f(t)$ is a function defined for all $t \geq 0$, its **Laplace transform** is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. This operation results in a function of s, (i.e., $F(s)$), and denoted as $\mathcal{L}(f)$ [2]:

[2] This symbol is known as cursive L.

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) \, dt \tag{6.1}$$

> Here we need to assume $f(t)$ **has** an integral[3] i(i.e., it is finite). This assumption is usually satisfied in **practical** engineering applications.

[3] Some functions have no integration such as e^{x^n} where $n > 1$. Interestingly the integral of e^{x^2} is called the error function and is an important function in error correction.

Not only is the result $F(s)$ called the **Laplace transform**, but the operation just described, which gives $F(s)$ from a given $f(t)$, is also called the **Laplace transform**. It is an integral transform:

$$F(s) = \int_0^\infty k(s, t) \, f(t) \, dt$$

with a kernel[4] defined as $k(s, t) = e^{-st}$:

[4] Here kernel can be used to define any number of operations for a different transform.

> Laplace transform is called an integral transform because it transforms a function in one space to a function in another space by a process of integration which involves a **kernel**.

The kernel is a function of the variables in two (2) spaces and defines the transform. Furthermore, the given function $f(t)$ in Eq. (6.1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$:

$$f(t) = \mathcal{L}^{-1}(F). \tag{6.2}$$

Note Eq. (6.1) and Eq. (6.2) together imply:

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f, \quad \mathcal{L}(\mathcal{L}^{-1}(F)) = F$$

Notation

Original functions depend on t and their transforms on s. Original functions are denoted by **lowercase letters** and their transforms by the same letters in **capital**, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$.

Exercise 6.1: Introduction to Laplace Transform

Let $f(t) = 1$ when $t \geq 0$.

Find $F(s)$.

Solution

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt$$
$$= -\frac{1}{s}e^{-st}\bigg|_0^\infty = \frac{1}{s} \qquad (s > 0)$$

Such an integral is called an improper integral and, is evaluated according to the rule:

$$int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt.$$

Therefore our convention notation means:

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^T$$
$$= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}e^{0}\right] = \frac{1}{s}$$

where $(s > 0)$ ■

Exercise 6.2: Laplace Transform of an Exponential Function

Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant.

Find $\mathcal{L}(f)$.

Solution

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{a-s} e^{-(s-a)t}\bigg|_0^\infty$$

Therefore, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad ■.$$

Linearity

The Laplace transform is a **linear operation**. This means, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, The following statement holds true[5]:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

[5]This is only valid when we add two operations. Multiplying them would not give a correct equivalence.

Exercise 6.3: Hyperbolic Functions

Find the transforms of:

$$\cosh at \quad \text{and} \quad \sinh at$$

Solution

As $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we can obtain the definitions of them using the exponential function definition from an earlier example.

$$\begin{aligned}\mathcal{L}(\cosh at) &= \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) \\ &= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2} \\ \mathcal{L}(\sinh at) &= \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) \\ &= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \quad \blacksquare\end{aligned}$$

Exercise 6.4: Cosine and Sine

Derive the following formulas:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution

We start by write $L_e = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain:

$$\begin{aligned}L_e &= \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{1}{s} - \frac{\omega}{s} L_s, \\ L_s &= \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} L_c.\end{aligned}$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain[6]:

$$\begin{aligned}L_c &= \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \quad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \quad L_c = \frac{s}{s^2 + \omega^2} \quad \blacksquare \\ L_s &= \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \quad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \quad L_s = \frac{\omega}{s^2 + \omega^2} \quad \blacksquare\end{aligned}$$

[6]Of course, this is not the only method as one can just use one of the operations twice over tho get the same result

6.2.1 Replacing s by s - a in the Transform

The Laplace transform has an advantageous property. If we know the transform of $f(t)$, we can immediately get that of $e^{at} f(t)$. Let's write this in a formal theorem.

Theorem: *s-Shifting*

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s-a)$ (where $s - a > k$).

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\} \tag{6.3}$$

Exercise 6.5: Damped Vibrations

Using the below definitions:

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2} \quad \text{and} \quad \mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

Find the inverse of the transform:

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}.$$

Solution

Applying the inverse transform, and using its linearity, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t) \quad \blacksquare$$

Theorem: *Existence and Uniqueness of Laplace Transforms*

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies:

$$\mathbf{p}f(t) \leq Me^{kt}$$

For all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s \geqq k$.

6.3 Transforming Derivatives and Integrals

> The Laplace transform is a method of solving ODEs and IVPs.

The idea is to replace operations of calculus on functions by operations of algebra. Roughly, differentiation of $f(t)$ will correspond to multiplication of $\mathcal{L}(f)$ by s and integration of $f(t)$ to division of $\mathcal{L}(f)$ by s.

To solve ODEs, we must first consider the Laplace transform of derivatives.

You might have encountered this idea previously in logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms. To simplify calculations was one of the main reasons that logarithms were invented.

Theorem: *Derivatives*

First and Second Order Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

These hold if $f(t)$ is continuous for all $t \equiv 0$ and satisfies the growth restriction and $f'(t)$, $f''(t)$ are piece-wise continuous on every finite interval on the semi-axis $t \equiv 0$.

Higher Order Derivatives

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}(f^{(n)}) = s^{n}\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

6.3.1 Laplace Transform a Function Integral

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

Theorem: *Laplace Transform of an Integral*

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{therefore} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Exercise 6.6: Inverse using Integration

Find the inverse of the following functions

$$\frac{1}{s(s^2+\omega^2)} \quad \text{and} \quad \frac{1}{s^2(s^2+\omega^2)}$$

Solution

Using a standard Laplace Transform table we obtain the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^1 \frac{\sin \omega t}{\omega}\, d\tau = \frac{1}{\omega^2}(1-\cos \omega t).$$

The second one we obtain as the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos \omega\tau) d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin \omega\tau}{\omega^2}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin \omega\tau}{\omega^3}.$$

6.3.2 Differential Equations with Initial Values

It's time to discuss how the Laplace Transform method solves ODEs and IVPs. Consider the following IVP:

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are **constants**. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained.

In Laplace's method we do three (3) steps:

Step 1 Setting up the subsidiary equation This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming, namely:

$$[s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2 Solution of the subsidiary equation by algebra We divide by $s^2 + as + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

> Q is often denoted by H, but we need H much more frequently for other purposes[7].

[7] For example in control theory and signal processing.

This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$. Therefore

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

> Q depends neither on $r(t)$ nor on the initial conditions (but only on a and b).

Step 3 Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$ We reduce our $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables so that we obtain the solution $y(t)$.

> **Exercise** 6.7: IVP - A Basic Laplace Transform

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution

Step 1: Using a Standard Laplace Transform we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2 Y - sy(0) - y'(0) - Y = 1/s^2,$$

therefore:

$$(s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2: The transfer function is $Q = 1/(s^2 - 1)$, and becomes

$$Y = (s+1)Q + \frac{1}{s^2} Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}.$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2} \right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t \quad \blacksquare$$

The solution method to the aforementioned example can be seen in **Fig.** 6.2 .

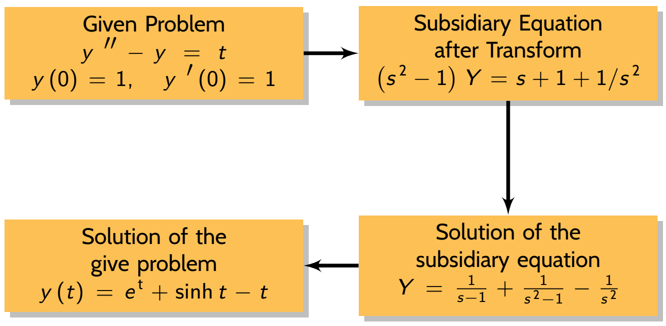

Figure 6.2.: The operational steps of laplace transform used in generating a IVP solution.

Exercise 6.8: Comparison with Previous Methods

Solve the IVP:

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0$$

Solution

We see that the subsidiary equation is:

$$s^2 Y - 0.16s + sY - 0.16 + 9Y = 0,$$

Therefore

$$\left(s^2 + s + 9\right) Y = 0.16\,(s+1)$$

The solution to the algebraic equation is:

$$Y = \frac{0.16\,(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}.$$

Therefore by the first shifting theorem and the formulas for cos and sin in from this Laplace Transform table we obtain:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y) \\ &= e^{-t/2}\left(0.16 \cos \sqrt{\frac{35}{4}}\, t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin \sqrt{\frac{35}{4}}\, t\right) \\ &= e^{-0.5\,t}\,(0.16 \cos 2.96t + 0.027 \sin 2.96t) \quad \blacksquare \end{aligned}$$

Exercise 6.9: Shifted Data

As a note for this question, shifted data means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$.

For such a problem, set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied.

For instance, solve:

$$y'' + y = 2t, \qquad y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \qquad y'\left(\tfrac{1}{4}\pi\right) = 2 - \sqrt{2}.$$

Solution

We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem becomes:

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{1}{4}\pi\right)$$
$$\tilde{y}(0) = \frac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}.$$

where $\tilde{y}\left(\tilde{t}\right) = y(t)$. Using a standard Laplace Transform table and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the **shifted** initial value problem is:

$$s^2\tilde{Y} - s\left(\frac{1}{2}\pi\right) - \left(2 - \sqrt{2}\right) + \tilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\pi\right)}{s}$$

Therefore:

$$\left(s^2 + 1\right)\tilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\right)}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this *algebraically* for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{\left(s^2+1\right)s^2} + \frac{\frac{1}{2}\pi}{\left(s^2+1\right)s} + \frac{\frac{1}{2}\pi s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}$$

The inverse of the first two terms can be seen from a previous example (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned}
\tilde{y} &= \mathcal{L}^{-1}\left(\tilde{Y}\right) \\
&= 2\left(\tilde{t} - \sin\tilde{t}\right) + \frac{1}{2}\pi\left(1 - \cos\tilde{t}\right) \\
&\quad + \frac{1}{2}\pi\cos\tilde{t} + \left(2 - \sqrt{2}\right)\sin\tilde{t} \\
&= 2\tilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\tilde{t}.
\end{aligned}$$

Now $\tilde{t} = t - 1/4\pi$, $\sin\tilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is:

$$y = 2t - \sin t + \cos t \quad \blacksquare$$

6.4 Unit Step Function (t - Shifting)

This section and the next one are extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach of Chap. 2. The reason is that we shall introduce two auxiliary functions, the unit step function or Heaviside function[8] $u(t-a)$ (below) and Dirac's delta $\delta(t-a)$. These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (for example, a hammer hitting an object).

[8] Named after Oliver Heaviside, an English self-taught mathematician and physicist who invented a new technique for solving differential equations (equivalent to the Laplace transform), independently developed vector calculus, and rewrote Maxwell's equations in the form commonly used today.

6.4.1 Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t > a \end{cases} \qquad \text{where} \qquad a \geq 0.$$

The transform of $u(t-a)$, heaviside function, follows directly from the defining integral for the Laplace transform:

$$\mathcal{L}\left\{u(t-a)\right\} = \int_0^\infty e^{-st}u(t-a)\,\mathrm{d}t = \int_0^\infty e^{-st}\cdot 1\,\mathrm{d}t = -\frac{e^{-st}}{s}\Bigg|_{t-a}^{\infty};$$

here the integration begins at $t = a(\geq 0)$ as $u(t-a)$ is 0 for $t < a$. This definition allows us the following simplification:

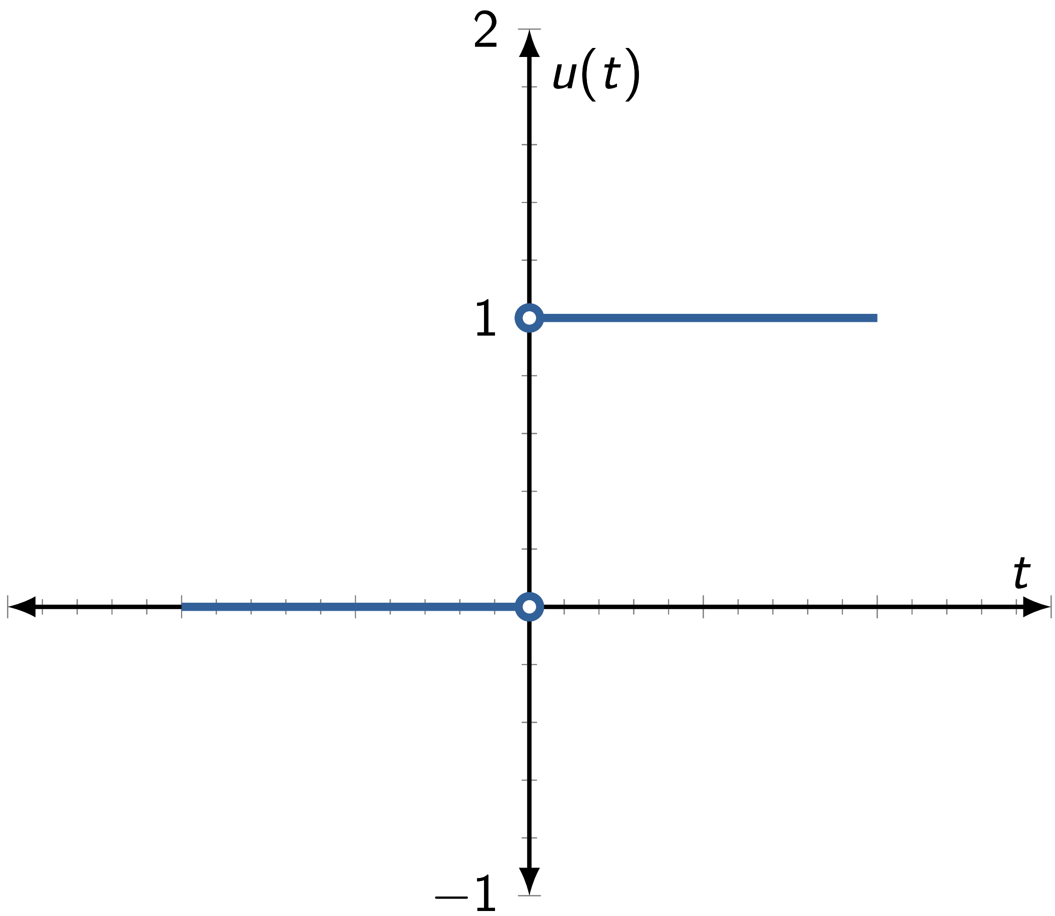

Figure 6.3.

$$\mathcal{L}\left\{u\left(t-a\right)\right\} = \frac{e^{-as}}{s} \qquad \text{where} \qquad s > 0 \tag{6.4}$$

The unit step function is a typical engineering function made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either `off` or on Multiplying functions $f(t)$ with $u(t-a)$, we can produce all sorts of effects.

Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.

6.4.2 Time Shifting (t-Shifting): Replacing t by t - a in f(t)

The first shifting theorem ("s-shifting") in Sec. 6.1 concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s-a) = \mathcal{L}\{e^{2at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t-a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

The Second Shifting Theorem

If $f(t)$ has the transform $F(s)$, then the "shifted function"

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if} t < a \\ f(t-a) & \text{if} t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)\,. \tag{6.5}$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\{\,e^{-as}F(s)\}. \tag{6.6}$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. The transform of 5 sin t is $F(s) = 5/(s^2+1)$, therefore the shifted function 5 sin $(t-2)u(t-2)$ shown in Fig. 120C has the transform

$$e^{-2s}F(s) = 5e^{-2s}/(s^2+1).$$

Exercise 6.10: Use of Unit Step Functions

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if} 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if} 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if} \quad t > \frac{1}{2}\pi. \end{cases}$$

(Fig. 122)

Solution

Step 1 In terms of unit step functions,

$$f(t) = 2(1-u(t-1)) + \frac{1}{2}t^2(u(t-1) - u(t-\frac{1}{2}\pi)) + (\cos f u(t-\frac{1}{2}\pi)).$$

Indeed, $2(1-u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2 To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1-u(t-1))$ remains as it is and gives the transform $2(1-e^{-a})/s$. Then

$$\mathcal{R}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{R}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-a}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t-\frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\right\}$$

$$= \left(\frac{1}{s^2} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{\cos\theta\, u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t - a)$ is inconvenient, replace it by

$$\mathcal{L}\{f(t)u(t - a)\} = e^{-at}\mathcal{L}\{f(t + a)\}.$$

Exercise 6.11: Application of Both Shifting Theorems

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-2s}}{(s+2)^2}.$$

Solution

Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} as $1/s^{-2}$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the *first shifting theorem*. Therefore by the *second shifting theorem* (t-shifting),

$$\begin{aligned} f(t) &= \frac{1}{\pi}\sin\big(\pi(t-1)\big)\,u(t-1) \\ &+ \frac{1}{\pi}\sin\big(\pi(t-2)\big)\,u(t-2) \\ &+ (t-3)\,e^{-2(t-3)}\,u(t-3) \end{aligned}$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so the first and second terms cancel each other when $t > 2$. Therefore we obtain:

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 < t < 1, \\ -(\sin \pi t)/\pi & \text{if} \quad 1 < t < 2, \\ 0 & \text{if} \quad 2 < t < 3, \\ (t-3)\,e^{-2(t-3)} & \text{if} \quad t > 3 \end{cases}$$ ■

Exercise 6.12: Response of an RC-Circuit to a Single Rectangular Wave

Find the current $i(t)$ in the RC-circuit if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent[9] before the wave is applied.

[9] In this state, it means it is in a state or period of inactivity or dormancy.

Solution

The input is $V_0\left(u(t-a) - u(t-b)\right)$. Therefore the circuit is modeled by the integro-differential equation

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau = v(t) = V_0\left(u(t-a) - u(t-b)\right)$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}\left(e^{-as} - e^{-bs}\right)$$

Solving this equation algebraically for $I(s)$, we get:

$$I(s) = F(s)\left(e^{-as} - e^{-bs}\right) \quad \text{where} \quad F(s) = \frac{V_0 IR}{s + 1/(RC)} \quad \text{and} \quad \mathcal{L}^{-1}(F) = \frac{V_0}{R}e^{-t(1/RC)}$$

6.5 Dirac Delta Function

Imagine an air-plane making a hard landing, or a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in every-day life.

They are phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time. We can model such phenomena and problems by **Dirac's delta function**, and solve them very effecively by the Laplace transform.

To model situations of that type, we consider this function:

$$f_{\mathrm{k}}(t-a) = \begin{cases} 1/k & \text{if} \quad a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small.

In mechanics, the integral of a force acting over a time interval $a \leq t \leq a+k$ is called the impulse of the force. Similarly for electromotive forces $E(t)$ acting on circuits.

$$I_{\mathrm{k}} = \int_0^{\infty} f_{\mathrm{k}}(t-a)\,\mathrm{d}t = \int_a^{a+k} \frac{1}{k}\,\mathrm{d}t = 1$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_{k} as $k \to 0\,(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k \to 0} f_{\mathrm{k}}(t-a) \tag{6.7}$$

where $\delta(t-a)$ is called the Dirac delta function, or the **unit impulse function**.

Exercise 6.13: Mass-Sprint System Under a Square Wave

Determine the response of the damped mass-spring system under a square wave.

$$y'' + 3y' + 2y = u(t-1) - u(t-2),$$
$$y(0) = 0, \qquad y'(0) = 0.$$

Solution

nFrom (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

We have seen that $Y(s)$ is bounded from s-direction. Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$s^2Y + 3sY + 2Y = \frac{1}{s}\left(e^{-s} - e^{-2s}\right).$$ Solution $$Y(s) = \frac{1}{s(s^2+3s+2)}\left(e^{-s} - e^{-2s}\right)$$ $$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

6.6 Convolution

Convolution has to do with the multiplication of transforms. The situation is as follows. Addition of transforms provides no problem; we know that $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$.
Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product $\mathcal{L}(f)\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false. The transform of a product is generally different from the product of the transforms of the factors

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g)$$

in general.
To see this take $f = e^t$ and $g = 1$. Then $fg_e = e^t$, $\mathcal{L}(fg) = 1/(s-1)$, but $\mathcal{L}(f) = 1/(s-1)$ and $\mathcal{L}(1) = 1/s$ give $\mathcal{L}(f)\mathcal{L}(g) = 1/(s^2 - s)$.

According to the next theorem, the correct answer is that $\mathcal{L}(f)\mathcal{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f * g$ and defined by the integral.

$$h(t) = (f * g)(t) = \int_0^t f(t)\, g(t - \tau)\, \mathrm{d}\tau. \tag{6.8}$$

Exercise 6.14: Convolution - I

Let $H(s) = 1/\left[(s-a)\,s\right]$. Find $h(t)$.

Solution

$1/(s-a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{at}$ and $g(t-\tau) = 1$ we thus obtain from Eq. (6.8) the answer:

$$\begin{aligned} h(t) = e^{at} * 1 &= \int_0^t e^{a\tau} \cdot 1\, \mathrm{d}\tau \\ &= \frac{1}{a}\left(e^{at} - 1\right) \end{aligned}$$

To check the above result we can calculate:

$$\begin{aligned} H(s) = \mathcal{L}(h)(s) &= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) \\ &= \frac{1}{s} \cdot \frac{a}{s^2 - as} \\ &\frac{1}{s-a} \cdot \frac{1}{s} \\ &\mathcal{L}\left(e^{at}\right)\mathcal{L}(1) \quad \blacksquare \end{aligned}$$

Part II.

Linear Algebra & Vector Calculus

Chapter 7

Eigenvalue Problems

Table of Contents

7.1 Introduction

A matrix eigenvalue problem considers the following vector equation:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}. \tag{7.1}$$

Here $\mathbf{A}$ is a given **square matrix**[1], λ an unknown scalar, and $\mathbf{x}$ an unknown vector. In a matrix eigenvalue problem, the task is to determine λ's and $\mathbf{x}$'s that satisfy Eq. (7.1).

[1] Eigenvalue problems are only solvable for square matrices.

> As $\mathbf{x} = 0$ is always a solution for any λ, we only admit solutions with $\mathbf{x} \neq 0$ as the other solutions are considered trivial.

The solutions to Eq. (7.1) are given the following names:

1. The λ values satisfying Eq. (7.1) are called **eigenvalues of A**
2. The corresponding non-zero $\mathbf{x}$ values also satisfying Eq. (7.1) are called **eigenvectors of A**

From this simple looking vector equation comes a significant amount of relevant theory and numerous applications. Eigenvalue problems come up in engineering, physics, geometry, numerics, theoretical mathematics, biology, environmental science, urban planning, economics, psychology, and other areas.

7.2 The Eigenvalue Problem

7.2.1 Determining Eigenvalues and Eigenvectors

Let's consider multiplying **non-zero** vectors by a given square matrix[2], such as:

[2] Remember, it is a matrix with a size of $n \times n$.

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix}, \qquad \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}.$$

Our goal here is to see the influence the multiplication of the given matrix has on the vectors.

In the first case (LHS), we get a new vector with a different direction and different length when compared to the original vector. This is what usually happens and is of no interest here.

In the second case (RHS) something interesting happens. The multiplication produces a vector $[30 \quad 40]^T = 10\,[3 \quad 4]^T$. This means the new vector has the same direction as the original vector. The scale constant here, which we denote by λ is 10.

> The problem of systematically finding such λ's and non-zero vectors for a given square matrix will be major theme for this chapter. It is called the **matrix eigenvalue** problem or, more commonly among engineers, the **eigenvalue** problem.

Let's formulate our observation into a mathematical expression. Let $\mathbf{A} = \left[a_{jk}\right]$ be a given **non-zero** square matrix of dimension $n \times n$. Consider the following vector equation:

$$\mathbf{A}x = \lambda x. \tag{7.2}$$

The problem of finding non-zero **x**'s and λ's that satisfy equation Eq. (7.2.1) is called an eigenvalue problem.

Let's elaborate on this topic and introduce more terminology.

- A value of λ, for which Eq. (7.2.1) has a solution $\mathbf{x} \neq 0$, is called an **eigenvalue** or *characteristic value* of the matrix **A**. Another term for λ is a *latent* root.
- The corresponding solutions $\mathbf{x} \neq 0$ of Eq. (7.2.1) are called the **eigenvectors** or *characteristic vectors* of **A** corresponding to that eigenvalue λ.
- The set of all the eigenvalues of **A** is called the **spectrum** of **A**. We shall see that the spectrum consists of at least one eigenvalue and at most of n numerically different eigenvalues.
- The largest of the absolute values of the eigenvalues of **A** is called the *spectral radius* of **A**, a name to be motivated later.

7.2.2 The Process of Finding Eigenvalues and Eigenvectors

Now, with the new terminology for Eq. (7.2.1) out of the way, we can just say that the problem of determining the eigenvalues, and eigenvectors of a matrix is called an eigenvalue problem.[3]

[3] In this chapter we are considering an algebraic eigenvalue problem. More will be discussed in **Higher Mathematics II**.

Since, from the viewpoint of engineering applications, eigenvalue problems are the most important problems in connection with matrices and therefore let's start with a simple example.

Exercise 7.1: Determining Eigenvalues and Eigenvectors

Calculate the eigenvalue of the following matrix.

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

Solution

We start solving by first determining the eigenvalues:

$$\mathbf{A}x = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In component form:

$$\begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 \end{aligned}$$

Tidying up, we get the following:

$$\begin{aligned} (-5-\lambda)\, x_1 + 2x_2 &= 0 \\ 2x_1 + (-2-\lambda)\, x_2 &= 0. \end{aligned}$$

This can be written in matrix notation:

$$(\mathbf{A} - \lambda \mathbf{I})\, \mathbf{x} = 0$$

As Eq. (7.2.1) is $\mathbf{A}x - \lambda \mathbf{x} = \mathbf{A}x - \lambda \mathbf{I}\mathbf{x} = 0$, this gives the equation above. We see that this is a *homogeneous*[4] linear system. By Cramer's theorem, it has a non-trivial solution $\mathbf{x} \neq 0$ (an eigenvector of $\mathbf{A}$ we are looking for) if and only if its **coefficient determinant is zero**, that is,

$$\begin{aligned} D(\lambda) = \det(\mathbf{A} - \lambda I) &= \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\ &= (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0. \end{aligned}$$

We call $D(\lambda)$ the **characteristic determinant** or, if it is expanded, the **characteristic polynomial**, and $D(\lambda) = 0$ the **characteristic equation** of $\mathbf{A}$. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$.

These are the eigenvalues of $\mathbf{A}$.

Now we have calculated our eigenvalues, it is time for the eigenvectors. First, let's find the eigenvector of $\mathbf{A}$ corresponding to λ_1. This vector is obtained using $\lambda = \lambda_1 = -1$, that is,

$$\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - x_2 &= 0. \end{aligned}$$

A solution is $x_2 = 2x_1$, as we see from either of the two (2) equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We can do a check if our result is correct.

$$\mathbf{A}x_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\mathbf{x}_1 = \lambda_1 \mathbf{x}_1.$$

Time for the second equation. Eigenvector of $\mathbf{A}$ corresponding to λ_2. For $\lambda = \lambda_2 = -6$, it becomes

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0. \end{aligned}$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Therefore, an eigenvector of $\mathbf{A}$ corresponding to $\lambda_2 = -6$ is:

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

To be complete, let's do the check for this as well:

$$\begin{aligned} \mathbf{A}x_2 &= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} \\ &= (-6)\, \mathbf{x}_2 = \lambda \mathbf{x}_2 \quad \blacksquare \end{aligned}$$

[4] Remember, this means the RHS is 0.

Now we have an exercise under our belt, let's start with a gentle theory.

Theory 7.1: Eigenvalues

The eigenvalues of a square matrix $\mathbf{A}$ are the roots of the characteristic equation of $\mathbf{A}$. Therefore, a $n \times n$ matrix has **at least one eigenvalue** and at most n numerically different eigenvalues.

In an eigenvalue problem, the first action to take is to determine the eigenvalues. Once these are known, corresponding eigenvectors are obtained from the system of linear equations, for instance, by the Gauss elimination, where λ is the eigenvalue for which an eigenvector is wanted.

Eigenvectors have the following properties which are worth mentioning.

[5] This condition is imposed because we would have infinite solutions if not.

Theory 7.1: Eigenvectors & Eigenspaces

If $\mathbf{w}$ and $\mathbf{x}$ are eigenvectors of a matrix $\mathbf{A}$ corresponding to the same eigenvalue λ, so are $\mathbf{w}+\mathbf{x}$, provided $\mathbf{x} \neq -\mathbf{w}$[5], and $k\mathbf{x}$ for any $k \neq 0$. Therefore, the eigenvectors corresponding to one and the same eigenvalue λ of $\mathbf{A}$, together with 0, form a vector space, called the **eigenspace** of $\mathbf{A}$ corresponding to that λ.

Exercise 7.2: Multiple Eigenvalues

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Solution

For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^2 - \lambda^2 + 21\lambda + 45 = 0.$$

The roots (i.e., the eigenvalues of $\mathbf{A}$) are $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$. To find eigenvectors, we apply to Gauss elimination to the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, first with $\lambda = 5$ and then with $\lambda = -3$. For $\lambda = 5$ the characteristic matrix is

$$\begin{aligned} \mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} \\ &= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \quad \text{Row} - \text{reduction} \\ &= \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

[6] It has two (2) independent rows

As can be seen, it has *rank*[6] of two (2) .

Choosing $x_3 = -1$ we have $x_2 = 2$ from:

$$-\frac{24}{2}x_2 - \frac{48}{2}x_3 = 0$$

and then $x_3 = 1$ from:

$$-7x_1 + 2x_2 - 3x_3 = 0$$

Hence an eigenvector of $\mathbf{A}$ corresponding to $\lambda = 5$ is $\mathbf{x}_3 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For $\lambda = -3$ the characteristic matrix

$$\begin{aligned} \mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} \\ &= \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \quad \text{Row} - \text{reduction} \\ &= \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

It has a rank of one (1) . From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$, we obtain two linearly independent eigenvectors of $\mathbf{A}$ corresponding to $\lambda = -3$ [as they must exist by (5), Sec. 7.5, with rank $= 1$ and $n = 3$],

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}. \quad \blacksquare$$

The order M_λ of an eigenvalue λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ. The number m_λ of linearly independent eigenvectors corresponding to λ is called the **geometric multiplicity** of λ.

Thus m_λ is the dimension of the eigenspace corresponding to this λ.

In simple terms, **algebraic multiplicity** refers to the highest order an eigenvalue has, whereas **geometric multiplicity** is the dimension of the eigenvalue.

As the characteristic polynomial has degree n, the sum of all the algebraic multiplicities must equal n. In Example 2 for $\lambda = -3$ we have $m_\lambda = M_\lambda = 2$. In general, $m_\lambda \cong M_\lambda$, as can be shown. The difference $\Delta_\lambda = M_\lambda - m_\lambda$ is called the **defect** of λ. Thus $\Delta_{-3} = 0$ in Example 2, but positive defects Δ_λ can easily occur:

Exercise 7.3: Algebraic Multiplicity, Geometric Multiplicity, Positive Defect

The matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Has the following characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

From this equation we can see $\lambda = 0$ is an eigenvalue of algebraic multiplicity $M_0 = 2$ as the lambda value o λ_0 has the polynomial power of two (2) .

But its geometric multiplicity is only $m_0 = 1$, since eigenvectors result from:

$$-0x_1 + x_2 = 0$$

which makes $x_2 = 0$, in the form $\begin{bmatrix} x_1 & 0 \end{bmatrix}$. Hence for $\mathbf{A} = 0$ the defect is $\Delta_0 = 1$. ■

As a second example let's look at the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$$

Has the following characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = \left(3 - \lambda^2\right)^2 = 0$$

Hence $\lambda = 3$ is an eigenvalue of algebraic multiplicity $M_3 = 2$, but its geometric multiplicity is only $m_3 = 1$, since eigenvectors result from $0x_1 + 2x_2 = 0$ in the form $[x_1 \quad 0]^T$ ■.

Exercise 7.4: Real Matrices with Complex Eigenvalues and vectors

Given real polynomials may have complex roots[7], a real matrix may have complex eigenvalues and eigenvectors. For instance, the characteristic equation of the skew-symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(\mathbf{A} - \mathbf{A}\mathbf{I}) = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda_2 + 1 = 0.$$

It gives the eigenvalues of:

$$\lambda_1 = \mathbf{j} \quad \text{and} \quad \lambda_2 = -\mathbf{j}$$

Eigenvectors are obtained from:

$$-\mathbf{j}x_1 + x_2 = 0 \quad \text{and} \quad \mathbf{j}x_1 + x_2 = 0$$

respectively, and we can choose $x_1 = 1$ to get

$$\begin{bmatrix} 1 \\ \mathbf{j} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -\mathbf{j} \end{bmatrix} \quad ■$$

In the next section we shall need the following simple theorem.

[7] which occurs in conjugate pairs

Theory 7.4: Eigenvalues of the Transpose

The transpose $\mathbf{A}^T$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

7.3 Eigenvalue Applications

Let's look at some examples where eigenvalues and eigenvectors play an important role.

Exercise 7.5: Eigenvalues and Markov Processes

Markov processes as considered in the previous chapters lead to eigenvalue problems if we ask for the limit state of the process in which the state vector $\mathbf{x}$ is reproduced under the multiplication by the stochastic matrix $\mathbf{A}$. governing process, that is, $\mathbf{A}\mathbf{x} = \mathbf{x}$. Hence $\mathbf{A}$ should have the eigenvalue 1, and $\mathbf{x}$ should be a corresponding eigenvector. This is of practical interest because it shows the long-term tendency of the development modeled by the process.

In that example,

$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}.$$

$$\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $\mathbf{A}^{\mathsf{T}}$ has the eigenvalue 1, and the same is true for $\mathbf{A}$. An eigenvector $\mathbf{x}$ of $\mathbf{A}$ for $v\lambda = 1$ is obtained from:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix} \quad \text{row} - \text{reduction}$$

$$= \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} & 0 \\ 0 & -\frac{3}{20} & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

Taking $x_3 = 1$, we get $x_2 = 6$ from $-x_2/30 + x_3/5 = 0$ and then $x_1 = 2$ from $-3x_1/10 + x_2/10 = 0$. This gives $\mathbf{x} = [2 \;\; 6 \;\; 1]^T$. It means that in the long run, the ratio Commercial/industrial-Residential will approach 2:6-1, provided that the probabilities given by $\mathbf{A}$ remain

Exercise 7.6: Vibrating Systems of Two Masses on Two Springs

Mass-spring systems involving several masses and springs can be treated as eigenvalue problems. For instance, the following mechanical system is governed by the system of ODEs:

$$\begin{aligned} y_1'' &= -3y_1 - 2\left(y_1 - y_2\right) = -5y_1 + 2y_2 \\ y_1'' &= -2\left(y_2 - y_1\right) = 2y_1 - 2y_2 \end{aligned}$$

where y_1 and y_2 are the displacements of the masses from rest and primes denote derivatives with respect to time t. In vector form, this becomes

$$y'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We try a vector solution of the form

$$\mathbf{y} = \mathbf{x}e^{\omega t}.$$

This is suggested by a mechanical system of a single mass on a spring, whose motion is given by exponential functions (and sines and cosines). Substitution into (7) gives

$$\omega^2\mathbf{x}e^{\omega t} = A\mathbf{x}e^{\omega t}.$$

Dividing by $e^{\omega t}$ and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem:

$$\mathbf{A}x = \lambda\mathbf{x} \quad \text{where} \quad \lambda = \omega^2.$$

We see that $\mathbf{A}$ has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Consequently, $\omega = \pm\sqrt{-1} = \pm\mathbf{j}$ and $\sqrt{-6} = \pm\mathbf{j}\sqrt{6}$, respectively.

Corresponding eigenvectors are:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

From (8) we thus obtain the four complex solutions:

$$\mathbf{x}_1 e^{\pm\mathbf{j}t} = \mathbf{x}_1\left(\cos t \pm \mathbf{j}\sin t\right),$$

By addition and subtraction we get the four real solutions

$$\mathbf{x}_1\cos t, \quad \mathbf{x}_1\sin t, \quad \mathbf{x}_2\cos\sqrt{6}t, \quad \mathbf{x}_2\sin\sqrt{6}t.$$

A general solution is obtained by taking a linear combination of these,

$$\begin{aligned} \mathbf{y} = \mathbf{x}_1 &\left(a_1\cos t + b_1\sin t\right) \\ &+ \mathbf{x}_2\left(a_2\cos\sqrt{6}t + b_2\sin\sqrt{6}t\right) \end{aligned}$$

with arbitrary constants[8] a_1, b_1, a_2, b_2. By (10), the components of $\mathbf{y}$ are:

$$\begin{aligned} y_1 &= a_1\cos t + b_1\sin t + 2a_2\cos\sqrt{6}t + 2b_2\sin\sqrt{6}t \\ y_2 &= 2a_1\cos t + 2b_1\sin t - a_2\cos\sqrt{6}t - b_2\sin\sqrt{6}t. \end{aligned}$$

These functions describe harmonic oscillations of the two masses. Physically, this had to be expected because we have neglected damping ■

[8] to which values can be assigned by prescribing initial displacement and initial velocity of each of the two masses

7.4 Symmetric, Skew-Symmetric and Orthogonal Matrices

We consider three (3) classes of **real** square matrices, because of their remarkable properties, occur quite frequently in applications.

7.4.1 Necessary Definitions

A *real* square matrix $\mathbf{A} = \left[a_{jk}\right]$ is:

1. called **symmetric** if transposition leaves it unchanged,

$$\mathbf{A}^{\mathsf{T}} = \mathbf{A}, \qquad \text{thus} \qquad a_{kj} = a_{jk}, \tag{7.3}$$

2. skew-symmetric if transposition gives the negative of $\mathbf{A}$,

$$\mathbf{A}^{\mathsf{T}} = -\mathbf{A}, \qquad \text{thus} \qquad a_{kj} = -a_{jk}, \tag{7.4}$$

3. orthogonal if transposition gives the inverse of $\mathbf{A}$,

$$\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}. \tag{7.5}$$

Exercise 7.7: Symmetric, Skew-Symmetric, and Orthogonal Matrices

The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal, respectively, which can easily be verified based on the previous definitions.

Every skew-symmetric matrix has all main diagonal entries zero.

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}}) \quad \text{and} \quad \mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}}).$$

Which can be shown in the following example:

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R} + \mathbf{S} = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Theory 7.7: Eigenvalues of Symmetric and Skew-Symmetric Matrices

1. The eigenvalues of a symmetric matrix are real.
2. The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

7.4.2 Orthogonal Transformations and Matrices

Orthogonal transformations are transformations:

$$\mathbf{y} = \mathbf{A}x \qquad \text{where } \mathbf{A} \text{ is an orthogonal matrix.}$$

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n. For instance, the plane rotation through an angle θ is an orthogonal transformation.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a **rotation**[9].

[9] possibly combined with a reflection in a straight line or a plane, respectively.

The main reason for the importance of orthogonal matrices is as follows.

Theory 7.7: Invariance of Inner Product

An orthogonal transformation preserves the value of the **inner product** of vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n, defined by

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b} = [a_1 \quad \cdots \quad a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

That is, for any $\mathbf{a}$ and $\mathbf{b}$ in R^n, orthogonal $n \times n$ matrix $\mathbf{A}$, and $\mathbf{u} = \mathbf{Aa}$, $\mathbf{v} = \mathbf{Ab}$, we have $\mathbf{u}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{b}$. Therefore the transformation also preserves the **length** of any vector $\mathbf{a}$ in R^n given by:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\mathbf{a}^{\mathsf{T}}\mathbf{a}}.$$

Let's look at an example of a orthogonal matrix before we dive into more theories:

Exercise 7.8: An Orthogonal Matrix

Consider the following matrix:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let's check if this matrix above is orthogonal. First we need to look at its column vectors.

$$T_{e1} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad T_{e2} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Then we can check the orthogonality by:

$$\begin{aligned}
\langle T_{e1}, T_{e1}\rangle &= \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\
&= \cos^2\theta + \sin^2\theta = 1 \\
\langle T_{e1}, T_{e2}\rangle &= \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \sin\theta\cos\theta - \sin\theta\cos\theta = 0 \\
\langle T_{e2}, T_{e2}\rangle &= \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \cos^2\theta + \sin^2\theta = 1 \quad \blacksquare
\end{aligned}$$

Orthogonal matrices have further interesting properties as follows.

Theory 7.8: Orthonormality of Column and Row Vectors

A real square matrix is orthogonal if and only if its column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ (and also its row vectors) form an ***orthonormal system***, that is,

$$a_j \cdot a_k = a_j^{\mathsf{T}} a_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$

A simpler way to express this is let $\mathbf{Q}$ be a orthogonal matrix. Therefore the following identity will hold true:

$$\mathbf{Q}\mathbf{Q}^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{I}$$

Theory 7.8: Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value 1 or -1. For example:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$\det(\mathbf{T}) = \cos^2\theta + \sin^2\theta = 1$$

Theory 7.8: Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix **A** are real or complex conjugates in pairs and have absolute value 1[10].

For example:

$$\lambda = \pm 1 \quad \text{and} \quad \lambda = e^{i\phi}$$

[10]This means they all lie in the unit circle

7.5 Eigenbases, Diagonalisation and Quadratic Forms

So far we have emphasised properties of eigenvalues. But it is time to turn to general properties of **eigenvectors**.

Eigenvectors of an $n \times n$ matrix $\mathbf{A}$ may or may not form a basis for R^n.

If we are interested in a transformation $\mathbf{y} = \mathbf{x}\mathbf{A}$, such an eigenbasis[11], if it even exists, is of great advantage because then we can represent any $\mathbf{x}$ in R^n uniquely as a linear combination of the eigenvectors $\mathbf{x}_1, \cdots \mathbf{x}_n$, say,

[11]This means the basis of eigenvectors

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

And, denoting the corresponding (not necessarily distinct) eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \cdots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)$$
$$= c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_n\mathbf{A}\mathbf{x}_n$$
$$= c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n.$$

This shows that we have decomposed the complicated action of $\mathbf{A}$ on an arbitrary vector $\mathbf{x}$ into a sum of simple actions (multiplication by scalars) on the eigenvectors of $\mathbf{A}$.

This is the point of an eigenbasis.

Theory 7.8: Basis of Eigenvectors

If an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, then $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ for R^n.

Exercise 7.9: Eigenbasis, Non-distinct eigenvalues and non-existence

The matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

has a basis of eigenvectors of:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which corresponds to the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$ respectively. Even if not all real numbers are different, a matrix $\mathbf{A}$ may still provide an eigenbasis for R^n. On the other hand, $\mathbf{A}$ may not have enough linearly independent eigenvectors to make up a basis. For instance:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Has only one (1) eigenvector:

$$\begin{bmatrix} k \\ 0 \end{bmatrix} \qquad (k \neq 0, \text{ arbitrary}) \quad \blacksquare$$

Actually, eigenbases exist under much more general conditions than those in Theorem .

Theory 7.9: Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Exercise 7.10: Orthonormal Basis of Eigenvectors

Let's calculate the basis of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Solution

From previous example we know the eigenvectors:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Time to check if these are orthogonal:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^T \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot 1 + 1 \cdot (-1) = 0$$

Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ form an orthonormal set. To get the orthonormal basis we need normalise them.

$$\frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{\sqrt{1^2+(-1)^2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

7.5.1 Similarity of Matrices and Diagonalisation

Eigenbases also play a role in reducing a matrix $\mathbf{A}$ to a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$. This is done by a similarity transformation, which is defined as follows.

Similarity matrices play a pivotal role in computer graphics and solving differential equations.

Theory 7.10: Similarity Transformation

An $n \times n$ matrix $\mathbf{B}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{7.6}$$

for some non-singular[12] $n \times n$ matrix $\mathbf{P}$.

This transformation, which gives $\mathbf{B}$ from $\mathbf{A}$ is called a **similarity transformation**.

[12] Remember, a non-singular matrix is a square whose determinant is **not** zero.

The important concept of this transformation is that it preserves the eigenvalues of $\mathbf{A}$:

Theory 7.10: Eigenvalues and Eigenvectors of Similar Matrices

If $\mathbf{B}$ is similar to $\mathbf{A}$, then by extension $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$.

In addition, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{B}$ corresponding to the same eigenvalue.

Exercise 7.11: Eigenvalues and Vectors of Similar Matrices

Let us have the following matrices:

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

Then

$$\mathbf{B} = \overbrace{\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}}^{\mathbf{P}^{-1}} \overbrace{\begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}}^{\mathbf{P}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Here $\mathbf{P}^{-1}$ was by taking the inverse of the given matrix $\mathbf{P}$. We see that $\mathbf{B}$ has the following eigenvalues:

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 2$$

The characteristic equation of $\mathbf{A}$ is:

$$(6 - \lambda)(-1 - \lambda) + 12 = \lambda^2 - 5\lambda + 6 = 0$$

This is a polynomial of power 2 and has the roots

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 2$$

This proves $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$.

Let's check if we can prove these two (2) matrices have the eigenvector relation:

$$\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$$

Checking $\mathbf{A}$ we have the following equation:

$$(6 - \lambda)x_1 - 3x_2 = 0$$

For:

$$\begin{aligned} \lambda = 3 \quad &\text{gives} \quad 3x_1 - 3x_2 = 0 \quad \text{and} \quad \mathbf{x}_1 = [1 \quad 1]^T \\ \lambda = 2 \quad &\text{gives} \quad 4x_1 - 3x_3 = 0 \quad \text{and} \quad \mathbf{x}_2 = [3 \quad 4]^T \end{aligned}$$

Therefore we can do the check

$$\begin{aligned} \mathbf{y}_1 = \mathbf{P}^{-1}\mathbf{x}_1 &= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{y}_2 = \mathbf{P}^{-1}\mathbf{x}_2 &= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Indeed, these are eigenvectors of the diagonal matrix $\mathbf{B}$ ■

Perhaps we see that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the **columns** of $\mathbf{P}$. This suggests the general method of transforming a matrix $\mathbf{A}$ to diagonal form $\mathbf{D}$ by using $\mathbf{P} = \mathbf{X}$, the matrix with eigenvectors as columns.

By a suitable similarity transformation we can now transform a matrix $\mathbf{A}$ to a diagonal matrix $\mathbf{D}$ whose diagonal entries are the eigenvalues of $\mathbf{A}$. Let's formalise this idea into a theorem.

Theory 7.11: Diagonalization of a Matrix

If an $n \times n$ matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is **diagonal**, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal.

Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors. Also,

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \cdots).$$

Exercise 7.12: Diagonalisation of a Matrix

Diagonalise the following matrix:

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}.$$

Solution

The characteristic determinant gives the characteristic equation:

$$-\lambda^3 - \lambda^2 + 12\lambda = 0$$

The roots[13] are:

[13] eigenvalues of **A**

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 4 \quad \text{and} \quad \lambda_3 = 0$$

By the Gauss elimination applied to:

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = 0$$

with $\lambda = \lambda_1, \lambda_2, \lambda_3$ we find the eigenvectors and then $\mathbf{X}^{-1}$ by the Gauss-Jordan elimination. The results are:

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \; \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \; \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

The eigenvectors as column vector ($\mathbf{X}$) is therefore:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

And the inverse:

$$\mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Calculating $\mathbf{AX}$ and multiplying by $\mathbf{X}^{-1}$ from the left, we thus obtain

$$\begin{aligned} \mathbf{D} &= \mathbf{X}^{-1}\mathbf{AX} \\ &= \overbrace{\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}}^{\mathbf{X}^{-1}} \overbrace{\begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix}}^{\mathbf{AX}} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \blacksquare \end{aligned}$$

.

Exercise 7.13: Powers of a Matrix

Let:

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

Find a formula for $\mathbf{A}^k$ given that $\mathbf{A} = \mathbf{PDP}^{-1}$ where:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\text{and} \quad \mathbf{P}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Solution

The solution is as follows[14]:

[14] Remember, matrix operations are associative.

$$\begin{aligned} \mathbf{A}^2 &= \left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right) \\ &= \mathbf{PD}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{DP}^{-1} \\ &= \mathbf{PD}^2\mathbf{P}^{-1} \end{aligned}$$

For a power of three (3) we can do a similar calculation:

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} \\ \mathbf{A}^3 &= \left(\mathbf{PD}^2\mathbf{P}^{-1}\right)\left(\mathbf{PDP}^{-1}\right) \\ &= \mathbf{PD}^2\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{DP}^{-1} \\ &= \mathbf{PD}^3\mathbf{P}^{-1} \end{aligned}$$

In general (for k) we can write:

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1} \quad \blacksquare$$

7.5.2 Quadratic Forms and Transformation to Principle Axis

By definition, a **quadratic form** (Q) in the components $x_1, \cdots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$\begin{aligned} Q = \mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \end{aligned}$$

$\mathbf{A} = \left[a_{jk}\right]$ is called the coefficient matrix of the form. We may assume that $\mathbf{A}$ is symmetric, because we can take off-diagonal terms together in pairs and write the result as a sum of two (2) equal terms.

Exercise 7.14: Quadratic Forms and Symmetric Coefficient Matrix

Let,

$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$
$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$
$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

Here $4 + 6 = 10 = 5 + 5$. From the corresponding symmetric matrix $C = \begin{bmatrix} c_{jk} \end{bmatrix}$, where $c_{jk} = 1/2 \left(a_{jk} + a_{kj} \right)$, therefore

$$c_{11} = 3, \; c_{12} = c_{21} = 5, \; c_{22} = 2$$

we get the following result:

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$
$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$
$$= 3x_1^2 + 10x_1x_2 + 2x_2^2 \quad \blacksquare$$

Quadratic forms occur in physics and geometry, for instance, in connection with conic sections[15] and quadratic surfaces (cones, etc.). Their transformation to principal axes is an important practical task related to the diagonalisation of matrices, as follows.

[15] ellipses $x_1^2/a_2 + x_2^2/b_2 = 1$, etc.

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathsf{T}}\mathbf{x}. \tag{7.7}$$

If we set $\mathbf{X}^{\mathsf{T}}\mathbf{x} = \mathbf{y}$, then, given $\mathbf{X}^{\mathsf{T}} = \mathbf{X}^{-1}$, we have $\mathbf{X}^{-1}\mathbf{x} = \mathbf{y}$ which we can write:

$$\mathbf{x} = \mathbf{X}\mathbf{y}. \tag{7.8}$$

Furthermore, in Eq. (7.7) we have $\mathbf{x}^{\mathsf{T}}\mathbf{X} = \left(\mathbf{X}^{\mathsf{T}}\mathbf{x}\right)^{T} = \mathbf{y}^{\mathsf{T}}$ and $\mathbf{X}^{\mathsf{T}}\mathbf{x} = \mathbf{y}$, so that Q becomes simply:

$$Q = \mathbf{y}^{\mathsf{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \tag{7.9}$$

The aforementioned equation proves the following theorem.

Theory 7.14: Principle Axes Theorem

The substitution Eq. (7.8) transforms a quadratic form:

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{\mathrm{jk}} x_{\mathrm{j}} x_{\mathrm{k}} \qquad \left(a_{\mathrm{kj}} = a_{\mathrm{jk}}\right)$$

to the principal axes form or canonical form (10), where $\lambda_1, \cdots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the (symmetric!) matrix A, and X is an orthogonal matrix with corresponding eigenvectors x1, Á , xn, respectively, as column vectors.

Exercise 7.15: Transformation to Principle Axes

Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128.$$

Solution

We have $Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$, where:

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives the characteristic equation:

$$(17-\lambda)^2 - 15^2 = 0$$

It has the roots:

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 32$$

Hence (10) becomes

$$Q = 2y_1^2 + 32y_2^2.$$

We see that $Q = 128$ represents the ellipse:

$$2y_1^2 + 32y_2^2 = 128$$

that is,

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

To know the direction of the principal axes in the x_1x_2-coordinates, we have to determine normalised eigenvectors from $(\mathbf{A} - \lambda\mathbf{I}) = 0$ with $\lambda = \lambda_1 = 2$ and $\lambda = \lambda_2 = 32$ and then use (9). We get

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

hence

$$\mathbf{x} = \mathbf{X}\mathbf{y} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and

$$x_1 = y_1/\sqrt{2} - y_2/\sqrt{2}$$
$$x_2 = y_1/\sqrt{2} + y_2/\sqrt{2}.$$

This is a 45° rotation. Our results agree with those in Sec. 8.2, Example 1, except for the notations. See also, Fig. 160 in that example.

7.6 Complex Matrices

The three (3) classes of matrices in previously have **complex counterparts** which are of practical interest in certain applications, for instance, in quantum mechanics.

Theory 7.15: Notation

$\overline{\mathbf{A}} = \left[\overline{a}_{\mathrm{jk}}\right]$ is obtained from $\mathbf{A} = \left[a_{\mathrm{jk}}\right]$ by replacing each entry of $a_{\mathrm{jk}} = \alpha + \mathrm{j}\beta$ (where $\alpha, \ \beta$ is real) with its complex conjugate $\overline{a}_{\mathrm{jk}} = \alpha - \mathrm{j}\beta$.

In addition, $\overline{\mathbf{A}}^{\mathsf{T}} = \left[\overline{a}_{\mathrm{kj}}\right]$ is the transpose of $\overline{\mathbf{A}}$, hence the conjugate transpose of $\mathbf{A}$.

Exercise 7.16: Notations

Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3+4\mathrm{j} & 1-\mathrm{j} \\ 6 & 2-5\mathrm{j} \end{bmatrix}$$

then we can derive the following definitions:

$$\overline{\mathbf{A}} = \begin{bmatrix} 3-4\mathrm{j} & 1+\mathrm{j} \\ 6 & 2+5\mathrm{j} \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{A}}_{\mathsf{T}} = \begin{bmatrix} 3-4\mathrm{j} & 6 \\ 1+\mathrm{j} & 2+5\mathrm{j} \end{bmatrix} \quad \blacksquare$$

Theory 7.16: Hermitian, Skew-Hermitian and Unitary Matrix

A square matrix of $\mathbf{A} = \left[a_{kj}\right]$ is called:

1. **Hermitian** if $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}$ which means $\overline{a}_{kj} = a_{jk}$,
2. **Skew-Hermitian** if $\overline{\mathbf{A}}^{\mathsf{T}} = -\mathbf{A}$ which means $\overline{a}_{kj} = -a_{jk}$,
3. **Unitary** if $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$.

The first two classes are named after *Charles Hermite*.

From the aforementioned definitions we see the following:

- If $\mathbf{A}$ is Hermitian, main diagonal entries must satisfy $\overline{a}_{jj} = a_{jj}$, which means they are real.
- If $\mathbf{A}$ is skew-Hermitian, then $\overline{a}_{jj} = -a_{jj}$. If we set $a_{jj} = \alpha + \mathbf{j}\beta$, this becomes $\alpha - \mathbf{j}\beta = -(\alpha + \mathbf{j}\beta)$. Hence $\alpha = 0$, so that a_{jj} must be pure imaginary or 0.

If a Hermitian matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}$. Therefore a real Hermitian matrix is a **symmetric matrix**.

Similarly, if a skew-Hermitian matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = -\mathbf{A}$. This makes a real skew-Hermitian matrix is a skew-symmetric matrix.

Finally, if a unitary matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$. Hence a real unitary matrix is an orthogonal matrix.

This shows that Hermitian, skew-Hermitian, and unitary matrices generalize symmetric, skew-symmetric, and orthogonal matrices, respectively.

7.6.1 Eigenvalues of Complex Matrices

It is quite remarkable that the matrices under consideration have spectra that can be characterised in a general way as follows:

Theory 7.16: Complex Matrix Eigenvalues

- The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.
- The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.
- The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value 1.

Key properties of orthogonal matrices generalise to unitary matrices in a remarkable way. To see this, instead of R^n we now use the **complex vector space** C^n of all complex vectors with n complex numbers as components, and complex numbers as scalars. For such complex vectors the **inner product** is defined by[16]:

[16] note the overbar for the complex conjugate.

$$\mathbf{a} \cdot \mathbf{b} = \overline{\mathbf{a}}^{\mathsf{T}}\mathbf{b} \tag{7.10}$$

The **length** or **norm** of such a complex vector is a *real* number defined as:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\overline{\mathbf{a}}^{\mathsf{T}}\mathbf{a}} = \sqrt{\overline{a}_1 a_1 + \cdots + \overline{a}_n a_n} = \sqrt{\left|a_1\right| + \cdots + \left|a_n\right|}. \tag{7.11}$$

Theory 7.16: Complex - Invariance of Inner Product

A **unitary transformation**, that is, $\mathbf{y} = \mathbf{A}\mathbf{x}$ with a unitary matrix $\mathbf{A}$, preserves the value of the inner product Eq. (7.10), hence also the norm Eq. (7.11).

Theory 7.16: Unitary System

A *unitary system* is a set of complex vectors satisfying the following relationship:

$$\mathbf{a}_j \cdot \mathbf{a}_k = \overline{\mathbf{a}}_j^{\mathsf{T}} a_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k. \end{cases}$$

Theory 7.16: Unitary Systems of Column and Row Vectors

A complex square matrix is unitary if and only if its column vectors (and also its row vectors) form a unitary system.

Theory 7.16: Determinant of a Unitary Matrix

Let $\mathbf{A}$ be a unitary matrix. Then its determinant has absolute value one, that is, $|\det \mathbf{A}| = 1$.

Chapter 8

Vector Calculus

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8.1 Vector Algebra

Walking 5 kilometers north and then 12 kilometers east, you will have gone a total of 17 kilometers, but you're not 13 kilometers from where you set out, which is only 7. We need a set of mathematics principles to describe quantities like this, which evidently do not add in the ordinary way.

The reason they don't, is **displacements** have *direction* as well as *magnitude*, and it is essential to take both into account when you combine them. Such objects are called **vectors**.

> Examples include: velocity, acceleration, force, momentum ...

By contrast, quantities that have magnitude but no direction are called **scalars**.

> Examples include: mass, charge, density, temperature, ..

We shall use **boldface** ($\mathbf{A}$, $\mathbf{B}$, and so on) for vectors and ordinary type for scalars. The magnitude of a vector $\mathbf{A}$ is written $|\mathbf{A}|$ or, more simply, A. In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowed indicates its direction.

Minus $\mathbf{A}$ ($-\mathbf{A}$) is a vector with the same magnitude as $\mathbf{A}$ but of opposite direction.

> Vectors have magnitude and direction but *not location*

We define four (4) vector operations: addition and three kinds of multiplication.

(i) Addition of two vectors: Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$. The sum, $\mathbf{A}+\mathbf{B}$, is the vector from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$. Addition is *commutative*:

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A},$$

5 kilometers east followed by 12 kilometers north gets you to the same place as 12 kilometers north followed by 5 kilometers east. Addition is also *associative*:

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})\,.$$

To subtract a vector, add its opposite:

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})\,.$$

(ii) Multiplication by a scalar: Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged. This means if a is negative, the direction is reversed. Scalar multiplication is *distributive*:

$$a\,(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}.$$

(iii) Dot product of two vectors: The dot product of two (2) vectors is defined by

$$\mathbf{A}\cdot\mathbf{B}\equiv AB\cos\theta \tag{8.1}$$

where θ is the angle they form when placed tail-to-tail.

$\mathbf{A} \cdot \mathbf{B}$ is itself a *scalar*[1]

[1] hence the alternative name **scalar product**

The dot product is *commutative*,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A},$$

and *distributive*,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \tag{8.2}$$

Geometrically, $\mathbf{A} \cdot \mathbf{B}$ is the product of A times the projection of $\mathbf{B}$ along $\mathbf{A}$ (or the product of B times the projection of $\mathbf{A}$ along $\mathbf{B}$). If the two vectors are parallel, then $\mathbf{A} \cdot \mathbf{B} = AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{8.3}$$

Theory 8.0: Orthogonality Criterion

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

Exercise 8.1: Dot Product of Two Vectors

Find the inner product and the lengths of $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$ as well as the angle between these vectors;

Solution

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$$
$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{5}$$
$$|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} = \sqrt{14}$$

$$\begin{aligned}\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ &= \arccos(-0.11952) = 1.69061 \\ &= 96.865^\circ \quad \blacksquare\end{aligned}$$

(iv) Cross product of two vectors: The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta\, \hat{n} \tag{8.4}$$

where $\hat{n}$ is a **unit vector**[2] pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. Of course, there are *two* directions perpendicular to any plane: **in** and **out**.

[2] it is a vector of magnitude 1

Exercise 8.2: Calculating Cross-Product

Find $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ with $\mathbf{a} = [1, 1, 0]$ and $\mathbf{b} = [3, 0, 0]$.

Solution

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{x} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \hat{y} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \hat{z} = -3\mathbf{k} = [0, 0, -3] \quad \blacksquare$$

The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{n}$.

$\mathbf{A} \times \mathbf{B}$ is itself a *vector* and it is also known as **vector product**.

The cross product is *distributive*,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + \mathbf{A} \times \mathbf{C} \tag{8.5}$$

but **NOT** commutative:

$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B}) \tag{8.6}$$

If two (2) vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = 0,$$

for any vector **A**.

8.1.1 Vector Component Forms

In the previous section, we defined the four (4) vector operations in abstract form, without reference to any particular coordinate system.

In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector **components**. Let $\hat{x}$, $\hat{y}$, $\hat{z}$ be unit vectors parallel to the x, y, and z axes, respectively.

An arbitrary vector **A** can be expanded in terms of these **basis vectors**:

$$\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

The symbols A_x, A_y, A_z, are the components of **A**. In geometrical terms they are the **projections** of **A** along the three (3) coordinate axes (i.e., $A_x = \mathbf{A} \cdot \hat{x}$, $A_y = \mathbf{A} \cdot \hat{y}$, $A_z = \mathbf{A} \cdot \hat{z}$). We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) + \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) \\ &= (A_x + B_x)\ \hat{x} + \left(A_y + B_y\right)\ \hat{y} + (A_z + B_z)\ \hat{z}\end{aligned}$$

The operation rules can be summarised as follows:

(i) To add vectors, add like components.

$$a\mathbf{A} = (aA_x)\ \hat{x} + \left(aA_y\right)\ \hat{y} + (aA_z)\ \hat{z}$$

(ii) To multiply by a scalar, multiply each component.

As $\hat{x}$, $\hat{y}$, $\hat{z}$ are mutually perpendicular unit vectors, the following properties are valid:

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \qquad \hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

Accordingly,

$$\mathbf{A} \cdot \mathbf{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \cdot \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right)$$

(iii) To calculate the dot product, multiply like components, and add. In particular,

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Similarly,

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0,$$

$$\hat{x} \times \hat{y} = -\hat{y} \times \hat{x} = \hat{z},$$
$$\hat{y} \times \hat{z} = -\hat{z} \times \hat{y} = \hat{x},$$
$$\hat{z} \times \hat{x} = -\hat{x} \times \hat{z} = \hat{y}.$$

Therefore,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \times \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) \\ &= \left(A_y B_z - A_z B_y\right) \hat{x} + (A_z B_x - A_x B_z) \hat{y} + \left(A_x B_y - A_y B_x\right) \hat{z}.\end{aligned}$$

This expression can be written more neatly as a **determinant**:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(iv) To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\mathbf{A}$ (in component form), and whose third row is $\mathbf{B}$.

Exercise 8.3: Vector Addition. Multiplication by Scalars

With respect to a given coordinate system, let

$$\mathbf{a} = [4, 0, 1] \quad \text{and} \quad \mathbf{b} = [2, -5, \tfrac{1}{2}].$$

Calculate $2\mathbf{a} - 2\mathbf{b}$.

Solution

Then $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \tfrac{4}{2}]$, and

$$2\,(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{2}] = [4, 10, \tfrac{4}{2}] = 2\mathbf{a} - 2\mathbf{b}.$$

8.1.2 Triple Products

As the cross product of two (2) vectors is itself a vector, it can be dotted or crossed with a 3^{rd} vector to form a *triple* product.

(i) Scalar triple product: Evidently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

for they all correspond to the same value. Note that ”alphabetical” order is preserved. The ”nonalphabetical” triple products,

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}),$$

have the opposite sign. In component form,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Note that the dot and cross can be interchanged:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C},$$

however, the placement of the parentheses is critical:

$(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$ is a meaningless expression. You can't make a cross product from a scalar and a vector.

(ii) **Vector triple product:** $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$$

Notice that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=-\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

is an entirely **different vector** (*cross-products are not associative*). All *higher* vector products can be similarly reduced, often by repeated application , so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})&=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}),\\ \mathbf{A}\times\left[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\right]&=\mathbf{B}\left[\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})\right]-(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D}).\end{aligned}$$

8.1.3 Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates $(x,\ y,\ z)$. The vector to that point from the origin ($\mathcal{O}$) is called the **position vector**:

$$\mathbf{r}\equiv x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$$

Throughout this course, $\mathbf{r}$ will be used to measure **distance**. Its magnitude:

$$r=\sqrt{x^2+y^2+z^2}$$

is the distance from the origin, and

$$\hat{r}=\frac{\mathbf{r}}{r}=\frac{(x)\ \hat{x}+(y)\ \hat{y}+(z)\ \hat{z}}{\sqrt{x^2+y^2+z^2}}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from $(x,\ y,\ z)$ to $(x+dx,\ y+dy,\ z+dz)$, is

$$\mathrm{d}\mathbf{l}=(\mathrm{d}x)\ \hat{x}+(\mathrm{d}y)\ \hat{y}+(\mathrm{d}z)\ \hat{z}.$$

Exercise 8.4: Components and Length of a Vector

The vector **a** has the initial point P: $(4,\ 0,\ 2)$ and terminal point Q: $(6,\ -1,\ 2)$. Find its mangnitude:

Solution

$$a_1=6-4=2,\qquad a_2=-1-0=-1,\qquad a_1=2-2=0.$$

Hence $\mathbf{a}=[2,\ -1,\ 0]$. The length

$$|\mathbf{a}|=\sqrt{2^2+(-1)^2+0^2}=\sqrt{5}.$$

If we choose $(-1,\ 5,\ 8)$ as the initial point of **a**, the corresponding terminal point is $(1,\ 4,\ 8)$.
If we choose the origin $(0,\ 0,\ 0)$ as the initial point of **a**, the corresponding terminal point is $(2,\ -1,\ 0)$; its coordinates equal the components of **a**. This suggests that we can determine each point in space with a vector. ■

8.2 Differential Calculus

8.2.1 Ordinary Derivatives

Assume a function of one variable: $f(x)$. Therefore, what does the derivative, df/dx, do. It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, $\mathrm{d}x$:

$$\mathrm{d}f = \left(\frac{df}{dt}\right) \mathrm{d}x$$

If we increment x by an infinitesimal amount dx, then f changes by an amount df.

the derivative is the proportionality factor

Geometrically, the derivative df/dt is the *slope* of the graph of f versus x.

8.2.2 Gradient

Assume a function of three (3) variables, for example, the temperature $T(x, y, z)$ in the lecture room. Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot. We want to generalise the notion of "derivative" to functions like T, which depend not on *one* but on *three* variables.

A derivative tells us **how fast the function varies**, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move:

1. If we go straight up, then the temperature will probably increase fairly rapidly,
2. If we move horizontally, it may not change much at all.

In fact, the question "How fast does T vary?" has an infinite number of answers, one for each direction we might choose to explore.
Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states:

$$\mathrm{d}T = \left(\frac{dT}{dx}\right) \mathrm{d}x + \left(\frac{dT}{dy}\right) \mathrm{d}y + \left(\frac{dT}{dz}\right) \mathrm{d}z$$

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. We can write the aforementioned equation as a dot product:

$$\mathrm{d}T = \left(\frac{dT}{dx}\hat{x} + \frac{dT}{dy}\hat{y} + \frac{dT}{dz}\hat{z}\right) \cdot \left((\mathrm{d}x)\hat{x} + (\mathrm{d}y)\hat{y} + (\mathrm{d}z)\hat{z}\right) = (\nabla T) \cdot (\mathrm{d}\mathbf{l}),$$

where

$$\nabla T \equiv \frac{dT}{dx}\hat{x} + \frac{dT}{dy}\hat{y} + \frac{dT}{dz}\hat{z}$$

is the **gradient** of T. Note that ∇T is a **vector quantity**, with three (3) components.

This is the generalized derivative we have been looking for.

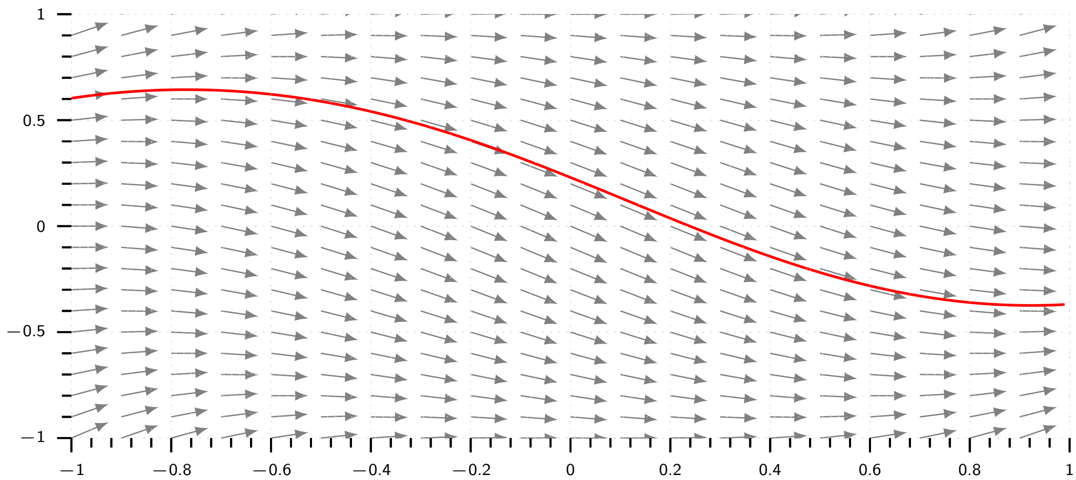

Figure 8.1.

Exercise 8.5: Finding Vector Components

Find the components of the vector **v** with given initial point P and terminal point Q. Find $|\mathbf{v}|$ and unit vector $\hat{v}$.

$$
\begin{array}{llll}
P\,(3,\ 2,\ 0)\,, & Q\,(5,\ -2,\ 2)\,, & P\,(1,\ 1,\ 1)\,, & Q\,(-4,\ -4,\ -4) \\
P\,(1,\ 0,\ 1.2)\,, & Q\,(0,\ 0,\ 6.2)\,, & P\,(2,\ -2,\ 0)\,, & Q\,(0,\ 4,\ 6) \\
P\,(4,\ 3,\ 2)\,, & Q\,(-4,\ -3,\ 2)\,, & P\,(0,\ 0,\ 0)\,, & Q\,(6,\ 8,\ 10)
\end{array}
$$

Given the components of a vector $\mathbf{v} = \left[v_x,\ v_y,\ v_z\right]$ and a particular initial point P, find the corresponding terminal point Q and the length of **v** (i.e.,$|\mathbf{v}|$).

$$
\begin{array}{llll}
\mathbf{v} = [3,\ -1,\ 0]\,; & P\,(4,\ 6,\ 0)\,, & \mathbf{v} = [8,\ 4,\ 2]\,; & P\,(-8,\ -4,\ -2)\,, \\
\mathbf{v} = [0.25,\ 2,\ 0.75]\,; & P\,\{\cdot\}\ 0,\ -0.5,\ 0, & \mathbf{v} = [3,\ 2,\ 6]\,; & P\,(4,\ 6,\ 0)\,, \\
\mathbf{v} = [4,\ 2,\ -2]\,; & P\,(4,\ 6,\ 0)\,, & \mathbf{v} = [3,\ -3,\ 3]\,; & P\,(4,\ 6,\ 0)\,,
\end{array}
$$

Solution

The solution is as follows:

$$\mathbf{v} = (5-3)\ \hat{x} + (-2-2)\ \hat{y} + (2-0)\ \hat{z} = (2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(2)\ \hat{x} + (-4)\ \hat{y} + (2)\ \hat{z}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\ \hat{x} + \left(-\frac{2}{\sqrt{6}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{6}}\right)\ \hat{z} \quad \blacksquare$$
$$\mathbf{v} = (-4-1)\ \hat{x} + (-4-1)\ \hat{y} + (-4-1)\ \hat{z} = (-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-5)\ \hat{x} + (-5)\ \hat{y} + (-5)\ \hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\ \hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\ \hat{z} \quad \blacksquare$$
$$\mathbf{v} = (0-1)\ \hat{x} + (0-0)\ \hat{y} + (6.2-1.2)\ \hat{z} = (-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-1)\ \hat{x} + (0)\ \hat{y} + (5)\ \hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\ \hat{x} + (0)\ \hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\ \hat{z} \quad \blacksquare$$
$$\mathbf{v} = (0-2)\ \hat{x} + (4-(-2))\ \hat{y} + (6-0)\ \hat{z} = (-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-2)\ \hat{x} + (6)\ \hat{y} + (6)\ \hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\ \hat{x} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{y} + \left(\frac{3}{\sqrt{19}}\right)\ \hat{z} \quad \blacksquare$$
$$\mathbf{v} = (-4-4)\ \hat{x} + (-3-3)\ \hat{y} + (2-2)\ \hat{z} = (-8)\ \hat{x} + (-6)\ \hat{y} + (0)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-6)\ \hat{x} + (-8)\ \hat{y} + (0)\ \hat{z}}{10} = \left(-\frac{3}{5}\right)\ \hat{x} + \left(-\frac{4}{5}\right)\ \hat{y} + (0)\ \hat{z} \quad \blacksquare$$
$$\mathbf{v} = (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}.$$
$$\hat{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\ \hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\ \hat{y} + \left(\frac{1}{\sqrt{2}}\right)\ \hat{z} \quad \blacksquare$$

Previously we have defined $\mathbf{v} = Q - P$. Here we have $\mathbf{v}$ and P. To calculate Q we only need to add individual components of the vector with the initial point P.

$$Q = \mathbf{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare$$
$$Q = \mathbf{v} + P = (8+(-8))\ \hat{x} + (4+(-4))\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare$$
$$Q = \mathbf{v} + P = (0.25+0)\ \hat{x} + (2+(-0.5))\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare$$
$$Q = \mathbf{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare$$
$$Q = \mathbf{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare$$
$$Q = \mathbf{v} + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z},$$
$$|\mathbf{v}| = \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare$$

Exercise 8.6: Vector Addition and Scalar Multiplication

- Let $\mathbf{a} = [2,\ 1,\ 0]$, $\mathbf{b} = [-4,\ 2,\ 5]$ and $\mathbf{c} = [0,\ 0,\ 3]$. Calculate the following vector operations:

$$\begin{array}{lll} 2\mathbf{a}, & -\mathbf{a}, & -1/2\mathbf{a}, \\ 5(\mathbf{a}-\mathbf{c}), & 5\mathbf{a}-5\mathbf{c}, & (3\mathbf{a}-5\mathbf{b})+2\mathbf{c}, \\ 3\mathbf{a}+(-5\mathbf{b}+2\mathbf{c}), & \mathbf{a}+2\mathbf{b}, & 2\mathbf{b}+\mathbf{a}. \end{array}$$

  reset

- Find the dot product (i.e., $\mathbf{a} \cdot \mathbf{b}$) on the lengths of $\mathbf{a} = [1,\ 2,\ 0]$ and $\mathbf{b} = [3,\ -2,\ 1]$ as well as the angle (θ) between vectors.
- Let $\mathbf{a} = [2,\ 1,\ 4]$, $\mathbf{b} = [-4,\ 0,\ 3]$ and $\mathbf{c} = [3,\ -2,\ 1]$. Find the following descriptions.

$$\begin{array}{ll} |\mathbf{a}|, |\mathbf{b}|, |\mathbf{c}|, & \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}),\ \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}, \\ \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}, & 4\mathbf{a}\cdot 3\mathbf{c}, 12\mathbf{a}\cdot\mathbf{c}, \\ |\mathbf{b}+\mathbf{c}|, |\mathbf{b}|+|\mathbf{c}|, & \mathbf{a}\cdot\mathbf{c}, |\mathbf{a}||\mathbf{c}|. \end{array}$$

  reset

- Let $\mathbf{a} = [1,\ 1,\ 1]$, $\mathbf{b} = [2,\ 3,\ 1]$ and $\mathbf{c} = [-1,\ 1,\ 0]$. Find the angle between the following:

$$(\mathbf{a}-\mathbf{c}),\ \text{and},\ (\mathbf{b}-\mathbf{c}), \qquad (\mathbf{a}),\ \text{and},\ (\mathbf{b}-\mathbf{c}).$$

- Find the vector product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a} = [1,\ 1,\ 0]$ and $\mathbf{b} = [3,\ 0,\ 0]$.
- Let $\mathbf{a} = [1, 2, 0]$, $\mathbf{b} = [3,\ -4, 0]$, $\mathbf{c} = [3, 5, 2]$, $\mathbf{d} = [6, 2, 0]$. Calculate the cross product of:

$$\begin{array}{ll} \mathbf{a}\times\mathbf{b},\quad \mathbf{b}\times\mathbf{a}, & \mathbf{a}\times\mathbf{c},\quad |\mathbf{a}\times\mathbf{c}|,\quad \mathbf{a}\cdot\mathbf{c}, \\ (\mathbf{c}+\mathbf{d})\times\mathbf{d},\quad \mathbf{c}\times\mathbf{d}, & \mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{b}, \\ (\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{a}), & (\mathbf{a}\times\mathbf{b})\times\mathbf{c},\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}). \end{array}$$

8.2.3 The Del Operator

The gradient has the formal appearance of a vector, ∇, multiplying a scalar T:

$$\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) T$$

The term in parentheses is called **del** operator:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{y}\frac{\partial}{\partial z}$$

Del is **NOT** a vector, in the usual sense. It doesn't mean much until we provide it with a function to act upon. Furthermore, it does not "multiply" T; rather, it is an instruction to differentiate what follows. To be precise, then, we say that ∇ is a vector operator that acts upon T, not a vector that multiplies T.

With this qualification, though, $\mathbf{V}$ mimics the behaviour of an ordinary vector in virtually every way; almost anything that can be done with other vectors can also be done with ∇

Now, an ordinary vector $\mathbf{A}$ can multiply in three (3) ways:

1. By a scalar a: $\mathbf{A}a$;
2. By a vector $\mathbf{B}$, via the dot product: $\mathbf{A} \cdot \mathbf{B}$;
3. By a vector $\mathbf{B}$ via the cross product: $\mathbf{A} \times \mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function **v**, via the dot product: $\nabla \cdot \mathbf{v}$ (divergence)
3. On a vector function **v**, via the cross product: $\nabla \times \mathbf{v}$ (curl).

It is time to examine the other two vector derivatives: divergence and curl.

8.2.4 Divergence

From the definition of ∇ we construct the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\left(\frac{\partial}{\partial x}\right)\hat{x} + \left(\frac{\partial}{\partial y}\right)\hat{y} + \left(\frac{\partial}{\partial z}\right)\hat{z}\right) \cdot \left((v_x)\hat{x} + \left(v_y\right)\hat{y} + (v_z)\hat{z}\right) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

Observe that the divergence of a vector function **v** is itself a scalar $\nabla \cdot \mathbf{v}$.

8.2.5 Curl

From the definition of ∇ we construct the curl:

Exercise 8.7: Curl Example

Find the curl ($\nabla\times$) of the following functions.

$$\mathbf{v} = (y)\hat{x} + \left(2x^2\right)\hat{y} + (0)\hat{z}, \qquad \mathbf{v} = (y^n)\hat{x} + (z^n)\hat{y} + (x^n)\hat{z},$$
$$\mathbf{v} = (\sin y)\hat{x} + (\cos z)\hat{y} + (-\tan x)\hat{z}, \qquad \mathbf{v} = \left(x^2 - z\right)\hat{x} + (xe^z)\hat{y} + (xy)\hat{z}.$$

Solution

The curl ($\nabla\times$) of the functions are as follows:

$$\begin{aligned}
f(x, y, z) &= (y)\hat{x} + \left(2x^2\right)\hat{y} + (0)\hat{z}, \\
\nabla \times f &= (0)\hat{x} + (0)\hat{y} + (-1 + 4x)\hat{z}. \\
f(x, y, z) &= (y^n)\hat{x} + (z^n)\hat{y} + (x^n)\hat{z}, \\
\nabla \times f &= \left(-nz^{n-1}\right)\hat{x} + \left(-nx^{n-1}\right)\hat{y} + \left(-ny^{n-1}\right)\hat{z}. \\
f(x, y, z) &= (\sin y)\hat{x} + (\cos z)\hat{y} + (-\tan x)\hat{z}, \\
\nabla \times f &= (\sin z)\hat{x} + \left(\sec^2 x\right)\hat{y} + (-\cos y)\hat{z}. \\
f(x, y, z) &= \left(x^2 - z\right)\hat{x} + (xe^z)\hat{y} + (xy)\hat{z}, \\
\nabla \times f &= (x - e^z x)\hat{x} + (-1 - y)\hat{y} + (e^z)\hat{z}.
\end{aligned}$$

8.2.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of rules, such as the sum rule:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k\frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla \cdot (\mathbf{A}+\mathbf{B}) = (\nabla \cdot \mathbf{A}) + (\nabla \cdot \mathbf{B}),$$

$$\nabla \times (\mathbf{A}+\mathbf{B}) = (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{B}),$$

and

$$\nabla(kf) = k\nabla f, \qquad \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}), \qquad \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

$$fg \quad \text{(product of two scalar functions)}, \; \mathbf{A} \cdot \mathbf{B} \quad \text{(dot product of two vector functions)},$$

and two ways to make a vector:

$$f\mathbf{A} \quad \text{(scalar times vector)},$$

$$\mathbf{A} \times \mathbf{B} \quad \text{(cross product of two vectors)}.$$

Accordingly, there are six product rules, two for gradients:
(i)

$$\mathbf{V}(fg) = fvVg + gvVf,$$

(ii) $\mathbf{V}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\mathbf{V} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{V} \times \mathbf{A}) + (\mathbf{A} \cdot \mathbf{V})\mathbf{B} + (\mathbf{B} \cdot \mathbf{V})\mathbf{A}$, two for divergences:
(iii) $\mathbf{V} \cdot (f\mathbf{A}) = f(\mathbf{V} \cdot \mathbf{A}) + \mathbf{A} \cdot (\mathbf{V}f)$,
(iv) $\mathbf{V} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{V} \times \mathbf{A}) - \mathbf{A} \cdot (\mathbf{V} \times \mathbf{B})$,
and two for curls:
(v)

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f),$$

(vi)

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$$

You will be using these product rules so frequently that I have put them inside the front cover for easy reference. The proofs come straight from the product rule for ordinary derivatives. For instance,

$$\nabla \cdot (f\mathbf{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right)$$

$$= (\nabla f) \cdot \mathbf{A} + f (\nabla \cdot \mathbf{A}) .$$

It is also possible to formulate three quotient rules:

$$\nabla \left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},$$
$$\nabla \cdot \left(\frac{\mathbf{A}}{g}\right) = \frac{g (\nabla \cdot \mathbf{A}) - \mathbf{A} \cdot (\nabla g)}{g^2},$$
$$\nabla \times \left(\frac{\mathbf{A}}{g}\right) = \frac{g (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla g)}{g^2}.$$

However, since these can be obtained quickly from the corresponding product rules, there is no point in listing them separately.

8.2.7 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\nabla \cdot \mathbf{v}$ by applying ∇ twice, we can construct five (5) types of 2nd derivatives.

The gradient ∇T is a vector, so we can take the divergence and curl of it:

1. Divergence of gradient: $\nabla \cdot (\nabla T)$.
2. Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\mathbf{V} \cdot \mathbf{v}$ is a scalar, therefore all we can do is take its gradient:

3. Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$.

The curl $\nabla \times \mathbf{v}$ is a vector, so we can take its divergence and curl:

4. Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.
5. Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

This exhausts the possibilities, and in fact not all of them give anything new. Let's consider them one at a time:

Divergence of a Gradient

This object, which we write as $\nabla^2 T$ for short, is called the **Laplacian** of T, which will be our focus later.

> The Laplacian of a scalar T is a scalar.

Occasionally, we will use the Laplacian of a vector, $\nabla^2 \mathbf{v}$. By this we mean a **vector** quantity whose x-component is the Laplacian of v_x, and so on.

$$\nabla^2 \mathbf{v} \equiv \left(\nabla^2 v_x\right) \hat{x} + \left(\nabla^2 v_y\right) \hat{y} + \left(\nabla^2 v_z\right) \hat{z}$$

This is nothing more than a convenient extension of the meaning of ∇^2.

Exercise 8.8: Laplacian of a Vector

Calculate the Laplacian of the following functions:

$$(i) \quad T_a = x_2 + 3xy + 3z + 4, \quad (ii) \quad T_b = \sin x \sin y \sin z,$$
$$(iii) \quad T_c = e^{-5x} \sin 4y \cos 3z, \qquad (iv) \quad \mathbf{v} = \left(x^2\right) \hat{x} + \left(3xz^2\right) \hat{y} + (-2xz) \hat{z}.$$

Solution

The solution to the Laplacian of the functions are as follows:

$$\text{(i)} \quad \frac{\partial^2 T_a}{\partial x^2} = 2; \frac{\partial^2 T_a}{\partial y^2} = 0; \frac{\partial^2 T_a}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 T_a = 2 \quad \blacksquare$$
$$\text{(ii)} \quad \frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -3T_b \quad \rightarrow \quad \nabla^2 T_b = -3T_b = 3 \sin x \sin y \sin z \quad \blacksquare$$
$$\text{(iii)} \quad \frac{\partial^2 T_c}{\partial x^2} = 25T_c;$$
$$\frac{\partial^2 T_c}{\partial y^2} = -16T_c; \quad \frac{\partial^2 T_c}{\partial z^2} = -9T_c \quad \rightarrow \quad \nabla^2 T_c = 0 \quad \blacksquare$$
$$\text{(iii)} \quad \frac{\partial^2 v_x}{\partial x^2} = 2; \frac{\partial^2 v_x}{\partial y^2} = 0; \frac{\partial^2 v_x}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_x = 0,$$
$$\frac{\partial^2 v_y}{\partial x^2} = 0; \frac{\partial^2 v_y}{\partial y^2} = 0; \frac{\partial^2 v_y}{\partial z^2} = 6 \quad \rightarrow \quad \nabla^2 v_y = 6x,$$
$$\frac{\partial^2 v_z}{\partial x^2} = 0; \frac{\partial^2 v_z}{\partial y^2} = 0; \frac{\partial^2 v_z}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_z = 0,$$
$$\nabla^2 \mathbf{v} = 2\hat{x} + 6x\hat{y} \quad \blacksquare$$

Curl of a Gradient

The curl of a gradient is **always** zero:

$$\nabla \times (\nabla T)$$

This is an **important fact**, which will be used repeatedly. Without going into too much detail into the proof, it relies on the following relation:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$

If you think I'm being fussy, test your intuition on this one:

Gradient of Divergence

This operation rarely occurs in physical applications, and it has not been given any special name of its own.

Notice that $\nabla(\nabla \cdot \mathbf{v})$ is not the same as the Laplacian of a vector:

$$\nabla^2 = (\nabla \cdot \nabla) \neq \nabla(\nabla \cdot \mathbf{v})$$

Divergence of a Curl

Like the curl of a gradient, is always zero:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

Curl of a Curl

As you can check from the definition of ∇:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}.$$

So curl-of-curl gives nothing new; the first term is just number Divergence of a Curl, and the second is the Laplacian.

Really, then, there are just two kinds of second derivatives:

1. the Laplacian,
2. gradient-of-divergence

> It is possible to work out 3rd derivatives, but fortunately second derivatives suffice for practically all physical applications.

8.3 Integral Calculus

8.3.1 Line, Surface, and Volume Integrals

In electrodynamics, we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**, which will be the focus of this section.

a **Line Integrals** an expression of the form:

$$\int_a^b \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector, and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$. If the path forms a closed loop (i.e., if $\mathbf{b} = \mathbf{a}$), We put a circle on the integral sign:

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

At each point on the path, we take the dot product of $\mathbf{v}$ (evaluated at that point) with the displacement $d\mathbf{l}$ to the next point on the path.

> A good example of a line integral is the work done by a force $\mathbf{F}$:
>
> $$W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{l}$$

Ordinarily, the value of a line integral depends critically on the path taken from $\mathbf{a}$ to $\mathbf{b}$, but there is an important special class of vector functions for which the line integral is independent of path and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A force that has this property is called **conservative**.)

> **Exercise** 8.9: Fluid Flow
>
> A fluid's velocity field is $\mathbf{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z}$.

Find the flow along the helix $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (t)\ \hat{z}$ with a range of $0 \leq t \leq \pi/2$.

Solution

We first evaluate $\mathbf{F}$ on the curve:

$$\mathbf{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z} = (\cos t)\ \hat{x} + (t)\ \hat{y} + (\sin t)\ \hat{z} \qquad \text{Substitute } \mathbf{x} = \cos \mathbf{t},\ \mathbf{z} = \mathbf{t},\ \mathbf{y} = \sin \mathbf{t}.$$

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\mathbf{F} \cdot \left(d\boldsymbol{\ell}/dt\right)$ from $t = 0$ to $t = \pi/2$:

$$\begin{aligned} \mathbf{F} \cdot \frac{d\boldsymbol{\ell}}{dt} &= (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1), \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

Which makes,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{dy}{dx}^{\ell} [t]\, \mathrm{d}t = \int_{0}^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\, dt, \\ &= \left[\frac{\cos^2 t}{2} + t \sin t \right] \Bigg|_{0}^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare \end{aligned}$$

Exercise 8.10: Circulation of a Field

Find the circulation of the field $\mathbf{F} = (x - y)\ \hat{x} + x\, \hat{y}$ around the circle $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$ with a range of $0 \leq t \leq 2\pi$.

Solution

On the circle, $\mathbf{F} = (x - y)\ \hat{x} + (x)\ \hat{y} + (0)\ \hat{z} = (\cos t - \sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}$ and

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then

$$\mathbf{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1},$$

Gives.

$$\begin{aligned} \text{Circulation} &= \int_{0}^{2\pi} \mathbf{F} \cdot \frac{dy}{dx}^{\ell} [t] dt = \int_{0}^{2\pi} (1 - \sin t \cos t)\, dt \\ &= \left[t - \frac{\sin^2 t}{2} \Bigg|_{0}^{2\pi} \right] = 2\pi \quad \blacksquare \end{aligned}$$

b. **Surface Integrals:** A surface integral is an expression of the form:

$$\int_{\mathcal{S}} \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$

where $\mathbf{v}$ is a vector function, and the integral is over a specified surface $\mathcal{S}$. Here $\mathrm{d}a$ is an infinitesimal patch of area, with direction perpendicular to the surface. There are, two (2) directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous.
If the surface is closed (forming a "ballon"), we put a circle on the integral sign

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$

Tradition dictates that "outward" is positive, but for open surfaces it's arbitrary.

As an example, if $\mathbf{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \mathbf{v} \cdot \mathrm{d}\mathbf{a}$ represents the total mass per unit time passing through the surface.

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface and is determined entirely by the boundary line. An important task will be to characterize this special class of functions.

Exercise 8.11: Double Integrals

Find the following double integrals:

$$\int_0^1 \int_x^{2x} (x+y)^2 \, dy dx, \qquad \int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx dy,$$
$$\int_0^3 \int_x^3 \cosh(x+y) \, dy dx, \qquad \int_0^1 \int_0^{y^3} \exp y^4 \, dx dy.$$

Solution

The solution to integrations are as follows:

$$\begin{aligned}
\int_0^1 \int_x^{2x} (x+y)^2 \, dy dx &= \int_0^1 \int_x^{2x} x^2 + 2xy + y^2 dy dx, \\
&= \int_0^1 \left[yx^2 + xy^2 + \frac{y^3}{3} \right] \Bigg|_x^{2x} dx, \\
&= \int_0^1 \left(4x^3 + \frac{7x^3}{3} \right) dx, \\
&= \left[4x^3 + \frac{7x^4}{12} \right] \Bigg|_0^1 = \frac{19}{12} \quad \blacksquare \\
\int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx dy &= \int_0^1 \left[x - x^2 y \right] \Bigg|_y^{\sqrt{y}} dy, \\
&= \int_0^1 \left[\left(\sqrt{y} - y^2 \right) - \left(y - y^3 \right) \right] dy = \int_0^1 \left[y^3 + \sqrt{y} - y^2 - y \right] dy, \\
&= \left[\frac{y^4}{4} + \frac{2}{3} y^{3/2} - \frac{y^3}{3} - \frac{y^2}{2} \right] \Bigg|_0^1, \\
&= \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} \right) - (0) = \frac{1}{12} \quad \blacksquare \\
\int_0^3 \int_x^3 \cosh(x+y) \, dy dx &= \int_0^1 \left[\sinh(x+y) \right] \Bigg|_x^3 dx = \int_0^1 \left[\sinh(3+x) - \sinh(2x) \right] dx
\end{aligned}$$

c. **Volume Integrals** A volume integral is an expression of the form:

$$\int_{\mathcal{V}} T \, \mathrm{d}\tau$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$\mathrm{d}\tau = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

As an example, if T is the density of a substance (which might vary from point to point), then the volume integral would give the total mass.

Occasionally we shall encounter volume integrals of vector functions:

$$\int \mathbf{v}\, d\tau = \int \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right) d\tau = \hat{x} \int v_x\, d\tau + \hat{y} \int v_y\, d\tau + \hat{z} \int v_z\, d\tau.$$

As the unit vectors ($\hat{x}$, $\hat{y}$, $\hat{z}$) are constants, they come outside the integral.

8.3.2 The Fundamental Theorem of Calculus

Assume $f(x)$ is a function of one (1) variable. The **fundamental theorem of calculus** says:

Theory 8.11: Calculus Theorem

the integral of a derivative over some region is given by the value of the function at the end points (boundaries)

$$\int_a^b \left(\frac{df}{dx} \right) dx = f(x) - f(a) \qquad \text{or} \qquad \int_a^b F(x)\, dx = f(x) - f(a)$$

In vector calculus there are three species of derivative (gradient, divergence, and curl,) and each has its own "fundamental theorem," with essentially the same format. I don't plan to prove these theorems here; rather, I will explain what they mean, and try to make them plausible. Proofs are given in Appendix A.

8.3.3 The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point **a**, move a small distance $d\mathbf{l}_1$. The function T will change by an amount:

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$

Now move an additional small displacement $d\mathbf{l}_2$. The incremental change in T will be:

$$dT = (\nabla T) \cdot d\mathbf{l}_2$$

In this manner, proceeding by infinitesimal steps, we make the journey to point **b**. At each step we compute the gradient of T (at that point) and dot it into the displacement $dv1_{\dots}$ this gives us the change in T.

Theory 8.11: Gradient Theorem

The total change in T in going from **a** to **b** (along the path selected) is:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Similar to "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (**a** and **b**).
Assumee you want to measure the height of GrossGlockner. You could climb the mountain from base, or take the high alpine road, or take a helicopter ride all the way up to top. Regardless or the options you take, you should get the same answer either way (that's the fundamental theorem).

Theorem 1: Incliendent,
Incidentally, as we found in Ex. 1.6, line integrals ordinarily depend on the path taken from **a** to **b**. But the right side of Eq. 1.55 makes no reference to the path–only to the end points. Evidently, gradients have the special property that their line integrals are path independent:
Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from **a** to **b**.
Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

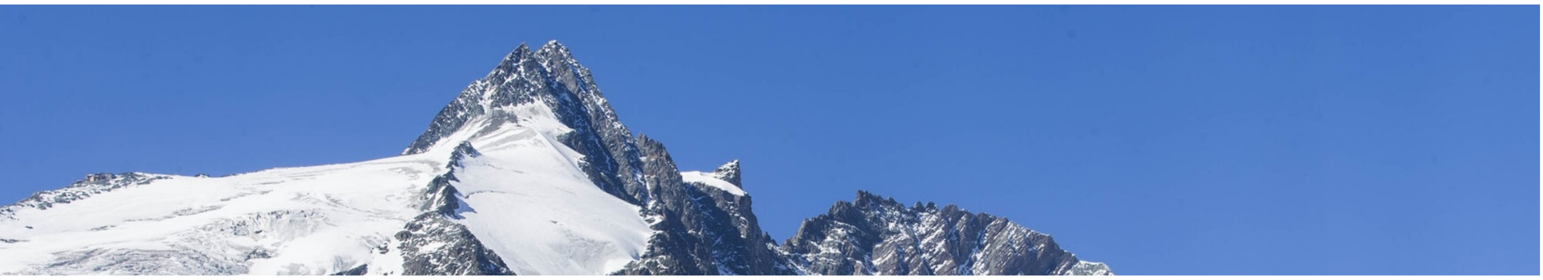

Figure 8.2.: To measure the height of a mountain, it doesn't matter what way you take, as long as you know the base and the top, you will know the height.

The Fundamental Theorem for Divergences

this theorem has at least three special names: **Gauss's theorem, Green's theorem,** or simply the **divergence theorem**, The fundamental theorem for divergences states that:

> **Theory** 8.11: Divergence Theorem
>
> the integral of a derivative (in this case the divergence) over a region (in this case the volume, $\mathcal{V}$) is equal to the value of the function at the boundary (in this case the surface$\mathcal{S}$ that bounds the volume).
>
> $$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) \, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot \, d\mathbf{a}.$$

The boundary term is itself an integral, more specifically, a surface integral. This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

To create an analogy, if **v** represents the flow of an incompressible fluid, then the flux **v** is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the *spreading out* of the vectors from a point, a place of high divergence is like a tap, pouring out liquid. If we have a bunch of tap in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In fact, there are two (2) ways we could determine how much is being produced:

a. we could count up all the faucets, recording how much each puts out

b. we could go around the boundary, measuring the flow at each point, and add it all up

You get the same answer either way:

$$\int \text{(faucets within the volume)} = \oint \text{(flow out through the surface)}$$

> **Exercise** 8.12: Divergence Theorem - I
>
> Evaluate both sides of the Divergence theorem for the expanding vector field $\mathbf{F} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}$ over the sphere $x^2 + y^2 + z^2 = a^2$
>
> **Solution**
>
> The outer unit normal to S, calculated from the gradient of $f\ \{\cdot\}\ x,\ y,\ z = x^2 + y^2 + z^2 - a^2$, is:
>
> $$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{a}. \qquad \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{a}^2, \text{ on, } \mathbf{S}$$
>
> Therefore:
>
> $$(\mathbf{F} \cdot \hat{n}) \, da = \frac{x^2 + y^2 + z^2}{a} \, da = \frac{a^2}{a} \, da = a \, da.$$

This in turn gives us:

$$\iint_S (\mathbf{F} \cdot \hat{n}) \, da = \iint_S a \, da = a \iint_S da = a \left(4\pi a^2\right) = 4\pi a^3. \qquad \text{Area of, } \mathbf{S}, \text{ is, } 4\pi \mathbf{a}^2$$

The divergence of **F** is:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

So,

$$\iiint_V (\nabla \cdot \mathbf{v}) \, d\tau = \iiint_V 3 d\tau = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3 \quad \blacksquare$$

Exercise 8.13: Divergence Theorem - II

Check the divergence theorem for the function:

$$\mathbf{v} = \left(r^2 \cos\theta\right) \hat{r} + \left(r^2 \cos\phi\right) \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right) \hat{\phi}.$$

using as your volume one octant of the sphere of radius R.

Solution

It is always useful to write the theorem we are going to work on:

$$\underbrace{\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{v}) \, dV}_{\text{Divergence integral}} = \underbrace{\iint_{\mathcal{S}} \mathbf{v} \cdot \mathbf{n} da}_{\text{Outward flux}}.$$

First solve the left hand side of the equation:

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \cos\theta\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta r^2 \cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right), \\
&= \frac{1}{r^2} 4r^3 \cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2 \cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right), \\
&= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\int (\nabla \cdot \mathbf{v}) \, d\tau &= \int (4r\cos\theta) \, r^2 \sin\theta dr \, d\theta \, d\phi = 4\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{\pi/2} d\phi, \\
&= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \blacksquare
\end{aligned}$$

Now it is time to solve the right hand side of the question. As we are aware from the shape, an octant of the sphere has 4 sides to it: the curved surface $xyz \to \mathbf{a}_1$, and $xz \to \mathbf{a}_2$, $yz \to \mathbf{a}_3$ and $xy \to \mathbf{a}_4$. These are

$$\begin{aligned}
d\mathbf{a}_1 &= \hat{r} dl_\theta dl_\phi = \hat{r} R^2 \sin\theta \, d\phi \, d\theta, & d\mathbf{a}_2 &= dl_r dl_\theta = -\hat{\phi} r dr \, d\theta, \\
d\mathbf{a}_3 &= \hat{\phi} dl_r dl_\theta = \hat{\phi} r dr \, d\theta, & d\mathbf{a}_4 &= dl_r dl_\phi = \hat{\theta} r dr d\theta. \quad (\theta = \pi/2)
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{v} \cdot \, d\mathbf{a} &= \iint_{S_1} \mathbf{v} \cdot \, d\mathbf{a} + \iint_{S_2} \mathbf{v} \cdot \, d\mathbf{a} + \iint_{S_3} \mathbf{v} \cdot \, d\mathbf{a} + \iint_{S_4} \mathbf{v} \cdot d\mathbf{a}, \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \left[r^2\cos\theta \, \hat{r} + r^2\cos\phi \, \hat{\theta} - r^2\cos\theta\sin\phi \, \hat{\phi}\right]\Big|_{r=R} \cdot \left(\hat{r} R^2 \sin\theta d\phi d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta \, \hat{r} + r^2\cos\phi \, \hat{\theta} - r^2\cos\theta\sin\phi \, \hat{\phi}\right]\Big|_{\phi=0} \cdot \left(-\hat{\phi} r dr d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta \, \hat{r} + r^2\cos\phi \, \hat{\theta} - r^2\cos\theta\sin\phi \, \hat{\phi}\right]\Big|_{\phi=\pi/2} \cdot \left(\hat{\phi} r dr d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta \, \hat{r} + r^2\cos\phi \, \hat{\theta} - r^2\cos\theta\sin\phi \, \hat{\phi}\right]\Big|_{\theta=\pi/2} \cdot \left(\hat{\theta} r dr \, d\theta\right),
\end{aligned}$$

Time to do some integration.

$$\begin{aligned}\iint_S \mathbf{v}\cdot d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2}\left[R^2\cos\theta\,\hat{r}+R^2\cos\phi\,\hat{\theta}-R^2\cos\theta\sin\phi\,\hat{\phi}\right]\cdot\left(\hat{r}R^2\sin\theta d\phi d\theta\right)\\ &+\int_0^{\pi/2}\int_0^{R}\left[r^2\cos\theta\,\hat{r}+r^2(1)\,\hat{\theta}-(0)\sin\phi\,\hat{\phi}\right]\cdot\left(-\hat{\phi}rdrd\theta\right)\\ &+\int_0^{\pi/2}\int_0^{R}\left[r^2\cos\theta\,\hat{r}+(0)\phi\,\hat{\theta}-r^2\cos\theta(1)\,\hat{\phi}\right]\cdot\left(\hat{\phi}rdrd\theta\right)\\ &+\int_0^{\pi/2}\int_0^{R}\left[(0)\,\hat{r}+r^2\cos\phi\,\hat{\theta}-(0)\,\hat{\phi}\right]\cdot\left(\hat{\theta}rdrd\theta\right).\end{aligned}$$

Final touches and cleaning up,

$$\begin{aligned}\iint_S \mathbf{v}\cdot\,\mathrm{d}\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2}R^4\sin\theta\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\theta+0-\overbrace{\int_0^{\pi/2}\int_0^{R}r^3\cos\theta dr\,\mathrm{d}\theta+\int_0^{\pi/2}\int_0^{R}r^3\cos\theta dr d\phi}^{=0},\\ &= R^4\left(\int_0^{\pi/2}d\phi\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\right),\\ &= R^4\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right),\\ &= \frac{\pi R^4}{4}\quad\blacksquare\end{aligned}$$

8.3.4 The Fundamental Theorem for Curls

The fundamental theorem for curls, also known as **Stokes' theorem**, states:

Theory 8.13: Stokes' Theorem

the integral of a derivative over a region ($\mathcal{S}$) is equal to the value of the function at the boundary ($\mathcal{P}$).

$$\int_{\mathcal{S}}(\nabla\times\mathbf{v})\cdot\,\mathrm{d}\mathbf{a}=\oint_{\mathcal{P}}\mathbf{v}\cdot d\mathbf{l}.$$

Similar to the divergence theorem, the boundary term is itself an integral. Specifically, a *closed line integral*.

Remember the curl measures the *twist* of the vectors **v**. Think of a region of high curl as a whirlpool, where if you put a wheel there, it will rotate. Now, the integral of the curl over some surface (or, more precisely, the *flux* of the curl through the surface) represents the *total amount of swirl*, and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$\oint\mathbf{v}\cdot\,\mathrm{d}\mathbf{l}$ is sometimes called the **circulation** of **v**.

There seems to be an ambiguity in Stokes' theorem: concerning the boundary line integral:

Which way are we supposed to go around (clockwise or counterclockwise)?

The answer is that it doesn't matter which way you go **as long as you are consistent**, for there is an additional sign ambiguity in the surface integral:

*Which way does d**a** point?*

For a closed surface (i.e., the divergence theorem), d**a** points in the direction of the outward normal. But for an open surface, which way would be defined as out? Consistency in Stokes' theorem is given by the right-hand rule. If your rings point in the direction of the line integral, then your thumb fixes the direction of d**a**.

Ordinary, a flux integral depends critically on what surface you integrate over, but this is **not** the case with curls. For Stokes' theorem says that $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ is equal to the line integral of $\mathbf{v}$ around the boundary, and the latter makes no reference to the specific surface you choose.

Preposition I $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Preposition II $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

These corollaries are analogous to those for the gradient theorem.

Exercise 8.14: Surface Area of an Implicit Surface

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

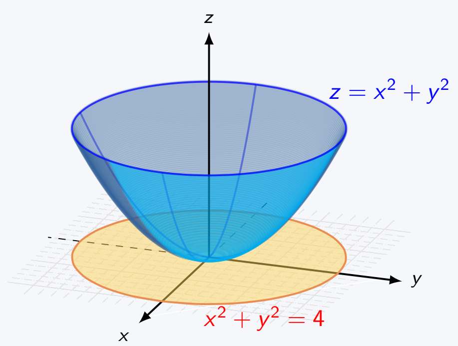

Figure 8.3.: We will calculate the area of the parabolic surface in Example 14.

Solution

We sketch the surface S and the region R below it in the xy-plane (**Fig.** 8.3). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, ans R is the disk $x^2 + y^2 \leq 4$ in the xy-plane.

To get a unit vector normal (i.e., $\hat{n}$) to the plane R, we can take $\hat{n} = \hat{z}$. At any point (x, y, z) on the surface, we have:

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= (2x)\ \hat{x} + (2y)\ \hat{y} + (-1)\ \hat{z} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \hat{n}| &= |\nabla F \cdot \hat{z}| = |-1| = 1. \end{aligned}$$

In the region R, the area is defined to be $dA = dx\,dy$. Therefore:

$$\begin{aligned} \text{Surface Area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{n}|}\, dA \\ &= \iint_{x^2+y^2 \leq 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(4r^2 + 1\right)^{3/2} \Big|_0^2 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(17^{3/2} - 1\right) d\theta \\ &= \frac{\pi}{6} \left(17\sqrt{17} - 1\right) \quad \blacksquare \end{aligned}$$

Exercise 8.15: Stokes Theorem Over a Hemisphere

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\mathbf{F} = (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z}$.

Tip: Parametrisation of a circle is: $x = r\cos\theta,\ y = r\sin\theta$ and $da = \frac{3}{z} dA$)

Solution

The start by calculating the counter-clockwise circulation around C using the following parametrisation:

$$\boldsymbol{\ell}(\theta) = (3\cos\theta)\ \hat{x} + (3\sin\theta)\ \hat{y} + (0)\ \hat{z}, \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

Using this we can calculate the counter-clockwise circulation.

$$
\begin{aligned}
d\boldsymbol{\ell} &= (-3\sin\theta\, d\theta)\ \hat{x} + (3\cos\theta\, d\theta)\ \hat{y} + (0)\ \hat{z}, \\
\mathbf{F} &= (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z} \\
&= (3\sin\theta)\ \hat{x} + (-3\cos\theta)\ \hat{y} + (0)\ \hat{z}, \\
\mathbf{F}\cdot\ d\boldsymbol{\ell} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta, \\
\oint_C \mathbf{F}\cdot d\boldsymbol{\ell} &= \int_0^{2\pi} -9 d\theta = -18\pi.
\end{aligned}
$$

For the curl of integral we have:

$$
\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (0-0)\ \hat{x} + (0-0)\ \hat{y} + (-1-1)\ \hat{z} = -2\,\hat{z} \\
\hat{n} = \frac{\nabla S}{|\nabla S|} &= \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{3} \qquad \textbf{Unit normal}
\end{aligned}
$$

Now it is time to define the area of integration (da):

$$
\begin{aligned}
da &= \frac{|\nabla S|}{|\nabla S\cdot\ \hat{z}|} dA \\
&= \frac{|(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}|}{2z} \\
&= \frac{2\overbrace{\sqrt{x^2+y^2+z^2}}^{a}}{2z} \\
&= \frac{3}{z} dA, \\
\nabla\times\mathbf{F}\cdot\mathbf{n}da &= -\frac{2z}{3}\frac{3}{z} dA = -2dA
\end{aligned}
$$

The cardinal direction $\hat{z}$ comes from being the direction **perpendicular** to the surface (S).

$$
\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}da = \iint_{x^2+y^2\le 9} -2dA = -18\pi
$$

The circulation around the circle equals the integral of the curl over the hemisphere ■

8.4 Curvilinear Coordinates

8.4.1 Spherical Coordinate System

It is possible to label a point P in Cartesian coordinates $(x,\ y,\ z)$, but sometimes it is more convenient to use **spherical** coordinates (r,θ,ϕ); r is the distance from the origin (the magnitude of the position vector r), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates can be read from Fig. 8.4.

$$
x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.
$$

Fig. 8.4 also shows three unit vectors, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, pointing in the direction of increase of the corresponding coordinates.

They constitute an **orthogonal** (mutually perpendicular) basis set, similar to $\hat{x}$, $\hat{y}$, $\hat{z}$), and any vector $\mathbf{A}$ can be expressed in terms of them, in the usual way:

$$\mathbf{A} = (A_r)\ \hat{r} + (A_\theta)\ \hat{\theta} + \left(A_\phi\right)\ \hat{\phi}$$

Here, A_r, A_θ, A_ϕ are the radial, polar, and azimuthal components of vector $\mathbf{A}$. In terms of the Cartesian unit vectors:

$$\begin{aligned}
\hat{r} &= (\sin\theta\cos\phi)\ \hat{r} + (\sin\theta\sin\phi)\ \hat{\theta} + (\cos\theta)\ \hat{\phi},\\
\hat{\theta} &= (\cos\theta\cos\phi)\ \hat{r} + (\cos\theta\sin\phi)\ \hat{\theta} + (-\sin\theta)\ \hat{\phi},\\
\hat{\phi} &= (-\sin\phi)\ \hat{r} + (\cos\phi)\ \hat{\theta} + (0)\ \hat{\phi}.
\end{aligned}$$

An infinitesimal displacement in the $\hat{r}$ direction is simply $\mathrm{d}r$, just as an infinitesimal element of length in the $\hat{x}$ direction is $\mathrm{d}x$:

$$\mathrm{d}l_r = \mathrm{d}r$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (Fig. 1.38b) is not just $\mathrm{d}\theta$ rather,

$$\mathrm{d}l_\theta = r\,\mathrm{d}\theta$$

Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (Fig. 1.38c) is

$$\mathrm{d}l_\phi = r\sin\theta\,\mathrm{d}\phi$$

Thus the general infinitesimal displacement $\mathrm{d}\mathbf{l}$ is:

$$\mathrm{d}\mathbf{l} = (\mathrm{d}r)\ \hat{r} + (r\,\mathrm{d}\theta)\ \hat{\theta} + (r\sin\theta)\ \hat{\phi}$$

This plays the role $\mathrm{d}\mathbf{l} = (\mathrm{d}x)\ \hat{x} + (\mathrm{d}y)\ \hat{y} + (\mathrm{d}z)\ \hat{z}$ plays in Cartesian coordinates. The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three (3) infinitesimal displacements:

$$\mathrm{d}\tau = \mathrm{d}l_r\,\mathrm{d}l_\theta\,\mathrm{d}l_\phi = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

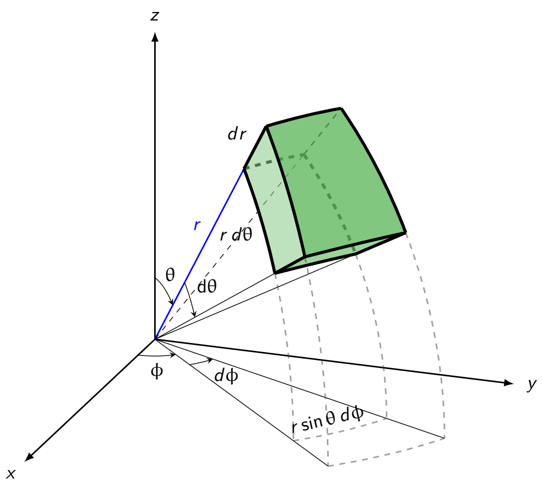

Figure 8.4.: The physics convention. Spherical coordinates (r,θ , ϕ) as commonly used: (ISO 80000-2:2019): radial distance r (slant distance to origin), polar angle θ (theta) (angle with respect to positive polar axis), and azimuthal angle φ (phi) (angle of rotation from the initial meridian plane)

Operator	Mathematical Definition
Gradient	$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$
Divergence	$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$
Curl	$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$
Laplacian	$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$

Table 8.1.: Defined mathematical operations in spherical coordinate system.

It is not possible to give a general expression for surface elements $\mathrm{d}\mathbf{a}$, since these depend on the orientation of the surface. We simply have to analyze the geometry for any given case, which goes for Cartesian and curvilinear coordinates.

Integrating over the surface of a sphere, for instance, makes r constant, whereas θ and ϕ change:

$$\mathrm{d}\mathbf{a}_1 = \mathrm{d}l_\theta\,\mathrm{d}l_\phi\,\hat{r} = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{r}$$

On the other hand, if the surface lies in the xy plane, making θ is constant, while r and ϕ vary:

$$\mathrm{d}\mathbf{a}_2 = \mathrm{d}l_r\,\mathrm{d}l_\phi\,\hat{\theta} = r\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{\theta}$$

Finally: r ranges from 0 to ∞, ϕ from 0 to 2π, and θ from 0 to π.
Up to know, we only talked about the geometry of spherical coordinates. Now let's translate the vector derivatives (gradient, divergence, curl, and Laplacian) into r, θ, ϕ notation.
Here, then, are the vector derivatives in spherical coordinates:

Exercise 8.16: Volume of A Sphere

Find the volume of a sphere of radius R.

Solution

The derivation is as follows:

$$\begin{aligned} V &= \int \mathrm{d}\tau = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi, \\ &= \left(\int_0^R r^2\,\mathrm{d}r\right)\left(\int_0^\pi \sin\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right) = \left(\frac{R^3}{3}\right)(2)(2\pi) = \frac{4}{3}\pi R^3 \quad \blacksquare \end{aligned}$$

8.4.2 Cylindrical Coordinates

The cylindrical coordinates (s, ϕ, z) of a point P are defined in **Fig.** 8.5 . Observe that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is:

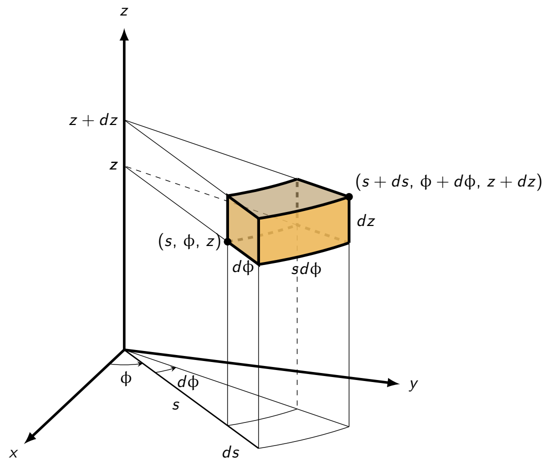

Figure 8.5.: A 3D representation of the cylindrical coordinate system.

$$x = s\cos\phi \qquad y = s\sin\phi \qquad z = z.$$

The unit vectors are:

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad \hat{z} = \hat{z}$$

The infinitesimal displacements are

$$\mathrm{d}l_s = \mathrm{d}s \qquad \mathrm{d}l_\phi = s\,\mathrm{d}\phi, \qquad \mathrm{d}l_z = \mathrm{d}z$$

which makes:

$$\mathrm{d}\mathbf{l} = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{z}.$$

and the volume element is

$$\mathrm{d}\tau' = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$$

The range of s is $(0, \infty)$, ϕ is from 0 to 2π and z is from $-\infty$ to $+\infty$.

8.5 Dirac Delta Function

8.5.1 A Mathematical Anomaly

Consider the following vector function:

$$\mathbf{v} = \frac{1}{r^2}\,\hat{r}$$

At every location, $\mathbf{v}$ is directed radially outward which can be seen in **Fig.** 8.6 . Let's calculate its divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

This is interesting as this calculation gives us a unforeseen solution. Let's look at this closer. Suppose we integrate over a sphere of radius R, centered at the origin. The surface integral is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \left(\frac{1}{R^2}\,\hat{r}\right) \cdot \left(R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\hat{r}\right) = \left(\int_0^{\pi} \sin\theta \,\mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = 4\pi$$

But the volume integral, $\int \nabla \cdot \mathbf{v}\,\mathrm{d}\tau$, is zero if we assume the aforementioned calculation to be true.

Does this mean that the divergence theorem is false? What's going on here?

> There seems to be a contradiction.

The source of the problem is the point $r = 0$, where $\mathbf{v}$ **blows up**. It is quite true that $\nabla \cdot \mathbf{v} = 0$ everywhere except the origin, but right at the origin is the situation is more complicated. Observe, the surface integral is independent of R. If the divergence theorem is right, we should expect $\int \nabla \cdot \mathbf{v}\,\mathrm{d}\tau = 4\pi$ for any non-zero vector and the origin.

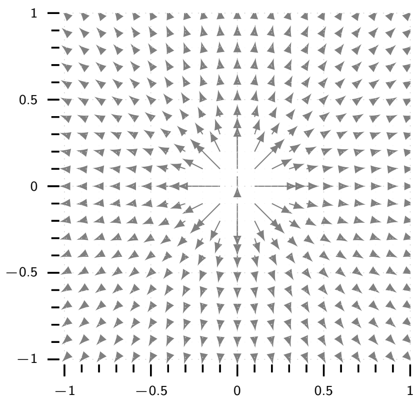

Figure 8.6.: Visually it is obvious there is positive divergence, yet with our current definition of divergence it seems there is a contradiction.

This means the value of 4π must be coming from the point $r = 0$. Therefore, $\nabla \cdot \mathbf{v}$ has the unique property that it vanishes everywhere except at one point, and yet its integral is 4π.

No normal function behaves like that.

> To wrap our heads around this property think of **density**.
>
> The density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite–namely, the mass of the particle.)

What we have stumbled upon is called the **Dirac delta function**. It arises in numerous branches of theoretical physics and plays a central role in the theory of electrodynamics.

8.5.2 The 1D Dirac Delta Function

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow spike, with an area of one (1). This approach to the infinitesimal width can be seen in **Fig.** 8.7 .

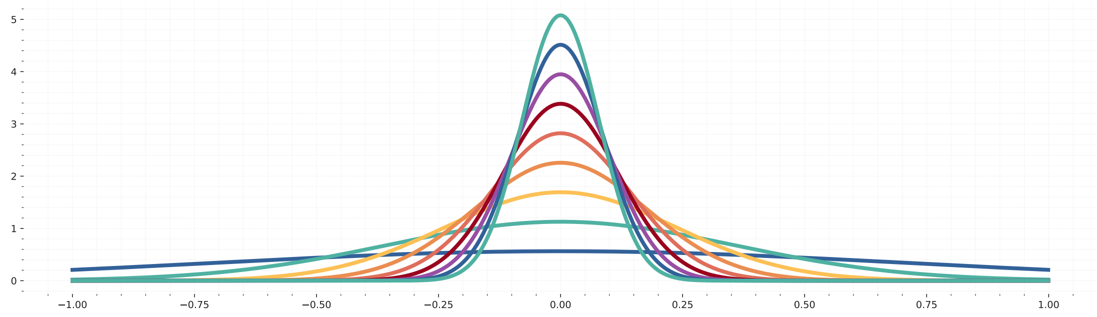

Figure 8.7.: A visual representation of a 1D Dirac Delta Function. Think of it as a distribution function being squeezed to an infinitely small width.

That is to say:

$$\delta(x) = \begin{cases} 0 & \text{if} \quad x \neq 0, \\ \infty & \text{if} \quad x = 0. \end{cases}$$

and in an integral form:

$$\int_{-\infty}^{\infty} \delta(x) \, \mathrm{d}x = 1 \tag{8.7}$$

In a strict sense of definition, $\delta(x)$ is not a function at all, as its value is not finite at $x = 0$. In literature it is known as a **generalized function**[3]

[3]The concept of a generalised function is just anything which generalises what it means to be a function.

If $f(x)$ is some ordinary function, then the product $f(x)\,\delta(x)$ is zero everywhere except at $x = 0$.

It follows that:

$$f(x)\,\delta(x) = f(0)\,\delta(x).$$

The product is zero anyway except at $x = 0$. This allows us to replace $f(x)$ with the value it assumes at the origin.

In particular

$$\int_{-\infty}^{\infty} f(x)\,\delta(x) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0).$$

Under an integral, the delta function picks out the value of $f(x)$ at $x = 0$[4]. Of course, we can shift the spike from $x = 0$ to some other point, $x = a$:

[4]Here the integral need not run from $-\infty$ to $+\infty$; it is sufficient that the domain extend across the delta function, and $-\epsilon$ to $+\epsilon$ would do as well.

$$\delta(x-a) = \begin{cases} 0 & \text{if} \quad x \neq a, \\ \infty & \text{if} \quad x = a. \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(x-a) \, \mathrm{d}x = 1.$$

which becomes:

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a),$$

and finally generalises to:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)dx = f(a)\,. \tag{8.8}$$

Although δ itself is not a legitimate function, integrals over δ are perfectly acceptable. In fact, think of the delta function as something that is always intended for use under an integral sign. In particular, two (2) expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if:

$$\int_{-\infty}^{\infty} f(x)\,D_1(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,D_2(x)\,\mathrm{d}x,$$

for all **ordinary** functions $f(x)$.

Exercise 8.17: A Simple Dirac Integral

Evaluate the following integral:

$$\int_0^3 x^3 \delta(x-2)\,\mathrm{d}x$$

Solution

The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$.

Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Exercise 8.18: 1D Dirac Delta

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right)\delta(x-3)\,dx, \tag{a}$$

$$\int_0^5 \cos x\,\delta(x-\pi)\,\mathrm{d}x, \tag{b}$$

$$\int_0^3 x^3 \delta(x+1)\,\mathrm{d}x, \tag{c}$$

$$\int_{-\infty}^{+\infty} \ln(x+3)\,\delta(x+2)\,\mathrm{d}x. \tag{d}$$

Solution

The solution are as follows:

(a) $3\left(3^2\right) - 2(3) - 1 = 27 - 6 - 1 = 20$ ■

(b) $\cos\pi = -1$ ■

(c) 0 ■

(d) $\ln(-2+3) = \ln 1 = 0$ ■

8.5.3 The 3D Dirac Delta Function

Once we have defined the 1D Dirac, it is trivial to generalise it to 3D:

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)\,,$$

and similar to 1D, 3D Dirac is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\mathbf{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

And, the general form is:

$$\int_{\text{all space}} f(\mathbf{r})\,\delta^3(\mathbf{r}-\mathbf{a})\,\mathrm{d}\tau = f(\mathbf{a})\,. \tag{8.9}$$

As in the 1D case, integration with δ picks out the value of the function f at the location of the spike.

We can fix the paradox introduced in the beginning of Section 8.5. Remember, the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, however, its integral over any volume containing the origin is a constant. These are precisely the defining conditions for the Dirac delta function; evidently

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}) \quad \blacksquare \tag{8.10}$$

8.6 Vector Field Theory

8.6.1 Helmholtz Theorem

As an example, electricity and magnetism are generally expressed as electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ and like many physical laws, such as vortices in a fluid or the flow of a gas in an open environment, these are most compactly expressed as **differential equations**. As $\mathbf{E}$ and $\mathbf{B}$ are **vectors**, the differential equations naturally involve vector derivatives:

divergence and *curl*.

This formulation raises an interesting question:

To what extent is a vector function determined by its divergence and curl?

To study this case lets assume a vector of $\mathbf{F}$. if the divergence of $\mathbf{F}$ is a specified (scalar) function D,

$$\nabla \cdot \mathbf{F} = D,$$

and the curl of $\mathbf{F}$ is a specified (vector) function $\mathbf{C}$,

$$\nabla \times \mathbf{F} = \mathbf{C},$$

and for consistency, we assume $\mathbf{C}$ to have **NO** divergence,

$$\nabla \cdot \mathbf{C} = 0,$$

Remember, the divergence of a curl is **ALWAYS** zero.

Using this knowledge, is it possible to determine the function $\mathbf{F}$?

Without knowing more information, it is not really possible. There are many functions whose divergence and curl are both zero everywhere. Some examples are:

$$\begin{aligned}
\mathbf{F} &= 0, \\
\mathbf{F} &= (y)\ \hat{x} + (zx)\ \hat{y} + (xy)\ \hat{z}, \\
\mathbf{F} &= (\sin x \cosh y)\ \hat{x} + (-\cos x \sinh y)\ \hat{y} + (.)\ \hat{z}
\end{aligned}$$

If you recall the beginning of **Higher Mathematics I**, to solve a differential equation with a particular solution, you must also be supplied with appropriate **boundary conditions**.

In electrodynamics, for example, we typically require that the fields go to zero at infinity[5]. With that extra information, the **Helmholtz theorem** guarantees the field is uniquely determined by its divergence and curl.

[5] To make calculations easier (a.k.a assume the cow is a sphere)

8.6.2 Potentials

If the curl of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be written as the **gradient of a scalar potential** (V):

$$\nabla \times \mathbf{F} = 0 \iff \mathbf{F} = -\nabla V$$

> The minus sign is purely conventional.

That's the essential burden of the following theorem:

> **Theory** 8.18: Zero Curl Fields
>
> The following conditions are equivalent.
>
> (i) $\nabla \times \mathbf{F} = 0$ everywhere,
>
> (ii) $\int_a^b \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points,
>
> (iii) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop,
>
> (iv) $\mathbf{F}$ is the gradient of some scalar function: $\mathbf{F} = -\nabla V$.

The potential is **NOT** unique as any constant can be added to V, since this will not affect its gradient.

If the divergence of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be expressed as the curl of a **vector potential** ($\mathbf{A}$):

$$\nabla \cdot \mathbf{F} = 0 \iff \mathbf{F} = \nabla \times \mathbf{A}$$

That's the main conclusion of the following theorem:

> **Theory** 8.18: Zero Divergence Fields
>
> The following conditions are equivalent:
>
> (i) . $\nabla \cdot \mathbf{F} = 0$ everywhere.
>
> (ii) $f\mathbf{F} \cdot d\mathbf{a}$ is independent of surface, for any given boundary line.
>
> (iii) $f\mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.
>
> (iv) $\mathbf{F}$ is the curl of some vector function: $\mathbf{F} = \nabla \times \mathbf{A}$.

The vector potential is **NOT** unique as the gradient of any scalar function can be added to $\mathbf{A}$ without affecting the curl, given the curl of a gradient is zero.

Incidentally, in all cases, a vector field $\mathbf{F}$ can be written as the gradient of a scalar plus the curl of a vector.

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$