Topics on Fundamental Science

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The document is designed with no intention of publication and has only been designed for education purposes.

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Part I

Ordinary Differential Equations

Physics is written in this grand book . . . which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.

(Galilei, Galileo: Il Saggiatore, Chapter 6)

Chapter 1

First-Order Ordinary Differential Equations

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1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a mathematical expression in terms of its variables, functions, and equations. Such an expression is known as a **mathematical model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**. Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,

- find new values,
- interpret result in a physical terms.

This is all done to understand the behaviour of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout the chapter of our book.

An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually written as $y(x)$. The equation may also contain y itself, known functions of x, and constants.

For example all the equation shown below are classified as ODE:

$$y' = \sin x, \qquad y'' + 9y = e^{-3x}, \qquad y'y'' - \frac{5}{4}y = 0.$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from Partial Differential Equation (PDE)s, which involve **partial** derivatives of an unknown function of two or more variables[1]. For instance, a PDE with unknown function u of two (2) variables x and y is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

[1] The topic of PDE will be the focus of **Higher Mathematics II**.

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

For now, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F\left(x, y, y'\right) = 0, \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE:

$$x^{-4}y' - 3y^2 = 0 \qquad \text{where} \qquad x \neq 0$$

can be written explicitly as $y' = 3x^4y^2$.

1.1.1 The Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y\left(x_0\right) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

An ODE, together with an initial condition, is called an **initial value problem**.

Theory 1.1: Initial Value Problem

In multi-variable calculus, an Initial Value Problem (IVP) is an ODE together with an **initial condition** which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Exercise 1.1 An Initial Value Problem

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

SOLUTION The general solution is:

$$y(x) = ce^{3x}$$

From the solution and the initial condition:

$$y(0) = ce^{0} = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation. Visually the solution is plotted as follows ■

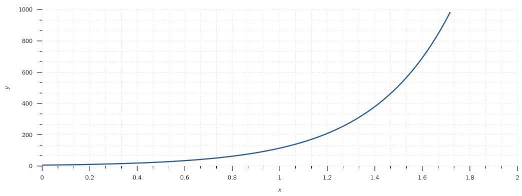


Figure 1.1: A Solution to the initial value problem.

Exercise 1.2 Radioactive Decay

Given 0,5 g of a radioactive substance, find the amount present at any later time. The decay of Radium is measured to be $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$.

SOLUTION We know $y(t)$ is the substance amount still present at t. Using the law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is **positive**, and due of the minus, we get *decay*. We know k which the question has given as $k = 1{,}4 \times 10^{-11}\,\mathrm{s}^{-1}$. Now the given initial amount is 0,5 g, and we can call the corresponding instant $t = 0$. We have the initial condition $y(0) = 0.5$, which is the instant the process begins. Therefore, the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the ODE is an exponential decay and has the general solution:

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition which gives $y(0) = c = 0.5$. Therefore:

$$y(t) = 0.5e^{-kt} \quad ■$$

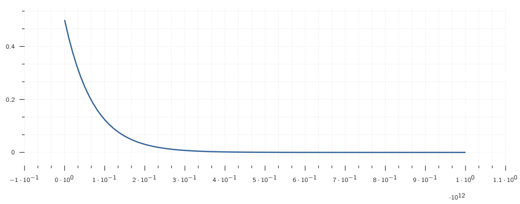


Figure 1.2: A Solution to the radioactive decay.

Information: Prey-Predator Model

The *Lotka - Volterra* equations, are a pair of 1st-order non-linear differential equations, used to describe the dynamics of biological systems in which two (**2**) species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = -\gamma y + \delta xy,$$

where:

- x, y is the population density of prey, predator,
- dy/dt growth rates of the two populations,
- t represents time;
- α, β are the maximum prey per capita growth rate, and the effect of of predators on the prey death rate.
- γ, δ are the predator's per capita death rate, and the effect of of prey on the predator's growth rate.

Figure 1.3: Bunny, natures fast food.

NOTE All parameters are positive and real.

The solution of the differential equations is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping.

1.2 Separable Ordinary Differential Equations

Many practically useful ODEs can be reduced to the following form:

$$g(y)\,y' = f(x) \tag{1.4}$$

using only algebraic manipulations. We can then do integration on both sides with respect to x, obtaining the following expression:

$$\int g(y)\,y'\,\mathrm{d}x = \int f(x)\;\mathrm{d}x + c. \tag{1.5}$$

As a gentle reminder, c here is an integration constant. On the Left Hand Side (LHS) we can switch to y as the variable of integration.

By calculus, we know the relation $y'\,\mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\;\mathrm{d}y = \int f(x)\;\mathrm{d}x + c. \tag{1.6}$$

If f and g are continuous functions,[2] the integrals in Eq. (1.6) **exist**, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

[2] a continuous function is a function such that a small variation of the argument induces a small variation of the value of the function.

Exercise 1.3 Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian-Italian border, caused a scientific sensation.

When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

NOTE The half-life of carbon is 5175 years.

SOLUTION Radioactive decay is governed by the ODE $y' = ky$ as we have discussed previously. By separation and integration

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t,\;\; \ln|y| = kt + c,\;\; y = y_0 e^{kt},\;\; y_0 = e^0.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \quad e^{kH} = 0.5,$$
$$k = \frac{\ln 0.5}{H} = \frac{0.693}{5715} = -0.0001213.$$

we then use the ratio 52.5% to determine the time:

$$e^{k\,t} = e^{-0.0001\,213t} = 0.525,$$
$$t = \frac{\ln 0.525}{-0.0001213} = 5312 \quad \blacksquare$$

Exercise 1.4 A Bell Shaped Curve

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

SOLUTION By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution:

$$y = 1.8e^{-x^2}$$

This is a particular solution, representing a bell-shaped curve ■

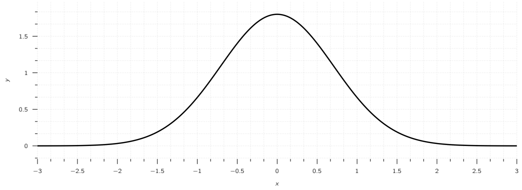


Figure 1.4: A Solution to the Separable ODE.

Exercise 1.5 Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

SOLUTION The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx \qquad \text{By integration} \qquad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x+c)$$

Note It is important to introduce the constant c when the integration is performed.

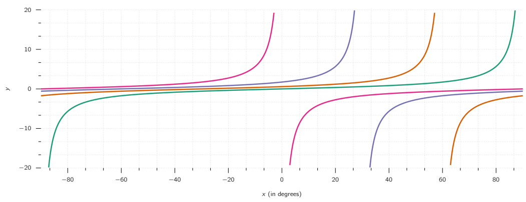


Figure 1.5: A Solution to the Separable ODE.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is **NOT** a solution, when $c \neq 0$ ■

1.2.1 Reduction to Separable Form

Certain **non-separable** ODEs can be made separable by transformations which introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right). \tag{1.7}$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Exercise 1.6 Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

SOLUTION To get the usual explicit form, we start by dividing the given equation by $2xy$ which gives,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then we simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

We see in the last equation that we can now separate the variables,

$$\frac{2u\,\mathrm{d}u}{1+u^2} = -\frac{\mathrm{d}x}{x} \qquad \text{and by integration} \qquad \ln\left(1+u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

We now take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx \qquad \text{therefore we can get} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen below ■

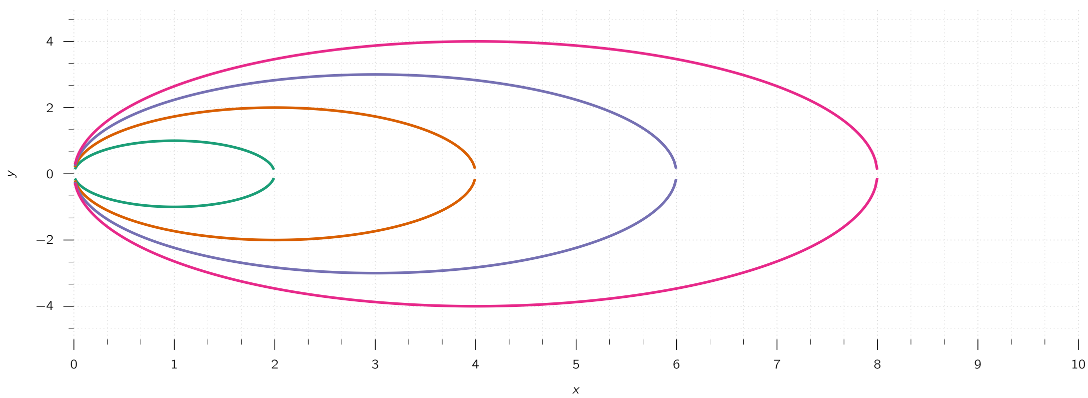


Figure 1.6: A Solution to the Separable ODE.

1.3 Exact Ordinary Differential Equations

If we remember from calculus courses, if a function $u(x, y)$ has continuous partial derivatives, its differential[3] is:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

From this it follows that if $u(x, y) = c =$ is constant, then $du = 0$.

[3] i.e., total differential.

As an example, let's have a look at the function:

$$u = x + x^2 y^3 = c$$

Finding its factors:

$$du = \left(1 + 2xy^3\right) dx + 3x^2 y^2 dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE in the form $M(x, y) + N(x, y)\, y' = 0$, written as:

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1.8}$$

is called an **exact differential equation** if the differential form $M(x, y)\, dx + N(x, y)\, dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{1.9}$$

of some function $u(x, y)$. Then Eq. (1.8) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.8) in the form:

$$u(x, y) = c \tag{1.10}$$

Comparing Eq. (1.8) and Eq. (1.9), we see that Eq. (1.8) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N. \tag{1.11}$$

From this we can derive a formula for checking whether Eq. (1.8) is exact or not, as follows.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.11),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \, \partial x}, \tag{1.12}$$
$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \, \partial y}. \tag{1.13}$$

By the assumption of continuity the two second partial derivatives are **equal**. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.14}$$

This condition is not only necessary but also sufficient for Eq. (1.8) to be an exact differential equation.

If Eq. (1.8) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way. From Eq. (1.12) we have by integration with respect to x:

$$u = \int M \, \mathrm{d}x + k(y), \tag{1.15}$$

in this integration, y is to be regarded as a **constant**, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u / \partial y$ from Eq. (1.15), use Eq. (1.11) (a) to get dk/dy, and integrate dk/dy to get k.

Formula Eq. (1.15) was obtained from Eq. (1.12).

It is valid to use **either** of them and arrive at the same result.

Then, instead of Eq. (1.15), we first have by integration with respect to y

$$u = \int N \, \mathrm{d}y + l(x). \tag{1.16}$$

To determine $l(x)$, we derive $\partial u / \partial x$ from , use Eq. (1.12) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Exercise 1.7 Exact ODE - An Initial Value Problem

Solve the initial value problem:

$$(\cos y \sinh x + 1) \, \mathrm{d}x - \sin y \cosh x \, \mathrm{d}y = 0, \qquad \text{with} \qquad y(1) = 2.$$

SOLUTION Let's begin by verifying the given equation is **exact**:

$$M(x, y) = (\cos y \sinh x + 1),$$
$$N(x, y) = -\sin y \cosh x.$$

We now apply our criteria:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin y \sinh x$$

This shows the given ODE is **exact**. We find u. For a change, let us use Eq. (1.16):

$$u = -\int \sin y \cosh x dy + l(x) = \cos y \cosh x + l(x).$$

From this:

$$\partial u / \partial x = \cos y \sinh x + \frac{dl}{dx} = u = \cos y \sinh x + 1$$

Therefore $dl/dx = 1$ and by integration,

$$l(x) = x + c^*.$$

This gives the general solution

$$u(x, y) = \cos y \cosh x + x = c.$$

From the initial condition:

$$\cos 2 \cosh 1 + 1 = 0.358 = c$$

Therefore the answer is:

$$\cos y \cosh x + x = 0.358 \quad \blacksquare$$

Exercise 1.8 An Exact ODE

Solve the following ODE:

$$\cos(x + y)\ dx + \left(3y^2 + 2y + \cos(x + y)\right)\ dy = 0.$$

SOLUTION

The solution is as follows:

Test for exactness First check if our equation is **exact**, try to convert the equation of the form Eq. (1.8):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

This proves our equation to be exact.

Implicit General Solution From Eq. (1.15), we obtain by integration:

$$u = \int M\,dx + k(y)$$
$$= \int \cos(x + y)\ dx + k(y)$$
$$= \sin(x + y) + k(y) \qquad (1.17)$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (1.13), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Therefore $\frac{dk}{dy} = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$. Inserting this result into Eq. (1.17) and observing Eq. (1.10), we obtain:

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c \quad \blacksquare$$

Exercise 1.9 The Breakdown of Exactness

Check the exactness of the following ODE:

$$-y\,dx + x\,dy = 0$$

SOLUTION The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

Let us show that in such a case the present method does **NOT** work.

$$u = \int M\,dx + k(y) = -xy + k(y),$$
$$\frac{\partial u}{\partial y} = -x + \frac{\partial k}{\partial y}.$$

Now, $\partial u / \partial y$ should equal $N = x$, as required for this equation to be exact. However, this is impossible because $k(y)$ can depend only on y $\blacksquare$

1.4 Linear Ordinary Differential Equations

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A 1$^{\text{st}}$-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)\,y = r(x) \tag{1.18}$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

1.4.1 Homogeneous Linear Ordinary Differential Equations

We want to solve Eq. (1.18) in some interval $a < x < b$, let's call it J, and we begin with the simpler special case where $r(x)$ is zero for all x in J.[4] Then the ODE given in Eq. (1.18) becomes:

[4] This is sometimes written $r(x) = 0$.

$$y' + p(x)\,y = 0 \tag{1.19}$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{\mathrm{d}y}{y} = -p(x)\,\mathrm{d}x, \qquad \text{therefore} \qquad \ln|y| = -\int p(x)\,\mathrm{d}x + c^*.$$

Taking exponents on both sides, the general solution of the homogeneous ODE Eq. (1.19) is,

$$y(x) = ce^{-\int p(x)\,\mathrm{d}x} \qquad \left(c = \pm e^{\,c^*} \quad \text{when} \quad y \neq 0\right) \tag{1.20}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

1.4.2 Non-Homogeneous Linear Ordinary Differential Equations

We now solve Eq. (1.18) in the case that $r(x)$ in Eq. (1.18) is **NOT** everywhere zero in the interval J considered. Then the ODE Eq. (1.18) is called non-homogeneous. It turns out that in this case, Eq. (1.18) has a useful property. Namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ as follows.

We multiply Eq. (1.18) by $F(x)$, obtaining:

$$Fy' + pFy = rF. \tag{1.21}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{therefore} \qquad pF = F'.$$

By separating variables, $\mathrm{d}F/F = p\,\mathrm{d}x$. By integration, writing $h = \int p\,\mathrm{d}x$,

$$\ln|F| = h = \int p\,\mathrm{d}x, \qquad \text{therefore} \qquad F = e^{h}.$$

With this F and $h' = p$, Eq. (1.21) becomes:

$$e^{h}y' + h'e^{h}y = e^{h}y' + \left(e^{h}\right)'y = \left(e^{h}y\right)' = re^{h} \quad \text{by integration} \quad e^{h}y = \int e^{h}r\,\mathrm{d}x + c$$

Dividing by e^{h}, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^{h}r\,\mathrm{d}x + c\right), \qquad h = \int p(x)\,\mathrm{d}x. \tag{1.22}$$

This reduces solving Eq. (1.18) to the generally simpler task of evaluating integrals.[5] The structure of Eq. (1.22) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing Eq. (1.22) as a sum of two terms,

[5]For ODEs for which this is still difficult, we may have to use a numeric method for integrals or for the ODE itself.

$$y(x) = e^{-h}\int e^{h}r\,\mathrm{d}x + ce^{-h},$$

Exercise 1.10 A Non Homogeneous Ordinary Differential Equation

Solve the initial value problem of the following equation:

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

SOLUTION Here we define the parameters as:

$$p = \tan x, \quad r = \sin 2x = 2\sin x\cos x,$$

and

$$h = \int p\,\mathrm{d}x = \int \tan x\,\mathrm{d}x = \ln|\sec x|.$$

From this we see that in Eq. (1.22),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x,$$
$$e^{h}r = (\sec x)(2\sin x\cos x) = 2\sin x,$$

and the general solution of our equation is:

$$\begin{aligned} y(x) &= \cos x\left(2\int \sin x\,\mathrm{d}x + c\right), \\ &= c\cos x - 2\cos^2 x. \end{aligned}$$

From this and the initial condition

$$1 = c\cdot 1 - 2\cdot 1^2, \qquad \text{therefore} \qquad c = 3,$$

and the solution of our initial value problem is:

$$y = 3\cos x - 2\cos^2 x$$

Here $3\cos x$ is the response to the initial data, and $-2\cos^2 x$ is the response to the input $\sin 2x$ ■

Chapter 2

Second-Order Ordinary Differential Equations

Table of Contents

2.1 Introduction

A second order ODE is a specific type of differential equation which consists of a derivative of a function of **order 2** and no other higher-order derivative of the function appears in the equation. These equations have significant engineering applications such as in the study of mechanical and electrical vibrations, wave motion, and heat conduction.

A second-order ODE is called **linear**, if it can be written[1] as:

[1] in its standard form.

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.1}$$

Of course, we can extend most of what we learned in the study of first-order ODE and describe:

homogeneous if $r(x) = 0$,

non-homogeneous if otherwise.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs. For example:

$$y'' = 25y - e^{-x}\cos x \qquad \text{non-homogeneous linear}$$

$$y'' + \frac{1}{x}y\prime + y = 0 \qquad \text{homogeneous linear}$$

$$y''y + \left(y''\right)^2 = 0 \qquad \text{non-linear}$$

2.1.1 The Principle of Superposition

For the homogeneous form the backbone of finding a useful solution is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1 y_1 + c_2 y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or **linearity principle**.

Theory 2.2: Fundamental Theorem for the Homogeneous Linear ODE

For a homogeneous linear ODE of the form:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

any linear combination of two (2) solutions on an open interval I is again a solution of on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions. This theorem is only applicable to **homogeneous** form.

While the iron is hot, lets do a couple of exercises to begin studying 2$^{\text{nd}}$-order ODEs:

Exercise 2.1 **A Superposition of Solutions**

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0, \quad \text{for all } x.$$

SOLUTION By differentiation and substitution, we obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)$$
$$= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad ■$$

Exercise 2.2 **Example of a Non-Homogeneous Linear ODE**

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 1$$

SOLUTION Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1 + \sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

While the methodology is same as before, it is worth mentioning here the small difference. For a second-order homogeneous linear ODE, an initial value problem consists of two (2) initial conditions:[2]

[2] This makes sense as to properly evaluate a 2nd differential equation, we need two values.

$$y(x_0) = K_0 \qquad y'(x_0) = K_1. \tag{2.2}$$

The Eq. (2.2) are used to determine the two (2) arbitrary constants c_1 and c_2 in a general solution

Exercise 2.3 **An Initial Value Problem**

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

SOLUTION **General Solution** From the previous examples, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Particular Solution

We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.

Exercise 2.4 Reduction of Order

Find a basis of solutions of the ODE

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

SOLUTION Inspection shows $y_1 = x$ is a solution as $y_1{}' = 1$ and $y_1{}'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u,$$
$$y'' = u''x + 2u'$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0,$$
$$(x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely,

$$(x^2 - x)w' + (x - 2)w = 0.$$

Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx,$$
$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We don't need constant of integration as we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain n

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \quad u = \int v\,dx = \ln|x| + \frac{1}{x},$$
$$\text{hence} \quad y_2 = ux = x\ln|x| + 1$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained as basic of solutions, valid for all positive x ■

Information: Differential Equations - Early Beginnings

Differential equations have been a major branch of pure and applied mathematics since mid 17th century.

"Differential equations" began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's "fluxional equations" in the 1670s. Applications were made largely to geometry and mechanics with particular interest in optimisation. Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, which lead to partial differential equations. Generalisation of isoperimetrical problems led to the calculus of variations.

New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace. Development of the general theory of solutions included singular ones, functional solutions and those by infinite series. Many applications were made to mechanics, especially to astronomy and continuous media [1].

2.2 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.3}$$

We start to solve the above equation by starting:

$$y = e^{\lambda x} \tag{2.4}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.3) gives:

$$\left(\lambda^2 + a\lambda + b\right) e^{\lambda x} = 0.$$

Therefore if λ is a solution of the important **characteristic** equation,[3]

$$\lambda^2 + a\lambda + b = 0 \tag{2.5}$$

[3] This is also known as an auxiliary equation.

then the exponential function Eq. (2.4) is a solution of the ODE given in Eq. (2.3). Now from algebra we recall the roots of the quadratic equation:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \tag{2.6}$$

Using Eq. (2.5) and Eq. (2.6) we can see that

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

as solutions to Eq. (2.3). From algebra we further know that the quadratic equation Eq. (2.5) may have three (3) kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, which are shown in **Tbl.** 2.1.

Exercise 2.5 IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

SOLUTION **General Solution** The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1,$$

$$\text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^{x} + c_2 e^{-2x}.$$

Particular Solution As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

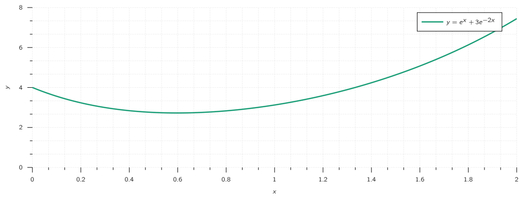


Figure 2.1: Solution to Case of distinct real roots.

Exercise 2.6 IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

SOLUTION The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = \left(\lambda + 2\right)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = \left(c_1 + c_2 x\right) e^{-0.5\, x}$$

We need its derivative:

$$y\prime = c_2 e^{-0.5\, x} - 0.5 \left(c_1 + c_2 x\right) e^{-0.5\, x}$$

From this and the initial conditions we obtain:

$$y\,(0) = c_1 = 3.0,$$

$$y\prime\,(0) = c_2 - 0.5 c_1 = 3.5,$$

$$c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\, x}$$

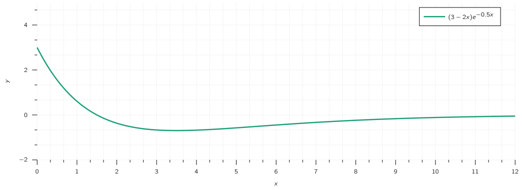


Figure 2.2: Solution to case of double roots.

Case	Condition	Roots of	Basis	General Solution
I	$a^2 - 4b > 0$	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
		$\left(\lambda_1, \lambda_2\right)$	$e^{\lambda_2 x}$	
II	$a^2 - 4b = 0$	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x) e^{-ax/2}$
		$\left(\lambda = -1/2a\right)$	$xe^{-ax/2}$	
III	$a^2 - 4b < 0$	Complex Conjugate	$e^{-ax/2} \cos \omega x$	$y = e^{-ax/2} \left(A \cos \omega x + B \sin \omega x\right)$
		$\lambda_1 = -1/2a + j\omega$	$e^{-ax/2} \sin \omega x$	
		$\lambda_2 = -1/2a - j\omega$		

Table 2.1: Possible roots of the characteristic equation based on the discriminant value.

Exercise 2.7 **IVP: Case of Complex Roots**

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

SOLUTION

General Solution The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathrm{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Particular Solution The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

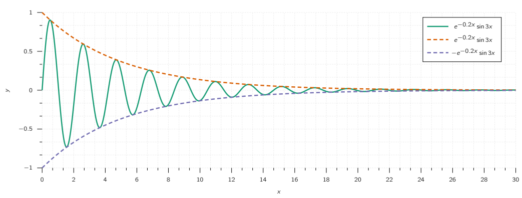


Figure 2.3: Solution to case of complex roots.

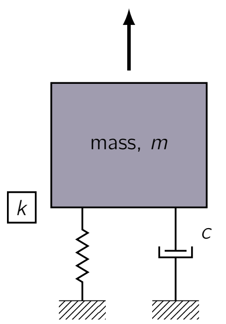
2.2.1 A Study of Damped System

Linear ODEs with constant coefficients have important applications in mechanics, and one of the important system to study is spring-mass-damper system[4], which has the following important component:

$$F_2 = -cy'.$$

Using this damping we can define the ODE of the damped mass-spring system:

$$my'' + cy' + ky = 0. \tag{2.7}$$



[4] A spring mass damper system.

This can physically be done by connecting the ball to a bowl containing a liquid. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

This is generally a good approximation for small velocities.

The constant c is called the damping constant.

The damping force $F_2 = -cy'$ acts **against** the motion. Therefore for a downward motion we have $y' > 0$ which for positive c makes F negative,[5] as it should be.

[5] an upward force.

Similarly, for an upward motion we have $y' > 0$ which, for $c > 0$ makes F_2 positive.[6]

[6] a downward force.

The ODE Eq. (2.7) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \lambda_2 = -\alpha - \beta, \qquad \text{where} \qquad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}. \quad (2.8)$$

Depending on the amount of damping present-whether a lot of damping, a medium amount of damping or little damping-three types of motions occur, respectively. A summary of its behaviour is shown in **Tbl.** 2.2.

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.2: The three cases of behaviour depending on the condition.

A Deeper Look into the Three Cases

Case I: Over-damping If $c^2 > 4mk$, then λ_1 and λ_2 are said to be **distinct real roots**. In this case, the corresponding general solution becomes:[7]

$$y(t) = c_1 e^{-(\alpha - \beta)\, t} + c_2 e^{-(\alpha + \beta)\, t}. \qquad (2.9)$$

[7] In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.9) are negative as $\alpha > 0$ and $\beta > 0$ with:

$$\beta^2 = \alpha^2 - k/m < \alpha^2$$

Therefore both terms in Eq. (2.9) approach zero as $t \to \infty$. Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position (i.e., $y = 0$). A graphical representation of this behaviour can be seen in **Fig. ??**.

Case II: Critical-Damping Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III) and occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$.

Then the corresponding general solution of Eq. (2.7) is:

$$y(t) = \left(c_1 + c_2 t\right) e^{-\alpha t}. \qquad (2.10)$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha\, t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. **Fig. ??** shows typical forms of **Fig. ??**.

The graph above looks almost like those in the previous figure.

Case III: Under-Damping This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in Eq. (2.8) is no longer real but pure **imaginary**, which we write as,

$$\beta = \mathbf{j}\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad \text{where} \qquad \beta > 0.$$

The asterisk $(*)$ is used to differentiate from ω which is used predominantly in electrical engineering do describe angular frequency.

The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + \mathbf{j}\omega^*, \qquad \lambda_2 = -\alpha - \mathbf{j}\omega^*.$$

with $\alpha = c/2m$. The corresponding general solution is:

$$y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right) \tag{2.11}$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$ This represents **damped oscillations**. Their curve lies between the two dashed curves:

$$y = Ce^{-\alpha t} \qquad \text{and} \qquad y = -Ce^{-\alpha t}$$

The frequency of the under-damping process is $\omega^*/2\pi$ Hz. Based on the equation, we see that the smaller c is,[8] the larger is ω^* and the more rapid the oscillations become.

[8] As long as it is bigger than 0.

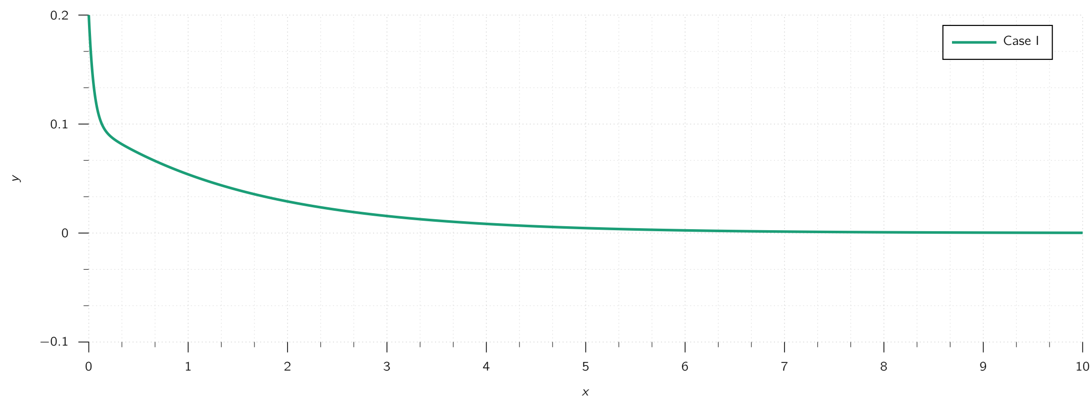


Figure 2.4: Standard behaviour of an over-damped system.

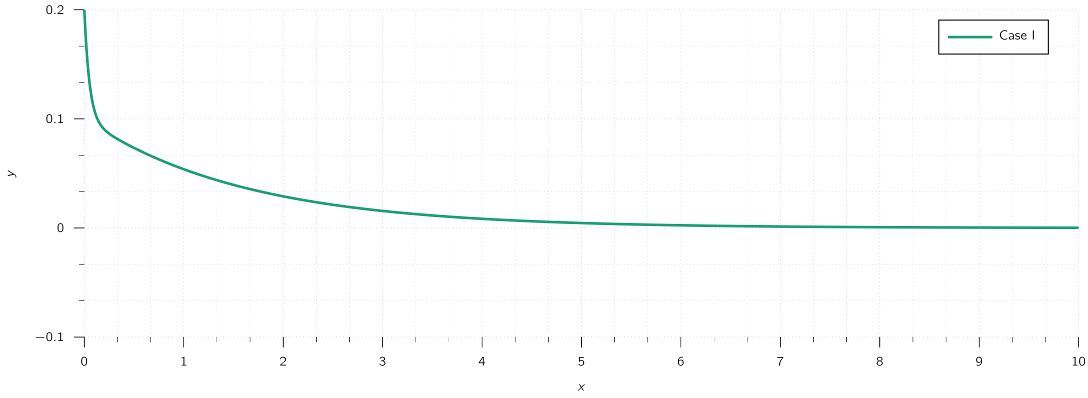


Figure 2.5: Standard behaviour of an critical system.

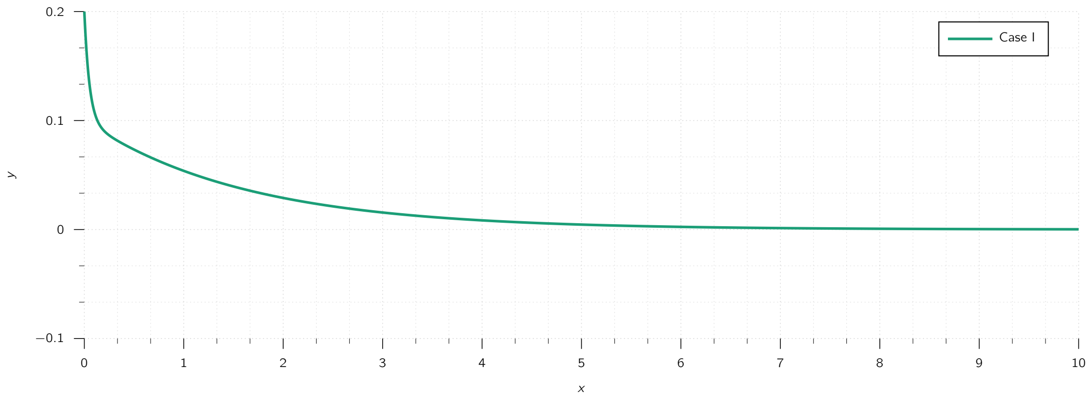


Figure 2.6

2.3 Euler-Cauchy Equations

Without much prior literature, let's get to the point. These class of equations have the form:[9]

$$x^2 y'' + axy' + by = 0 \tag{2.12}$$

To solve do the following substitutions:

$$y = x^{m}, \qquad y' = mx^{m-1}, \qquad y'' = m\,(m-1)\,x^{m-2}.$$

Which gives:

$$x^2 m\,(m-1)\,x^{m-2} + axmx^{m-1} + bx^{m} = 0.$$

$y = x^{m}$ is a good choice as it produces a common factor x^{m}.

Simplifying the equation gives us the **auxiliary** equation.

$$m^2 + (a-1)m + b = 0. \tag{2.13}$$

$y = x^{m}$ is a solution of Eq. (2.12) if and only if m is a root of Eq. (2.13).

The roots of Eq. (2.13) are:

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}.$$

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^{m} \ln x + c_2 x^{m}$
	(m)	
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta \mathrm{j}$ and $m_2 = \alpha - \beta \mathrm{j}$	$\alpha = \mathrm{Re}\,(m)$ and $\beta = \mathrm{Im}\,(m)$

Table 2.3: Possible solutions of the Euler-Cauchy based on the m value.

Complex conjugate roots are of minor practical importance for practical purposes.

[9]**Augustin-Louis Cauchy** *(1789 - 1857)*

A French mathematician, engineer, and physicist. He was one of the first to rigorously state and prove the key theorems of calculus (thereby creating real analysis), pioneered the field complex analysis, and the study of permutation groups in abstract algebra. Cauchy also contributed to a number of topics in mathematical physics, notably continuum mechanics.

Exercise 2.8 A General Solution in the Case of Different Real Roots

Solve the following ODE:

$$x^2y'' + 1.5xy' - 0.5y = 0$$

SOLUTION This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

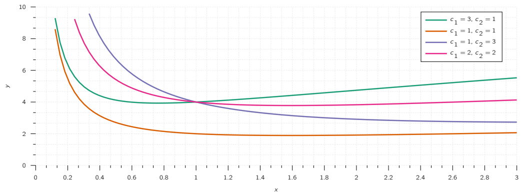


Figure 2.7: Solution to A General Solution in the Case of Different Real Roots with different constants.

Exercise 2.9 A General Solution in the Case of a Double Root

Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

SOLUTION Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

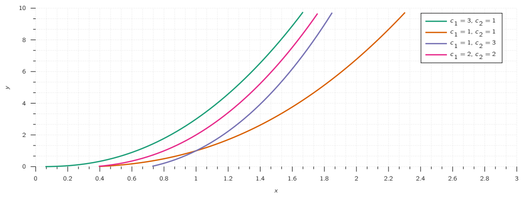


Figure 2.8: Solution to A General Solution in the Case of a Double Root with different constants.

Exercise 2.10 Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of the *Euler-Cauchy equation* $rv'' + 2v' = 0$.

SOLUTION The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

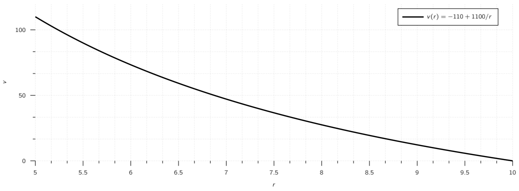


Figure 2.9: Solution to the Electric Potential Field Between Two Concentric Spheres.

2.4 Non-Homogeneous ODEs

To start, they have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.14}$$

where $r(x) \neq 0$. A general solution of Eq. (2.14) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.15}$$

and a **particular solution** of Eq. (2.14). These two (2) new terms **general solution** of Eq. (2.14) and **particular solution** of Eq. (2.14) are defined as follows:

Theory 2.3: General Solution and Particular Solution

A general solution of the non-homogeneous ODE Eq. (2.14) on an open interval I is a solution of the form:

$$y(x) = y_h(x) + y_p(x)\,. \tag{2.16}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.15) on I and y_p is any solution of Eq. (2.14) on I containing **no arbitrary constants**. A particular solution of Eq. (2.14) on I is a solution obtained from Eq. (2.16) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

2.4.1 Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.14) or an initial value problem for Eq. (2.14), we have to solve the homogeneous ODE Eq. (2.15) or an initial value problem for and find any solution y_p of Eq. (2.14), so that we obtain a general solution Eq. (2.16) of Eq. (2.14).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.17}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.

Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Theory 2.4: Choice Rules for the Method of Undetermined Coefficients

Basic Rule

If $r(x)$ in Eq. (2.17) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.17).

Modification Rule

If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.17), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule

If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x} (K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x} (K \cos \omega x + M \sin \omega x)$

Table 2.4: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.14) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.14) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Exercise 2.11 Application of the Basic Rule A

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

SOLUTION **General Solution of the Homogeneous ODE** The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

Solution of the non-Homogeneous ODE First we try

$y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

Looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0,$$
$$y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Therefore:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, therefore $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$
$$\text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

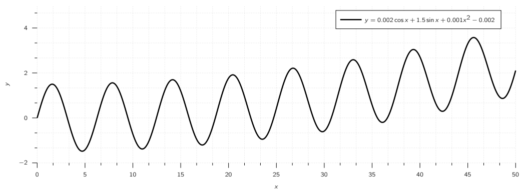


Figure 2.10: Solution to Application of the basic rule A.

Exercise 2.12 Application of the Basic Rule B

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x},$$
$$y(0) = 1, \qquad y'(0) = 0.$$

SOLUTION **General solution of the homogeneous ODE** The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$$

Therefore the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\, e^{-1.5x}$$

Solution y_p of the non-homogeneous ODE The function $e^{-1.5x}$ on the Right Hand Side (RHS) would normally require the choice $Ce^{-1.5x}$. However, we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Which means, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose:

$$y_p = Cx^2e^{-1.5x}, \quad \text{then}$$
$$y_p' = C(2x - 1.5x^2)e^{-1.5x},$$
$$y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2e^{-1.5x}$$

Step 3. Solution of the initial value problem Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2v)e^{-1.2x}$$

$$-10xe^{-1.2x} + 7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$ and gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

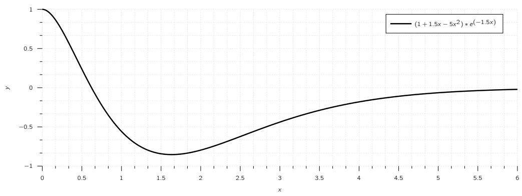


Figure 2.11: Solution to Application of the basic rule B.

Exercise 2.13 Application of the Basic Rule C

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

SOLUTION **The General Solution** The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1e^{-x/2} + c_2e^{-3x/2}.$$

The Particular Solution We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

therefore

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE is

$$y = c_1e^{-x/2} + c_2e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Solution of the initial value problem From y, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

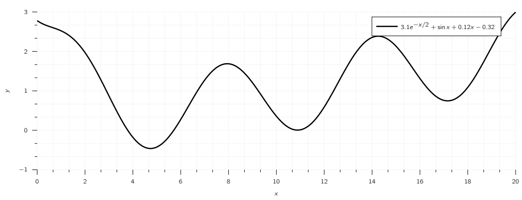


Figure 2.12: Solution to Application of the basic rule C.

2.5 A Study of Forced Oscillations and Resonance

Previously we considered vertical motions of a mass-spring system[10] and modelled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.18}$$

[10] as in the vibration of a mass m on an elastic spring

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass-spring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia $my\prime\prime$, the damping force $cy\prime$ (if $c < 0$), and the spring force ky, a restoring force.

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.18) into:

$$my'' + cy' + ky = r(t). \tag{2.19}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.20}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.20) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.18) plus any solution y_p of Eq. (2.20). To find y_p, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.21}$$

By differentiating this function (remember the chain rule) we obtain:

$$\begin{aligned} y_p' &= -\omega a \sin \omega t + \omega b \cos \omega t, \\ y_p'' &= -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t. \end{aligned}$$

Substituting y_p, $y_p\prime$, $y_p\prime\prime$, into Eq. (2.20) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2)equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.22}$$

$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.23}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega\, c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $\left(km\omega^2\right)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.24}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.20) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.18) and y_p is given by Eq. (2.21) with coefficients Eq. (2.24).

2.5.1 Solving Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I\, \mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t. \tag{2.25}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.25) is the sum $I = I_{\text{h}} + I_{\text{p}}$, where I_{h} is a general solution of the homogeneous ODE corresponding to Eq. (2.25) and I_{p} is a particular solution of Eq. (2.25). We first determine I_{p} by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned}
I_p &= a \cos \omega t \; + \; b \sin \omega t, \\
I'_p &= \omega(-a \sin \omega t \; + \; b \cos \omega t), \\
I''_p &= \omega^2(-a \cos \omega t \; - \; b \sin \omega t).
\end{aligned}$$

into (1). Then we collect the cosine terms and equate them to E0v cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned}
L\omega^2(-a) + R\omega b + a/C &= E_0 \omega \qquad \text{(Cosine terms)} \\
L\omega^2(-b) + R\omega(-a) + b/C &= 0 \qquad \text{(Sine terms).}
\end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.26}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$\begin{aligned}
-Sa + Rb &= E_0, \\
-Ra - Sb &= 0.
\end{aligned}$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 \; + \; R^2)a \; = \; E_0 S, \qquad (R^2 \; + \; S^2)b \; = \; E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 \; + \; S^2}, \qquad b = \frac{E_0 R}{R^2 \; + \; S^2}. \tag{2.27}$$

Equation (2) with coefficients a and b given by Eq. (2.27) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage.

Using Eq. (2.27), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.28}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $(R^2 + S^2)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.28) and whose frequency is that of the input (i.e., voltage).

Exercise 2.14 Harmonic Oscillation of an Undamped Mass-Spring System

If a mass-spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

SOLUTION Hooke's law:

$$F_1 = -ky \tag{2.29}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0{,}48\,\text{Hz}$$

From Eq. (2.29) and the initial conditions, $y(0) = A = 0{,}16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$.

Hence the motion is

$$y(t) = 0.16\cos 3t\,\text{m} \quad \blacksquare$$

Exercise 2.15 Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped Mass-Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\mathrm{kg \cdot s^{-1}}$
- $c = 60\,\mathrm{kg \cdot s^{-1}}$
- $c = 10\,\mathrm{kg \cdot s^{-1}}$

SOLUTION It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass-Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0{,}16\,\mathrm{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9\,t} + 0.18e^{-\,t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3\,t}, \qquad \text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3\,t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y\prime(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96\mathrm{j}$$

This gives the general solution:

$$y = e^{-0.5\,t}\left(A\cos 2.96t + B\sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$\begin{aligned} y\prime = e^{-0.5\,t}(&-0.5A\cos 2.96t - 0.5B\sin 2.96t \\ &- 2.96A\sin 2.96t + 2.96B\cos 2.96t) \end{aligned}$$

Hence $y\prime(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution:

$$\begin{aligned} y &= e^{-0.5\,t}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right) \\ &= 0.162e^{-0.5\,t}\cos\left(2.96t - 0.17\right) \quad \blacksquare \end{aligned}$$

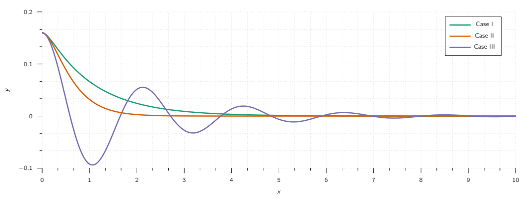


Figure 2.13: Three cases of damped motion.

Exercise 2.16 Studying a RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\,\Omega$, $L = 0{,}9\,\text{H}$, $C = 0{,}01\,\text{F}$, which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Note Assume that current and capacitor charge are 0 when $t = 0$.

SOLUTION **The General solution** Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t.$$

$$0.1I'' + 11I' + 100I = 110 \cdot 377\cos 377t.$$

Therefore the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1e^{-10t} + c_2e^{-100t}.$$

The Particular solution We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from:

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Therefore in our present case, a general solution of the nonhomogeneous ODE is:

$$I(t) = c_1e^{-10t} + c_2e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

Particular solution satisfying the initial conditions How to use Q(0) 0? We finally determine c_1 and c_2 from the in initial conditions I(0) = 0 and Q(0) = 0.

From the first condition and the general solution we have:

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796\cdot 377 = 0 \qquad \text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 30$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

> You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

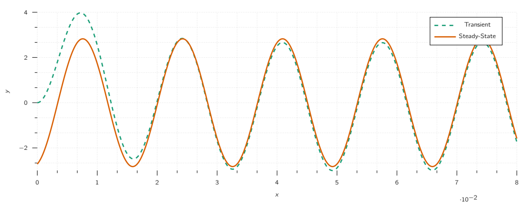


Figure 2.14: A comparison of the actual solution and the steady-state values.

Glossary

IVP Initial Value Problem. 7, 9

LHS Left Hand Side. 9

ODE Ordinary Differential Equation. 6, 7, 9–12, 14–21, 23, 29, 31, 32, 38

PDE Partial Differential Equation. 6

RHS Right Hand Side. 31

Bibliography

[1] T. Archibald, C. Fraser, and I. Grattan-Guinness, "The history of differential equations, 1670–1950," *Oberwolfach reports*, vol. 1, no. 4, pp. 2729–2794, 2005.