Lecture Book
M.Sc Higher Mathematics I

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version 2024.0.0.2

Current version is 2024.0.0.2.

This document includes the contents of Higher Mathematics I taught at MCI. This document is the part of Higher Mathematics.

All relevant code of the document is done using *SageMath* v10.3 and Python v3.12.5.

This document was compiled with LuaTEX and all editing were done using
GNU Emacs using AUCTEX and org-mode package.

This document is based on the books on the topics of ODEs, Linear Algebra and Vector Calculus, which includes *Introduction to Electrodynamics* by D. Griffiths (4th Edition), *Advanced Engineering Mathematics* by E. Kreyszig (9th Edition), *Thomas Calculus* by G. B. Thomas (12th Edition), *Linear Algebra and its Applications* by D. Lay et. al, *Calculus: A Complete Course* by R. A. Adams (5th Edition), and *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus* by H. E. Schey (4th Edition).

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Ordinary Differential Equations

Chapter 1.

First-Order Ordinary Differential Equations

1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a *mathematical expression* in terms of: variables, functions, equations. Such an expression is known as a mathematical **model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**.

Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,
- find new values,
- interpret result in a physical terms.

This is all done to understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter. An ODE is an equation containing one or several derivatives of an unknown function, usually $y(x)$. The equation may also contain y itself, known functions of x, and constants. For example all the equation shown below are classified as ODE.

$$\begin{aligned} y' &= \cos x \\ y'' + 9y &= e^{-2x} \\ y'y''' - \tfrac{3}{2}y'^2 &= 0. \end{aligned}$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from partial differential equations (PDEs), which involve **partial** derivatives of an unknown function of two or more variables.

> The topic of Partial Differential Equation (PDE) will be the focus of **Higher Mathematics II**.

For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...
In this part of the chapter, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F(x, y, y') = 0 \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

1.1.1 Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

> Geometrically, this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE, together with an initial condition, is called an **initial value problem**.

Initial Value Problem

In multivariable calculus, an initial value problem (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Example Initial Value Problem - A 1

Solve the initial value problem:

$$y\prime = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

Solution Initial Value Problem - B

The general solution is:

$$y(x) = ce^{3x}$$

From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

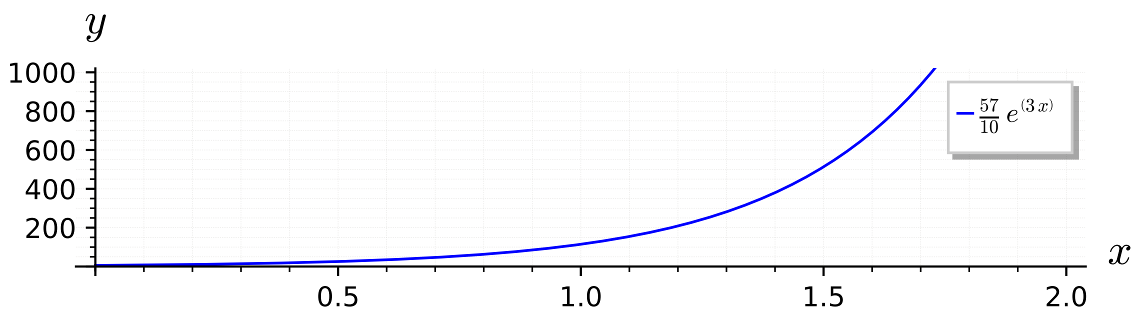

Figure 1.1.: Solution to the exercise "*Initial Value Problem -A*"

Example Radioactive Decay 2

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

The decay of Radium is measured to be $k = 1.4 \cdot 10^{-11}\,\text{sec}^{-1}$.

Solution Radioactive Decay

Setting Up a Mathematical Model

$y(t)$ is the amount of substance still present at t. By the physical law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives is the following:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is positive, so that, because of the minus, we get *decay*. The value of k is known from experiments for various radioactive substances which the question has given as $k = 1.4 \cdot 10^{-11}\,\text{sec}^{-1}$. Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

We have the initial condition $y(0) = 0.5$. This is the instant at which our observation of the process begins. In motivates the original condition which however, is also used when the independent variable is not time or when we choose a t other than $t = 0$.
Hence the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

Mathematical Solution

We conclude the ODE is an exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is:

$$y(t) = 0.5e^{-kt} \quad \blacksquare$$

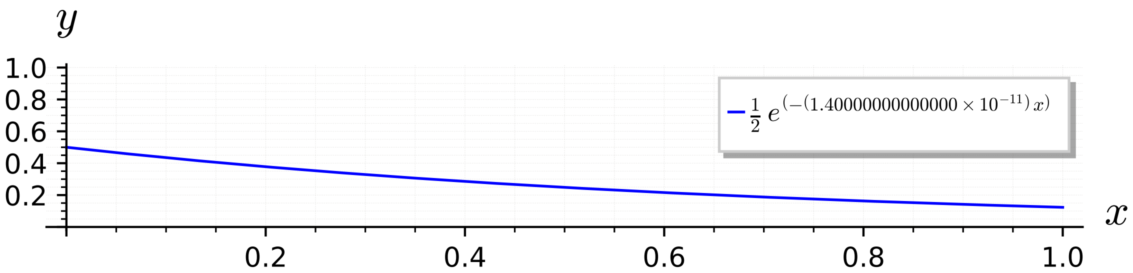

Figure 1.2.: Solution to the exercise "*Radioactive Decay*". The x-scale is 1e11.

1.2 Separable ODEs

Many practically useful ODEs can be reduced to the following form:

$$g(y)\, y' = f(x) \tag{1.4}$$

using *algebraic manipulations*. We can then integrate on both sides with respect to x, obtaining the following expression:

$$\int g(y)\, y'\, dx = \int f(x)\, dx + c. \tag{1.5}$$

On the left we can switch to y as the variable of integration. By calculus, we know the relation $y'\, dx = dy$, so that:

$$\int g(y)\, dy = \int f(x)\, dx + c. \tag{1.6}$$

If f and g are **continuous functions**, the integrals in Eq. (1.6) exist, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

Example Separable ODE 3

Solve the following ODE:

$$y' = 1 + y^2$$

Solution Separable ODE

The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c)$$

It is important to introduce the constant c when the integration is performed.

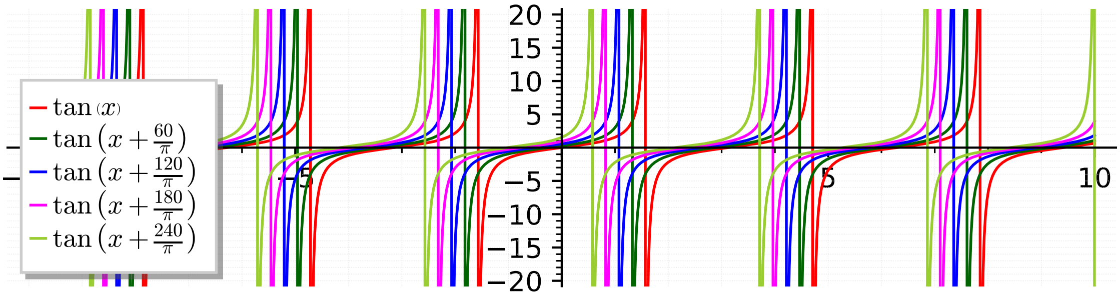

Figure 1.3.: Results with different c values.

Example IVP: Bell-Shaped Curve 4

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

Solution IVP: Bell-Shaped Curve

By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + c\,, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve. The plot of the solution is given in Figure 1.4.

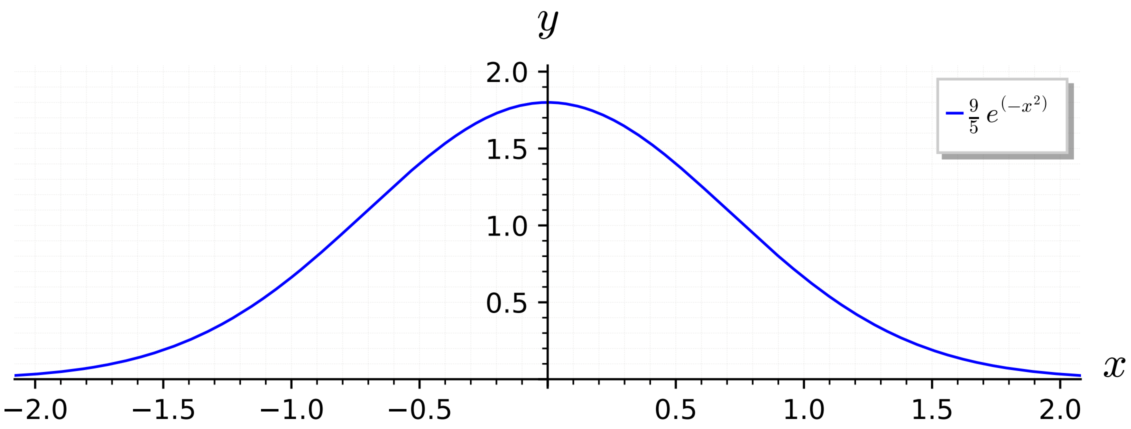

Figure 1.4.: Solution plot for the exercise: *IVP: Bell-Shaped Curve*.

Example Radiocarbon Dating 5

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian–Italian border, caused a scientific sensation. When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

> The half-life of carbon is 5175 years.

Solution Radiocarbon Dating

Radioactive decay is governed by the ODE $y' = ky$ as we have developed previously. By separation and integration:

Figure 1.5.: Ötzi was found in the Öetztal Alps in Southern Tirol near the Austrian–Italian border

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0\,e^{kt} \qquad (y_0 = e^0).$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Ötzi died,

$$e^{k\tau} = e^{-0.0001\,213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001\,213} = 5312. \qquad \blacksquare$$

Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right).$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests that we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Example **Reduction to Separable Form** 6

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

Solution **Reduction to Separable Form**

To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis, which can be seen in Figure 1.6.

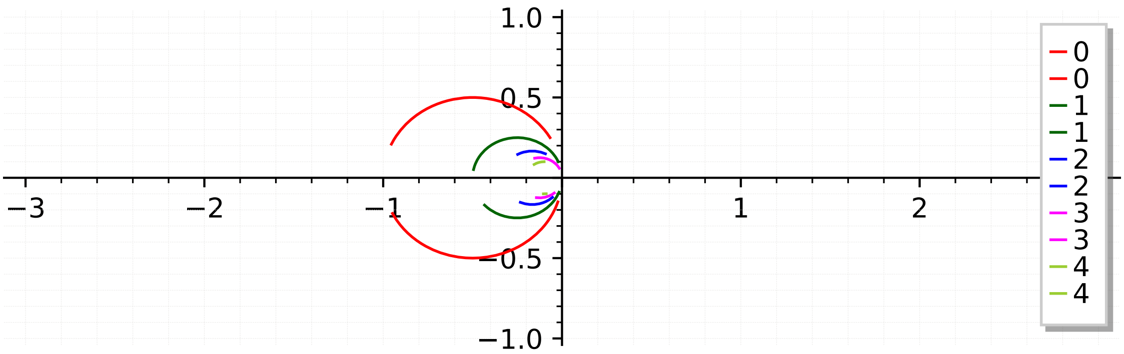

Figure 1.6.: Solution to *Reduction to Separable form* which is a family of solutions.

1.3 Exact ODEs

1.3.1 Integrating Factors

Recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (i.e., **total differential**) is:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c =$ const, then $du = 0$. As an example, let's have a look at the function $u = x + x^2y^3 = c$. Finding its factors:

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows. A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as:

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.7}$$

is called an **exact differential equation** if the differential form $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{1.8}$$

of some function $u(x, y)$. Then Eq. (1.7) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.7) in the form:

$$u(x, y) = c \tag{1.9}$$

Comparing Eq. (1.7) and Eq. (1.8), we see that Eq. (1.7) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)}\quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)}\quad \frac{\partial u}{\partial y} = N \tag{1.10}$$

From this we can derive a formula for checking whether Eq. (1.7) is exact or not, as follows.
Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.10),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.11}$$

This condition is not only necessary but also sufficient for Eq. (1.7) to be an exact differential equation.

If Eq. (1.7) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way.
From (4a) we have by integration with respect to x:

$$u = \int M\, dx + k(y), \tag{1.12}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.12), use (4b) to get dk/dy, and integrate dk/dy to get k.
Formula Eq. (1.12) was obtained from (4a).

It is valid to use **either** of them and arrive at the same result.

Then, instead of (6), we first have by integration with respect to y

$$u = \int N\, dy + l(x).$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Example Initial Value Problem 7

Solve the initial value problem

$$(\cos y \sinh x + 1)\, dx - \sin y \cosh x\, dy = 0, \qquad y(1) = 2.$$

Solution Initial Value Problem

Verify that the given ODE is **exact**. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x\, dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = u = \cos y \sinh x + 1$. Therefore $dl/dx = 1$ by integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$.
Therefore the answer is $\cos y \cosh x + x = 0.358$.

Example An Exact ODE 8

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0.$$

Solution An Exact ODE

Step 1. Test for exactness. Our equation is of the form (1) with

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

Example An Exact ODE 9

Solve the following ODE:

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{1.13}$$

Step 1 - Test for exactness

First check if our equation is **exact**, try to convert the equation of the form Eq. (1.7):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

This proves our equation to be exact.

Step 2 - Implicit General Solution

From Eq. (1.12), we obtain by integration:

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y) \tag{1.14}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Therefore $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into Eq. (1.14) and observing Eq. (1.9), we obtain:

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c \quad \blacksquare$$

Example Breakdown of Exactness 10

Check the exactness of the following ODE:

$$-y\,dx + x\,dy = 0$$

Solution Breakdown of Exactness

The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\partial M/\partial y = -1 \qquad \partial N/\partial x = 1$$

Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + ky = -xy + ky, \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$).

1.4 Linear ODEs

1.4.1 Introduction

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A first-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)y = r(x)$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

Homogeneous Linear ODE

We want to solve in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) = 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\,Ip(x)dx} \qquad (c = \pm e^{c\pi} \quad \text{when} \quad y \geq 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution**$y(x) = 0$ for all x in that interval.

Non-Homogeneous Linear ODE

We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called nonhomogeneous. It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the previous section or we can proceed directly, as follows. We multiply (1) by $F(x)$, obtaining

$$Fy' + pFy = rF.$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\ dx$. By integration, writing $h = \int p\ dx$,

$$\ln|F| = h = \int p\ dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h'e^h y = e^h y' + (e^h)'y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r \, dx + c.$$

Dividing by e^h, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^h r \, dx + c\right), \qquad h = \int p(x) \, dx.$$

This reduces solving (1) to the generally simpler task of evaluating integrals. For ODEs for which this is still difficult, you may have to use a numeric method for integrals from Sec. 19.5 or for the ODE itself from Sec. 21.1. We mention that h has nothing to do with $h(x)$ in Sec. 1.1 and that the constant of integration in h does not matter; see Prob. 2.
The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

$$y(x) = e^{-h} \int e^h r \, dx + ce^{-h},$$

Example First-Order ODE, General Solution Initial Value Problem 11

Solve the ODE Initial Value Problem

Solution First-Order ODE, General Solution Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$h = \int p \, dx = \int \tan x \, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x \, dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c - 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$.

Chapter 2.

Second-Order Ordinary Differential Equations

2.1 Introduction

A second-order ODE is called **linear**, if it can be written (in its standard form) as:

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.1}$$

- when $r(x) = 0$ it is homogeneous,
- else it is **non-homogeneous**.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs.
An example of a **non-homogeneous linear** equation is:

$$y'' = 25y - e^{-x}\cos x$$

An example of a **homogeneous linear** equation is:

$$y'' + \frac{1}{x}y' + y = 0$$

An example of **non-linear**ODE is:

$$y''y + (y')^2 = 0$$

2.1.1 Superposition Principle

For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1y_1 + c_2y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or linearity principle

Fundamental Theorem for the Homogeneous Linear ODE (2)
For a homogeneous linear ODE (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Example **Homogeneous Linear ODEs: Superposition of Solutions** 12

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0,$$

for all x.

Solution **Homogeneous Linear ODEs: Superposition of Solutions**

Verify by differentiation and substitution. We obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$\begin{aligned}&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\ &= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare\end{aligned}$$

Example **Example of a Non-homogeneous Linear ODE** 13

Verify the functions $y = 1+\cos x$ and $y = 1+\sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 0$$

Solution **Example of a Non-homogeneous Linear ODE**

Testing out the first function

$$(1+\cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$y'' + y = 1$$
$$-\sin x + 1 + \cos x \neq 1 \quad \blacksquare$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$(1+\sin x)'' = -\cos x$$
$$y'' + y = 1$$
$$-\cos x + 1 + \sin x \neq 1 \quad \blacksquare$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

For a second-order homogeneous linear ODE (2) an initial value problem consists of (2) and two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

The conditions (4) are used to determine the two arbitrary constants c1 and c2 in a general solution

Example Initial Value Problem 14

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution Initial Value Problem

Step 1: General Solution
From Example 1, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Step 2: Particular Solution
We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos$ 0 = 1 and $\sin$ 0 = 0,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

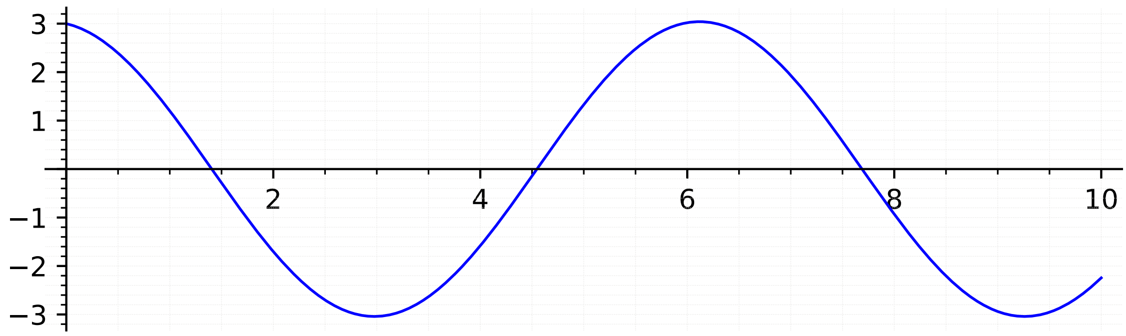

Figure 2.1.: Solution to the initial value problem exercise.

2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.
For an example please look at Exercise 2.1.6.

2.1.4 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.2}$$

Solve by starting

$$y = e^{\lambda x}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.2) gives:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic** equation (or auxiliary equation),

$$\lambda^2 + a\lambda + b = 0$$

then the exponential function (2) is a solution of the ODE (1). Now from algebra we recall the roots of the quadratic equation

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

Case	Roots of	Basis	General Solution
I	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
	(λ_1, λ_2)	$e^{\lambda_2 x}$	
II	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x) e^{-ax/2}$
	$(\lambda = -1/2a)$	$xe^{-ax/2}$	
III	Complex Conjugate	$e^{-ax/2} \cos \omega x$	$y = e^{-ax/2} (A \cos \omega x + B \sin \omega x)$
	$\lambda_1 = -1/2a + \mathbf{j}\omega$	$e^{-ax/2} \sin \omega x$	
	$\lambda_2 = -1/2a - \mathbf{j}\omega$		

Table 2.1.: Possible roots of the characteristic equation based on the discriminant value.

2.1.5 Euler-Cauchy Equations

Has the following form:

$$x^2 y'' + axy' + by = 0 \tag{2.3}$$

To solve do the following substitutions:

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Which gives:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

> $y = x^m$ is a good choice as it produces a common factor x^m.

Simplifying the equation produces the **auxiliary** equation.

$$m^2 + (a-1)m + b = 0. \tag{2.4}$$

> $y = x^m$ is a solution of Eq. (2.3) if and only if m is a root of Eq. (2.4).

The roots of Eq. (2.4) are:

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}.$$

> Complex conjugate roots are of minor practical importance for practical purposes.

Example **General Solution in the Case of Different Real Roots** 15

Solve the following ODE:

$$x^2 y'' + 1.5xy' - 0.5y = 0$$

Solution **General Solution in the Case of Different Real Roots**

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^{m} \ln x + c_2 x^{m}$
	(m)	
III	Complex Conjugate	$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$
	$m_1 = \alpha + \beta\mathbf{j}$	$\alpha = \mathrm{Re}(m)$
	$m_2 = \alpha - \beta\mathbf{j}$	$\beta = \mathrm{Im}(m)$

Table 2.2.: Possible solutions of the Euler-Cauchy based on the m value.

This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

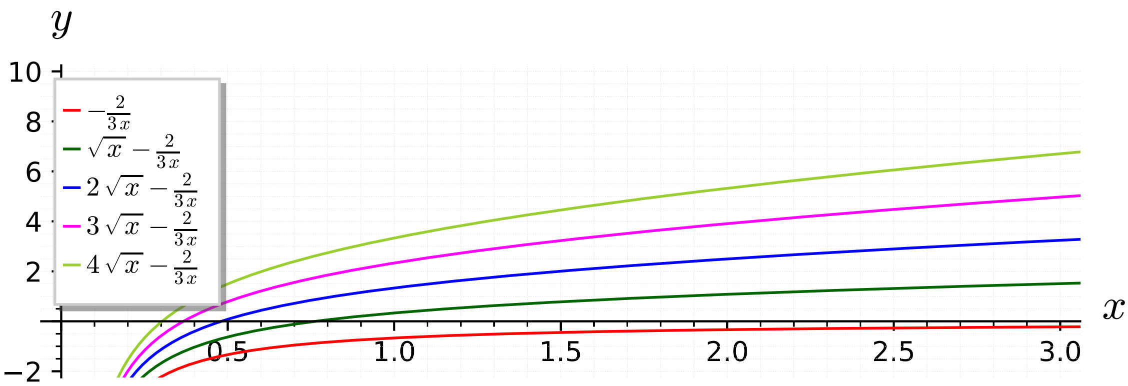

Figure 2.2.: Solution to the example "*General Solution in the Case of Different Real Roots*"

Example **General Solution in the Case of a Double Root** 16

Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

Solution **General Solution in the Case of a Double Root**

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

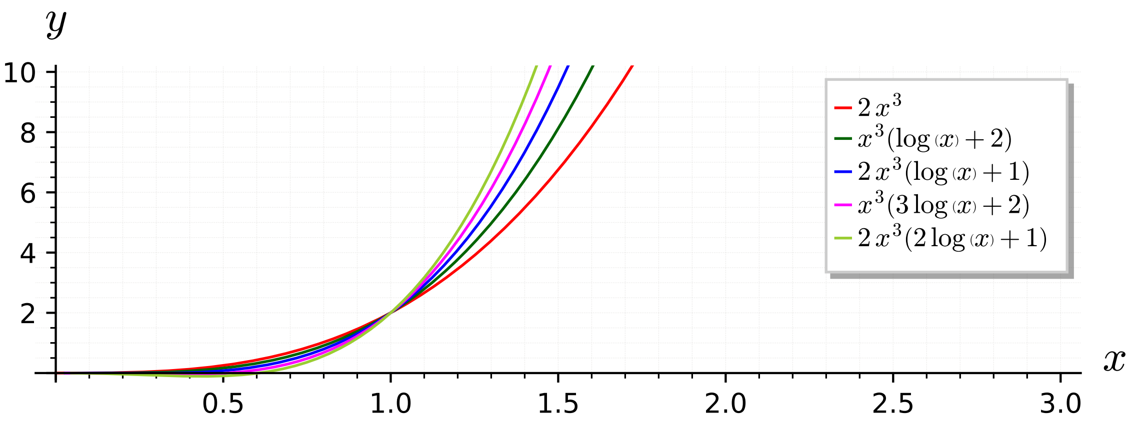

Figure 2.3.: Solution to the example "*General Solution in the Case of a Double Root*"

Example BVP: Electric Potential Field Between Two Concentric Spheres 17

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii r_1 = 5 cm and r_2 = 10 cm kept at potentials v_1 = 110 V and v_2 = 0, respectively.

> $v = v(r)$ is a solution of the *Euler–Cauchy equation* $rv'' + 2v' = 0$.

Solution BVP: Electric Potential Field Between Two Concentric Spheres

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

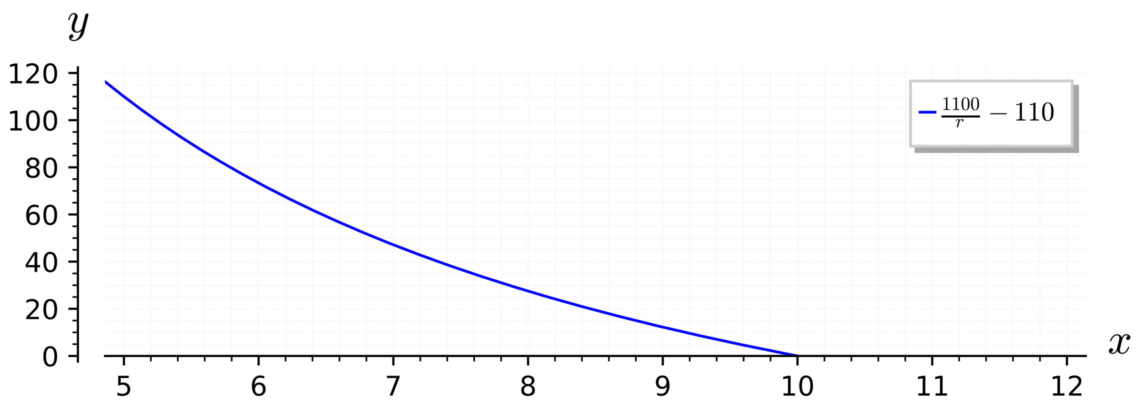

Figure 2.4.: Solution to the BVP: Electric Potential Field Between Two Concentric Spheres exercise.

2.1.6 Non-homogeneous ODEs

They have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.5}$$

where $r(x) \neq 0$. a general solution of Eq. (2.5) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.6}$$

and a **particular solution** of Eq. (2.5). These two new terms **general solution** of Eq. (2.5) and **particular solution** of Eq. (2.5) are defined as follows:

General Solution and Particular Solution
A general solution of the nonhomogeneous ODE Eq. (2.5) on an open interval I is a solution of the form:

$$y(x) = y_h(x) + y_p(x)\,. \tag{2.7}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.6) on I and y_p is any solution of Eq. (2.5) on I containing **no arbitrary constants**. A particular solution of Eq. (2.5) on I is a solution obtained from Eq. (2.7) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.5) or an initial value problem for Eq. (2.5), we have to solve the homogeneous ODE Eq. (2.6) or an initial value problem for and find any solution y_p of Eq. (2.5), so that we obtain a general solution Eq. (2.7) of Eq. (2.5).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.8}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.
Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Choice Rules for the Method of Undetermined Coefficients

Basic Rule: If $r(x)$ in Eq. (2.8) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.8).

Modification Rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.8), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule: If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	**Choice for** $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Table 2.3.: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.5) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.5) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction.

A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Model Damped System

To our previous undamped model $my'' = -ky$ we now add the damping force:

$$F_2 = -cy',$$

therefore, the ODE of the damped mass–spring system is:

$$my'' + cy' + ky = 0. \tag{2.9}$$

This can physically be done by connecting the ball to a dashpot. Assume this damping force to be proportional to the velocity $y\prime = dy/dt$.

> This is generally a good approximation for small velocities.

The constant c is called the damping constant.
Let us show that c is positive.
The damping force $F_2 = -cy'$ acts **against** the motion; hence for a downward motion we have $y\prime > 0$ which for positive c makes F negative (an upward force), as it should be.
Similarly, for an upward motion we have $y\prime < 0$ which, for $c > 0$ makes F_2 positive (a downward force).
The ODE Eq. (2.9) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

Depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.4.: A Detailed look into the scientific method.

Case I: Over-damping

If $c^2 > 4mk$, then λ_1 and λ_2 are **distinct real roots**. In this case the corresponding general solution is:

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}. \tag{2.10}$$

> In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.10) are negative because $\alpha > 0$ and $\beta > 0$ and:

$$[][2] = \alpha^2 - k/m < \alpha^2$$

Hence both terms in Eq. (2.10) approach zero as $t \to \infty$.
Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Below are the results for some typical initial conditions.

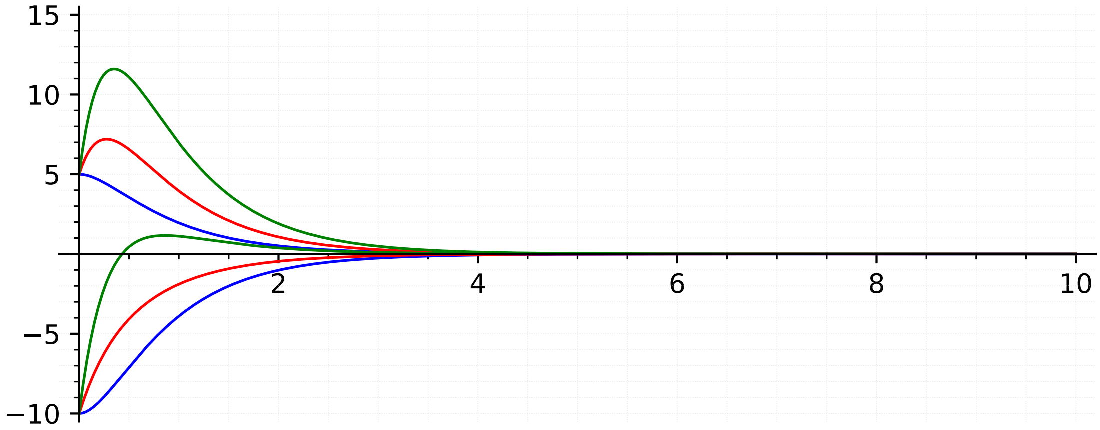

Figure 2.5.

Case II: Critical-Damping

Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III). Occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of Eq. (2.9) is:

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{2.11}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.
If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.
The Figure below shows typical forms of Eq. (2.11).

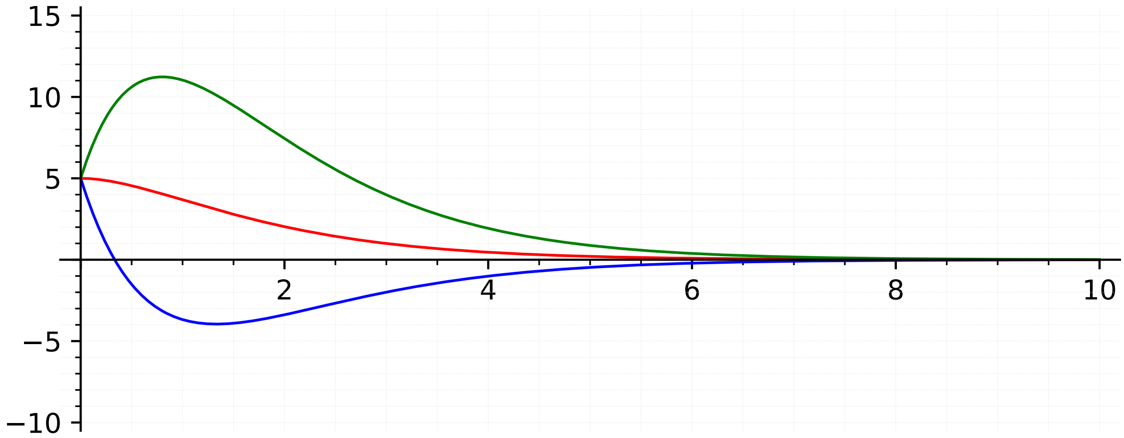

Figure 2.6.

The graph above looks almost like those in the previous figure.

Case III: Under-Damping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 = 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (> 0).$$

This is to differentiate from ω which is used predominantly in electrical engineering.

The roots of the characteristic equation are now complex conjugates,

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos(\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$ This represents **damped oscillations**. Their curve lies between the dashed curves: The roots of the characteristic equation are now complex conjugates,2m at
The frequency is $\omega * /2\pi$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (o) is, the larger is v* and the more rapid the oscillations become. If c approaches o, v >(2p) is the natural frequency of the system. o

Model Modelling: Forced Oscillations and Resonance

Previously we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring) and modeled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.12}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass–spring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia $my\prime\prime$, the damping force $cy\prime$ (if $c < 0$), and the spring force ky, a restoring force.

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.12) into:

$$my'' + cy' + ky = r(t). \tag{2.13}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.14}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.14) is the sum of a general solution y_{h} of the homogeneous ODE Eq. (2.12) plus any solution y_{p} of Eq. (2.14). To find y_{p}, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.15}$$

By differentiating this function (remember the chain rule) we obtain:

$$y_{\text{p}}' = -\omega a \sin \omega t + \omega b \cos \omega t,$$
$$y_{\text{p}}'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_{p}, $y_{\text{p}}\prime$, $y_{\text{p}}\prime\prime$, into Eq. (2.14) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega c b] \cos \omega t + [-\omega c a + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2) equations:

$$\left(k - m\omega^2\right) a + \omega c b = F_0, \tag{2.16}$$
$$-\omega c a + \left(k - m\omega^2\right) b = 0. \tag{2.17}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(km\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.18}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.14) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.12) and y_p is given by Eq. (2.15) with coefficients Eq. (2.18).

Model Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I \,\mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t. \tag{2.19}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.19) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to Eq. (2.19) and I_p is a particular solution of Eq. (2.19). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a \cos \omega t + b \sin \omega t, \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t), \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t). \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to Eov cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned} L\omega^2(-a) + R\omega b + a/C &= E_0 \omega \qquad \text{(Cosine terms)} \\ L\omega^2(-b) + R\omega(-a) + b/C &= 0 \qquad \text{(Sine terms)}. \end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.20}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$-Sa + Rb = E_0,$$
$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.21}$$

Equation (2) with coefficients a and b given by Eq. (2.21) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage. Using Eq. (2.21), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.22}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $(R^2 + S^2)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.22) and whose frequency is that of the input (i.e., voltage).

Example **Reduction of Order if a Solution Is Known** 18

Find a basis of solutions of the ODE:

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

Solution **Reduction of Order if a Solution Is Known**

Inspection shows that $y_1 = x$ is a solution because $y_1{\prime} = 1$ and $y_1{''} = 0$, so that the first term vanishes identically and the second and third terms cancel.
The idea of the method is to substitute

$$\begin{aligned}
y &= uy_1 = ux, \\
y' &= u'x + ux' = u'x + u, && \text{(Chain Rule)} \\
y'' &= (u'x + u)\prime = u\prime\prime x + u\prime x\prime + u\prime = u\prime\prime x + 2u\prime. && \text{(Chain Rule)}
\end{aligned}$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and xu cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u\prime$, namely:

$$\left(x^2 - x\right) v\prime + (x - 2)\, v = 0$$

Separation of variables and integration gives:

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}\, dx = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx, \qquad \ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}$$

We need no constant of integration because we want to obtain a particular solution.
Taking exponents and integrating again, we obtain:

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\, dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ \ln|x| + 1.$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are **linearly independent**.

> This means their quotient is not constant.

we have obtained a basis of solutions, valid for all positive x. ∎

Example **IVP: Case of Distinct Real Roots** 19

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution IVP: Case of Distinct Real Roots

Step 1. General Solution

The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular Solution

As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

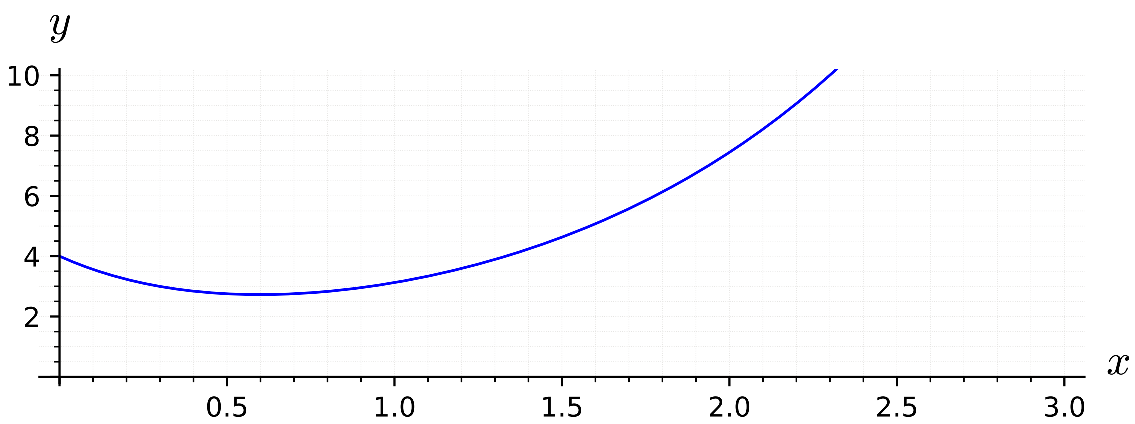

Figure 2.7.: The curve begins with a negative slope in agreement with the initial conditions.

Example IVP: Case of Real Double Roots 20

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

Solution IVP: Case of Real Double Roots

The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = (\lambda + 2)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = (c_1 + c_2 x)\, e^{-0.5\,x}$$

We need its derivative:

$$y\prime = c_2 e^{-0.5\,x} - 0.5\,(c_1 + c_2 x)\, e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$y\,(0) = c_1 = 3.0, \quad y\prime\,(0) = c_2 - 0.5 c_1 = 3.5, \quad c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\,x}$$

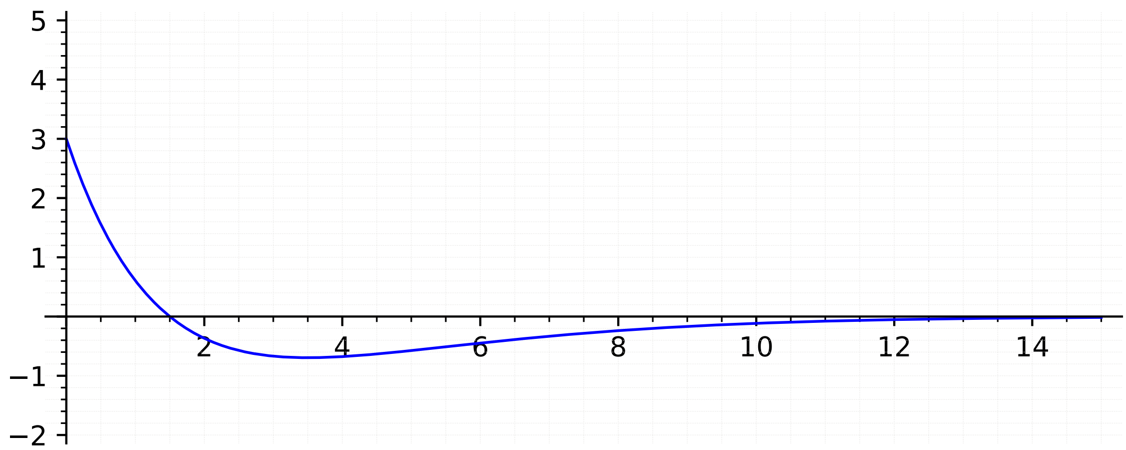

Figure 2.8.: Solution to the case of a double root.

Example IVP: Case of Complex Roots 21

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution IVP: Case of Complex Roots

Step 1. General Solution

The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathbf{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Step 2. Particular Solution

The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x}\sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

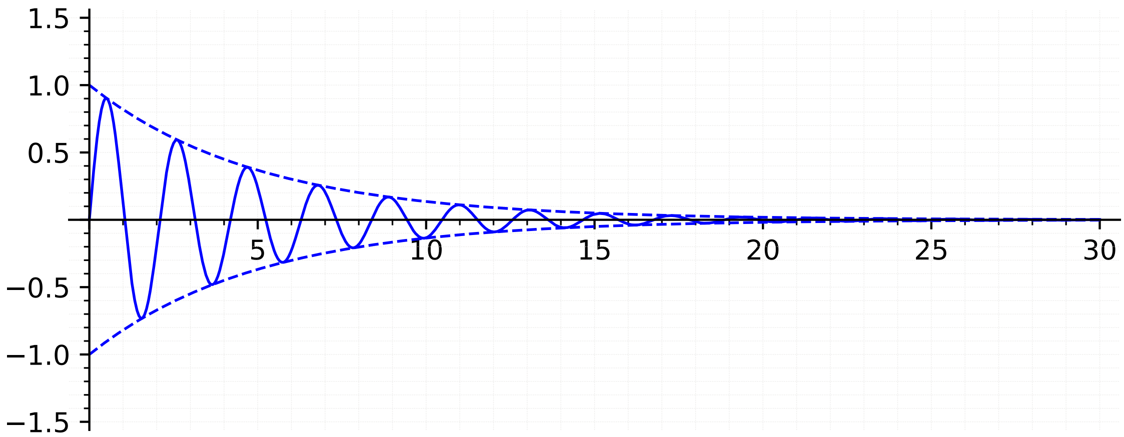

Figure 2.9.: Solution to the complex roots - initial value problem exercise.

Example Harmonic Oscillation of an Undamped Mass–Spring System 22

If a mass–spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

Solution Harmonic Oscillation of an Undamped Mass–Spring System

Hooke's law:

$$F_1 = -ky \tag{2.23}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\mathrm{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0.48\,\mathrm{Hz}$$

From Eq. (2.23) and the initial conditions, $y(0) = A = 0.16\,\mathrm{m}$ and $y\prime(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16 \cos 3t\,\mathrm{m} \quad ■$$

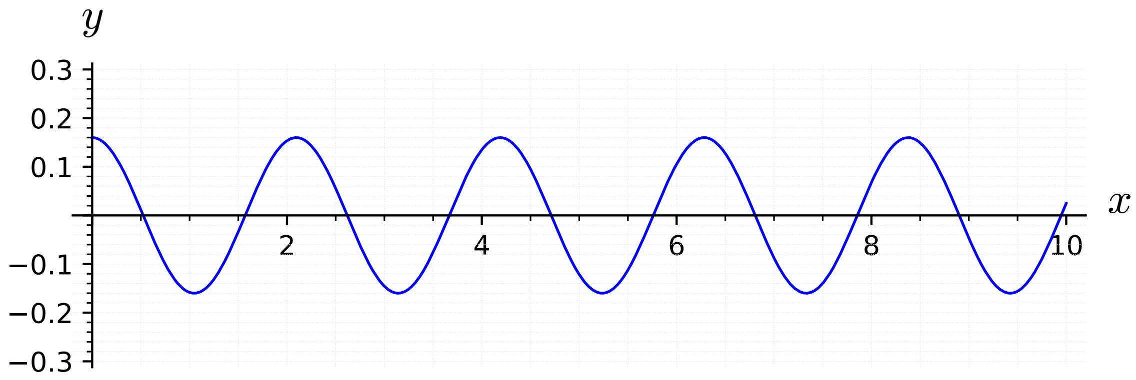

Figure 2.10.: The harmonic oscillation on a string.

Example **The Three Cases of Damped Motion** 23

How does the motion in *Harmonic Oscillation of an Undamped Mass–Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y\prime(0) = 0$ as before?

- $c = 100\,\mathrm{kg} \cdot \mathrm{s}^{-1}$
- $c = 60\,\mathrm{kg} \cdot \mathrm{s}^{-1}$
- $c = 10\,\mathrm{kg} \cdot \mathrm{s}^{-1}$

Solution **The Three Cases of Damped Motion**

It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I

With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass–Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16\,\mathrm{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02,\ c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02 e^{-9t} + 0.18 e^{-t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II

The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10 (\lambda + 3)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = (c_1 + c_2 t)\, e^{-3t}, \qquad \text{we also need} \qquad y\prime = (c_2 - 3c_1 - 3c_2 t)\, e^{-3t}$$

The initial conditions give $y(0) = c_1 = 0.16,\ y\prime(0) = c_2 - 3c_1 = 0,\ c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III

The model is now:

$$10y\prime\prime + 10y\prime + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96\mathbf{j}$$

This gives the general solution:

$$y = e^{-0.5t} (A \cos 2.96t + B \sin 2.96t)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$y\prime = e^{-0.5t} (-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t)$$

Hence $y\prime(0) = -0.5A + 2.96B = 0,\ B = 0.5A/2.96 = 0.027$. This give the solution:

$$y = e^{-0.5t} (0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162 e^{-0.5t} \cos(2.96t - 0.17) \quad \blacksquare$$

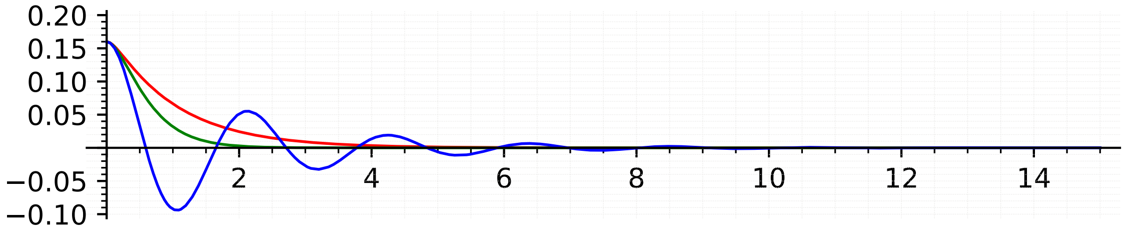

Figure 2.11.: The solution curve to the Three Cases of Damped Motion exercise

Example **Studying the RLC Circuit** 24

Find the current $I(t)$ in an RLC-circuit with $R = 11$ (Ohms), $L = 0.9$ H (Henry), $C = 0.01$ F (Farad), which is connected to a source of $V(t) = 110 \sin(120\pi t)$.

Assume that current and capacitor charge are 0 when $t = 0$.

Solution **Studying the RLC Circuit**

Step 1. General solution of the homogeneous ODE

Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t.$$
$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Hence the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

Step 2. Particular solution I_p

We calculate the reactance S = 37.7 - 0.3 = 37.4 and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t$$

Step 3. Particular solution satisfying the initial conditions

How to use Q(0) 0? We finally determine c1 and c2 from the in initial conditions I(0) 0 and Q(0) 0. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.71 = 0$$

, hence $c_2 = 2.71 - c_1$

We turn to Q(0) 0. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0$$

, so that $I'(0) = 0$ Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0$$

, hence by (7), $-10c_1 = 100(2.71 - c_1) - 300.1$. The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

Example Application of the Basic Rule (a) 25

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

Solution Application of the Basic Rule (a)

Step 1: General Solution of the Homogeneous ODE

The ODE $y'' + y = 0$ has the general solution

$$y_h = A \cos x + B \sin x.$$

Step 2: Solution of the non-Homogeneous ODE

First try $y_p = Kx^2$ and also $y_p'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) **must be the same on both sides**. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

The looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0, \quad \text{Then} \quad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Hence:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$$

Step 3. Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \quad \text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

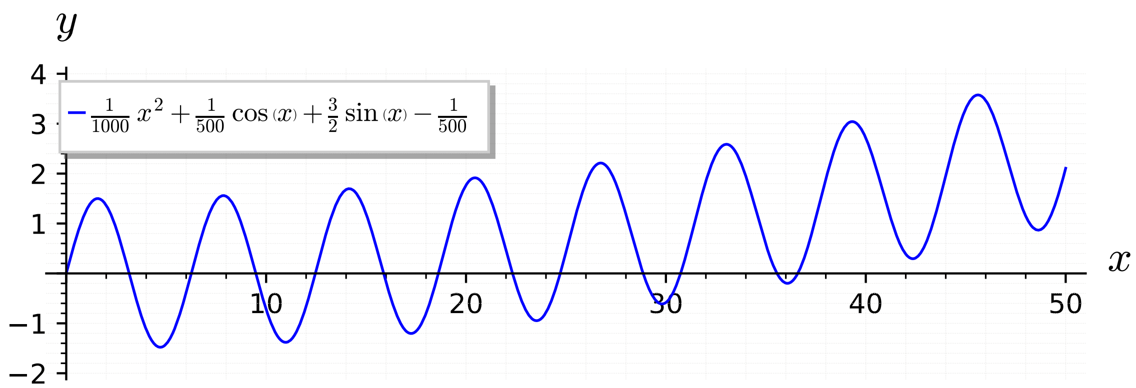

Figure 2.12.: Solution to Method of Undetermined Coefficients exercise.

Example Application of the Modification Rule (b) 26

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution Application of the Modification Rule (b)

Step 1.General solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\,e^{-1.5x}$$

Step 2.Solution y_p of the non-homogeneous ODE

The function $e^{-1.5x}$ on the RHS would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2e^{-1.5x}, \quad \text{then} \quad y_p' = C(2x - 1.5x^2)e^{-1.5x}, y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2-6x+2.25x^2)+3C(2x-1.5x^2)+2.25Cx^2=-10$$

Comparing the coefficients of x^2, x, x^0 gives $0=0, 0=0, 2C=-10$, hence $C=-5$. This gives the solution $y_p=-5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y=y_h+y_p=(c_1+c_3)e^{-1.5x}-5x^2e^{-1.5x}$$

Step 3. Solution of the initial value problem

Setting $x=0$ in y and using the first initial condition, we obtain $y(0)=c_1=1$. Differentiation of y gives:

$$y'=(c_2-1.5c_1-1.5c_2v)e^{-1.2x}-10xe^{-1.2x}+7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0)=c_2-1.5c_1=0$. Hence $c_2=1.5c_1=1.5$. This gives the answer

$$y=(1+1.5x)e^{-1.5x}-5x^2e^{-1.5x}=(1+1.5x-5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x=0.6217$ (where $1+1.5x-5x^2=0$) and approaches the axis from below as x increases.

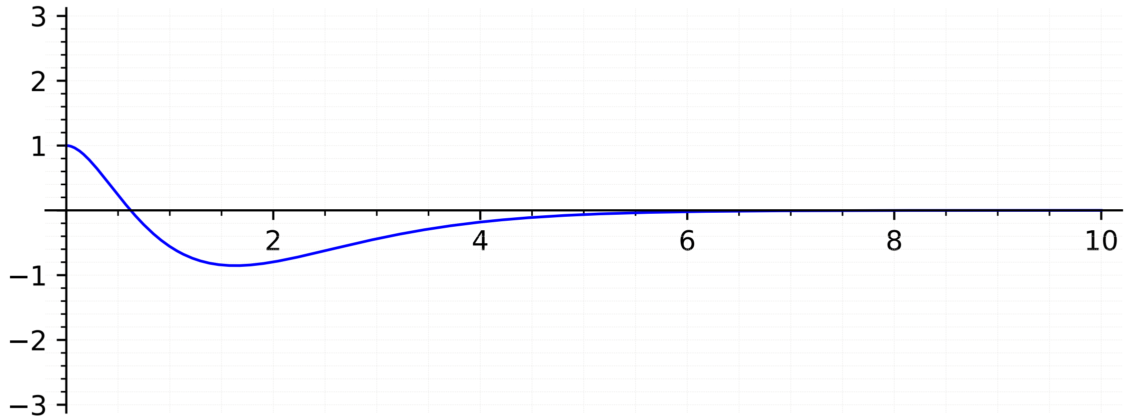

Figure 2.13.

Example Application of the Sum Rule (c) 27

Solve the initial value problem

$$y''+2y'+0.75y=2\cos x-0.25\sin x+0.09x, \qquad y(0)=2.78, \qquad y'(0)=-0.43.$$

Solution Application of the Sum Rule (c)

Step 1.General Solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}.$$

Step 2. Particular Solution of the non-homogeneous ODE

We write:

$$y_\text{p} = y_\text{p1} + y_\text{p2}$$

and following the table,

$$y_\text{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_\text{p2} = K_1 x + K_0$$

Differentiating gives:

$$y_\text{p1}\prime = -K\sin x + M\cos x, \quad y_\text{p1}\prime\prime = -K\cos x - M\sin x, \quad y_\text{p2}\prime = 1, \quad y_\text{p2}\prime\prime = 0.$$

Substitution of y_p1 into the ODE in (7) gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

hence $K = 0$ and $M = 1$. Substituting y_p2 into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0, \qquad \text{thus} \qquad K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE in (7) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Step 3. Solution of the initial value problem

From y, $y\prime$ and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78, \qquad y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

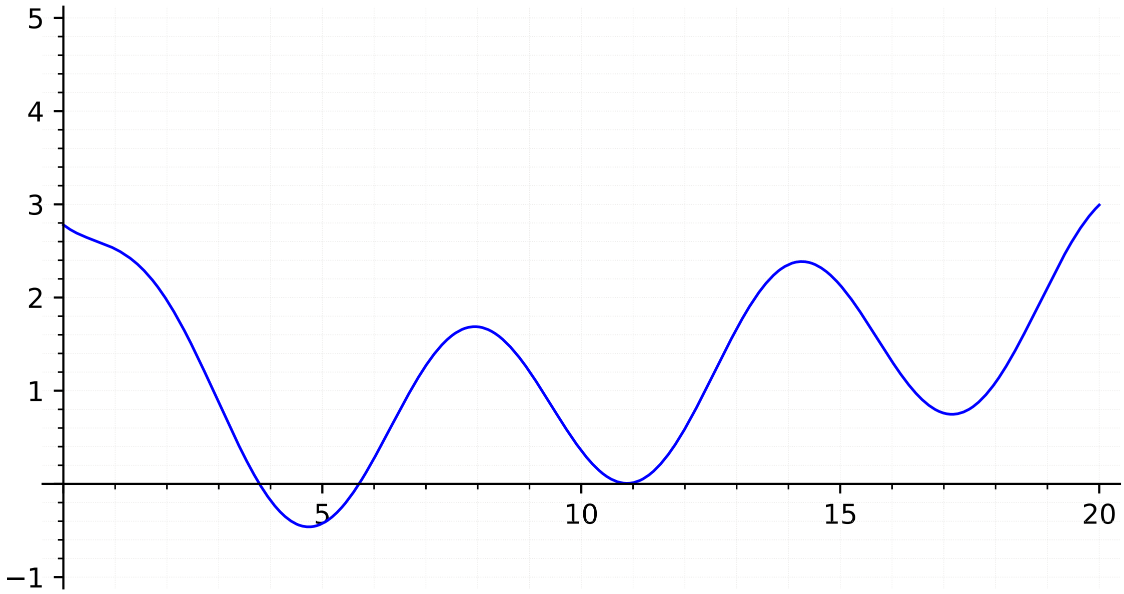

Figure 2.14.: Solution of Application of the Sum Rule (c)

Chapter 3.

Higher-Order Ordinary Differential Equations

3.1 Homogeneous Linear ODEs

Recall from **First-Order ODEs** that an ODE is of n^{th} if the n^{th} derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F(x, y, y', \cdots, y^{(n)}) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

> $y^{(n)}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.

> An n^{th}-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.
A **solution** of an n^{th}-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to n^{th}-order homogeneous linear ODEs as following theorems.

Fundamental Theorem for the Homogeneous Linear ODE (2)
For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

This does not hold for a nonhomogeneous or non-linear ODE.

General Solution, Basis, Particular Solution
A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \quad \text{arbitrary}) \tag{3.3}$$

where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.

That is, these solutions are linearly independent on I, as defined below.
A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

Linear Independence and Dependence
Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on } I \tag{3.4}$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \ldots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n-1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (3.4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Example Linear Dependence 28

Show that the functions $y_1 = x^2, y_2 = 5x, y_3 = 2x$ are linearly dependent on any interval.

Solution Linear Dependence

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Example General Solution 29

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{(where} \quad y^{\text{iv}} = d^4y/dx^4\text{).}$$

Solution General Solution

Similar to Chapter 2 we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1e^{-2\mu} + c_2e^{-\nu} + c_3e^{\nu} + c_4e^{2\mu}$$

provided those four solutions are linearly independent ■

Example Initial Value Problem for a Third-Order Euler–Cauchy Equation 30

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \qquad y'(1) = 1, \qquad y''(1) = -4.$$

Solution **Initial Value Problem for a Third-Order Euler–Cauchy Equation**

General solution

As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular solution

The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
&\text{(a)} \quad y(1) = c_1 + \quad c_2 + c_3 = \quad 2 \\
&\text{(b)} \quad y'(1) = c_1 + 2c_2 + \quad 3c_3 = 1 \\
&\text{(c)} \quad y'(1) = 2c_2 + \quad 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(n)} = d^n y/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \qquad \cdots, \qquad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Example Distinct Real Roots 31

Solve the following ODE:

$$y''' - 2y'' - y' + 2y = 0$$

Solution Distinct Real Roots

The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

Example Simple Complex Roots 32

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0, \qquad y(0) = 4, \qquad y'(0) = 11, \qquad y''(0) = -299$$

Solution Simple Complex Roots

The characteristic equation is:

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10\mathbf{j}$.

Therefore, a general solution and its derivatives (obtained by differentiation) are:

$$\begin{aligned} y &= c_1 e^x + A \cos 10x + B \sin 10x, \\ y' &= c_1 e^x - 10A \sin 10x + 10B \cos 10x, \\ y'' &= c_1 e^x - 100A \cos 10x - 100B \sin 10x. \end{aligned}$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$\text{(a) } c_1 + A = 4, \qquad \text{(b) } c_1 + 108 = 11, \qquad \text{(c) } c_1 - 1004 = -299$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^x + 3 \cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.

Multiple Real Roots

If a real double root occurs ($\lambda_1 = \lambda_2$) then $y_1 = y_2$ in Eq. (3.3), and we take y_1 and xy_1 as corresponding linearly independent solutions.
More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2 e^{\lambda x}, \quad \cdots, \quad x^{m-1} e^{\lambda x}$$

Example Real Double and Triple Roots 33

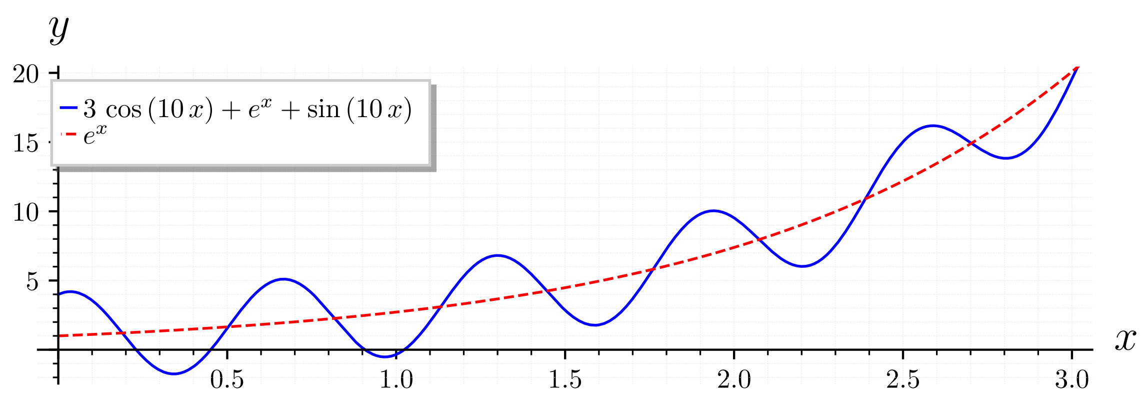

Figure 3.1.

Solve the following ODE:

$$yv - 3yiv + 3y/// - y// = 0$$

Solution **Real Double and Triple Roots**

The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda} = \gamma - i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x, \quad e^{\gamma x} \sin \omega x, \quad x e^{\gamma x} \cos \omega x, \quad x e^{\gamma x} \sin \omega x$$

The first two of these result from $e^{\lambda x}$ and $\overline{e}^{\lambda x}$ as before, and the second two from $x e^{\lambda x}$ and $x e^{\overline{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}[(A_1 + A_2 x) \cos \omega x + (B_1 + B_2 x) \sin \omega x].$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\gamma x} \cos \omega x$, $x^2 e^{\gamma x} \sin \omega x$, and so on.

3.1.3 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of nth order. We write them in standard form:

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a **general solution** of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Thus Eq. (3.7) has no singular solutions. An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

> The ideas of proof are the same as those for $n = 2$.

Example **IVP - Modification Rule** 34

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \qquad y'(0) = -3, \qquad y''(0) = -47$$

Solution **IVP - Modification Rule**

Step 1

The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2)e^{-x} \end{aligned}$$

Step 2

If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and Cx^6e^{-x}. The Modification Rule calls for

$$y_p = Cx^3e^{-x}$$

Then

$$\begin{aligned} y_p{}' &= C(3x^2 - x^3)e^{-x}, \\ y_p{}'' &= C(6x - 6x^2 + x^3)e^{-x}, \\ y_p{}''' &= C(6 - 18x + 9x^2 - x^3)e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2e^{-x}$.

Step 3

We now write down $y = y_\text{h} + y_\text{p}$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$\begin{aligned}
y &= y_\text{h} + y_\text{p} = (c_1 + c_2 + c_3x^2)e^{-x} + 5x^3e^{-x}, & y(0) &= c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, & y'(0) &= -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, & y''(0) &= 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2e^{-x}$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.

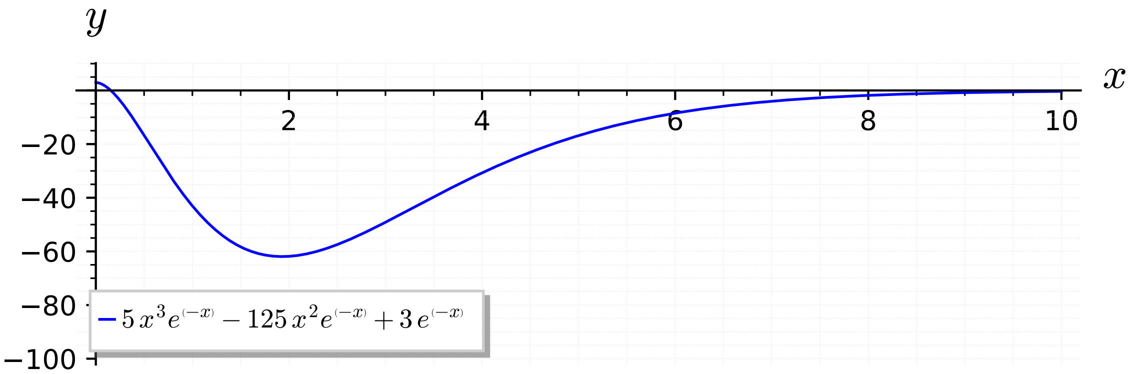

Figure 3.2.: Solution to the example *"IVP - Modification Rule"*.

Model Elastic Beam

Whereas second-order ODEs have various applications, of which we have discussed some of the more important ones (i.e., RLC Circuit, Mass-Damper system), higher order ODEs have much fewer engineering applications.

> An important fourth-order ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge.

A related application of vibration of beams does not fit in here since it leads to PDEs.

Problem Description

Consider a beam B of length L and constant (e.g., **rectangular**) cross section and homogeneous elastic material (e.g., **level**).
We assume under its own weight the beam is bent so little that it is certainly straight. If we apply a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.
Its axis is curved into the so-called **elastic curve** (or **deflection curve**).
It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ (determining the tangent direction of C) are small. Then, by calculus:

$$k = y''/(1 + y''x)^{1/2} \approx y''$$

Therefore:

$$M(x) = EIy\prime\prime(x)$$

EI is the constant of proportionality. E Young's modulus of elasticity of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.
Elasticity theory shows further that $M\prime\prime(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy\text{iv} = f(x)$$

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows and shown in Fig. 77.
* Simply supported

$$y = y'' = 0 \text{at} x = 0 \text{and} L$$

$$y = y' = 0 \text{at} x = 0 \text{and} L$$

$$\text{(C)Clamped at} x = 0, \text{free at} x = L$$

$$y(0) = y'(0) = 0, y''(L) = y'''(L) = 0.$$

The boundary condition $y = 0$ means no displacement at that point, $y'' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Solution Derivation

Let us apply this to the uniformly loaded simply supported beam in Fig. 76. The load is $f(x) = f_0 =$ const. Then (8) is

$$y^{iv} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Two integrations give

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

$y''(0) = 0$ gives $c_2 = 0$. Then $y''(L) = L(\frac{1}{2}kL + c_1) = 0$, $c_1 = -kL/2$ (since $L \neq 0$). Hence

$$y'' = \frac{k}{2}(x^2 - Lx).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution

$$y = \frac{f_0}{24EI}(x^4 - 2Lx^3 + L^3 x).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L-x)$.

Verify this by setting $x = u + L/2$ and show that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0\,(x = L/2)$ is:

$$\frac{5f_0L^4}{(16 \cdot 24EI)}$$

Recall that the positive direction points downward.

Chapter 4.

Systems of ODEs

4.1 Introduction

In this chapter of our book, we introduce a different way of looking at systems of ODEs. The method consists of examining the general behaviour of whole families of solutions of ODEs in the phase plane, called the **phase plane** method.

Phase Plane

A visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two state variables, say (x, y), or (q, p) etc.

It gives information on the stability of solutions. This approach to systems of ODEs is a qualitative method because it depends only on the nature of the ODEs and does not require the actual solutions. This can be very useful because it is often difficult or even impossible to solve systems of ODEs. In contrast, the approach of actually solving a system is known as a **quantitative** method.

Qualitative Method

The qualitative analysis of ODEs is to be able to say something about the behavior of solutions of the equations, without solving them explicitly.

> The phase plane method has many applications in control theory, circuit theory, population dynamics and so on.

4.1.1 System of ODEs as Models in Engineering

Time to see how systems of ODEs are of practical importance. We first illustrate how systems of ODEs can serve as models in various applications. Then we show how a higher order ODE (with the highest derivative standing alone on one side) can be reduced to a first-order system.

Example **Mixing Problem Involving Two Tanks** 35

A mixing problem involving a single tank is modeled by a single ODE which can be extended to two sets of equations.
Tanks T_1 and T_2 contain initially 100 L of water each, In T_1 the water is pure, whereas 150 kg of fertilizer are dissolved in T_2. By circulating liquid at rate of 2 l/min and stirring the amount of fertiliser $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain at least half as much fertiliser as there will be left in T_2?

> Assume the mixture is uniform.

Solution **Mixing Problem Involving Two Tanks**

Setting Up the Model

As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. Therefore:

$$y_1' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_2 - \frac{2}{100}y_1 \qquad (Tank\, T_1),$$
$$y_2' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_1 - \frac{2}{100}y_2 \qquad (Tank\, T_2).$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs:

$$y_1\prime = -0.02y_1 + 0.02y_2,$$
$$y_2\prime\prime = 0.02y_1 - 0.02y_2.$$

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix $\mathbf{A}$ this becomes:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y}\,, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

General Solution

As for a single equation, we try an exponential function of t,

$$\mathbf{y} = \mathbf{x}e^{\lambda t}. \qquad \text{Then} \qquad \mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{x}e^{\lambda t}. \tag{1}$$

Dividing the last equation $\lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x}.$$

We need nontrivial solutions (solutions that are not identically zero). Hence we have to look for eigenvalues and eigenvectors of $\mathbf{A}$. The eigenvalues are the solutions of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = (-0.02 - \lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0 \tag{4.1}$$

We see that $\lambda_1 = 0$ and $\lambda_2 = -0.04$.

> $\lambda = 0$ can very well happen but don't get mixed up. It is eigenvectors that must not be zero.

Eigenvectors are obtained as $\lambda = 0$ and $\lambda = -0.04$. For our present $\mathbf{A}$ this gives:

$$-0.02x_1 + 0.02x_2 = 0 \qquad \text{and} \qquad (-0.02 + 0.04)x_1 + 0.02x_2 = 0,$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From Eq. (1) and the superposition principle, we thus obtain a solution:

> This principle continues to hold for systems of homogeneous linear ODEs

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \tag{4.2}$$

where c_1 and c_2 are arbitrary constants.

Use of initial conditions

The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and Eq. (4.2) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer:

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)}e^{-0.04t} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

In components,

$$\begin{aligned} y_1 &= 75 - 75e^{-0.04\,t} && \text{Tank } T_1\text{, lower curve,} \\ y_2 &= 75 + 75e^{-0.04\,t} && \text{Tank } T_2\text{, upper curve.} \end{aligned}$$

Figure 4.1 shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 kg.

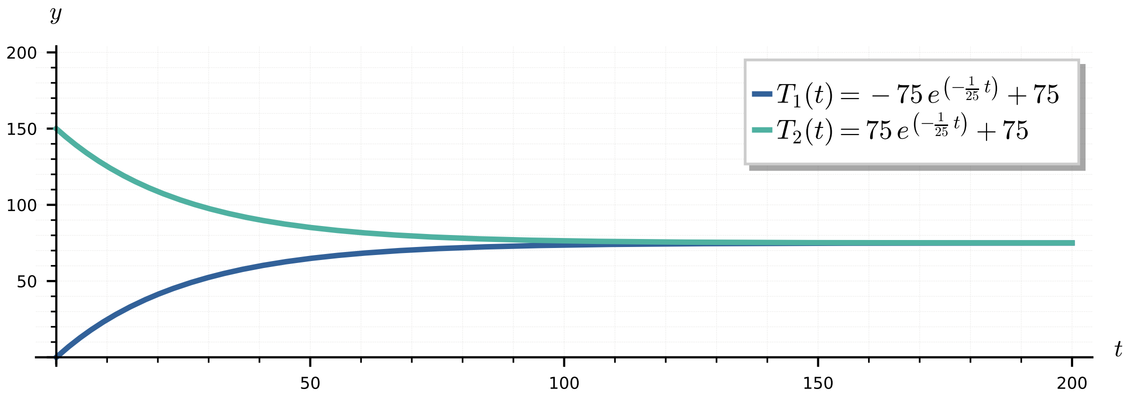

Figure 4.1.: Result of the *Mixing Problem Involving Two Tanks*. As can be seen two tanks converge on a singular value as time goes on.

Answer

T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 kg. Therefore:

$$y_1 = 75 - 75e^{-0.04t} = 50, \qquad e^{-0.04t} = \frac{1}{3}, \qquad t = (\ln 3)/0.04 = 27.5$$

Hence the fluid should circulate for roughly half an hour ■

Example Electrical Network 36

Find the currents $I_1(t)$ and $I_2(t)$ in the network.

> Assume all currents and charges zero at $t = 0$, the instant when the switch is **closed**.

Solution Electrical Network

Setting up the mathematical model

The model of this network is obtained from Kirchhoff's Voltage Law.

Kirchhoff's Voltage Law

> The sum of the voltage differences around any closed loop in a circuit must be zero. A loop in a circuit is any path which ends at the same point at which it starts.

Let $I_1(t)$ and $I_2(t)$ be the currents in the left (L) and right (R) loops, respectively.
In (L), the voltage drops are:

$$L_1' I_1' = (1) I_1' \; V \qquad \text{Over Inductor}$$
$$R_1 (I_1 - I_2) = 4\,(I_1 - I_2) \; V \qquad \text{Over Resistor}$$

The difference is caused by I_1 and I_2 flowing through the resistor in **opposite** directions.

By Kirchhoff's Voltage Law the sum of these drops equals the voltage of the battery:

$$I_1' + 4(I_1 - I_2) = 12$$

Cleaning the aforementioned equation creates our first ODE:

$$I_1' = -4I_1 + 4I_2 + 12. \tag{4.3}$$

In (R), the voltage drops are:

$$R_2 I_2 = 6 I_2 \, [\mathrm{V}]$$
$$R_1 (I_2 - I_1) = 4(I_2 - I_1) \, [\mathrm{V}]$$
$$(I/C) \int I_2 \, dt = 4 \int I_2 \, dt \, [\mathrm{V}] \, .$$

As there is no voltages sources in the (R) loop, the voltage sum **MUST** be zero.

$$6I_2 + 4(I_2 - I_1) + 4 \int I_2 \, dt = 0 \qquad \text{or} \qquad 10I_2 - 4I_1 + 4 \int I_2 \, dt = 0.$$

Division by 10 and differentiation gives $I_2' - 0.4I_1' + 0.4I_2 = 0$.

To simplify the solution process, we first get rid of $0.4I_1'$, which by Eq. (4.3) equals $0.4(-4I_1+4I_2+12)$. Substitution into the present ODE gives

$$I_2' = 0.4I_1' - 0.4I_2 = 0.4(-4I_1 + 4I_2 + 12) - 0.4I_2$$

and by simplification

$$I_2' = -1.6I_1 + 1.2I_2 + 4.8. \tag{4.4}$$

In matrix form, Eq. (4.3) and Eq. (4.4) are (we write **J** since **I** is the unit matrix)

$$\mathbf{J}' = \mathbf{A}\,\mathbf{J} +, \qquad \text{where} \qquad \mathbf{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}. \tag{6}$$

General Solution

As we have a vector , this is a **non-homogeneous** system, and we try to proceed as for a single ODE, solving first the homogeneous system $\mathbf{J}' = \mathbf{A}\,\mathbf{J}$ (thus $\mathbf{J}' - \mathbf{A}\,\mathbf{J} = v0$) by substituting $\mathbf{J} = \mathbf{x}e^{\lambda t}$. This gives

$$\mathbf{J}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t} \qquad \text{hence} \qquad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Hence, to obtain a non-trivial solution, we again need the eigenvalues and eigenvectors. For the present matrix **A** they are:

$$\lambda_1 = -2, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

Hence a *general solution* of the homogeneous system is:

$$\mathbf{J}_{\text{h}} = c_1\mathbf{x}^{(1)}e^{-2\,t} + c_2\mathbf{x}^{(2)}e^{-0.8\,t}.$$

For a particular solution of the nonhomogeneous system Eq. (6), since is constant, we try a constant column vector $\mathbf{J}_{\text{p}} = \mathbf{a}$ with components a_1, a_2. Then $\mathbf{J}_p' = 0$, and substitution into Eq. (6) gives $\mathbf{A}\,\mathbf{a} + = 0$, in components,

$$-4.0a_1 + 4.0a_2 + 12.0 = 0$$

$$-1.6a_1 + 1.2a_2 + \ 4.8 = 0.$$

The solution is $a_1 = 3, a_2 = 0$; thus $\mathbf{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

$$\mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a}; \tag{4.5}$$

in components,

$$I_1 = 2c_1e^{-2t} + c_2e^{-0.8t} + 3$$

$$I_2 = c_1e^{-2t} + 0.8c_2e^{-0.8t}$$

The initial conditions give

$$I_1(0) = 2c_1 + \quad c_2 + 3 = 0$$

$$I_2(0) = \quad c_1 + 0.8c_2 \qquad = 0.$$

Hence $c_1 = -4$ and $c_2 = 5$. As the solution of our problem we thus obtain

$$\mathbf{J} = -4\mathbf{x}^{(1)}e^{-2t} + 5\mathbf{x}^{(2)}e^{-0.0t} + \mathbf{a} \tag{4.6}$$

In components:

$$I_1 = -8e^{-2t} + 5e^{-0.8t} + 3$$
$$I_2 = -4e^{-2t} + 4e^{-0.8t}.$$

Now comes an important idea, on which we shall elaborate further. Figure 4.2 shows $I_1(t)$ and $I_2(t)$ as two (2) separate curve. Figure 4.3 shows those two currents as a **single curve** $[I_1(t), I_2(t)]$ in the $I_1 I_2$-plane.

This is a parametric representation with time as the parameter t. It is often important to know in which sense such a curve is traced. This can be indicated by an arrow in the sense of increasing t. The I_1, I_2-plane is called the **phase plane** of our system Eq. (6), and the curve in 4.3 is called a trajectory.

In following chapters we will see that such *phase plane representations* are far more important than graphs because they will give a much better qualitative overall impression of the general behavior of whole families of solutions, not merely of one solution as in the present case. ■

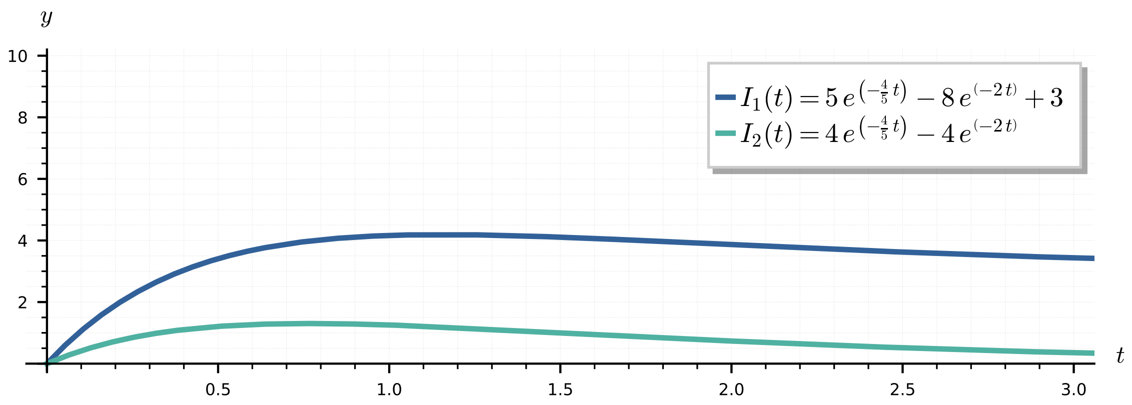

Figure 4.2.: The solution for the two currents (I_1, I_2) flowing through the circuit as time (t) passes.

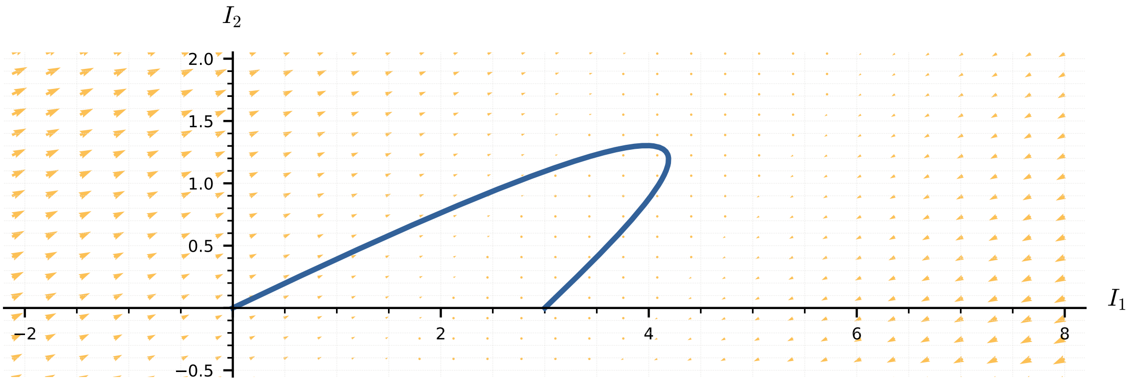

Figure 4.3.: The trajectory of the I_1, I_2 superimposed on its phase plane.

4.1.2 Conversion of an n-th Order ODE to a System

An nth-order ODE of the general form can be converted to a system of n first-order ODEs. This permits the study and solution of single ODEs by methods for systems, and opens a way of including the theory of higher order ODEs into that of first-order systems.

Theorem: *Conversion of an ODE*

An nth-order ODE:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{4.7}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}. \tag{4.8}$$

This system is of the form

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned} \tag{4.9}$$

While the iron is hot, let's look at an example.

Example Mass on a Spring 37

To gain confidence in the conversion method, let us apply it to an old friend of ours, modelling the free motions of a mass on a spring with value given as $m = 1$, $c = 2$, and $k = 0.75$.

$$my'' + cy' + ky = 0 \qquad \text{or} \qquad y'' = -\left(\frac{c}{m}\right) y, - \left(\frac{k}{m}\right) y.$$

Solution Mass on a Spring

For this ODE given in the question can be written in the form ofEq. (4.7), making the system shown Eq. (4.8) as **linear** and **homogeneous**, applying to our system in question.

$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2.$$

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is:

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Entering the values of $m = 1, c = 2$, and $k = 0.75$, produces:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0$$

This gives the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$.
Eigenvectors follow from the first equation in $\mathbf{A} - \lambda\mathbf{I} = 0$, which is $-\lambda x_1 + x_2 = 0$.

$\lambda_1 = 0.5$ Produces $0.5x_1 + x_2 = 0$, which have solutions $x_1 = 2$, $x_2 = -1$.

$\lambda_2 = -1.5$ Produces $1.5x_1 + x_2 = 0$, which have solutions $x_1 = 1$, $x_2 = -1.5$.

These eigenvectors $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \quad \text{give} \quad \mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1e^{-0.5t} + c_2e^{-1.5t}$$

which is the expected solution. The second component is its derivative:

$$y_2 = y_1' = y' = -c_1e^{-0.5t} - 1.5c_2e^{-1.5t} \quad \blacksquare$$

4.1.3 Linear Systems

Extending the notion of a linear ODE, we call a linear system if it is linear in $y_1, \cdots, y_n$; that is, if it can be written

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t). \end{aligned} \tag{4.10}$$

As a vector equation this becomes

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} + \tag{4.11}$$

where:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ . & \cdots & . \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

This system is called **homogeneous** if $= 0$, so that it is:

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} \tag{4.12}$$

If $\neq 0$, then Eq. (4.12) is called **non-homogeneous**.

4.2 Constant-Coefficient Systems

4.2.1 Phase Plane Method

Continuing, we now assume that our **homogeneous** linear system:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{4.13}$$

under discussion has **constant coefficients**, so that the $n \times n$ matrix $\mathbf{A} = [a_{\text{jk}}]$ has entries not depending on t. We want to solve Eq. (4.13). Now a single ODE $y' = ky$ has the solution $y = Ce^{\text{kt}}$. So let us try:

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \tag{4.14}$$

Substitution into Eq. (4.13) gives:

$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t} = \mathbf{A}\,\mathbf{y} = \mathbf{A}\,\mathbf{x} e^{\lambda t}.$$

Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**:

$$\mathbf{A}\,\mathbf{x} = \lambda \mathbf{x} \tag{4.15}$$

Thus the nontrivial solutions of Eq. (4.13) (i.e., non-zero vectors solutions) are of the form Eq. (4.14), where λ is an eigenvalue of λ and $\mathbf{x}$ is a corresponding eigenvector.

We assume that λ has a **linearly independent** set of n eigenvectors. This holds in most applications, in particular if $\mathbf{A}$ is symmetric ($a_{\text{kj}} = a_{\text{jk}}$) or skew-symmetric ($a_{\text{kj}} = -a_{\text{jk}}$) or has n different eigenvalues.

Let those eigenvectors be $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(\text{n})}$ and let them correspond to eigenvalues $\lambda_1, \cdots, \lambda_n$ (which may be all different, or some–or even all–may be equal). Then the corresponding solutions Eq. (4.14) are

$$\mathbf{y}^{(1)} = \mathbf{x}^{(1)} e^{\lambda_1 t}, \quad \cdots, \quad \mathbf{y}^{(n)} = \mathbf{x}^{(n)} e^{\lambda_n t}. \tag{4.16}$$

Their Wronskian $W = W(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)})$ is given by

$$W = (\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ x_2^{(1)} e^{\lambda_1 t} & \cdots & x_2^{(n)} e^{\lambda_n t} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} e^{\lambda_1 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t + \cdots + \lambda_n t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$

On the right, the exponential function is never zero, and the determinant is not zero either because its columns are the n linearly independent eigenvectors. This proves the following theorem, whose assumption is true if the matrix $\mathbf{A}$ is symmetric or skew-symmetric, or if the n eigenvalues of $\mathbf{A}$ are all different.

Theorem: General Solution

If the constant matrix **A** in the system Eq. (4.13) has a linearly independent set of n eigenvectors, then the corresponding solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ in Eq. (4.16) form a basis of solutions of Eq. (4.13), and the corresponding general solution is:

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \cdots + c_n \mathbf{x}^{(n)} e^{\lambda_n t} \tag{4.17}$$

Example **Type I: Improper Node -Trajectories in the Phase Plane** 38

Find and graph solutions of the system.

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= -3y_1 + y_2, \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

Solution **Type I: Improper Node -Trajectories in the Phase Plane**

To see what is going on, let us find and graph solutions of the system. It is always a good idea to start with known solutions. Substituting $\mathbf{y} = \mathbf{x}e^{\lambda t}$ and $\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t}$ and dropping the exponential function (as they exist both on the LHS and RHS we can eliminate them) we get $\mathbf{A}\,\mathbf{x} = \lambda \mathbf{x}$.

The characteristic equation is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$.
Eigenvectors are then obtained from:

$$(-3-\lambda)x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\mathbf{x}^{(1)} = [1 \quad 1]^\mathsf{T}$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\mathbf{x}^{(2)} = [1 \quad -1]^\mathsf{T}$.

This gives the general solution:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4}.$$

Figure below shows a phase portrait of some of the trajectories (to which more trajectories could be added if so desired).

> The two straight trajectories correspond to $c_1 = 0$ and c_2 and the others to other choices of c_1, c_2.

4.2.2 Critical Points of the System

The point $\mathbf{y} = 0$ in Figure seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from Eq. (4.13) we obtain:

$$\frac{dy}{dt} = \frac{y_2' \, dt}{y_1' \, dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}. \tag{4.18}$$

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: (0, 0), where the right side of Eq. (4.18) becomes 0/0.

> This point P_0, at which dy_2/dy_1 becomes **undetermined** and called a **critical point** of Eq. (4.18).

Five Types of Critical Points

There are five types of critical points depending on the geometric shape of the trajectories near them. These are: (1) improper nodes, (2) proper nodes, (3) saddle points, (4) centres, and (5) spiral points.

Let's look at them with examples.

Example Type II: Proper Node 39

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction at P_0 there is a trajectory having as its limiting direction.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}, \qquad \text{Therefore} \qquad y_1' = y_1 \qquad \text{and} \qquad y_2' = y_2$$

has a proper node at the origin with the matrix being the **identity matrix**. Its characteristic equation $(1 - \lambda)^2 = 0$ has the root $\lambda = 1$.

> Any $\mathbf{x} \neq 0$ is an eigenvector.

and we can take $[1 \quad 0]^\mathsf{T}$ and $[0 \quad 1]^\mathsf{T}$.

Hence, a general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \qquad \text{or} \qquad \begin{array}{l} y_1 = c_1 e^t, \\ y_2 = c_2 e^t. \end{array} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1 \quad \blacksquare$$

Example Type III: Saddle Node 40

A **saddle point** is a critical point P_0 at which there are two (2) incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_1 \\ y_1' &= -y_2 \end{aligned}$$

has a saddle point at the origin.

Its characteristic equation $(1-\lambda)(-1-\lambda) = 0$ has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$.

For $\lambda = 1$ in eigenvector $[1 \ 0]^T$ is obtained from the second row of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, that is, $0x_1 + (-1-1)\,x_2 = 0$.

For $\lambda_2 = -1$, the first row gives $[0 \ \ 1]^T$. Hence a general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^{t} \\ y_2 &= c_2 e^{-t} \end{aligned} \qquad \text{or} \qquad y_1 y_2 = \text{const.}$$

This is a family of **hyperbolas** ■.

Example Type IV: Centre Node 41

A **centre** is a critical point that is enclosed by infinitely many closed trajectories.

The system:

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \qquad \text{Therefore} \qquad y_1' = y_2 \qquad \text{and} \qquad y_2' = -4y_1 \tag{4.19}$$

has a center at the origin.

The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2\mathbf{j}$ and $-2\mathbf{j}$. For $2\mathbf{j}$, an eigenvector follows from the first equation $-2\mathbf{j}\,x_1 + x_2 = 0$ of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, which can be, $[1 \quad 2\mathbf{j}]^{\mathsf{T}}$.

For $\lambda = -2\mathbf{j}$ that equation is $-(-2\mathbf{j})x_1 + x_2 = 0$ and gives, say, $[1 \quad -2\mathbf{j}]^{\mathsf{T}}$. Hence a complex general solution is:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2\mathbf{j} \end{bmatrix} e^{2\mathbf{j}\,t} + c_2 \begin{bmatrix} 1 \\ -2\mathbf{j} \end{bmatrix} e^{-2\mathbf{j}\,t}, \qquad \text{therefore} \qquad \begin{aligned} y_1 &= c_1 e^{2\mathbf{j}\,t} + c_2 e^{-2\mathbf{j}\,t}, \\ y_2 &= 2\mathbf{j}\, c_1 e^{2\mathbf{j}\,t} - 2\mathbf{j}\, c_2 e^{-2\mathbf{j}\,t}. \end{aligned} \tag{4.20}$$

A real solution is obtained from Eq. (4.20) by the Euler formula or from Eq. (4.19).

Namely, we can create a relation of $-4y_1y_1^2$.

$$-4y_1y_1' = y_2y_2' \qquad \text{By Integration} \qquad 2y_1^2 + \frac{1}{2}{y_2}^2 = \text{const.}$$

This is a family of ellipses enclosing the center at the origin. ■

Example **Type V: Spiral Point** 42

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$.

> or tracing these spirals in the opposite sense, away from P_0.

The system:

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}, \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \tag{4.21}$$

has a spiral point at the origin.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which gives the eigenvalues $-1 + \mathbf{j}$ and $-1 - \mathbf{j}$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For $\lambda = -1 + \mathbf{j}$ this becomes $-\mathbf{j}\,x_1 + x_2 = 0$ and we can take $[1 \quad \mathbf{j}]^T$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - \mathbf{j}$ is $[1 \quad -\mathbf{j}]^T$.

This gives the **complex** general solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ \mathbf{j} \end{bmatrix} e^{(-1+\mathbf{j})t} + c_2 \begin{bmatrix} 1 \\ -\mathbf{j} \end{bmatrix} e^{(-1-\mathbf{j})t}$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. We multiply the first equation in Eq. (4.21) by y_1, the second by y_2- and add, obtaining:

$$y_1y_1' + y_2{y_2}' = -\left(y_1^2 + {y_2}^2\right).$$

We now introduce polar coordinates r, t, where $r^2 = y_1^2 + {y_2}^2$. Differentiating this with respect to t gives:

$$2rr' = 2y_1y_1' + 2y_2{y_2}'$$

Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real c this is a spiral.

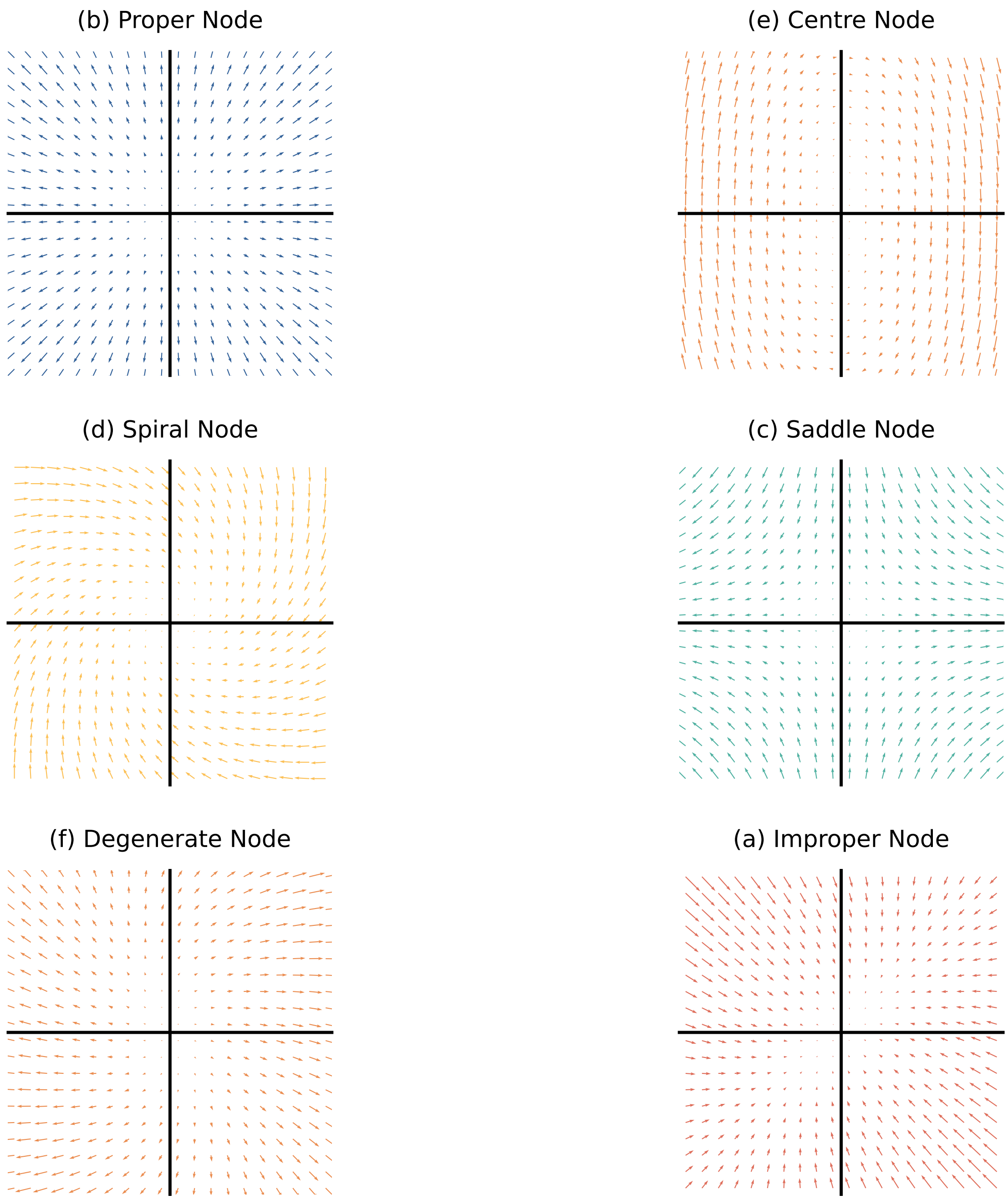

Figure 4.4.: Types of possible systems encountered in the ODE System analysis.

4.3 Criteria for Critical Points & Stability

Continuing our discussion of homogeneous linear systems with **constant coefficients** Eq. (4.13). Let us review where we are. From the previous section we have,

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}, \qquad \text{in components, } y_1' = a_{11}y_1 + a_{12}y_2$$

$$y_2' = a_{21}y_1 + a_{22}y_2. \tag{4.22}$$

From the examples in the last section, we have seen that we can obtain an overview of families of solution curves if we represent them parametrically as $\mathbf{y}(t) = [y_1(t) \quad y_2(t)]^{\mathsf{T}}$ and graph them as curves in the y_1y_2-plane, called the **phase plane**.
Such a curve is called a **trajectory** of Eq. (4.13), and their totality is known as the **phase portrait** of Eq. (4.13).

Now we have seen that solutions are of the form:

$$\mathbf{y}(t) = \mathbf{x}e^{\lambda t}. \qquad \text{Substitution into(1)gives} \qquad \mathbf{y}'(t) = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\,\mathbf{y} = \mathbf{A}\,\mathbf{x}e^{\lambda t}$$

Dropping the common factor $e^{\lambda t}$, we arrive at a similar equation.

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x} \tag{4.23}$$

> $\mathbf{y}(t)$ is a (nonzero) solution of Eq. (4.3) if λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ a corresponding eigenvector.

Our examples in the last section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system Eq. (4.3) defined as a point at which $d\mathbf{y}_2/d\mathbf{y}_1$ becomes **undetermined** (i.e., 0/0).

$$\frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \tag{4.24}$$

Also recall from there are various types (5) of critical points.

What is new here, how these types of critical points are related to the eigenvalues. The latter are solutions $\lambda = \lambda_1$ and λ_2 of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det\mathbf{A} = 0. \tag{4.25}$$

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with coefficients p, q and discriminant Δ given by:

$$p = a_{11} + a_{22}, \qquad q = \det\mathbf{A} = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta = p^2 - 4q. \tag{4.26}$$

From algebra we know that the solutions of this equation are

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}).$$

Furthermore, the product representation of the equation gives

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Hence p is the sum and q the product of the eigenvalues. Also $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ from (6). Together,

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2.$$

This gives the criteria in Table 4.1 for classifying critical points. A derivation will be indicated later in this section. Critical points may also be classified in terms of their **stability**. Stability concepts

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	**Comments**
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite signs
Centre	$p = 0$	$q > 0$		Pure imaginary
Spiral		$q \neq 0$	$\Delta < 0$	Complex (not pure imaginary)

Table 4.1.: Eigenvalue Criteria for Critical Points.

are fundamental for engineering purposes where it means, a small change of a physical system at some instant changes the behavior of the system only slightly at all future times t.

Stable Unstable Attractive

A critical point P_0 of Eq. (4.3) is called **stable** if, roughly, all trajectories of Eq. (4.3) that at some instant are close to P_0 remain close to P_0 at all future times, or in another way if for every disk D_ϵ of radius $\epsilon > 0$ with center P_0 there is a disk D_δ of radius $\delta > 0$ with center P_0 such that every trajectory of Eq. (4.3) that has a point P_1 in D_δ has all its points corresponding to $t \equiv t_1$ in D_ϵ.
P_0 is called **unstable** if P_0 is not stable.
P_0 is called **stable and attractive** if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \to \infty$.

In general term this can be written in a following table.

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$
Stable and attractive	$q < 0$	$q > 0$
Stable	$q \leq 0$	$q > 0$
Unstable	either $q \leq 0$	or $q > 0$

Table 4.2.: Stability criteria for critical points.

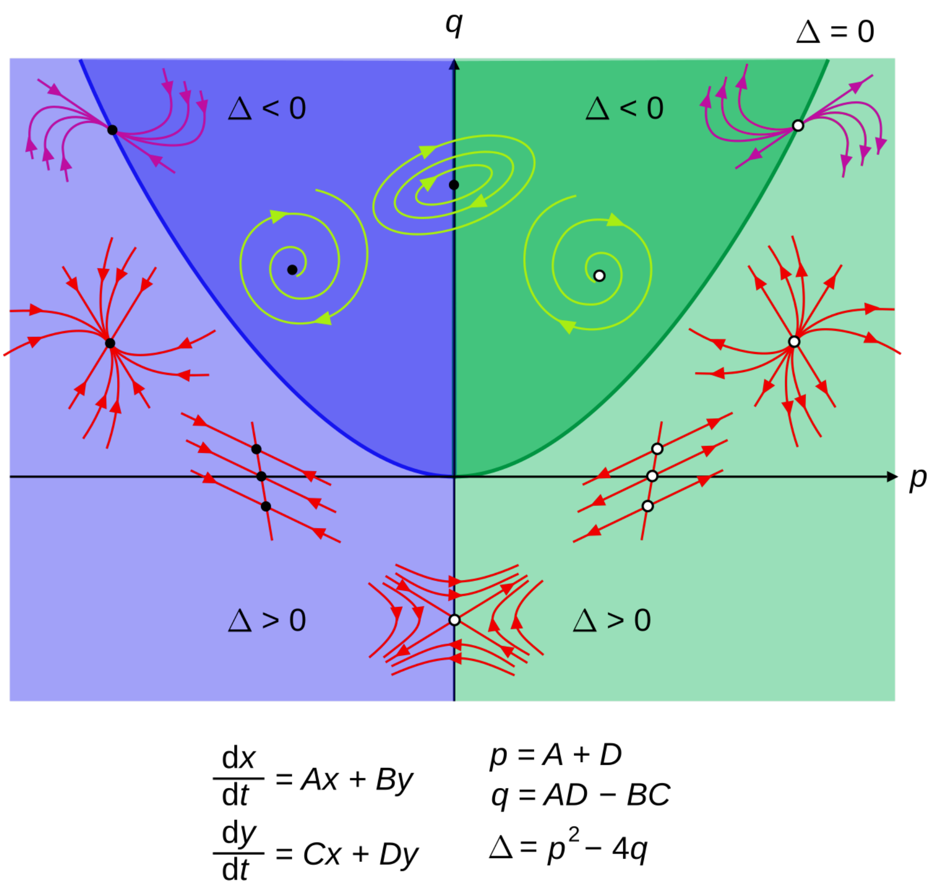

Figure 4.5.: A diagram showing the stability criteria.

Example **Free Motions of a Mass-Spring System** 43

What kind of critical point does the following equation have ?

$$my'' + c'y' + ky = 0$$

Solution Free Motions of a Mass-Spring System

First, division by m gives:

$$y'' = -(k/m)y - (c/m)y'$$

To get a system, set $y_1 = y, y_2 = y'$. Then $y_2' = y'' = -(k/m)y_1 - (c/m)y_2$. Therefore:

$$y' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} y, \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{bmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{bmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

We can see that:

$$p = -c/m, \qquad q = k/m, \qquad \Delta = (c/m)^2 - 4k/m$$

From this we obtain the following results.

Note that in the last three cases the discriminant $\varDelta$ plays an essential role.

No Damping $c = 0,\ p = 0,\ q > 0$ and has a centre.

Under damping $c^2 < 4mk$, $p < 0$, $q > 0$, $\varDelta < 0$ and has a stable and attractive spiral point.

Critical Damping $c^2 = 4mk$, $p < 0$, $q > 0$, $\varDelta = 0$ and has a stable and attractive node.

Overdamping $c^2 > 4mk$, $p < 0$, $q > 0$, $\varDelta > 0$ and has a stable and attractive node.

4.4 Qualitative Methods for Non-Linear Systems

Qualitative methods are methods of obtaining qualitative information on solutions *without actually solving a system*. These methods are particularly valuable for systems whose solution by analytic methods is difficult or impossible.
This is the case for many practically important **non-linear systems**.

$$y' = f(y), \quad \text{therefore} \quad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned} \tag{4.27}$$

Here we will extend the previously discussed phase plane methods, from linear systems to nonlinear systems Eq. (4.27). We assume that Eq. (4.27) is autonomous, that is, the independent variable t does not occur explicitly.

> All examples in the last section are autonomous.

We shall, again exhibit entire families of solutions.

> This is an advantage over numeric methods, which give only one (approximate) solution at a time.

For this analysis we need to employ the previously defined concepts of **phase plane** (the y_1-y_2-plane), **trajectories** (solution curves of Eq. (4.27) in the phase), the **phase portrait** of Eq. (4.27) (the totality of these trajectories), and **critical points** of Eq. (4.27) points (y_1, y_2) at which both $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are zero).
Now Eq. (4.27) may have several critical points. Our approach shall be to discuss one critical point after another. If a critical point P_0 is not at the origin, then, for technical convenience, we shall move this point to the origin before analyzing the point.
More formally, if P_0: (a, b) is a critical point with (a, b) **NOT** at the origin (0, 0), then we apply the translation:

$$\overline{y_1} = y_1 - a, \qquad \overline{y_2} = y_2 - b,$$

which moves P_0 to $(0, 0)$ as desired. Thus we can assume P_0 to be the origin $(0, 0)$, and for simplicity we continue to write y_1, y_2 (instead of $\overline{y}_1, \overline{y}_2$). We also assume that P_0 is **isolated**, that is, it is the only critical point of Eq. (4.27) within a (sufficiently small) disk with center at the origin.

4.4.1 Linearisation of Non-Linear Systems

How to determine the kind and stability of a critical point P_0: $(0, 0)$ of Eq. (4.27)?

In most cases this can be done by linearisation of Eq. (4.27) near P_0, writing Eq. (4.27) as $\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \mathbf{A}\,\mathbf{y} + \mathbf{h}(\mathbf{y})$ and dropping $\mathbf{h}(\mathbf{y})$, as follows.
Since P_0 is critical, $f_1(0,0) = 0$, $f_2(0,0) = 0$, so that f_1 and f_2 have no constant terms and we can write

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} + \mathbf{h}(\mathbf{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2). \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2). \end{aligned} \tag{4.28}$$

A is constant as Eq. (4.27) is autonomous.

Linearisation

If f_1 and f_2 in Eq. (4.27) are continuous and have continuous partial derivatives in a neighborhood of the critical point P_0: $(0, 0)$, and if $\det \mathbf{A} \neq 0$ in Eq. (4.28), then the kind and stability of the critical point of Eq. (4.27) are the same as those of the linearized system*

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{4.29}$$

Exceptions occur if $\mathbf{A}$ has equal or pure imaginary eigenvalues; then Eq. (4.27) may have the same kind of critical point as (3) or a spiral point.

Example Linearisation of a Free Un-damped Pendulum 44

A pendulum consists of a body of mass m (the bob) and a rod of length L. Determine the locations and type of the critical points.

> Assume that the mass of the rod and an reference are negligible.

Solution Linearisation of a Free Un-damped Pendulum

Setting Up the Mathematical Model

Let θ denote the *angular displacement*, measured counterclockwise from the equilibrium position. The weight of the bob is mg, where g is the acceleration of gravity.

This causes a restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's 2nd law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the**acceleration**.
Therefore, the resultant of these two forces is zero, and we obtain as the mathematical model:

$$mL\theta'' + mg \sin\theta = 0.$$

Dividing this by mL, we have:

$$\theta'' + k \sin\theta = 0 \qquad \text{with} \qquad \left(k = \frac{g}{L}\right). \tag{4.30}$$

When θ is very small, we can approximate $\sin\,\theta$ rather accurately by θ and obtain as an approximate solution $A\,\cos\,\sqrt{k}\,+\,B\,\sin\,\sqrt{k}$, but the *exact* solution for any θ is not an **elementary function**.

Critical Points ($\pm 2\pi n,\ 0$) and Linearisation

To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from Eq. (4.30) we obtain a nonlinear system Eq. (4.27) of the form:

$$
\begin{aligned}
y_1' &= f_1(y_1, y_2) = y_2,\\
y_2' &= f_2(y_2, y_1) = -k \sin y_1.
\end{aligned}
$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi,\ 0)$, where $n = 0,\ \pm 1,\ \pm 2,\ \cdots$.

We consider $(0,\ 0)$. Since the Maclaurin series is

$$
\sin y_1 = y_1 - \tfrac{1}{6} y_1^2 + - \cdots \approx y_1,
$$

Maclaurin Series

A Maclaurin series is a Taylor series expansion of a function about 0,

$$
f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots
$$

Maclaurin series are named after the Scottish mathematician *Colin Maclaurin*.

the linearized system at $(0,\ 0)$ is:

$$
\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \mathbf{y} \qquad \text{Therefore} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -k y_1. \end{aligned}
$$

To apply our criteria in Sec. 4.4 we calculate:

$$
\begin{aligned}
p &= a_{11} + a_{22} = 0,\\
q &= \det(\mathbf{A}) = k = g/L \qquad (> 0),\\
\Delta &= p^2 - 4q = -4k.
\end{aligned}
$$

From this and Table 4.1(c) in Sec. 4.4 we conclude that $(0, 0)$ is a **centre**, which is **always stable**. Since $\sin\theta = \sin y_1$ is periodic with period of 2π.

This means the critical points $(n\pi,\ 0)$, $n = \pm 2,\ \pm 4,\ \cdots$, are all centres.

Critical Points ($\pm(2n-1)\pi,\ 0$) and Linearisation

We now consider the critical point $(\pi, 0)$, setting:

$$y_1 = \theta - \pi$$
$$y_2 = (\theta - \pi)'$$

Then in Eq. (4.30), we can apply the MacLaurin series:

$$\sin\ \theta = \sin\ (y_1 + \pi) = -\sin\ y_1 = -y_1 + \frac{1}{2}y_1^2 - + \cdots = -y_1$$

and the linearised system at $(\pi, 0)$ is now

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\mathbf{y} \qquad \text{Thus} \qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= ky_1. \end{aligned}$$

We see that:

$$p = 0,$$
$$q = -k \qquad (< 0)\,,$$
$$\Delta = -4q = 4k.$$

Hence, by Table 4.1(b), this gives a saddle point, which is always unstable.

Because of periodicity, the critical points $(n\pi,\ 0)$, $n = \pm 1,\ \pm 3,\ \cdots$, are all **saddle points**.

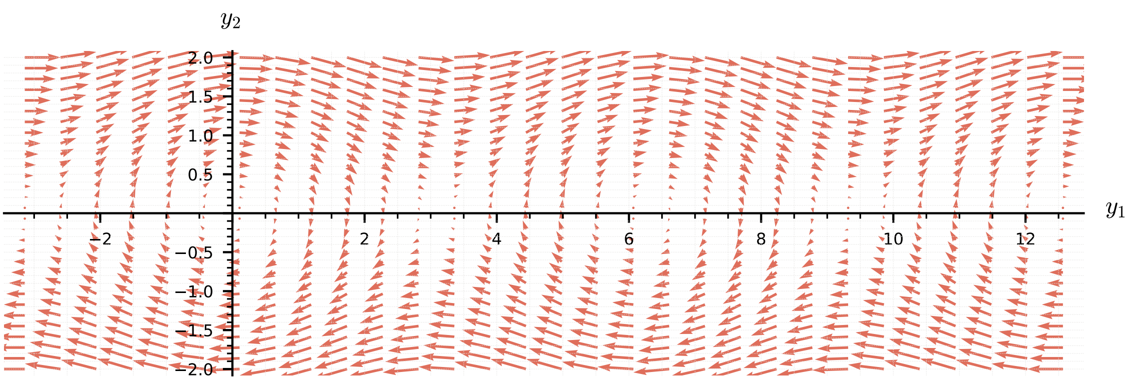

Figure 4.6.

Example Linearisation of a Damped Pendulum 45

To gain further experience in investigating critical points, as another practically Important, let us see how the previous example changes when we add a damping term $c\theta$, (damping proportional to the angular velocity) to equation Eq. (4.30), so that it becomes:

$$\theta'' + c\theta' + k\sin\theta = 0$$

where $k > 0$ and $c \geq 0$ (which includes our previous case of no damping, $c = 0$).

Solution Linearisation of a Damped Pendulum

First we start by setting $\theta = y_1, \theta'$ as before, we obtain the nonlinear system (use $\theta'' = y_2$),

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 - cy_2. \end{aligned}$$

We see the critical points have the same locations as the example before, namely, $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, ... To analyse this system, we start with analysing $(0, 0)$. Linearising $\sin y_1 \approx y_1$ as in the previous example, we get the linearised system at $(0, 0)$.

$$y' = \mathbf{Ay} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\mathbf{y} \quad \text{therefore} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2 \end{aligned}$$

This is identical with the system in previous example, except for the positive factor m (and except for the physical meaning of y_1). Hence for $c = 0$ (no damping) we have a centre, for small damping we have a spiral point, and so on.

We now consider the critical point $(\pi, 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = y_2$ and linearise

$$\sin\theta = \sin(y_1 + \pi) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi, 0)$:

$$\mathbf{y}' = \mathbf{Ay} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix}\mathbf{y}, \quad \text{therefore} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2. \end{aligned}$$

For our criteria, we calculate:

$$\begin{aligned} p &= a_{11} + a_{22} = -c \\ q &= \det\mathbf{A} = -k \\ \Delta &= p^2 - 4q = c^2 + 4k \end{aligned}$$

This gives the following results for the critical point $(\pi, 0)$.

No Damping $c > 0, p = 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 3b.

Damping $c > 0, p < 0, q < 0, \Delta > 0$, a saddle point, Sec. Fig. 94.

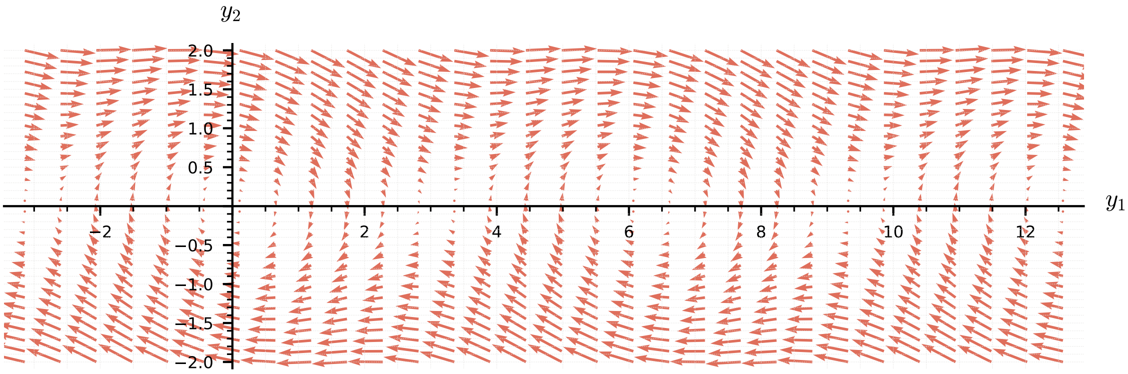

Figure 4.7.: The vector field of a damped medium with $k = 0.5,\ c = 0.2$.

As $\sin y_1$ is periodic with period of 2π, the critical points $(\pm 2\pi,\ 0)$, $(\pm 4\pi,\ 0)$, ... are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished. ■

Model Self-Sustained Oscillations - Van der Pol Equation

There are physical systems such that for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system.

In other words, large oscillations will be damped, whereas for small oscillations there is *negative damping* (feeding of energy into the system). For physical reasons we expect such a system to approach a periodic behaviour, which will thus appear as a closed trajectory in the phase plane, called a **limit cycle**.

An ODE describing such vibrations is the famous **van der Pol equation**.

$$y'' - \mu\left(1 - y^2\right) y' + y = 0$$

It first occurred in the study of electrical circuits containing vacuum tubes.

Vacuum Tube

A vacuum tube, electron tube, valve (British usage), or tube (North America) is a device that controls electric current flow in a high vacuum between electrodes to which an electric potential difference has been applied.

For $\mu = 0$ this equation becomes $y'' + y = 0$ and so with harmonic oscillations. If we define $\mu > 0$, then the damping term has the factor $-\mu\,(1 - y^2)$. This is a consequence for small oscillations, when $y^2 < 1$, so that we have negative damping, is zero for $y^2 = 1$ (no imaginary), and is positive if $y^2 > 1$ (positive damping. Loss of energy).

If μ is small, we expect a limit cycle almost a circle because then our equation differs but finite from $y'' + y =$. If μ is large, the limit cycle will probably look different.
Setting $y = y_1$, $y' = y_2$ and using $y'' = (dy_2/dy_1)y_2$ as in (8), we have from (10)

$$\frac{dy_2}{dy_1}y_2 - \mu(1 - y_1^2)y_2 + y_1 = 0.$$

The isoclines in the y_1y_2-plane (the phase plane) are the curves $dy_2/dy_1 = K =$ const, that is,

$$\frac{dy_2}{dy_1} = \mu(1 - y_1^2) - \frac{y_1}{y_2} = K.$$

Solving algebraically for y_2, we see that the icoclines are given by

$$y_2 = \frac{y_1}{\mu(1 - y_1^2) - K}$$

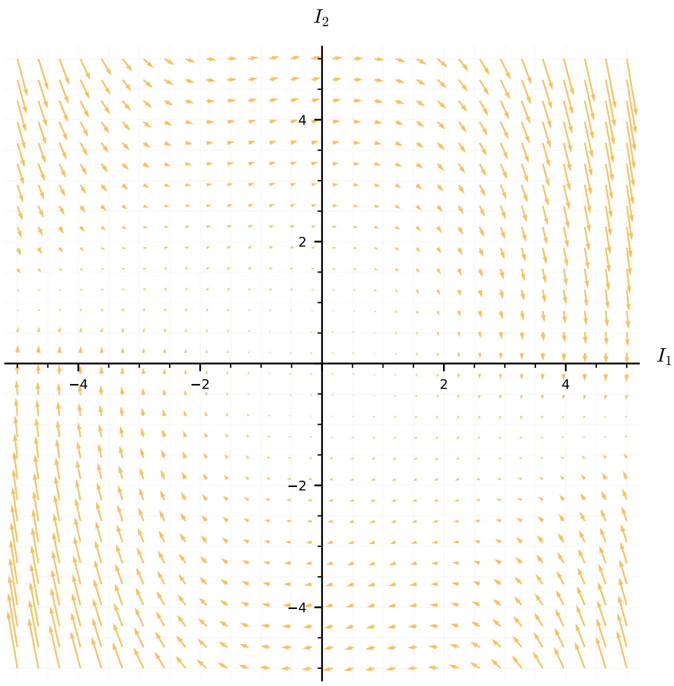

Figure 4.8.: Vector plot of the van der pol equation.

Chapter 5.

Special Functions for ODEs

5.1 Introduction

Linear ODEs with **constant coefficients** can be solved by *algebraic* methods, and their solutions are elementary functions known from calculus.

Elementary Functions

A function of a single variable, (real or complex) defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, and their inverses (e.g., $\arcsin$, $\log$, $x^{1/n}$).

ODEs with **variable coefficients**, however, it is more complicated, and their solutions may be non-elementary functions which means we cant write the solution with explicit functions.
For engineering applications where explicit solutions are not possible, *Legendre's*, *Bessel's*, and the *hypergeometric* equations are important ODEs of this kind.
We will look at the two (2) standard methods for solving ODEs:

Power Series Gives the solution in terms of a power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Frobenius Method Gives the solution in power series (similar to the power series, multiplied by a logarithmic term $\ln x$ or a fractional power x^r.

5.2 Power Series Method

The power series method is the standard method for solving linear ODEs with *variable* coefficients. It gives solutions in the form of **power series**.

> The power series method is used for computing values, graphing curves, proving formulas, and exploring properties of solutions

Remember, the **power series** (in powers of $x - x_0$) is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots . \tag{5.1}$$

Here, x is a variable and $a_0, a_1, a_2, \cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **centre** of the series. For $x_0 = 0$, we obtain a power series in powers of x:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \tag{5.2}$$

For the duration of the chapter we will assume all variables and constants are real.

The term **power series** usually refers to a series of the form Eq. (5.1), but does not include series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the *Legendre* and *Bessel equations* for integer values.

Example Power Series Solution 46

Solve the following ODE:

$$y' - y = 0$$

Solution Power Series Solution

First insert:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by term-wise differentiation:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1} \tag{5.3}$$

We put these values into the ODE:

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

Then we collect like powers of x, finding:

$$(a_1 - a_0) + (2a_2 - a_1)\, x + (3a_3 - a_2)\, x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0 \cdots$$

Solving these equations, express $a_1, a_2, \cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \qquad \cdots \qquad a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = ae^x. \quad \blacksquare$$

We may now generalise this idea. For a given ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{5.4}$$

First represent $p(x)$, $q(x)$ by power series in powers of x.

If $p(x)$, $q(x)$ are polynomials, and then nothing needs to be done in this first step.

Next we assume a solution in the form of a power series with unknown coefficients and insert it as well as Eq. (5.3) and:

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x_2 + \cdots = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} \tag{5.5}$$

into the ODE. Then we **collect same powers of** x and equate the sum of the coefficients of each occurring power of x to zero (0), starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of Eq. (5.3) successively.

Example A Special Legendre Function 47

Solve the following ODE:

$$(1-x^2)\, y'' - 2xy' + 2y = 0$$

These equations usually occur in models with spherical symmetry.

Solution A Special Legendre Function

Substitute Eq. (5.2), Eq. (5.3), and Eq. (5.5) into the ODE, $(1-x^2)\, y''$ gives two (2) series:

- For y'',
- For $-x^2 y''$.

For the term $-2xy'$ use Eq. (5.3) and in $2y$ use Eq. (5.2). Write like powers of x vertically aligned for easy viewing. This gives:

$$\begin{aligned}
y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\
-xy'' &= - 2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots \\
-2xy' &= - 2a_1 x - 4a_5 x^2 - 6a_9 x^3 - 8a_4 x^4 - \cdots \\
2y &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots .
\end{aligned}$$

Add terms of like powers of x. For each power x^0, x, x^2 equate the sum obtained to zero. Denote these sums by [0] (constant terms), [1] (first power of x), and so on:

This gives the solution

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) \quad \blacksquare$$

a_0, a_1 remain **arbitrary**.

Sum	Power	Equation
[0]	x^0	$a_2 = -a_0$
[1]	x	$a_3 = 0$
[2]	x^2	$14a_4 = 4a_2, \qquad a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
[3]	x^3	$a_5 = 0 \qquad \text{since} \qquad a_3 = 0$
[4]	x^4	$30a_6 = 18a_4, \qquad a_6 = \frac{18}{30}a_4 = \frac{18}{30}\left(-\frac{1}{3}\right)a_0 = -\frac{1}{5}a_0$

Table 5.1.: Coefficient table for the example ”*A Special Legendre Function*”.

Therefore, this is a **general solution** consisting of two (2) solutions: x and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$.

These two (2) solutions are members of families of families called *Legendre polynomials* $P_n(\cdot)\,x$ and *Legendre functions* $Q_1(\cdot)\,x$.

Here we have $x = P_1(\cdot)\,x$ and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(\cdot)\,x$

> The minus is by convention.

The index 1 is called the *order* of these functions and here the order is 1.

5.3 Legendre's Equation

5.3.1 Legendre Polynomials ($P_n(x)$)

Legendre's differential equation:

$$(1-x^2)\,y'' - 2xy' + n\,(n+1)\,y = 0 \tag{5.6}$$

is an important ODE in physics. It arises in numerous problems, particularly in boundary value problems for spheres.
The equation involves a **parameter** n, whose value depends on the physical or engineering problem. Therefore Eq. (5.6) is actually a whole family of ODEs. For $n = 1$ we solved it in the previous example.

Any solution of Eq. (5.6) is called a **Legendre function**.

The study of these and other higher functions not occurring in calculus is called the theory of special functions.

Dividing Eq. (5.6) by $1 - x^2$, we obtain the standard form:

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{n\,(n+1)}{(1-x^2)}y = 0$$

We see that the coefficients $-2x/(1-x^2)$ and $n\,(n+1)/(1-x^2)$ of the new equation are analytic at $x = 0$, so the power series method is applicable for this equation.

Substituting:

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{5.7}$$

and its derivatives into Eq. (5.6), and denoting the constant $n\,(n+1)$ simply as k, we obtain the following:

$$(1-x^2)\sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By splitting the first expression as two (2) separate series we have the equation:

$$\sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} - \sum_{m=2}^{\infty} m\,(m-1)\,a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x_n in all four (4) series, set $m-2 = s$ (therefore $m = s+2$) in the first series and simply write s instead of m in the other three series. This gives:

$$\sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s - \sum_{s=2}^{\infty} s(s-1) a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

Note the first series the summation begins with $s = 0$.

As this equation with the right side 0 must be an identity in x if Eq. (5.7) is to be a solution of Eq. (5.7), the sum of the coefficients of each power of x on the must be zero.

Now x^0 occurs in the first and fourth series only, and gives:

remember $k = n(n+1)$

$$x^0 \qquad 2 \cdot 1 a_2 + n(n+1) a_0 = 0, \tag{5.8}$$

$$x^1 \qquad 3 \cdot 2 a_3 + [-2 + n(n+1)] a_1 = 0, \tag{5.9}$$

$$x^2, x_3, \cdots \qquad (s+2)(s+1) a_{s+2} + [-s(s-1) - 2s + n(n+1)] a_s = 0. \tag{5.10}$$

The expression in the brackets $[\cdots]$ can be simplified to $(n-s)(n+s+1)$.
Solving Eq. (5.8) for a_2 and Eq. (5.9) for a_3 as well as Eq. (5.10) for a_{s+2}, we obtain the **general formula**:

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad \text{for} \qquad s = 0, 1, 2, \cdots \tag{5.11}$$

This is called a **recurrence relation** or **recursion formula**. It gives each coefficient in terms of the 2$^{\text{nd}}$ one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively:

$$\begin{aligned}
a_2 &= -\frac{n(n+1)}{2 \cdot 1} a_0 \\
a_3 &= -\frac{(n-1)(n+2)}{3 \cdot 2} a_1 \\
a_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3} a_2 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0 \\
a_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4} a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

By inserting these expressions for the coefficients into Eq. (5.7) we obtain:

$$y(x) = a_0 y_1(x) + a_1 y_2(x). \tag{5.12}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2\cdot 1}x^2 + \frac{(n-2)\,n\,(n+1)\,(n+3)}{4!}x^4 - \cdots \tag{5.13}$$

$$y_2(x) = x - \frac{(n-1)\,(n+2)}{3\cdot 2}x^3 + \frac{(n-3)\,(n-1)\,(n+2)\,(n+4)}{5!}x^5 - \cdots \tag{5.14}$$

These series converge for $|x| < 1$. As Eq. (5.13) contains **even** powers of x only, while Eq. (5.14) contains **odd** powers of x only, the ratio y_1/y_2 is not a **constant**, This means y_1 and y_2 are not proportional and are thus linearly independent solutions.
Therefore Eq. (5.12) is a general solution of Eq. (5.6) on the interval $-1 < x < 1$.

> $x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardised ODE are no longer analytic.

Polynomial Solutions

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions.

For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials. For *Legendre's equation* this happens when the parameter n is a non-negative integer because the of Eq. (5.11) is zero for $s = n$, so that $a_{n+2} = 0,\ a_{n+4} = 0,\ a_{n+6} = 0, \cdots$. Therefore if n is even, $y_1(x)$ reduces to a polynomial of degree n.

If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of such constants is done as follows.

We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n\,(n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \qquad \text{where} \qquad n \text{ is a positive integer.} \tag{5.15}$$

and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from Eq. (5.11), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)\,(s+1)}{(n-s)\,(n+s+1)}a_{s+2} \qquad (s \leq n-2) \tag{5.16}$$

The choice Eq. (5.15) makes $p_n(1) = 1$ for every n which makes our lives easier. From Eq. (5.16) with $s = n-2$ and Eq. (5.15) we obtain:

$$a_{n-2} = -\frac{n\,(n-1)}{2\,(n-1)}a_n = -\frac{n\,(n-1)}{2\,(2n-1)}\cdot\frac{(2n)!}{2^n(n!)^2}$$

Using $(2n)! = 2n\,(2n-1)\,(2n-2)!$ in the numerator and $n! = n\,(n-1)!$ and $n! = n\,(n-1)\,(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n\,(n-1)\,2n\,(2n-1)\,(2n-2)!}{2\,(2n-1)\,2^n n\,(n-1)!n\,(n-1)\,(n-2)!}.$$

$n\,(n-1)\,2n\,(2n-1)$ cancels out, which we get:

$$a_{n-2} = -\frac{(2n-2)!}{2^n\,(n-1)!\,(n-2)!}$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}\,a_{n-2}$$
$$= \frac{(2n-4)!}{2^n 2!\;(n-2)!\;(n-4)!}$$

and so on, and in general, when $n - 2m \geq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\;(n-m)!\;(n-2m)!}. \tag{5.17}$$

The resulting solution of Legendre's differential equation Eq. (5.6) is called the *Legendre*

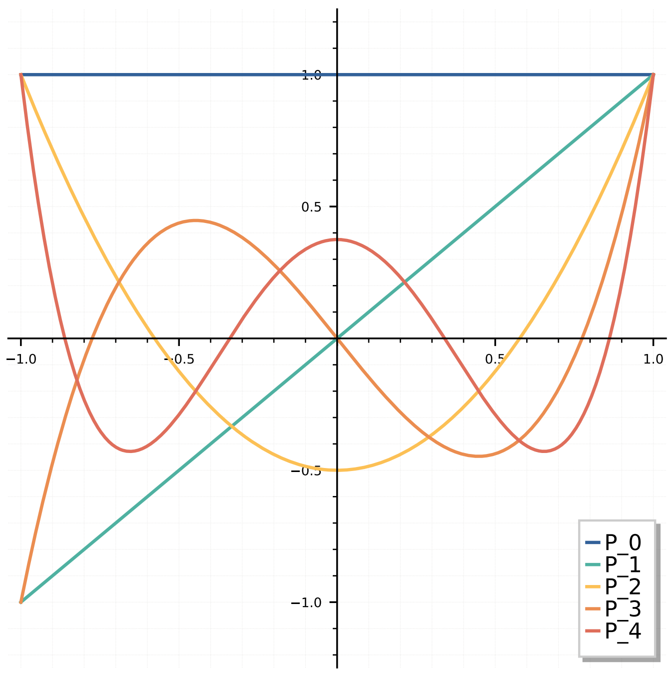

Figure 5.1.

polynomial of degree n and is denoted by $P_n(x)$.:

From Eq. (5.17) we obtain:

$$P_n(x) = \sum_{m-0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \tag{5.18}$$

$$= \frac{(2m)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots \tag{5.19}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x \\ &P_2(x) = \tfrac{1}{2}(3x^2 - 1), && P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \\ &P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

The Legendre polynomials $P_n(x)$ are o**rthogonal** on the interval $-1 \leqq x \leqq 1$, a basic property to be defined and used in making up ”Fourier-Legendre series” which will be the focus for *Higher Mathematics II*.

5.4 Extended Power Series: Frobenius Method

Several 2nd-order ODEs are important for engineering applications.

> One of the famous ones **Bessel Equation** will be our focus in the continuing section.

Unfortunately, these practical 2nd-order s have coefficients that are not analytic, but are possible to solve via series method (power series times a logarithm or times a fractional power of x, etc.).

The following theorem permits an extension of the power series method.

The new method is called the **Frobenius method**.

Theorem: *Frobenius Method*

Let $b(x)$ and $c(x)$ be any functions defined **analytic** at $x = 0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{5.20}$$

has **at least one solution** that can be represented in the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{5.21}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE Eq. (5.20) also has a 2nd solution (such that these two solutions are linearly independent) that may be similar to Eq. (5.21) (with a different r and different coefficients) or may contain a logarithmic term.

To see this theorem in action, let's look at the Bessel's equation.

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad \text{where } \nu \text{ is a parameter}$$

is of the form Eq. (5.20) with:

$$b(x) = 1 \qquad c(x) = x^2 - \nu^2 \qquad \text{analytic at } x = 0 \qquad .$$

> This form allows us to use the Frobenius method.

This ODE could **NOT** be handled in full generality by the power series method as these functions are known as hyper-geometric differential equations. Therefore, this equation (also known as hypergeometric differential equation) requires the Frobenius method.

> In Eq. (5.21) we have a power series times a single power of x whose exponent r is not restricted to be a non-negative integer.

Regular and Singular Points

The following terms are practical and commonly used. A **regular point** of the ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a **regular point** of the ODE:

$$\widetilde{h}(x)y'' + \widetilde{p}(x)y'(x) + \widetilde{q}(x)y = 0,$$

is an x_0 at which $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$ are analytic and $\widetilde{h}(x_0) \neq 0$ (so what we can divide by $\widetilde{h}$ and get the previous standard form). Then the power series method can be applied. If x_0 is not a regular point, it is called a **singular point**.

5.4.1 Indicial Equation

Time to explain the *Frobenius method* for solving Eq. (5.20) which is the Bessel equation. Multiplication of Eq. (5.20) by x^2 gives the more convenient form which can be worked upon:

$$x^2 y'' + x b(x)\, y' + c(x)\, y = 0 \tag{5.22}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \qquad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

> If both $b(x)$ and $c(x)$ are polynomials, no actions are needed.

Then we differentiate Eq. (5.21) term by term, finding:

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \\ &= x^{r-1}[r a_0 + (r+1) a_1 x + \cdots] \\ y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} \\ &= x^{r-2}[r(r-1) a_0 + (r+1) r a_1 x + \cdots]. \end{aligned} \tag{5.23}$$

By inserting all these series into Eq. (5.22) we obtain:

$$\begin{aligned} x^{r} &+ (b_0 + b_1 x + \cdots)x^{\tau}(r a_0 + \cdots) \\ &+ (c_0 + c_1 x + \cdots)x^{\tau}(a_0 + a_1 x + \cdots) = 0. \end{aligned} \tag{5.24}$$

We now equate the sum of the coefficients of each power x^{r}, x^{r+1}, x^{r+2}, $\ldots$ to zero. This presents a system of equations involving the unknown coefficients a_m. The smallest power is x^{r} and the corresponding equation is:

$$[r(r-1) + b_0 r + c_0]a_0 = 0$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives:

$$r(r-1) + b_0 r + c_0 = 0 \tag{5.25}$$

This important quadratic equation is called the **indical equation** of the ODE Eq. (5.22).

Its role is as follows.

The Frobenius method present a basis of solutions. One of the two solutions will always be of the form Eq. (5.23), where r is a root of Eq. (5.25). The other solution will be of a form indicated by the indicial equation.

There are three (3) cases:

Case 1 Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2 A double root.

Case 3 Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are related to the *Euler-Cauchy equation*, the simplest ODE of the form Eq. (5.20).

Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i\mathrm{Im}\, r_1$ is imaginary, so it cannot be a real integer.

> Case 2 we must have a logarithm, whereas in Case 3 we may or may not.

Theorem: *Frobenius Method II - The Three Cases*

Assume the ODE in Eq. (5.22) satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation Eq. (5.25).

Then we have the following three (3) cases:

Case 1. Distinct Roots Not Differing by an Integer
A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.26}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.27}$$

with coefficients obtained successively from Eq. (5.24) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$.
A basis is

$$y_1(x) = x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{5.28}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^{r}(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0) \tag{5.29}$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5.30}$$

(of the same general form as before) and

$$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{5.31}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

5.4.2 Typical Applications

Technically, the *Frobenius method* is similar to the power series method, once the roots of the indicial equation have been determined.

However, Eq. (5.26)-Eq. (5.31) merely indicate the general form of a basis, and a 2nd solution can often be obtained more rapidly by reduction of order.

Example **Euler-Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm** 48

Solve the following ODE:

$$x^2y'' + b_0xy' + c_0y = 0 \qquad (b_0,\ c_0 \text{ constant})$$

Solution **Euler-Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm**

Substitution of $y = x^r$ gives the auxiliary equation:

$$r(r-1) + b_0r + c_0 = 0,$$

which is the indicial equation. For different roots $r_1,\ r_2$ we get a basis $y_1 = x^{r_1},\ y_2 = x^{r_2}$, and for a double root r we get a basis $x^r,\ x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

Example **Example of Case II - Double Root** 49

Solve the ODE

$$x(x-1)y'' + (3x-1)y'' + y = 0 \tag{5.32}$$

Solution **Example of Case II - Double Root**

Writing Eq. (5.32) in the standard form Eq. (5.22):

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

we see it satisfies the assumptions in **Theorem 1** (i.e., analytic as $x = 0$). By inserting Eq. (5.23) and its derivatives Eq. (5.23) into Eq. (5.32) we obtain:

$$\begin{aligned} &\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r-1} \\ &+3\sum_{m=0}^{\infty}(m+r)a_mx^{m+r} - \sum_{m=0}^{\infty}(m+r)a_mx^{m+r-1} + \sum_{m=0}^{\infty}a_mx^{m+r} = 0 \end{aligned} \tag{5.33}$$

The smallest power is x^{r-1}, occurring in the 2$^{\text{nd}}$ and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]\,a_0 = 0, \qquad \text{therefore} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.
First Solution
Insert this value $r = 0$ into Eq. (5.33) and equate the sum of the coefficients of the power x^s to zero, obtaining:

$$s(s-1)a_s - (s+1)a_{s+1} + 3a_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

Second Solution

We get a 2nd independent solution y_2 by the method of reduction of order, substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9). Sec. 2.1, which we shall use in this example, instead of stating reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of y' in (11) $_instandardform. Bypartialfractions,$

$$-\int p\,dx = -\int \frac{3x-1}{3(x-1)}\,dx = -\int \left(\frac{2}{x-1} + \frac{1}{x}\right) dx = -2\ln(x-1) - \ln x.$$

Hence (9), Sec. 2.1, becomes

$$u' = U = y_1^{-2} e^{-\int y\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = w_1 = \frac{\ln x}{1-x}.$$

y_1 and y_2 are shown in Fig. 109. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$).

The Frobenius method solves the **hypergeometric equation**, whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

5.5 Bessel's Function

One of the most important ODEs in applied mathematics in **Bessel's equation** which it's form is shown as:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \tag{5.34}$$

Converting this to the traditional *Frobenius* form:

$$y'' + \frac{1}{x} y' + \frac{1 - \nu^2}{x^2} y = 0$$

$$b(x) = 1 \qquad c(x) = 1 - \nu^2.$$

where the parameter ν is a given **real number** which is positive or zero.

> Bessel's equation often in problems showing cylindrical symmetry or membranes.

According to the *Frobenius theory*, it has a solution of the form:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{5.35}$$

Substituting Eq. (5.35) and its 1st and 2nd derivatives into Bessel's equation, we obtain:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - v^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero.

> Note that this power x^{s+r} corresponds to $m = s$ in the first, 2nd, and fourth series, and to $m = s - 2$ in the third series.

Therefore, for $s = 0$ and $s = 1$, the third series does not contribute since $m \geq 0$. For $s = 2, 3, \cdots$ all four series contribute, so that we get a general formula for all these s. We find:

$$r(r-1)a_0 + ra_0 - v^2 a_0 = 0 (s = 0) \tag{5.36}$$

$$(r+1)ra_1 + (r+1)a_1 - v^2 a_1 = 0 (s = 1) \tag{5.37}$$

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - v^2 a_s = 0 (s = 2, 3, \cdots) \tag{5.38}$$

From Eq. (5.36) we obtain the **indicial equation** by dropping a_0.

$$(r + \nu)(r - \nu) = 0 \tag{5.39}$$

The roots are $r_1 = \nu\,(\geq 0)$ and $r_2 = -\nu$.

Coefficient Recursion for $r = r_1 = \nu$

For $r = \nu$, Eq. (5.37) reduces to $(2\nu + 1)\, a_1 = 0$. Therefore $a_1 = 0$ as $\nu \geq 0$. Substituting $r = \nu$ in Eq. (5.38) and combining the three terms containing $a_s = 0$ gives simply:

$$(s + 2\nu)\, s a_s + a_{s-2} = 0 \tag{5.40}$$

As $a_1 = 0$ and $v \equiv 0$, it follows from Eq. (5.40), $a_3 = 0, a_5 = 0, \cdots$. Hence we have to deal only with even-numbered coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (5.40) becomes:

$$(2m + 2\nu)\, 2m a_{2m} + a_{2m-2} = 0$$

Solving for a_{2m} gives the recursion formula

$$a_{2m} = -\frac{1}{2^2 m\,(v + m)} a_{2m-2} \qquad m = 1,\, 2,\, \cdots \tag{5.41}$$

From Eq. (5.41) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(v\, +\, 1)}$$
$$a_4 = -\frac{a_2}{2^2 2(v\, +\, 2)} = \frac{a_0}{2^4 2!\,(v\, +\, 1)(v\, +\, 2)}$$

and so on, and in general:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v\, +\, 1)(v\, +\, 2)\cdots(v\, +\, m)}, \qquad m = 1, 2, \cdots . \tag{5.42}$$

5.5.1 Bessel Functions (J_n) for Integers

> Integer values of ν are denoted by n, which is the standard mathematical notation.

For $\nu = n$ the relation Eq. (5.42) becomes:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n\, +\, 1)(n\, +\, 2)\cdots(n\, +\, m)}, \qquad m = 1,\, 2, \cdots \tag{5.43}$$

a_0 is still arbitrary, so that the series Eq. (5.35) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function.

Accordingly, we have to make a choice.

The choice $a_0 = 1$ would be possible. A simpler series Eq. (5.35) could be obtained if we could absorb the growing product $(n+1)(n+2)\cdots(n+m)$ into a factorial function $(n+m)!$ What should be our choice? Our choice should be:

$$a_0 = \frac{1}{2^n n!} \tag{5.44}$$

because then $n!\,(n+1)\cdots(n+m) = (n+m)!$ in Eq. (5.43), so that Eq. (5.43) simply becomes:

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n+m)!},\ m = 1, 2, \cdots \tag{5.45}$$

By inserting these coefficients into Eq. (5.35) and remembering that $c_1 = 0$, $c_3 = 0$, $\cdots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} \qquad (n \geqq 0). \tag{5.46}$$

$J_n(x)$ is called the **Bessel function of the first kind** of order n. The series Eq. (5.46) converges for all x, as the ratio test shows.

> $J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

Example Bessel Function J_0 and J_1 50

Please calculate the bessel functions of $J_0(x)$ and $J_1(x)$.

Solution Bessel Function J_0 and J_1

For $n = 0$ we obtain from Eq. (5.46) the *Bessel function* of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} - \dots \tag{5.47}$$

which looks similar to a cosine. For $n = 1$ we obtain in the *Bessel function* of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2} + \frac{x^5}{2^5 2!3} - \frac{x^7}{2^7 3!4} - \dots \tag{5.48}$$

which looks similar to a sine (Fig. 110). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing z. Heuristically, n^2/x^2 in (1) in standard form (1) divided by x^2] is zero (if $n = 0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y' + y = 0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

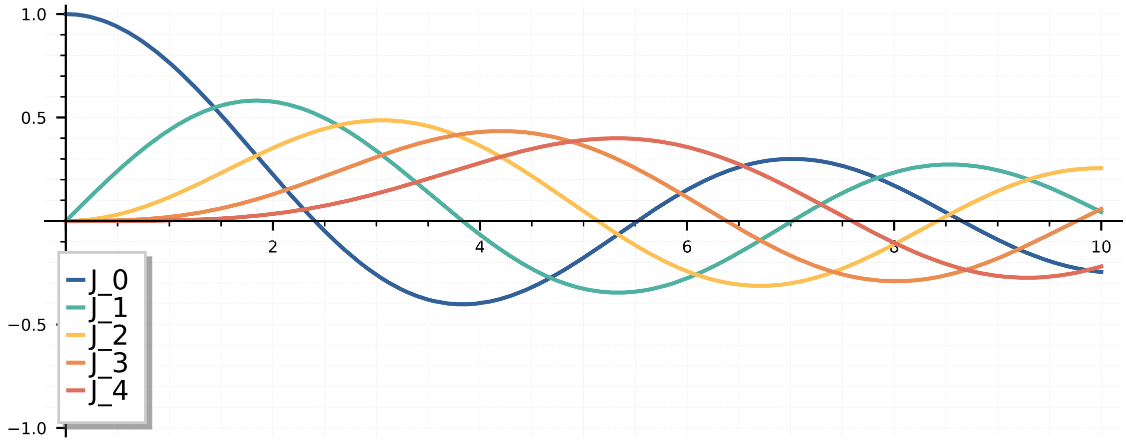

Figure 5.2.: Bessel functions of the first kind.

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

where $\sim$ is read "asymptotically equal" and means that for fixed n the quotient of the two sides approaches 1 as $x \to \infty$ $\frac{x^2}{2^2(1!)^2}$.

Formula (14) is surprisingly accurate even for smaller x (>0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of f_0, you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

Chapter 6.

Laplace Transform

6.1 Introduction

Laplace transform are important for any engineer as it makes solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. There are numerous applications such as:

- electrical networks,
- springs,
- mixing problems,
- signal processing,

and other areas of engineering and physics. The process of solving an ODE using Laplace transform consists of three (3) steps:

Part 1 The given ODE is transformed into an algebraic equation, called the **subsidiary equation**.

Part 2 The subsidiary equation is solved by purely algebraic manipulations.

Part 3 The solution in Step 2 is transformed back, resulting in the solution of the given problem.

> The idea of Laplace transforms converting an ODE to an algebraic problem.

Laplace Transform has two (2) major advantages over the previous methods.

1. **Problems are solved more directly:** Initial value problems are solved without first determining a general solution. Nonhomogeneous ODEs are solved without first solving the corresponding homogeneous ODE.
2. **Solving Discontinuities:** More importantly, the use of the unit step function (**Heaviside** function) and Dirac's **delta** make the method particularly powerful for problems with inputs with discontinuities or represent short impulses or complicated periodic functions.

6.2 First Shifting Theorem (s-Shifting)

Laplace transform, when applied to a function, **changes the function into a new function** by using a process involving integration.

If $f(t)$ is a function defined for all $t \geq 0$, its **Laplace transform** is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, (i.e., $F(s)$), and is denoted by $\mathcal{L}(f)$:

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \tag{6.1}$$

> Here we need to assume $f(t)$ **has** an integral (i.e., it is finite). This assumption is usually satisfied in **practical** engineering applications.

Not only is the result $F(s)$ called the **Laplace transform**, but the operation just described, which gives $F(s)$ from a given $f(t)$, is also called the **Laplace transform**. It is an integral transform:

$$F(s) = \int_0^\infty k(s, t) f(t)\, \mathrm{d}t$$

with a kernel defined as $k(s, t) = e^{-st}$:

> Laplace transform is called an integral transform because it transforms a function in one space to a function in another space by a process of integration which involves a **kernel**.

The kernel is a function of the variables in two (2) spaces and defines the transform. Furthermore, the given function $f(t)$ in Eq. (6.1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$:

$$f(t) = \mathcal{L}^{-1}(F)\,. \tag{6.2}$$

Note that Eq. (6.1) and Eq. (6.2) together imply:

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f, \quad \mathcal{L}(\mathcal{L}^{-1}(F)) = F$$

Notation

Original functions depend on t and their transforms on s. Original functions are denoted by **lowercase letters** and their transforms by the same letters in **capital**, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

Example Introduction to Laplace Transform 51

Let $f(t) = 1$ when $t \geq 0$.

Find $F(s)$.

Solution Introduction to Laplace Transform

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad (s > 0)$$

Such an integral is called an **improper integral** and, is evaluated according to the rule:

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt.$$

Therefore our convention notation means:

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0}\right] = \frac{1}{s} \qquad (s > 0) \quad \blacksquare$$

Example Laplace Transform of an Exponential Function 52

Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant.

Find $\mathcal{L}(f)$.

Solution Laplace Transform of an Exponential Function

From Eq. (6.1) we obtain by integration:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{a-s} e^{-(s-a)t}\Big|_0^\infty$$

Therefore, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad \blacksquare.$$

Linearity

The Laplace transform is a **linear operation**. This means, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, The following statement holds true:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Example Hyperbolic Functions 53

Find the transforms of:

$$\cosh at \qquad \text{and} \qquad \sinh at$$

Solution Hyperbolic Functions

As $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we can obtain the definitions of them using the exponential function definition from an earlier example.

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \quad \blacksquare$$

Example Cosine and Sine 54

Derive the following formulas:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \qquad \text{and} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution Cosine and Sine

We write $L_e = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain:

$$L_e = \int_0^\infty e^{-st} \cos \omega t \; \mathrm{d}t = \frac{e^{-st}}{-s} \cos \omega t \bigg|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \; \mathrm{d}t = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \; \mathrm{d}t = \frac{e^{-st}}{-s} \sin \omega t \bigg|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \; \mathrm{d}t = \frac{\omega}{s} L_c.$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain:

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2}.$$

6.2.1 Replacing s by s - a in the Transform

The Laplace transform has the very useful property that, if we know the transform of $f(t)$, we can immediately get that of $e^{at} f(t)$, as follows.

***Theorem:** s-Shifting*

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s-a)$ (where $s - a > k$).

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\}.$$

Example Damped Vibrations 55

Using the below definitions:

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2} \qquad \text{and} \qquad \mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

Find the inverse of the transform:

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}.$$

Solution Damped Vibrations

Applying the inverse transform, and using its linearity, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t) \quad \blacksquare$$

Existence and Uniqueness of Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies:

$$|f(t)| \leq Me^{kt}$$

For all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s \geqq k$.

6.3 Transform and Derivatives and Integrals

> The Laplace transform is a method of solving ODEs and Initial Value Problem (IVP)s.

The idea is to replace operations of calculus on functions by operations of algebra. Roughly, differentiation of $f(t)$ will correspond to multiplication of $\mathcal{L}(f)$ by s and integration of $f(t)$ to division of $\mathcal{L}(f)$ by s.

To solve ODEs, we must first consider the Laplace transform of derivatives. You might have encountered this idea previously in logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms.

> To simplify calculations was one of the main reasons that logarithms were invented.

Theorem: *Derivatives*

First and Second Order Derivatives
The transforms of the first and second derivatives of $f(t)$ satisfy:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

These hold if $f(t)$ is continuous for all $t \equiv 0$ and satisfies the growth restriction and $f'(t)$, $f''(t)$ are piece-wise continuous on every finite interval on the semi-axis $t \equiv 0$.

Higher Order Derivatives
Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}(f^{(n)}) = s^{\text{n}}\mathcal{L}(f) - s^{\text{n-1}}f(0) - s^{\text{n-2}}f'(0) - \cdots - f^{(n-1)}(0).$$

6.3.1 Laplace Transform a Function Integral

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

Theorem: *Laplace Transform of an Integral*

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s} F(s), \qquad \text{therefore} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\}.$$

Example Inverse using Integration 56

Find the inverse of the following functions

$$\frac{1}{s(s^2+\omega^2)} \qquad \text{and} \qquad \frac{1}{s^2(s^2+\omega^2)}$$

Solution Inverse using Integration

Using a standard Laplace Transform table we obtain the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^1 \frac{\sin \omega t}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

The second one we obtain as the following:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega \tau) d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin \omega \tau}{\omega^2}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin \omega \tau}{\omega^3}.$$

6.3.2 Differential Equations with Initial Values

It's time to discuss how the Laplace Transform method solves ODEs and IVPs. Consider the following IVP:

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are **constants**. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained.

In Laplace's method we do three (3) steps:

Step 1 Setting up the subsidiary equation This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming, namely:

$$[s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2 Solution of the subsidiary equation by algebra We divide by $s^2 + as + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

Q is often denoted by H, but we need H much more frequently for other purposes.

This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$. Therefore

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Q depends neither on $r(t)$ nor on the initial conditions (but only on a and b).

Step 3 Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$ We reduce our $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables so that we obtain the solution $y(t)$.

Example IVP - A Basic Laplace Transform 57

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution IVP - A Basic Laplace Transform

Step 1.: Using a Standard Laplace Transform we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and becomes

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}.$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t \quad \blacksquare$$

Example Comparison with Previous Methods 58

Solve the IVP:

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Solution Comparison with Previous Methods

We see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{therefore} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2+s+9} = \frac{0.16(s+\frac{1}{2}) + 0.08}{(s+\frac{1}{2})^2 + \frac{35}{4}}.$$

Therefore by the first shifting theorem and the formulas for cos and sin in from a Laplace Transform table we obtain:

$$\begin{aligned} y(t) = L^{-1}(Y) &= e^{-t/2}\left(0.16\cos\sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\sqrt{\frac{35}{4}}t\right) \\ &= e^{-0.8t}(0.16\cos 2.96t + 0.027\sin 2.96t) \quad \blacksquare \end{aligned}$$

Example Shifted Data 59

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$.

For such a problem, set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied.

For instance, solve

$$y'' + y = 2t, \qquad y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution Shifted Data

We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem becomes:

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \frac{1}{4}\pi), \qquad \tilde{y}(0) = \frac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using a standard Laplace Transform table and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the **shifted** initial value problem is

$$s^2\tilde{Y} - s\cdot\frac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{therefore} \qquad (s^2+1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2+1)s^2} + \frac{\frac{1}{2}\pi}{(s^2+1)s} + \frac{\frac{1}{2}\pi s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}.$$

The inverse of the first two terms can be seen from a previous example (with $\omega = 1$), and the last two terms give cos and sin,

$$\widetilde{y} = \mathcal{L}^{-1}(\widetilde{Y}) = 2(\widetilde{t} - \sin\widetilde{t}) + \frac{1}{2}\pi(1 - \cos\widetilde{t}) + \frac{1}{2}\pi\cos\widetilde{t} + (2 - \sqrt{2})\sin\widetilde{t}$$

$$= 2\widetilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\widetilde{t}.$$

Now $\widetilde{t} = t - \frac{1}{4}\pi$, $\sin\widetilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t \quad \blacksquare$$

Chapter 7.

Appendix

7.1 List of Common Integration Operations

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

$$\int \frac{1}{x}dx = \ln|x|$$

$$\int udv = uv - \int vdu$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, \ a \neq b$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x}$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x}+\sqrt{x+a}\right]$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b}$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|\right]$$

$$\int \sqrt{x^3(ax+b)}dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|$$

$$\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2 \pm a^2}\right|$$

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\int x\sqrt{x^2 \pm a^2}dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x+\sqrt{x^2 \pm a^2}\right|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}}dx = -\sqrt{a^2 - x^2}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2\ln\left|x+\sqrt{x^2 \pm a^2}\right|$$

$$\int \sqrt{ax^2+bx+c}dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx^+c)}\right|$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\Big(2\sqrt{a}\sqrt{ax^2+bx+c} \times \left(-3b^2+2abx+8a(c+ax^2)\right) +3(b^3-4abc)\ln\left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c}\right|\Big)$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a}\right) \ln(ax+b) - x, a \neq 0$$

$$\int \ln(x^2+a^2)\ \mathrm{dx} = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x$$

$$\int \ln(x^2-a^2)\ \mathrm{dx} = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x$$

$$\int \ln\left(ax^2+bx+c\right) dx = \frac{1}{a}\sqrt{4ac-b^2} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x\right) \ln\left(ax^2+bx+c\right)$$

$$\int x \ln(ax+b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax+b)$$

$$\int x \ln\left(a^2 - b^2x^2\right) dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right) \ln\left(a^2 - b^2x^2\right)$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a}\sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}\left(i\sqrt{ax}\right), \quad \text{where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int x e^x dx = (x-1)e^x$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^x dx = \left(x^2 - 2x + 2\right) e^x$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$$

$$\int x^3 e^x dx = \left(x^3 - 3x^2 + 6x - 6\right) e^x$$

$$\int x^n e^{ax}\ \mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\ \mathrm{d}x$$

$$\int x^n e^{ax}\ \mathrm{d}x = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\ \mathrm{d}t$$

$$\int e^{ax^2}\ \mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(ix\sqrt{a}\right)$$

$$\int e^{-ax^2}\ \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(x\sqrt{a}\right)$$

$$\int x e^{-ax^2}\ \mathrm{d}x = -\frac{1}{2a} e^{-ax^2}$$

$$\int x^2 e^{-ax^2}\ \mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right]$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right]$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \tan ax dx = -\frac{1}{a}\ln\cos ax$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right)$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax$$

$$\int \sec^3 x \, \mathrm{dx} = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, n \neq 0$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x|$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, n \neq 0$$

$$\int \sec x \csc x dx = \ln|\tan x|$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos x dx = 2x \cos x + \left(x^2 - 2\right) \sin x$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$$

$$\int x^n cosx dx = -\frac{1}{2}(i)^{n+1} \left[\Gamma(n+1, -ix) \right.$$
$$\left. +(-1)^n \Gamma(n+1, ix)\right]$$

$$\int x^n cosax dx = \frac{1}{2}(ia)^{1-n} \left[(-1)^n \Gamma(n+1, -iax) \right.$$
$$\left. -\Gamma(n+1, ixa)\right]$$

$$\int x \sin x dx = -x \cos x + \sin x$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2 \sin x dx = \left(2 - x^2\right) \cos x + 2x \sin x$$

$$\int x^2 \sin ax dx = \frac{2 - a^2x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2}$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right]$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x)$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a} \sinh ax$$

$$\int e^{ax} \cosh bx dx =$$
$$\begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases}$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax$$

$$\int e^{ax} \sinh bx dx =$$
$$\begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases}$$

$$\int e^{ax} \tanh bx dx =$$
$$\begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx}\right] & \\ \qquad -\frac{1}{a} e^{ax} {}_2F_1 \left[\frac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} & a = b \end{cases}$$

$$\int \tanh ax \, dx = \frac{1}{a} \ln \cosh ax$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[a \sin ax \cosh bx \right.$$
$$\left. + b \cos ax \sinh bx\right]$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cos ax \cosh bx + \right.$$
$$\left. a \sin ax \sinh bx\right]$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[-a \cos ax \cosh bx + \right.$$
$$\left. b \sin ax \sinh bx\right]$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cosh bx \sin ax - a \cos ax \sinh bx \right]$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} \left[-2ax + \sinh 2ax \right]$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} \left[b \cosh bx \sinh ax - a \cosh ax \sinh bx \right]$$

7.2 Common Laplace Transforms

$f(t)$	$\mathcal{L}f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \dots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\dfrac{ae^{at} - be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at} \sin kt$	$\dfrac{k}{(s-a)^2 + k^2}$
$e^{at} \cos kt$	$\dfrac{s-a}{(s-a)^2 + k^2}$
$e^{at} \sinh kt$	$\dfrac{k}{(s-a)^2 - k^2}$
$e^{at} \cosh kt$	$\dfrac{s-a}{(s-a)^2 - k^2}$
$t \sin kt$	$\dfrac{2ks}{(s^2 + k^2)^2}$
$t \cos kt$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$
$t \sinh kt$	$\dfrac{2ks}{(s^2 - k^2)^2}$
$t \cosh kt$	$\dfrac{s^2 - k^2}{(s^2 - k^2)^2}$
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$

Glossary

IVP Initial Value Problem. 3, 113, 114, 116

ODE Ordinary Differential Equation. 3, 5, 7, 8, 84, 87, 89–108, 113, 114

PDE Partial Differential Equation. 3, 8