Lecture Book
M.Sc Higher Mathematics I

Daniel T. McGuiness, PhD

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The current maintainer of this work along with the primary lecturer
is D. T. McGuiness, PhD (dtm@mci4me.at).

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First-Order Ordinary Differential Equations

1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a *mathematical expression* in terms of: variables, functions, equations. Such an expression is known as a mathematical **model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**.

Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,
- find new values,
- interpret result in a physical terms.

This is all done to understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter. An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually $y(x)$. The equation may also contain y itself, known functions of x, and constants. For example all the equation shown below are classified as ODE.

$$\begin{aligned} y' &= \cos x \\ y'' + 9y &= e^{-2x} \\ y'y''' - \tfrac{3}{2}y'^2 &= 0. \end{aligned}$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from partial differential equations (PDEs), which involve **partial** derivatives of an unknown function of two or more variables.

> The topic of Partial Differential Equation (PDE) will be the focus of **Higher Mathematics II**.

For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

In this part of the chapter, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F(x, y, y') = 0 \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

1.1.1 Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

> Geometrically, this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE, together with an initial condition, is called an **initial value problem**.

Initial Value Problem In multivariable calculus, an initial value problem (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Example Initial Value Problem - A 1

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

Solution Initial Value Problem - B

The general solution is:

$$y(x) = ce^{3x}$$

From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

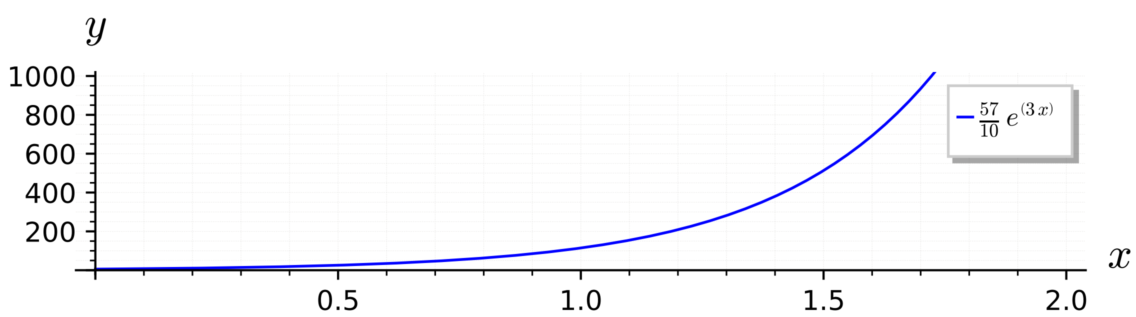

Figure 1.1.: Solution to the exercise "*Initial Value Problem -A*"

Example Radioactive Decay 2

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

> The decay of Radium is measured to be $k = 1.4 \cdot 10^{-11}\,\text{sec}^{-1}$.

Solution Radioactive Decay

Setting Up a Mathematical Model

$y(t)$ is the amount of substance still present at t. By the physical law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives is the following:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is positive, so that, because of the minus, we get *decay*. The value of k is known from experiments for various radioactive substances which the question has given as $k = 1.4 \cdot 10^{-11}\,\text{sec}^{-1}$. Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

We have the initial condition $y(0) = 0.5$. This is the instant at which our observation of the process begins. In motivates the original condition which however, is also used when the independent variable is not time or when we choose a t other than $t = 0$. Hence the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5.$$

Mathematical Solution

We conclude the ODE is an exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is:

$$y(t) = 0.5e^{-kt} \quad \blacksquare$$

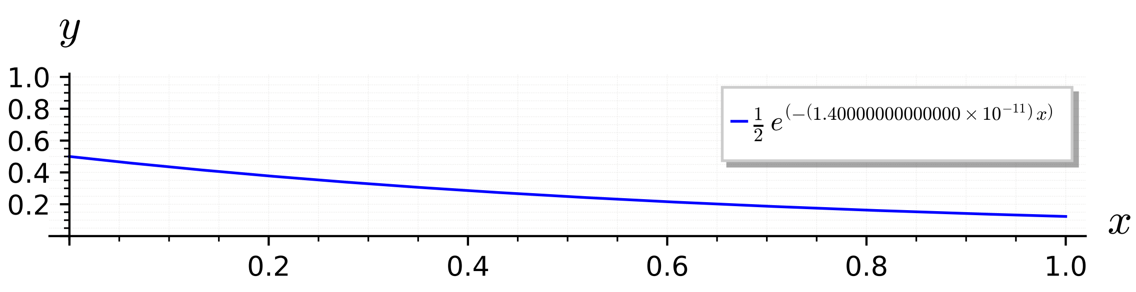

Figure 1.2.: Solution to the exercise "*Radioactive Decay*". The x-scale is 1e11.

1.2 Separable ODEs

Many practically useful ODEs can be reduced to the following form:

$$g(y)\,y' = f(x) \tag{1.4}$$

using *algebraic manipulations*. We can then integrate on both sides with respect to x, obtaining the following expression:

$$\int g(y)\,y'\,dx = \int f(x)\,dx + c. \tag{1.5}$$

On the left we can switch to y as the variable of integration. By calculus, we know the relation $y'\,dx = dy$, so that:

$$\int g(y)\,dy = \int f(x)\,dx + c. \tag{1.6}$$

If f and g are **continuous functions**, the integrals in Eq. (1.6) exist, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

Example Separable ODE 3

Solve the following ODE:

$$y' = 1 + y^2$$

Solution Separable ODE

The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

It is important to introduce the constant c when the integration is performed.

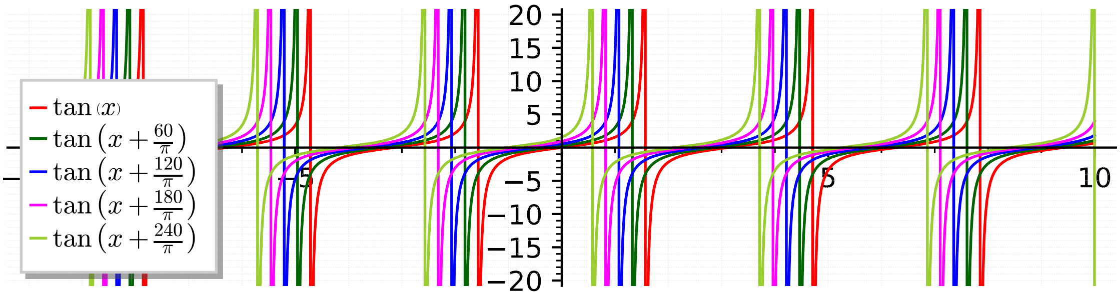

Figure 1.3.: Results with different c values.

Example IVP: Bell-Shaped Curve 4

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

Solution IVP: Bell-Shaped Curve

By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve. The plot of the solution is given in Figure 1.4.

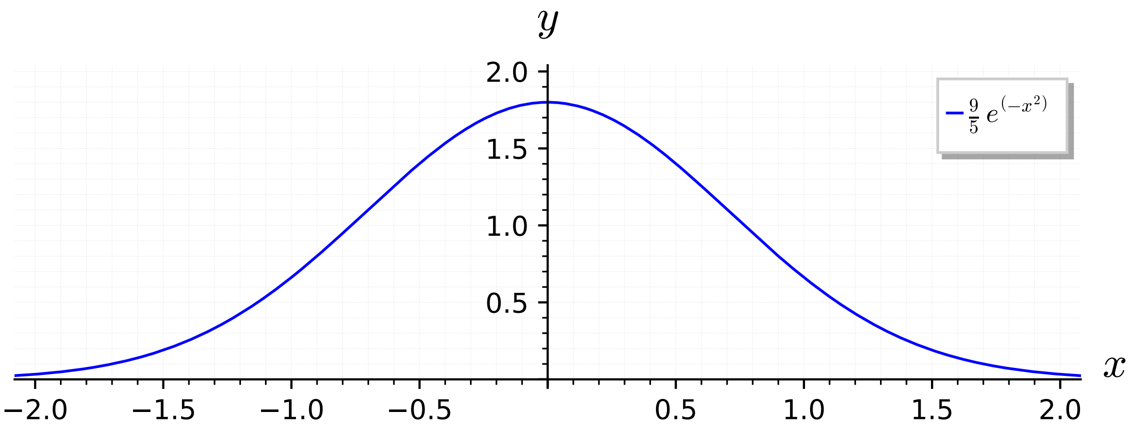

Figure 1.4.: Solution plot for the exercise: *IVP: Bell-Shaped Curve*.

Example Radiocarbon Dating 5

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian–Italian border, caused a scientific sensation. When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

> The half-life of carbon is 5175 years.

Solution **Radiocarbon Dating**

Radioactive decay is governed by the ODE $y' = ky$ as we have developed previously. By separation and integration:

Figure 1.5.: Ötzi was found in the Öetztal Alps in Southern Tirol near the Austrian–Italian border

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0\,e^{kt} \qquad (y_0 = e^0).$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Ötzi died,

$$e^{k\tau} = e^{-0.0001\,213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001\,213} = 5312. \qquad \blacksquare$$

Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right).$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests that we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Example **Reduction to Separable Form** 6

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

Solution **Reduction to Separable Form**

To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis, which can be seen in Figure 1.6.

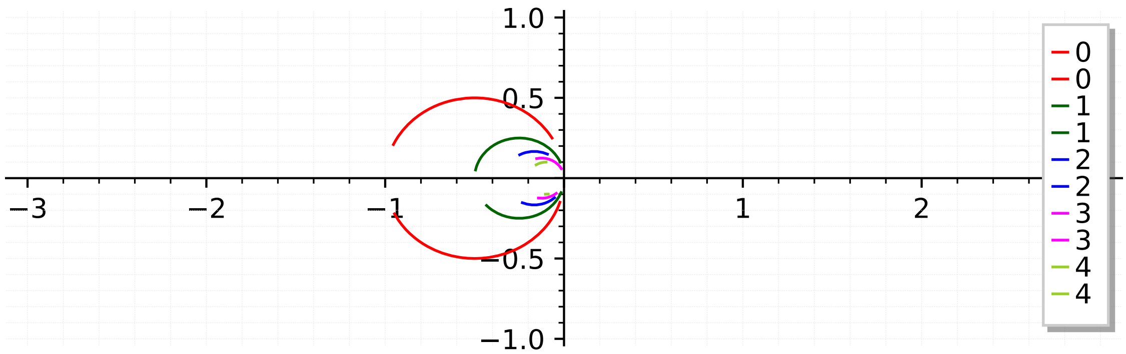

Figure 1.6.: Solution to *Reduction to Separable form* which is a family of solutions.

1.3 Exact ODEs

1.3.1 Integrating Factors

Recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (i.e., **total differential**) is:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c = \text{const}$, then $du = 0$. As an example, let's have a look at the function $u = x + x^2y^3 = c$. Finding its factors:

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows. A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as:

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.7}$$

is called an **exact differential equation** if the differential form $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{1.8}$$

of some function $u(x, y)$. Then Eq. (1.7) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.7) in the form:

$$u(x, y) = c \tag{1.9}$$

Comparing Eq. (1.7) and Eq. (1.8), we see that Eq. (1.7) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N \tag{1.10}$$

From this we can derive a formula for checking whether Eq. (1.7) is exact or not, as follows. Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.10),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x},$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.11}$$

> This condition is not only necessary but also sufficient for Eq. (1.7) to be an exact differential equation.

If Eq. (1.7) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way.
From (4a) we have by integration with respect to x:

$$u = \int M\, dx + k(y), \tag{1.12}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.12), use (4b) to get dk/dy, and integrate dk/dy to get k.
Formula Eq. (1.12) was obtained from (4a).

> It is valid to use **either** of them and arrive at the same result.

Then, instead of (6), we first have by integration with respect to y

$$u = \int N\, dy + l(x).$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Example **Initial Value Problem** 7

Solve the initial value problem

$$(\cos y \sinh x + 1)\, dx - \sin y \cosh x\, dy = 0, \qquad y(1) = 2.$$

Solution **Initial Value Problem**

Verify that the given ODE is **exact**. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x \, dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = u = \cos y \sinh x + 1$. Therefore $dl/dx = 1$ by integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$.
Therefore the answer is $\cos y \cosh x + x = 0.358$.

Example An Exact ODE 8

Solve

$$\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))\, dy = 0.$$

Solution An Exact ODE

Step 1. Test for exactness. Our equation is of the form (1) with

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

Example An Exact ODE 9

Solve the following ODE:

$$\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))\, dy = 0. \tag{7}$$

Step 1 - Test for exactness

First check if our equation is **exact**, try to convert the equation of the form Eq. (1.7):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x+y),$$
$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

This proves our equation to be exact.

Step 2 - Implicit General Solution

From Eq. (1.12), we obtain by integration:

$$u = \int M\,dx + k(y) = \int \cos(x+y)\,dx + k(y) = \sin(x+y) + k(y) \tag{1.13}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Therefore $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into Eq. (1.13) and observing Eq. (1.9), we obtain:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c \quad \blacksquare$$

Example Breakdown of Exactness 10

Check the exactness of the following ODE:

$$-y\,dx + x\,dy = 0$$

Solution Breakdown of Exactness

The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\partial M/\partial y = -1 \qquad \partial N/\partial x = 1$$

Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + ky = -xy + ky, \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$).

1.4 Linear ODEs

1.4.1 Introduction

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A first-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)y = r(x)$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

Homogeneous Linear ODE

We want to solve in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) = 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-Ip(x)dx} \qquad (c = \pm e^{c\pi} \quad \text{when} \quad y \geq 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution**$y(x) = 0$ for all x in that interval.

Non-Homogeneous Linear ODE

We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called nonhomogeneous. It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the previous section or we can proceed directly, as follows. We multiply (1) by $F(x)$, obtaining

$$Fy' + pFy = rF.$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h'e^h y = e^h y' + (e^h)'y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r\,dx + c.$$

Dividing by e^h, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx.$$

This reduces solving (1) to the generally simpler task of evaluating integrals. For ODEs for which this is still difficult, you may have to use a numeric method for integrals from Sec. 19.5 or for the ODE itself from Sec. 21.1. We mention that h has nothing to do with $h(x)$ in Sec. 1.1 and that the constant of integration in h does not matter; see Prob. 2.
The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

$$y(x) = e^{-h}\int e^h r\,dx + ce^{-h},$$

Example First-Order ODE, General Solution Initial Value Problem 11

Solve the ODE Initial Value Problem

Solution First-Order ODE, General Solution Initial Value Problem

Solve the initial value problem

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.Here $p = \tan x$, $r = \sin 2x = 2\,\sin x \cos x$, and

$$h = \int p\,dx = \int \tan x\,dx = \ln\,|\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2\,\sin x \cos x) = 2\,\sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\,dx \,+\, c \right) = c\,\cos x \,-\, 2\cos^2 x.$$

From this and the initial condition, $1 = c \,-\, 1 \,-\, 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3\,\cos x - 2\,\cos^2 x$. Here $3\,\cos x$ is the response to the initial data, and $-2\,\cos^2 x$ is the response to the input $\sin 2x$.

Chapter 2.

Appendix

2.1 List of Common Integration Operations

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int u dv = uv - \int v du$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b|$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2}\ln|a^2+x^2|$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln|a+x|$$

$$\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

Integrals with Roots

$$\int \sqrt{x-a} dx = \frac{2}{3}(x-a)^{3/2}$$

$$\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a}$$

$$\int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x}$$

$$\int x\sqrt{x-a} dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$$

$$\int \sqrt{ax+b} dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$$

$$\int (ax+b)^{3/2} dx = \frac{2}{5a}(ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3}(x \mp 2a)\sqrt{x \pm a}$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a}$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln \left[\sqrt{x} + \sqrt{x+a}\right]$$

$$\int x\sqrt{ax+b} dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b}$$

$$\int \sqrt{x(ax+b)} dx = \frac{1}{4a^{3/2}} \left[(2ax+b)\sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right]$$

$$\int \sqrt{x^3(ax+b)} dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right|$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \sqrt{ax^2 + bx + c} dx = \frac{b + 2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx^+ c)} \right|$$

$$\int x\sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \left(2\sqrt{a}\sqrt{ax^2 + bx + c} \times \left(-3b^2 + 2abx + 8a(c + ax^2)\right) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| \right)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right|$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right|$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a}\right) \ln(ax+b) - x, a \neq 0$$

$$\int \ln(x^2+a^2)\ \mathrm{dx} = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x$$

$$\int \ln(x^2-a^2)\ \mathrm{dx} = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x$$

$$\int \ln \left(ax^2+bx+c\right) dx = \frac{1}{a}\sqrt{4ac-b^2} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x\right) \ln \left(ax^2+bx+c\right)$$

$$\int x \ln(ax+b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax+b)$$

$$\int x \ln \left(a^2 - b^2x^2\right) dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right) \ln \left(a^2 - b^2x^2\right)$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \mathrm{erf}\left(i\sqrt{ax}\right), \quad \text{where } \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int x e^x dx = (x-1)e^x$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^x dx = \left(x^2 - 2x + 2\right) e^x$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$$

$$\int x^3 e^x dx = \left(x^3 - 3x^2 + 6x - 6\right) e^x$$

$$\int x^n e^{ax}\ \mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\ \mathrm{d}x$$

$$\int x^n e^{ax}\ \mathrm{d}x = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\ \mathrm{d}t$$

$$\int e^{ax^2}\ \mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \mathrm{erf}\left(ix\sqrt{a}\right)$$

$$\int e^{-ax^2}\ \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}} \mathrm{erf}\left(x\sqrt{a}\right)$$

$$\int x e^{-ax^2}\ \mathrm{d}x = -\frac{1}{2a} e^{-ax^2}$$

$$\int x^2 e^{-ax^2}\ \mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \mathrm{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right]$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right]$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \tan ax dx = -\frac{1}{a}\ln \cos ax$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right)$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln \cos ax + \frac{1}{2a}\sec^2 ax$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax$$

$$\int \sec^3 x \, \mathrm{dx} = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, n \neq 0$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x|$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, n \neq 0$$

$$\int \sec x \csc x dx = \ln|\tan x|$$

Products of Trigonometric Functions and Monomials

$$\int x\cos x dx = \cos x + x\sin x$$

$$\int x\cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

$$\int x^2\cos x dx = 2x\cos x + \left(x^2-2\right)\sin x$$

$$\int x^2\cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$$

$$\int x^n cos x dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix) + (-1)^n\Gamma(n+1,ix)\right]$$

$$\int x^n cos ax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax) - \Gamma(n+1,ixa)\right]$$

$$\int x\sin x dx = -x\cos x + \sin x$$

$$\int x\sin ax dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2\sin x dx = \left(2-x^2\right)\cos x + 2x\sin x$$

$$\int x^2\sin ax dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$$

$$\int x^n\sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix) - (-1)^n\Gamma(n+1,-ix)\right]$$

Products of Trigonometric Functions and Exponentials

$$\int e^x\sin x dx = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\int e^{bx}\sin ax dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$$

$$\int e^x\cos x dx = \frac{1}{2}e^x(\sin x + \cos x)$$

$$\int e^{bx}\cos ax dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$$

$$\int xe^x\sin x dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$$

$$\int xe^x\cos x dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax$$

$$\int e^{ax}\cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a\neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax$$

$$\int e^{ax}\sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a\neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases}$$

$$\int e^{ax}\tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}{}_2F_1\left[1+\dfrac{a}{2b},1,2+\dfrac{a}{2b},-e^{2bx}\right] & \\ \quad -\dfrac{1}{a}e^{ax}{}_2F_1\left[\dfrac{a}{2b},1,1E,-e^{2bx}\right] & a\neq b \\ \dfrac{e^{ax}-2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases}$$

$$\int \tanh ax\, dx = \frac{1}{a}\ln\cosh ax$$

$$\int \cos ax\cosh bx dx = \frac{1}{a^2+b^2}\left[a\sin ax\cosh bx + b\cos ax\sinh bx\right]$$

$$\int \cos ax\sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax\cosh bx + a\sin ax\sinh bx\right]$$

$$\int \sin ax\cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax\cosh bx + b\sin ax\sinh bx\right]$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cosh bx \sin ax - a \cos ax \sinh bx \right]$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} \left[-2ax + \sinh 2ax \right]$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} \left[b \cosh bx \sinh ax - a \cosh ax \sinh bx \right]$$

2.2 Common Laplace Transforms

$f(t)$	$f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) -$
	$\cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\dfrac{ae^{at} - be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at} \sin kt$	$\dfrac{k}{(s-a)^2 + k^2}$
$e^{at} \cos kt$	$\dfrac{s-a}{(s-a)^2 + k^2}$
$e^{at} \sinh kt$	$\dfrac{k}{(s-a)^2 - k^2}$
$e^{at} \cosh kt$	$\dfrac{s-a}{(s-a)^2 - k^2}$
$t \sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
$t \cos kt$	$\dfrac{s^2 - k^2}{(s^2+k^2)^2}$
$t \sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
$t \cosh kt$	$\dfrac{s^2 - k^2}{(s^2-k^2)^2}$
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$

Chapter 3.

Glossary

Glossary

ODE Ordinary Differential Equation. 1, 5, 6

PDE Partial Differential Equation. 1, 6