Lecture Book
M.Sc Higher Mathematics I

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Contents

Part I.

Ordinary Differential Equations

Chapter 1.

First-Order Ordinary Differential Equations

1.1 Introduction to Modelling

To solve an engineering problem, we first need to formulate the problem as a *mathematical expression* in terms of: variables, functions, equations. Such an expression is known as a mathematical **model** of the problem.

> The process of setting up a model, solving it mathematically, and interpreting results in physical or other terms is called **mathematical modeling**.

Properties which are common, such as velocity (v) and acceleration (a), are derivatives and a model is an equation usually containing derivatives of an unknown function.

Such a model is called a **differential equation**.

Of course, we then want to find a solution which:

- satisfies our equation,
- explore its properties,
- graph our equation,
- find new values,
- interpret result in a physical terms.

This is all done to understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter. An Ordinary Differential Equation (ODE) is an equation containing one or several derivatives of an unknown function, usually $y(x)$. The equation may also contain y itself, known functions of x, and constants. For example all the equation shown below are classified as ODE.

$$\begin{aligned} y' &= \cos x \\ y'' + 9y &= e^{-2x} \\ y'y''' - \tfrac{3}{2}y'^2 &= 0. \end{aligned}$$

Here, y' means dy/dx, $y'' = d^2y/dx^2$ and so on. The term ordinary distinguishes from partial differential equations (PDEs), which involve **partial** derivatives of an unknown function of two or more variables.

> The topic of Ordinary Differential Equation (ODE) will be the focus of **Higher Mathematics II**.

For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

An ODE is said to be **order**-n if the n$^{\text{th}}$ derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, ...

In this part of the chapter, we shall consider **First-Order-ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Therefore we can write them as:

$$F(x, y, y') = 0 \tag{1.1}$$

or often in the form

$$y' = f(x, y).$$

This is called the explicit form, in contrast to the implicit form presented in Eq. (1.1). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

1.1.1 Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c.

> Geometrically, this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE, together with an initial condition, is called an **initial value problem**.

Initial Value Problem

In multivariable calculus, an initial value problem (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain.

Therefore, if the ODE is **explicit**, $y' = f(x, y)$, the initial value problem is of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.2}$$

Example Initial Value Problem - A 1

Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7$$

Solution Initial Value Problem - B

The general solution is:

$$y(x) = ce^{3x}$$

From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

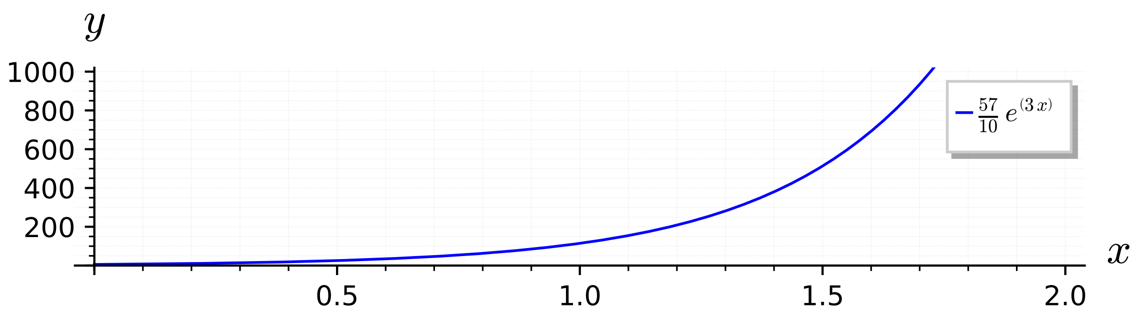

Figure 1.1.: Solution to the exercise "*Initial Value Problem -A*"

Example Radioactive Decay 2

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

> The decay of Radium is measured to be $k = 1.4 \cdot 10^{-11}\,\mathrm{sec}^{-1}$.

Solution Radioactive Decay

Setting Up a Mathematical Model

$y(t)$ is the amount of substance still present at t. By the physical law of decay, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives is the following:

$$\frac{dy}{dt} = -ky \tag{1.3}$$

where the constant k is positive, so that, because of the minus, we get *decay*. The value of k is known from experiments for various radioactive substances which the question has given as $k = 1.4 \cdot 10^{-11}\,\text{sec}^{-1}$. Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

We have the initial condition $y(0) = 0.5$. This is the instant at which our observation of the process begins. In motivates the original condition which however, is also used when the independent variable is not time or when we choose a t other than $t = 0$. Hence the mathematical model of the physical process is the initial value problem.

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5$$

Mathematical Solution

We conclude the ODE is an exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(0) = ce^{-kt}.$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is:

$$y(t) = 0.5e^{-kt} \quad \blacksquare$$

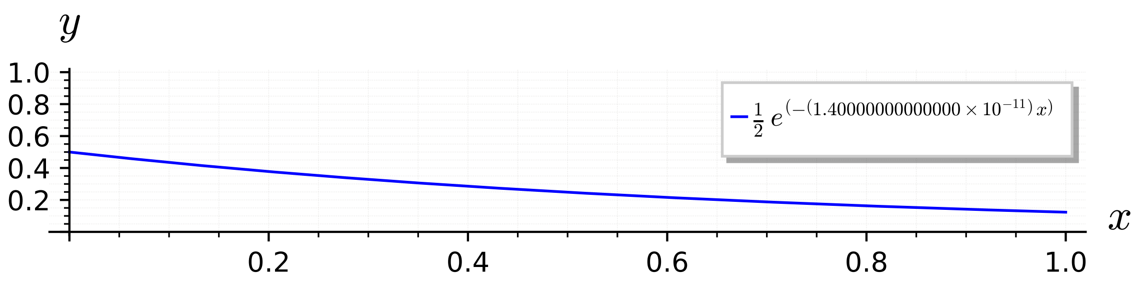

Figure 1.2.: Solution to the exercise "*Radioactive Decay*". The x-scale is 1e11.

1.2 Separable ODEs

Many practically useful ODEs can be reduced to the following form:

$$g(y)\, y' = f(x) \tag{1.4}$$

using *algebraic manipulations*. We can then integrate on both sides with respect to x, obtaining the following expression:

$$\int g(y)\, y'\, dx = \int f(x)\, dx + c. \tag{1.5}$$

On the left we can switch to y as the variable of integration. By calculus, we know the relation $y'\, dx = dy$, so that:

$$\int g(y)\, dy = \int f(x)\, dx + c. \tag{1.6}$$

If f and g are **continuous functions**, the integrals in Eq. (1.6) exist, and by evaluating them we obtain a general solution of Eq. (1.6). This method of solving ODEs is called the **method of separating variables**, and Eq. (1.4) is called a **separable equation**, as Eq. (1.6) the variables are now separated. x appears only on the right and y only on the left.

Example Separable ODE 3

Solve the following ODE:

$$y' = 1 + y^2$$

Solution Separable ODE

The given ODE is separable because it can be written:

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x+c)$$

> It is important to introduce the constant c when the integration is performed.

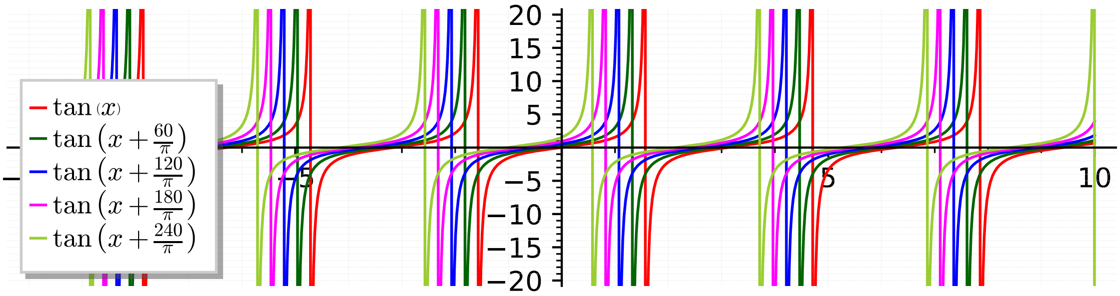

Figure 1.3.: Results with different c values.

Example IVP: Bell-Shaped Curve 4

Solve the following ODE:

$$y' = -2xy, \qquad y(0) = 1.8$$

Solution IVP: Bell-Shaped Curve

By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + c, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Therefore the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve. The plot of the solution is given in Figure 1.4.

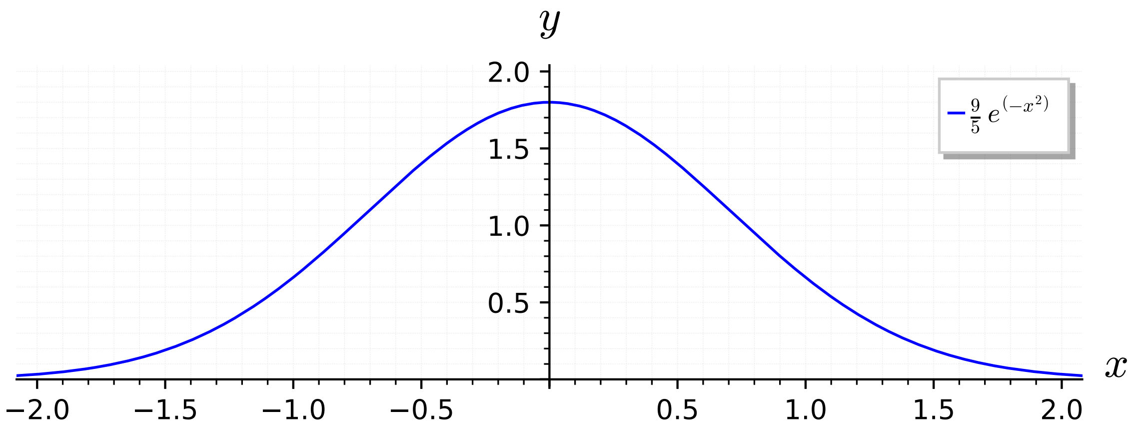

Figure 1.4.: Solution plot for the exercise: *IVP: Bell-Shaped Curve*.

Example Radiocarbon Dating 5

In September 1991 the famous Iceman (Ötzi), a mummy from the Stone Age found in the ice of the Öetztal Alps in Southern Tirol near the Austrian–Italian border, caused a scientific sensation. When did Ötzi approximately live and life if the ratio of carbon-14 to carbon-12 in the mummy is 52.5% of that of a living organism?

> The half-life of carbon is 5175 years.

Solution **Radiocarbon Dating**

Radioactive decay is governed by the ODE $y' = ky$ as we have developed previously. By separation and integration:

Figure 1.5.: Ötzi was found in the Öetztal Alps in Southern Tirol near the Austrian–Italian border

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0\,e^{kt} \qquad (y_0 = e^0).$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Ötzi died,

$$e^{k\tau} = e^{-0.0001\,213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001\,213} = 5312. \qquad \blacksquare$$

Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function (i.e., u). This method can be applied to the following form:

$$y' = f\left(\frac{y}{x}\right).$$

Here, f is any differentiable function of y/x, such as $\sin(y/x)$, (y/x), and so on. The form of such an ODE suggests that we set $y/x = u$. This makes the conversion:

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u.$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Example **Reduction to Separable Form** 6

Solve the following ODE:

$$2xy' = y^2 - x^2.$$

Solution **Reduction to Separable Form**

To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1+u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1+u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c$

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis, which can be seen in Figure 1.6.

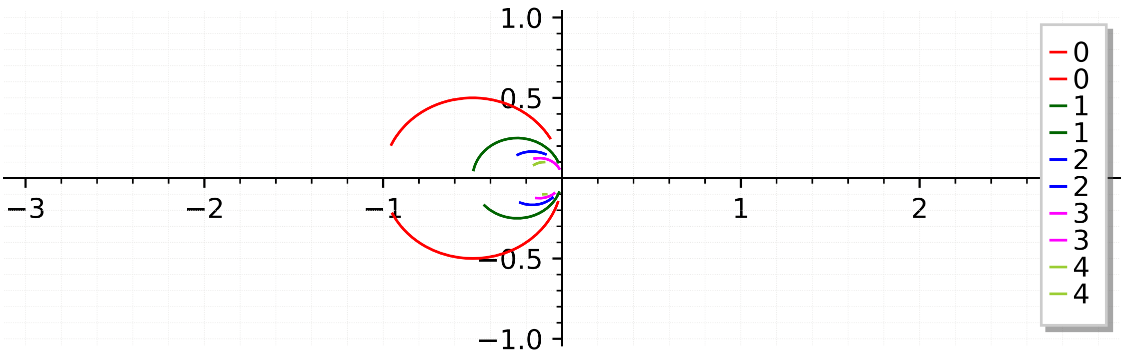

Figure 1.6.: Solution to *Reduction to Separable form* which is a family of solutions.

1.3 Exact ODEs

1.3.1 Integrating Factors

Recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (i.e., **total differential**) is:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c = \text{const}$, then $du = 0$. As an example, let's have a look at the function $u = x + x^2y^3 = c$. Finding its factors:

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows. A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as:

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.7}$$

is called an **exact differential equation** if the differential form $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential:

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{1.8}$$

of some function $u(x, y)$. Then Eq. (1.7) can be written

$$du = 0$$

By integration we immediately obtain the general solution of Eq. (1.7) in the form:

$$u(x, y) = c \tag{1.9}$$

Comparing Eq. (1.7) and Eq. (1.8), we see that Eq. (1.7) is an exact differential equation if there is some function $u(x, y)$ such that:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N \tag{1.10}$$

From this we can derive a formula for checking whether Eq. (1.7) is exact or not, as follows. Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections.

Then by partial differentiation of Eq. (1.10),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \, \partial x},$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \, \partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \blacksquare \tag{1.11}$$

> This condition is not only necessary but also sufficient for Eq. (1.7) to be an exact differential equation.

If Eq. (1.7) is proved to be **exact**, the function $u(x, y)$ can be found by inspection or in the following systematic way.
From (4a) we have by integration with respect to x:

$$u = \int M \, dx + k(y), \tag{1.12}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine $k(y)$, derive $\partial u/\partial y$ from Eq. (1.12), use (4b) to get dk/dy, and integrate dk/dy to get k.
Formula Eq. (1.12) was obtained from (4a).

> It is valid to use **either** of them and arrive at the same result.

Then, instead of (6), we first have by integration with respect to y

$$u = \int N \, dy + l(x).$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

Example **Initial Value Problem** 7

Solve the initial value problem

$$(\cos y \sinh x + 1) \, dx - \sin y \cosh x \, dy = 0, \qquad y(1) = 2.$$

Solution **Initial Value Problem**

Verify that the given ODE is **exact**. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x \, dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = u = \cos y \sinh x + 1$. Therefore $dl/dx = 1$ by integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$.
Therefore the answer is $\cos y \cosh x + x = 0.358$.

Example An Exact ODE 8

Solve

$$\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))\, dy = 0.$$

Solution An Exact ODE

Step 1. Test for exactness. Our equation is of the form (1) with

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

Example An Exact ODE 9

Solve the following ODE:

$$\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))\, dy = 0. \tag{1.13}$$

Step 1 - Test for exactness

First check if our equation is **exact**, try to convert the equation of the form Eq. (1.7):

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Therefore:

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

This proves our equation to be exact.

Step 2 - Implicit General Solution

From Eq. (1.12), we obtain by integration:

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y) \tag{1.14}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining:

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Therefore $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into Eq. (1.14) and observing Eq. (1.9), we obtain:

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c \quad \blacksquare$$

Example Breakdown of Exactness 10

Check the exactness of the following ODE:

$$-y\,dx + x\,dy = 0$$

Solution Breakdown of Exactness

The above equation is **NOT** exact as $M = -y$ and $N = x$, so that:

$$\partial M/\partial y = -1 \qquad \partial N/\partial x = 1$$

Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + ky = -xy + ky, \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed.

If we wrote $\arctan y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$).

1.4 Linear ODEs

1.4.1 Introduction

Linear ODEs are prevalent in many engineering and natural science and therefore it is important for us to study them. A first-order ODE is said to be **linear** if it can be brought into the form:

$$y' + p(x)y = r(x)$$

If it cannot brought into this form, it is classified as **non-linear**.

Linear ODEs are linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be any given functions of x.

In engineering, $r(x)$ is generally called the input and $y(x)$ is called the output or response.

Homogeneous Linear ODE

We want to solve in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) = 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0$$

and is called homogeneous. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-Ip(x)dx} \qquad (c = \pm e^{c\pi} \quad \text{when} \quad y \geq 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution**$y(x) = 0$ for all x in that interval.

Non-Homogeneous Linear ODE

We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called nonhomogeneous. It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the previous section or we can proceed directly, as follows. We multiply (1) by $F(x)$, obtaining

$$Fy' + pFy = rF.$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h' e^h y = e^h y' + (e^h)' y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r\,dx + c.$$

Dividing by e^h, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx.$$

This reduces solving (1) to the generally simpler task of evaluating integrals. For ODEs for which this is still difficult, you may have to use a numeric method for integrals from Sec. 19.5 or for the ODE itself from Sec. 21.1. We mention that h has nothing to do with $h(x)$ in Sec. 1.1 and that the constant of integration in h does not matter; see Prob. 2.
The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

$$y(x) = e^{-h}\int e^h r\,dx + ce^{-h},$$

Example First-Order ODE, General Solution Initial Value Problem 11

Solve the ODE Initial Value Problem

Solution First-Order ODE, General Solution Initial Value Problem

Solve the initial value problem

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.Here $p = \tan x$, $r = \sin 2x = 2\,\sin x \cos x$, and

$$h = \int p\,dx = \int \tan x\,dx = \ln\,|\sec x|.$$

From this we see that in (4),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h} r = (\sec x)(2\,\sin x \cos x) = 2\,\sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\,dx \, + \, c \right) = c\,\cos x \, - \, 2\cos^2 x.$$

From this and the initial condition, $1 = c \, - \, 1 \, - \, 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3\,\cos x - 2\,\cos^2 x$. Here $3\,\cos x$ is the response to the initial data, and $-2\,\cos^2 x$ is the response to the input $\sin 2x$.

Chapter 2.

Second-Order Ordinary Differential Equations

2.1 Introduction

A second-order ODE is called **linear**, if it can be written (in its standard form) as:

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.1}$$

- when $r(x) = 0$ it is homogeneous,
- else it is **non-homogeneous**.

The functions $p(x)$ and $q(x)$ are called the **coefficients** of the ODEs.
An example of a **non-homogeneous linear** equation is:

$$y'' = 25y - e^{-x}\cos x$$

An example of a **homogeneous linear** equation is:

$$y'' + \frac{1}{x}y' + y = 0$$

An example of **non-linear**ODE is:

$$y''y + (y')^2 = 0$$

2.1.1 Superposition Principle

For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.

$$y = c_1y_1 + c_2y_2$$

This is called a linear combination of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or linearity principle

Fundamental Theorem for the Homogeneous Linear ODE (2)
For a homogeneous linear ODE (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Example **Homogeneous Linear ODEs: Superposition of Solutions** 12

Verify the function $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE:

$$y'' + y = 0,$$

for all x.

Solution **Homogeneous Linear ODEs: Superposition of Solutions**

Verify by differentiation and substitution. We obtain,

$$(\cos x)'' = -\cos x,$$

Therefore:

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly:

$$(\sin x)'' = -\sin x,$$

and we would arrive in a similar result.

$$y'' + y = (\sin x)'' + \sin x = -\sin x + \sin x = 0$$

To further elaborate on the result, multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives:

$$\begin{aligned}&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\ &= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0 \quad \blacksquare\end{aligned}$$

Example **Example of a Non-homogeneous Linear ODE** 13

Verify the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions to the following non-homogeneous linear ODE.

$$y'' + y = 0$$

Solution **Example of a Non-homogeneous Linear ODE**

Testing out the first function

$$(1+\cos x)'' = -\sin x$$

Plugging this to the main equation gives:

$$\begin{aligned} y'' + y &= 1 \\ -\sin x + 1 + \cos x &\neq 1 \quad \blacksquare \end{aligned}$$

The first equation is **NOT** the solution to the ODE. Trying the second one:

$$\begin{aligned} (1+\sin x)'' &= -\cos x \\ y'' + y &= 1 \\ -\cos x + 1 + \sin x &\neq 1 \quad \blacksquare \end{aligned}$$

The second function is also **NOT** a solution.

2.1.2 Initial Value Problem

For a second-order homogeneous linear ODE (2) an initial value problem consists of (2) and two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

The conditions (4) are used to determine the two arbitrary constants c1 and c2 in a general solution

Example Initial Value Problem 14

Solve the initial value problem:

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution Initial Value Problem

Step 1: General Solution
From Example 1, we know the function $\cos x$ and $\sin x$ are solutions to the ODE, and therefore we can write:

$$y = c_1 \cos x + c_2 \sin x$$

This will turn out to be a general solution as defined below.

Step 2: Particular Solution

We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, as $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x \quad \blacksquare$$

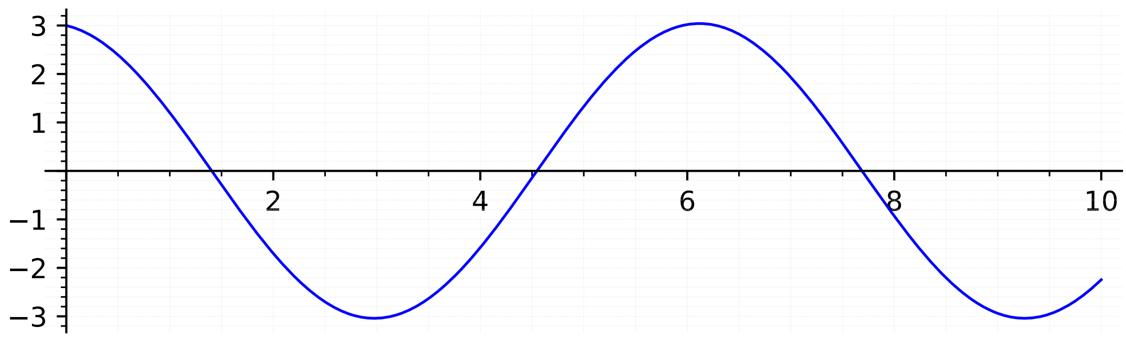

Figure 2.1.: Solution to the initial value problem exercise.

2.1.3 Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of reduction of order.
For an example please look at Exercise 2.1.6.

2.1.4 Homogeneous Linear ODEs

Consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{2.2}$$

Solve by starting

$$y = e^{\lambda x}$$

Taking the derivatives of the aforementioned function gives:

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

Plugging these values to Eq. (2.2) gives:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic** equation (or auxiliary equation),

$$\lambda^2 + a\lambda + b = 0$$

then the exponential function (2) is a solution of the ODE (1). Now from algebra we recall the roots of the quadratic equation

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

Case	Roots of	Basis	General Solution
I	Distict real	$e^{\lambda_1 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
	(λ_1, λ_2)	$e^{\lambda_2 x}$	
II	Real Double Root	$e^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
	$(\lambda = -1/2a)$	$xe^{-ax/2}$	
III	Complex Conjugate	$e^{-ax/2}\cos\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$
	$\lambda_1 = -1/2a + \mathbf{j}\omega$	$e^{-ax/2}\sin\omega x$	
	$\lambda_2 = -1/2a - \mathbf{j}\omega$		

Table 2.1.: Possible roots of the characteristic equation based on the discriminant value.

2.1.5 Euler-Cauchy Equations

Has the following form:

$$x^2 y'' + axy' + by = 0 \tag{2.3}$$

To solve do the following substitutions:

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Which gives:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

> $y = x^m$ is a good choice as it produces a common factor x^m.

Simplifying the equation produces the **auxiliary** equation.

$$m^2 + (a - 1)m + b = 0. \tag{2.4}$$

$y = x^m$ is a solution of Eq. (2.3) if and only if m is a root of Eq. (2.4).

The roots of Eq. (2.4) are:

$$m_1 = \frac{1}{2}(1 - a) + \sqrt{\frac{1}{4}(1 - a)^2 - b}, \qquad m_2 = \frac{1}{2}(1 - a) - \sqrt{\frac{1}{4}(1 - a)^2 - b}.$$

Case	Roots of	General Solution
I	Distict real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
	(m_1, m_2)	
II	Real Double Root	$y = c_1 x^m \ln x + c_2 x^m$
	(m)	
III	Complex Conjugate	$y = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x)$
	$m_1 = \alpha + \beta \mathbf{j}$	$\alpha = \mathrm{Re}(m)$
	$m_2 = \alpha - \beta \mathbf{j}$	$\beta = \mathrm{Im}(m)$

Table 2.2.: Possible solutions of the Euler-Cauchy based on the m value.

Complex conjugate roots are of minor practical importance for practical purposes.

Example **General Solution in the Case of Different Real Roots** 15

Solve the following ODE:

$$x^2 y'' + 1.5xy' - 0.5y = 0$$

Solution **General Solution in the Case of Different Real Roots**

This equation can be classified as **Euler-Cauchy equation** and it has an auxiliary equation $m^2 + 0.5m - 0.5 = 0$. Based on this equation, the roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution.

$$y = c_1 \sqrt{x} + \frac{c_2}{x} \qquad (x > 0) \quad \blacksquare$$

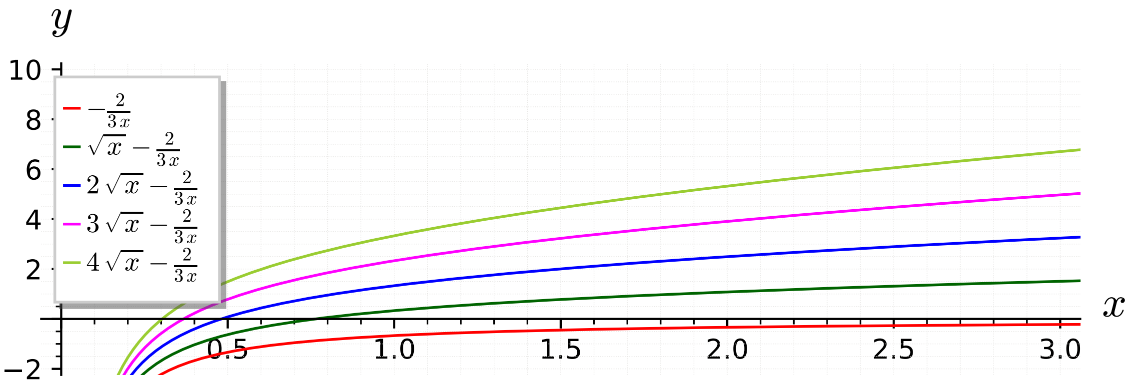

Figure 2.2.: Solution to the example "*General Solution in the Case of Different Real Roots*"

Example General Solution in the Case of a Double Root 16

Solve the following ODE:

$$x^2y'' - 5xy' + 9y = 0$$

Solution General Solution in the Case of a Double Root

Based on its format it can be classified as an **Euler-Cauchy** equation with an auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is:

$$y = (c_1 + c_2 \ln x)\, x^2. \quad \blacksquare$$

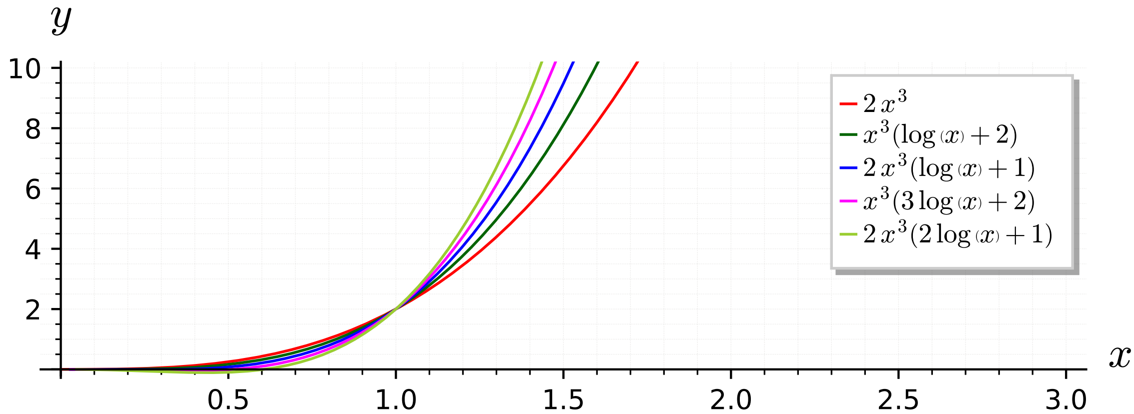

Figure 2.3.: Solution to the example "*General Solution in the Case of a Double Root*"

Example BVP: Electric Potential Field Between Two Concentric Spheres 17

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii r_1 = 5 cm and r_2 = 10 cm kept at potentials v_1 = 110 V and v_2 = 0, respectively.

> $v = v(r)$ is a solution of the *Euler–Cauchy equation* $rv'' + 2v' = 0$.

Solution BVP: Electric Potential Field Between Two Concentric Spheres

The auxiliary equation is:

$$m^2 + m = 0$$

It has the roots 0 and -1. This gives the general solution of:

$$v(r) = c_1 + c_2/r$$

From the boundary conditions (i.e., the potentials on the spheres), we obtain:

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$ which gives the final equation:

$$v(r) = -110 + 1100/r \quad \blacksquare$$

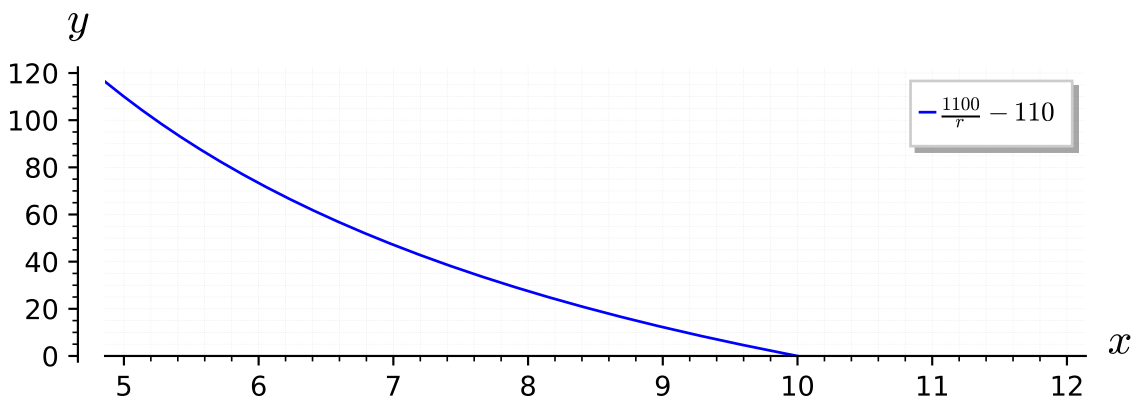

Figure 2.4.: Solution to the BVP: Electric Potential Field Between Two Concentric Spheres exercise.

2.1.6 Non-homogeneous ODEs

They have the form:

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2.5}$$

where $r(x) \neq 0$. a general solution of Eq. (2.5) is the sum of a general solution of the corresponding homogeneous ODE:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2.6}$$

and a **particular solution** of Eq. (2.5). These two new terms **general solution** of Eq. (2.5) and **particular solution** of Eq. (2.5) are defined as follows:

> **General Solution and Particular Solution**
> A general solution of the nonhomogeneous ODE Eq. (2.5) on an open interval I is a solution of the form:
>
> $$y(x) = y_h(x) + y_p(x)\,. \tag{2.7}$$
>
> here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE Eq. (2.6) on I and y_p is any solution of Eq. (2.5) on I containing **no arbitrary constants**. A particular solution of Eq. (2.5) on I is a solution obtained from Eq. (2.7) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Method of Undetermined Coefficients

To solve the non-homogeneous ODE Eq. (2.5) or an initial value problem for Eq. (2.5), we have to solve the homogeneous ODE Eq. (2.6) or an initial value problem for and find any solution y_p of Eq. (2.5), so that we obtain a general solution Eq. (2.7) of Eq. (2.5).

This method is called **method of undetermined coefficients**.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with constant coefficients a and b.

$$y'' + ay' + by = r(x) \tag{2.8}$$

when $r(x)$ is:

- an exponential function,
- a cosine or sine,
- sums or products of such functions

These functions have derivatives similar to $r(x)$ itself.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.
Table below shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Choice Rules for the Method of Undetermined Coefficients

Basic Rule: If $r(x)$ in Eq. (2.8) is one of the functions in the first column in Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into Eq. (2.8).

Modification Rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to Eq. (2.8), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Sum Rule: If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$C\,e^{\gamma x}$
kx^n where $(n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Table 2.3.: Method of Undetermined Coefficients.

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of Eq. (2.5) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of Eq. (2.5) with $r = r_1 + r_2$. (Verify!)

The method is **self-correcting**. A false choice for y_p or one with too few terms will lead to a contradiction.

A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Model **Damped System**

To our previous undamped model $my'' = -ky$ we now add the damping force:

$$F_2 = -cy',$$

therefore, the ODE of the damped mass–spring system is:

$$my'' + cy' + ky = 0. \tag{2.9}$$

This can physically be done by connecting the ball to a dashpot. Assume this damping force to be proportional to the velocity $y' = dy/dt$.

This is generally a good approximation for small velocities.

The constant c is called the damping constant.
Let us show that c is positive.
The damping force $F_2 = -cy'$ acts **against** the motion; hence for a downward motion we have $y' > 0$ which for positive c makes F negative (an upward force), as it should be.
Similarly, for an upward motion we have $y' < 0$ which, for $c > 0$ makes F_2 positive (a downward force).
The ODE Eq. (2.9) homogeneous linear and has constant coefficients. We can solve it by deriving its characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

As this is a quadratic equation, its roots are:

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

Depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case I: Over-damping

If $c^2 > 4mk$, then λ_1 and λ_2 are **distinct real roots**. In this case the corresponding general solution is:

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}. \tag{2.10}$$

Case	Condition	Description	Type
I	$c^2 > 4mk$	Distinct real roots λ_1, λ_2	Overdamping
II	$c^2 = 4mk$	A real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Table 2.4.: A Detailed look into the scientific method.

In this case, damping takes out energy so quickly without the body **oscillating**.

For $t > 0$ both exponents in Eq. (2.10) are negative because $\alpha > 0$ and $\beta > 0$ and:

$$[][2] = \alpha^2 - k/m < \alpha^2$$

Hence both terms in Eq. (2.10) approach zero as $t \to \infty$.
Practically, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Below are the results for some typical initial conditions.

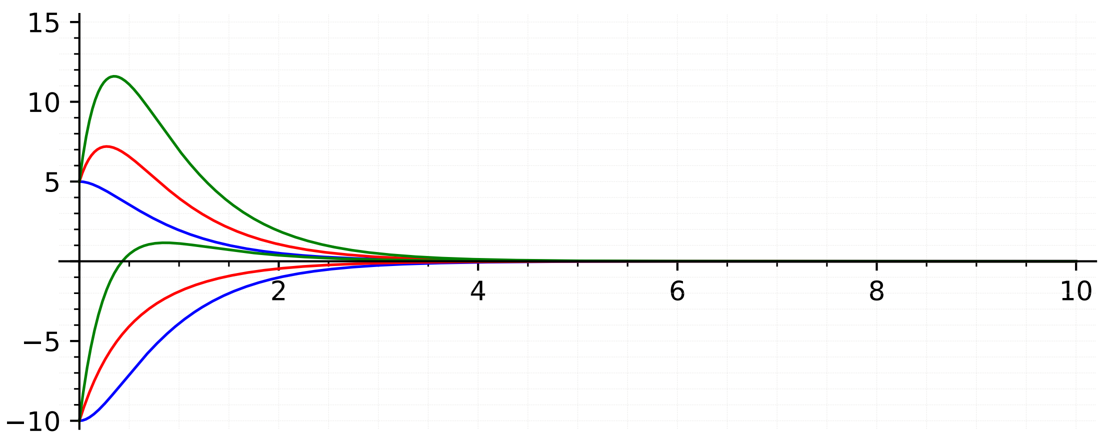

Figure 2.5.

Case II: Critical-Damping

Critical damping is the border case between non-oscillatory motions (Case I) and oscillations (Case III). Occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of Eq. (2.9) is:

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{2.11}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.
The Figure below shows typical forms of Eq. (2.11).

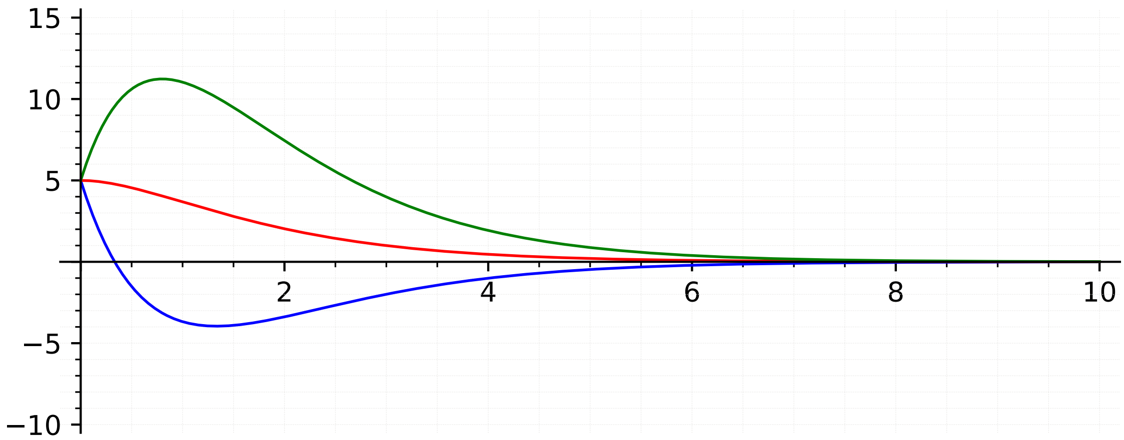

Figure 2.6.

> The graph above looks almost like those in the previous figure.

Case III: Under-Damping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 = 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (> 0).$$

> This is to differentiate from ω which is used predominantly in electrical engineering.

The roots of the characteristic equation are now complex conjugates,

$$y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t) = Ce^{-\alpha t}\cos(\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$ This represents **damped oscillations**. Their curve lies between the dashed curves: The roots of the characteristic equation are now complex conjugates,2m at
The frequency is $\omega * /2\pi$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (0) is, the larger is v* and the more rapid the oscillations become. If c approaches 0, v >(2p) is the natural frequency of the system. 0

Model Modelling: Forced Oscillations and Resonance

Previously we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring) and modeled it by the homogeneous linear ODE:

$$my'' + cy' + ky = 0. \tag{2.12}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. The previous mass–spring system exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia my'', the damping force cy' (if $c < 0$), and the spring force ky, a restoring force.

Now extend our model by including an additional force, that is, the external force $r(t)$, on the RHS. This turns Eq. (2.12) into:

$$my'' + cy' + ky = r(t). \tag{2.13}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a forced motion with forcing function $r(t)$, which is also known as input or driving force, and the solution $y(t)$ to be obtained is called the **output or the response** of the system to the driving force.

Of special interest are periodic external forces, and we shall consider a driving force of the form:

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the non-homogeneous ODE:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.14}$$

Its solution will allow us to model resonance.

Solving the Non-homogeneous ODE

We know that a general solution of Eq. (2.14) is the sum of a general solution y_h of the homogeneous ODE Eq. (2.12) plus any solution y_p of Eq. (2.14). To find y_p, we use the **method of undetermined coefficients**, starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{2.15}$$

By differentiating this function (remember the chain rule) we obtain:

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$
$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_p, $y_p{}'$, $y_p{}''$, into Eq. (2.14) and collecting the cos and the sin terms, we get:

$$[(k - m\omega^2)a + \omega cb]\cos\omega t + [-\omega ca + (k - m\omega^2)b]\sin\omega t = F_0 \cos\omega t.$$

The cos terms on both sides **must be equal**, and the coefficient of the sin term on the left must be zero since there is no sine term on the right. This gives the two (2) equations:

$$\left(k - m\omega^2\right) a + \omega cb = F_0, \tag{2.16}$$

$$-\omega ca + \left(k - m\omega^2\right) b = 0. \tag{2.17}$$

for determining the unknown coefficients a, b. This is a **linear system**. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining:

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get:

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(km\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$, then $k = m\omega_0^2$ we obtain:

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{2.18}$$

We thus obtain the general solution of the nonhomogeneous ODE Eq. (2.14) in the form

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE Eq. (2.12) and y_p is given by Eq. (2.15) with coefficients Eq. (2.18).

Model Electric Circuits

Let's study a simple RLC Circuit. These circuits occurs as a basic building block of large electric networks in computers and elsewhere. An RLC-circuit is obtained from an RL-circuit by adding a *capacitor*.

> A capacitor is a passive, electrical component that has the property of storing electrical charge, that is, electrical energy, in an electrical field.

$$LI' + RI + \frac{1}{C}\int I\,\mathrm{d}t = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate the above equation respect to t:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t. \tag{2.19}$$

This shows that the current in an RLC-circuit is obtained as the solution of the non-homogeneous second-order ODE with **constant coefficients**.

Solving the ODE for the Current

A general solution of Eq. (2.19) is the sum $I = I_\mathrm{h} + I_\mathrm{p}$, where I_h is a general solution of the homogeneous ODE corresponding to Eq. (2.19) and I_p is a particular solution of Eq. (2.19). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a\cos \omega t + b \sin \omega t, \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t), \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t). \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to Eov cos vt on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned} L\omega^2(-a) + R\omega b + a/C &= E_0 \omega \qquad \text{(Cosine terms)} \\ L\omega^2(-b) + R\omega(-a) + b/C &= 0 \qquad \text{(Sine terms)}. \end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called **reactance**:

> reactance, in electricity, measure of the opposition that a circuit or a part of a circuit presents to electric current insofar as the current is varying or alternating

$$S = \omega L - \frac{1}{\omega C} \tag{2.20}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives:

$$-Sa + Rb = E_0,$$
$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding.Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives:

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve this for a and b :

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{2.21}$$

Equation (2) with coefficients a and b given by Eq. (2.21) is the desired particular solution I_p of the non-homogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal input voltage.

Using Eq. (2.21), we can write I_p in terms of **physically visible** quantities, namely, amplitude I_0 and phase lag θ of the current behind voltage, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{2.22}$$

where:

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $(R^2 + S^2)$ is called **impedance**. Our formula shows that the impedance equals the ratio $E_0/I[0]$. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the apparent resistance.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

where λ_1, λ_2 are the roots of the characteristic equation of:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = \alpha + \beta$, where:

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this, it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time.

Hence the transient current $I = I_h + I_p$ tends to the steady-state current Ip, and after some time the output will practically be a harmonic oscillation, which is given by Eq. (2.22) and whose frequency is that of the input (i.e., voltage).

Example **Reduction of Order if a Solution Is Known** 18

Find a basis of solutions of the ODE:

$$\left(x^2 - x\right) y'' - xy' + y = 0.$$

Solution **Reduction of Order if a Solution Is Known**

Inspection shows that $y_1 = x$ is a solution because $y_1{}' = 1$ and $y_1{}'' = 0$, so that the first term vanishes identically and the second and third terms cancel.
The idea of the method is to substitute

$$\begin{aligned} y &= uy_1 = ux, \\ y' &= u'x + ux' = u'x + u, && \text{(Chain Rule)} \\ y'' &= (u'x + u)' = u''x + u'x' + u' = u''x + 2u'. && \text{(Chain Rule)} \end{aligned}$$

into the ODE. This gives:

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

ux and xu cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely:

$$\left(x^2 - x\right) v' + (x - 2)\, v = 0$$

Separation of variables and integration gives:

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}\, dx = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx, \qquad \ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}$$

We need no constant of integration because we want to obtain a particular solution.
Taking exponents and integrating again, we obtain:

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\, dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ \ln|x| + 1.$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are **linearly independent**.

> This means their quotient is not constant.

we have obtained a basis of solutions, valid for all positive x. ■

Example **IVP: Case of Distinct Real Roots** 19

Solve the initial value problem:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution **IVP: Case of Distinct Real Roots**

Step 1. General Solution

The characteristic equation is:

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are:

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2.$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular Solution

As we obtained the general solution with the initial conditions:

$$y(0) = c_1 + c_2 = 4, \qquad y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 3$ and $c_2 = 3$. This gives the answer:

$$y = e^{x} + 3e^{-2x} \quad \blacksquare$$

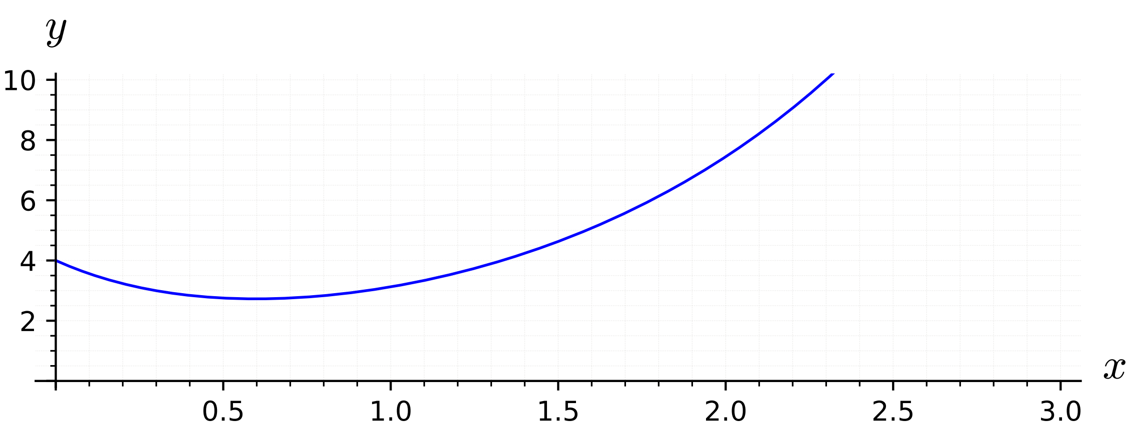

Figure 2.7.: The curve begins with a negative slope in agreement with the initial conditions.

Example IVP: Case of Real Double Roots 20

Solve the initial value problem:

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3, \qquad y'(0) = -3.5.$$

Solution IVP: Case of Real Double Roots

The characteristic equation is:

$$\lambda^2 + \lambda + 0.25 = (\lambda + 2)^2 = 0,$$

It has the double root $\lambda = -0.5$. This gives the general solution:

$$y = (c_1 + c_2 x)\, e^{-0.5\,x}$$

We need its derivative:

$$y' = c_2 e^{-0.5\,x} - 0.5\,(c_1 + c_2 x)\, e^{-0.5\,x}$$

From this and the initial conditions we obtain:

$$y\,(0) = c_1 = 3.0, \quad y'\,(0) = c_2 - 0.5c_1 = 3.5, \quad c_2 = -2$$

The particular solution of the initial value problem is:

$$y = (3 - 2x)\, e^{-0.5\,x}$$

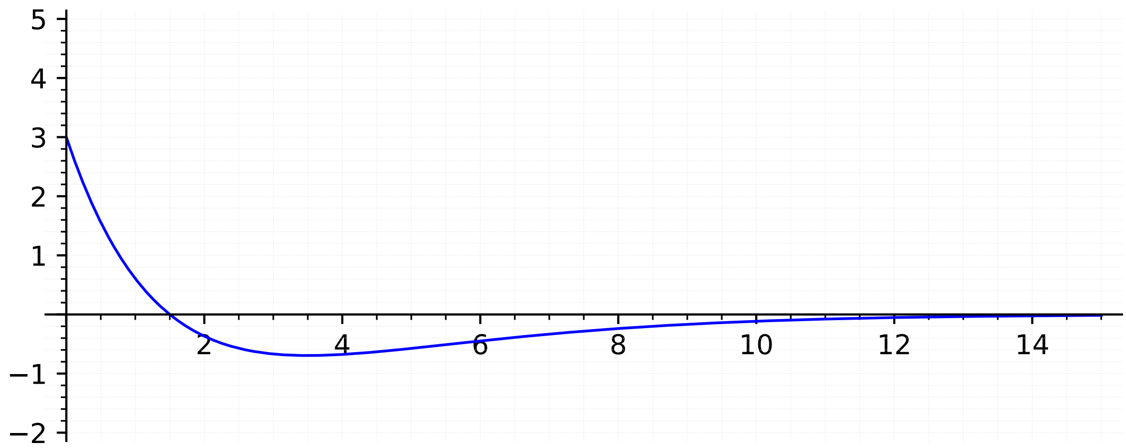

Figure 2.8.: Solution to the case of a double root.

Example **IVP: Case of Complex Roots** 21

Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution **IVP: Case of Complex Roots**

Step 1. General Solution
The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

It has the roots of $-0.2 \pm 3\mathbf{j}$. Hence $\omega = 3$ and the general solution is:

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Step 2. Particular Solution
The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x)$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$, therefore:

$$y = e^{-0.2x} \sin 3x.$$

The Figure shows y and $-e^{-0.2x}$ and $e^{-0.2x}$ (dashed), between which y oscillates. Such "damped vibrations" have important mechanical and electrical applications.

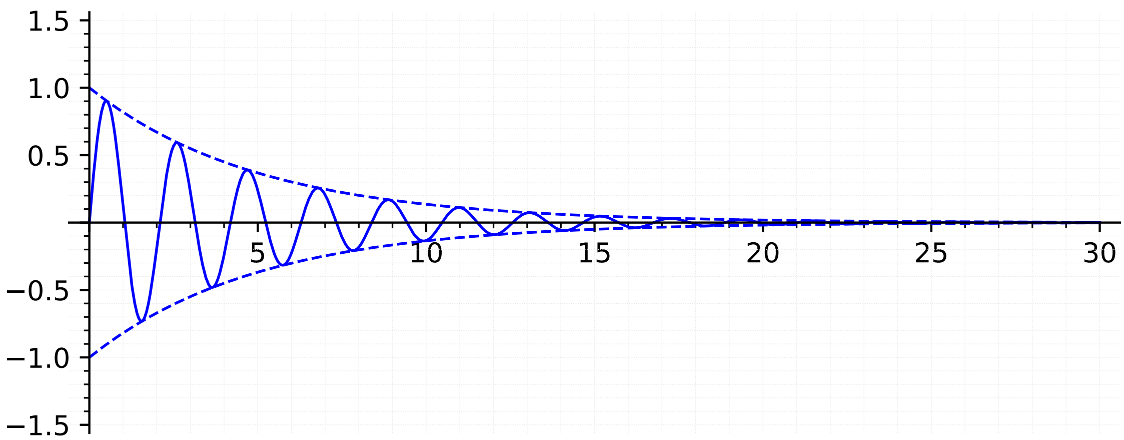

Figure 2.9.: Solution to the complex roots - initial value problem exercise.

Example **Harmonic Oscillation of an Undamped Mass–Spring System** 22

If a mass–spring system with an iron ball of weight W = 98 N can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

Solution **Harmonic Oscillation of an Undamped Mass–Spring System**

Hooke's law:

$$F_1 = -ky \tag{2.23}$$

with W as the force and 1.09 meter as the stretch gives $W = 1.09k$. Therefore

$$k = \frac{W}{1.09} = \frac{98}{1.09} = 90$$

whereas the mass (m) is:

$$m = \frac{W}{g} = \frac{98}{9.8} = 10\,\mathrm{kg}$$

This gives the frequency:

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{3}{2\pi} = 0.48\,\mathrm{Hz}$$

From Eq. (2.23) and the initial conditions, $y(0) = A = 0.16\,\mathrm{m}$ and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16\cos 3t\,\mathrm{m} \quad \blacksquare$$

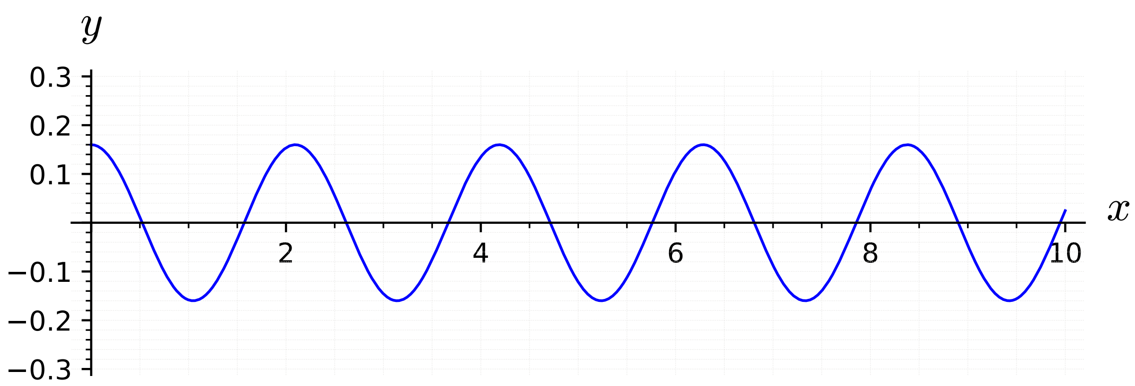

Figure 2.10.: The harmonic oscillation on a string.

Example **The Three Cases of Damped Motion** 23

How does the motion in *Harmonic Oscillation of an Undamped Mass–Spring System* change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

- $c = 100\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 60\,\mathrm{kg}\cdot\mathrm{s}^{-1}$
- $c = 10\,\mathrm{kg}\cdot\mathrm{s}^{-1}$

Solution **The Three Cases of Damped Motion**

It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

Case I

With $m = 10$ and $k = 90$, as in *Harmonic Oscillation of an Undamped Mass–Spring System*, the model is the initial value problem:

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16\,\mathrm{m}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90$. It has the roots $\lambda_1 = -9$ and $\lambda_2 = -1$. This gives the general solution:

$$y = c_1 e^{-9t} + c_2 e^{-t} \quad \text{We also need} \quad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$ and $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is:

$$y = -0.02e^{-9t} + 0.18e^{-t} \quad \blacksquare$$

It approaches 0 as $t \to 0$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

Case II

The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form:

$$10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$$

It has the double root $\lambda_1 = \lambda_2 = -3$. Hence the corresponding general solution is:

$$y = (c_1 + c_2 t)\, e^{-3\,t}, \qquad \text{we also need} \qquad y' = (c_2 - 3c_1 - 3c_2 t)\, e^{-3\,t}$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$ Hence in the critical case the solution is:

$$y = (0.16 + 0.48t)\, e^{-3\,t} \quad \blacksquare$$

It is always positive and decreases to 0 in a monotone fashion.

Case III

The model is now:

$$10y'' + 10y' + 90y = 0.$$

As $c = 10$ is smaller critical c, we will see oscillations. The characteristic equation is:

$$10\lambda^2 + 10\lambda + 90 = 10\left[\left(\lambda + \frac{1}{2}\right)^2 + 9 - \frac{1}{4}\right] = 0$$

This equation has complex roots:

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96\mathbf{j}$$

This gives the general solution:

$$y = e^{-0.5\,t}\,(A\cos 2.96t + B\sin 2.96t)$$

Therefore $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5\,t}\,(-0.5A\cos 2.96t - 0.5B\sin 2.96t - 2.96A\sin 2.96t + 2.96B\cos 2.96t)$$

Hence $y'(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This give the solution:

$$y = e^{-0.5\,t}\,(0.16\cos 2.96t + 0.027\sin 2.96t) = 0.162e^{-0.5\,t}\cos(2.96t - 0.17) \quad \blacksquare$$

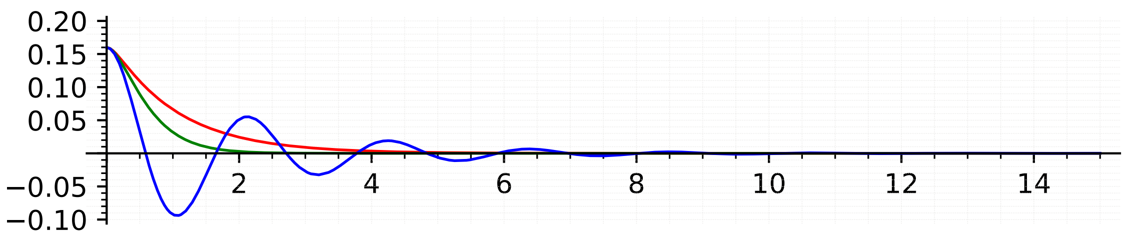

Figure 2.11.: The solution curve to the Three Cases of Damped Motion exercise

Example Studying the RLC Circuit 24

Find the current $I(t)$ in an RLC-circuit with $R = 11$ (Ohms), $L = 0.9$ H (Henry), $C = 0.01$ F (Farad), which is connected to a source of $V(t) = 110\sin(120\pi t)$.

Assume that current and capacitor charge are 0 when $t = 0$.

Solution Studying the RLC Circuit

Step 1. General solution of the homogeneous ODE

Substituting R, L, C and the derivative $V(t)$, we obtain:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega cos \omega t.$$
$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Hence the homogeneous ODE is:

$$0.1I'' + 11I' + 100I = 0$$

Its **characteristic equation** is:

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is:

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

Step 2. Particular solution I_p

We calculate the reactance S = 37.7 - 0.3 = 37.4 and the steady-state current:

$$I_p(t) = acos377t + bsin377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t$$

Step 3. Particular solution satisfying the initial conditions

How to use Q(0) 0? We finally determine c1 and c2 from the in initial conditions I(0) 0 and Q(0) 0. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.71 = 0$$

, hence $c_2 = 2.71 - c_1$

We turn to Q(0) 0. The integral in (1r) equals I dt Q(t); see near the beginning of this section. Hence for t 0, Eq. (1r) becomes

$$L'(0) + R \cdot 0 = 0$$

, so that $I'(0) = 0$ Differentiating (6) and setting t 0, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0$$

, hence by (7), $-10c_1 = 100(2.71 - c_1) - 300.1$. The solution of this and (7) is c1 0.323, c2 3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71cos377t + 0.796sin377t \quad \blacksquare$$

> You may get slightly different values depending on the rounding.

Figure below shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly.

Thus after a very short time the current will practically execute harmonic oscillations of the input frequency.

Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824sin(377t - 1.29) \quad \blacksquare$$

Example Application of the Basic Rule (a) 25

Solve the initial value problem

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

Solution Application of the Basic Rule (a)

Step 1: General Solution of the Homogeneous ODE

The ODE $y'' + y = 0$ has the general solution

$$y_h = A \cos x + B \sin x.$$

Step 2: Solution of the non-Homogeneous ODE

First try $y_p = Kx^2$ and also $y_p{}'' = 2K$. By substitution:

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficient of each power of x (x^2 and x^0) **must be the same on both sides**. Therefore:

$$K = 0.001, \quad \text{and} \quad 2K = 0, \qquad \text{a contradiction.}$$

The looking at the table suggests the choice:

$$y_p = K_2x^2 + K_1x + K_0, \quad \text{Then} \quad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have:

$$K_2 = 0.001, \qquad K_1 = 0, \qquad 2K_2 + K_0 = 0$$

Hence:

$$K_0 = -2K_2 = -0.002$$

This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$$

Step 3. Solution of the initial value problem.

Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \quad \text{and} \quad y'(0) = B = 1.5.$$

This gives the answer in the along with the Figure below.

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002 \quad \blacksquare$$

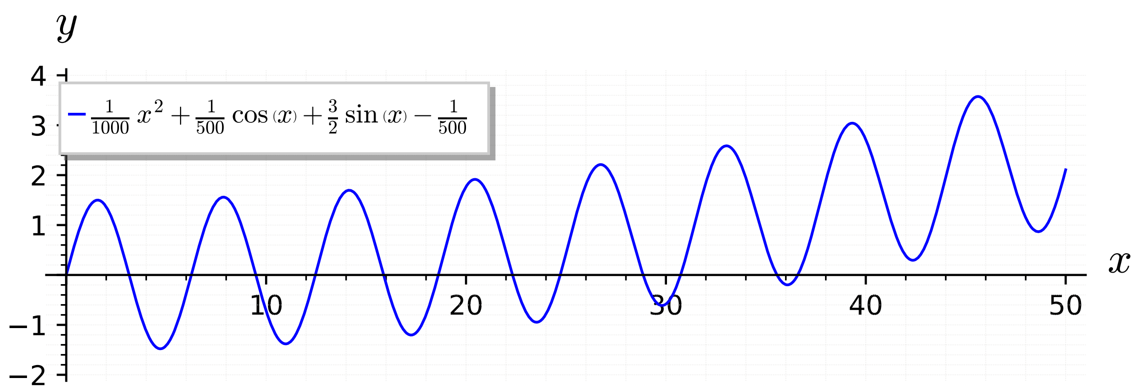

Figure 2.12.: Solution to Method of Undetermined Coefficients exercise.

Example Application of the Modification Rule (b) 26

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution Application of the Modification Rule (b)

Step 1.General solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution:

$$y_h = (c_1 + c_2 y)\, e^{-1.5x}$$

Step 2.Solution y_p of the non-homogeneous ODE

The function $e^{-1.5x}$ on the RHS would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2e^{-1.5x}, \quad \text{then} \quad y_p' = C(2x - 1.5x^2)e^{-1.5x}, y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^5e^{-1.5x}$. Hence the given ODE has the general solution:

$$y = y_h + y_p = (c_1 + c_3)e^{-1.5x} - 5x^2e^{-1.5x}$$

Step 3. Solution of the initial value problem

Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives:

$$y' = (c_2 - 1.5c_1 - 1.5c_2v)e^{-1.2x} - 10xe^{-1.2x} + 7.5x^2e^{-1.2x}$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x} \quad \blacksquare$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

Example Application of the Sum Rule (c) 27

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) = -0.43.$$

Solution Application of the Sum Rule (c)

Step 1.General Solution of the homogeneous ODE

The characteristic equation of the homogeneous ODE is:

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$$

which gives the solution:

$$y_h = c_1e^{-x/2} + c_2e^{-3x/2}.$$

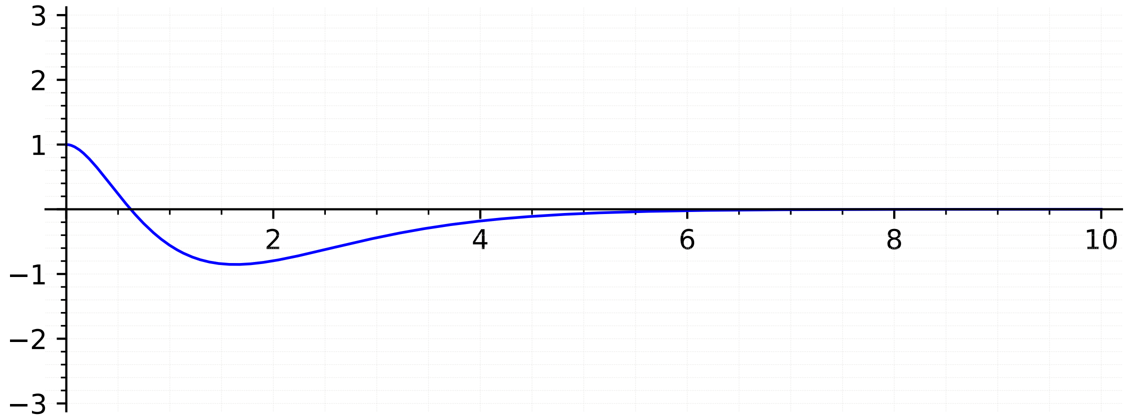

Figure 2.13.

Step 2. Particular Solution of the non-homogeneous ODE

We write:

$$y_{\mathrm{p}} = y_{\mathrm{p1}} + y_{\mathrm{p2}}$$

and following the table,

$$y_{\mathrm{p1}} = K \cos x + M \sin x \qquad \text{and} \qquad y_{\mathrm{p2}} = K_1 x + K_0$$

Differentiating gives:

$$y_{\mathrm{p1}}' = -K \sin x + M \cos x, \quad y_{\mathrm{p1}}'' = -K \cos x - M \sin x, \quad y_{\mathrm{p2}}' = 1, \quad y_{\mathrm{p2}}'' = 0.$$

Substitution of y_{p1} into the ODE in (7) gives, by comparing the cosine and sine terms:

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25$$

hence $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x and x^0 terms gives:

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0, \qquad \text{thus} \qquad K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE in (7) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32 \quad \blacksquare$$

Step 3. Solution of the initial value problem

From y, y' and the initial conditions we obtain:

$$y(0) = c_1 + c_2 - 0.32 = 2.78, \qquad y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1,\ c_2 = 0$. This gives the solution of the IVP:

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32. \quad \blacksquare$$

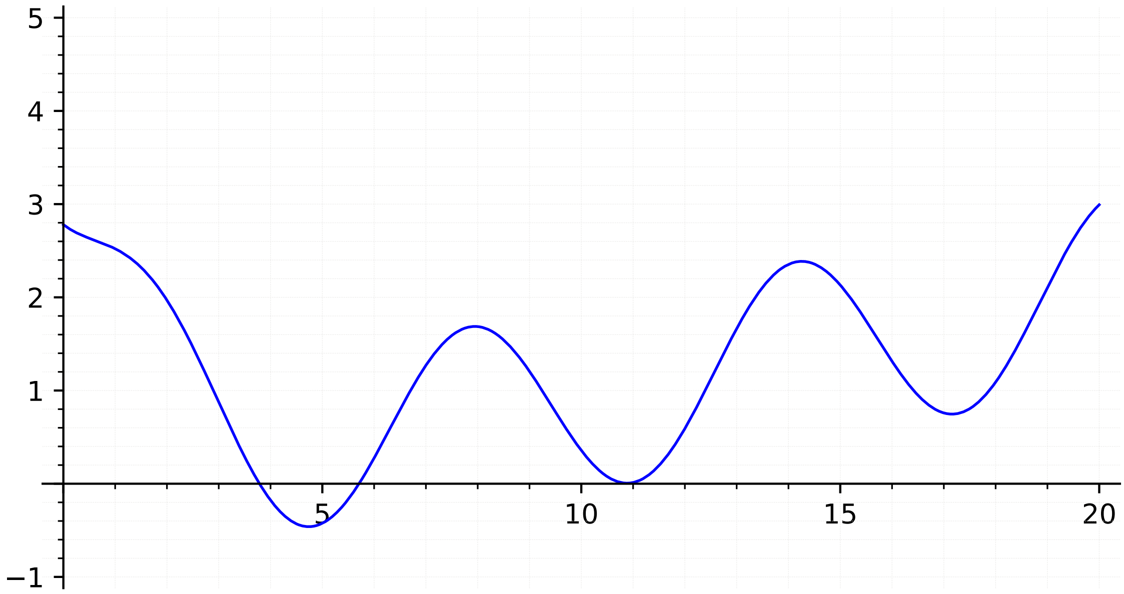

Figure 2.14.: Solution of Application of the Sum Rule (c)

Chapter 3.

Higher-Order Ordinary Differential Equations

3.1 Homogeneous Linear ODEs

Recall from **First-Order ODEs** that an ODE is of nth if the nth derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Therefore, based on the previous definition, the ODE has the form:

$$F(x, y, y', \cdots, y^{(n)}) = 0,$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{3.1}$$

(For $n = 2$ this is Eq. (3.1) in **Second-Order ODE** with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the RHS are any given functions of x, and y is unknown.

> $y^{(n)}$ has a coefficient of 1 which we call the **standard form**.

If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.

> An nth-order ODE that cannot be written in the form Eq. (3.1) is called **non-linear**.

If $r(x)$ is zero, in some open interval I, then Eq. (3.1) becomes:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.2}$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **non-homogeneous**. These definitions are the same as the ones were discussed in **Second-Order ODEs**.
A **solution** of an nth-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that's defined and n times differentiable on I.

Superposition and General Solution

The basic superposition or linearity principle discussed in **Second-Order ODEs** extends to nth-order homogeneous linear ODEs as following theorems.

Fundamental Theorem for the Homogeneous Linear ODE (2)
For a homogeneous linear ODE Eq. (3.2), sums and constant multiples of solutions on some open interval I are again solutions on I.

This does not hold for a nonhomogeneous or non-linear ODE.

General Solution, Basis, Particular Solution
A **general solution** of Eq. (3.2) on an open interval I is a solution of Eq. (3.2) on I of the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \quad \text{arbitrary}) \tag{3.3}$$

where $y_1, \cdots, y_n$ is a **fundamental system** of solutions of Eq. (3.2) on I.

That is, these solutions are linearly independent on I, as defined below.
A **particular solution** of Eq. (3.2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in Eq. (3.3).

Linear Independence and Dependence
Consider n functions $y_1(x), \cdots, y_n(x)$ defined on some interval I. These functions are called **linearly independent** on I if the equation:

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on } I \tag{3.4}$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

If and only if $y_1, \ldots, y_n$ are linearly dependent on I, we can express one of these functions on I as a **linear combination** of the other $n - 1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not).

This motivates the term linearly dependent. For instance, if Eq. (3.4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination:

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Example Linear Dependence 28

Show that the functions $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$ are linearly dependent on any interval.

Solution Linear Dependence

By inspection it can be seen that $y_2 = 0y_1 + 2.5y_2$. This relation of solutions proves linear dependence on any interval ■

Example General Solution 29

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{(where} \quad y^{\text{iv}} = d^4y/dx^4).$$

Solution General Solution

Similar to Chapter 2 we substitute $y = e^{4x}$. Omitting the common factor e^{4x}, we obtain the characteristic equation:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2\mu} + c_2 e^{-\nu} + c_3 e^{\nu} + c_4 e^{2\mu}$$

provided those four solutions are linearly independent ■

Example Initial Value Problem for a Third-Order Euler–Cauchy Equation 30

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \qquad y'(1) = 1, \qquad y''(1) = -4.$$

Solution **Initial Value Problem for a Third-Order Euler–Cauchy Equation**

General solution

As in Chapter 2, try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$. We can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I.

> In general one shall need a numerical method, such as Newton's to find the roots of the equation.

Hence a general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) we not continuous.

Particular solution

The derivatives are $y' = c_1 + 2c_2 x + 3c_3 x^2$ and $y'' = 2c_2 + 6c_3 x$. From this, and y and the initial conditions, we get by setting $x = 1$

$$\begin{aligned}
&\text{(a)} \quad y(1) = c_1 + \quad c_2 + c_3 = \quad 2 \\
&\text{(b)} \quad y'(1) = c_1 + 2c_2 + \quad 3c_3 = 1 \\
&\text{(c)} \quad y'(1) = 2c_2 + \quad 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule, or by elimination, which is simple, which gives the answer:

$$y = 2x + x^2 - x^3 \quad \blacksquare$$

3.1.1 Wronskian: Linear Independence of Solutions

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. From Chapter 2 we know how Wronskian work. This idea can be extended to n$^{\text{th}}$-order. This extended criterion uses the W of n solutions $y_1, \cdots, y_n$ defined as the n$^{\text{th}}$-order determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

> Note that W depends on x since $y_1, \cdots, y_n$ do. The criterion states that these solutions form a basis if and only if W is not zero.

3.1.2 Homogeneous Linear ODEs with Constant Coefficients

We proceed along the lines of Sec. 2.2, and generalize the results from n 2 to arbitrary n. We want to solve an nth-order homogeneous linear ODE with constant coefficients, written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(n)} = d^n y/dx^n$, etc. As in Sec. 2.2, we substitute $y = e^{\lambda x}$ to obtain the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 y = 0$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We can have distinct real roots, simple complex roots, multiple roots, and multiple complex roots, respectively. This will be shown next and illustrated by examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \qquad \cdots, \qquad y_m = e^{\lambda_m x} \tag{3.5}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{3.6}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

Example **Distinct Real Roots** 31

Solve the following ODE:

$$y''' - 2y'' - y' + 2y = 0$$

Solution **Distinct Real Roots**

The characteristic equation is:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

It has the roots $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

> If you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation.

The corresponding general solution Eq. (3.4) is:

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} \quad \blacksquare$$

Simple Complex Roots

If complex roots occur, they must **occur in conjugate pairs** as coefficients of Eq. (3.1) are real. Therefor, if $\lambda = \gamma + i\omega$ is a simple root of Eq. (3.2), so is the conjugate $\overline{\lambda} = \gamma - i\omega$, and two (2) corresponding linearly independent solutions are:

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

Example **Simple Complex Roots** 32

Solve the initial value problem:

$$y''' - y'' + 100y' - 100y = 0, \qquad y(0) = 4, \qquad y'(0) = 11, \qquad y''(0) = -299$$

Solution **Simple Complex Roots**

The characteristic equation is:

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10\mathbf{j}$.

Therefore, a general solution and its derivatives (obtained by differentiation) are:

$$\begin{aligned} y &= c_1 e^{x} + A \cos 10x + B \sin 10x, \\ y' &= c_1 e^{x} - 10A \sin 10x + 10B \cos 10x, \\ y'' &= c_1 e^{x} - 100A \cos 10x - 100B \sin 10x. \end{aligned}$$

From this and the initial conditions we obtain, by setting $x = 0$,

$$\text{(a) } c_1 + A = 4, \qquad \text{(b) } c_1 + 108 = 11, \qquad \text{(c) } c_1 - 1004 = -299$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is:

$$y = e^x + 3\cos 10x + \sin 10x \quad \blacksquare$$

This gives the solution curve, which oscillates about e^x.

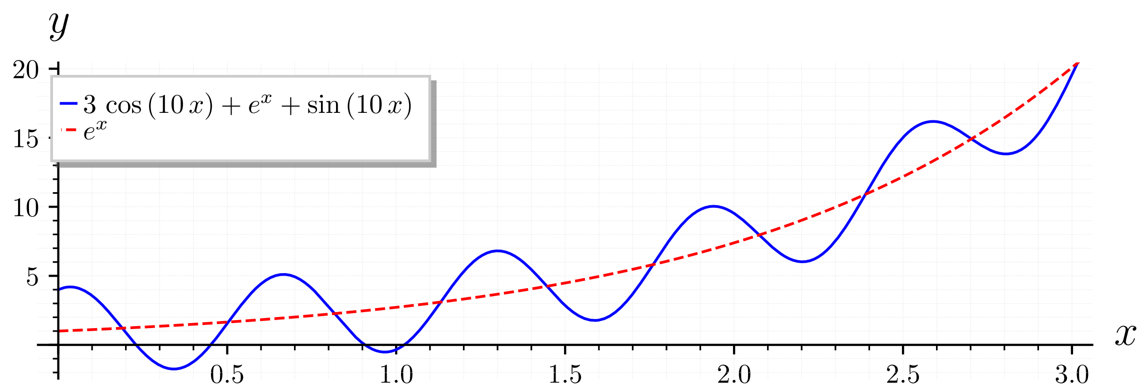

Figure 3.1.

Multiple Real Roots

If a real double root occurs ($\lambda_1 = \lambda_2$) then $y_1 = y_2$ in Eq. (3.3), and we take y_1 and xy_1 as corresponding linearly independent solutions.
More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2 e^{\lambda x}, \quad \cdots, \quad x^{m-1} e^{\lambda x}$$

Example Real Double and Triple Roots 33

Solve the following ODE:

$$y^{\text{v}} - 3y^{\text{iv}} + 3y''' - y'' = 0$$

Solution Real Double and Triple Roots

The characteristic equation is:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

and has the roots $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \lambda_0 = 1$, and the answer is

$$y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^x \quad \blacksquare$$

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots as discussed previously. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root**, so is the conjugate $\overline{\lambda} = \gamma - i\omega$.

Corresponding linearly independent solutions are

$$e^{\gamma x}\cos\omega x, \quad e^{\gamma x}\sin\omega x, \quad xe^{\gamma x}\cos\omega x, \quad xe^{\gamma x}\sin\omega x$$

The first two of these result from $e^{\lambda x}$ and $\overline{e}^{\lambda x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\overline{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}[(A_1 + A_2 x)\cos\omega x + (B_1 + B_2 x)\sin\omega x].$$

For **complex triple roots** (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\gamma x}\cos\omega x$, $x^2 e^{\gamma x}\sin\omega x$, and so on.

3.1.3 Non-Homogeneous Linear ODEs

We now turn from homogeneous to non-homogeneous linear ODEs of nth order. We write them in standard form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{3.7}$$

with $y^{(n)} = d^n y/dx^n$ as the first term, and $r(x) \neq 0$. As for second-order ODEs, a general solution of Eq. (3.7) on an open interval I of the x-axis is of the form:

$$y(x) = y_h(x) + y_p(x). \tag{3.8}$$

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a **general solution** of the corresponding homogeneous ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3.9}$$

on I. Also, y_p is any solution of Eq. (3.7) on I containing no arbitrary constants. If Eq. (3.7) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of Eq. (3.7) exists and includes all solutions. Thus Eq. (3.7) has no singular solutions. An **initial value problem** for Eq. (3.7) consists of Eq. (3.7) and n **initial conditions**:

$$y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with x_0 in I Under those continuity assumptions it has a unique solution.

> The ideas of proof are the same as those for $n = 2$.

Example IVP - Modification Rule 34

Solve the initial value problem:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \qquad y'(0) = -3, \qquad y''(0) = -47$$

Solution IVP - Modification Rule

Step 1

The characteristic equation is:

$$\lambda^2 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$$

It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is:

$$\begin{aligned} y_\mathrm{h} &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2) e^{-x} \end{aligned}$$

Step 2

If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has **NO** solution. Try Cxe^{-x} and Cx^6e^{-x}. The Modification Rule calls for

$$y_p = Cx^3e^{-x}$$

Then

$$\begin{aligned} {y_\mathrm{p}}' &= C(3x^2 - x^3)e^{-x}, \\ {y_\mathrm{p}}'' &= C(6x - 6x^2 + x^3)e^{-x}, \\ {y_\mathrm{p}}''' &= C(6 - 18x + 9x^2 - x^3)e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$, giving $y_p = 5x^2e^{-x}$.

Step 3

We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y'(0)$ and the third initial condition:

$$
\begin{aligned}
y &= y_h + y_p = (c_1 + c_2 + c_3 x^2)e^{-x} + 5x^3 e^{-x}, && y(0) = c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, && y'(0) = -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, && y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}
$$

Hence the answer to our problem is:

$$y = (3 - 25x^2)e^{-x} + 5x^2 e^{-x}$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$.

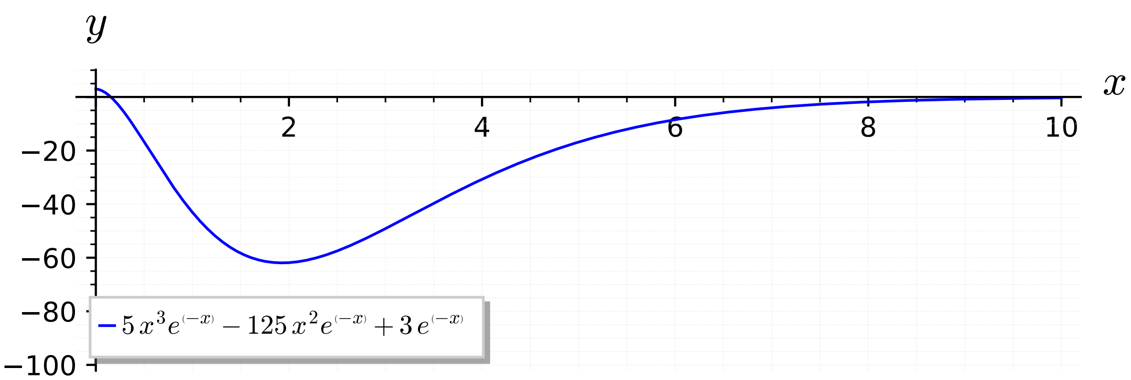

Figure 3.2.: Solution to the example "*IVP - Modification Rule*".

Model Elastic Beam

Whereas second-order ODEs have various applications, of which we have discussed some of the more important ones (i.e., RLC Circuit, Mass-Damper system), higher order ODEs have much fewer engineering applications.

> An important fourth-order ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge.

A related application of vibration of beams does not fit in here since it leads to PDEs.

Problem Description

Consider a beam B of length L and constant (e.g., **rectangular**) cross section and homogeneous elastic material (e.g., **level**).
We assume under its own weight the beam is bent so little that it is certainly straight. If we apply a load to B in a vertical plane through the axis of symmetry (the x-axis), B is bent.
Its axis is curved into the so-called **elastic curve** (or **deflection curve**).
It is shown in elasticity theory, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and y' is symmetric $y'(x)$ (determining the tangent direction of C) are small. Then, by calculus:

$$k = y''/(1 + y''x)^{1/2} \approx y''$$

Therefore:

$$M(x) = EIy''(x)$$

EI is the constant of proportionality. E Young's modulus of elasticity of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis.
Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length.
Together,

$$EIy^{\mathrm{iv}} = f(x)$$

Boundary Conditions

In applications the most important supports and corresponding boundary conditions are as follows and shown in Fig. 77.
* Simply supported

$$y = y'' = 0 \text{at} x = 0 \text{and} L$$

$$y = y' = 0 \text{at} x = 0 \text{and} L$$

$$\text{(C)Clamped at} x = 0, \text{free at} x = L$$

$$y(0) = y'(0) = 0, y''(L) = y'''(L) = 0.$$

The boundary condition $y = 0$ means no displacement at that point, $y'' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Solution Derivation

Let us apply this to the uniformly loaded simply supported beam in Fig. 76. The load is $f(x) = f_0 = \text{const}$. Then (8) is

$$y^{iv} = k, \qquad k = \frac{f_0}{EI}.$$

This can be solved simply by calculus. Two integrations give

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2,$$

$y''(0) = 0$ gives $c_2 = 0$. Then $y''(L) = L(\frac{1}{2}kL + c_1) = 0$, $c_1 = -kL/2$ (since $L \neq 0$). Hence

$$y'' = \frac{k}{2}(x^2 - Lx).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution

$$y = \frac{f_0}{24EI}(x^4 - 2Lx^3 + L^3 x).$$

As the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be **symmetric** with respect to $L/2$, that is, $y(x) = y(L - x)$.

Verify this by setting $x = u + L/2$ and show that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can observe the maximum deflection in the middle at $u = 0$ $(x = L/2)$ is:

$$\frac{5f_0L^4}{(16 \cdot 24EI)}$$

> Recall that the positive direction points downward.

Chapter 4.

Appendix

4.1 List of Common Integration Operations

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

$$\int \frac{1}{x}dx = \ln|x|$$

$$\int udv = uv - \int vdu$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x}$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a}$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln \left[\sqrt{x} + \sqrt{x+a}\right]$$

$$\int x\sqrt{ax+b} dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b}$$

$$\int \sqrt{x(ax+b)} dx = \frac{1}{4a^{3/2}} \left[(2ax+b)\sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right]$$

$$\int \sqrt{x^3(ax+b)} dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right|$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \sqrt{ax^2 + bx + c} dx = \frac{b + 2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx^+ c)} \right|$$

$$\int x\sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \left(2\sqrt{a}\sqrt{ax^2 + bx + c} \times \left(-3b^2 + 2abx + 8a(c + ax^2)\right) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| \right)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right|$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right|$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a}\right) \ln(ax+b) - x, a \neq 0$$

$$\int \ln(x^2+a^2)\, \mathrm{d}x = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x$$

$$\int \ln(x^2-a^2)\, \mathrm{d}x = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x$$

$$\int \ln\left(ax^2+bx+c\right) dx = \frac{1}{a}\sqrt{4ac-b^2} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x\right) \ln\left(ax^2+bx+c\right)$$

$$\int x \ln(ax+b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax+b)$$

$$\int x \ln\left(a^2 - b^2x^2\right) dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right) \ln\left(a^2 - b^2x^2\right)$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a}\sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}\left(i\sqrt{ax}\right), \quad \text{where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int x e^{x} dx = (x-1) e^{x}$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{x} dx = \left(x^2 - 2x + 2\right) e^{x}$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$$

$$\int x^3 e^{x} dx = \left(x^3 - 3x^2 + 6x - 6\right) e^{x}$$

$$\int x^n e^{ax}\, \mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, \mathrm{d}x$$

$$\int x^n e^{ax}\, \mathrm{d}x = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\, \mathrm{d}t$$

$$\int e^{ax^2}\, \mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(ix\sqrt{a}\right)$$

$$\int e^{-ax^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(x\sqrt{a}\right)$$

$$\int x e^{-ax^2}\, \mathrm{d}x = -\frac{1}{2a} e^{-ax^2}$$

$$\int x^2 e^{-ax^2}\, \mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a} \cos ax$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \; {}_2F_1 \left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right]$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$

$$\int \cos ax dx = \frac{1}{a} \sin ax$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times {}_2F_1 \left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right]$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1 \left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right)$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan \frac{x}{2}\right)$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax$$

$$\int \sec^3 x \, \mathrm{dx} = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x|$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x, n \neq 0$$

$$\int \csc x dx = \ln\left|\tan \frac{x}{2}\right| = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln|\csc x - \cot x|$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x, n \neq 0$$

$$\int \sec x \csc x dx = \ln|\tan x|$$

Products of Trigonometric Functions and Monomials

$$\int x\cos x dx = \cos x + x\sin x$$

$$\int x\cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

$$\int x^2\cos x dx = 2x\cos x + \left(x^2-2\right)\sin x$$

$$\int x^2\cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$$

$$\int x^n cosx dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix) + (-1)^n\Gamma(n+1,ix)\right]$$

$$\int x^n cosax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax) - \Gamma(n+1,ixa)\right]$$

$$\int x\sin x dx = -x\cos x + \sin x$$

$$\int x\sin ax dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2\sin x dx = \left(2-x^2\right)\cos x + 2x\sin x$$

$$\int x^2\sin ax dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$$

$$\int x^n\sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix) - (-1)^n\Gamma(n+1,-ix)\right]$$

Products of Trigonometric Functions and Exponentials

$$\int e^x\sin x dx = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\int e^{bx}\sin ax dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$$

$$\int e^x\cos x dx = \frac{1}{2}e^x(\sin x + \cos x)$$

$$\int e^{bx}\cos ax dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$$

$$\int xe^x\sin x dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$$

$$\int xe^x\cos x dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax$$

$$\int e^{ax}\cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax$$

$$\int e^{ax}\sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases}$$

$$\int e^{ax}\tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}{}_2F_1\left[1+\dfrac{a}{2b},1,2+\dfrac{a}{2b},-e^{2bx}\right] & \\ \quad -\dfrac{1}{a}e^{ax}{}_2F_1\left[\dfrac{a}{2b},1,1E,-e^{2bx}\right] & a \neq b \\ \dfrac{e^{ax}-2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases}$$

$$\int \tanh ax\, dx = \frac{1}{a}\ln\cosh ax$$

$$\int \cos ax\cosh bx dx = \frac{1}{a^2+b^2}\left[a\sin ax\cosh bx + b\cos ax\sinh bx\right]$$

$$\int \cos ax\sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax\cosh bx + a\sin ax\sinh bx\right]$$

$$\int \sin ax\cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax\cosh bx + b\sin ax\sinh bx\right]$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b \cosh bx \sin ax - a \cos ax \sinh bx\right]$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right]$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}\left[b \cosh bx \sinh ax - a \cosh ax \sinh bx\right]$$

4.2 Common Laplace Transforms

$f(t)$	$\mathcal{L}f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) -$
	$\cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
t^n $(n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
t^x $(x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\dfrac{ae^{at} - be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at} \sin kt$	$\dfrac{k}{(s-a)^2 + k^2}$
$e^{at} \cos kt$	$\dfrac{s-a}{(s-a)^2 + k^2}$
$e^{at} \sinh kt$	$\dfrac{k}{(s-a)^2 - k^2}$
$e^{at} \cosh kt$	$\dfrac{s-a}{(s-a)^2 - k^2}$
$t \sin kt$	$\dfrac{2ks}{(s^2 + k^2)^2}$
$t \cos kt$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$
$t \sinh kt$	$\dfrac{2ks}{(s^2 - k^2)^2}$
$t \cosh kt$	$\dfrac{s^2 - k^2}{(s^2 - k^2)^2}$
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\text{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$