Topics on Fundamental Science

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The document is designed with no intention of publication and has only been designed for education purposes.

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Part Contents

Physics is written in this grand book . . . which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.

(Galilei, Galileo: Il Saggiatore, Chapter 6)

Chapter 1

First-Order Ordinary Differential Equations

Table of Contents

1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a mathematical expression in terms of its variables, functions, and equations. Such an expression is known as a **mathematical model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**. Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,

- find new values,
- interpret result in a physical terms.

This is all done to understand the behaviour of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout the chapter of our book.

An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually written as $y(x)$. The equation may also contain y itself, known functions of x, and constants.

For example all the equation shown below are classified as ODE:

$$y' = \sin x, \qquad y'' + 9y = e^{-3x}, \qquad y'y'' - \frac{5}{4}y = 0.$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from Partial Differential Equation (PDE)s, which involve **partial** derivatives of an unknown function of two or more variables[1]. For instance, a PDE with unknown function u of two (2) variables x and y is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

[1] The topic of PDE will be the focus of **Higher Mathematics II**.

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

For now, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F\left(x, y, y'\right) = 0, \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE:

$$x^{-4}y' - 3y^2 = 0 \qquad \text{where} \qquad x \neq 0$$

can be written explicitly as $y' = 3x^4y^2$.

1.1.1 The Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y\left(x_0\right) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

An ODE, together with an initial condition, is called an **initial value problem**.

Theory 1.1: Initial Value Problem

In multi-variable calculus, an Initial Value Problem (IVP) is an ODE together with an **initial condition** which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Exercise 1.1 An Initial Value Problem

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

SOLUTION The general solution is:

$$y(x) = ce^{3x}$$

From the solution and the initial condition:

$$y(0) = ce^0 = c = 5.7$$

Hence the initial value problem has the solution:

$$y(x) = 5.7e^{3x}$$

This is a particular solution which can be checked by entering it back into the main equation. Visually the solution is plotted as follows ■

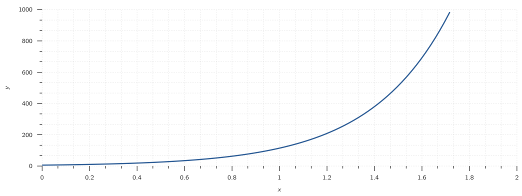


Figure 1.1: A Solution to the initial value problem.

Exercise 1.2 Radioactive Decay

Given 0,5 g of a radioactive substance, find the amount present at any later time. The decay of Radium is measured to be $k = 1{,}4 \times 10^{-11}\,\text{s}^{-1}$.

SOLUTION We know $y(t)$ is the substance amount still present at t. Using the law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is **positive**, and due of the minus, we get *decay*. We know k which the question has given as $k = 1{,}4 \times 10^{-11}\,\text{s}^{-1}$. Now the given initial amount is 0,5 g, and we can call the corresponding instant $t = 0$. We have the initial condition $y(0) = 0.5$, which is the instant the process begins. Therefore, the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

We conclude the ODE is an exponential decay and has the general solution:

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition which gives $y(0) = c = 0.5$. Therefore:

$$y(t) = 0.5e^{-kt} \quad ■$$

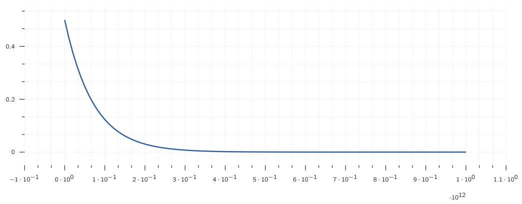


Figure 1.2: A Solution to the radioactive decay.

Information: Prey-Predator Model

The *Lotka - Volterra* equations, are a pair of 1st-order non-linear differential equations, used to describe the dynamics of biological systems in which two (**2**) species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = -\gamma y + \delta xy,$$

where:

- x, y is the population density of prey, predator,
- dy/dt growth rates of the two populations,
- t represents time;
- α, β are the maximum prey per capita growth rate, and the effect of of predators on the prey death rate.
- γ, δ are the predator's per capita death rate, and the effect of of prey on the predator's growth rate.

Figure 1.3: Bunny, natures fast food.

NOTE All parameters are positive and real.

The solution of the differential equations is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping.

1.2 Separable Ordinary Differential Equations

Many practically useful ODEs can be reduced to the following form:

$$g(y)\,y' = f(x) \tag{1.4}$$

using only algebraic manipulations. We can then do integration on both sides with respect to x, obtaining the following expression:

$$\int g(y)\,y'\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + c. \tag{1.5}$$

As a gentle reminder, c here is an integration constant. On the Left Hand Side (LHS) we can switch to y as the variable of integration.

By calculus, we know the relation $y'\,\mathrm{d}x = \mathrm{d}y$, so that:

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + c. \tag{1.6}$$

If f and g are continuous functions,[2] the integrals in Eq. (1.6) **exist**, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

[2] a continuous function is a function such that a small variation of the argument induces a small variation of the value of the function.

Exercise 1.3 Radiocarbon Dating

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian-Italian border, caused a scientific sensation.

When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

NOTE The half-life of carbon is 5175 years.

SOLUTION Radioactive decay is governed by the ODE $y' = ky$ as we have discussed previously. By separation and integration

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t, \quad \ln|y| = kt + c, \quad y = y_0 e^{kt}, \quad y_0 = e^0.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5,$$
$$k = \frac{\ln 0.5}{H} = \frac{0.693}{5715} = -0.0001213.$$

we then use the ratio 52.5% to determine the time:

$$e^{kt} = e^{-0.0001\,213t} = 0.525,$$
$$t = \frac{\ln 0.525}{-0.0001213} = 5312 \quad \blacksquare$$

Exercise 1.4 A Bell Shaped Curve

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

SOLUTION By separation and integration,

$$\frac{\mathrm{d}y}{y} = -2x\,\mathrm{d}x, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution:

$$y = 1.8e^{-x^2}$$

This is a particular solution, representing a bell-shaped curve ■

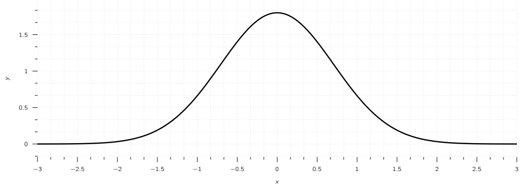


Figure 1.4: A Solution to the Separable ODE.

Exercise 1.5 Separable ODE

Solve the following ODE:

$$y' = 1 + y^2$$

SOLUTION The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx \qquad \text{By integration} \qquad ,$$

$$\arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

Note It is important to introduce the constant c when the integration is performed.

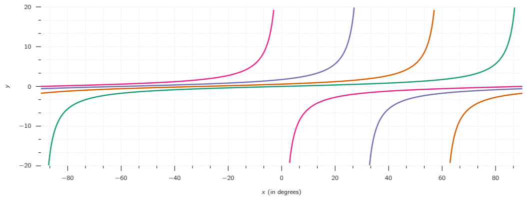


Figure 1.5: A Solution to the Separable ODE.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is **NOT** a solution, when $c \neq 0$ ■

1.2.1 Reduction to Separable Form

Certain **non-separable** ODEs can be made separable by transformations which introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right). \tag{1.7}$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Exercise 1.6 Reduction to Separable Form

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

SOLUTION To get the usual explicit form, we start by dividing the given equation by $2xy$ which gives,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then we simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

We see in the last equation that we can now separate the variables,

$$\frac{2u\,\mathrm{d}u}{1+u^2} = -\frac{\mathrm{d}x}{x} \quad \text{and by integration} \quad \ln\left(1+u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

We now take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx \qquad \text{therefore we can get} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centres on the x-axis, which can be seen below ■

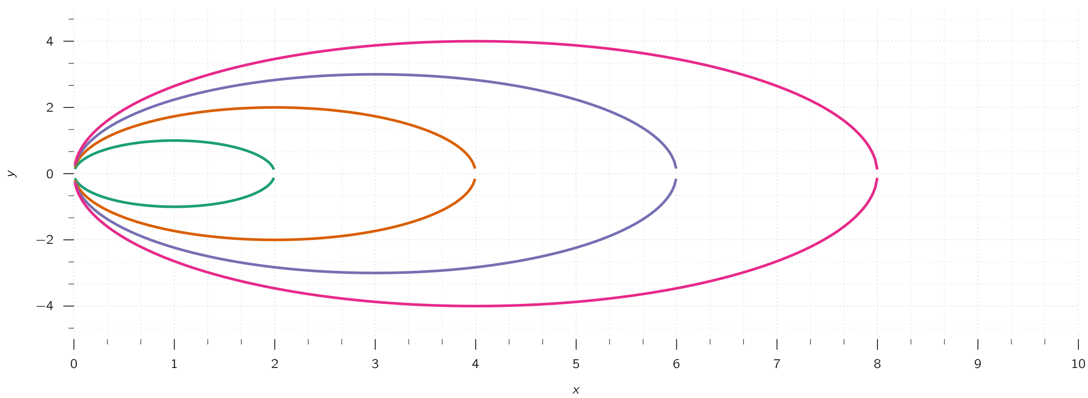


Figure 1.6: A Solution to the Separable ODE.

Information: Reason Behind the Leibniz Notation

To some of you, it might look archaic as to why the notation is written as d^2y/dx^2.

Purely symbolically, if we accept that $\mathrm{d}y = f'(x)\,\mathrm{d}x$, and treat $\mathrm{d}x$ as a constant, then:

$$\mathrm{d}^2y = \mathrm{d}(\mathrm{d}y) = \mathrm{d}(f'(x)\,dx) = dx\,d(f'(x)) = dx\,f''(x)\,dx = f''(x)\,(dx)^2,$$

As to where this notation actually comes from, though: My guess is that it comes from a time when mathematicians primarily thought of and as "infinitesimal quantities."

It is just customary to write $\mathrm{d}x^2$to denote $(\mathrm{d}x)^2$ in all common theories of calculus,

1.3 Exact Ordinary Differential Equations

If we remember from calculus courses, if a function $u(x, y)$ has continuous partial derivatives, its differential[3] is:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

From this it follows that if $u(x, y) = c =$ is constant, then $du = 0$.

[3] i.e., total differential.

As an example, let's have a look at the function:

$$u = x + x^2 y^3 = c$$

Finding its factors:

$$du = \left(1 + 2xy^3\right) dx + 3x^2 y^2 dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE in the form $M(x, y) + N(x, y) y' = 0$, written as:

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1.8}$$

is called an **exact differential equation** if the differential form $M(x, y)\, dx + N(x, y)\, dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{1.9}$$

of some function $u(x, y)$. Then Eq. (**1.8**) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (**1.8**) in the form:

$$u(x, y) = c \tag{1.10}$$

Comparing Eq. (**1.8**) and Eq. (**1.9**), we see that Eq. (**1.8**) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N. \tag{1.11}$$

From this we can derive a formula for checking whether Eq. (**1.8**) is exact or not, as follows.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.11),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x}, \tag{1.12}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{1.13}$$

By the assumption of continuity the two second partial derivatives are **equal**. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.14}$$

This condition is not only necessary but also sufficient for Eq. (1.8) to be an exact differential equation.

If Eq. (1.8) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way. From Eq. (1.12) we have by integration with respect to x:

$$u = \int M \,\mathrm{d}x + k(y), \tag{1.15}$$

in this integration, y is to be regarded as a **constant**, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.15), use Eq. (1.11) (a) to get dk/dy, and integrate dk/dy to get k.

Formula Eq. (1.15) was obtained from Eq. (1.12).

It is valid to use **either** of them and arrive at the same result.

Then, instead of Eq. (1.15), we first have by integration with respect to y

$$u = \int N \,\mathrm{d}y + l(x). \tag{1.16}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from , use Eq. (1.12) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Exercise 1.7 Exact ODE - An Initial Value Problem

Solve the initial value problem:

$$(\cos y \sinh x + 1)\ \mathrm{d}x - \sin y \cosh x \,\mathrm{d}y = 0,$$
$$\text{with} \qquad y(1) = 2.$$

SOLUTION Let's begin by verifying the given equation is **exact**:

$$M(x, y) = (\cos y \sinh x + 1),$$

$$N(x, y) = -\sin y \cosh x.$$

We now apply our criteria:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin y \sinh x$$

This shows the given ODE is **exact**. We find u. For a change, let us use Eq. (**1.16**):

$$u = -\int \sin y \cosh x dy + l(x) = \cos y \cosh x + l(x).$$

From this:

$$\partial u/\partial x = \cos y \sinh x + \frac{dl}{dx} = u = \cos y \sinh x + 1$$

Therefore $dl/dx = 1$ and by integration,

$$l(x) = x + c^*.$$

This gives the general solution

$$u(x, y) = \cos y \cosh x + x = c.$$

From the initial condition:

$$\cos 2 \cosh 1 + 1 = 0.358 = c$$

Therefore the answer is:

$$\cos y \cosh x + x = 0.358 \quad \blacksquare$$

Exercise 1.8 An Exact ODE

Solve the following ODE:

$$\cos(x + y)\ dx + \left(3y^2 + 2y + \cos(x + y)\right)\ dy = 0.$$

SOLUTION

The solution is as follows:

Test for exactness First check if our equation is **exact**, try to convert the equation of the form Eq. (**1.8**):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

This proves our equation to be exact.

Implicit General Solution From Eq. (**1.15**), we obtain by integration:

$$\begin{aligned} u &= \int M\, dx + k(y) \\ &= \int \cos(x + y)\ dx + k(y) \\ &= \sin(x + y) + k(y) \end{aligned} \qquad (1.17)$$

To find $k(y)$, we differentiate this formula with respect to y and use formula Eq. (**1.13**), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Therefore $\frac{dk}{dy} = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^\star$. Inserting this result into Eq. (**1.17**) and observing Eq. (**1.10**), we obtain:

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c \quad \blacksquare$$

Exercise 1.9 The Breakdown of Exactness

Check the exactness of the following ODE:

$$-y\, dx + x\, dy = 0$$

SOLUTION The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

Let us show that in such a case the present method does **NOT** work.

$$u = \int M\, dx + k(y) = -xy + k(y),$$

$$\frac{\partial u}{\partial y} = -x + \frac{\partial k}{\partial y}.$$

Now, $\partial u/\partial y$ should equal $N = x$, as required for this equation to be exact. However, this is impossible because $k(y)$ can depend only on y $\blacksquare$

1.4 Linear Ordinary Differential Equations

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A 1st-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)\, y = r(x) \tag{1.18}$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

1.4.1 Homogeneous Linear Ordinary Differential Equations

We want to solve Eq. (1.18) in some interval $a < x < b$, let's call it J, and we begin with the simpler special case where $r(x)$ is zero for all x in J.[4] Then the ODE given in Eq. (1.18) becomes:

[4] This is sometimes written $r(x) = 0$.

$$y' + p(x)\, y = 0 \tag{1.19}$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{\mathrm{d}y}{y} = -p(x)\, \mathrm{d}x, \qquad \text{therefore} \qquad \ln|y| = -\int p(x)\, \mathrm{d}x + c^*.$$

Taking exponents on both sides, the general solution of the homogeneous ODE Eq. (1.19) is,

$$y(x) = ce^{-\int p(x)\, \mathrm{d}x} \qquad \left(c = \pm e^{\,c^*} \qquad \text{when} \qquad y \neq 0\right) \tag{1.20}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

1.4.2 Non-Homogeneous Linear Ordinary Differential Equations

We now solve Eq. (1.18) in the case that $r(x)$ in Eq. (1.18) is **NOT** everywhere zero in the interval J considered. Then the ODE Eq. (1.18) is called non-homogeneous. It turns out that in this case, Eq. (1.18) has a useful property. Namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ as follows.

We multiply Eq. (1.18) by $F(x)$, obtaining:

$$Fy' + pFy = rF. \tag{1.21}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{therefore} \qquad pF = F'.$$

By separating variables, $\mathrm{d}F/F = p\,\mathrm{d}x$. By integration, writing $h = \int p\,\mathrm{d}x$,

$$\ln|F| = h = \int p\,\mathrm{d}x, \qquad \text{therefore} \qquad F = e^{h}.$$

With this F and $h' = p$, Eq. (1.21) becomes:

$$e^{h}y' + h'e^{h}y = e^{h}y' + \left(e^{h}\right)'y = \left(e^{h}y\right)' = re^{h} \quad \text{by integration} \quad e^{h}y = \int e^{h}r\,\mathrm{d}x + c$$

Dividing by e^{h}, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^{h}r\,\mathrm{d}x + c\right), \qquad h = \int p(x)\,\mathrm{d}x. \tag{1.22}$$

This reduces solving Eq. (1.18) to the generally simpler task of evaluating integrals.[5] The structure of Eq. (1.22) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing Eq. (1.22) as a sum of two terms,

[5] For ODEs for which this is still difficult, we may have to use a numeric method for integrals or for the ODE itself.

$$y(x) = e^{-h}\int e^{h}r\,\mathrm{d}x + ce^{-h},$$

Exercise 1.10 A Non Homogeneous Ordinary Differential Equation

Solve the initial value problem of the following equation:

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

SOLUTION Here we define the parameters as:

$$p = \tan x, \quad r = \sin 2x = 2\sin x\cos x,$$

and

$$h = \int p\,\mathrm{d}x = \int \tan x\,\mathrm{d}x = \ln|\sec x|.$$

From this we see that in Eq. (1.22),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x,$$
$$e^{h}r = (\sec x)(2\sin x\cos x) = 2\sin x,$$

and the general solution of our equation is:

$$\begin{aligned} y(x) &= \cos x\left(2\int \sin x\,\mathrm{d}x + c\right), \\ &= c\cos x - 2\cos^2 x. \end{aligned}$$

From this and the initial condition

$$1 = c\cdot 1 - 2\cdot 1^2, \qquad \text{therefore} \qquad c = 3,$$

and the solution of our initial value problem is:

$$y = 3\cos x - 2\cos^2 x$$

Here $3\cos x$ is the response to the initial data, and $-2\cos^2 x$ is the response to the input $\sin 2x$ ■

Chapter 2

Second-Order Ordinary Differential Equations

Table of Contents

2.1 Introduction

A second order ODE is a specific type of differential equation which consists of a derivative of a function of **order 2** and no other higher-order derivative of the function appears in the equation. These equations have significant engineering applications such as in the study of mechanical and electrical vibrations, wave motion, and heat conduction.

A second-order ODE is called **linear**, if it can be written[1] as:

[1] in its standard form.

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2.1}$$

Of course, we can extend most of what we learned in the study of first-order ODE and describe:

homogeneous if $r(x) = 0$,

non-homogeneous if otherwise.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs. For example:

$$y'' = 25y - e^{-x}\cos x \qquad \text{non-homogeneous linear}$$

$$y'' + \frac{1}{x}y\prime + y = 0 \qquad \text{homogeneous linear}$$

$$y''y + \left(y''\right)^2 = 0 \qquad \text{non-linear}$$

2.1.1 The Principle of Superposition

For the homogeneous form the backbone of finding a useful solution is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1 y_1 + c_2 y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or **linearity principle**.

Theory 2.2: Fundamental Theorem for the Homogeneous Linear ODE

For a homogeneous linear ODE of the form:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

any linear combination of two (2) solutions on an open interval I is again a solution of on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions. This theorem is only applicable to **homogeneous** form.

While the iron is hot, lets do a couple of exercises to begin studying 2$^{\text{nd}}$-order ODEs:

Exercise 2.1 A Superposition of Solutions

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0, \quad \text{for all } x.$$

SOLUTION By differentiation and substitution, we obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)$$

$$= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare$$

Exercise 2.2 **Example of a Non-Homogeneous Linear ODE**

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 1$$

SOLUTION Testing out the first function

$$(1 + \cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1 + \sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

While the methodology is same as before, it is worth mentioning here the small difference. For a second-order homogeneous linear ODE, an initial value problem consists of two (2) initial conditions:[2]

[2] This makes sense as to properly evaluate a 2nd differential equation, we need two values.

$$y(x_0) = K_0 \qquad y'(x_0) = K_1. \tag{2.2}$$

The Eq. (2.2) are used to determine the two (2) arbitrary constants c_1 and c_2 in a general solution

Exercise 2.3 **An Initial Value Problem**

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

SOLUTION **General Solution** From the previous examples, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Particular Solution

We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.

Exercise 2.4 **Reduction of Order**

Find a basis of solutions of the ODE

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

SOLUTION Inspection shows $y_1 = x$ is a solution as $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u,$$
$$y'' = u''x + 2u'$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0,$$
$$(x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely,

$$(x^2 - x)w' + (x - 2)w = 0.$$

Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx,$$
$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We don't need constant of integration as we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain n

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x},$$
$$\text{hence} \quad y_2 = ux = x\ln|x| + 1$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained as basic of solutions, valid for all positive x ■

Information: Differential Equations - Early Beginnings

Differential equations have been a major branch of pure and applied mathematics since mid 17$^{\text{th}}$ century.

“Differential equations” began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's “fluxional equations” in the 1670s. Applications were made largely to geometry and mechanics with particular interest in optimisation. Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, which lead to partial differential equations. Generalisation of isoperimetrical problems led to the calculus of variations.

New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace. Development of the general theory of solutions included singular ones, functional solutions and those by infinite series. Many applications were made to mechanics, especially to astronomy and continuous media [1].

2.2 Homogeneous Linear ODEs

Consider 2$^{\text{nd}}$-order homogeneous linear ODEs whose coefficients a and b are **constant**.

$$y'' + ay' + by = 0. \tag{2.3}$$

We start to solve the above equation by starting:

$$y = e^{\lambda x} \tag{2.4}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.3) gives:

$$\left(\lambda^2 + a\lambda + b\right) e^{\lambda x} = 0.$$

Therefore if λ is a solution of the important **characteristic** equation,[3]

$$\lambda^2 + a\lambda + b = 0, \tag{2.5}$$

then the exponential function Eq. (2.4) is a solution of the ODE given in Eq. (2.3). Now from algebra we recall the roots of the quadratic equation:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \tag{2.6}$$

Using Eq. (2.5) and Eq. (2.6) we can see that

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

as solutions to Eq. (2.3). From algebra we further know that the quadratic equation Eq. (2.5) may have three (3) kinds of roots, depending on the sign of the discriminant[4] $(a^2 - 4b)$, which are shown in **Tbl.** 2.1.

[3] This is also known as an auxiliary equation.

[4] A function of the coefficients of a polynomial equation whose value gives information about the roots of the polynomial.

Exercise 2.5 IVP: Case of Distinct Real Roots

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

SOLUTION **General Solution** The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1,$$

$$\text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Particular Solution As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

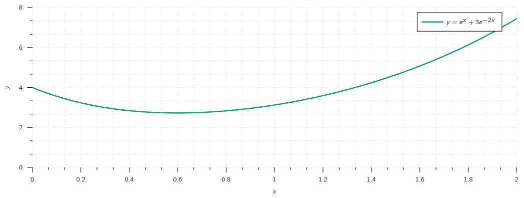


Figure 2.1: Solution to Case of distinct real roots.

Exercise 2.6 IVP: Case of Real Double Roots

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

SOLUTION The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = \left(\lambda + 2\right)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = \left(c_1 + c_2 x\right) e^{-0.5\,x}$$

We need its derivative:

$$y\prime = c_2 e^{-0.5\,x} - 0.5\left(c_1 + c_2 x\right) e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$y\left(0\right) = c_1 = 3.0,$$

$$y\prime\left(0\right) = c_2 - 0.5c_1 = 3.5,$$

$$c_2 = -2$$

The particular solution of the initial value problem is:

$$y = \left(3 - 2x\right) e^{-0.5\,x}$$

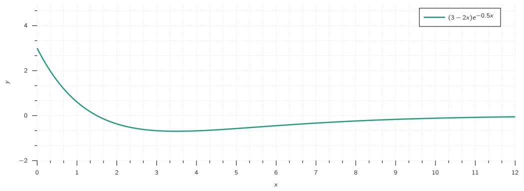


Figure 2.2: Solution to case of double roots.

Case	Condition	Roots of	Basis	General Solution
I	$a^2 - 4b > 0$	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
		$\left(\lambda_1, \lambda_2\right)$	$e^{\lambda_2 x}$	
II	$a^2 - 4b = 0$	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
		$\left(\lambda = -1/2a\right)$	$xe^{-ax/2}$	
III	$a^2 - 4b < 0$	Complex Conjugate	$e^{-ax/2}\cos\omega x$	$y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$
		$\lambda_1 = -1/2a + \mathrm{j}\omega$	$e^{-ax/2}\sin\omega x$	
		$\lambda_2 = -1/2a - \mathrm{j}\omega$		

Table 2.1: Possible roots of the characteristic equation based on the discriminant value.

Exercise 2.7 IVP: Case of Complex Roots

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

SOLUTION

General Solution The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathrm{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Particular Solution The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2\,x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows y and $-e^{-0.2\,x}$ and $e^{-0.2\,x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

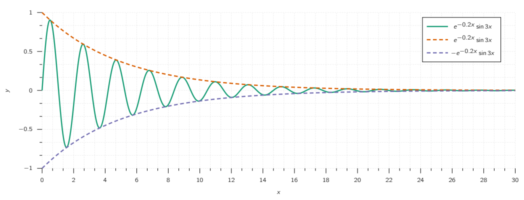


Figure 2.3: Solution to case of complex roots.

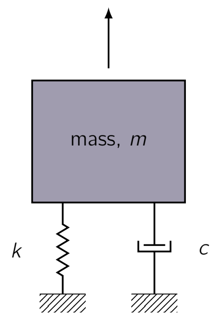
2.2.1 A Study of Damped System

Linear ODEs with constant coefficients have important applications in mechanics, and one of the important system to study is spring-mass-damper system[5], which has the following important component:

$$F_2 = -cy'.$$

Using this damping we can define the ODE of the damped mass-spring system:

$$my'' + cy' + ky = 0. \tag{2.7}$$



[5] A spring mass damper system.

This can physically be done by connecting the ball to a bowl containing a liquid. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

> This is generally a good approximation for small velocities.

The constant c is called the damping constant.

The damping force $F_2 = -cy'$ acts **against** the motion. Therefore for a downward motion we have $y' > 0$ which for positive c makes F negative,[6] as it should be.

[6] an upward force.

Similarly, for an upward motion we have $y' > 0$ which, for $c > 0$ makes F_2 positive.[7]

[7] a downward force.

The ODE Eq. (2.7) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \lambda_2 = -\alpha - \beta, \qquad \text{where} \qquad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}. \quad (2.8)$$

Depending on the amount of damping present-whether a lot of damping, a medium amount of damping or little damping-three types of motions occur, respectively. A summary of its behaviour is shown in **Tbl.** 2.2.

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.2: The three cases of behaviour depending on the condition.

A Deeper Look into the Three Cases

Case I: Over-damping If $c^2 > 4mk$, then λ_1 and λ_2 are said to be **distinct real roots**. In this case, the corresponding general solution becomes:[8]

$$y(t) = c_1 e^{-(\alpha - \beta)\ t} + c_2 e^{-(\alpha + \beta)\ t}. \qquad (2.9)$$

[8]In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (**2.9**) are negative as $\alpha > 0$ and $\beta > 0$ with:

$$\beta^2 = \alpha^2 - k/m < \alpha^2$$

Therefore both terms in Eq. (**2.9**) approach zero as $t \to \infty$. Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position (i.e., $y = 0$). A graphical representation of this behaviour can be seen in

Case II: Critical-Damping Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III) and occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$.

Then the corresponding general solution of Eq. (**2.7**) is:

$$y(t) = \left(c_1 + c_2 t\right) e^{-\alpha t}. \qquad (2.10)$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha\ t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. **Fig. ??** shows typical forms of

The graph above looks almost like those in the previous figure.

Case III: Under-Damping This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in Eq. (**2.8**) is no longer real but pure **imaginary**, which we write as,

$$\beta = \mathbf{j}\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad \text{where} \qquad \beta > 0.$$

The asterisk $(*)$ is used to differentiate from ω which is used predominantly in electrical engineering do describe angular frequency.

The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + \mathbf{j}\omega^*, \qquad \lambda_2 = -\alpha - \mathbf{j}\omega^*.$$

with $\alpha = c/2m$. The corresponding general solution is:

$$y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right) \tag{2.11}$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$ This represents **damped oscillations**. Their curve lies between the two dashed curves:

$$y = Ce^{-\alpha t} \qquad \text{and} \qquad y = -Ce^{-\alpha t}$$

The frequency of the under-damping process is $\omega^*/2\pi$ Hz. Based on the equation, we see that the smaller c is,[9] the larger is ω^* and the more rapid the oscillations become.

[9] As long as it is bigger than 0.

2.3 Euler-Cauchy Equations

Without much prior literature, let's get to the point. These class of equations have the form:[10]

$$x^2 y'' + axy' + by = 0 \tag{2.12}$$

To solve do the following substitutions:

$$y = x^{\mathrm{m}}, \qquad y' = mx^{\mathrm{m-1}}, \qquad y'' = m(m-1)\,x^{\mathrm{m-2}}.$$

Which gives:

$$x^2 m(m-1)\,x^{\mathrm{m-2}} + axmx^{\mathrm{m-1}} + bx^{\mathrm{m}} = 0.$$

$y = x^{m}$ is a good choice as it produces a common factor x^{m}.

Simplifying the equation gives us the **auxiliary** equation.

$$m^2 + (a-1)m + b = 0. \tag{2.13}$$

$y = x^{m}$ is a solution of Eq. (2.12) if and only if m is a root of Eq. (2.13).

The roots of Eq. (2.13) are:[11]

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}.$$

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^{m} \ln x + c_2 x^{m}$
	(m)	
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta \mathrm{j}$ and $m_2 = \alpha - \beta \mathrm{j}$	$\alpha = \mathrm{Re}(m)$ and $\beta = \mathrm{Im}(m)$

Table 2.3: Possible solutions of the Euler-Cauchy based on the m value.

Complex conjugate roots are of minor practical importance for practical purposes.

[10] **Augustin-Louis Cauchy**
(1789 - 1857)

A French mathematician, engineer, and physicist. He was one of the first to rigorously state and prove the key theorems of calculus (thereby creating real analysis), pioneered the field complex analysis, and the study of permutation groups in abstract algebra. Cauchy also contributed to a number of topics in mathematical physics, notably continuum mechanics.

[11] Unlike other types of equations, these class of ODEs generally are used as an intermediary step to solve more complex problems, such as PDEs, however these equations on their own is used in some areas, such as solving Laplace's equation in polar coordinates, modeling time-harmonic vibrations of a thin elastic rod, and in Black-Scholes models for option pricing [2].

Exercise 2.8 A General Solution in the Case of Different Real Roots

Solve the following ODE:

$$x^2y'' + 1.5xy' - 0.5y = 0$$

SOLUTION This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

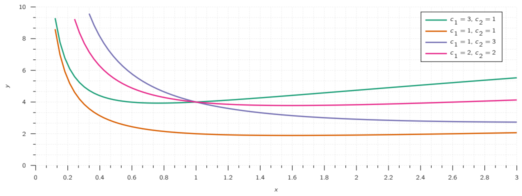


Figure 2.4: Solution to A General Solution in the Case of Different Real Roots with different constants.

Exercise 2.9 A General Solution in the Case of a Double Root

Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

SOLUTION Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

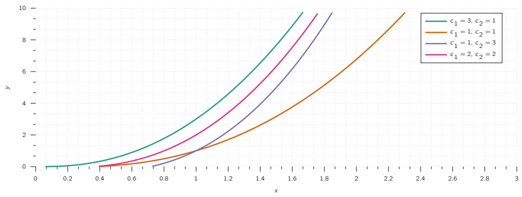


Figure 2.5: Solution to A General Solution in the Case of a Double Root with different constants.

Exercise 2.10 Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

$v = v(r)$ is a solution of the *Euler-Cauchy equation* $rv'' + 2v' = 0$.

SOLUTION The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

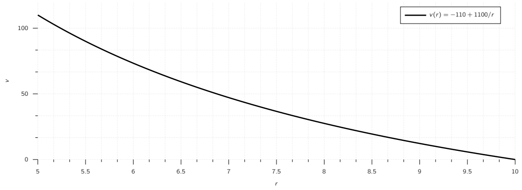


Figure 2.6: Solution to the Electric Potential Field Between Two Concentric Spheres.

2.4 Non-Homogeneous ODEs

To start, they have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.14}$$

where $r(x) \neq 0$. A general solution of Eq. (2.14) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.15}$$

and a **particular solution** of Eq. (2.14). These two (2) new terms **general solution** of Eq. (2.14) and **particular solution** of Eq. (2.14) are defined as follows:

Theory 2.3: General Solution and Particular Solution

A general solution of the non-homogeneous ODE Eq. (2.14) on an open interval I is a solution of the form:

$$y(x) = y_h(x) + y_p(x)\,. \tag{2.16}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.15) on I and y_p is any solution of Eq. (2.14) on I containing **no arbitrary constants**. A particular solution of Eq. (2.14) on I is a solution obtained from Eq. (2.16) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

2.4.1 Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.14) or an initial value problem for Eq. (2.14), we have to solve the homogeneous ODE Eq. (2.15) or an initial value problem for and find any solution y_p of Eq. (2.14), so that we obtain a general solution Eq. (2.16) of Eq. (2.14).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.17}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.

Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Theory 2.4: Choice Rules for the Method of Undetermined Coefficients

Basic Rule

If $r(x)$ in Eq. (**2.17**) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (**2.17**).

Modification Rule

If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (**2.17**), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule

If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 2.4: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (**2.14**) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (**2.14**) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Exercise 2.11 Application of the Basic Rule A

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

SOLUTION **General Solution of the Homogeneous ODE** The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

Solution of the non-Homogeneous ODE First we try

$y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \quad \text{a contradiction.}$$

Looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0,$$
$$y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \quad K_1 = 0, \quad 2K_2 + K_0 = 0$$

Therefore:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, therefore $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$
$$\text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

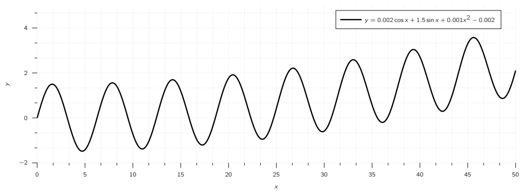


Figure 2.7: Solution to Application of the basic rule A.

Exercise 2.12 Application of the Basic Rule B

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x},$$
$$y(0) = 1, \quad y'(0) = 0.$$

SOLUTION **General solution of the homogeneous ODE** The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$$

Therefore the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2y)\,e^{-1.5x}$$

Solution y_p of the non-homogeneous ODE The function $e^{-1.5x}$ on the Right Hand Side (RHS) would normally require the choice $Ce^{-1.5x}$. However, we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Which means, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose:

$$y_p = Cx^2e^{-1.5x}, \quad \text{then}$$
$$y_p' = C(2x - 1.5x^2)e^{-1.5x},$$
$$y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2e^{-1.5x}$$

Step 3. Solution of the initial value problem Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2v)e^{-1.2x}$$

$$-10xe^{-1.2x} + 7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$ and gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} =$$
$$(1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

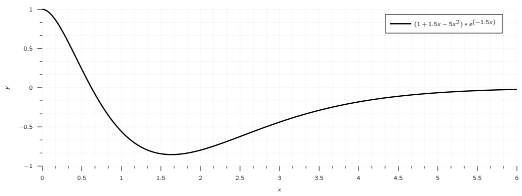


Figure 2.8: Solution to Application of the basic rule B.

Exercise 2.13 Application of the Basic Rule C

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,$$
$$y(0) = 2.78, \qquad y'(0) = -0.43.$$

SOLUTION **The General Solution** The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1e^{-x/2} + c_2e^{-3x/2}.$$

The Particular Solution We write:

$$y_p = y_{p1} + y_{p2}$$

and following the table,

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1x + K_0$$

Differentiating gives:

$$y_{p1}' = -K\sin x + M\cos x,$$
$$y_{p1}'' = -K\cos x - M\sin x,$$
$$y_{p2}' = 1,$$
$$y_{p2}'' = 0.$$

Substitution of y_{p1} into the ODE gives, by comparing the cosine and sine terms:

$$-K+2M+0.75K = 2, \qquad -M-2K+0.75M = -0.25$$

Therefore $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0,$$

therefore

$$K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE is

$$y = c_1e^{-x/2} + c_2e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Solution of the initial value problem From y, y' and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78,$$
$$y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

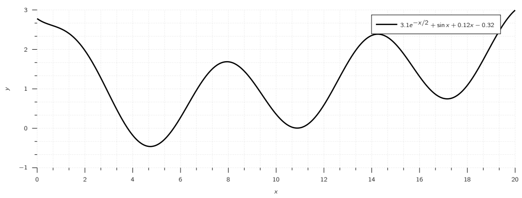


Figure 2.9: Solution to Application of the basic rule C.

2.5 A Study of Forced Oscillations and Resonance

Previously we considered vertical motions of a mass-spring system[12] and modelled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.18}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass-spring system exhibited only free motion. This means no external forces[13] but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia my'', the damping force cy' (if $c < 0$), and the spring force ky, a restoring force.

[12] as in the vibration of a mass m on an elastic spring

[13] i.e., outside forces

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.18) into:

$$my'' + cy' + ky = r(t). \tag{2.19}$$

Mechanically, this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the output or the response of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t. \qquad [\, F_0>0, \omega>0]$$

Then we have the non-homogeneous ODE:

$$\boxed{my'' + cy' + ky = F_0 \cos \omega t \tag{2.20}}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.20) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.18) plus any solution y_p of Eq. (2.20). To find y_p, we use the **method of undetermined coefficients**, starting from

$$\boxed{y_p(t) = a \cos \omega t + b \sin \omega t \tag{2.21}}$$

By differentiating this function we obtain:

$$\begin{aligned} y_p' &= -\omega a \sin \omega t + \omega b \cos \omega t, \\ y_p'' &= -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t. \end{aligned}$$

Substituting y_p, $y_p\prime$, $y_p\prime\prime$, into Eq. (**2.20**) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2) equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.22}$$

$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.23}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega\, c$ and add the results, obtaining:

$$\left(k - m\omega^2\right)^2 a + \omega_2 c^2 a = F_0 \left(k - m\omega^2\right).$$

Similarly, to eliminate a, multiply the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $\left(km\omega^2\right)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.24}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (**2.20**) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (**2.18**) and y_p is given by Eq. (**2.21**) with coefficients Eq. (**2.24**).

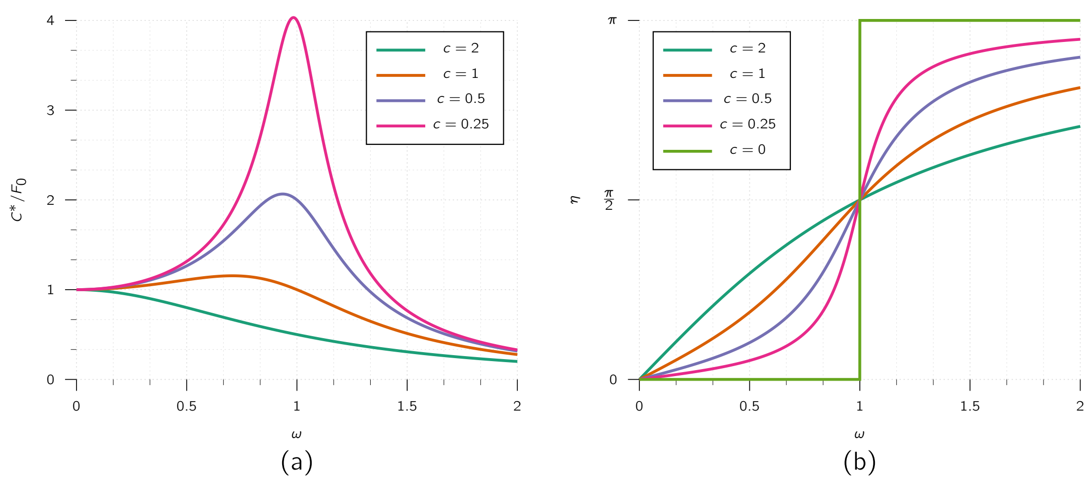
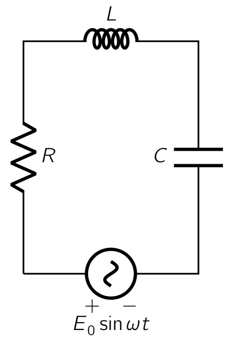


Figure 2.10 (a) Amplification (C^*/F_0) as a function of ω for $k = 1, m = 1$, and various values of the damping constant (c) (b) Phase lag (η) as a function of ω for $k = 1, m = 1$, and various values of the damping constant (c)

2.5.1 Solving Electric Circuits

Let's study a simple RLC Circuit.[14] These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a capacitor.



[14]A simple RLC circuit waiting to be examined.

> A capacitor is a passive, electrical component which has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I \, \mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation". To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega \cos \omega t \tag{2.25}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous 2nd-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.25) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to Eq. (2.25) and I_p is a particular solution of Eq. (2.25). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$I_p = a \cos \omega t + b \sin \omega t, \tag{2.26}$$

$$I_p' = \omega \left(-a \sin \omega t + b \cos \omega t\right), \tag{2.27}$$

$$I_p'' = \omega^2 \left(-a \cos \omega t - b \sin \omega t\right). \tag{2.28}$$

into Eq. (2.25). Then we collect the cosine terms and equate them to $E_0 \omega \cos \omega t$ on the right, and we equate the sine terms to zero as there is no sine term on the right,

$$L\omega^2(-a) + R\omega b + a/C = E_0\omega \qquad \text{[Cosine terms]}$$

$$L\omega^2(-b) + R\omega(-a) + b/C = 0 \qquad \text{[Sine terms]}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:[15]

$$S = \omega L - \frac{1}{\omega C} \tag{2.29}$$

[15]reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating.

Dividing the previous two (2) equations by ω, ordering them, and substituting S gives:

$$-Sa + Rb = E_0,$$

$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding. Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-\left(S^2 + R^2\right) a = E_0 S, \qquad \left(R^2 + S^2\right) b = E_0 R.$$

We can solve this for a and b:

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.30}$$

Eq. (**2.26**) with coefficients a and b given by Eq. (**2.30**) is the desired particular solution I_p of the non-homogeneous ODE in Eq. (**2.25**) governing the current I in an RLC-circuit with sinusoidal input voltage.

Using Eq. (**2.30**), we can write I_p in terms of **physically visible** quantities, namely, amplitude (I_0) and phase lag (θ) of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \quad \text{where} \quad I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \quad \tan\theta = -\frac{a}{b} = \frac{S}{R}. \tag{2.31}$$

The quantity $\left(R^2 + S^2\right)$ is called **impedance**.[16] Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

[16] a measure of the total opposition to electrical current flow in an Alternating Current (AC) circuit, combining resistance and reactance.

A general solution of the homogeneous equation corresponding to Eq. (**2.25**) is:

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current I_p, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (**2.31**) and whose frequency is that of the input (i.e., voltage).

Exercise 2.14 Harmonic Oscillation of an Undamped Mass-Spring System

If a mass-spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

SOLUTION Hooke's law:

$$F_1 = -ky \tag{2.32}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\text{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0{,}48\,\text{Hz}$$

From Eq. (2.32) and the initial conditions, $y(0) = A = 0{,}16\,\text{m}$ and $y\prime(0) = \omega_0 B = 0$.

Hence the motion is

$$y(t) = 0.16 \cos 3t\,\text{m} \quad \blacksquare$$

Exercise 2.15 Three Cases of Damped Motion

How does the motion in *Harmonic Oscillation of an Undamped Mass-Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\text{kg}\cdot\text{s}^{-1}$
- $c = 60\,\text{kg}\cdot\text{s}^{-1}$
- $c = 10\,\text{kg}\cdot\text{s}^{-1}$

SOLUTION It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass-Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0{,}16\,\text{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9\,t} + 0.18e^{-\,t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10\left(\lambda + 3\right)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = \left(c_1 + c_2 t\right) e^{-3\,t},$$

$$\text{we also need} \qquad y\prime = \left(c_2 - 3c_1 - 3c_2 t\right) e^{-3\,t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y\prime(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96j$$

This gives the general solution:

$$y = e^{-0.5\,t}\left(A \cos 2.96t + B \sin 2.96t\right)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$\begin{aligned} y\prime = e^{-0.5\,t}(&-0.5A \cos 2.96t - 0.5B \sin 2.96t \\ &- 2.96A \sin 2.96t + 2.96B \cos 2.96t) \end{aligned}$$

Hence $y\prime(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution:

$$\begin{aligned} y &= e^{-0.5\,t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) \\ &= 0.162e^{-0.5\,t} \cos(2.96t - 0.17) \quad \blacksquare \end{aligned}$$

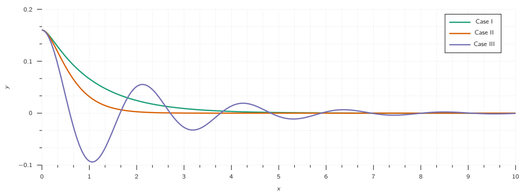


Figure 2.11: Three cases of damped motion.

Exercise 2.16 Studying a RLC Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\,\Omega$, $L = 0{,}9\,\mathrm{H}$, $C = 0{,}01\,\mathrm{F}$, which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Note Assume that current and capacitor charge are 0 when $t = 0$.

SOLUTION **The General solution** Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega cos\omega t.$$

$$0.1I'' + 11I' + 100I = 110 \cdot 377\cos 377t.$$

Therefore the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1e^{-10t} + c_2e^{-100t}.$$

The Particular solution We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from:

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71,$$

$$b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Therefore in our present case, a general solution of the nonhomogeneous ODE is:

$$I(t) = c_1e^{-10t} + c_2e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

Particular solution satisfying the initial conditions How to use Q(0) 0? We finally determine c_1 and c_2 from the in initial conditions I(0) = 0 and Q(0) = 0.

From the first condition and the general solution we have:

$$I(0) = c_1 + c_2 - 2.71 = 0 \qquad \text{hence} \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0 \qquad \text{so that} \qquad I'(0) = 0$$

Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0$$

$$\text{hence} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

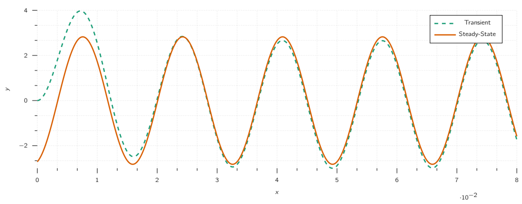


Figure 2.12: A comparison of the actual solution and the steady-state values.

Information: An Analogy Between Electrical And Mechanical Quantities

Entirely different systems can have the same mathematical model.

For instance, an electric RLC-circuit and for a mass-spring system, written in the following form

$$LI'' + RI' + \frac{1}{C}I = E_0\omega\cos\omega t \qquad \text{and} \qquad my'' + cy' + ky = F_0\cos\omega t$$

are the same, from a model perspective. **Tbl.** 2.5 shows the analogy between the various quantities involved. The inductance (L) corresponds to the mass (m) as an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance R corresponds to the damping constant c, and a resistor causes loss of energy, just as a damping dashpot does. And so on.

Electrical	**Mechanical**
Inductance (L)	Mass (m)
Resistance (R)	Damping Constant (c)
Reciprocal of Capacitance ($1/C$)	Spring modulus (k)
Derivative of EMF	Driving Force
Current (I)	Displacement ($y(t)$)

Table 2.5: An analogy between electrical and mechanical quantities

This analogy is **strictly quantitative** in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

The practical importance of this analogy is almost obvious. The analogy may be used for constructing an "electrical model" of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

Chapter 3

Higher-Order Ordinary Differential Equations

Table of Contents

3.1 Homogeneous Linear ODEs

Let's do a small revisit, and recall from First-Order ODEs that an ODE is of n^{th} if the n^{th} derivative $y^{(\mathrm{n})} = d^{\mathrm{n}}y/dx^{n}$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F\left(x, y, y', \cdots, y^{(\mathrm{n})}\right) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(\mathrm{n})} + p_{\mathrm{n}-1}(x)y^{(\mathrm{n-1})} + \cdots + p_1(x)y' + p_0(x)y = r(x)\,. \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{\mathrm{n}-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

> $y^{(\mathrm{n})}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_{\mathrm{n}}(x)y^{(\mathrm{n})}$, divide by $p_{\mathrm{n}}(x)$ to get this form.

An n$^{\text{th}}$-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.

A **solution** of an n$^{\text{th}}$-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to n$^{\text{th}}$-order homogeneous linear ODEs as following theorems.

Theory 3.5: Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

This does not hold for a nonhomogeneous or non-linear ODE.

Theory 3.6: General Solution, Basis, Particular Solution

A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \ldots, c_n \quad \text{arbitrary}) \tag{3.3}$$

where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.

That is, these solutions are linearly independent on I, as defined below.

A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

Theory 3.7: Linear Independence and Dependence

Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on} I \tag{3.4}$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \dots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n-1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (**3.4**) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Exercise 3.1 Linear Dependence

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

SOLUTION By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Exercise 3.2 A General Solution

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{where} \qquad y^{\text{iv}} = \frac{d^4 y}{dx^4}$$

SOLUTION Similar to Chapter 2 we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Exercise 3.3 Initial Value Problem for a Third-Order Euler-Cauchy Equation

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0,$$
$$y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4.$$

SOLUTION **General Solution** As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular Solution The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
&(a) \quad y(1) = c_1 + c_2 + c_3 = 2 \\
&(b) \quad y'(1) = c_1 + 2c_2 + 3c_3 = 1 \\
&(c) \quad y'(1) = 2c_2 + 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

> Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(n)} = d^n y/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \quad \cdots, \quad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Exercise 3.4 Distinct Real Roots

Solve the following ODE:

$$y_{///} - 2y'' - y' + 2y = 0$$

SOLUTION The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots:

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

Exercise 3.5 Simple Complex Roots

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0,$$
$$y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299.$$

SOLUTION The characteristic equation is:

$$\lambda_3 - \lambda_2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10j$. Therefore, a general solution and its derivatives obtained by differentiation are:

$$y = c_1 e^{x} + A \cos 10x + B \sin 10x,$$
$$y' = c_1 e^{x} - 10A \sin 10x + 10B \cos 10x,$$
$$y'' = c_1 e^{x} - 100A \cos 10x - 100B \sin 10x.$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$\text{(a)} \quad c_1 + A = 4,$$
$$\text{(b)} \quad c_1 + 108 = 11,$$
$$\text{(c)} \quad c_1 - 1004 = -299.$$

We solve this system for the unknowns A, B, and c_1. Equation (a) minus Equation (c) gives $101A = 303$, therefore $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^{x} + 3 \cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.

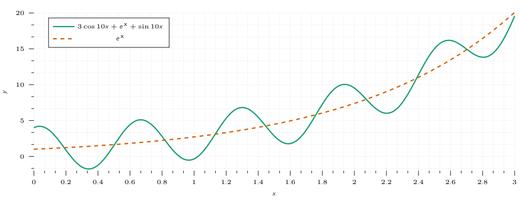


Figure 3.1: Solution to the question Simple Complex Roots

Multiple Real Roots

If a real double root occurs ($\lambda_1 = \lambda_2$) then $y_1 = y_2$ in Eq. (**3.3**), and we take y_1 and xy_1 as corresponding linearly independent solutions.

More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2 e^{\lambda x}, \quad \dots, \quad x^{m-1} e^{\lambda x}$$

Exercise 3.6 Real Double and Triple Roots

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y^{\text{iv}} - y'' = 0$$

SOLUTION The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + \left(c_3 + c_4 x + c_5 x^2\right) e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda} = \gamma - i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x, \quad e^{\gamma x} \sin \omega x, \quad xe^{\gamma x} \cos \omega x, \quad xe^{\gamma x} \sin \omega x$$

The first two of these result from $e^{\lambda x}$ and $\overline{e}^{\lambda x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\overline{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}.$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\gamma x} \cos \omega x$, $x^2 e^{\gamma x} \sin \omega x$, and so on.

3.2 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of n$^{\text{th}}$ order. As usual with other versions we looked at, we write them in standard form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} +, \cdots, +p_1(x)y' + p_0(x)y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a general solution of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Therefore Eq. (3.7) has no singular solutions.

An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

The ideas of proof are the same as those for $n = 2$.

Exercise 3.7 IVP: Modification Rule

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x},$$
$$y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47$$

SOLUTION **Step 1** The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2) e^{-x} \end{aligned}$$

Step 2 If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and $Cx^6 e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}$$

Then

$$\begin{aligned} y_p' &= C\left(3x^2 - x^3\right) e^{-x}, \\ y_p'' &= C\left(6x - 6x^2 + x^3\right) e^{-x}, \\ y_p''' &= C\left(6 - 18x + 9x^2 - x^3\right) e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$\begin{aligned} C\left(6 - 18x + 9x^2 - x^3\right) &+ 3C(6x - 6x^2 + x^3) \\ &+ 3C(3x^2 - x^3) \\ &+ Cx^3 = 30. \end{aligned}$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2e^{-x}$.

Step 3 We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$y = y_h + y_p = (c_1 + c_2 + c_3x^2)e^{-x} + 5x^3e^{-x},$$
$$y(0) = c_1 = 3$$
$$y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x},$$
$$y'(0) = -3 + c_2 = -3, \quad c_2 = 0$$
$$y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x},$$
$$y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2e^{-x} \quad \blacksquare$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.

3.2.1 Application: Modelling an Elastic Beam

While 2^{nd}-order ODEs have numerous applications, of which we have discussed some of the more important ones,[1] higher order ODEs have much fewer engineering applications.

[1] i.e., RLC Circuit, mass-damper system, . . .

For electrical engineers, the order of a circuit can be increased by adding an inductor or a capacitor to the circuit. However, in those cases using computational resources to calculate the circuit behaviour (i.e., ngSpice, OpenModelica) would be an easier path.

For mechanical engineers, an important 4^{th} ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge. This equation is also known as Euler-Bernoulli beam theory.[2]

[2] Daniel Bernoulli (1700 - 1782) was a Swiss mathematician and physicist and was one of the many prominent mathematicians in the Bernoulli family from Basel. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics.

> A related application of vibration of beams does not fit in here since it leads to PDEs.

Describing the Problem

Let us consider a beam B of length L and constant, **rectangular**, cross section and homogeneous elastic material (e.g., **level**),

We assume, under its own weight the beam is bent so little that it is **straight**. Applying a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.

Its axis is curved into the so-called elastic curve[3].

[3] This is known as deflection curve

It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ are small.[4] Then, by the principles of calculus:

[4] The goal here is to determine the tangent direction of C

$$k = \frac{y''}{\sqrt{1 + y''x}} \approx y''$$

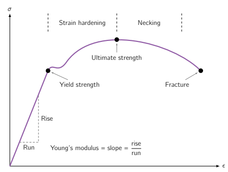
Therefore:

$$M(x) = EIy''(x)$$

where EI is the constant of proportionality. E Young's modulus of elasticity[5] of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.

Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy^{\text{iv}} = f(x) \tag{3.10}$$



[5]Young's modulus (or the Young modulus) is a mechanical property of solid materials that measures the tensile or compressive stiffness when the force is applied lengthwise. It is the elastic modulus for tension or axial compression. Young's modulus is defined as the ratio of the stress (force per unit area) applied to the object and the resulting axial strain (displacement or deformation) in the linear elastic region of the material.

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows:

Table 3.1: Different practical boundary conditions applicable to the problem.

Condition	Mathematical Definition
Simply supported	$y = y'' = 0$ at $x = 0$ and $x = L$
Clamped at both ends	$y = y' = 0$ at $x = 0$ and L
Clamped at $x = 0$, free at $x = L$	$y(0) = y'(0) = 0$ and $y''(L) = y'''(L) = 0$.

The boundary condition $y = 0$ means no displacement at that point, $y' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Information: Shear Force

In solid mechanics, shearing forces are unaligned forces acting on one part of a body in a specific direction, and another part of the body in the opposite direction. When the forces are collinear (aligned with each other), they are called tension forces or compression forces. Shear force can also be defined in terms of planes: "If a plane is passed through a body, a force acting along this plane is called a shear force or shearing force."

Deriving the Solution

Let us apply this to the uniformly loaded simply supported beam. The load is $f(x) = f_0 = \text{const}$. Then Eq. (3.10) is

$$y^{\text{iv}} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Integrating the above expression twice gives:

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

Time to apply the boundary conditions we have set. $y''(0) = 0$ gives $c_2 = 0$. Then

$$y''(L) = L\left(\frac{1}{2}kL + c_1\right) = 0, \quad c_1 = -k\frac{L}{2} \qquad (\text{since} \quad L \neq 0)$$

Therefore:

$$y'' = \frac{k}{2}\left(x^2 - Lx\right).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right),$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution:

$$y = \frac{f_0}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L - x)$. We can verify this by setting $x = u + L/2$ and see that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0$ (i.e., $x = L/2$) is:

$$\frac{5f_0 L^4}{(16 \cdot 24EI)} \quad \blacksquare$$

> Recall that the positive direction points downward.

Chapter 4

Systems of ODEs

Table of Contents

4.1 Looking at Connected ODEs

We now introduce a different way of looking at systems of ODEs. The method consists of examining the general behaviour of families of solutions of ODEs in the phase plane, called the **phase plane** method.

Theory 4.8: Phase Plane

A visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two (2) state variables, for example (x, y), or (q, p) etc.

It gives information on the stability of solutions. This approach to systems of ODEs is a qualitative method as it depends only on the nature of the ODEs and does **NOT** require the actual solutions. This can be very useful because as is often difficult or sometimes even impossible to solve systems of ODEs. In contrast, the approach of actually solving a system is known as a **quantitative** method.

Theory 4.9: Qualitative Method

The qualitative analysis of ODEs is to be able to say something about the behavior of solutions of the equations, without solving them explicitly.

The phase plane method has numerous applications in control theory, circuit analysis theory, population dynamics and so on.

4.1.1 System of ODEs as Engineering Models

Time to see how systems of ODEs are of practical importance. We start by first illustrating how systems of ODEs can serve as models in different applications. Then we show how a higher order[1] ODE can be reduced to a first-order system. The following two (2) examples will look at a system of ODEs as a fluid mechanics problem and then an electrical engineering problem.

[1] with the highest derivative standing alone on one side.

Case Study: Mixing of Two Tanks

To get a better understanding of how multiple ODEs work together, let's look at a case involving two (2) tanks. A mixing problem involving a single tank is modeled by a single ODE which can be extended to two (2) sets of equations.

Let's assume two (2) Tanks T_1 and T_2 containing initially 100 L of water each, In T_1 the water is pure, whereas 150 kg of fertilizer are dissolved in T_2. By circulating liquid at rate of $2\,\mathrm{L}\cdot\mathrm{min}^{-1}$ and stirring the amount of fertiliser $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

> How long should we let the liquid circulate so that T_1 will contain at least half as much fertiliser as there will be left in T_2?

For simplicity, let us also assume the mixture is uniform.

Setting Up the Model As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. Therefore:

$$y_1' = \frac{2}{100}y_2 - \frac{2}{100}y_1 \quad \text{Tank 1,} \qquad \text{and} \qquad y_2' = \frac{2}{100}y_1 - \frac{2}{100}y_2 \quad \text{Tank 2.}$$

Therefore the mathematical model of our mixture problem is the system of 1st ODEs:

$$\begin{aligned} y_1' &= -0.02y_1 + 0.02y_2, \\ y_2' &= +0.02y_1 - 0.02y_2. \end{aligned} \qquad \text{As a vector form} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

with this information, matrix $\mathbf{A}$ this becomes:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

General Solution As for a single equation, we try an exponential function of t,

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \quad \text{and} \quad \mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}. \tag{4.1}$$

Dividing the last equation $\lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A}\boldsymbol{x} e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}.$$

We need nontrivial solutions. Therefore we have to look for eigenvalues and eigenvectors of $\boldsymbol{A}$. The eigenvalues are the solutions of the characteristic equation

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} = (-0.02-\lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$. $\lambda = 0$ can very well happen but don't get mixed up. It is eigenvectors which must **NOT** be zero. Eigenvectors are obtained as $\lambda = 0$ and $\lambda = -0.04$.

For our present $\boldsymbol{A}$ this gives:

$$-0.02x_1 + 0.02x_2 = 0 \qquad \text{and} \qquad (-0.02 + 0.04)\,x_1 + 0.02x_2 = 0,$$

respectively. Therefore $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two (2) eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

> This principle continues to hold for systems of homogeneous linear ODEs.

From Eq. (**4.1**) and the superposition principle, we therefore obtain a solution:

$$\begin{aligned} \boldsymbol{y} &= c_1 \boldsymbol{x}^{(1)} e^{\lambda_1 t} + c_2 \boldsymbol{x}^{(2)} e^{\lambda_2 t} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned} \tag{4.2}$$

where c_1 and c_2 are arbitrary constants.

Use of Initial Conditions The initial conditions are $y_1(0) = 0$[2] and $y_2(0) = 150$. From this and Eq. (**4.2**) with $t = 0$ we obtain

[2] Remember, no fertilizer in tank T_1.

$$\boldsymbol{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer:

$$\begin{aligned} \boldsymbol{y} &= 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)} e^{-0.04t} \\ &= 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \end{aligned}$$

In components,

$$
\begin{aligned}
y_1 &= 75 - 75e^{-0.04\ t} && \text{Tank } T_1,\ \text{lower curve,}\\
y_2 &= 75 + 75e^{-0.04\ t} && \text{Tank } T_2,\ \text{upper curve.}
\end{aligned}
$$

Figure below shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 kg.

Calculating the Answer Here, T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 kg. Therefore:

$$
\begin{aligned}
& & y_1 &= 75 - 75e^{-0.04t} = 50,\\
e^{-0.04t} &= \tfrac{1}{3}, \qquad \text{and} & t &= (\ln 3)/0.04 = 27.5
\end{aligned}
$$

Hence the fluid should circulate for roughly half an hour ■

4.1.2 Conversion of an n-th Order ODE to a System

A n$^{\text{th}}$-order ODE of the general form can be converted to a system of n 1$^{\text{st}}$-order ODEs. This allows the study and solution of single ODEs by methods for systems, and opens a way of including the theory of higher order ODEs into that of 1$^{\text{st}}$-order systems.

Theory 4.10: Conversion of an ODE

An nth-order ODE:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{4.3}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}. \tag{4.4}$$

This system is of the form

$$
\begin{aligned}
y_1' &= y_2\\
y_2' &= y_3\\
&\vdots\\
y_{n-1}' &= y_n\\
y_n' &= F(t, y_1, y_2, \cdots, y_n).
\end{aligned}
\tag{4.5}
$$

While the iron is hot, let's look at an example which we should be familiar with.

Exercise 4.1 Mass on a String

To gain confidence in the conversion method, let us apply it to an old problem of ours:

modelling the free motions of a mass on a spring with value given as $m = 1$, $c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \qquad \text{or}$$

$$y'' = -\left(\frac{c}{m}\right)y' - \left(\frac{k}{m}\right)y.$$

SOLUTION For this ODE given in the question can be written in the form ofEq. (**4.3**), making the system shown Eq. (**4.4**) as **linear** and **homogeneous**, applying to our system in question.

$$y_1' = y_2$$

$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2.$$

Setting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I}\right) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Entering the values of $m = 1$, $c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)\,(\lambda + 1.5) = 0$$

This gives the eigenvalues:

$$\lambda_1 = -0.5 \qquad \text{and} \qquad \lambda_2 = -1.5$$

Eigenvectors follow from the first equation in $\boldsymbol{A} - \lambda \boldsymbol{I} = 0$, which is $-\lambda x_1 + x_2 = 0$. $\lambda_1 = 0.5$ produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$. $\lambda_2 = -1.5$ produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$. These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

Which gives:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t} \quad \blacksquare$$

4.1.3 Linear Systems

Extending the notion of a linear ODE, we call a linear system if it is linear in 1[1], $\cdots$, 1[n]. That is, if it can be written in the form:

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t). \end{aligned} \tag{4.6}$$

As a vector equation this becomes

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} + \boldsymbol{g} \tag{4.7}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

This system is called **homogeneous** if $\boldsymbol{g} = 0$, so it is:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} \tag{4.8}$$

If $\boldsymbol{g} \neq 0$, then Eq. (**4.8**) is called, like other ODEs, **non-homogeneous**.

4.2 Constant Coefficient Systems

4.2.1 The Phase Plane Method

Continuing, we now assume our **homogeneous** linear system is of the form:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \tag{4.9}$$

We shall assume the aforementioned equations has constant coefficients, so the $n \times n$ matrix $\boldsymbol{A} = [a_{\mathrm{jk}}]$ has entries which are **NOT** dependent on t. We want to solve Eq. (4.9). Now we know from previous examples, a single ODE $y' = ky$ has the solution $y = Ce^{\mathrm{kt}}$. So let us try:

$$\boldsymbol{y} = \boldsymbol{x}e^{\lambda t} \tag{4.10}$$

Substitution into Eq. (4.9) gives:

$$\boldsymbol{y}' = \lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}.$$

Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{4.11}$$

Therefore the non-trivial solutions of Eq. (4.9) are[3] of the form Eq. (4.10), where λ is an eigenvalue of λ and $\boldsymbol{x}$ is a corresponding eigenvector.

[3] i.e., non-zero vectors solutions.

We assume λ has a **linearly independent** set of n eigenvectors. This holds in most applications, particularly if $\boldsymbol{A}$ is symmetric ($a_{\mathrm{kj}} = a_{\mathrm{jk}}$) or skew-symmetric ($a_{\mathrm{kj}} = -a_{\mathrm{jk}}$) or has n different eigenvalues.

Let those eigenvectors be $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(\mathrm{n})}$ and let them correspond to eigenvalues $1[1], \cdots, 1[n]$.[4] Then the corresponding solutions Eq. (4.10) are given as:

[4] which may be all different, or some, or even all, may be equal.

$$\boldsymbol{y}^{(1)} = \boldsymbol{x}^{(1)} e^{\lambda_1 t}, \quad \cdots, \quad \boldsymbol{y}^{(\mathrm{n})} = \boldsymbol{x}^{(\mathrm{n})} e^{\lambda_n t}. \tag{4.12}$$

Their Wronskian $W = W(y^{(1)}, \dots, y^{(n)})$ is given by

$$W = \left(\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(\mathrm{n})}\right) = \begin{vmatrix} \boldsymbol{x}_1^{(1)} e^{\lambda_1 t} & \cdots & \boldsymbol{x}_1^{(\mathrm{n})} e^{\lambda_n t} \\ \boldsymbol{x}_2^{(1)} e^{\lambda_1 t} & \cdots & \boldsymbol{x}_2^{(\mathrm{n})} e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_{\mathrm{n}}^{(1)} e^{\lambda_1 t} & \cdots & \boldsymbol{x}_{\mathrm{n}}^{(\mathrm{n})} e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t + \cdots + \lambda_n t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(\mathrm{n})} \\ x_2^{(1)} & \cdots & x_2^{(\mathrm{n})} \\ \vdots & \ddots & \vdots \\ x_{\mathrm{n}}^{(1)} & \cdots & x_{\mathrm{n}}^{(\mathrm{n})} \end{vmatrix}$$

Theory 4.11: General Solution

If the constant matrix $\boldsymbol{A}$ in the system Eq. (4.9) has a **linearly independent** set of n eigenvectors, then the corresponding solutions $\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(\mathrm{n})}$ in Eq. (4.12) form a basis of solutions[5] of Eq. (4.9),

[5] a set of linearly independent vectors or functions which can be combined to form any other solution in a given solution space, such as the solution set of a linear system or differential equation.

and the corresponding general solution is:

$$y = c_1 x^{(1)} e^{\lambda_1 t} + \cdots + c_n x^{(n)} e^{\lambda_n t} \tag{4.13}$$

In the aforementioned theorem, On the RHS, the exponential function is never zero, and the determinant is **NOT** zero either because its columns are the n linearly independent eigenvectors. This proves the theorem, whose assumption is true if the matrix $\boldsymbol{A}$ is symmetric or skew-symmetric, or if the n eigenvalues of $\boldsymbol{A}$ are all different.

Theory 4.12: Symmetric, Skew, Orthogonal

A *real* square matrix $\boldsymbol{A} = \left[a_{jk}\right]$ can have the following properties which can prove to be useful. A matrix is called:

1. **symmetric** if transposition leaves it unchanged,

$$\boldsymbol{A}^{\mathsf{T}} = \boldsymbol{A}, \qquad \text{where} \qquad a_{kj} = a_{jk} \qquad \text{[Symmetric]}$$

2. **skew-symmetric** if transposition gives the negative of $\boldsymbol{A}$,

$$\boldsymbol{A}^{\mathsf{T}} = -\boldsymbol{A}, \qquad \text{where} \qquad a_{kj} = -a_{jk}, \qquad \text{[Skew-Symmetric]}$$

3. **orthogonal** if transposition gives the inverse of $\boldsymbol{A}$,

$$\boldsymbol{A}^{\mathsf{T}} = \boldsymbol{A}^{-1}. \qquad \text{[Orthogonal]}$$

4.2.2 The Critical Points of a System

The point $\boldsymbol{y} = 0$ in the previous example seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from Eq. (**4.9**) we obtain:

$$\frac{dy_2}{dy_1} = \frac{y_2' \, dt}{y_1' \, dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}. \tag{4.14}$$

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: $(0, 0)$, where the RHS of Eq. (**4.14**) becomes $0/0$.

P_0, at which dy_2/dy_1 becomes **undetermined** is called a **critical point** of Eq. (**4.14**).

4.2.3 The Five Types of Critical Points

There are five (5) types of critical points depending on the geometric shape of the trajectories near them. These are:

Improper, Proper, Saddle, Centre, Spiral.

Let's look at them with examples.

Improper Node

Let's start with a simple exercise and find solutions of the following system:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \boldsymbol{y}, \qquad \text{where} \qquad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

To see what is going on, let us find the solutions of the system and it is always a good idea to start with known solutions.

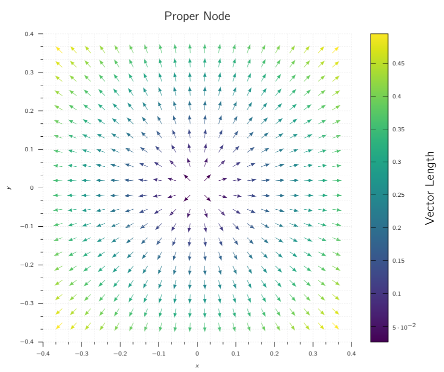


Figure 4.1 A visual description of the improper node phase-plane.

Substituting $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ and $\boldsymbol{y}' = \lambda\boldsymbol{x}e^{\lambda t}$ and dropping the exponential function, as they exist both on the LHS and RHS we can eliminate them, we get $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

The characteristic equation is:

$$\det\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right) = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from:

$$(-3-\lambda)\,x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\boldsymbol{x}^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\boldsymbol{x}^{(2)} = [1 \quad -1]^{\mathsf{T}}$.

This gives the general solution:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad \blacksquare$$

Proper Node

Next on our list is Proper Node. Let's see the behaviour of a proper node by finding the solutions of the following system:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \boldsymbol{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

Let us find the solutions of the system. As usual, it is always a good idea to start with known solutions. Start by substituting $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ and $\boldsymbol{y}' = \lambda \boldsymbol{x} e^{\lambda t}$ and dropping the exponential function, as they exist both on the LHS and RHS we can eliminate them, we get $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$.

The characteristic equation is:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

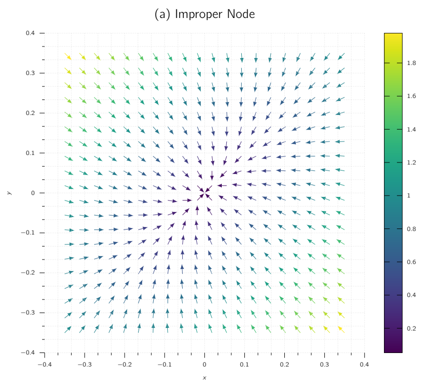


Figure 4.2 A visual description of the proper node phase-plane.

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from:

$$(-3-\lambda)\, x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\boldsymbol{x}^{(1)} = [1 \quad 1]^T$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\boldsymbol{x}^{(2)} = [1 \quad -1]^T$.

This gives the general solution:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \boldsymbol{y}^{(1)} + c_2 \boldsymbol{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad \blacksquare$$

Saddle Point

A **saddle point** is a critical point (P_0) at which there are two (2) incoming trajectories, two (2) outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

To see this behaviour, let's study the following system:

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{y}, \quad \text{therefore} \quad \begin{cases} y_1' & = y_1 \\ y_1' & = -y_2 \end{cases}$$

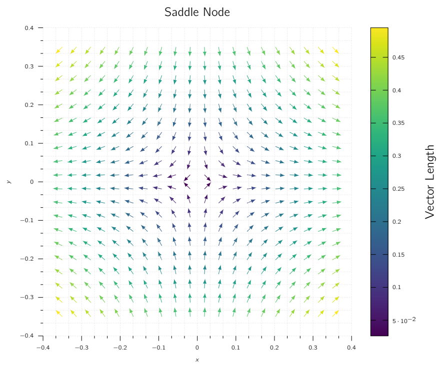


Figure 4.3 A visual description of saddle point.

The equation has a saddle point at the origin with its characteristic equation,

$$(1-\lambda)(-1-\lambda) = 0.$$

has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ in eigenvector $[1\ 0]^T$ is obtained from the second row of $(\boldsymbol{A} - \lambda \mathbf{I})\mathbf{x} = 0$, that is, $0x_1 + (-1-1)\, x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0 \quad 1]^T$. Hence a general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{aligned} y_1 &= c_1 e^{t} \\ y_2 &= c_2 e^{-t} \end{aligned} \quad \text{or} \quad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas**

Centre Node

A **centre** is a critical point that is enclosed by infinitely many closed trajectories. Let's study the following system:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y, \therefore y_1' = y_2 \quad \text{and} \quad y_2' = -4y_1 \tag{4.15}$$

The equation has a center at the origin.

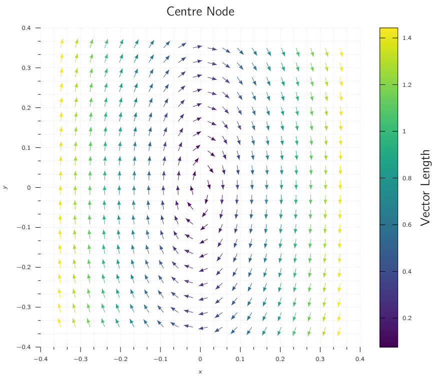


Figure 4.4: A visual description of the centre node.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathbf{j}$ and $-2\mathbf{j}$. For $2\mathbf{j}$, an eigenvector follows from the first equation $-2\mathbf{j}\,x_1 + x_2 = 0$ of $(\boldsymbol{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which can be, $[1 \quad 2\mathbf{j}]^{\mathsf{T}}$.

For $\lambda = -2\mathbf{j}$ that equation is $-(-2\mathbf{j})x_1 + x_2 = 0$ and gives, say, $[1 \quad -2\mathbf{j}]^{\mathsf{T}}$. Hence a complex general solution is:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 2\mathbf{j} \end{bmatrix} e^{2\mathbf{j}\,t} + c_2 \begin{bmatrix} 1 \\ -2\mathbf{j} \end{bmatrix} e^{-2\mathbf{j}\,t}, \qquad \text{therefore} \qquad \begin{aligned} y_1 &= c_1 e^{2\mathbf{j}\,t} + c_2 e^{-2\mathbf{j}\,t}, \\ y_2 &= 2\mathbf{j}\,c_1 e^{2\mathbf{j}\,t} - 2\mathbf{j}\,c_2 e^{-2\mathbf{j}\,t}. \end{aligned} \tag{4.16}$$

A real solution is obtained from Eq. (**4.16**) by the Euler formula or from Eq. (**4.15**).

Namely, we can create a relation of $-4y_1y_1^2$.

$$-4y_1y_1' = y_2y_2' \qquad \text{By Integration} \qquad 2y_1^2 + \frac{1}{2}y_2^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. ■

[6] or tracing these spirals in the opposite sense, away from (P_0).

Spiral Node

A **spiral point** is a critical point (P_0) about which the trajectories spiral, approaching (P_0) as $t \to \infty$.[6]

The system:

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y, \quad \text{and} \quad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.17}$$

has a spiral point at the origin.

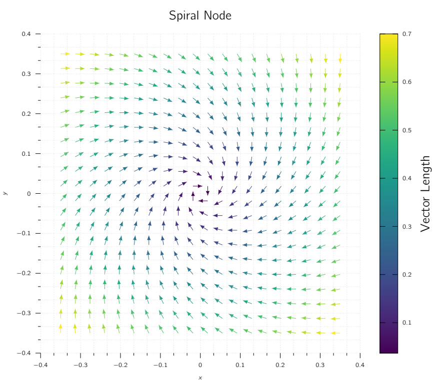


Figure 4.5 A visual description of the spiral node.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1+\mathbf{j}$ and $-1-\mathbf{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$.

For $\lambda = -1 + \mathbf{j}$ this becomes $-\mathbf{j}\, x_1 + x_2 = 0$ and we can take $[1 \quad \mathbf{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathbf{j}$ is $[1 \quad -\mathbf{j}]^T$.

This gives the **complex** general solution:

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ \mathbf{j} \end{bmatrix} e^{(-1+\mathbf{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathbf{j} \end{bmatrix} e^{(-1-\mathbf{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (**4.17**) by y_1, the second by y_2- and add, obtaining:

$$y_1 y_1' + y_2 y_2' = -\left(y_1^2 + y_2^2\right).$$

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + y_2^2$. Differentiating this with respect to t gives:

$$2rr' = 2y_1 y_1' + 2y_2 y_2'$$

Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real c this is a spiral.

4.3 Criteria for Critical Points and Stability

Continuing our discussion of homogeneous linear systems with **constant coefficients** given in Eq. (4.9), let us review where we are. From the previous section we have,

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \boldsymbol{y}, \qquad \text{in components,} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.18}$$

From the examples in the last section, we have seen that we can obtain an overview of families of solution curves if we represent them parametrically as $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T$ and graph them as curves in the y_1y_2-plane, which is called its **phase plane**.

Such a curve is called a **trajectory** of Eq. (4.9), and their totality is known as the **phase portrait** of Eq. (4.9).

Now, we have seen that solutions are of the form:

$$\boldsymbol{y}(t) = \boldsymbol{x}e^{\lambda t} \qquad \text{substitution to Eq. (4.18) gives} \qquad \boldsymbol{y}'(t) = \lambda \boldsymbol{x}e^{\lambda t} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}e^{\lambda t}$$

Dropping the common factor ($e^{\lambda t}$), we arrive at a similar equation.

$$\boldsymbol{A}\mathbf{x} = \lambda\mathbf{x} \tag{4.19}$$

> $\boldsymbol{y}(t)$ is a solution of Eq. (4.18) if λ is an eigenvalue of $\boldsymbol{A}$ and $\mathbf{x}$ a corresponding eigenvector.[7]

[7] Of course the solution has to be non-trivial which means a non-zero solution.

Our examples in the previous section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system Eq. (4.18) defined as a point at which dy_2/dy_1 becomes **undetermined**.[8]

[8] In this case the undetermined behaviour comes from dividing 0 by 0 which is a big no-no in mathematics.

$$\frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \tag{4.20}$$

Also recall from there are various types (5) of critical points.

What is new here, how these types of critical points are related to the eigenvalues. The latter are solutions $\lambda = \lambda_1$ and λ_2 of the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det A = 0. \tag{4.21}$$

This is a quadratic equation ($\lambda^2 - p\lambda + q = 0$) with coefficients p, q, and discriminant Δ which are calculated as:

$$p = a_{11} + a_{22}, \qquad q = \det A = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta = p^2 - 4q. \tag{4.22}$$

From algebra we know that the solutions of this equation are

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}).$$

Furthermore, the product representation of the equation gives

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Hence p is the sum and q the product of the eigenvalues. Also $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ from (6). Together,

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2.$$

This gives the criteria in Table 4.1 for classifying critical points. A derivation will be indicated later in this section. Critical points may also be classified in terms of their **stability**. Stability concepts

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite signs
Centre	$p = 0$	$q > 0$		Pure imaginary
Spiral		$q \neq 0$	$\Delta < 0$	Complex (not pure imaginary)

Table 4.1: Eigenvalue Criteria for Critical Points.

are fundamental for engineering purposes where it means, a small change of a physical system at some instant changes the behavior of the system only slightly at all future times t.

Information: Stable Unstable Attractive

A critical point P_0 of Eq. (**4.18**) is called **stable** if, roughly, all trajectories of Eq. (**4.18**) that at some instant are close to P_0 remain close to P_0 at all future times, or in another way if for every disk D_ϵ of radius $\epsilon > 0$ with center P_0 there is a disk D_δ of radius $\delta > 0$ with center P_0 such that every trajectory of Eq. (**4.18**) that has a point P_1 in D_δ has all its points corresponding to $t \equiv t_1$ in D_ϵ.

P_0 is called **unstable** if P_0 is not stable.

P_0 is called **stable and attractive** if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \to \infty$.

In general term this can be written in a following table.

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$
Stable and attractive	$q < 0$	$q > 0$
Stable	$q \leq 0$	$q > 0$
Unstable	either $q \leq 0$	or $q > 0$

Table 4.2: Stability criteria for critical points.

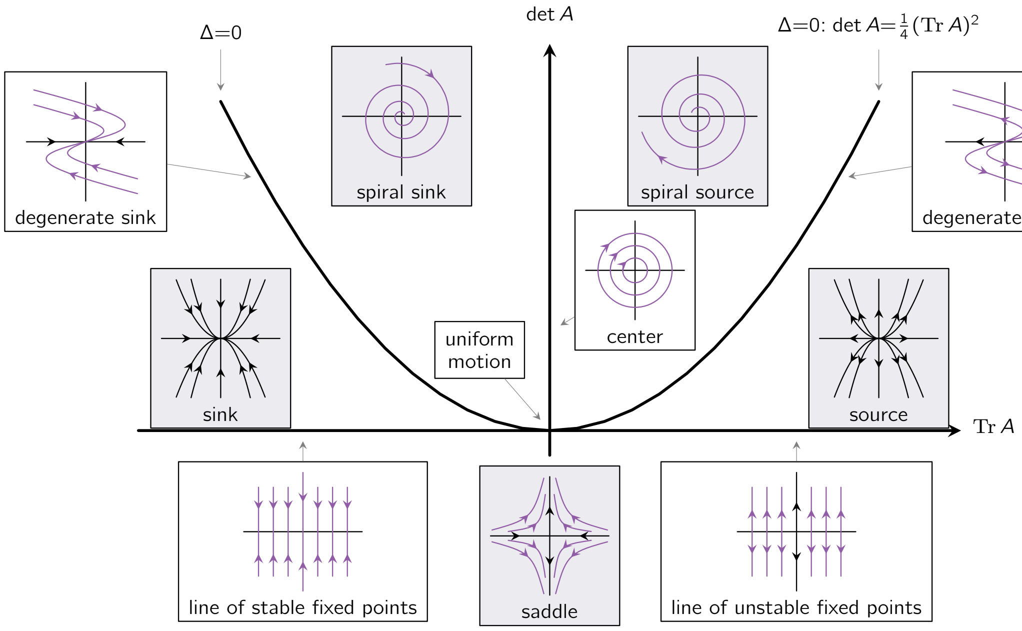


Figure 4.6 The Poincaré Phase-Plot diagram showcasing different behaviours. The shaded diagrams represent the region within the plot whereas the white boxes represent the behaviour when it is on the line.

4.3.1 Model: A Mass Damper Spring System

Let's look back at what we have learned and apply our newfound methodology to the following equation.

$$my'' + cy' + ky = 0$$

As we know this is a 2nd order ODE explaining the mass-damper-spring system. To solve this we start with division by m which gives:

$$y'' = -\left(\frac{k}{m}\right) y - \left(\frac{c}{m}\right) y'.$$

To get a system out of this equation, we set:

$$y_1 = y \qquad \text{and} \qquad y_2 = y'.$$

Which, when inserted to our mass-damper-spring equation:

$$y_2' = y'' = \left(\frac{k}{m}\right) y_1 - \left(\frac{c}{m}\right) y_2$$

Therefore we can derive our set of equations are as follows:

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \mathbf{y}, \qquad \det\left(\mathbf{A} - \lambda \mathbf{I}\right) = \begin{bmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{bmatrix} = \lambda^2 + \left(\frac{c}{m}\right)\lambda + \left(\frac{k}{m}\right) = 0$$

We can see that:

$$p = -\frac{c}{m}, \qquad q = \frac{k}{m}, \qquad \Delta = \left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)$$

From this along with tables **Tbl.** 4.2, and **Tbl.** 4.1, we obtain the following results and it is worth stressing that in the last three (3) cases the discriminant Δ plays an essential role.

Behaviour	Coefficient				Stability Type	Type
	c	p	q	Δ		
No Damping	$c = 0$	$p = 0$	$q > 0$		Stable	Centre
Under Damping	$c^2 < 4mk$	$p < 0$	$q > 0$	$\Delta < 0$	Stable	Spiral
Critical Damping	$c^2 = 4mk$	$p < 0$	$q > 0$	$\Delta = 0$	Stable	Attractive
Overdamping	$c^2 > 4mk$	$p < 0$	$q > 0$	$\Delta > 0$	Stable	Attractive

Table 4.3: The behaviour of the mass-damper-spring system

4.4 Qualitative Methods for Non-Linear Systems

Figure 4.7: An interesting application of using ODEs is to determine the behaviour of circuits containing vacuum tubes [1].

Qualitative methods are methods of obtaining qualitative information on solutions *without actually solving a system*. These methods are particularly valuable for systems whose solution by analytic methods is difficult or impossible.[9]

This is the case for many practically important **non-linear systems**.

$$y' = f(y), \qquad \text{therefore} \qquad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned} \tag{4.23}$$

Here we will extend the previously discussed *phase plane* methods, from linear systems to non-linear systems Eq. (4.23). We assume that Eq. (4.23) is autonomous, that is, the independent variable t does **NOT** occur explicitly.[10]

[9] It originated from the works of *Henri Poincaré* and *Aleksandr Lyapunov*. There are relatively few differential equations that can be solved explicitly, but using tools from analysis and topology, one can "solve" them in the qualitative sense, obtaining information about their properties.

[10] When the variable is time, they are also called time-invariant systems.

For example, The equation $y' = (2 - y)\,y$ is autonomous, since the independent variable x does not explicitly appear in the equation.

All examples in the previous section are **autonomous**.

We shall, again exhibit entire families of solutions.

This is an advantage over numeric methods, which give only one (approximate) solution at a time.

For this analysis we need to employ the previously defined concepts of **phase plane** (the y_1-y_2-plane), **trajectories** (solution curves of Eq. (4.23) in the phase), the **phase portrait** of Eq. (4.23) (the totality of these trajectories), and **critical points** of Eq. (4.23) points (y_1, y_2) at which both $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are zero).

Now Eq. (4.23) may have several critical points. Our approach shall be to discuss one critical point after another. If a critical point P_0 is **NOT** at the origin, then, for technical convenience, we shall move this point to the origin before analysing the point. More formally:

> if P_0: (a, b) is a critical point with (a, b) **NOT** at the origin $(0, 0)$, then we apply the translation:
>
> $$\bar{y}_1 = y_1 - a, \qquad \text{and} \qquad \bar{y}_2 = y_2 - b,$$
>
> which moves P_0 to $(0, 0)$ as desired.

Therefore we can assume P_0 to be the origin $(0, 0)$, and for simplicity we continue to write y_1, y_2.[11] We also assume that P_0 is **isolated**, that is, it is the only critical point of Eq. (4.23) within a (sufficiently small) disk with center at the origin.

[11] We write this instead of $\bar{y}_1$, $\bar{y}_2$

4.4.1 Linearisation of Non-Linear Systems

How to determine the kind and stability of a critical point P_0: $(0, 0)$ of Eq. (**4.23**)?

In most cases this can be done by linearisation of Eq. (**4.23**) near P_0, writing Eq. (**4.23**) as $\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y})$ and dropping $\mathbf{h}(\mathbf{y})$, as follows.

Given P_0 is critical, $f_1(0, 0) = 0$, $f_2(0, 0) = 0$, so that f_1 and f_2 have no constant terms and we can write

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2). \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2). \end{aligned} \tag{4.24}$$

A is constant as Eq. (**4.23**) is autonomous.

Theory 4.13: Linearisation

If f_1 and f_2 given in Eq. (**4.23**) are **continuous** and have **continuous partial derivatives** in a neighbourhood of the critical point P_0 : $(0, 0)$, and if $\det \mathbf{A} \neq 0$ in Eq. (**4.24**), then the kind and stability of the critical point of Eq. (**4.23**) are the same as those of the **linearized system**.

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{therefore} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.25}$$

Exceptions occur if $\mathbf{A}$ has **equal** or **pure imaginary** eigenvalues. If that is the case, then Eq. (**4.23**) may have the same kind of critical point as Eq. (**4.25**) or a spiral point.

4.4.2 Model: Linearisation of an Undamped Pendulum

A simple to study linearisation is to study a pendulum. For our case, A pendulum consists of a body of mass m (the bob) and a rod of length L. Based on this information we are tasked with determining the locations and the types of the critical points. To keep things simple, we will assume all types of resistance are negligible

Let us begin tackling the problem:

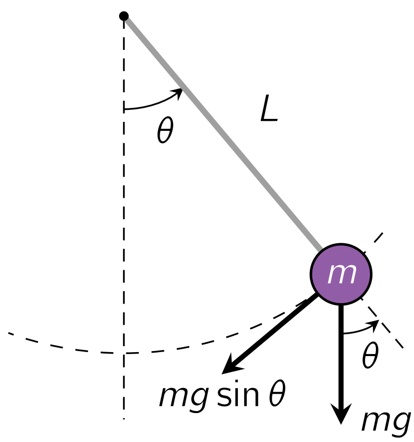


Figure 4.8
A simple pendulum in motion.

Setting the Model Let θ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is mg, where g is the acceleration of gravity ($\mathrm{m \cdot s^{-2}}$).

This action creates restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2nd law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the **acceleration**.

Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg\sin\theta = 0.$$

Dividing this by mL, we have our model:

$$\theta'' + k\sin\theta = 0 \qquad \text{where} \qquad \left(k = \frac{g}{L}\right). \tag{4.26}$$

> When θ is very small,[12] we can approximate $\sin\theta$ rather accurately as θ and obtain as an approximate solution $A\cos\sqrt{k} + B\sin\sqrt{k}$, but the *exact* solution for any θ is **NOT** an **elementary function**.

[12] This value needs to be much less than 1 rad which we write it as: $\theta \ll 1$

Finding the Critical Points and Linearisation To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. From this definition, using Eq. (4.26) we obtain a non-linear system Eq. (4.23) of the form:

$$\begin{aligned} y_1' &= f_1\left(y_1, y_2\right) = y_2, \\ y_2' &= f_2\left(y_1, y_2\right) = -k\sin y_1. \end{aligned}$$

The right sides are both zero (0) when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi, 0)$, where $n = 0, \pm 1, \pm 2, \cdots$.

We consider $(0, 0)$. Since the Maclaurin series[13] for sin is:

$$\sin y_1 = y_1 - \frac{1}{6}y_1^3 + - \cdots \approx y_1,$$

[13] **Colin Maclaurin** (*1698 - 1746*)

a Scottish mathematician who made important contributions to geometry and algebra. He is also known for being a child prodigy and holding the record for being the youngest professor. The Maclaurin series, a special case of the Taylor series, is named after him.

Theory 4.14: Maclaurin Series

A Maclaurin series is a Taylor series expansion of a function about 0,

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Maclaurin series are named after the Scottish mathematician *Colin Maclaurin*.

the linearized system at $(0, 0)$ is:

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\boldsymbol{y} \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1. \end{aligned}$$

To apply our stability criteria discussed previously, of we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = 0, \\ q &= \det(\boldsymbol{A}) = k = g/L \qquad (> 0), \\ \Delta &= p^2 - 4q = -4k. \end{aligned}$$

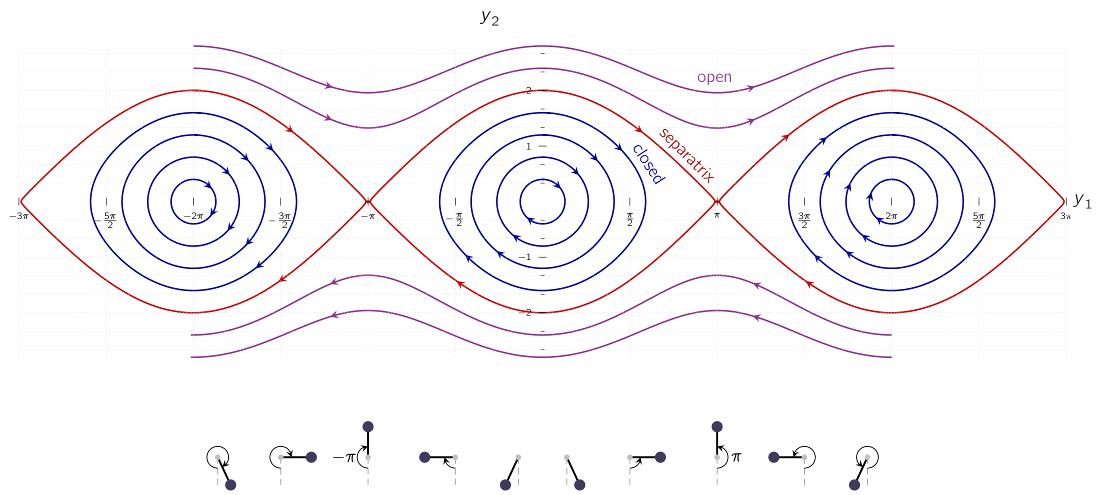


Figure 4.9 The Phase plane of a simple pendulum motion. Please observe the two (2) types of behaviour the system has which are centre node and saddle. If the system exhibits centre mode, it is considered a stable, whereas a saddle point, which can also be seen from the small pendulum diagrams below, are clearly unstable. Here the word seperatrix means the boundary which separates the two (2) modes of behaviour.

From this and **Tbl.** 4.1, we conclude that $(0, 0)$ is a **centre**, which is **always stable**. Since $\sin\theta = \sin y_1$ is periodic with period of 2π, the critical points $(n\pi, 0)$, $n = \pm 2, \pm 4, \cdots$, are all centres.

Critical Points and Linearisation We now consider the critical point $(\pi, 0)$. We start by setting:

$$y_1 = \theta - \pi \qquad \text{and} \qquad y_2 = (\theta - \pi)' = \theta'$$

Then in Eq. (**4.26**), we can apply the *MacLaurin series*:

$$\sin\theta = \sin\left(y_1 + \pi\right) = -\sin y_1 = -y_1 + \frac{1}{2} y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} \boldsymbol{y} \qquad \text{Thus} \qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= k y_1. \end{aligned}$$

From here we can determine the coefficients for the quadratic equation:

$$p = 0, \qquad q = -k \qquad (< 0), \qquad \Delta = -4q = 4k.$$

Hence, by consulting **Tbl.** 4.1, we see that this gives a saddle point, which is always unstable. Because of periodicity, the critical points $(n\pi, 0)$, $n = \pm 1, \pm 3, \cdots$, are all **saddle points**. The phase-plane for this method can be seen in **Fig.** 4.9.

4.4.3 Model: Linearisation of A Damped Pendulum

To gain further experience in investigating critical points, as another practically important case, let us see how the previous example changes when we add a damping term[14] $c\theta$, to equation Eq. (**4.26**), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

[14] This means the damping is proportional to the angular velocity.

where $k > 0$ and $c \geq 0$.[15]

[15] which includes our previous case of no damping, $c = 0$

First we start by setting $\theta = y_1$, θ' as before, we obtain the nonlinear system where we will use $\theta'' = y_2$:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k \sin y_1 - c y_2. \end{aligned}$$

We see the critical points have the same locations as the example before, namely, $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, $\ldots$. To analyse this system, we start with analysing $(0, 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0, 0)$.

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \mathbf{y} \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2 \end{aligned}$$

This is identical with the system in our previous modelling, except for the positive factor m.[16] Therefore for $c = 0$, meaning no damping, we have a centre, for small damping we have a spiral point, and so on.

[16] and except for the physical meaning of y_1

We now consider the critical point $(\pi, 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin\left(y_1 + \pi\right) = -\sin y_1 \approx -y_1.$$

This gives the new linearised system at $(\pi, 0)$:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \mathbf{y}, \qquad \text{therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= k y_1 - c y_2. \end{aligned}$$

For our criteria, we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = -c \\ q &= \det \mathbf{A} = -k \\ \Delta &= p^2 - 4q = c^2 + 4k \end{aligned}$$

This gives the following results for the critical point $(\pi, 0)$.

No Damping $c > 0, p = 0, q < 0, \Delta > 0$, a saddle point, and

Damping $c > 0, p < 0, q < 0, \Delta > 0$, which is a saddle point.

As $\sin y_1$ is periodic with period of 2π, the critical points $(\pm 2\pi, 0)$, $(\pm 4\pi, 0)$, $\ldots$ are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished. ■

4.4.4 Model: Self-Sustained Oscillations - Van der Pol Equation

There are physical systems such that for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system.

In other words, large oscillations will be damped, whereas for small oscillations there is *negative damping*.[17] For physical reasons we expect such a system to approach a periodic behaviour, which will thus appear as a closed trajectory in the phase plane, called a **limit cycle**.

An ODE describing such vibrations is the famous **van der Pol equation**.[18]

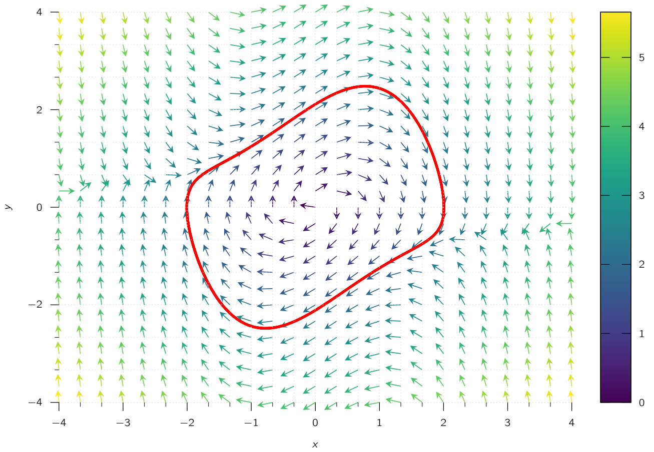


Figure 4.10

$$y'' - \mu\left(1 - y^2\right)y' + y = 0$$

It first occurred in the study of electrical circuits containing vacuum tubes.

Information: Vacuum Tube

A vacuum tube, electron tube, valve (British usage), or tube (North America) is a device that controls electric current flow in a high vacuum between electrodes to which an electric potential difference has been applied. An example of a vacuum tube is shown in **Fig.** 4.7.

For $\mu = 0$ this equation becomes $y'' + y = 0$ and so with harmonic oscillations. If we define $\mu > 0$, then the damping term has the factor $-\mu\left(1 - y^2\right)$. This is a consequence for small oscillations, when $y^2 < 1$, so that we have negative damping, is zero for $y^2 = 1$ (no imaginary), and is positive if $y^2 > 1$ (positive damping. Loss of energy).

If μ is small, we expect a limit cycle almost a circle because then our equation differs but finite from $y'' + y =$. If μ is large, the limit cycle will probably look different.

Setting $y = y_1$, $y' = y_2$ and using $y'' = \left(dy_2/dy_1\right)y_2$ as in (8), we have from (10)

$$\frac{dy_2}{dy_1}y_2 - \mu(1 - y_1^2)y_2 + y_1 = 0.$$

The isoclines in the y_1y_2-plane (the phase plane) are the curves $dy_2/dy_1 = K = \text{const}$, that is,

$$\frac{dy_2}{dy_1} = \mu(1 - y_1^2) - \frac{y_1}{y_2} = K.$$

Solving algebraically for y_2, we see that the icoclines are given by

$$y_2 = \frac{y_1}{\mu\left(1 - y_1^2\right) - K}$$

[17] feeding of energy **into** the system.

[18] The *Van der Pol oscillator* was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol while he was working at Philips [2]. Van der Pol found stable oscillations [3], which he subsequently called relaxation-oscillations [4] and are now known as a type of limit cycle, in electrical circuits employing vacuum tubes. When these circuits are driven near the limit cycle, they become entrained, i.e., the driving signal pulls the current along with it.

Van der Pol and his colleague, van der Mark, reported in the September 1927 issue of *Nature* that at certain drive frequencies an irregular noise was heard [5], which was later found to be the result of deterministic chaos.

Chapter Bibliography

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Chapter 5

Special Functions for ODEs

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5.1 Defining Special Functions

Linear ODEs with **constant coefficients** can be solved by algebraic methods, and their solutions are elementary functions known from calculus.

Definition: Elementary Functions

A function of a single variable which can be either **real** or **complex**, defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, and their inverses.

ODEs with **variable coefficients**, however, is more complicated, and their solutions may be non-elementary which means we can't write the solution explicitly.[1] We will look at the two (2) standard methods for solving ODEs:

Power Series Gives the solution in terms of a power series:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Frobenius Method Gives the solution in power series, multiplied by $\ln x$ or x^r.

[1] For engineering applications where explicit solutions are **NOT** possible, *Legendre's*, *Bessel's*, and the *hypergeometric* equations are important ODEs to know.

5.2 The Method of Power Series

The power series method is the foundational method for solving linear ODEs with variable coefficients where gives solutions in the form of a power series.[2] Remember, the **power series**, in powers of $x - x_0$, is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2 + \cdots . \tag{5.1}$$

[2] The power series method is used for computing values, graphing curves, proving formulas, and exploring properties of solutions and generally approximating solutions.

Here, x is a variable and $a_0, a_1, \cdots$ are constants, called the **coefficients** of the series. Here, x_0 is a constant, called the **centre** of the series. For $x_0 = 0$, we obtain a power series in powers of x:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \tag{5.2}$$

For the duration of the chapter we will assume all variables and constants are **real**.

The term **power series** usually refers to a series of the form Eq. (5.1), but does **NOT** include series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the *Legendre* and *Bessel equations* for integer values.

Exercise 5.1 Power Series Solution

Solve the following ODE:

$$y' - y = 0$$

SOLUTION First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \tag{5.3}$$

We put these values into the ODE:

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

Then we collect like powers of x, finding:

$$\left(a_1 - a_0\right) + \left(2a_2 - a_1\right) x + \left(3a_3 - a_2\right) x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express $a_1, a_2, \cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^{x}. \quad \blacksquare$$

Based on this, we may now generalise this idea. For a given ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{5.4}$$

First represent $p(x)$, $q(x)$ by power series in powers of x.

> If $p(x)$, $q(x)$ are polynomials, then we don't have to do anything in this first step.

Next, we assume a solution in the form of a power series with **unknown coefficients** and insert it as well as Eq. (5.3) and:

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x_2 + \cdots = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} \tag{5.5}$$

into the ODE. Then we **collect same powers of** x and equate the sum of the coefficients of each occurring power of x to zero (0), starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of Eq. (5.3) successively.

Now let's work on a problem which generally applies to conditions with spherical symmetry.

Exercise 5.2 A Special Legendre Function

Solve the following ODE:

$$\left(1 - x^2\right) y'' - 2xy' + 2y = 0$$

SOLUTION Substitute Eq. (5.2), Eq. (5.3), and Eq. (5.5) into the ODE, $\left(1 - x^2\right) y''$ gives two (2) series: for y'', and for $-x^2 y''$. For the term $-2xy'$ use Eq. (5.3) and in $2y$ use Eq. (5.2). Write like powers of x vertically aligned for easy viewing. This gives:

$$\begin{aligned}
y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\
-xy'' &= \qquad\qquad\qquad - 2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots \\
-2xy' &= \qquad - 2a_1 x - 4a_5 x^2 - 6a_9 x^3 - 8a_4 x^4 - \cdots \\
2y &= 2a_0 + 2a_1 x - 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots .
\end{aligned}$$

Add terms of like powers of x. For each power x^0, x, x^2 equate the sum obtained to zero. Denote these sums by 0 (constant terms), 1 (first power of x), and so on and write it down to the following table:

Sum	Power	Equation
0	x^0	$a_2 = -a_0$
1	x	$a_3 = 0$
2	x^2	$14a_4 = 4a_2, \qquad a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
3	x^3	$a_5 = 0 \qquad \text{since} \qquad a_3 = 0$
4	x^4	$30a_6 = 18a_4, \qquad a_6 = \frac{18}{30}a_4 = \frac{18}{30}\left(-\frac{1}{3}\right)a_0 = -\frac{1}{5}a_0$

This gives the solution

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) \quad \blacksquare$$

Note: a_0, a_1 remain **arbitrary**.

Therefore, this is a **general solution** consisting of two (2) solutions: x and

$$1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$$

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(x)$ and *Legendre functions* $Q_1(x)$. Here we have

$$x = P_1(x)$$

and

$$1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(x)$$

Note: The minus is by convention. The index 1 is called the *order* of these functions and here the order is 1. ■

5.3 Legendre's Equation

5.3.1 The Polynomials of Legendre

Legendre's[3] differential equation:

$$\left(1-x^2\right)y'' - 2xy' + n\left(n+1\right)y = 0, \tag{5.6}$$

is an important ODE in physics. It arises in numerous problems, particularly in boundary value problems for spheres. The equation involves a parameter n, whose value depends on the physical or engineering problem. Therefore Eq. (5.6) is actually a whole family of ODEs. For $n = 1$ we solved it in the previous example.

[3]Adrien-Marie Legendre *(1752 - 1833)*

A French mathematician who made numerous contributions to mathematics. Well-known and important concepts such as the *Legendre polynomials* and *Legendre transformation* are named after him [1].

Any solution of Eq. (5.6) is called a **Legendre function**.

The study of these and other higher functions **NOT** occurring in calculus is called the theory of special functions.

Dividing Eq. (5.6) by $1 - x^2$, we obtain the standard form:

$$y'' - \frac{2x}{\left(1-x^2\right)}y' + \frac{n\left(n+1\right)}{\left(1-x^2\right)}y = 0$$

We see the coefficients $-2x/\left(1-x^2\right)$ and $n\left(n+1\right)/\left(1-x^2\right)$ of the new equation are analytic[4] at $x = 0$, so the power series method is applicable for this equation. Substituting:

[4]This means the function has a derivative at a given point.

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{5.7}$$

and its derivatives into Eq. (5.6), and denoting the constant $n\left(n+1\right)$ simply as k, we obtain the following:

$$\left(1-x^2\right)\sum_{m=2}^{\infty} m\left(m-1\right)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By splitting the first expression as two (2) separate series we have the equation:

$$\sum_{m=2}^{\infty} m\left(m-1\right)a_m x^{m-2} - \sum_{m=2}^{\infty} m\left(m-1\right)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x_n in all four (4) series, set $m - 2 = s$ (therefore $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives:

$$\sum_{s=0}^{\infty}\left(s+2\right)\left(s+1\right)a_{s+2}x^s - \sum_{s=2}^{\infty} s\left(s-1\right)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

Note the first series the summation begins with $s = 0$.

As this equation with the right side 0 must be an identity in x if Eq. (**5.7**) is to be a solution of Eq. (**5.7**), the sum of the coefficients of each power of x on the LHS must be zero.

Now x^0 occurs in the first and fourth series only, and gives:

remember $k = n(n+1)$

$$x^0 \qquad 2 \cdot 1 a_2 + n(n+1)\, a_0 = 0, \tag{5.8}$$

$$x^1 \qquad 3 \cdot 2 a_3 + \left[-2 + n(n+1)\right] a_1 = 0, \tag{5.9}$$

$$x^2, x^3, \cdots \qquad (s+2)(s+1)\, a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right] a_s = 0. \tag{5.10}$$

The expression in the brackets $[\cdots]$ can be simplified to $(n-s)(n+s+1)$.

Solving Eq. (**5.8**) for a_2 and Eq. (**5.9**) for a_3 as well as Eq. (**5.10**) for a_{s+2}, we obtain the **general formula**:

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad \text{for} \qquad s = 0, 1, 2, \cdots \tag{5.11}$$

This is called a **recurrence relation** or **recursion formula**.[5] It gives each coefficient in terms of the 2nd one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively:

[5] an equation according to which the n^{th} term of a sequence of numbers is equal to some combination of the previous terms.

$$\begin{aligned}
a_2 &= -\frac{n(n+1)}{2 \cdot 1} a_0 & & \\
a_3 &= -\frac{(n-1)(n+2)}{3 \cdot 2} a_1 &&= \frac{(n-2)(n+1)}{3!} a_1 \\
a_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3} a_2 &&= \frac{(n-2)(n+1)(n+3)\, n}{4!} a_0 \\
a_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4} a_3 &&= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 \\
&\vdots && \vdots
\end{aligned}$$

By inserting these expressions for the coefficients into Eq. (**5.7**) we obtain:

$$y(x) = a_0 y_1(x) + a_1 y_2(x). \tag{5.12}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\, n(n+1)(n+3)}{4!} x^4 - \cdots \tag{5.13}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots \tag{5.14}$$

These series converge for $|x| < 1$. As Eq. (**5.13**) contains **even** powers of x only, while Eq. (**5.14**) contains **odd** powers of x only, the ratio y_1/y_2 is not a **constant**, This means y_1 and y_2 are not proportional and are therefore are linearly independent solutions.

Therefore Eq. (**5.12**) is a general solution of Eq. (**5.6**) on the interval $-1 < x < 1$.

$x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardised ODE are no longer analytic.

5.3.2 Polynomial Solutions

The reduction of power series to polynomials is a great advantage because we now have solutions for all x, without convergence restrictions.

For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials. For *Legendre's equation* this happens when the parameter n is a non-negative integer because the RHS of Eq. (**5.11**) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0, \cdots$.

Therefore if n is even, $y_1(x)$ reduces to a polynomial of degree n.

If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of such constants is done as follows.

We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad \text{where} \qquad n \text{ is a positive integer.} \tag{5.15}$$

and $a_n = 1$ if $n = 0$. Then we calculate the other coefficients from Eq. (**5.11**), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leq n-2) \tag{5.16}$$

The choice Eq. (**5.15**) makes $p_n(1) = 1$ for every n which makes our lives easier. From Eq. (**5.16**) with $s = n - 2$ and Eq. (**5.15**) we obtain:

$$a_{n-2} = -\frac{n(n-1)}{2(n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2}$$

Using $(2n)! = 2n(2n-1)(2n-2)!$ in the numerator and $n! = n(n-1)!$ and $n! = n(n-1)(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n(n-1)\,2n(2n-1)(2n-2)!}{2(2n-1)\,2^n n(n-1)!\,n(n-1)(n-2)!}.$$

$n(n-1)\,2n(2n-1)$ cancels out, which we get:

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!} \quad \text{and} \quad a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

and so on, and in general, when $n - 2m \geq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!(n-2m)!}. \tag{5.17}$$

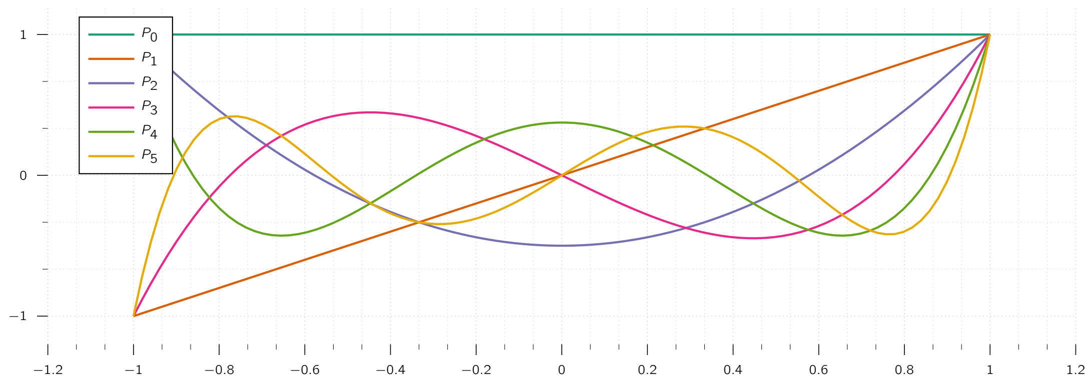

Figure 5.1
The first six Legendre polynomials.

The resulting solution of Legendre's differential equation Eq. (**5.6**) is called the *Legendre polynomial of degree n* and is denoted by $P_n(x)$.:

From Eq. (**5.17**) we obtain:

$$P_n(x) = \sum_{m-0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m} \tag{5.18}$$

$$= \frac{(2m)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!\,(n-1)!\,(n-2)!} x^{n-2} + - \cdots \tag{5.19}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are

$$P_0(x) = 1, \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

The Legendre polynomials $P_n(x)$ are **orthogonal** on the interval $-1 \leqq x \leqq 1$, a basic property to be defined and used in making up "Fourier-Legendre series" which will be the focus for *Higher Mathematics II*.

5.4 Extending the Power Series using Frobenius Method

Several 2nd-order ODEs are important for engineering applications.

One of the famous ones **Bessel Equation** will be our focus in the continuing section.

Unfortunately, these practical 2nd-order ODEs have coefficients that are **NOT** analytic, but are possible to solve via series method.[6] The following theorem permits an extension of the power series method.

This method is called the Frobenius method.[7]

Theory 5.15: Frobenius Method

Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{5.20}$$

has **at least one solution** that can be represented in the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{5.21}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE Eq. (**5.20**) also has a 2nd solution, such that these two (**2**) solutions are linearly independent, which may be similar to Eq. (**5.21**), with a different r and different coefficients, or may contain a logarithmic term.

To see this theorem in action, let's look at the Bessel's equation.

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad \text{where } \nu \text{ is a parameter}$$

is of the form Eq. (**5.20**) with:

$$b(x) = 1 \qquad c(x) = x^2 - \nu^2 \qquad \text{analytic at } x = 0 \qquad .$$

This form allows us to use the Frobenius method.

This ODE could **NOT** be handled in full generality by the power series method as these functions are known as *hyper-geometric differential equations*. Therefore, this equation, requires the Frobenius method.

In Eq. (**5.21**) we have a power series times a single power of x whose exponent r is not restricted to be a non-negative integer.

[6] power series times a logarithm or times a fractional power of x, etc.

[7] **Ferdinand Georg Frobenius** *(1849 - 1917)*

A German mathematician, best known for his contributions to the theory of elliptic functions, differential equations, number theory, and to group theory. He is known for the famous determinantal identities, known as *Frobenius-Stickelberger formulae*, governing elliptic functions, and for developing the theory of biquadratic forms.

Regular and Singular Points

The following terms are practical and commonly used. A **regular point** of the ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a **regular point** of the ODE:

$$\widetilde{h}(x)y'' + \widetilde{p}(x)y'(x) + \widetilde{q}(x)y = 0,$$

is an x_0 at which $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$ are analytic and $\widetilde{h}(x_0) \neq 0$.[8] Then the power series method can be applied. If x_0 is not a regular point, it is called a **singular point**.

[8] so what we can divide by $\widetilde{h}$ and get the previous standard form.

5.4.1 Indicial Equation

Time to explain the Frobenius method for solving Eq. (5.20) which is the Bessel equation. Multiplication of Eq. (5.20) by x^2 gives the more convenient form which can be worked upon:

$$x^2 y'' + xb(x)\, y' + c(x)\, y = 0 \tag{5.22}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \qquad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

> If both $b(x)$ and $c(x)$ are polynomials, no actions are needed.

Then we differentiate Eq. (5.21) term by term, finding:

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \\ &= x^{r-1}[r a_0 + (r+1) a_1 x + \cdots] \\ y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} \\ &= x^{r-2}[r(r-1) a_0 + (r+1) r a_1 x + \cdots]. \end{aligned} \tag{5.23}$$

By inserting all these series into Eq. (5.22) we obtain:

$$x^r \left[r(r-1) a_0 + \cdots \right] + (b_0 + b_1 x + \cdots) x^\tau (r a_0 + \cdots) \tag{5.24}$$

$$+ (c_0 + c_1 x + \cdots) x^\tau (a_0 + a_1 x + \cdots) = 0. \tag{5.25}$$

We now equate the sum of the coefficients of each power x^r, x^{r+1}, x^{r+2}, ... to zero. This presents a system of equations involving the unknown coefficients a_m. The smallest power is x^r and the corresponding equation is:

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives:

$$r(r-1) + b_0 r + c_0 = 0 \tag{5.26}$$

This important quadratic equation is called the **indical equation** of the ODE Eq. (**5.22**). Its role is as follows.

The Frobenius method present a basis of solutions. One of the two solutions will always be of the form Eq. (**5.23**), where r is a root of Eq. (**5.26**). The other solution will be of a form indicated by the indicial equation.

There are three (3) cases:

Case 1 Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2 A double root.

Case 3 Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are related to the *Euler-Cauchy equation*, the simplest ODE of the form Eq. (**5.20**).

Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ as

$$r_1 - r_2 = r_1 - \bar{r}_1 = 2\mathbf{j}\,\boldsymbol{Im}\left(r_1\right)$$

is imaginary, so it cannot be a real integer.

> Case 2 we must have a logarithm, whereas in Case 3 we may or may not.

Theory 5.16: Frobenius Method II - The Three Cases

Assume the ODE in Eq. (**5.22**) satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation Eq. (**5.26**).

Then we have the following three (**3**) cases:

Case 1. Distinct Roots Not Differing by an Integer

A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \tag{5.27}$$

and

$$y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right) \tag{5.28}$$

with coefficients obtained successively from Eq. (**5.24**) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.

A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{5.29}$$

(of the same general form as before) and

$$y_2(x) = y_1(x)\ln x + x^r(A_1x + A_2x^2 + \cdots) \qquad (x > 0) \tag{5.30}$$

Case 3. Roots Differing by an Integer.

A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \tag{5.31}$$

(of the same general form as before) and

$$y_2(x) = ky_1(x)\ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots) \tag{5.32}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

5.4.2 Typical Applications

Technically, the *Frobenius method* is similar to the power series method, once the roots of the indicial equation have been determined.

However, Eq. (5.27) - Eq. (5.32) merely indicate the general form of a basis, and a 2$^{\text{nd}}$ solution can often be obtained more rapidly by reduction of order.

Exercise 5.3 Frobenius Method of Euler-Cauchy

Solve the following ODE:

$x^2y'' + b_0xy' + c_0y = 0$ where b_0, c_0 are constant.

As can be seen this equation obeys the form of Euler-Cauchy. Here we will solve the problem using the Frobenius method.

SOLUTION

Substitution of $y = x^r$ gives the auxiliary equation:

$$r(r-1) + b_0r + c_0 = 0,$$

which is the **indicial equation**. For different roots r_1, r_2 we get a basis

$$y_1 = x^{r_1}, \; y_2 = x^{r_2},$$

and for a double root r we get a basis x^r, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

The Frobenius method solves the **hypergeometric equation**, whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

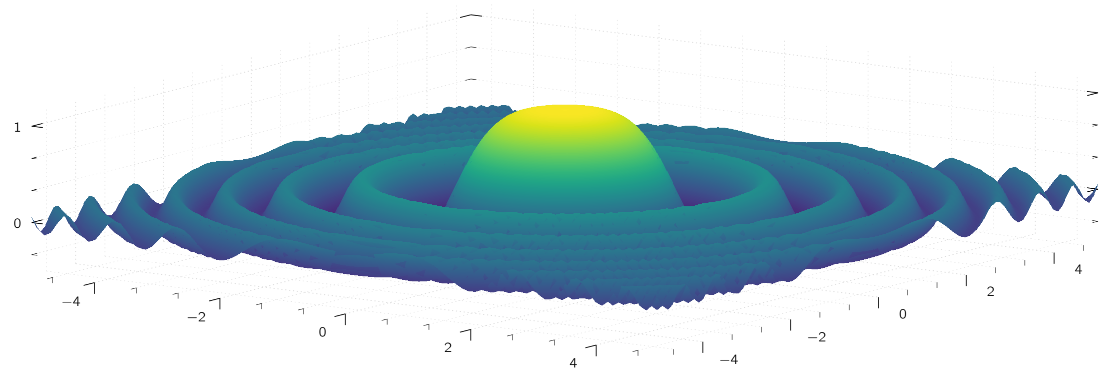

Figure 5.2
Bessel functions describe the radial part of vibrations of a circular membrane.

5.5 Bessel's Function

One of the most important ODEs in applied mathematics in **Bessel's equation** which it's form is shown as:[9]

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \qquad (5.33)$$

[9]Friedrich Wilhelm Bessel *(1784 - 1846)* was a German astronomer, mathematician, physicist, and geodesist. He was the first astronomer who determined reliable values for the distance from the Sun to another star by the method of parallax. Certain important mathematical functions were first studied systematically by Bessel and were named Bessel functions in his honour.

Converting this to the traditional Frobenius form:

$$y'' + \frac{1}{x}y' + \frac{1-\nu^2}{x^2}y = 0 \qquad \text{where} \qquad b(x) = 1 \qquad c(x) = 1 - \nu^2,$$

and the parameter ν is a given **real number** which is either positive or zero.

> Bessel's equation appears often in problems showing cylindrical symmetry or membranes such as modeling heat flow in cylindrical objects, the distribution of potential in an electrostatic field, and in hydrodynamics in the irrotational motion of an incompressible fluid.

5.5.1 Deriving the Solution

According to the Frobenius theory, it has a solution of the form:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad [\, a_0 \neq 0] \qquad (5.34)$$

Substituting Eq. (5.34) and its 1st and 2nd derivatives into Bessel's equation, we obtain:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero.

> Note that this power x^{s+r} corresponds to $m = s$ in the first, 2nd, and fourth series, and to $m = s - 2$ in the third series.

Therefore, for $s = 0$ and $s = 1$, the third series does not contribute since $m \geq 0$. For $s = 2, 3, \ldots$ all four series contribute, so that we get a general formula for all these s. We find:

$$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \qquad (s = 0) \qquad (5.35)$$

$$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \qquad (s = 1) \qquad (5.36)$$

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 \qquad (s = 2, 3, \ldots) \qquad (5.37)$$

From Eq. (**5.35**) we obtain the **indicical equation** by dropping a_0.

$$(r+\nu)(r-\nu) = 0 \qquad (5.38)$$

The roots are $r_1 = \nu\,(\geq 0)$ and $r_2 = -\nu$.

Coefficient Recursion for $r = r_1 = \nu$ For $r = \nu$, Eq. (**5.36**) reduces to $(2\nu+1)a_1 = 0$. Therefore $a_1 = 0$ as $\nu \geq 0$. Substituting $r = \nu$ in Eq. (**5.37**) and combining the three terms containing $a_s = 0$ gives simply:

$$(s+2\nu)sa_s + a_{s-2} = 0 \qquad (5.39)$$

As $a_1 = 0$ and $v \equiv 0$, it follows from Eq. (**5.39**), $a_3 = 0, a_5 = 0, \cdots$. Hence we have to deal only with even-numbered coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (**5.39**) becomes:

$$(2m+2\nu)2ma_{2m} + a_{2m-2} = 0$$

Solving for a_{2m} gives the recursion formula

$$a_{2m} = -\frac{1}{2^2 m(v+m)} a_{2m-2} \qquad m = 1, 2, \cdots \qquad (5.40)$$

From Eq. (**5.40**) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(v+1)}$$

$$a_4 = -\frac{a_2}{2^2 2(v+2)} = \frac{a_0}{2^4 2!\,(v+1)(v+2)}$$

and so on, and in general:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v+1)(v+2)\cdots(v+m)}, \qquad m = 1, 2, \cdots. \qquad (5.41)$$

5.5.2 Bessel Functions (J_n) for Integers

Integer values of ν are denoted by n, which is the standard mathematical notation.

For $\nu = n$ the relation Eq. (**5.41**) becomes:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n+1)(n+2)\cdots(n+m)}, \qquad m = 1, 2, \cdots \tag{5.42}$$

a_0 is still arbitrary, so that the series Eq. (**5.34**) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function.

Accordingly, we have to make a choice.

The choice $a_0 = 1$ would be possible. A simpler series Eq. (**5.34**) could be obtained if we could absorb the growing product $(n+1)(n+2)\cdots(n+m)$ into a factorial function $(n+m)!$ What should be our choice? Our choice should be:

$$a_0 = \frac{1}{2^n n!} \tag{5.43}$$

because then $n!\,(n+1)\cdots(n+m) = (n+m)!$ in Eq. (**5.42**), so that Eq. (**5.42**) simply becomes:

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n+m)!}, \qquad \text{where} \qquad m = 1, 2, \ldots \tag{5.44}$$

By inserting these coefficients into Eq. (**5.34**) and remembering that $c_1 = 0$, $c_3 = 0$, $\cdots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} \qquad (n \geqq 0). \tag{5.45}$$

$J_n(x)$ is called the **Bessel function of the first kind** of order n. The series Eq. (**5.45**) converges for all x, as the ratio test shows.

$J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

Exercise 5.4 Working with Bessel Functions

Please calculate the bessel functions of $J_0(x)$ and $J_1(x)$.

SOLUTION

For $n = 0$ we obtain from Eq. (**5.45**) the *Bessel function* of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} - \ldots \tag{5.46}$$

which looks similar to a cosine. For $n = 1$ we obtain in the *Bessel function* of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2} + \frac{x^5}{2^5 2! 3} - \frac{x^7}{2^7 3! 4} - \ldots \tag{5.47}$$

which looks similar to a sine. But the zeros of these functions are not completely regularly spaced and the height of the "waves" decreases with increasing z. Heuristically, n^2/x^2 in Eq. (**5.33**) in standard form (i.e., Eq. (**5.33**) divided by x^2) is zero (if $n=0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y'+y=0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.48}$$

where $\sim$ is read "asymptotically equal" and means that for fixed n the quotient of the two sides approaches 1 as $x \to \infty$ $\frac{x^2}{2^2(1!)^2}$.

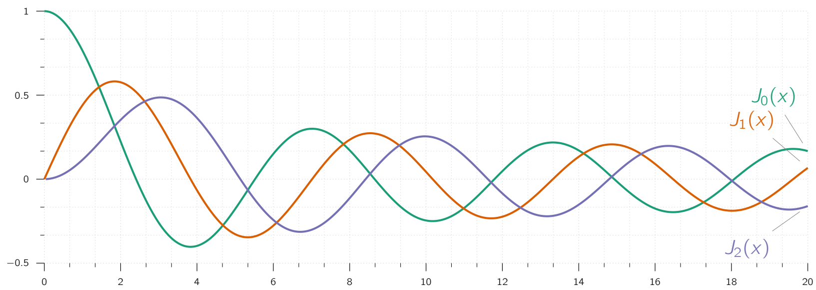

Figure 5.3

Formula Eq. (**5.48**) is surprisingly accurate even for smaller x (>0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of f_0, you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

Chapter Bibliography

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Chapter 6

Laplace Transform

Table of Contents

6.1 Introduction

Laplace[1] transform are important for any engineer as it makes solving **linear** ODEs, related initial value problems, and systems of linear ODEs, much easier.

There are numerous applications which can be significantly simplified such as:

electrical networks, springs, mixing problems, signal processing,

and other areas of engineering and physics. The process of solving an ODE using Laplace transform consists of three (3) steps:

1. The given ODE is transformed into an algebraic equation, called the **subsidiary equation**.
2. The subsidiary equation is solved by purely algebraic manipulations.
3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.

The diagram explaining the thought process can be seen in **Fig.** 6.1.

[1]**Pierre-Simon, Marquis de Laplace**
(1749 - 1827)

A French polymath, a scholar whose work has been instrumental in the fields of physics, astronomy, mathematics, engineering, statistics, and philosophy. He summarized and extended the work of his predecessors in his five-volume *Celestial Mechanics*. This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. Laplace also popularised and further confirmed Sir Isaac Newton's work. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace.

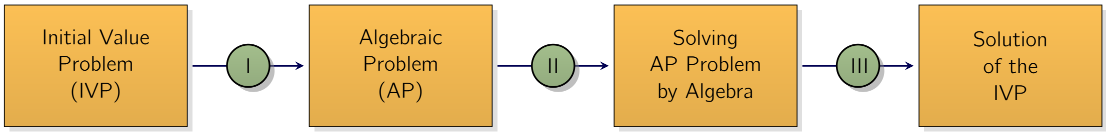


Figure 6.1: A block diagram showcasing the methodology of the Laplace transform.

> The essential idea of Laplace transforms converting an ODE to an algebraic problem.

Laplace Transform has two (2) major advantages over the previous methods.

Problems are solved more directly

IVPs are solved without first determining a general solution. Non-homogeneous ODEs are solved without first solving the corresponding homogeneous ODE.

Solving Discontinuities

More importantly, the use of the unit step function (**Heaviside** function) and Dirac's **delta**[2] make the method particularly powerful for problems with inputs with discontinuities or represent short impulses or complicated periodic functions.

[2] Both these functions play pivotal roles in solving engineering problems, particularly control theory, signal processing, and electrodynamics.

6.2 First Shifting Theorem (s-Shifting)

The key operation of Laplace transform is the following:

> Laplace transform, when applied to a function, **changes the function into a new function** by using a process involving integration.

If $f(t)$ is a function defined for all $t \geq 0$, its **Laplace transform** is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. This operation results in a function of s, (i.e., $F(s)$), and denoted as $\mathcal{L}(f)$:[3]

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{6.1}$$

Here we need to assume $f(t)$ **has** an integral[4] (i.e., it is finite). This assumption is usually satisfied for. **practical** engineering applications.

Not only is the result $F(s)$ called the **Laplace transform**, but the operation just described, which gives $F(s)$ from a given $f(t)$, is also called the **Laplace transform**. It is an integral transform:[5]

$$F(s) = \int_0^{\infty} k(s, t)\, f(t)\, dt$$

with a kernel[6] defined as $k(s, t) = e^{-st}$:

Laplace transform is called an integral transform as it transforms a function in one space to a function in another space by a process of integration which involves a **kernel**.

The kernel is a function of the variables in two (2) spaces and defines the transform. Furthermore, the given function $f(t)$ in Eq. (**6.1**) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$:

$$f(t) = \mathcal{L}^{-1}(F). \tag{6.2}$$

Note Eq. (**6.1**) and Eq. (**6.2**) together imply:

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f, \quad \mathcal{L}(\mathcal{L}^{-1}(F)) = F$$

Information: Notation

Original functions depend on t and their transforms on s. Original functions are denoted by **lowercase letters** and their transforms by the same letters in **capital**:

> For example, $F(s)$ denotes the transform of $f(t)$, and $Y(s)$, the transform of $y(t)$.

[3] This symbol is known as cursive L ($\mathcal{L}$).

[4] Some functions have no integration such as e^{x^n} where $n > 1$. Interestingly the integral of e^{x^2} is called the error function and is an important function in error correction.

[5] an integral transform is a type of transform which maps a function from its original function space into another function space via integration, where some of the properties of the original function might be more easily characterized and manipulated than in the original function space. The transformed function can generally be mapped back to the original function space using the inverse transform.

[6] Here kernel can be used to define any number of operations for a different transform.

Exercise 6.1 Introduction to Laplace Transform

Let $f(t) = 1$ when $t \geq 0$ using this information, please find $F(s)$.

SOLUTION

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^{\infty} e^{-st}\, \mathrm{d}t = -\frac{1}{s} e^{-st} \bigg|_0^{\infty} = \frac{1}{s} \qquad (s > 0)$$

Such an integral is called an improper integral and, is evaluated according to the rule:

$$\int_0^{\infty} e^{-st} f(t)\, \mathrm{d}t = \lim_{T \to \infty} \int_0^{T} e^{-st} f(t)\, \mathrm{d}t.$$

Therefore our convention notation means:

$$\int_0^{\infty} e^{-st}\, \mathrm{d}t = \lim_{T \to \infty} \left[-\frac{1}{s} e^{-st} \right] \bigg|_0^{T} = \lim_{T \to \infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0} \right] = \frac{1}{s}$$

where $(s > 0)$ ■

Exercise 6.2 Laplace Transform of an Exponential Function

Let $f(t) = e^{at}$ when $t \geq 0$, where a is given as a constant. Using this information, please find $\mathcal{L}(f)$.

SOLUTION

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}\left(e^{at}\right) = \int_0^{\infty} e^{-st} e^{at}\, \mathrm{d}t = \frac{1}{a - s} e^{-(s-a)t} \bigg|_0^{\infty}$$

Therefore, when $s - a > 0$,

$$\mathcal{L}\left(e^{at}\right) = \frac{1}{s - a} \quad ■.$$

After these exercises, we could say:

> *Must we go on in this fashion and obtain the transform of one function after another directly from the definition?*

with the answer being, **NOT** necessarily.

We can obtain new transforms from known ones by the use of the many general properties of the Laplace transform. Above all, the Laplace transform is a linear operation, just as are differentiation and integration. Before we continue, lets formalise this as a theory:

Theory 6.17: Linearity

The Laplace transform is a **linear operation**.

This means, for any functions, say $f(t)$ and $g(t)$, whose transforms exist and any **constants** a and b the transform of $af(t) + bg(t)$ exists, The following statement holds true[7]:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

[7] This is only valid when we **add** two (2) operations. Multiplying them would **NOT** give a correct equivalence.

Exercise 6.3 Hyperbolic Functions

Find the transforms of the following hyperbolic functions:

$$\cosh at \qquad \text{and} \qquad \sinh at$$

SOLUTION

We know the following relations exist for the hyperbolic functions:

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

$$\text{and} \qquad \sinh at = \frac{1}{2}(e^{at} - e^{-at})$$

Using this, we can obtain the definitions of them using the exponential function definition from an earlier example.

$$\begin{aligned}\mathcal{L}(\cosh at) &= \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) \\ &= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2} \\ \mathcal{L}(\sinh at) &= \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) \\ &= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \quad \blacksquare\end{aligned}$$

Exercise 6.4 Cosine and Sine

Derive the following formulas:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \qquad \text{and} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

SOLUTION

We start by write $L_e = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain:

$$\begin{aligned}L_e &= \int_0^\infty e^{-st} \cos \omega t \ dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \ dt = \frac{1}{s} - \frac{\omega}{s} L_s, \\ L_s &= \int_0^\infty e^{-st} \sin \omega t \ dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \ dt = \frac{\omega}{s} L_c.\end{aligned}$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain[8]:

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2} \quad \blacksquare$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2} \quad \blacksquare$$

[8] Of course, this is not the only method as one can just use one of the operations twice over tho get the same result

6.2.1 Replacing s by s - a in the Transform

The Laplace transform has an advantageous property whereby if we know the transform of $f(t)$, we can immediately get that of $e^{at} f(t)$. Let's write this in a formal theorem.

Theory 6.18: s-Shifting

If $f(t)$ has the transform $F(s)$, where $s > k$ for some k, then $e^{at} f(t)$ has the transform $F(s-a)$, where $s - a > k$.

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

Let's test this idea with a simple exercise:

Exercise 6.5 **Damped Vibrations**

Using the below definitions:

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2} \quad \text{and} \quad \mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

Find the inverse of the transform:

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}.$$

SOLUTION

Applying the inverse transform, and using its linearity, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t) \quad \blacksquare$$

6.2.2 Existence and Uniqueness

This is **NOT** a big practical problem as in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts.

A function $f(x)$ has a Laplace transform if it does **NOT** grow too fast, say, if for all $t \geq 0$ and some constants M and k it satisfies the growth restriction:[9]

$$|f(x)| \leq Me^{kt}.$$

Here, $f(x)$ need **NOT** be continuous, but it **NOT** be too bad. The technical term is piecewise continuity. $f(x)$ is piecewise continuous on a finite interval $a \leq t \leq b$ where f is defined, if this interval can be divided into finitely many subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives finite jumps as in Fig X as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

[9] This sometimes called "growth of exponential order", which may be misleading given it hides that the exponent must be kt, not kt^2 or similar.

Theory 6.19: Existence and Uniqueness of Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies:

$$\boldsymbol{p}f(t) \leq Me^{\mathrm{kt}}$$

For all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ **exists** for all $s \geqq k$.

6.3 Transforming Derivatives and Integrals

The Laplace transform is a powerful method of solving ODEs and IVPs.

The idea is to replace operations of calculus on functions by operations of algebra. Roughly, differentiation of $f(t)$ will correspond to multiplication of $\mathcal{L}(f)$ by s and integration of $f(t)$ to division of $\mathcal{L}(f)$ by s.[10]

[10] This operational behaviour may remind us of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms. To simplify calculations was one of the main reasons that logarithms were invented.

To solve ODEs, we must first consider the Laplace transform of derivatives.

Theory 6.20: Derivatives

First and Second Order Derivatives

The transforms of the 1st and 2nd derivatives of $f(t)$ satisfy:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

These hold true if $f(t)$ is **continuous** for all $t \equiv 0$ and **satisfies the growth restriction** and $f'(t)$, $f''(t)$ are **piece-wise continuous** on every finite interval on the semi-axis $t \equiv 0$.

Higher Order Derivatives

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}(f^{(n)}) = s^{\text{n}}\mathcal{L}(f) - s^{\text{n-1}}f(0) - s^{\text{n-2}}f'(0) - \cdots - f^{(n-1)}(0).$$

6.3.1 Laplace Transform a Function Integral

Differentiation and integration are inverse operations, and so are multiplication and division. Given differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

Theory 6.21: Laplace Transform of an Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is **piecewise continuous** for $t \geq 0$ and **satisfies a growth restriction**.

Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left(\int_0^t \tau(\cdot)\, \mathrm{d}\tau\right) = \frac{1}{s}F(s), \qquad \text{therefore} \qquad \int_0^t f(\tau)\, \mathrm{d}\tau = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

Exercise 6.6 **Inverse Using Integrations**

Find the inverse of the following functions

$$\frac{1}{s(s^2+\omega^2)} \quad \text{and} \quad \frac{1}{s^2(s^2+\omega^2)}$$

SOLUTION

Using a standard Laplace Transform table we obtain the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^1 \frac{\sin\omega t}{\omega}\,d\tau = \frac{1}{\omega^2}(1-\cos\omega t).$$

The second one we obtain as the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^2}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}.$$

6.3.2 Differential Equations with Initial Values

It's time to discuss how the Laplace Transform method solves ODEs and IVPs. To start with this method, let's consider the following IVP:

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are **constants**. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained.[11].

[11] As we can see from this equation, this is a non-homogeneous 2nd-order ODE

In Laplace's method we do three (3) steps:

Setting up the Subsidiary Equation This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming, namely:

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s).$$

Solution of the Subsidiary Equation using Algebra $s^2 + as + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{s^2+as+b} = \frac{1}{(s+\frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

Q is often denoted by H, but we need H much more frequently for other purposes[12].

[12] For example in control theory and signal processing.

This gives the solution

$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$. Therefore

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

> Q depends neither on $r(t)$ nor on the initial conditions (but only on a and b).

We reduce our $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables so that we obtain the solution[13] $y(t)$.

Given Problem
$y'' - y = t$
$y(0) = 1, \quad y'(0) = 1$

↓

Subsidiary Equation after Transform
$(s^2 - 1)\,Y = s + 1 + 1/s^2$

↓

Solution of the subsidiary equation
$Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$

Solution of the give problem
$y(t) = e^t + \sinh t - t$

[13] The operational steps of laplace transform used in generating a IVP solution.

Exercise 6.7 Inversion of Y

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

SOLUTION

Step 1: Using a Standard Laplace Transform we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2 Y - sy(0) - y'(0) - Y = 1/s^2,$$

therefore:

$$(s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2: The transfer function is $Q = 1/(s^2 - 1)$, and becomes

$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}.$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &= e^t + \sinh t - t \quad \blacksquare \end{aligned}$$

Exercise 6.8 Comparison with Previous Methods

Solve the IVP:

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0$$

SOLUTION

We see that the subsidiary equation is:

$$s^2 Y - 0.16s + sY - 0.16 + 9Y = 0,$$

Therefore

$$\left(s^2 + s + 9\right) Y = 0.16\,(s+1)$$

The solution to the algebraic equation is:

$$Y = \frac{0.16\,(s+1)}{s^2+s+9} = \frac{0.16(s+\frac{1}{2}) + 0.08}{(s+\frac{1}{2})^2 + \frac{35}{4}}.$$

Therefore by the first shifting theorem and the formulas for cos and sin in from this Laplace Transform table we obtain:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y) \\ &= e^{-t/2}\left(0.16\cos\sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\sqrt{\frac{35}{4}}t\right) \\ &= e^{-0.5t}(0.16\cos 2.96t + 0.027\sin 2.96t) \quad \blacksquare \end{aligned}$$

Exercise 6.9 Shifted Data

As a note for this question, shifted data means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$.

For such a problem, set $t = \widetilde{t} + t_0$, so that $t = t_0$ gives $\widetilde{t} = 0$ and the Laplace transform can be applied.

For instance, solve:

$$y'' + y = 2t, \qquad y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \qquad y'\left(\tfrac{1}{4}\pi\right) = 2 - \sqrt{2}.$$

SOLUTION

We have $t_0 = \frac{1}{4}\pi$ and we set $t = \widetilde{t} + \frac{1}{4}\pi$. Then the problem becomes:

$$\widetilde{y}'' + \widetilde{y} = 2\left(\widetilde{t} + \frac{1}{4}\pi\right)$$
$$\widetilde{y}(0) = \frac{1}{2}\pi, \qquad \widetilde{y}'(0) = 2 - \sqrt{2}.$$

where $\widetilde{y}\left(\widetilde{t}\right) = y(t)$. Using a standard Laplace Transform table and denoting the transform of $\widetilde{y}$ by $\widetilde{Y}$, we see that the subsidiary equation of the **shifted** initial value problem is:

$$s^2\widetilde{Y} - s\left(\frac{1}{2}\pi\right) - \left(2 - \sqrt{2}\right) + \widetilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\pi\right)}{s}$$

Therefore:

$$\left(s^2 + 1\right)\widetilde{Y} = \frac{2}{s^2} + \frac{\left(\frac{1}{2}\right)}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this *algebraically* for $\widetilde{Y}$, we obtain

$$\widetilde{Y} = \frac{2}{\left(s^2+1\right)s^2} + \frac{\frac{1}{2}\pi}{\left(s^2+1\right)s} + \frac{\frac{1}{2}\pi s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}$$

The inverse of the first two terms can be seen from a previous example (with $\omega = 1$), and the last two terms give cos and sin,

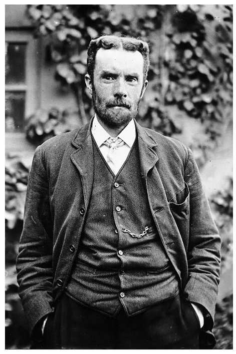
$$\begin{aligned}
\widetilde{y} &= \mathcal{L}^{-1}\left(\widetilde{Y}\right) \\
&= 2\left(\widetilde{t} - \sin\widetilde{t}\right) + \frac{1}{2}\pi\left(1 - \cos\widetilde{t}\right) \\
&+ \frac{1}{2}\pi\cos\widetilde{t} + \left(2 - \sqrt{2}\right)\sin\widetilde{t} \\
&= 2\widetilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\widetilde{t}.
\end{aligned}$$

Now $\widetilde{t} = t - 1/4\pi$, $\sin\widetilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is:

$$y = 2t - \sin t + \cos t \quad \blacksquare$$

6.4 Unit Step Function (t-Shifting)

The following two functions are extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach we discussed in the previous chapters. The reason is that we shall introduce two (2) auxiliary functions, the unit step function or Heaviside function[14] $u(t-a)$ (below) and Dirac's delta $\delta(t-a)$. These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (for example, a hammer hitting an object).



[14] Oliver Heaviside *(1850 - 1925)*

An English self-taught mathematician and physicist who invented a new technique for solving differential equations (equivalent to the Laplace transform), independently developed vector calculus, and rewrote Maxwell's equations in the form commonly used today.

[15] where we can leave it undefined

6.4.1 Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$,[15] and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t > a \end{cases} \qquad \text{where} \qquad a \geq 0.$$

The transform of $u(t-a)$, heaviside function, follows directly from the defining integral for the Laplace transform:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_0^\infty e^{-st} \cdot 1\, dt = -\frac{e^{-st}}{s}\Bigg|_{t-a}^{\infty};$$

here the integration begins at $t = a (\geq 0)$ as $u(t-a)$ is zero (0) for $t < a$. This definition allows us the following simplification:

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad \text{where} \qquad s > 0 \tag{6.3}$$

The unit step function is a typical engineering function made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either `off` or on Multiplying functions $f(t)$ with $u(t-a)$, we can produce all sorts of effects.

Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.

6.4.2 Time Shifting (t-Shifting): Replacing t by t - a in f(t)

The first shifting theorem ("s-shifting") concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s-a) = \mathcal{L}\{e^{2at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t-a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

Theory 6.22: The Second Shifting Theorem

Time Shifting

If $f(t)$ has the transform $F(s)$, then the **shifted function**

$$\widetilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ f(t-a) & \text{if} \quad t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)\,.$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. The transform of 5 sin t is $F(s) = 5/(s^2+1)$, therefore the shifted function 5 sin $(t-2)u(t-2)$ has the transform

$$e^{-2s}F(s) = 5e^{-2s}/(s^2+1).$$

Exercise 6.10 Use of Unit Step Function

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if} 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if} 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if} \quad t > \frac{1}{2}\pi. \end{cases}$$

(Fig. 122)

SOLUTION

Step 1 In terms of unit step functions,

$$f(t) = 2(1-u(t-1)) + \frac{1}{2}t^2(u(t-1) - u(t-\frac{1}{2}\pi)) + (\cos f u(t-\frac{1}{2}\pi)).$$

Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2 To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1-u(t-1))$ remains as it is and gives the transform $2(1-e^{-a})/s$. Then

$$\mathcal{R}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{R}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\Bigg\} = \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-a}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\Bigg\}$$

$$= \left(\frac{1}{s^2} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{\cos\theta\, u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-at}\mathcal{L}\{f(t+a)\}.$$

Exercise 6.11 Application of Both Shifting Theorems

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-2s}}{(s+2)^2}.$$

SOLUTION

Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin\pi t)/\pi$, $(\sin\pi t)/\pi$, and te^{-2t} as $1/s^{-2}$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the *first shifting theorem*. Therefore by the *second shifting theorem* (t-shifting),

$$f(t) = \frac{1}{\pi}\sin\left(\pi(t-1)\right)u(t-1)$$

$$+\frac{1}{\pi}\sin\left(\pi(t-2)\right)u(t-2)$$

$$+(t-3)e^{-2(t-3)}u(t-3)$$

Now $\sin(\pi t - \pi) = -\sin\pi t$ and $\sin(\pi t - 2\pi) = \sin\pi t$, so the first and second terms cancel each other when $t > 2$. Therefore we obtain:

$$\begin{aligned} f(t) &= 0 && \text{if} && 0 < t < 1, \\ -(\sin\pi t)/\pi & && \text{if} && 1 < t < 2, \\ 0 & && \text{if} && 2 < t < 3, \\ (t-3)e^{-2(t-3)} & && \text{if} && t > 3 \end{aligned}$$ ■

Exercise 6.12 Response of a RC-Circuit to a Singular Rectangular Wave

Find the current $i(t)$ in the RC-circuit if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent[16] before the wave is applied.

[16] In this state, it means it is in a state or period of inactivity or dormancy.

SOLUTION

The input is $V_0 \left(u(t-a) - u(t-b) \right)$. Therefore the circuit is modeled by the integro-differential equation

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C} \int_0^t i(\tau)\, d\tau = v(t) = V_0 \left(u(t-a) - u(t-b) \right)$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s} \left(e^{-as} - e^{-bs} \right)$$

Solving this equation algebraically for $I(s)$, we get:

$$I(s) = F(s) \left(e^{-as} - e^{-bs} \right) \qquad \text{where} \qquad F(s) = \frac{V_0 I R}{s + 1/(RC)} \qquad \text{and} \qquad \mathcal{L}^{-1}(F) = \frac{V_0}{R} e^{-t\left(1/RC\right)}$$

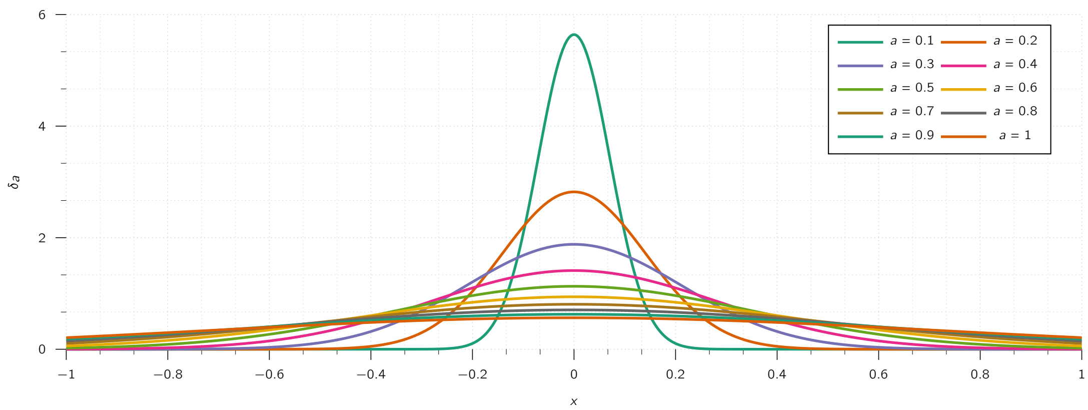


Figure 6.2 Plotting of the Dirac Delta Function.

6.5 Dirac Delta Function

Imagine an air-plane making a hard landing, or a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in everyday life.

They are phenomena of an impulsive nature where actions of forces-mechanical, electrical, etc. are applied over short intervals of time. We can model such phenomena and problems by **Dirac's delta function**, and solve them very effecively by the Laplace transform.

To model situations of that type, we consider this function:

$$f_{\mathrm{k}}(t-a) = \begin{cases} 1/k & \text{if} \quad a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small.

In mechanics, the integral of a force acting over a time interval $a \leq t \leq a + k$ is called the impulse of the force. Similarly for electromotive forces $E(t)$ acting on circuits.

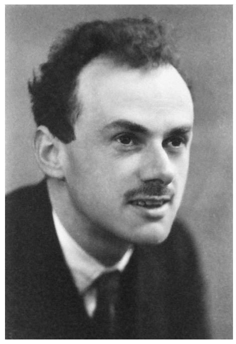
$$I_{\mathrm{k}} = \int_0^{\infty} f_{\mathrm{k}}(t-a)\, \mathrm{d}t = \int_a^{a+k} \frac{1}{k}\, \mathrm{d}t = 1$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_{k} as $k \to 0\,(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k \to 0} f_{\mathrm{k}}(t-a)$$

where $\delta(t-a)$ is called the Dirac delta function, or the **unit impulse function**.[17]

Exercise 6.13 **Mass-Spring System Under a Square Wave**



[17]**Paul A. M. Dirac** *(1902 - 1984)*

was a British theoretical physicist who is considered to be one of the founders of quantum mechanics. Dirac laid the foundations for both quantum electrodynamics and quantum field theory. He was the Lucasian Professor of Mathematics at the University of Cambridge from 1932 to 1969 and a professor of physics at Florida State University from 1970 to 1984. Dirac shared the 1933 Nobel Prize in Physics with *Erwin Schrödinger* for the discovery of new productive forms of atomic theory.

Determine the response of the damped mass-spring system under a square wave.

$$y'' + 3y' + 2y = u(t-1) - u(t-2),$$
$$y(0) = 0, \qquad y'(0) = 0.$$

SOLUTION

nFrom (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \frac{1}{s}\left(e^{-s} - e^{-2s}\right). \qquad \text{Solution} \qquad Y(s) = \frac{1}{s(s^2+3s+2)}\left(e^{-s} - e^{-2s}\right).$$

We have seen that $Y(s)$ is bounded from s-direction.

Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

6.6 Convolution

Convolution[18] has to do with the multiplication of transforms. The situation is as follows.

Addition of transforms provides no problem as we know that:

$$\mathcal{L}(f+g)=\mathcal{L}(f)+\mathcal{L}(g).$$

Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product $\mathcal{L}(f)\,\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false.

The transform of a product is generally different from the product of the transforms of the factors:

$$\mathcal{L}(fg)\neq\mathcal{L}(f)\,\mathcal{L}(g) \qquad \text{[in general]}$$

To see this useful property, let's take $f=e^{t}$ and $g=1$.

Then $fg=e^{t}$, $\mathcal{L}(fg)=1/(s-1)$, however $\mathcal{L}(f)=1/(s-1)$ and $\mathcal{L}(1)=1/s$ give $\mathcal{L}(f)\,\mathcal{L}(g)=1/\left(s^2-s\right)$.

According to the following theorem, the correct answer is $\mathcal{L}(f)\,\mathcal{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f*g$ and defined by the integral shown as:

$$h(t)=(f*g)(t)=\int_0^t f(t)\,g(t-\tau)\,\mathrm{d}\tau. \tag{6.4}$$

Theory 6.23: The Theorem of Convolution

If two (2) functions, for example, f and g satisfy the assumption in given in **existence theorem**, so that their transforms (F and G) exist, their product $H=FG$ is the transform of h given by Eq. (6.4).

Exercise 6.14 Convolution - I

Let $H(s)=1/\left[(s-a)\,s\right]$. Find $h(t)$.

SOLUTION

$1/(s-a)$ has the inverse $f(t)=e^{at}$, and $1/s$ has the inverse $g(t)=1$. With $f(\tau)=e^{at}$ and $g(t-\tau)=1$ we thus obtain from Eq. (6.4) the answer:

$$h(t)=e^{at}*1=\int_0^t e^{a\tau}\cdot 1\,\mathrm{d}\tau$$

$$=\frac{1}{a}\left(e^{at}-1\right)$$

To check the above result we can calculate:

$$\begin{aligned}
H(s)=\mathcal{L}(h)(s)&=\frac{1}{a}\left(\frac{1}{s-a}-\frac{1}{s}\right)\\
&=\frac{1}{s}\cdot\frac{a}{s^2-as}\\
&\frac{1}{s-a}\cdot\frac{1}{s}\\
&\mathcal{L}\left(e^{at}\right)\mathcal{L}(1) \quad \blacksquare
\end{aligned}$$

[18] Convolution has wide applications in various fields such as signal processing, image processing, probability, and physics, where it is used for filtering signals, blurring or sharpening images, detecting edges, and solving differential equations. In artificial intelligence, convolutional neural networks use convolution to design kernels for tasks like image recognition.

For example, a big application of convolution is to use in blurring images such as shown above where Gaussian filters is applied [1].

Part II

Linear Algebra & Vector Calculus

Part Contents

Mathematics is the art of reducing any problem to linear algebra.

(William Stein - Mathematician, the lead developer of SageMath.)

Chapter 7

Vector Calculus

Table of Contents

7.1 Vector Algebra

7.1.1 Vector Operations

Walking 5 km north and then 12 km east, we will have gone a total of 17 km, but if we carefully observe our current position, we will realise that we're not 17 km from where we set out, but only 13. To describe these quantities, We need a set of mathematics principles, which evidently do **NOT** add in the ordinary way.

The reason they don't, is **displacements** have *direction* as well as *magnitude*, and it is essential to take both into account when we combine them. Such objects are called **vectors**.

Examples include: velocity, acceleration, force, momentum ...

By contrast, quantities that have magnitude but no direction are called **scalars**.

Examples include: mass, charge, density, temperature, ...

We shall use boldface ($\boldsymbol{A}$, $\boldsymbol{B}$, and so on) for vectors and normal-font for scalars. The magnitude of a vector $\boldsymbol{A}$ is written $|\boldsymbol{A}|$ or, more simply, A. In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrow indicates its direction.

Minus $\boldsymbol{A}$ ($-\boldsymbol{A}$) is a vector with the same magnitude as $\boldsymbol{A}$ but of opposite direction.

> Vectors have magnitude and direction but *not location*.

Here we will define four (4) vector operations: addition and three kinds of multiplication.

Addition of Two Vectors Place the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$. Their sum, $\boldsymbol{A}+\boldsymbol{B}$, is the vector from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$. Addition is *commutative*:

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A},$$

5 kilometers east followed by 12 kilometers north gets us to the same place as 12 kilometers north followed by 5 kilometers east. Addition is also *associative*:

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}).$$

To subtract a vector, add its opposite

$$\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B}).$$

Multiplication by a Scalar Value Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged. This means if a is negative, the direction is reversed. Scalar multiplication is *distributive*:

$$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B}.$$

Dot Product of Two Vectors The dot product of two vectors is defined by:

$$\boldsymbol{A}\cdot\boldsymbol{B}\equiv AB\cos\theta \tag{7.1}$$

where θ is the angle they form when placed tail-to-tail.

> $\boldsymbol{A}\cdot\boldsymbol{B}$ is itself a scalar.[1]

[1] which is why its alternative name is **scalar product**.

The dot product is *commutative*,

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A},$$

and *distributive*,

$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}, \tag{7.2}$$

Geometrically, $\boldsymbol{A} \cdot \boldsymbol{B}$ is the product of A times the projection of $\boldsymbol{B}$ along $\boldsymbol{A}$.[2] If the two vectors are parallel, then $\boldsymbol{A} \cdot \boldsymbol{B} = AB$. In particular, for any vector $\boldsymbol{A}$,

[2] Or the product of B times the projection of $\boldsymbol{A}$ along $\boldsymbol{B}$.

$$\boldsymbol{A} \cdot \boldsymbol{A} = A^2 \tag{7.3}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular, then $\boldsymbol{A} \cdot \boldsymbol{B} = 0$.

Cross Product of Two Vectors The cross product of two (2) vectors is defined by:

$$\boldsymbol{A} \times \boldsymbol{B} \equiv AB \sin\theta \, \hat{\boldsymbol{n}} \tag{7.4}$$

where $\hat{\boldsymbol{n}}$ is a **unit vector** (vector of magnitude 1) pointing perpendicular to the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$. Of course, there are *two* directions perpendicular to any plane: **in** and **out**.

The ambiguity is resolved by the **right-hand rule**: let our fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then our thumb indicates the direction of $\hat{\boldsymbol{n}}$.

> $\boldsymbol{A} \times \boldsymbol{B}$ is itself a *vector* and it is also known as **vector product**.

The cross product is *distributive*,

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C}) \tag{7.5}$$

but **NOT** commutative:

$$(\boldsymbol{B} \times \boldsymbol{A}) = -(\boldsymbol{A} \times \boldsymbol{B}) \tag{7.6}$$

If two (2) vectors are parallel, their cross product is zero. In particular,

$$\boldsymbol{A} \times \boldsymbol{A} = 0,$$

for any vector $\boldsymbol{A}$.

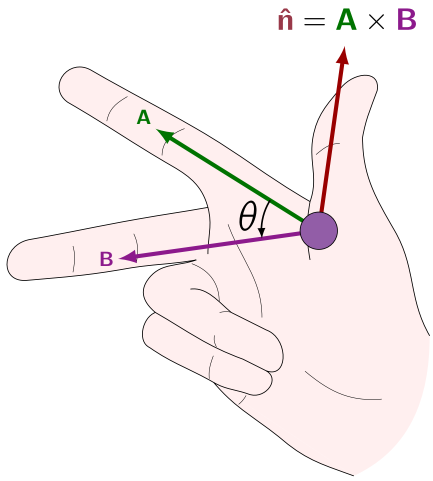


Figure 7.1: A mnemonic, used to define the orientation of axes in three-dimensional space and to determine the direction of the cross product of two vectors, as well as to establish the direction of the force on a current-carrying conductor in a magnetic field.

7.1.2 Vector Component Forms

In the previous section, we defined the four (4) vector operations in abstract form, without reference to any particular coordinate system.

In practice, it is often easier to set up Cartesian coordinates (x, y, z) and work with vector **components**. Let $\hat{x}$, $\hat{y}$, $\hat{z}$ be unit vectors parallel to the x, y, and z axes, respectively. An arbitrary vector $\boldsymbol{A}$ can be expanded in terms of these **basis vectors**:

$$\boldsymbol{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

The symbols A_x, A_y, A_z, are the components of $\boldsymbol{A}$. In geometrical terms they are the **projections** of $\boldsymbol{A}$ along the three (3) coordinate axes (i.e., $A_x = \boldsymbol{A} \cdot \hat{x}$, $A_y = \boldsymbol{A} \cdot \hat{y}$, $A_z = \boldsymbol{A} \cdot \hat{z}$).

We can now reformulate each of the four (4) vector operations as a rule for manipulating components:

$$\begin{aligned}\boldsymbol{A} + \boldsymbol{B} &= \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) + \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) \\ &= (A_x + B_x)\ \hat{x} + \left(A_y + B_y\right)\ \hat{y} + (A_z + B_z)\ \hat{z}\end{aligned}$$

The operation rules are as follows:

i. To add vectors, add like components.

$$a\boldsymbol{A} = (aA_x)\ \hat{x} + \left(aA_y\right)\ \hat{y} + (aA_z)\ \hat{z}$$

ii. To multiply by a scalar, multiply each component. As $\hat{x}$, $\hat{y}$, $\hat{z}$ are mutually perpendicular unit vectors, the following properties are valid:

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \qquad \hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

Accordingly,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \cdot \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right)$$

iii. To calculate the dot product, multiply like components, and add. In particular,

$$\boldsymbol{A} \cdot \boldsymbol{A} = A_x^2 + A_y^2 + A_z^2, \qquad \text{so} \qquad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Similarly the following relations can be derived:

$$\begin{aligned}\hat{x} \times \hat{x} &= \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0, \\ \hat{x} \times \hat{y} &= -\hat{y} \times \hat{x} = \hat{z}, \\ \hat{y} \times \hat{z} &= -\hat{z} \times \hat{y} = \hat{x}, \\ \hat{z} \times \hat{x} &= -\hat{x} \times \hat{z} = \hat{y}.\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{A} \times \boldsymbol{B} &= \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \times \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) \\ &= \left(A_y B_z - A_z B_y\right)\ \hat{x} + (A_z B_x - A_x B_z)\ \hat{y} + \left(A_x B_y - A_y B_x\right)\ \hat{z}.\end{aligned}$$

This expression look unruly but we can tidy it up and write it more neatly as a **determinant**:

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{7.7}$$

iv. To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\boldsymbol{A}$ (in component form), and whose third row is $\boldsymbol{B}$.

7.1.3 Triple Products

As the cross product of two (2) vectors is itself a vector, it can be dotted or crossed with a 3rd vector to form a **triple** product.

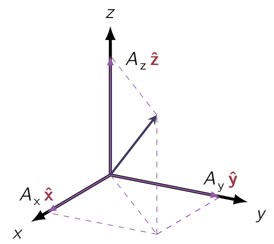


Figure 7.2: The decomposition of the vector $\boldsymbol{A}$ into its components.

Scalar Triple Product Evidently,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}),$$

for they all correspond to the same value.

The **alphabetical** order is preserved.

Alternatively, the **non-alphabetical** triple products,

$$\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{C})=\boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{A}),$$

have the **opposite** sign. In component form,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}.$$

A final point worth mentioning is the dot and cross can be interchanged:

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C},$$

however, it is important to stress it out, the placement of the parentheses is **critical**:

Information: Cross Product of Two Scalars

$(\boldsymbol{A}\cdot\boldsymbol{B})\times\boldsymbol{C}$ is a meaningless expression. We can't make a cross product from a scalar and a vector.

Vector Triple Product The vector triple product can be simplified by the **BAC-CAB** rule:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\,.$$

Please observe that the following triple product

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=-\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=-\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C})+\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})$$

is an entirely different vector.[3] All *higher* vector products can be similarly reduced, often by repeated application, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})&=(\boldsymbol{A}\cdot\boldsymbol{C})\,(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})\,(\boldsymbol{B}\cdot\boldsymbol{C})\,,\\ \boldsymbol{A}\times\left[\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D})\right]&=\boldsymbol{B}\left[\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{D})\right]-(\boldsymbol{A}\cdot\boldsymbol{B})\,(\boldsymbol{C}\times\boldsymbol{D})\,.\end{aligned}$$

[3] To reiterate cross-products are not associative.

7.1.4 Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin ($\mathcal{O}$) is called the **position vector**:

$$\boldsymbol{r} \equiv x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

Throughout this course, $\boldsymbol{r}$ will be used to measure **distance**. Its magnitude:

$$r = \sqrt{x^2 + y^2 + z^2}$$

is the distance from the origin, and

$$\hat{r} = \frac{\boldsymbol{r}}{r} = \frac{(x)\;\hat{x} + (y)\;\hat{y} + (z)\;\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + \mathrm{d}x,\, y + \mathrm{d}y,\, z + \mathrm{d}z)$, is

$$\mathrm{d}\boldsymbol{l} = (\,\mathrm{d}x)\;\hat{x} + (\,\mathrm{d}y)\;\hat{y} + (\,\mathrm{d}z)\;\hat{z}.$$

In electrodynamics, one frequently encounters problems involving two (2) points:

source point ($\boldsymbol{r}'$), where an electric charge is located

field point ($\boldsymbol{r}$), at which we are calculating the electric or magnetic field

To make these redundant calculations easier to handle, let's use the following short-hand notation:

$$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}'$$

Its magnitude is:

$$\mathscr{r} = \left|\boldsymbol{r} - \boldsymbol{r}'\right|$$

and a unit vector in the direction from $\boldsymbol{r}'$ to $\boldsymbol{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}$$

In Cartesian coordinates our new short-notations would be as following:

$$\boldsymbol{\mathscr{r}} = \left(x - x'\right)\;\hat{x} + \left(y - y'\right)\;\hat{y} + \left(z - z'\right)\;\hat{z}$$
$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$
$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\left(x - x'\right)\;\hat{x} + \left(y - y'\right)\;\hat{y} + \left(z - z'\right)\;\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

7.2 Differential Calculus

7.2.1 Ordinary Derivatives

Assume a function of just one (1) variable: $f(x)$. Therefore, the question we need to answer is what does the derivative, df/dx, do and convey?

It tells us how rapidly the function $f(x)$ **varies** when we change the argument x by an infinitesimal amount, $\mathrm{d}x$:

$$\mathrm{d}f = \left(\frac{df}{dt}\right) \mathrm{d}x$$

If we increment x by an infinitesimal amount $\mathrm{d}x$, then f changes by an amount $\mathrm{d}f$.[4]

[4] Here we can think the derivative as the proportionality factor.

> Geometrically, the derivative df/dt is the *slope* of the graph of f versus x.

7.2.2 Gradient

Assume a function which accepts three (3) variables. As an example, lets take the temperature $T(x, y, z)$ in the lecture room. Start out in one corner, and set up a system of cardinal directions. Then for each point (x, y, z) in the room, T gives the temperature at that spot. We want to generalise the notion of derivative to functions like T, which depend **NOT** on one but on three variables.

A derivative tells us **how fast the function varies**, if we move a little distance. But this time the situation is more complicated, because it depends on what **direction** we move:

- Going straight up, the temperature will probably increase fairly rapidly,
- Moving horizontally, it may not change much at all.

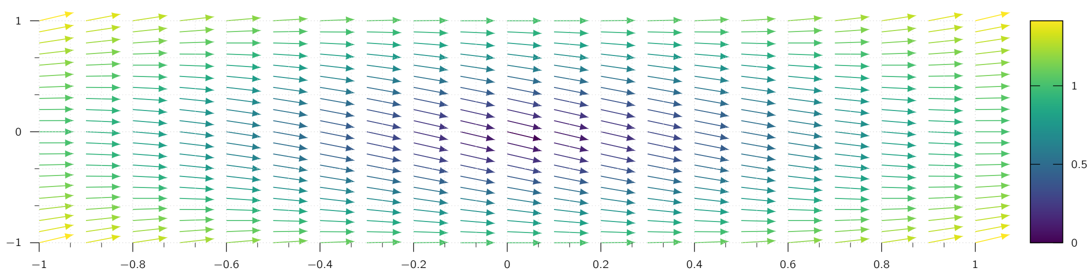


Figure 7.3: An example of a gradient field. Here the field itself is plotted by using arrows to designate the direction of the gradient. To explain the magnitude of the gradient it is either shown by the length of the arrow or by imposing colour on to the plot, which the latter has been used here. Extending this plot to our example, we can see that most of the high temperature resides on the edges of the room whereas the centre remains cool.

In fact, the question of "How fast does T vary?" can have an infinite number of answers, one for each direction we might choose to explore. Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states:

$$\mathrm{d}T = \left(\frac{\partial T}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial T}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial T}{\partial z}\right)\mathrm{d}z$$

This tells us how T changes when we alter all three (3) variables by the infinitesimal amounts of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. We can write the aforementioned equation as a dot product:

$$\mathrm{d}T = \left(\frac{dT}{dx}\,\hat{x} + \frac{dT}{dy}\,\hat{y} + \frac{dT}{dz}\,\hat{z}\right)\cdot\left((\mathrm{d}x)\,\hat{x} + (\mathrm{d}y)\,\hat{y} + (\mathrm{d}z)\,\hat{z}\right) = (\nabla T)\cdot(\mathrm{d}\boldsymbol{l}),$$

where

$$\nabla T \equiv \frac{dT}{dx}\,\hat{x} + \frac{dT}{dy}\,\hat{y} + \frac{dT}{dz}\,\hat{z}$$

is the **gradient** of T. Note that ∇T is a **vector quantity**, with three (3) components.[5]

[5] This is the generalised derivative we have been looking for.

Exercise 7.1 Finding Vector Components - I

Find the components of the vector $\boldsymbol{v}$ with given initial point P and terminal point Q. Find $|\boldsymbol{v}|$ and unit vector $\hat{v}$.

$$\begin{aligned}
&P(3,2,0), && Q(5,-2,2), && P(1,1,1), && Q(-4,-4,-4)\\
&P(1,0,1.2), && Q(0,0,6.2), && P(2,-2,0), && Q(0,4,6)\\
&P(4,3,2), && Q(-4,-3,2), && P(0,0,0), && Q(6,8,10)
\end{aligned}$$

SOLUTION The solution is as follows:

$$\begin{aligned}
\boldsymbol{v} &= (5-3)\,\hat{x} + (-2-2)\,\hat{y} + (2-0)\,\hat{z} = (2)\,\hat{x} + (-4)\,\hat{y} + (2)\,\hat{z},\\
|\boldsymbol{v}| &= \sqrt{(2)^2 + (-4)^2 + (2)^2} = 2\sqrt{6}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(2)\,\hat{x} + (-4)\,\hat{y} + (2)\,\hat{z}}{2\sqrt{6}} = \left(\frac{1}{\sqrt{6}}\right)\hat{x} + \left(-\frac{2}{\sqrt{6}}\right)\hat{y} + \left(\frac{1}{\sqrt{6}}\right)\hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (-4-1)\,\hat{x} + (-4-1)\,\hat{y} + (-4-1)\,\hat{z} = (-5)\,\hat{x} + (-5)\,\hat{y} + (-5)\,\hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-5)\,\hat{x} + (-5)\,\hat{y} + (-5)\,\hat{z}}{5\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}\right)\hat{x} + \left(-\frac{1}{\sqrt{3}}\right)\hat{y} + \left(-\frac{1}{\sqrt{3}}\right)\hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (0-1)\,\hat{x} + (0-0)\,\hat{y} + (6.2-1.2)\,\hat{z} = (-1)\,\hat{x} + (0)\,\hat{y} + (5)\,\hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-1)^2 + (0)^2 + (5)^2} = \sqrt{26}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-1)\,\hat{x} + (0)\,\hat{y} + (5)\,\hat{z}}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}\right)\hat{x} + (0)\,\hat{y} + \left(-\frac{5}{\sqrt{26}}\right)\hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (0-2)\,\hat{x} + \big(4-(-2)\big)\,\hat{y} + (6-0)\,\hat{z} = (-2)\,\hat{x} + (6)\,\hat{y} + (6)\,\hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-2)^2 + (6)^2 + (6)^2} = 2\sqrt{19}.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-2)\,\hat{x} + (6)\,\hat{y} + (6)\,\hat{z}}{2\sqrt{19}} = \left(-\frac{1}{\sqrt{19}}\right)\hat{x} + \left(\frac{3}{\sqrt{19}}\right)\hat{y} + \left(\frac{3}{\sqrt{19}}\right)\hat{z} \quad \blacksquare\\
\boldsymbol{v} &= (-4-4)\,\hat{x} + (-3-3)\,\hat{y} + (2-2)\,\hat{z} = (-8)\,\hat{x} + (-6)\,\hat{y} + (0)\,\hat{z},\\
|\boldsymbol{v}| &= \sqrt{(-8)^2 + (-6)^2 + (0)^2} = 10.\\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(-6)\,\hat{x} + (-8)\,\hat{y} + (0)\,\hat{z}}{10} = \left(-\frac{3}{5}\right)\hat{x} + \left(-\frac{4}{5}\right)\hat{y} + (0)\,\hat{z} \quad \blacksquare
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{v} &= (6-0)\ \hat{x} + (8-0)\ \hat{y} + (10-0)\ \hat{z} = (6)\ \hat{x} + (8)\ \hat{y} + (10)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(6)^2 + (8)^2 + (10)^2} = 10\sqrt{2}. \\
\hat{v} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{(6)\ \hat{x} + (8)\ \hat{y} + (0)\ \hat{z}}{10\sqrt{2}} = \left(\frac{3}{5\sqrt{2}}\right)\hat{x} + \left(\frac{4}{5\sqrt{(2)}}\right)\hat{y} + \left(\frac{1}{\sqrt{2}}\right)\hat{z} \quad \blacksquare
\end{aligned}
$$

Exercise 7.2 Finding Vector Components - II

Given the components of a vector $\boldsymbol{v} = \left[v_x,\ v_y,\ v_z\right]$ and a particular initial point P, find the corresponding terminal point Q and the length of $\boldsymbol{v}$ (i.e., $|\boldsymbol{v}|$).

$$
\begin{aligned}
&\boldsymbol{v} = [3,\ -1,\ 0]\,; && P(4,\ 6,\ 0)\,, && \boldsymbol{v} = [8,\ 4,\ 2]\,; && P(-8,\ -4,\ -2)\,, \\
&\boldsymbol{v} = [0.25,\ 2,\ 0.75]\,; && P\{\cdot\}\ 0,\ -0.5,\ 0, && \boldsymbol{v} = [3,\ 2,\ 6]\,; && P(4,\ 6,\ 0)\,, \\
&\boldsymbol{v} = [4,\ 2,\ -2]\,; && P(4,\ 6,\ 0)\,, && \boldsymbol{v} = [3,\ -3,\ 3]\,; && P(4,\ 6,\ 0)\,,
\end{aligned}
$$

SOLUTION Previously we have defined $\boldsymbol{v} = Q - P$. Here we have $\boldsymbol{v}$ and P. To calculate Q we only need to add individual components of the vector with the initial point P.

$$
\begin{aligned}
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (-1+6)\ \hat{y} + (0+0)\ \hat{z} = (7)\ \hat{x} + (5)\ \hat{y} + (0)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10} \quad \blacksquare \\
Q &= \boldsymbol{v} + P = \big(8 + (-8)\big)\ \hat{x} + \big(4 + (-4)\big)\ \hat{y} + (-2+2)\ \hat{z} = (0)\ \hat{x} + (0)\ \hat{y} + (0)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(8)^2 + (4)^2 + (2)^2} = 2\sqrt{21} \quad \blacksquare \\
Q &= \boldsymbol{v} + P = (0.25+0)\ \hat{x} + \big(2 + (-0.5)\big)\ \hat{y} + (0.75+0)\ \hat{z} = (0.25)\ \hat{x} + (1.5)\ \hat{y} + (0.75)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(0.25)^2 + (1.5)^2 + (0.75)^2} = \sqrt{74}/4 \quad \blacksquare \\
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (2+6)\ \hat{y} + (6+0)\ \hat{z} = (7)\ \hat{x} + (8)\ \hat{y} + (6)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(7)^2 + (8)^2 + (6)^2} = \sqrt{149} \quad \blacksquare \\
Q &= \boldsymbol{v} + P = (4+4)\ \hat{x} + (2+6)\ \hat{y} + (-2+0)\ \hat{z} = (8)\ \hat{x} + (8)\ \hat{y} + (-2)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(8)^2 + (8)^2 + (-2)^2} = 2\sqrt{33} \quad \blacksquare \\
Q &= \boldsymbol{v} + P = (3+4)\ \hat{x} + (-3+6)\ \hat{y} + (3+0)\ \hat{z} = (7)\ \hat{x} + (3)\ \hat{y} + (3)\ \hat{z}, \\
|\boldsymbol{v}| &= \sqrt{(7)^2 + (3)^2 + (3)^2} = 2\sqrt{67} \quad \blacksquare
\end{aligned}
$$

7.2.3 The Del Operator

The gradient has the formal appearance of a vector, (∇),[6] multiplying a scalar T:

$$
\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) T
$$

The term in parentheses is called **del** operator:

$$
\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{y}\frac{\partial}{\partial z} \tag{7.8}
$$

Del is **NOT** a vector, in the usual sense. It doesn't mean much until we provide it with a function to act upon. Furthermore, it does **NOT** "multiply" T. Rather, it is an instruction to differentiate

$$
\triangleleft = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},
$$

[6] The History of Nabla

The operator (∇), was originally introduced by William Rowan Hamilton (1805-1865). Hamilton wrote the operator as a rotated nabla and it was P. G. Tait who established An upside-down delta as the conventional symbol in *An Elementary Treatise on Quaternions* (1867) [1]. Tait was also responsible for establishing the term *nabla*.

David Wilkins suggests Hamilton may have used the nabla as a general purpose symbol or abbreviation for whatever operator he wanted to introduce at any time. In 1837 Hamilton used the nabla, in its modern orientation, as a symbol for any arbitrary function in *Trans. R. Irish Acad. XVII. 236.* (OED.) [2] He used the nabla to signify a permutation operator in *"On the Argument of Abel, respecting the Impossibility of expressing a Root of any General Equation above the Fourth Degree, by any finite Combination of Radicals and Rational Functions,* (1839).

Hamilton used the rotated nabla $(\triangleleft)$, for the vector differential operator in the *Proceedings of the Royal Irish Academy* (1846). Hamilton also used the rotated nabla as the vector differential operator in *On Quaternions; or on a new System of Imaginaries in Algebra*. For more information, please visit here.

what follows. To be precise, then, we say that ∇ is a vector operator which acts upon T, not a vector that multiplies T.

With this qualification, though, ∇ mimics the behaviour of an ordinary vector in virtually every way. Almost anything that can be done with other vectors can also be done with ∇. Now, an ordinary vector $\boldsymbol{A}$ can multiply in three (3) ways:

1. By a scalar a: $\boldsymbol{A}a$;
2. By a vector $\boldsymbol{B}$, via the dot product: $\boldsymbol{A} \cdot \boldsymbol{B}$;
3. By a vector $\boldsymbol{B}$ via the cross product: $\boldsymbol{A} \times \boldsymbol{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\boldsymbol{v}$, via the dot product: $\nabla \cdot \boldsymbol{v}$ (divergence)
3. On a vector function $\boldsymbol{v}$, via the cross product: $\nabla \times \boldsymbol{v}$ (curl).

It is time to examine the other two (2) vector derivatives: divergence and curl.

Divergence From the definition of ∇ we construct the divergence as follows:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \left(\left(\frac{\partial}{\partial x}\right) \hat{x} + \left(\frac{\partial}{\partial y}\right) \hat{y} + \left(\frac{\partial}{\partial z}\right) \hat{z} \right) \cdot \left(\left(v_x\right) \hat{x} + \left(v_y\right) \hat{y} + \left(v_z\right) \hat{z} \right) \\
&= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.
\end{aligned}
$$

> Observe that the divergence of a vector function $\boldsymbol{v}$ is itself a scalar $\nabla \cdot \boldsymbol{v}$.

Let's try to visualise this concept. The name divergence is well chosen, for $\nabla \cdot \boldsymbol{v}$ is a measure of how much $\boldsymbol{v}$ spreads out[7] from the initial point.

[7] i.e., diverges

For example, a vector function in **Fig.** 7.4 has a large (positive) divergence,[8] two functions in **Fig.** 7.4

[8] if the arrows pointed in, it would be a negative divergence

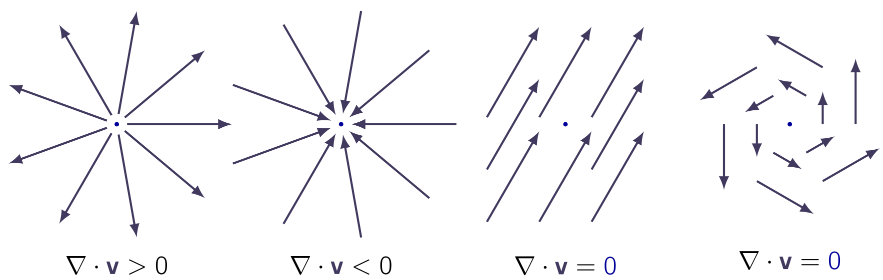


Figure 7.4: Visual description of the divergence operation.

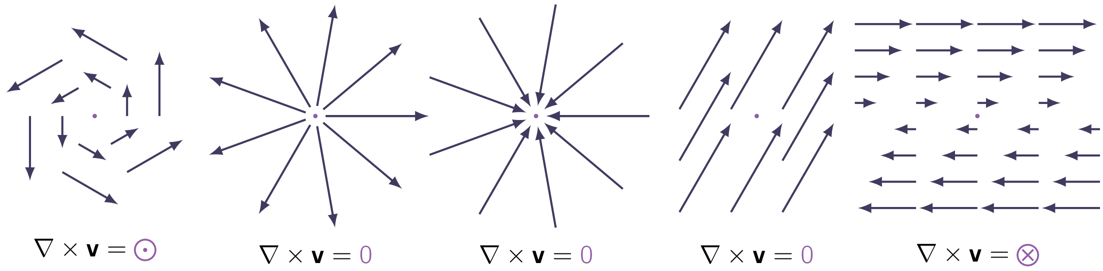


Figure 7.5: Different behaviours of the curl operation.

has zero divergence, and one function has a negative divergence.

As an example, imagine standing at the edge of a pond. Sprinkle some sawdust on the surface.

1. If the material spreads out, then we dropped it at a point of positive divergence;
2. If it collects together, we dropped it at a point of negative divergence.

Curl From the definition of ∇ we construct the curl:

$$\mathbf{V} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$
$$= \hat{\mathbf{x}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right).$$

As with divergence, the name curl is also well chosen, for $\nabla \times \boldsymbol{v}$ is a measure of how much $\boldsymbol{v}$ swirls around the point in question. Therefore the functions in Fig. 1.18 all have zero curl, whereas the functions in **Fig.** 7.5 all have a substantial curl sans one, pointing in the $\hat{z}$ direction, as the natural right-hand rule would suggest.

To finish these two (2) important operations, let's again imagine we are standing at the edge of a pond. Float a small flower, if it starts to rotate, then we placed it at a point of nonzero curl. A whirlpool would be a region of large curl.

Exercise 7.3 An Example of a Curl

Find he curl ($\nabla \times$) of the following functions.

$$\boldsymbol{v} = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z}, \qquad \boldsymbol{v} = \left(y^n\right)\ \hat{x} + \left(z^n\right)\ \hat{y} + \left(x^n\right)\ \hat{z},$$
$$\boldsymbol{v} = (\sin y)\ \hat{x} + (\cos z)\ \hat{y} + (-\tan x)\ \hat{z}, \qquad \boldsymbol{v} = \left(x^2 - z\right)\ \hat{x} + \left(xe^z\right)\ \hat{y} + (xy)\ \hat{z}.$$

SOLUTION The curl ($\nabla \times$) of the functions are as follows:

$$f(x, y, z) = (y)\ \hat{x} + \left(2x^2\right)\ \hat{y} + (0)\ \hat{z},$$
$$\nabla \times f = (0)\ \hat{x} + (0)\ \hat{y} + (-1 + 4x)\ \hat{z}.$$

$$
\begin{aligned}
f(x, y, z) &= \left(y^n\right) \hat{x} + \left(z^n\right) \hat{y} + \left(x^n\right) \hat{z}, \\
\nabla \times f &= \left(-nz^{n-1}\right) \hat{x} + \left(-nx^{n-1}\right) \hat{y} + \left(-ny^{n-1}\right) \hat{z}. \\
f(x, y, z) &= (\sin y)\, \hat{x} + (\cos z)\, \hat{y} + (-\tan x)\, \hat{z}, \\
\nabla \times f &= (\sin z)\, \hat{x} + \left(\sec^2 x\right) \hat{y} + (-\cos y)\, \hat{z}. \\
f(x, y, z) &= \left(x^2 - z\right) \hat{x} + \left(xe^{z}\right) \hat{y} + (xy)\, \hat{z}, \\
\nabla \times f &= \left(x - e^{z}x\right) \hat{x} + (-1 - y)\, \hat{y} + \left(e^{z}\right) \hat{z}.
\end{aligned}
$$

7.2.4 Product Rules

The calculation of ordinary derivatives is facilitated by a number of rules which are as follows:

Table 7.1: Product rules of ordinary derivatives

Sum Rule	$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
Constant Multiplication	$\frac{d}{dx}(kf) = k\frac{df}{dx}$
Product Rule	$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$
Quotient Rule	$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

Similar relations hold for the vector derivatives. Thus,

$$
\begin{aligned}
\nabla(f+g) &= \nabla f + \nabla g, \\
\nabla \cdot (\boldsymbol{A} + \boldsymbol{B}) &= (\nabla \cdot \boldsymbol{A}) + (\nabla \cdot \boldsymbol{B}), \\
\nabla \times (\boldsymbol{A} + \boldsymbol{B}) &= (\nabla \times \boldsymbol{A}) + (\nabla \times \boldsymbol{B}),
\end{aligned}
$$

and

$$
\nabla(kf) = k\nabla f, \qquad \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}), \qquad \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}), \tag{7.9}
$$

The product rules, on the other hand, are not quite so simple. There are two (2) ways to construct a scalar as the product of two functions:

$$
\begin{aligned}
fg &\quad \text{(product of two scalar functions)}, \\
\mathbf{A} \cdot \mathbf{B} &\quad \text{(dot product of two vector functions)},
\end{aligned}
$$

and two ways to make a vector:

$$
\begin{aligned}
f\mathbf{A} &\quad \text{(scalar times vector)}, \\
\mathbf{A} \times \mathbf{B} &\quad \text{(cross product of two vectors)}.
\end{aligned}
$$

Accordingly, there are six (6) product rules, which are given in **Tbl.** 7.2.

Table 7.2: Product rules of vector operations

Gradient I	$\nabla(fg) = f\nabla g + g\nabla f$
Gradient II	$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$
Divergence I	$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f)$
Divergence II	$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
Curl I	$\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f)$,
Curl II	$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$

The proofs for rules given in **Tbl.** 7.2 come straight from the product rule for ordinary derivatives. As an example:

$$\begin{aligned}
\nabla \cdot (f\boldsymbol{A}) &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\
&= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right) \\
&= (\nabla f) \cdot \boldsymbol{A} + f(\nabla \cdot \boldsymbol{A}).
\end{aligned}$$

It is also possible to formulate three quotient rules:

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \frac{g\nabla f - f\nabla g}{g^2}, \\
\nabla \cdot \left(\frac{\boldsymbol{A}}{g}\right) &= \frac{g(\nabla \cdot \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla g)}{g^2}, \\
\nabla \times \left(\frac{\boldsymbol{A}}{g}\right) &= \frac{g(\nabla \times \boldsymbol{A}) + \boldsymbol{A} \times (\nabla g)}{g^2}.
\end{aligned}$$

However, given these can be obtained quickly from the previously mentioned product rules, there is no point in listing them separately and is left for the reader as exercise.

7.2.5 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\nabla \cdot \boldsymbol{v}$ by applying ∇ twice, we can construct five (5) types of 2nd derivatives.

The gradient ∇T is a vector, so we can take the divergence and curl of it:

1. Divergence of gradient: $\nabla \cdot (\nabla T)$.
2. Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \boldsymbol{v}$ is a scalar, therefore all we can do is take its gradient:

3. Gradient of divergence: $\nabla(\nabla \cdot \boldsymbol{v})$.

The curl $\nabla \times \boldsymbol{v}$ is a vector, so we can take its divergence and curl:

4. Divergence of curl: $\nabla \cdot (\nabla \times \boldsymbol{v})$.
5. Curl of curl: $\nabla \times (\nabla \times \boldsymbol{v})$.

This exhausts the possible combinations, and in fact **NOT** all of them give anything new. Let's consider them one at a time:

Divergence of a Gradient

$$\begin{aligned}\nabla \cdot (\nabla T) &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.\end{aligned}$$

This object, which we write as $\nabla^2 T$ for short, is called the Laplacian[9] of T, which will be our focus later.

> The Laplacian of a scalar T is a scalar value.

Occasionally, we will use the Laplacian of a vector, $\nabla^2 \boldsymbol{v}$. By this we mean a **vector** quantity whose x-component is the Laplacian of v_x, and so on.

$$\nabla^2 \boldsymbol{v} \equiv \left(\nabla^2 v_x\right)\hat{x} + \left(\nabla^2 v_y\right)\hat{y} + \left(\nabla^2 v_z\right)\hat{z}$$

This is nothing more than a convenient extension of the meaning of ∇^2.

Exercise 7.4 The Laplacian of a Vector

Calculate the Laplacian of the following functions:

$$(i) \quad T_a = x_2 + 3xy + 3z + 4, \qquad (ii) \quad T_b = \sin x \sin y \sin z,$$
$$(iii) \quad T_c = e^{-5x}\sin 4y \cos 3z, \qquad (iv) \quad \boldsymbol{v} = \left(x^2\right)\hat{x} + \left(3xz^2\right)\hat{y} + (-2xz)\hat{z}.$$

SOLUTION The solution to the Laplacian of the functions are as follows:

(i) $\dfrac{\partial^2 T_a}{\partial x^2} = 2;\ \dfrac{\partial^2 T_a}{\partial y^2} = 0;\ \dfrac{\partial^2 T_a}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 T_a = 2 \ \blacksquare$

(ii) $\dfrac{\partial^2 T_b}{\partial x^2} = \dfrac{\partial^2 T_b}{\partial y^2} = \dfrac{\partial^2 T_b}{\partial z^2} = -3T_b \quad \rightarrow \quad \nabla^2 T_b = -3T_b = 3\sin x \sin y \sin z \ \blacksquare$

(iii) $\dfrac{\partial^2 T_c}{\partial x^2} = 25T_c;$

$\dfrac{\partial^2 T_c}{\partial y^2} = -16T_c;\quad \dfrac{\partial^2 T_c}{\partial z^2} = -9T_c \quad \rightarrow \quad \nabla^2 T_c = 0 \ \blacksquare$

(iii) $\dfrac{\partial^2 v_x}{\partial x^2} = 2;\ \dfrac{\partial^2 v_x}{\partial y^2} = 0;\ \dfrac{\partial^2 v_x}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_x = 0,$

[9] A differential operator given by the divergence of the gradient of a scalar function on Euclidean space. It is usually denoted by the symbols $\nabla \cdot \nabla$, ∇^2 or Δ. In a Cartesian coordinate system, the Laplacian is given by the sum of second partial derivatives of the function with respect to each independent variable.

The Laplace operator is named after the French mathematician *Pierre-Simon de Laplace (1749-1827)*, who first applied the operator to the study of celestial mechanics: the Laplacian of the gravitational potential due to a given mass density distribution is a constant multiple of that density distribution. Solutions of Laplace's equation $\Delta f = 0$ are called harmonic functions and represent the possible gravitational potentials in regions of vacuum.

The Laplacian occurs in many differential equations describing physical phenomena. Poisson's equation describes electric and gravitational potentials; the diffusion equation describes heat and fluid flow; the wave equation describes wave propagation; and the Schrödinger equation describes the wave function in quantum mechanics. In image processing and computer vision, the Laplacian operator has been used for various tasks, such as blob and edge detection.

$$\frac{\partial^2 v_y}{\partial x^2} = 0; \frac{\partial^2 v_y}{\partial y^2} = 0; \frac{\partial^2 v_y}{\partial z^2} = 6 \quad \rightarrow \quad \nabla^2 v_y = 6x,$$
$$\frac{\partial^2 v_z}{\partial x^2} = 0; \frac{\partial^2 v_z}{\partial y^2} = 0; \frac{\partial^2 v_z}{\partial z^2} = 0 \quad \rightarrow \quad \nabla^2 v_z = 0,$$
$$\nabla^2 \boldsymbol{v} = 2\,\hat{x} + 6x\,\hat{y} \quad \blacksquare$$

Curl of a Gradient

The curl of a gradient is **always** zero:

$$\nabla \times (\nabla T)$$

This is an **important fact**, which will be used repeatedly. Without going into too much detail into the proof, it relies on the following relation:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$

Gradient of Divergence

This operation rarely occurs in physical applications, and it has not been given any special name of its own. Notice that $\nabla(\nabla \cdot \boldsymbol{v})$ is **NOT** the same as the Laplacian of a vector:

$$\nabla^2 = (\nabla \cdot \nabla) \neq \nabla(\nabla \cdot \boldsymbol{v})$$

Divergence of a Curl

Similar to the curl of a gradient, it is always zero:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0.$$

Curl of a Curl

As we can check from the definition of ∇:

$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}. \tag{7.10}$$

So curl-of-curl gives nothing new as the first term is just the divergence of a curl, and the second is the Laplacian. To put it short, then, there are just two kinds of second derivatives:[10]

1. Laplacian,
2. Gradient of divergence.

[10] It is possible to work out 3rd derivatives, but fortunately second derivatives suffice for practically all physical applications.

7.2.6 Line, Surface, and Volume Integrals

In electrodynamics, we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**, which will be the focus of this section.

Line Integrals Has an expression of the form:

$$\int_a^b \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

where $\boldsymbol{v}$ is a vector function, $d\boldsymbol{l}$ is the infinitesimal displacement vector, and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point a to point b. If the path forms a closed loop,[11]

[11] i.e., if $\boldsymbol{b} = \boldsymbol{a}$.

We put a circle on the integral sign:

$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

At each point on the path, we take the dot product of $\boldsymbol{v}$ (evaluated at that point) with the displacement $d\boldsymbol{l}$ to the next point on the path.

A good example of a line integral is the work done by a force $\boldsymbol{F}$:

$$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}$$

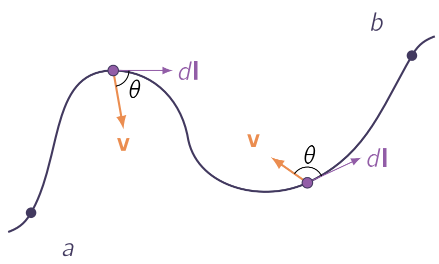


Figure 7.6: The method in which line integral is calculated. At each point the dot product of the vector ($\boldsymbol{v}$) is taken with the length vector ($\mathrm{d}\boldsymbol{l}$) which is always tangential to the point in which the integration is taken.

Ordinarily, the value of a line integral depends critically on the path taken from $\boldsymbol{a}$ to $\boldsymbol{b}$, but there is an important special class of vector functions for which the line integral is independent of path and is determined entirely by the end points.

It will be our business in due course to characterise this special class of vectors.

A force which has this property is called **conservative**.

Exercise 7.5 Fluid Flow

A fluid's velocity field is $\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z}$. Find the flow along the helix $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (t)\ \hat{z}$ with a range of $0 \leq t \leq \pi/2$.

SOLUTION We first evaluate $\boldsymbol{F}$ on the curve:

$$\boldsymbol{F} = (x)\ \hat{x} + (z)\ \hat{y} + (y)\ \hat{z} = (\cos t)\ \hat{x} + (t)\ \hat{y} + (\sin t)\ \hat{z} \qquad \text{Substitute } x = \cos t,\ z = t,\ y = \sin t.$$

and then find $d\boldsymbol{\ell}/dt$:

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then we integrate $\boldsymbol{F} \cdot \left(d\boldsymbol{\ell}/dt\right)$ from $t = 0$ to $t = \pi/2$:

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = (\cos t)\,(-\sin t) + (t)\,(\cos t) + (\sin t)\,(1)\,,$$

$$= -\sin t \cos t + t \cos t + \sin t.$$

Which makes,

$$\text{Flow} = \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\, dt = \int_0^{\pi/2} \left(-\sin t \cos t + t \cos t + \sin t\right)\, dt,$$

$$= \left[\frac{\cos^2 t}{2} + t \sin t\right]\Bigg|_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2} \quad \blacksquare$$

Exercise 7.6 Field Circulation

Find the circulation of the field $\boldsymbol{F} = (x - y)\ \hat{x} + x\,\hat{y}$ around the circle $\boldsymbol{\ell}(t) = (\cos t)\ \hat{x} + (\sin t)\ \hat{y} + (0)\ \hat{z}$ with a range of $0 \leq t \leq 2\pi$.

SOLUTION On the circle, $\boldsymbol{F} = (x - y)\ \hat{x} + (x)\ \hat{y} + (0)\ \hat{z} = (\cos t - \sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}$ and

$$\frac{d\boldsymbol{\ell}}{dt} = (-\sin t)\ \hat{x} + (\cos t)\ \hat{y} + (0)\ \hat{z}.$$

Then

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1},$$

Gives.

$$\text{Circulation} = \int_0^{2\pi} \boldsymbol{F} \cdot \frac{d\boldsymbol{\ell}}{dt}\, dt = \int_0^{2\pi} (1 - \sin t \cos t)\ dt$$

$$= \left[t - \frac{\sin^2 t}{2}\Bigg|_0^{2\pi}\right] = 2\pi \quad \blacksquare$$

Surface Integrals A surface integral is an expression of the form:

$$\int_S \boldsymbol{v} \cdot d\boldsymbol{a}$$

where $\boldsymbol{v}$ is a vector function, and the integral is over a specified surface $\mathcal{S}$. Here $d\boldsymbol{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface. There are, two (2) directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous.

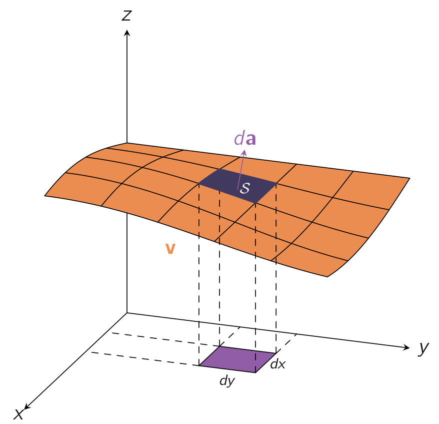


Figure 7.7: An visual description of the surface integral.

If the surface is closed,[12] we put a circle on the integral sign:

$$\oint \boldsymbol{v} \cdot d\boldsymbol{a}$$

[12] imagine it forming a balloon.

> Tradition dictates that **outward** is positive, but for open surfaces it's arbitrary.

As an example, if $\boldsymbol{v}$ describes the flow of a fluid,[13] then $\int \boldsymbol{v} \cdot d\boldsymbol{a}$ represents the total mass per unit time passing through the surface.

[13] Measured in mass per unit area per unit time.

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface and is determined entirely by the boundary line. An important task will be to characterise this special class of functions.

Volume Integrals A volume integral is an expression of the form:

$$\int_{\mathcal{V}} T \, d\tau$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx \, dy \, dz$$

As an example, if T is the density of a substance,[14] then the volume integral would give the total mass.

[14] This might vary from point to point.

Occasionally we shall encounter volume integrals of vector functions:

$$\int \mathbf{v} \, d\tau = \int \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right) d\tau = \hat{x} \int v_x \, d\tau + \hat{y} \int v_y \, d\tau + \hat{z} \int v_z \, d\tau.$$

As the unit vectors ($\hat{x}$, $\hat{y}$, $\hat{z}$) are constants, they can be taken outside the integral.

Exercise 7.7 **Double Integrals**

Find the following double integrals:

$$\int_0^1 \int_x^{2x} (x+y)^2 \, dy \, dx, \qquad \int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx \, dy,$$

$$\int_0^3 \int_x^3 \cosh(x+y) \, dy \, dx, \qquad \int_0^1 \int_0^{y^3} \exp y^4 \, dx \, dy.$$

SOLUTION

The solution to integrations are as follows:

$$\int_0^1 \int_x^{2x} (x+y)^2 \, dy \, dx = \int_0^1 \int_x^{2x} x^2 + 2xy + y^2 \, dy \, dx, = \int_0^1 \left[yx^2 + xy^2 + \frac{y^3}{3} \right] \Bigg|_x^{2x} dx,$$

$$= \int_0^1 \left(4x^3 + \frac{7x^3}{3} \right) dx, = \left[4x^3 + \frac{7x^4}{12} \right] \Bigg|_0^1 = \frac{19}{12} \quad \blacksquare$$

$$\int_0^1 \int_y^{\sqrt{y}} (1-2xy) \, dx \, dy = \int_0^1 \left[x - x^2 y \right] \Big|_y^{\sqrt{y}} dy,$$

$$= \int_0^1 \left[\left(\sqrt{y} - y^2 \right) - \left(y - y^3 \right) \right] dy = \int_0^1 \left[y^3 + \sqrt{y} - y^2 - y \right] dy,$$

$$= \left[\frac{y^4}{4} + \frac{2}{3} y^{3/2} - \frac{y^3}{3} - \frac{y^2}{2} \right] \Bigg|_0^1, = \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} \right) - (0) = \frac{1}{12} \quad \blacksquare$$

$$\int_0^3 \int_x^3 \cosh(x+y) \, dy \, dx = \int_0^1 \left[\sinh(x+y) \right] \Big|_x^3 dx = \int_0^1 \left[\sinh(3+x) - \sinh(2x) \right] dx$$

7.2.7 The Fundamental Theorems of Vector Calculus

Assume $f(x)$ is a function of one (1) variable. Based on this, the fundamental theorem of calculus says:

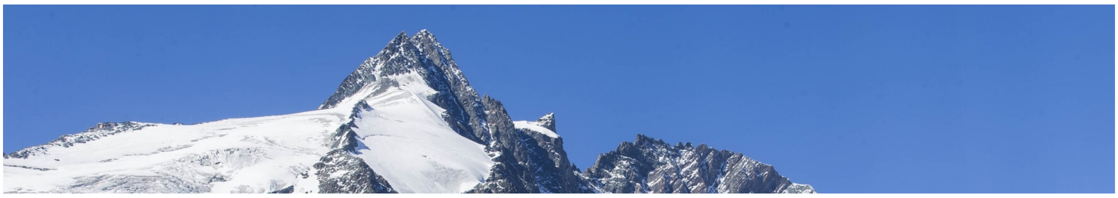


Figure 7.8: To measure the height of a mountain, it doesn't matter what way we take, as long as we know the base and the top, we will know the height.

Theory 7.24: Calculus Theorem

The **integral** of a **derivative** over some **region** is given by the **value of the function** at the end points (**boundaries**)

$$\int_a^b \left(\frac{df}{dx}\right) \, dx = f(x) - f(a) \qquad \text{or} \qquad \int_a^b F(x) \, dx = f(x) - f(a)$$

In vector calculus there are three (3) species of derivative[15] and each has its own "fundamental theorem" with essentially the same format. Our purpose here is to not prove these theorems here, but rather, understand what they mean.

[15] These are gradient, divergence, and curl

The Fundamental Theorem for Gradients Suppose we have a scalar function of three (3) variables $T(x, y, z)$. Starting at point **a**, move a small distance $\mathrm{d}\boldsymbol{l}_1$. The function T will change by an amount:

$$\mathrm{d}T = (\nabla T) \cdot \mathrm{d}\boldsymbol{l}_1$$

Now let's move an additional small displacement $\mathrm{d}\boldsymbol{l}_2$. The incremental change in T will be now:

$$\mathrm{d}T = (\nabla T) \cdot \mathrm{d}\boldsymbol{l}_2$$

In this manner, proceeding by infinitesimal steps, we make the journey to point $\boldsymbol{b}$. At each step we compute the gradient of T, at that point, and dot it into the displacement $\mathrm{d}\boldsymbol{l}$...this gives us the change in T.

Theory 7.25: Gradient Theorem

The total change in T in going from $\boldsymbol{a}$ to $\boldsymbol{b}$ (along the path selected) is:

$$\int_a^b (\nabla T) \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

Similar to "ordinary" fundamental theorem, this theorem says the integral[16] of a derivative, which here the gradient, is given by the value of the function at the boundaries which are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

[16] In this case it is a line integral

As an example, assume we want to measure the height of GrossGlockner. We could climb the mountain from base, or take the high alpine road, or take a helicopter ride all the way up to top. Regardless or the options we take, we should get the same answer either way.[17]

[17] This is the essence of the fundamental theorem

Theory 7.26: Line Independence of Gradient

Gradients have the special property that their line integrals are path independent:

- $\int_a^b (\nabla T) \cdot d\boldsymbol{l}$ is independent of the path taken from $\boldsymbol{a}$ to $\boldsymbol{b}$.
- $\oint (\nabla T) \cdot d\boldsymbol{l} = 0$, since the beginning and end points are identical, and hence $T(\boldsymbol{b}) - T(\boldsymbol{a}) = 0$.

The Fundamental Theorem for Divergences

This theorem has at least three (3) special names:

1. Gauss's theorem,
2. Green's theorem,
3. Divergence theorem.

The fundamental theorem for divergences states:

Theory 7.27: Divergence Theorem

the **integral** of a **derivative** (the **divergence**) over a **region** (in this case the **volume**, $\mathcal{V}$) is equal to the value of the function at the **boundary** (in this case the **surface** $\mathcal{S}$ bounding the volume).

$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{v}) \, \mathrm{d}\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}.$$

The boundary term is itself an integral, more specifically, a surface integral. This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a closed surface. To create an analogy, if $\boldsymbol{v}$ represents the flow of an incompressible fluid, then the flux $\boldsymbol{v}$ is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the *spreading out* of the vectors from a point, a place of high divergence is like a tap, pouring out liquid. If we have a bunch of tap in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In fact, there are two (2) ways we could determine how much is being produced:

i. we could count up all the faucets, recording how much each puts out

ii. we could go around the boundary, measuring the flow at each point, and add it all up

We get the same answer either way:

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Exercise 7.8 An Example of Divergence Theorem

Evaluate both sides of the Divergence theorem for the expanding vector field $\boldsymbol{F} = (x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}$ over the sphere $x^2 + y^2 + z^2 = a^2$

SOLUTION The outer unit normal to S, calculated from the gradient of $f\{\cdot\}\ x,\ y,\ z = x^2 + y^2 + z^2 - a^2$, is:

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{a}. \qquad \mathrm{x^2 + y^2 + z^2 = a^2 \text{ on S}}$$

Therefore:

$$(\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \frac{x^2 + y^2 + z^2}{a}\ \mathrm{d}a = \frac{a^2}{a}\ \mathrm{d}a = a\,\mathrm{d}a.$$

This in turn gives us:

$$\iint_S (\boldsymbol{F} \cdot \hat{n})\ \mathrm{d}a = \iint_S a\,\mathrm{d}a = a \iint_S \mathrm{d}a = a\left(4\pi a^2\right) = 4\pi a^3. \qquad \text{Area of S is 4ßa}^2$$

The divergence of $\boldsymbol{F}$ is:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

So,

$$\iiint_V (\nabla \cdot \boldsymbol{v})\ d\tau = \iiint_V 3\,d\tau = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3 \quad \blacksquare$$

Exercise 7.9 Divergence Theorem of an Octant of a Sphere

Check the divergence theorem for the function:

$$\boldsymbol{v} = \left(r^2 \cos\theta\right)\ \hat{r} + \left(r^2 \cos\phi\right)\ \hat{\theta} + \left(-r^2 \cos\theta \sin\phi\right)\ \hat{\phi}.$$

using as your volume one octant of the sphere of radius R.

SOLUTION It is always useful to write the theorem we are going to work on:

$$\underbrace{\iiint_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\ dV}_{\text{Divergence integral}} = \underbrace{\iint_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{n}\,da.}_{\text{Outward flux}}$$

First solve the left hand side of the equation:

$$\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta r^2\cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right), \\
&= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right), \\
&= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\int (\nabla \cdot \boldsymbol{v})\ d\tau &= \int (4r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\int_0^R r^3\, dr \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta \int_0^{\pi/2} d\phi, \\
&= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \blacksquare
\end{aligned}$$

Now it is time to solve the right hand side of the question. As we are aware from the shape, an octant of the sphere has 4 sides to it: the curved surface $xyz \to \boldsymbol{a}_1$, and $xz \to \boldsymbol{a}_2$, $yz \to \boldsymbol{a}_3$ and $xy \to \boldsymbol{a}_4$. These are

$$\begin{aligned}
d\boldsymbol{a}_1 &= \hat{r}\, dl_\theta\, dl_\phi = \hat{r}\, R^2 \sin\theta\, d\phi\, d\theta, & d\boldsymbol{a}_2 &= dl_r\, dl_\theta = -\hat{\phi}\, r\, dr\, d\theta, \\
d\boldsymbol{a}_3 &= \hat{\phi}\, dl_r\, dl_\theta = \hat{\phi}\, r\, dr\, d\theta, & d\boldsymbol{a}_4 &= dl_r\, dl_\phi = \hat{\theta}\, r\, dr\, d\theta. \quad \left(\theta = \pi/2\right)
\end{aligned}$$

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot d\boldsymbol{a} &= \iint_{S_1} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_2} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_3} \boldsymbol{v} \cdot d\boldsymbol{a} + \iint_{S_4} \boldsymbol{v} \cdot d\boldsymbol{a}, \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \left[r^2\cos\theta\ \hat{r} + r^2\cos\phi\ \hat{\theta} - r^2\cos\theta\sin\phi\ \hat{\phi}\right]\Bigg|_{r=R} \cdot \left(\hat{r}\, R^2 \sin\theta\, d\phi\, d\theta\right) \\
&+ \int_0^{\pi/2}\int_0^{R} \left[r^2\cos\theta\ \hat{r} + r^2\cos\phi\ \hat{\theta} - r^2\cos\theta\sin\phi\ \hat{\phi}\right]\Bigg|_{\phi=0} \cdot \left(-\hat{\phi}\, r\, dr\, d\theta\right)
\end{aligned}$$

$$+\int_0^{\pi/2}\int_0^R \left[r^2\cos\theta\ \hat{r} + r^2\cos\phi\ \hat{\theta} - r^2\cos\theta\sin\phi\ \hat{\phi}\right]\Big|_{\phi=\pi/2} \cdot \left(\hat{\phi}\, r\, dr\, d\theta\right)$$

$$+\int_0^{\pi/2}\int_0^R \left[r^2\cos\theta\ \hat{r} + r^2\cos\phi\ \hat{\theta} - r^2\cos\theta\sin\phi\ \hat{\phi}\right]\Big|_{\theta=\pi/2} \cdot \left(\hat{\theta}\, r\, dr\, d\theta\right),$$

Time to do some integration.

$$\iint_S \mathbf{v}\cdot d\mathbf{a} = \int_0^{\pi/2}\int_0^{\pi/2} \left[R^2\cos\theta\,\hat{r} + R^2\cos\phi\,\hat{\theta} - R^2\cos\theta\sin\phi\,\hat{\phi}\right] \cdot \left(\hat{r}\, R^2\sin\theta\, d\phi\, d\theta\right)$$

$$+\int_0^{\pi/2}\int_0^R \left[r^2\cos\theta\ \hat{r} + r^2(1)\ \hat{\theta} - (0)\sin\phi\ \hat{\phi}\right] \cdot \left(-\hat{\phi}\, r\, dr\, d\theta\right)$$

$$+\int_0^{\pi/2}\int_0^R \left[r^2\cos\theta\ \hat{r} + (0)\phi\ \hat{\theta} - r^2\cos\theta(1)\ \hat{\phi}\right] \cdot \left(\hat{\phi}\, r\, dr\, d\theta\right)$$

$$+\int_0^{\pi/2}\int_0^R \left[(0)\ \hat{r} + r^2\cos\phi\ \hat{\theta} - (0)\ \hat{\phi}\right] \cdot \left(\hat{\theta}\, r\, dr\, d\theta\right).$$

Final touches and cleaning up,

$$\iint_S \mathbf{v}\cdot d\mathbf{a} = \int_0^{\pi/2}\int_0^{\pi/2} R^4\sin\theta\cos\theta\, d\phi\, d\theta + 0 - \overbrace{\int_0^{\pi/2}\int_0^R r^3\cos\theta\, dr\, d\theta + \int_0^{\pi/2}\int_0^R r^3\cos\theta\, dr\, d\phi}^{=0},$$

$$= R^4\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right),$$

$$= R^4\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right),$$

$$= \frac{\pi R^4}{4} \quad \blacksquare$$

[18]Sir George Gabriel Stokes, 1st Baronet, *(1819 - 1903)*

was an Irish mathematician and physicist. Born in County Sligo, Ireland, Stokes spent his entire career at the University of Cambridge, where he served as the Lucasian Professor of Mathematics for 54 years, from 1849 until his death in 1903, the longest tenure held by the Lucasian Professor. As a physicist, Stokes made seminal contributions to fluid mechanics, including the Navier-Stokes equations; and to physical optics, with notable works on polarisation and fluorescence. As a mathematician, he popularised "Stokes' theorem" in vector calculus and contributed to the theory of asymptotic expansions. Stokes, along with Felix Hoppe-Seyler, first demonstrated the oxygen transport function of haemoglobin, and showed colour changes produced by the aeration of haemoglobin solutions.

7.2.8 The Fundamental Theorem for Curls

The fundamental theorem for curls, also known as **Stokes' theorem**, states:[18]

Theory 7.28: Stokes' Theorem

Integral of a **derivative** over a **region** ($\mathcal{S}$) is equal to the value of the function at **boundary** ($\mathcal{P}$).

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}.$$

Similar to the divergence theorem, the boundary term is itself an integral. Specifically, a *closed line integral*.

Remember the curl measures the *twist* of the vectors $\mathbf{v}$. Think of a region of high curl as a whirlpool, where if we put a wheel there, it will rotate. Now, the integral of the curl over some surface (or, more precisely, the *flux* of the curl through the surface) represents the *total amount of swirl*, and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$\oint \mathbf{v}\cdot d\mathbf{l}$ is sometimes called the **circulation** of $\mathbf{v}$.

There seems to be an ambiguity in Stokes' theorem: concerning the boundary line integral:

Which way are we supposed to go around?[19]

[19] clockwise or counterclockwise.

The answer is that it doesn't matter which way we go **as long as we are consistent**, for there is an additional sign ambiguity in the surface integral:

Which way does d$\boldsymbol{a}$ point?

For a closed surface,[20] d$\boldsymbol{a}$ points in the direction of the outward normal. But for an open surface, which way would be defined as out? Consistency in Stokes' theorem is given by the right-hand rule. If our rings point in the direction of the line integral, then our thumb fixes the direction of d$\boldsymbol{a}$.

[20] i.e., the divergence theorem.

Ordinary, a flux integral depends critically on what surface we integrate over, but this is **NOT** the case with curls. For Stokes' theorem says that $\int(\nabla \times \boldsymbol{v})\cdot d\boldsymbol{a}$ is equal to the line integral of $\boldsymbol{v}$ around the boundary, and the latter makes no reference to the specific surface we choose.

Preposition I $\int(\nabla \times \boldsymbol{v})\cdot d\boldsymbol{a}$ depends only on the boundary line, not on the particular surface used.

Preposition II $\oint(\nabla \times \boldsymbol{v})\cdot d\boldsymbol{a} = 0$ for any closed surface.

Exercise 7.10 Surface Area of an Implicit Surface

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

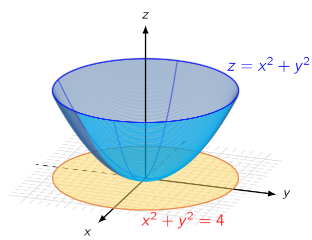


Figure 7.9: The paraboloid of "Surface Area of an Implicit Surface".

SOLUTION We sketch the surface S and the region R below it in the xy-plane (**Fig.** 7.9). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, ans R is the disk $x^2 + y^2 \leq 4$ in the xy-plane.

To get a unit vector normal (i.e., $\hat{n}$) to the plane R, we can take $\hat{n} = \hat{z}$. At any point (x, y, z) on the surface, we have:

$$F(x, y, z) = x^2 + y^2 - z$$
$$\nabla F = (2x)\,\hat{x} + (2y)\,\hat{y} + (-1)\,\hat{z}$$

$$\begin{aligned}|\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \hat{n}| &= |\nabla F \cdot \hat{z}| = |-1| = 1.\end{aligned}$$

In the region R, the area is defined to be $dA = dx\,dy$. Therefore:

$$\begin{aligned}\text{Surface Area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{n}|}\, dA \\ &= \iint_{x^2+y^2\leq 4} \sqrt{4x^2+4y^2+1}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{12}\left(4r^2+1\right)^{3/2}\Big|_0^2 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{12}\left(17^{3/2}-1\right)\, d\theta \\ &= \frac{\pi}{6}\left(17\sqrt{17}-1\right) \quad \blacksquare\end{aligned}$$

Exercise 7.11 Stokes' Theorem Over a Hemisphere

Evaluate Stokes's theorem for the hemisphere $S : x^2 + y^2 + z^2 = 9,\ z \geq 0$, its bounding circle $C : x^2 + y^2 = 9, z = 0$ and the field $\boldsymbol{F} = (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z}$.

Tip: Parametrisation of a circle is: $x = r\cos\theta,\ y = r\sin\theta$ and $da = \frac{3}{z} dA$)

SOLUTION The start by calculating the counter-clockwise circulation around C using the following parametrisation:

$$\boldsymbol{\ell}(\theta) = (3\cos\theta)\ \hat{x} + (3\sin\theta)\ \hat{y} + (0)\ \hat{z}, \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

Using this we can calculate the counter-clockwise circulation.

$$\begin{aligned}
d\boldsymbol{\ell} &= (-3\sin\theta\, d\theta)\ \hat{x} + (3\cos\theta\, d\theta)\ \hat{y} + (0)\ \hat{z}, \\
\boldsymbol{F} &= (y)\ \hat{x} + (-x)\ \hat{y} + (0)\ \hat{z} \\
&= (3\sin\theta)\ \hat{x} + (-3\cos\theta)\ \hat{y} + (0)\ \hat{z}, \\
\boldsymbol{F}\cdot d\boldsymbol{\ell} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta, \\
\oint_C \boldsymbol{F}\cdot d\boldsymbol{\ell} &= \int_0^{2\pi} -9\, d\theta = -18\pi.
\end{aligned}$$

For the curl of integral we have:

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (0-0)\ \hat{x} + (0-0)\ \hat{y} + (-1-1)\ \hat{z} = -2\,\hat{z}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{(x)\ \hat{x} + (y)\ \hat{y} + (z)\ \hat{z}}{3} \qquad \text{Unit normal}$$

Now it is time to define the area of integration (da):

$$\begin{aligned}
da &= \frac{|\nabla S|}{|\nabla S\cdot\hat{z}|}\, dA \\
&= \frac{\left|(2x)\ \hat{x} + (2y)\ \hat{y} + (2z)\ \hat{z}\right|}{2z} \\
&= \frac{2\overbrace{\sqrt{x^2+y^2+z^2}}^{a}}{2z} \\
&= \frac{3}{z} dA, \\
\nabla\times\boldsymbol{F}\cdot\boldsymbol{n}\, da &= -\frac{2z}{3}\frac{3}{z}\, dA = -2\, dA
\end{aligned}$$

The cardinal direction $\hat{z}$ comes from being the direction **perpendicular** to the surface (S).

$$\iint_S \nabla\times\boldsymbol{F}\cdot\boldsymbol{n}\, da = \iint_{x^2+y^2\leq 9} -2\, dA = -18\pi$$

The circulation around the circle equals the integral of the curl over the hemisphere ∎

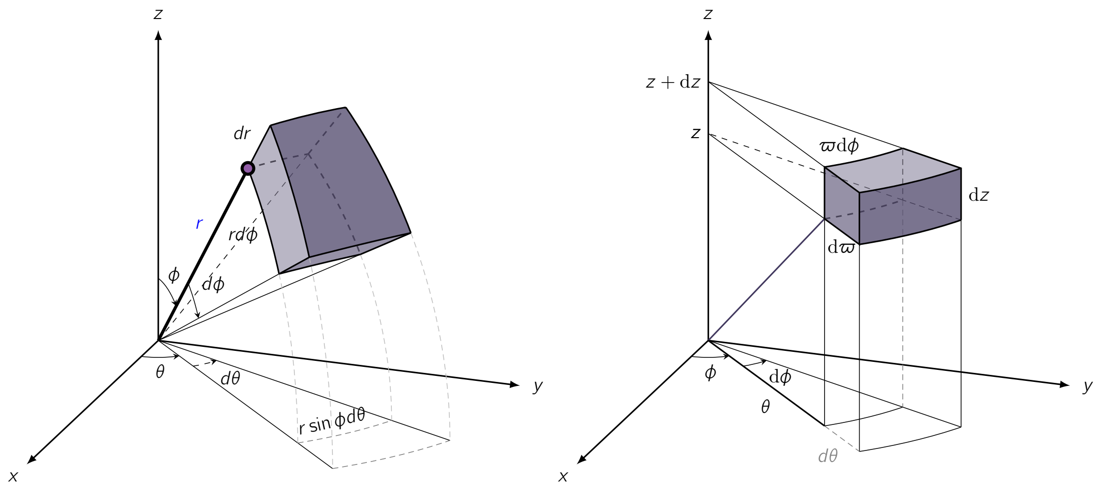


Figure 7.10: The two types of coordinate systems in question. (a) Spherical coordinate system (b) Spherical coordinate system.

7.3 Curvilinear Coordinates

7.3.1 Spherical Coordinate System

It is possible to label a point P in Cartesian coordinates (x, y, z), but sometimes it is more convenient to use **spherical** coordinates (r, θ, ϕ); r is the distance from the origin, θ is called the **polar angle**, and ϕ is the **azimuthal angle**. Their relation to Cartesian coordinates can be read from **Fig.** 7.10a.

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Fig. 7.10a also shows three unit vectors, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, pointing in the direction of increase of the corresponding coordinates.

They constitute an **orthogonal** basis set,[21] similar to $\hat{x}$, $\hat{y}$, $\hat{z}$, and any vector $\boldsymbol{A}$ can be expressed in terms of them, in the usual way:

[21] meaning mutually perpendicular.

$$\boldsymbol{A} = (A_r)\ \hat{r} + (A_\theta)\ \hat{\theta} + (A_\phi)\ \hat{\phi}$$

Here, A_r, A_θ, A_ϕ are the radial, polar, and azimuthal components of vector $\boldsymbol{A}$. In terms of the Cartesian unit vectors:

$$\begin{aligned}
\hat{r} &= (\sin\theta\cos\phi)\ \hat{r} + (\sin\theta\sin\phi)\ \hat{\theta} + (\cos\theta)\ \hat{\phi},\\
\hat{\theta} &= (\cos\theta\cos\phi)\ \hat{r} + (\cos\theta\sin\phi)\ \hat{\theta} + (-\sin\theta)\ \hat{\phi},\\
\hat{\phi} &= (-\sin\phi)\ \hat{r} + (\cos\phi)\ \hat{\theta} + (0)\ \hat{\phi}.
\end{aligned}$$

An infinitesimal displacement in the $\hat{r}$ direction is simply $\mathrm{d}r$, just as an infinitesimal element of length in the $\hat{x}$ direction is $\mathrm{d}x$:

$$\mathrm{d}l_r = \mathrm{d}r$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction is **NOT** just $d\theta$ but rather,

$$dl_\theta = r\, d\theta$$

Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction is

$$dl_\phi = r \sin\theta\, d\phi$$

Thus the general infinitesimal displacement $d\boldsymbol{l}$ is:

$$d\boldsymbol{l} = (dr)\, \hat{r} + (r\, d\theta)\, \hat{\theta} + (r \sin\theta)\, \hat{\phi}$$

This plays the role $d\boldsymbol{l} = (dx)\, \hat{x} + (dy)\, \hat{y} + (dz)\, \hat{z}$ plays in Cartesian coordinates. The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three (3) infinitesimal displacements:

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

> It is not possible to give a general expression for surface elements $d\boldsymbol{a}$, since these depend on the orientation of the surface. We simply have to analyse the geometry for any given case, which goes for Cartesian and curvilinear coordinates.

Integrating over the surface of a sphere, for instance, makes r constant, whereas θ and ϕ change:

$$d\boldsymbol{a}_1 = dl_\theta\, dl_\phi\, \hat{r} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

On the other hand, if the surface lies in the xy plane, making θ is constant, while r and ϕ vary:

$$d\boldsymbol{a}_2 = dl_r\, dl_\phi\, \hat{\theta} = r\, dr\, d\phi\, \hat{\theta}$$

Finally: r ranges from 0 to ∞, ϕ from 0 to 2π, and θ from 0 to π.

Exercise 7.12 The Volume of a Sphere

Find the volume of a sphere of radius R.

SOLUTION The derivation is as follows:

$$V = \int d\tau$$

$$\int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta\, dr\, d\theta\, d\phi,$$

$$= \left(\int_0^R r^2\, dr\right) \left(\int_0^\pi \sin\theta\, d\theta\right) \left(\int_0^{2\pi} d\phi\right)$$

$$= \left(\frac{R^3}{3}\right) (2)\, (2\pi) = \frac{4}{3}\pi R^3 \quad \blacksquare$$

7.3.2 Cylindrical Coordinates

The cylindrical coordinates (s, ϕ, z) of a point P are defined in **Fig.** 7.10b. Observe that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is:

$$x = s \cos\phi \qquad y = s \sin\phi \qquad z = z.$$

The unit vectors are:

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

The infinitesimal displacements are

$$dl_s = ds \qquad dl_\phi = s\,d\phi, \qquad dl_z = dz$$

which makes:

$$d\boldsymbol{l} = ds\,\hat{s} + s\,d\phi\,\boldsymbol{p\hat{h}i} + dz\,\hat{z}.$$

and the volume element is

$$d\tau\prime = s\,ds\,d\phi\,dz$$

The range of s is $(0, \infty)$, ϕ is from 0 to $2pi$ and z is from $-\infty$ to $+\infty$

7.4 Dirac Delta Function

7.4.1 A Mathematical Anomaly

Consider the following vector function:

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

At every location, $\mathbf{v}$ is directed radially outward which we can see in **Fig.** 7.11. Let's calculate its divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

This is interesting as this calculation gives us an unforeseen solution. Let's look at it closer. Suppose we integrate over a sphere of radius R, centred at the origin. The surface integral therefore is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot \left(R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi\, \hat{\mathbf{r}}\right) = \left(\int_0^{\pi} \sin\theta \,\mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = 4\pi$$

But the volume integral ($\int \nabla \cdot \mathbf{v}\,\mathrm{d}\tau$), should be zero (0) if we assume the aforementioned calculation to be true.

Does this mean that the divergence theorem is false?

The source of the problem lies at the point $r = 0$, where $\mathbf{v}$ **blows up**. It is true that $\nabla \cdot \mathbf{v} = 0$ everywhere except the origin, but right at the origin is the situation gets a little bit complicated.

Observe, the surface integral is independent of R. If the divergence theorem is to be true, we should expect

$$\int \nabla \cdot \mathbf{v}\,\mathrm{d}\tau = 4\pi,$$

for any non-zero vector and the origin.

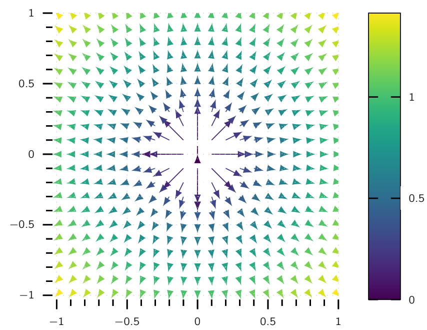


Figure 7.11: The vector plot of the "divergence problem".

This means the value of 4π must be coming from the point $r = 0$. Therefore, $\nabla \cdot \mathbf{v}$ has the unique property that it vanishes everywhere except at one point, and yet its integral is 4π.

No normal function behaves like that.

Information: An Analogy with Density

To wrap our heads around this property think of **density**.

The density of a point particle is zero except at the exact location of the particle, and yet its **integral** is finite. Namely, the mass of the particle.

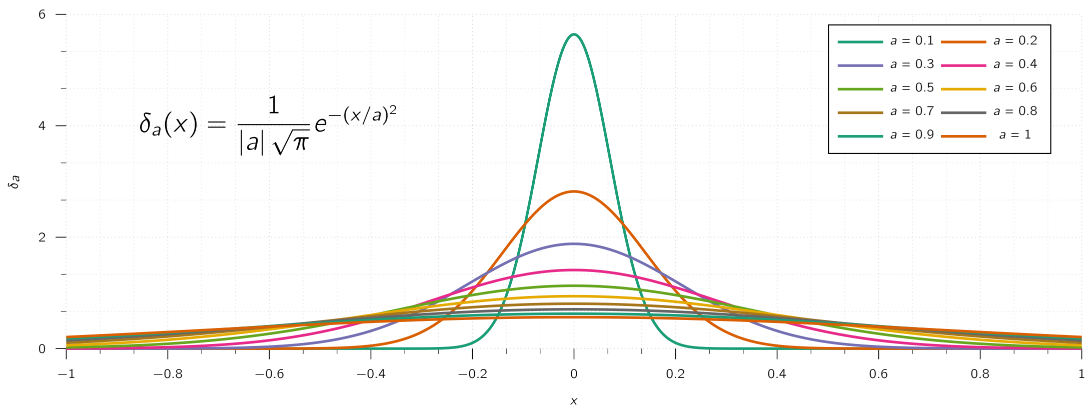


Figure 7.12: A visual representation of a 1D Dirac Delta Function. Think of it as a distribution function being squeezed to an infinitely small width.

What we have stumbled upon is called the **Dirac delta function**. It arises in numerous branches of theoretical physics and plays a central role in the theory of electrodynamics.

7.4.2 The 1D Dirac Delta Function

The one-dimensional Dirac delta function ($\delta(x)$), can be pictured as an infinitely high, infinitesimally narrow "spike", with area 1.

That is to say:

$$\delta(x) = \begin{cases} 0, & \text{if} \quad x \neq 0 \\ \infty, & \text{if} \quad x = 0 \end{cases}$$

and,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

In a strict sense of definition, $\delta(x)$ is **NOT** a function at all, as its value is not finite at $x = 0$. In literature it is known as a generalised function.[22]

If $f(x)$ is some "ordinary" function, then the product ($f(x)\delta(x)$) is zero everywhere except at $x = 0$. It follows that:

$$f(x)\delta(x) = f(0)\delta(x). \tag{7.11}$$

> The product is zero anyway except at $x = 0$. Based on this property, we may as well replace $f(x)$ by the value it assumes at the origin.

Particularly

[22] Objects extending the notion of functions on real or complex numbers. There is more than one recognised theory, for example the theory of distributions. Generalised functions are especially useful for treating discontinuous functions more like smooth functions, and describing discrete physical phenomena such as point charges.

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0). \tag{7.12}$$

Under an integral, then, the delta function "picks out" the value of $f(x)$ at $x = 0$.[23] Of course, we can shift the spike from $x = 0$ to some other point, $x = a$:

$$\delta(x-a) = \begin{cases} 0, & \text{if} \quad x \neq a \\ \infty, & \text{if} \quad x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty}\delta(x-a)\,dx = 1. \tag{7.13}$$

which turns Eq. (7.11) into:

$$f(x)\delta(x-a) = f(a)\delta(x-a), \tag{7.14}$$

and Eq. (7.12) generalises to:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a). \tag{7.15}$$

While δ itself is not a proper function, integrals over δ are perfectly acceptable. In fact, it's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions. say $D_1(x)$ and $D_2(x)$, are considered equal if:

$$\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx, \tag{7.16}$$

for all **ordinary** functions $f(x)$.

[23] Here and below, the integral need not run from $-\infty$ to $+\infty$; it is sufficient that the domain extend across the delta function, and $-\epsilon$ to $+\epsilon$ would do as well.

Exercise 7.13 A Simple Dirac Integral

Evaluate the following integral:

$$\int_0^3 x^3\delta(x-2)\,dx$$

SOLUTION The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Exercise 7.14 1D Dirac Delta Function

Evaluate the following integrals with Dirac delta functions:

$$\int_2^6 \left(3x^2 - 2x - 1\right)\delta(x-3)\,dx, \tag{xvii}$$

$$\int_0^5 \cos x\,\delta(x-\pi)\,dx, \tag{xviii}$$

$$\int_0^3 x^3\,\delta(x+1)\,dx, \tag{xix}$$

$$\int_{-\infty}^{+\infty} \ln(x+3)\,\delta(x+2)\,dx. \tag{xx}$$

SOLUTION The solution are as follows:

(a) $3\left(3^2\right) - 2(3) - 1 = 27 - 6 - 1 = 20$ ■

(b) $\cos\pi = -1$ ■

(c) 0 ■

(d) $\ln(-2+3) = \ln 1 = 0$ ■

7.4.3 The 3D Dirac Delta Function

Once we have defined the 1D Dirac, it is simple to generalise it to 3D with the following:

$$\delta^3(\boldsymbol{r}) = \delta(x)\,\delta(y)\,\delta(z)\,, \tag{7.21}$$

As we can see, it similar to 1D, where 3D Dirac is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\boldsymbol{r})d\tau = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1$$

And, the general form is:

$$\int_{\text{all space}} f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}\tau = f(\boldsymbol{a}). \tag{7.22}$$

As in the 1D case, integration with δ picks out the value of the function f at the location of the spike.

We can fix the paradox introduced in Section 7.4. Remember, the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, however, its integral over any volume containing the origin is a constant.

These are precisely the defining conditions for the Dirac delta function; evidently

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r})$$

Or in a more general fashion:

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}) \tag{7.23}$$

Differentiation here is with respect to $\boldsymbol{r}$, while $\boldsymbol{r}'$ is held constant.

7.5 Vector Field Theory

7.5.1 Helmholtz Theorem

Electricity and magnetism are generally expressed as electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ and like many physical laws, these are most compactly expressed as **differential equations**.

As $\boldsymbol{E}$ and $\boldsymbol{B}$ are **vectors**, the differential equations naturally involve vector derivatives: *divergence* and *curl*. Maxwell reduced the entire theory to four (4) fundamental equations, specifying respectively the divergence and the curl of $\boldsymbol{E}$ and $\boldsymbol{B}$.

This formulation raises an interesting question:

> To what extent is a vector function determined by its divergence and curl?

To study this case lets assume a vector of $\boldsymbol{F}$. If the divergence of $\boldsymbol{F}$ is a specified scalar function D,

$$\nabla \cdot \boldsymbol{F} = D,$$

and the curl of $\boldsymbol{F}$ is a specified vector function $\boldsymbol{C}$,

$$\nabla \times \boldsymbol{F} = \boldsymbol{C},$$

and for consistency, we assume $\boldsymbol{C}$ to have **NO** divergence,

$$\nabla \cdot \boldsymbol{C} = 0,$$

> Remember, the divergence of a curl is **ALWAYS** zero.

Using this knowledge, is it possible to determine the function $\boldsymbol{F}$?

Without knowing more information, it is not really possible. There are many functions whose divergence and curl are both zero everywhere.

Some examples are:

$$\begin{aligned}
\boldsymbol{F} &= 0, \\
\boldsymbol{F} &= (y)\ \hat{x} + (zx)\ \hat{y} + (xy)\ \hat{z}, \\
\boldsymbol{F} &= (\sin x \cosh y)\ \hat{x} + (-\cos x \sinh y)\ \hat{y} + (.)\ \hat{z}
\end{aligned}$$

If we recall **Higher Mathematics I**, to solve a differential equation with a particular solution, we must also be supplied with appropriate **boundary conditions**.

In electrodynamics we typically require the fields go to zero at infinity. With that extra information, the **Helmholtz theorem**[24] guarantees the field is uniquely determined by its divergence and curl.

[24] **Hermann Ludwig Ferdinand von Helmholtz** *(1821 - 1894)*

was a German physicist and physician who made significant contributions in several scientific fields, particularly hydrodynamic stability. The Helmholtz Association, the largest German association of research institutions, was named in his honour.

In physics, he is known for his theories on the conservation of energy and on the electrical double layer, work in electrodynamics, chemical thermodynamics, and on a mechanical foundation of thermodynamics. Although credit is shared with Julius von Mayer, James Joule, and Daniel Bernoulli-among others-for the energy conservation principles that eventually led to the first law of thermodynamics, he is credited with the first formulation of the energy conservation principle in its maximally general form.

7.5.2 Potentials

If the curl of a vector field ($\boldsymbol{F}$) vanishes everywhere, then $\boldsymbol{F}$ can be written as the **gradient of a scalar potential** (V):

$$\nabla \times \boldsymbol{F} = 0 \iff \boldsymbol{F} = -\nabla V$$

> The minus sign is purely conventional.

That's the essential idea of the following theorem:

Theory 7.29: Zero Curl Fields

The following conditions are **equivalent**.

i. $\nabla \times \boldsymbol{F} = 0$ everywhere,

ii. $\int_a^b \boldsymbol{F} \cdot d\boldsymbol{l}$ is independent of path, for any given end points,

iii. $\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$ for any closed loop,

iv. $\boldsymbol{F}$ is the gradient of some scalar function: $\boldsymbol{F} = -\nabla V$.

The potential is **NOT** unique as any constant can be added to V, since this will not affect its gradient.

If the divergence of a vector field ($\boldsymbol{F}$) vanishes everywhere, then $\boldsymbol{F}$ can be expressed as the curl of a **vector potential** ($\boldsymbol{A}$):

$$\nabla \cdot \boldsymbol{F} = 0 \iff \boldsymbol{F} = \nabla \times \boldsymbol{A}$$

That's the main conclusion of the following theorem:

Theory 7.30: Zero Divergence Fields

The following conditions are **equivalent**:

i. $\nabla \cdot \boldsymbol{F} = 0$ everywhere.

ii. $f \boldsymbol{F} \cdot d\boldsymbol{a}$ is independent of surface, for any given boundary line.

iii. $f \boldsymbol{F} \cdot d\boldsymbol{a} = 0$ for any closed surface.

iv. $\boldsymbol{F}$ is the curl of some vector function: $\boldsymbol{F} = \nabla \times \boldsymbol{A}$.

The vector potential is **NOT** unique as the gradient of any scalar function can be added to $\boldsymbol{A}$ without affecting the curl, given the curl of a gradient is zero.

Incidentally, in all cases, a vector field $\boldsymbol{F}$ can be written as the gradient of a scalar plus the curl of a vector.

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}$$

Chapter Bibliography

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2. Hamilton, W. R. On the Argument of Abel. *Transactions of the Royal Irish Academy* **18,** 171–259 (1839).

Acronyms

A

AC An electric current that periodically reverses direction and changes its magnitude continuously with time, in contrast to direct current (DC), which flows only in one direction.

I

IVP Initial Value Problem

L

LHS Left Hand Side

O

ODE Ordinary Differential Equation

P

PDE Partial Differential Equation

R

RHS Right Hand Side