Exam Higher Mathematics I Final

Neighbours

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Lecturer: Daniel T. McGuiness, Ph.D

SEMESTER: WS 2025
DATE: 16.01.2025
TIME: 09:00 - 10:30

First and Last Name

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Student Registration Number

. . . .

Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

Result:

_ _ _/ max. 100 points

Grade:

Student Cohort	MA-MECH-25-VZ
Study Programme	M.Sc Smart Technologies
Permitted Tools	One two-sided hand-written A4 paper is allowed.

Important Notes

Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

Writing materials

Do not use a pencil or red pen and write legibly.

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Question	Maximum Point	Received Point
Solving Differential Equations	60	
Validating Stokes' Theorem	20	
Creating a Set of ODEs	20	
Sum	100	

[Q1] Solving Differential Equations 60

Please solve the following equations. You are allowed to use any method as you see fit.

a. $2xy'' + (1+x)\,y' + y = 0$ (10)

b. $\cos x\left(e^{2y} - y\right) y' - e^{\,y}\sin 2x = 0, \quad \text{where} \quad y(0) = 0$ (10)

c. $4x^2y'' + 17y = 0 \qquad \text{where} \qquad y(1) = -1, \quad \text{and} \quad y'(1) = -\frac{1}{2}$ (20)

d. $y'' - 2y' - 3y = (x+1)^2 + 6xe^{2x} - 5 - (x-1)^2$ (20)

[A1] Solving Differential Equations 60

The solution to the questions are as follows:

1. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ gives

$$
\begin{aligned}
2xy'' + (1+x)y' + y &= 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\
&\quad + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r} \\
&= x^r\left[r(2r-1)c_0x^{-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right] \\
&= x^r\left[r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k\right],
\end{aligned}
$$

which implies $r(2r-1) = 0$ with its root being $r = 0$, and $r = 1/2$ Therefore it's roots are:

$$
y_1 = x^{1/2}\left(a_0 + a_1x + \cdots\right) \quad \text{and} \quad y_2 = x^0\left(A_0 + A_1x + \cdots\right) \quad \blacksquare
$$

2. Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y} dy = \frac{\sin 2x}{\cos x} dx.$$

Before integrating, we use termwise division on the left side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right side. Then

$$\int (e^y - ye^{-y}) dy = 2 \int \sin x dx$$

$$e^y + ye^{-y} + e^{-y} = -2 \cos x + c.$$

The initial condition $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x \quad \blacksquare$$

3. The y' term is missing in the given Cauchy-Euler equation; nevertheless, the substitution $y = x^m$ yields:

$$4x^2 y'' + 17y = x^m(4m(m-1) + 17) = x^m(4m^2 - 4m + 17) = 0$$

when $m_1 = \frac{1}{2} + 2i$ and $\beta = 2$, we see that the general solution of the differential equation on the interval $(0, \infty)$ is

$$y = x^{1/2}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$

By applying the initial conditions $y'(1) = 0$ to the foregoing solution and using $c_2 = 0$. Hence the solution of the initial-value problem is $y = -x^{1/2} \cos(2 \ln x)$ $\blacksquare$

4. The solution to this as follows:

Step 1 First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1 e^{-x} + c_2 e^{3x}$.

Step 2 Next, the presence of $r(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, since the derivative of the product $2xe^{2x}$ and e^{2x}. In other words, r is the sum of two basic kinds of functions:

$$r(x) = r_1(x) + r_2(x) = \text{polynomial + exponentials}.$$

Correspondingly, the superposition principle for nonhomogeneous equations suggests that we seek a particular solution:

$$y_p = y_{p_1} + y_{p_2}$$

where $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

into the given equation and grouping like terms gives

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}$$

From this identity we obtain the four equations

$$2C - 3E = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member is zero. Solving, we find $E = -\frac{4}{3}$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

Step 3 The general solution of the equation is

$$y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right) e^{2x} \quad \blacksquare$$

[Q2] Validating Stokes' Theorem 20

Assume the following vector function:

$$\boldsymbol{v} = \left(2xz + 3y^2\right) \hat{y} + \left(4yz^2\right) \hat{z}.$$

Check Stokes' theorem for the square surface shown: (20)

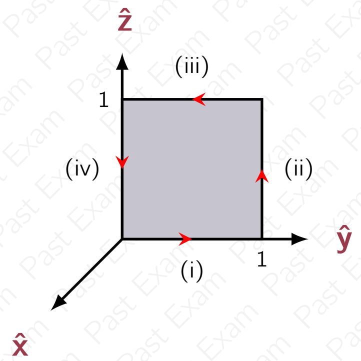

Figure 1: The diagram for the question "Validating Stokes' Theorem"

[A2] Validating Stokes' Theorem 20

Here

$$\nabla \times v = \left(4z^2 - 2x\right) \hat{x} + 2z\,\hat{z} \qquad \text{and} \qquad \mathrm{d}\boldsymbol{a} = \mathrm{d}y\,\mathrm{d}z\,\hat{x}.$$

Saying $\mathrm{d}\boldsymbol{a}$ points in the $\hat{x}$ direction, we are committing ourselves to a counterclockwise line integral. We could as well write $\mathrm{d}\boldsymbol{a} = -\,\mathrm{d}y\,\mathrm{d}z\,\hat{x}$, but then we would be obliged to go clockwise. Since $x = 0$ for this surface,

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\, dy\, dz = \frac{4}{3}$$

For line integral we break this up into four (4) segments:

$$x = 0, z = 0, \mathbf{v} \cdot d\mathbf{l} = 3y^2 dy, \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2 dy = 1,$$
$$x = 0, y = 1, \mathbf{v} \cdot d\mathbf{l} = 4z^2 dz, \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2 dz = \frac{4}{3},$$
$$x = 0, z = 1, \mathbf{v} \cdot d\mathbf{l} = 3y^2 dy, \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2 dy = -1,$$
$$x = 0, y = 0, \mathbf{v} \cdot d\mathbf{l} = 0, \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0\, dz = 0.$$

So

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}. \quad \blacksquare$$

[Q3] Creating a Set of ODEs 20

Please write down the *mass-damper-spring* system equation into a set of ODEs and find its characteristic equation. (20)

[A3] Creating a Set of ODEs 20

The equation for mass-damper-spring is:

$$my'' + cy' + ky = 0 \qquad \text{or} \qquad y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

For this ODE:

$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2$$

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad \blacksquare$$
