Tutorial Book

M.Sc Higher Mathematics II

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Version: 2025.SS

List of Questions

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Theory of Probability

[Q1] Assembling Computers

Sam is going to assemble a computer by himself. He has the choice of chips from 3 brands, a hard drive from 4, memory from 3, and an accessory bundle from 5 local stores.
How many different ways can Sam order the parts?

(**Answer:** 180)

[A1]

As $n_1 = 3$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$ there are:

$$n_1 \times n_2 \times n_3 \times n_4 = 3 \times 4 \times 3 \times 5 = 180 \quad \blacksquare$$

[Q2] Four Digits

How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

(**Answer:** 156)

[A2]

Since the number must be even, we have only $n_1 = 3$ choices for the units position. However, for a four-digit number the thousands position **cannot** be 0. Hence, we consider the units position in two (2) parts: 0 or not 0. If the units position is 0 (i.e., $n_1 = 1$), we have $n_2 = 5$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. Therefore, in this case we have a total of

$$n_1 n_2 n_3 n_4 = 1 \times 5 \times 4 \times 3 = 60,$$

even four-digit numbers. On the other hand, if the units position is not 0 (i.e., $n_1 = 2$), we have $n_2 = 4$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position.

In this situation, there are a total of

$$n_1 n_2 n_3 n_4 = 2 \times 4 \times 4 \times 3 = 96,$$

even four-digit numbers. Since the above two cases are mutually exclusive, the total number of even four-digit numbers can be calculated as 60 + 96 = 156 ■

[Q3] Choosing the President and Treasurer

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if:

i. there are no restrictions;
ii. A will serve only if he is president;
iii. B and C will serve together or not at all;
iv. D and E will not serve together?

(**Answer:** i. 2450, ii. 2401, iii., 2258, iv. 2448)

[A3]

i. The total number of choices of officers, without any restrictions, is

$$\frac{50!}{48!} = 2450 \quad \blacksquare$$

ii. Since A will serve only if they are president, we have two (2) situations here:

   a) A is selected as the president, which gives 49 possible outcomes for the treasurer's position

   b) Officers are selected from the remaining 49 people **without** A, which has the number of choices:

$$\frac{49!}{47!} = 2352$$

   Therefore, the total number of choices is 49 + 2352 = 2401 ■

iii. The number of selections when B and C serve together is 2. The number of selections when both B and C are not chosen is

$$\frac{48!}{46!} = 2255$$

   Therefore, the total number of choices in this situation is 2 + 2255 = 2257.

iv. The number of selections when D serves as an officer but not E is:

$$(2)\,(48) = 96,$$

where 2 is the number of positions D can take and 48 is the number of selections of the other officer from the remaining people in the club except E.

The number of selections when E serves as an officer but not D is also:

$$(2)\,(48) = 96,$$

The number of selections when both D and E are not chosen is:

$$\binom{48}{2} = 2256$$

Therefore, the total number of choices is:

$$(2)\,(96) + 2256 = 2448 \quad \blacksquare$$

[Q4] The Lineup

In a rugby training session, the defensive coordinator needs to have ten (10) players standing in a row. Among these ten (10) players, there are one (1) freshman, two (2) sophomores, four (4) juniors, and three (3) seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

(**Answer:** 12 600)

[A4]

We find that the total number of arrangements is:

$$\frac{10!}{1!2!3!4!} = 12\,600 \quad \blacksquare$$

[Q5] Arranging Accommodations

In how many ways can 7 graduate students be assigned to one triple and two double hotel rooms during a conference?

(**Answer:** 210)

[A5]

The total number of possible partitions would be

$$\binom{7}{3,\,2,\,2} = \frac{7!}{3!\;2!\;2!} = 210 \quad \blacksquare$$

[Q6] Buying Games

A young boy asks his mother to get 5 games from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

(**Answer:** 1200)

[A6]

The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = 120.$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = 10.$$

Using the multiplication rule with $n_1 = 120$ and $n_2 = 10$, we have

$$120 \times 10 = 1200 \quad \text{ways} \qquad \blacksquare$$

[Q7] Letter Soup

How many different letter arrangements can be made from the letters in the word STATISTICS ?

(**Answer:** 50400)

[A7]

We can use the classification idea we discussed:

$$\binom{10}{3,\ 3,\ 2,\ 1,\ 1} = \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!} = 50400 \quad \blacksquare$$

Here we have 10 total letters, with 2 letters (S, T) appearing 3 times each, letter I appearing twice, and letters A and C appearing once each.

[Q8] Flip of a Coin

A coin is tossed twice. What is the probability that at least 1 head occurs?

(**Answer:** $^3/_4$)

[A8]

The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of p to each sample point. Then:

$$4p = 1, \qquad \text{or} \qquad p = 1/4.$$

If A represents the event of **at least 1 head occurring**, then:

$$A = \{HH, HT, TH\} \qquad \text{and} \qquad P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \quad \blacksquare$$

[Q9] That is Not Fair!

A die is loaded in a way that an even number is **twice as likely** to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

(**Answer:** $^4/_9$)

[A9]

The sample space is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We assign a probability of p to each odd number and a probability of $2p$ to each even number. Since the sum of the probabilities must be 1, we have

$$9p = 1 \qquad \text{and} \qquad p = 1/9.$$

Therefore, probabilities of $^1/_9$ and $^2/_9$ are assigned to each odd and even number, respectively. Therefore,

$$E = \{1, 2, 3\} \qquad \text{and} \qquad P(E) = {}^1/_9 + {}^2/_9 + {}^1/_9 = {}^4/_9 \quad \blacksquare$$

[Q10] Who Wants to Help

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students.

If a person is **randomly** selected by the instructor to answer a question, find the probability that the student chosen is:

a. an industrial engineering major,

b. a civil engineering or an electrical engineering major.

(**Answer:** a. $^{25}/_{53}$, b. $^{18}/_{53}$)

[A10]

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively.

The total number of students in the class is 53, all of whom are equally likely to be selected.

a. Since 25 of the 53 students are majoring in industrial engineering, the probability of event I, selecting an industrial engineering major at random, is:

$$P(I) = \frac{25}{53}$$

b. Since 18 of the 53 students are civil or electrical engineering majors, it follows that:

$$P(C \cup E) = \frac{10}{53} + \frac{8}{53} = \frac{18}{53} \quad \blacksquare$$

[Q11] I Fold

In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

(**Answer:** 0.9×10^{-5})

[A11]

The number of ways of being dealt 2 aces from 4 cards is:

$$\binom{4}{2} = \frac{4!}{2!2!} = 6,$$

and the number of ways of being dealt 3 jacks from 4 cards is

$$\binom{4}{3} = \frac{4!}{3!2!} = 4, .$$

There are $n = (6)(4) = 24$ hands with 2 aces and 3 jacks. The total number of 5-card poker hands, all of which are equally likely, is

$$N = \binom{52}{5} = \frac{52!}{5!47!} = 2\,598\,960$$

Therefore, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is:

$$P(C) = \frac{24}{2\,598\,960} = 0.9 \times 10^{-5} \quad \blacksquare$$

[Q12] Any Colour as long as it is Black

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colours?

(**Answer:** 0.68)

[A12]

Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is:

$$P(G \cap W \cap R \cap B) = P(G) + P(W) + P(R) + P(B) = 0.09 + 0.15 + 0.21 + 0.23 = 0.68 \quad \blacksquare$$

[Q13] Check your Engine Light

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

(**Answer:** 0.69)

[A13]

Let E be the event that at least 5 cars are serviced. Now:

$$P(E) = 1 - P\left(E^{c}\right).$$

Since;

$$P(E) = 0.12 + 0.19 = 0.31,$$

it follows that

$$P(E) = 1 - 0.31 = 0.69 \quad \blacksquare$$

[Q14] Got to Catch My Flight

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane:

a. arrives on time, given that it departed on time, and

b. departed on time, given that it has arrived on time.

(**Answer:** a. 0.94, b. 0.95)

[A14]

a. The probability that a plane arrives on time, given that it departed on time is:

$$P\left(A|D\right) = \frac{P\left(D \cap A\right)}{P\left(D\right)} = \frac{0.78}{0.83} = 0.94 \quad \blacksquare$$

b. The probability that a plane departed on time, given that it has arrived on time, is

$$P\left(D|A\right) = \frac{P\left(D \cap A\right)}{P\left(A\right)} = \frac{0.78}{0.82} = 0.95 \quad \blacksquare$$

[Q15] The Calculation Looms

The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two (2) ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

(**Answer:** 0.08)

[A15]

Consider the events

L The length is defective,

T The texture is defective.

Given that the strip is length defective, the probability that this strip is texture defective is given by:

$$P\left(T|L\right) = \frac{P\left(T \cap L\right)}{P\left(L\right)} = \frac{0.008}{0.1} = 0.08$$

Thus, knowing the conditional probability provides considerably more information than merely knowing $P\left(T\right)$ $\blacksquare$

[Q16] Fuse In and Fuse Out

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

(**Answer:** $1/19$)

[A16]

We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective. Then we interpret $A \cap B$ as the event that A occurs and then B occurs after A has occurred.
The probability of first removing a defective fuse is 1/4. Then the probability of removing a second defective fuse from the remaining 4 is 4/19.
Hence:

$$P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19} \quad \blacksquare$$

[Q17] White or Black

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

(**Answer:** $^{38}/_{63}$)

[A17]

Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$.

Now:

$$\begin{aligned} P\Big((B_1 \cap B_2) \quad \text{or} \quad (W_1 \cap B_2)\Big) &= P\Big((B_1 \cap B_2)\Big) + P\Big((W_1 \cap B_2)\Big) \\ &= P(B_1)\, P(B_2|B_1) + P(W_1)\, P(B_2|W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63} \quad \blacksquare \end{aligned}$$

[Q18] It is my Turn to Draw!

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

(**Answer:** $^{8}/_{5525}$)

[A18]

First we define the events:

A_1 the first card is a red ace,

A_2 the second card is a 10 or a jack,

A_3 the third card is greater than 3 but less than 7.

Now:

$$P(A_1) = \frac{2}{52}, \qquad P(A_2|A_1) = \frac{8}{51}, \qquad \text{and} \qquad P(A_3|A_1 \cap A_2) = \frac{12}{50}$$

Using a Venn diagram we can derive the following mathematical expression:

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) \\ &= \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{78}{5525} \quad \blacksquare \end{aligned}$$

[Q19] A Firetruck is Red

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

(**Answer:** 0.9016)

[A19]

Let A and B represent the respective events that the fire engine and the ambulance are available. Then:

$$P(A \cap B) = P(A)\, P(B) = (0.98)(0.92) = 0.9016 \quad \blacksquare$$

[Q20] This Car went to Vegas and Back

If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

(**Answer:** $f(x) = \frac{1}{16}\binom{4}{x}, \qquad \text{for} \qquad x = 0, 1, 2, 3, 4.$)

[A20]

As the probability of selling a car with side airbags is 0.5, the $2^4 = 16$ points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16.
To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other.
This can be done in $\binom{4}{3} = 4$ ways. In general, the event of selling x models with side airbags and 4 x models without side airbags can occur in $\binom{4}{x}$ ways, where x can be 0, 1, 2, 3, or 4. Therefore the probability distribution is:

$$f(x) = \frac{1}{16}\binom{4}{x}, \qquad \text{for} \qquad x = 0,\ 1,\ 2,\ 3,\ 4 \quad \blacksquare$$

[Q21] Let me Get my Pen

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

(**Answer:** 1.7)

[A21]

Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad \text{where} \qquad x = 0, 1, 2, 3.$$

By doing the calculations we arrive at:

$$f(0) = \frac{1}{35}, \quad f(1) = \frac{12}{35}, \quad f(2) = \frac{18}{35}, \qquad \text{and} \quad f(3) = \frac{4}{35}.$$

Therefore:

$$\mu = E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right)(2)\left(\frac{18}{35}\right)(3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components $\blacksquare$

[Q22] I Have a Washing Machine

The total number of hours, measured in units of 100 hours, that a family runs a washing machine over a period of one year is a continuous random variable X that has the density function:

$$f(x) = \begin{cases} x & 0 < x < 1, \\ 2 - x & 1 \le x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that over a period of one year, a family runs their washing machine:

a. less than 120 hours,

b. between 50 and 100 hours.

(**Answer:** a. 0.68, b. 0.375)

[A22]

a. For less than 120 hours,

$$P(X < 1.2) = \int_0^1 x \, \mathrm{d}x + \int_1^{1.2} (2 - x) \, \mathrm{d}x = \left. \frac{x^2}{2} \right|_0^1 + \left. \left(2x - \frac{x^2}{2}\right) \right|_1^{1.2} = 0.68 \quad \blacksquare$$

b. For between 50 and 100 hours,

$$P(0.5 < X < 1) = \int_{0.5}^1 x \, \mathrm{d}x = \left. \frac{x^2}{2} \right|_{0.5}^1 = 0.375 \quad \blacksquare$$

[Q23] What Licence Plate Officer?

A witness to a traffic accident told the police, the license number contained the letters RLH followed by 3 digits, the first of which was a 5.
If the witness cannot recall the last 2 digits, but is **certain** that all 3 digits are different, find the maximum number of automobile registrations that the police may have to check.

(**Answer:** 72)

[A23]

As the first digit is a 5, there are $n_1 = 9$ possibilities for the second digit and then $n_2 = 8$ possibilities for the third digit. Therefore, by the multiplication rule there are $n_1 n_2 = 9 \cdot 8 = 72$ registrations to be checked $\blacksquare$

[Q24] Binomial Distribution

Calculate the probability of obtaining at least two (2) "six" in rolling a fair die 4 times.

[A24]

The answer is as follows:

$$p = P(A) = P(\text{six}) = \frac{1}{6}, \quad q = \frac{5}{6}, \quad \text{and} \quad n = 4.$$

The event at least two (2) "six" occurs if we obtain 2 or 3 or 4 "six" Hence the answer is:

$$\begin{aligned} p &= f(2) + f(3) + f(4) \\ &= \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 + \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 + \binom{4}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 \\ &= \frac{1}{6^4}(6 \cdot 25 + 4 \cdot 5 + 1) = \frac{171}{1296} = 13.2\% \quad \blacksquare \end{aligned}$$

2

Statistical Methods

[Q25] Maximum Likelihood of Poisson Distribution

Consider a Poisson distribution with probability mass function:

$$f\left(x|\mu\right) = \frac{e^{-\mu}\mu^{x}}{x!} \qquad \text{where} \qquad x = 0,\, 1,\, 2,\, \ldots.$$

Suppose that a random sample $x_1, x_2, \ldots, x_n$ is taken from the distribution. What is the maximum likelihood estimate of μ?

[A25]

The likelihood function is

$$L\left(x_1, x_2, \ldots, x_n; \mu\right) = \prod_{i=1}^{n} f\left(x_i|\mu\right) = \frac{e^{-n\mu}\sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i!}.$$

Now consider is logarithmic representation:

$$\ln L\left(x_1, x_2, \ldots, x_n; \mu\right) = -n\mu + \sum_{i=1}^{n} x_i \ln\mu - \ln\prod_{i=1}^{n} x_i!$$

And taking its partial derivative to the parameter gives:

$$\frac{\partial \ln L\left(x_1, x_2, \ldots, x_n; \mu\right)}{\partial\mu} = -n + \sum_{i=1}^{n}\frac{x_i}{\mu}.$$

Solving for $\hat{\mu}$, the maximum likelihood estimator, involves setting the derivative to zero and solving for the parameter. Therefore,

$$\hat{\mu} = \sum_{i=1}^{n}\frac{x_i}{n} = \hat{x}$$

If you were to test it, the second derivative of the log-likelihood function is **negative**, which implies that the solution above indeed is a maximum. As μ is the mean of the Poisson distribution, the sample average would certainly seem like a reasonable estimator ■.

[Q26] Maximum Likelihood of Gaussian Distribution

Find maximum likelihood estimates for $\theta_1 = \mu$ and $\theta_2 = \sigma$ in the case of the normal distribution.

[A26]

We obtain the likelihood function:

$$L = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma}\right)^n e^{-h}$$

$$\text{where} \qquad h = \frac{1}{2\sigma^2} \sum_{j=1}^{n} \left(x_j - \mu\right)^2 .$$

Taking logarithms, we have

$$\ln L = -n \ln \sqrt{2\pi} - n \ln \sigma - h.$$

The first equation for the parameters is $\dfrac{\partial \ln L}{\partial \mu} = 0$, written out:

$$\frac{\partial \ln L}{\partial \mu} = -\frac{\partial h}{\partial \mu} = \frac{1}{\sigma^2} \sum_{j=1}^{n} \left(x_j - \mu\right) = 0,$$

$$\text{therefore} \qquad \sum_{j=1}^{n} x_j - n\mu = 0.$$

The solution is the desired estimate $\hat{\mu}$ for μ: we find

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} x_j = \overline{x} .$$

The second equation for the parameter is $\partial \ln L / \partial \sigma = 0$, written out

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\partial h}{\partial \sigma} = -\frac{1}{\sigma} + \frac{1}{\sigma^3} \sum_{j=1}^{n} \left(x_j - \mu\right)^2 = 0.$$

Replacing μ by $\hat{\mu}$ and solving for σ^2, we obtain the estimate:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \left(x_j - \overline{x}\right)^2 \quad \blacksquare$$

[Q27] For Science

Suppose ten (10) rats are used in a biomedical study where they are injected with cancer cells and then given a cancer drug that is designed to increase their survival rate. The survival times, in months, are:

14 17 27 18 12 8 22 13 19 12

Assume exponential distribution applies which is given as:

$$f(x, \beta) = \begin{cases} \dfrac{1}{\beta} \exp \dfrac{x}{\beta}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give a maximum likelihood estimate of the mean survival time.

[A27]

We know that the probability density function for the exponential random variable X. Therefore, the log-likelihood function for the data, given $n = 10$, is:

$$\ln L\left(x_1, x_2, \ldots, x_{10}; \beta\right) = -10 \ln \beta - \frac{1}{\beta} \sum_{i=1}^{10} x_\mathrm{i}.$$

Setting

$$\frac{\partial \ln L}{\partial \beta} = -\frac{10}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^{10} x_\mathrm{i} = 0 \qquad \text{which means}$$

$$\hat{\beta} = \frac{1}{10} \sum_{i=1}^{10} x_\mathrm{i} = \hat{x} = 16.2 \quad \blacksquare$$

Evaluating the second derivative of the log-likelihood function at the value $\hat{\beta}$ above gives a negative value. As a result, the estimator of the parameter β, the population mean, is the sample average $\overline{x}$.

[Q28] Sampling the Population

It is known that a sample consisting of the values:

12 11.2 13.5 12.3 13.8 11.9

comes from a population with the density function:

$$f(x; \theta) = \begin{cases} \dfrac{\theta}{\theta^{\,x+1}}, & x > 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$. Find the maximum likelihood estimate of θ.

[A28]

The likelihood function of n observations from this population can be written as:

$$L\left(x_1, x_2, \ldots, x_{10}; \theta\right) = \prod_{i=1}^{n} \frac{\theta}{x_\mathrm{i}^{\theta+1}} = \frac{\theta^\mathrm{n}}{\left(\prod_{i=1}^{n} x_\mathrm{i}\right)^{\theta+1}},$$

which implies that

$$\ln L\left(x_1, x_2, \ldots, x_{10}; \theta\right) = n \ln \theta - (\theta + 1) \sum_{i=1}^{n} \ln x_\mathrm{i}.$$

Setting $0 = \partial \ln L / \partial \theta = {}^{n}/_{\theta} - \sum_{i=1}^{n} \ln(x_i)$ results in

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_\mathrm{i}} = 0.3970 \quad \blacksquare.$$

Since the second derivative of L is $-n/\theta^2$, which is always negative, the likelihood function does achieve its maximum value at $\hat{\theta}$.

[Q29] Confidence Interval for Mean with known Variance in Normal Distribution

Determine 95% confidence interval for the mean of a normal distribution with variance $\sigma^2 = 9$, using a sample of $n = 100$ values with mean $\overline{x} = 5$.

[A29]

1. First we define γ as 0.95 based on the 95% confidence.
2. Then looking at our reference table find the corresponding c which equals 1.960.
3. $\overline{x} = 5$ is given.
4. We need:

$$k = c\frac{\sigma}{\sqrt{n}} = 1.960\frac{3}{\sqrt{100}} = 0.588$$

Therefore

$$\overline{x} - k = 4.412 \qquad \text{and} \qquad \overline{x} + k = 5.588$$

and the confidence interval is:

$$\text{CONF}_{0.95}\left\{4.412 \leq \mu \leq 5.588\right\} \quad \blacksquare$$

[Q30] Sample Size Needed for a Confidence Interval of Prescribed Length

How large must n be in the Example **Confidence Interval for mean with known variance in Normal Distribution** to obtain a 95% confidence interval of length $L = 0.4$?

[A30]

The interval in Example **Confidence Interval for mean with known variance in Normal Distribution** has the length:

$$L = 2k = 2c\sigma/\sqrt{n}.$$

Solving for n, we obtain

$$n = \left(\frac{2c\sigma}{L}\right)^2$$

In the present case the answer is:

$$n = \left(\frac{2 \times 1.96 \times 3}{0.4}\right)^2 \approx 870 \quad \blacksquare$$

[Q31] Confidence Interval for Mean of Normal Distribution with Unknown Variance

The five (5) independent measurements of flash point of Diesel oil (D-2) gave the values (in °F):

144 147 146 142 144

If we assume normality, determine a 99% confidence interval for the mean.

[A31]

1. $\gamma = 0.99$ is required based on 99% confidence level.
2. $F(c) = \frac{1}{2}(1+\gamma) = 0.99$ and looking at the reference table with $n-1=4$ d.f., which gives $c = 4.60$.
3. Calculating the mean and the variance gives $\overline{x} = 144.6$ and $s = 3.8$,
4. $k = \sqrt{3.8} \times 4.60/\sqrt{5} = 4.01$. Therefore the confidence interval is:

$$\text{CONF}_{0.99}\{140.5 \leq \mu \leq 148.7\} \quad \blacksquare$$

If the variance σ^2 were known and equal to the sample variance s^2, therefore $\sigma^2 = 3.8$, then the Reference Table would give:

$$k = \frac{c\sigma}{\sqrt{n}} = 2.576\frac{\sqrt{3.8}}{\sqrt{3}} = 2.25$$

and

$$\text{CONF}_{0.99}\{142.35 \leq \mu \leq 146.85\} \quad \blacksquare$$

We see that the present interval is almost twice as long as that with a known variance $\sigma^2 = 3.8$.

[Q32] Confidence Interval for the Variance of the Normal Distribution

Determine a 95% confidence interval for the variance, based on the following sample (tensile strength of sheet steel in kg mm^{-2}, rounded to integer values)

$$89 \quad 84 \quad 87 \quad 81 \quad 89 \quad 86 \quad 91 \quad 90 \quad 78 \quad 89 \quad 87 \quad 99 \quad 83 \quad 89$$

[A32]

1. $\gamma = 0.95$ is required.
2. For $m = n - 1 = 13$ we find

$$c_1 = 5.01 \qquad \text{and} \qquad c_2 = 24.74.$$

3. $13s^2 = 326.9$
4. $13s^2/c_1 = 65.25 \qquad \text{and} \qquad 13s^2/c_2 = 13.21$
5. This makes the confidence interval as:

$$\text{CONF}_{0.09}\left\{13.21 \leq \sigma^2 \leq 65.25\right\} \quad \blacksquare$$

This is rather large, and for obtaining a more precise result, one would need a much larger sample.

[Q33] Test for the Mean of the Normal Distribution with Known Variance

Let X be a normal random variable with variance $\sigma^2 = 9$. Using a sample of size $n = 10$ with mean $\overline{x}$, test the hypothesis $\mu = \mu_0 = 24$ against the three (3) kinds of alternatives, namely,

(a) $\mu > \mu_0$ (b) $\mu < \mu_0$ (c) $\mu \neq \mu_0$

[A33]

We choose the significance level $\alpha = 0.05$ as it is customary at this point. An estimate of the mean will be obtained from:

$$\overline{X} = \frac{1}{n}\left(X_1 + \cdots + X_n\right).$$

If the hypothesis is true, $\overline{X}$ is normal with mean $\mu = 24$ and variance $\sigma^2/n = 0.9$. Therefore we may obtain the critical value c from X.

Right-Sided Test We determine c from

$$P\left(\overline{X} > c\right)_{\mu=24} = \alpha = 0.05$$

that is,

$$P\left(\hat{X} \leq c\right)_{\mu=24} = \Phi\left(\frac{c-24}{\sqrt{0.9}}\right) = 1 - \alpha = 0.95.$$

Reverse engineering **Table** 5.4 by looking for 0.95 percentile gives $(c-24)/\sqrt{0.9} = 1.645$, and $c = 25.56$, which is greater than μ_0. If $\overline{x} \leqq 25.56$, the hypothesis is accepted. If $\overline{x} > 25.56$, it is rejected ■

Left-Sided Test The critical value c is obtained from the equation

$$P\left(\hat{X} \leqq c\right)_{\mu-24} = \Phi\left(\frac{c-24}{\sqrt{0.9}}\right) = \alpha = 0.05.$$

Reverse engineering **Table** 5.4 by looking for 0.95 percentile gives $c = 24 - \sqrt{0.9} \times 1.645 = 22.44$. If $\hat{x} \geqq 22.44$, we accept the hypothesis. If $\hat{x} < 22.44$, we reject it ■

Two-Sided Test As the normal distribution is **symmetric**, we choose c_1 and c_2 equidistant from $\mu = 24$, say, $c_1 = 24 - k$ and $c_2 = 24 + k$, and determine k from:

$$P\left(24 - k \leqq \hat{X} \leqq 24 + k\right)_{\mu=24} = \Phi\left(\frac{k}{\sqrt{0.9}}\right) - \Phi\left(\frac{-k}{\sqrt{0.9}}\right) = 1 - \alpha = 0.95.$$

Looking for 0.975 in **Table** 5.4 gives $k/\sqrt{0.9} = 1.960$, therefore $k = 1.86$. This gives the values $c_1 = 24 - 1.86 = 22.14$ and $c_2 = 24 + 1.86 = 25.86$. If $\hat{x}$ is not smaller than c_1 and not greater than c_2, we accept the hypothesis. Otherwise, we reject it ■

[Q34] Test for the Mean of the Normal Distribution with Unknown Variance

The tensile strength of a sample of $n = 16$ manila ropes was measured. The sample mean was $\hat{x} = 4482$ kg, and the sample standard deviation was $s = 115$ kg. Assuming that the tensile strength is a normal random variable, test the hypothesis $\mu_0 = 4500$ kg against the alternative $\mu_1 = 4400$ kg. Here μ_0 may be a value given by the manufacturer, while μ_1 may result from previous experience.

[A34]

We choose the significance level a $\alpha = 5\%$. If the hypothesis is true, It follows that the random variable:

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{\overline{X} - 4500}{S/4}$$

has a t-distribution with $n - 1 = 15$ d.f. The test is left-sided. The critical value c is obtained from:

$$P(T < c)_{\mu_0} = \alpha = 0.05$$

Table 5.6 gives $c = -1.75$. As an observed value of T we obtain from the

$$t = \frac{4482 - 4500}{115/4} = -0.626.$$

We see that $t > c$ and accept the hypothesis ■

[Q35] Comparison of the Means of Two Normal Distributions

Using a sample $x_1, \cdots, x_n$ from a normal distribution with unknown mean μ_x and a sample $y_1, \cdots, y_n$ from another normal distribution with unknown mean μ_y, we want to test the hypothesis that the means are equal, $\mu_x = \mu_y$, against an alternative, $\mu_x \neq \mu_y$. The variances need not be known but are assumed to be equal.

105	108	86	103	103	107	124	105
89	92	84	97	103	107	111	97

[A35]

We find:

$$\overline{x} = 105.125 \qquad \overline{y} = 97.500 \qquad s_x^2 = 106.125 \qquad s_y^2 = 84.000.$$

We choose the significance level $\alpha = 5\%$. Using a two-sided test the cut-off points are 2.5% and 97.5%. The d.f. is calculated as:

$$n_1 + n_2 - 2 = 8 + 8 - 2 = 14$$

Using **Table** 5.6 with 14 d.f., the critical values are: $c_1 = -2.14$ and $c_2 = 2.14$. Using the following formula:

$$t_0 = \sqrt{n}\frac{\overline{x} - \overline{y}}{\sqrt{s_x^2 + s_y^2}}$$

Using $n_1 = n_2 = n = 8$ gives:

$$t_0 = \sqrt{8}\frac{7.625}{\sqrt{190.125}} = 1.56$$

Since $c_1 \leqq t_0 \leqq c_2$, we accept the hypothesis $\mu_x = \mu_y$ that under both conditions the mean output is the same.

[Q36] Sign Test for the Median

A median of the population is a solution $x = \mu$ of the equation $F(x) = 0.5$, where F is the distribution function of the population.

Suppose that eight (8) radio operators were tested, first in rooms without air-conditioning and then in air-conditioned rooms over the same period of time, and the difference of errors (unconditioned minus conditioned) were:

$$9 \quad 4 \quad 0 \quad 6 \quad 4 \quad 0 \quad 7 \quad 11$$

Test the hypothesis $\overline{\mu} = 0$ (that is, air-conditioning has no effect) against the alternative $\overline{\mu} > 0$ (that is, inferior performance in unconditioned rooms).

[A36]

We choose the significance level $\alpha = 5\%$. If the hypothesis is true, the probability p of a positive difference is the same as that of a negative difference. Hence in this case, $p = 0.5$, and the random variable

$$X = \text{Number of positive values among } n \text{ values.}$$

has a binomial distribution with $p = 0.5$. Our sample has eight (8) values. We omit the values 0, which do not contribute to the decision. Then six (6) values are left, all of which are positive. Since:

$$\begin{aligned} P(X = 6) &= \binom{6}{6} (0.5)^6 (0.5)^0 = 0.0156 \\ &= 1.56\% \end{aligned}$$

we have observed an event whose probability is very small if the hypothesis is true; in fact $1.56\% < \alpha = 5\%$. Hence we assert that the alternative $\hat{\mu} > 0$ is true. That is, the number of errors made in unconditioned rooms is significantly higher, so that installation of air conditioning should be considered ■

[Q37] Testing for Arbitrary Trend

A certain machine is used for cutting lengths of wire. The five (5) successive pieces had the lengths:

$$29 \quad 31 \quad 28 \quad 30 \quad 32.$$

Using this sample, test the hypothesis that there is no trend, that is, the machine does not have the tendency to produce longer and longer pieces or shorter and shorter pieces.

Assume that the type of machine suggests the alternative that there is positive trend, that is, there is the tendency of successive pieces to get longer.

[A37]

We count the number of **transpositions** in the sample, that is, the number of times a larger value precedes a smaller value:

$$\begin{aligned} &\text{29 precedes 28} && \text{1 transposition,} \\ &\text{31 precedes 28 and 30} && \text{2 transpositions.} \end{aligned}$$

The remaining three (3) sample values follow in ascending order. Hence in the sample there are $1 + 2 = 3$ transpositions. We now consider the random variable

$$T = \text{Number of transpositions.}$$

If the hypothesis is true (i.e., no trend), then each of the $5! = 120$ permutations of five (5) elements 1 2 3 4 5 has the same probability (1/120). We arrange these permutations according to their number of transpositions. From this we obtain

$$P\left(T \leqq 3\right) = \frac{1}{120} + \frac{4}{120} + \frac{9}{120} + \frac{15}{120} = \frac{29}{120} = 24\%.$$

We accept the hypothesis because we have observed an event that has a relatively large probability if the hypothesis is true.

Our method and those values refer to continuous distributions. Theoretically, we may then expect that all the values of a sample are different. Practically, some sample values may still be equal, because of rounding: If m values are equal, and $m(m-1)/4$ (= mean value of the transpositions in the case of the permutations of m elements), that is, $\frac{1}{2}$ for each pair of equal values, $\frac{1}{2}$ for each triple, etc ■

[Q38] Printed Circuit Boards

The number of defects in printed circuit board is hypothesized to follow a Poisson distribution. A random sample of $n = 60$ printed boards have been collected, and following number of defects were observed

Number of Defects	Observed Frequency
0	32
1	15
2	9
3	4

[A38]

The mean of the assumed Poisson distribution in this example is unknown and must be estimated from the sample data. The estimate of the mean number of defects per board is the sample average, that is:

$$(32 \times 0 + 15 \times 1 + 9 \times 2 + 4 \times 3)\ /60 = 0.75$$

From the Poisson distribution with parameter 0.75, we may compute p_{i}, the theoretical, hypothesized probability associated with the i^{th} class interval. Since each class interval corresponds to a particular number of defects, we may find the p_{i} as follows:

$$\begin{aligned}
p_1 &= P(X = 0) = \frac{e^{-0.75}(0.75)^0}{0!} = 0.472 \\
p_2 &= P(X = 1) = \frac{e^{-0.75}(0.75)^1}{1!} = 0.354 \\
p_3 &= P(X = 2) = \frac{e^{-0.75}(0.75)^2}{2!} = 0.133 \\
p_4 &= P(X \geqq 3) = 1 - (p_1 + p_2 + p_3) = 0.041
\end{aligned}$$

The expected frequencies are computed by multiplying the sample size $n = 60$ times the probabilities p_{i}. That is, $e_{\text{i}} = np_{\text{i}}$. The expected frequencies follow:

Number of Defects	Probability	Expected Frequency
0	0.472	28.32
1	0.354	21.24
2	0.133	7.98
3 (or more)	0.041	2.46

Since the expected frequency in the last cell is less than 3, we combine the last two cells:

NOTE: Categories with expected frequency is combined because the Chi-square test would not work if the frequency is less than 5. If the sample size is too small the chi-square value is over-estimated and if it is too large chi-square value is under-estimated. Hency why we combine with the category with the lowest frequency.

Since the expected frequency in the last cell is less than 3, we combine the last two cells:

Number of Defects	Probability	Expected Frequency
0	32	28.32
1	15	21.24
2 (or more)	13	10.44

Now, the chi-square test will have $k - p - 1 = 3 - 1 - 1 = 1$ degree of freedom, because the mean of the Poisson distribution was estimated from the data.

The hypothesis-testing procedure may now be applied using $\alpha = 0.05$,

1. The variable of interest is the form of the distribution of defects in printed circuit boards.
2. θ_0 The form of the distribution of defects is Poisson.
3. θ_1 The form of the distribution of defects is not Poisson.
4. Test statistic is:
$$\chi_0^2 = \sum_{j=1}^{k} \frac{\left(b_{\text{j}} - e_{\text{j}}\right)^2}{e_{\text{j}}}$$
5. Reject H_0 if $\chi_0^2 > \chi_{0.05,1}^2 = 3.84$.
6. Time to calculate χ_0^2:
$$\chi_0^2 = \frac{(32 - 28.32)^2}{28.32} + \frac{(15 - 21.24)^2}{21.24} + \frac{(13 - 10.44)^2}{10.44} = 2.94$$
7. As $\chi_0^2 = 2.94 < \chi_{0.05,1}^2 = 3.84$, we are unable to reject the null hypothesis that the distribution of defects in printed circuit boards is Poisson.

[Q39] Testing the Power Supply

A manufacturing engineer is testing a power supply used in a notebook computer and, using $\alpha = 0.05$, wishes to determine whether output voltage is adequately described by a normal distribution. Sample estimates of the mean and standard deviation of $\overline{x} = 5.04$ V and $s = 0.08$ V are obtained from a random sample of $n = 100$ units.

Is the data normally distributed?

[A39]

A common practice in constructing the class intervals for the frequency distribution used in the chi-square goodness-of-fit test is to choose the cell boundaries so that the expected frequencies $e_j = np_j$ are equal for all cells. To use this method, we want to choose the cell boundaries a_0, a_1,..., a_k for the k cells so that all the probabilities

$$p_{\mathrm{i}} = P\left(a_{\mathrm{j}-1} \leqq X \leqq a_{\mathrm{j}}\right) = \int_{a_{\mathrm{j}-1}}^{a_{\mathrm{j}}} f(x)\, \mathrm{d}x$$

are equal. Suppose we decide to use $k = 8$ cells. For the standard normal distribution, the intervals that divide the scale into eight equally likely segments are [0, 0.32), [0.32, 0.675) [0.675, 1.15), [1.15, ∞) and their four (4) "mirror image" intervals on the other side of zero. For each interval $p_i = 1/8 = 0.125$, so the expected cell frequencies are $e_j = np_j = 100(0.125) = 12.5$. The complete table of observed and expected frequencies is as follows:

Class Interval	Observed Frequency (b_{j})	Expected Frequency (e_{j})
$x < 4.948$	12	12.5
$4.948 \leqq x < 4.986$	14	12.5
$4.986 \leqq x < 5.014$	12	12.5
$5.014 \leqq x < 5.040$	13	12.5
$5.040 \leqq x < 5.066$	12	12.5
$5.066 \leqq x < 5.094$	11	12.5
$5.094 \leqq x < 5.132$	12	12.5
$5.132 \leqq x$	14	12.5

The boundary of the first class interval is $\overline{x} - 1.15s = 4,948$. The second class interval is $[\overline{x} - 1.15s, \overline{x} - 0.675s)$ and so forth. We may apply the hypothesis-testing procedure to this problem.

1. The variable of interest is the form of the distribution of power supply voltage.
2. θ_0 The form of the distribution is normal.
3. θ_1 The form of the distribution is not normal.
4. Test statistic is:
$$\chi_0^2 = \sum_{j=1}^{k} \frac{\left(b_{\mathrm{j}} - e_{\mathrm{j}}\right)^2}{e_{\mathrm{j}}}$$
5. Since two parameters in the normal distribution have been estimated, the chi-square statistic above will have $k - p - 1 = 8 - 2 - 1 = 5$ degrees of freedom. Therefore, we will reject χ_0 if $\chi_0^2 > \chi_{0.05,5}^2 = 11.07$.

6. Calculating χ_0^2:

$$
\begin{aligned}
\chi_0^2 &= \sum_{j=1}^{8} \frac{(b_j - e_j)^2}{e_j} \\
&= \frac{(12-12.5)^2}{12.5} + \frac{(14-12.5)^2}{12.5} + \cdots + \frac{(14-12.5)^2}{12.5} \\
&= 0.64
\end{aligned}
$$

7. Since $\chi_0^2 = 0.64 < \chi^2_{0.05,5} = 11.07$, we are unable to reject θ_0, and there is no strong evidence to indicate that output voltage is not normally distributed ■

[Q40] From the Same Cast

The two samples:

$$70 \quad 80 \quad 30 \quad 70 \quad 60 \quad 80$$

and

$$140 \quad 120 \quad 130 \quad 120 \quad 120 \quad 130 \quad 120$$

are values of the differences of temperatures (°C) of iron at two stages of casting, taken from two different crucibles.
Is the variance of the first population larger than that of the second? (Assume normality. Choose α = 5%.)

[A40]

We test the hypothesis $\sigma_x^2 = \sigma_y^2$ against the alternative $\sigma_x^2 > \sigma_y^2$. We proceed to calculate the variance of their given data as:

$$v_0 = \frac{s_x^2}{s_y^2} = \frac{350}{61.9} = 5.65.$$

For $\alpha = 5\%$ and $(5, 6)$ degrees of freedom, the reference Table gives the value 4.39.
As 5.65 is greater, we reject the hypothesis and assert that the variance of the first population is greater than that of the second ■

3

Fourier Analysis

[Q41] Periodic Rectangular Waves

Find the Fourier coefficients of the periodic function, which these kind of functions occur as **external** forces acting on mechanical systems, electromotive forces in electric circuits, etc., $f(x)$ in **Fig.** 3.1. The formula is

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x+2\pi) = f(x) \tag{3.1}$$

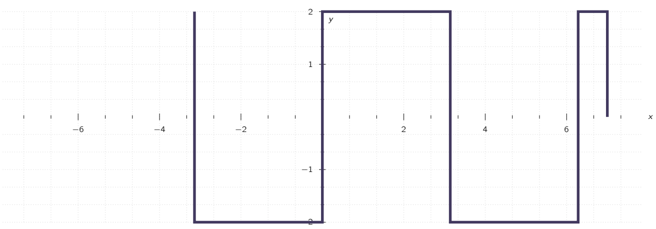

Figure 3.1: The given function $f(x)$ in Example

Note: The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.

[A41]

From Euler equations, we obtain $a_0 = 0$.

This can also be seen without integration by looking at the plot, as the area under the curve of $f(x)$ between $-\pi$ and π is zero. From the Euler equations, we obtain the coefficients $a_1, a_2, \dots$ of the cosine terms. As $f(x)$ is given by two (2) expressions, the integrals from $-\pi$ to π split into the following two (2) integrals:

$$a_{\mathrm{n}} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\cos nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0$$

As $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. We can see all these cosine coefficients are zero. That is, the Fourier series of Eq. (3.1) has no cosine terms, just sine terms, it is a **Fourier sine series** with coefficients $b_1, b_2, \cdots$ obtained from the Euler equations:

$$b_{\mathrm{n}} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]$$

As $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this gives us:

$$b_{\mathrm{n}} = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd} n, \\ 1 & \text{for even} n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd} n, \\ 0 & \text{for even} n. \end{cases}$$

Hence the Fourier coefficients b_0 of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_{n} are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}(\sin x + \frac{1}{2}\sin 3x + \frac{1}{3}\sin 5x + \cdots).$$

Their graphs in **Fig.** 3.2 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points. This is typical.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4\pi}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.

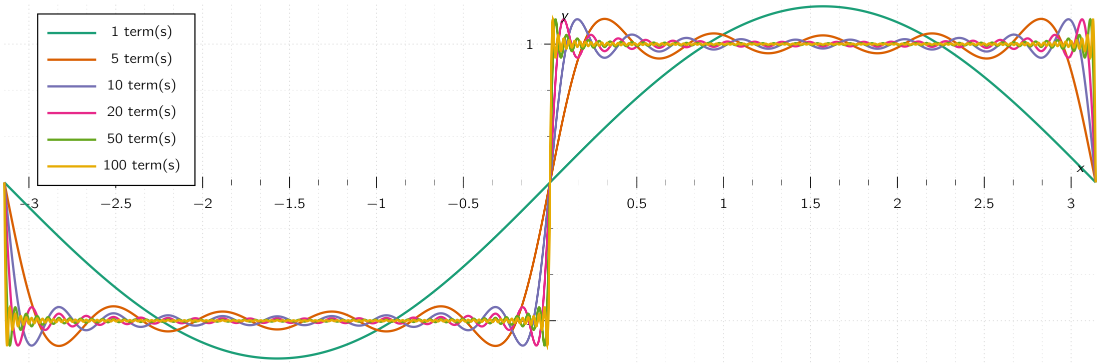

Figure 3.2: The consecutive sums of the corresponding Fourier series.

[Q42] Sawtooth Wave

Find the Fourier series of the function:

$$f(x) = x + \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

[A42]

We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Therefore, the Fourier coefficients a_n, b_n are those of f_1, except for a_n, which is π.
Given f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and:

$$b_n = \frac{2}{\pi}\int_0^\pi f_1(x)\sin nx \, dx = \frac{2}{\pi}\int_0^\pi x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx \, dx\right] = -\frac{2}{n}\cos n\pi$$

Therefore:

$$b_1 = 2, b_2 = -\frac{2}{2}, b_3 = -\frac{2}{4}, \cdots$$

and the Fourier series of $f(x)$ is:

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right) \quad \blacksquare$$

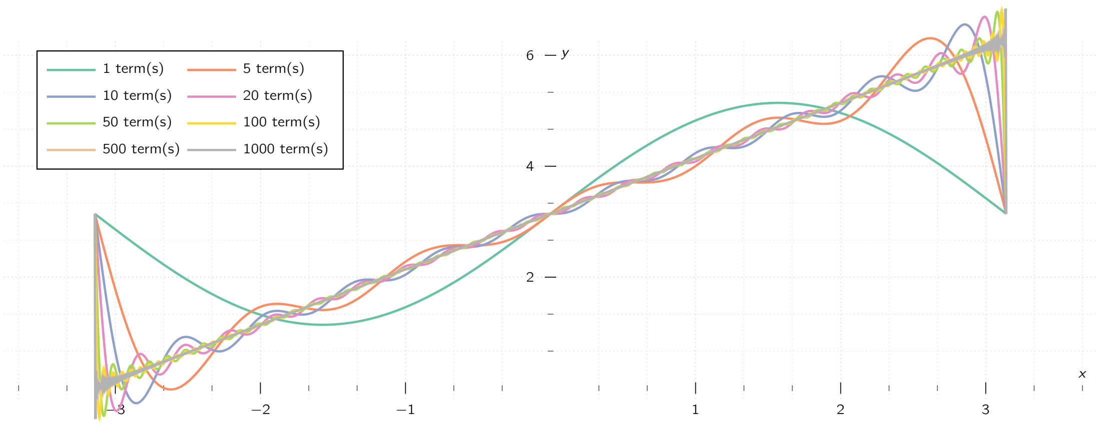

Figure 3.3: Generation of a saw-tooth wave using Fourier series.

[Q43] Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0 \quad \text{and} \quad y(\pi) = 0. \tag{3.2}$$

[A43]

From Eq. (**??**) and Eq. (**??**) we see that $p = 1$, $q = 0$, $r = 1$ in Eq. (**??**), and $a = 0$, $b = \pi$, $k_1 = l_1 = 1$, and $k_2 = l_2 = 0$ in Eq. (**??**). For negative $\lambda = -\nu^2$ a general solution of the ODE in Eq. (3.2) is:

$$y(x) = c_1 \exp \nu x + c_2 \exp -\nu x.$$

From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y = 0$, which is **NOT** an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = \nu^2$ a general solution is:

$$y(x) = A \cos \nu x + B \sin \nu x.$$

From the first boundary condition we obtain $y(0) = A = 0$. The second boundary condition then yields

$$y(\pi) = B \sin \pi\nu = 0, \qquad \text{thus} \qquad \nu = 0, \pm 1, \pm 2, \cdots.$$

For $v = 0$ we have $y = 0$. For $\lambda = v^2 = 1, 4, 9, 16, \cdots$, taking $B = 1$, we obtain

$$y(x) = \sin vx (v = \sqrt{\lambda} = 1, 2, \cdots).$$

Hence the eigenvalues of the problem are $\lambda = v^2$, where $v = 1, 2, \cdots$, and corresponding eigenfunctions are $y(x) = \sin vx$, where $v = 1, 2, \cdots$.

[Q44] Voltage Waveforms

Find the Fourier series of the function obtained by passing the voltage:

$$v(t) = V_0 \cos 100\pi t$$

through a half-wave rectifier

[A44]

The solution is as follows:

$$b_n = 0, \qquad \text{and} \qquad a_0 = \frac{V_0}{\pi}.$$

$$\begin{aligned}
a_n &= 100V_0 \int_{-1/200}^{1/200} \cos 100\pi t \cos 100n\pi t \, dt \\
&= 50V_0 \int_{-1/200}^{1/200} \cos 100(n+1)\pi t \, dt + 50V_0 \int_{-1/200}^{1/200} \cos 100(n-1)\pi t \, dt \\
&= \frac{V_0}{\pi} + \frac{V_0}{2} \cos 100\pi t \\
&+ \frac{2V_0}{\pi} \left(\frac{1}{1 \cdot 3} \cos 200\pi t - \frac{1}{3 \cdot 5} \cos 400\pi t + \frac{1}{5 \cdot 7} \cos 600\pi t \cdots \right) \quad \blacksquare
\end{aligned}$$

[Q45] RLC Waveform

Find the steady-state current $(I(t))$ of an series connected RLC circuit where:

$$R = 100\,\Omega \quad L = 10\,\text{H} \quad C = 1 \times 10^{-2}\,\text{F}$$

and $E(t)$ V as:

$$E(t) = \begin{cases} 100\left(\pi t + t^2\right) & \text{if} \quad -\pi < t < 0 \\ 100\left(\pi t - t^2\right) & \text{if} \quad 0 < t < \pi \end{cases}$$

and periodic with period 2π.

[A45]

The solution is as follows:

$$I = \sum_{n=1}^{\infty} \left(A_{\text{n}} \cos nt + B_{\text{n}} \sin nt \right),$$

where,

$$A_{\text{n}} = \frac{80\left(10 - n^2\right)}{\pi n^2 D_{\text{n}}} \quad \text{and} \quad B_{\text{n}} = \frac{800}{n\pi D_{\text{n}}} \quad \text{(where } n \text{ is odd.)}$$

$$A_{\text{n}} = 0 \quad \text{and} \quad B_{\text{n}} = 0 \quad \text{(where } n \text{ is even.)}$$

and of course,

$$B_{\text{n}} = \left(10 - n^2\right)^2 + 100n^2$$

Which gives us the solution:

$$\begin{aligned} I &= 1.266 \cos t + 1.406 \sin t + 0.003 \cos 3t + 0.094 \sin 3t \\ &- 0.006 \cos 5t + 0.019 \sin 5t - 0.003 \cos 7t + 0.006 \sin 7t \ldots \end{aligned} \quad \blacksquare$$

[Q46] A Half Wave Rectifier

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave.

Find the Fourier series of the resulting periodic function:

$$u(t) = \begin{cases} 0 & \text{if} \quad -L < t < 0, \\ E \sin \omega t & \text{if} \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

[A46]

As $u = 0$ when $-L < t < 0$, we obtain from (6.0), with t instead of x,

$$a_0 = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t \, \mathrm{d}\, t = \frac{E}{\pi}$$

and from Eq. (**??**), with $x = \omega t$ and $y = n\omega t$:

$$a_{\text{n}} = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t \, \mathrm{d}t = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} \left[\sin(1+n)\omega t + \sin(1-n)\omega t \right] \mathrm{d}t$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain:

$$\begin{aligned} a_{\text{n}} &= \frac{\omega E}{2\pi} \left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right] \Bigg|_0^{\pi/\omega} \\ &= \frac{E}{2\pi} \left(\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}. \right) \end{aligned}$$

If n is odd, this is equal to zero, and for even n we have:

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad \text{where} \qquad n = 2, 4\cdots$$

Similarly we find from Eq. (**??**), that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right)$$

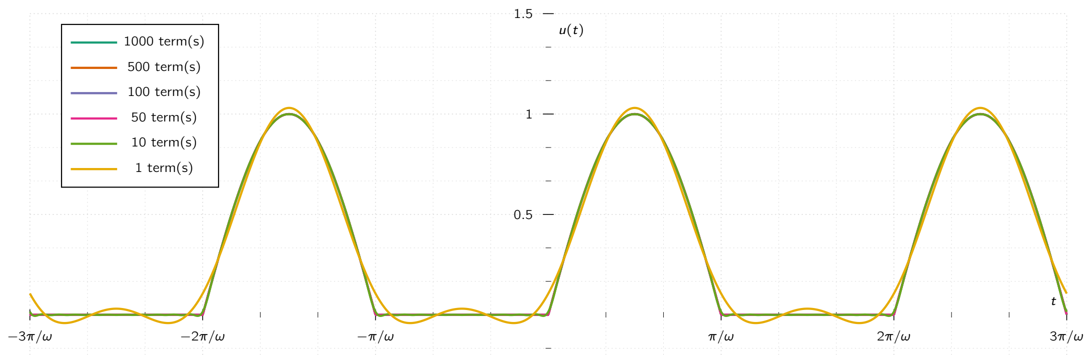

Figure 3.4: A Fourier series approximation of a half-wave rectification.

[Q47] Periodic Rectangular Waves

Find the Fourier series of the function:

$$f(x) = \begin{cases} -k & \text{if} \quad -2 < x < 0 \\ k & \text{if} \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

[A47]

Since $L = 2$, we have in Eq. (**??**) $\upsilon = \pi x/2$, that is:

$$g(\upsilon) = \frac{4k}{\pi}\left(\sin\upsilon + \frac{1}{3}\sin 3\upsilon + \frac{1}{5}\sin 5\upsilon + \cdots\right)$$

the present Fourier series:

$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right) \quad \blacksquare$$

[Q48] A Non Sinusoidal Forced Oscillation

Referring back to Eq. (**??**), let $m = 1\,\mathrm{g}$, $c = 0.05\,\mathrm{g\,s^{-1}}$, and $k = 25\,\mathrm{g\,s^{-2}}$, so Eq. (**??**) becomes:

$$y'' + 0.05y' + 25y = r(t) \tag{3.3}$$

where $r(t)$ is measured in $\mathrm{g\,cm\,s^{-2}}$. Let

$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if} \quad -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if} \quad 0 < t < \pi \end{cases} \qquad \text{and} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

[A48]

We represent $r(t)$ by a Fourier series, finding

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right). \tag{3.4}$$

Then we consider the Ordinary Differential Equation (ODE):

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad \text{where} \qquad n = 1, 3, \cdots, \tag{3.5}$$

whose right side is a single term of the series Eq. (3.4). From **Higher Mathematics I** we know that the steady-state solution $y_n(t)$ of Eq. (3.5) is of the form:

$$y_n = A_n \cos nt + B_n \sin nt. \tag{3.6}$$

By substituting this into Eq. (3.5) we find that:

$$A_n = \frac{4\left(25 - n^2\right)}{n^2\pi D_n}, \quad B_n = \frac{0.2}{n\pi D_n}, \qquad \text{where} \qquad D_n = \left(25 - n^2\right)^2 + (0.05n)^2 \tag{3.7}$$

Since the ODE Eq. (3.3) is linear, we may expect the steady-state solution to be in the form:

$$y = y_1 + y_3 + y_5 + \cdots \tag{3.8}$$

where y_n is given by Eq. (3.4) and Eq. (3.7). In fact, this follows readily by substituting Eq. (3.8) into Eq. (3.3) and using the Fourier series of $r(t)$, provided that termwise differentiation of Eq. (3.8) is permissible.

From Eq. (3.7) we find that the amplitude of Eq. (3.6) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}} \qquad \blacksquare$$

4

Partial Differential Equations

[Q49] Solving Like an ODE - I

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y.

[A49]

As there are no y-derivatives in the equation, we can solve this PDE like $u'' - u = 0$. If you recall from **Higher Mathematics I** we would have obtained a form of:

$$u = Ae^{x} + Be^{-x} \qquad \text{where A, B are constants.}$$

Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)\, e^{x} + B(y)\, e^{-x} \quad \blacksquare$$

with arbitrary functions A and B. As can be seen, we have a great variety of solutions.

[Q50] Solving Like an ODE - II

Find solutions of the following PDE.

$$u_{xy} = -u_{x}$$

[A50]

Setting $u_x = p$, we have:

$$\begin{aligned} p_y &= -p, & p_y/p &= -1, \\ \ln|p| &= -y + \tilde{c}(x), & p &= c(x)\, e^{-y}, \end{aligned}$$

and by integration with respect to x,

$$u(x, y) = f(x)\, e^{-y} + g(y)$$

$$\text{where} \qquad f(x) = \int c(x)\, \mathrm{d}x \quad \blacksquare$$

here, $f(x)$ and $g(y)$ are arbitrary.

[Q51] Triangular Initial Temperature

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is:

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < L/2, \\ L - x & \text{if} \quad L/2 < x < L, \end{cases}$$

[A51]

From

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

we get

$$B_n = \frac{2}{L} \left(\int_0^{L/2} x \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} (L - x) \sin \frac{n\pi x}{L} \, dx \right).$$

Integration gives $B_n = 0$ if n is even,

$$B_n = \frac{4L}{n^2\pi^2} \qquad (n = 1, 5, 9, \cdots) \qquad \text{and} \qquad B_n = -\frac{4L}{n^2\pi^2} \qquad (n = 3, 7, 11, \cdots).$$

Therefore the solution is:

$$u(x, t) = \frac{4L}{\pi^2} \left\{ \sin \frac{\pi x}{L} \exp\left[-\left(\frac{c\pi}{L}\right)^2 t \right] - \frac{1}{9} \sin \frac{3\pi x}{L} \exp\left[-\left(\frac{3c\pi}{L}\right)^2 t \right] + - \cdots \right\}. \quad \blacksquare$$

[Q52] Bar with Insulated Ends

Find a solution to the 1D heat equation with standard boundary and initial conditions replaced by the condition that both ends of the bar are insulated.

[A52]

Physical experiments show that the rate of heat flow is proportional to the gradient of the temperature. Therefore if the ends $x = 0$ and $x = L$ of the bar are insulated, so that no heat can flow through the ends, we have $\nabla u = u_x = \partial u / \partial x$ and the boundary conditions:

$$u_x(0, t) = 0 \qquad \text{and} \qquad u_x(L, t) = 0 \qquad \text{for all } t \qquad .$$

Given that,

$$u(x, t) = F(x) G(t) m$$

this gives:

$$u_x(0, t) = F'(0)G(t) = 0 \qquad \text{and} \qquad u_x(L, t) = F'(L)G(t) = 0.$$

Differentiating,

$$F(x) = A \cos px + B \sin px.$$

we have

$$F'(x) = -Ap\sin px + Bp\cos px$$

so that

$$F'(0) = Bp = 0 \qquad \text{and then} \qquad F'(L) = -Ap\sin pL = 0.$$

The second of these conditions gives $p = p_n = n\pi/L, (n = 0, 1, 2, \cdots)$. From this and (7) with $A = 1$ and $B = 0$ we get $F_n(x) = \cos(n\pi x/L), (n = 0, 1, 2, \cdots)$. With G_n as before, this yields the eigenfunctions

$$u_m(x,\ t) = F_n(x)G_m(t) = A_n \cos\frac{n\pi x}{L}\, e^{-\lambda t}$$

$$(n = 0,\ 1,\ \cdots)$$

corresponding to the eigenvalues $\lambda_n = cn\pi/L$. The latter are as before, but we now have the additional eigenvalue $\lambda_0 = 0$ and eigenfunction $u_0 = \text{const}$, which is the solution of the problem if the initial temperature $f(x)$ is constant. This shows the remarkable fact that a separation constant can very well be zero, and zero can be an eigenvalue.
Furthermore, whereas the solution $u_n(x,\ t)$ gave a Fourier sine series, we now get a Fourier cosine series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-2\pi} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right).$$

Its coefficients result from the initial condition,

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x),$$

That is:

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots \quad \blacksquare$$

[Q53] Temperature in an Infinite Bar

Find the temperature in the infinite bar if the initial temperature is:

$$f(x) = \begin{cases} U_0 = \text{const} & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

[A53]

From

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv.$$

we have

$$u(x,t) = \frac{U_0}{2c\sqrt{\pi t}} \left\{ \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv. \right.$$

If we introduce the above variable of integration z, then the integration over v from -1 to 1 corresponds to the integration over z from $(-1-x)/(2c\sqrt{t})$ to $(1-x)/(2c\sqrt{t})$, and

$$u\left(x,\, t\right) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/\left(2c\sqrt{t}\right)}^{(1-x)/\left(2c\sqrt{t}\right)} e^{-z^2}\, dz \qquad \text{where} \qquad t > 0.$$

Unfortunately this integral has no close form and therefore it cannot be expressed using elementary functions. The above equation is also called the **error function**.

[Q54] Sinusoidal Initial Temperature

Find the temperature $u\left(x,\, t\right)$ in a laterally insulated copper bar 80 cm long if the initial temperature $100 \sin\left(\pi x/80\right)$ °C and the ends are kept at 0 °C

How long will it take for the maximum temperature in the bar to drop to 50 °C?

Properties of Copper:

- density 8.92 g cm^{-3},
- specific heat 0.092 cal g^{-1} °C^{-1},
- thermal conductivity 0.95 cal cm^{-1} s^{-1} °C^{-1}

[A54]

The initial condition gives:

$$u\left(x,\, 0\right) = \sum_{n=1}^{\infty} B_{\mathrm{n}} \sin\frac{n\pi x}{80} = f\left(x\right) = 100\frac{\pi x}{80}.$$

Therefore, by inspection or using the following:

$$u\left(x,\, t\right) = \sum_{n=1}^{\infty} u_{\mathrm{n}}\left(x,\, t\right) = \sum_{n=1}^{\infty} B_{\mathrm{n}} \sin\frac{n\pi x}{L} e^{-\lambda_{\mathrm{n}}^2 t}$$

we get:

$$B_1 = 100, B_2 = B_2 = \cdots = 0$$

We also need:

$$\lambda_1^2 = c^2 \pi^2 / L^2,$$

where:

$$c^2 = \frac{K}{\sigma \rho} = \frac{0.95}{0.092 \cdot 8.92} = 1.158\,\mathrm{cm}^2\,\mathrm{s}^{-1}$$

Therefore we obtain:

$$\lambda_1^2 = 1.158 \cdot \frac{9.870}{80^2} = 0.001\,785\,\mathrm{s}^{-1}$$

Using this we can obtain the solution:

$$u(x,\ t) = 100 \sin \frac{\pi x}{80} e^{\text{-0.001875}\ t}$$

In addition, $100e^{\text{-0.001875}\ t} = 50$ when:

$$t = \frac{\ln 0.5}{-0.0018785} = 388\,\mathrm{s} \approx 6.5\,\mathrm{min} \quad \blacksquare$$

5 Reference Tables

Table 5.1: Particular values of the Bessel function.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0	0.0	3.0	-0.2601	0.3391	6.0	0.1506	-0.2767
0.1	0.9975	0.0499	3.1	-0.2921	0.3009	6.1	0.1773	-0.2559
0.2	0.99	0.0995	3.2	-0.3202	0.2613	6.2	0.2017	-0.2329
0.3	0.9776	0.1483	3.3	-0.3443	0.2207	6.3	0.2238	-0.2081
0.4	0.9604	0.196	3.4	-0.3643	0.1792	6.4	0.2433	-0.1816
0.5	0.9385	0.2423	3.5	-0.3801	0.1374	6.5	0.2601	-0.1538
0.6	0.912	0.2867	3.6	-0.3918	0.0955	6.6	0.274	-0.125
0.7	0.8812	0.329	3.7	-0.3992	0.0538	6.7	0.2851	-0.0953
0.8	0.8463	0.3688	3.8	-0.4026	0.0128	6.8	0.2931	-0.0652
0.9	0.8075	0.4059	3.9	-0.4018	-0.0272	6.9	0.2981	-0.0349
1.0	0.7652	0.4401	4.0	-0.3971	-0.066	7.0	0.3001	-0.0047
1.1	0.7196	0.4709	4.1	-0.3887	-0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	-0.3766	-0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.522	4.3	-0.361	-0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	-0.3423	-0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	-0.3205	-0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	-0.2961	-0.2566	7.6	0.2516	0.1592
1.7	0.398	0.5778	4.7	-0.2693	-0.2791	7.7	0.2346	0.1813
1.8	0.34	0.5815	4.8	-0.2404	-0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	-0.2097	-0.3147	7.9	0.1944	0.2192

Continued on next page

Table 5.1: Particular values of the Bessel function. (Continued)

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
2.0	0.2239	0.5767	5.0	-0.1776	-0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	-0.1443	-0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.556	5.2	-0.1103	-0.3432	8.2	0.1222	0.258
2.3	0.0555	0.5399	5.3	-0.0758	-0.346	8.3	0.096	0.2657
2.4	0.0025	0.5202	5.4	-0.0412	-0.3453	8.4	0.0692	0.2708
2.5	-0.0484	0.4971	5.5	-0.0068	-0.3414	8.5	0.0419	0.2731
2.6	-0.0968	0.4708	5.6	0.027	-0.3343	8.6	0.0146	0.2728
2.7	-0.1424	0.4416	5.7	0.0599	-0.3241	8.7	-0.0125	0.2697
2.8	-0.185	0.4097	5.8	0.0917	-0.311	8.8	-0.0392	0.2641
2.9	-0.2243	0.3754	5.9	0.122	-0.2951	8.9	-0.0653	0.2559

Table 5.2: Particular values of the Gamma function

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.0	1.0	1.2	0.9182	1.4	0.8873	1.6	0.8935	1.8	0.9314
1.02	0.9888	1.22	0.9131	1.42	0.8864	1.62	0.8959	1.82	0.9368
1.04	0.9784	1.24	0.9085	1.44	0.8858	1.64	0.8986	1.84	0.9426
1.06	0.9687	1.26	0.9044	1.46	0.8856	1.66	0.9017	1.86	0.9487
1.08	0.9597	1.28	0.9007	1.48	0.8857	1.68	0.905	1.88	0.9551
1.1	0.9514	1.3	0.8975	1.5	0.8862	1.7	0.9086	1.9	0.9618
1.12	0.9436	1.32	0.8946	1.52	0.887	1.72	0.9126	1.92	0.9688
1.14	0.9364	1.34	0.8922	1.54	0.8882	1.74	0.9168	1.94	0.9761
1.16	0.9298	1.36	0.8902	1.56	0.8896	1.76	0.9214	1.96	0.9837
1.18	0.9237	1.38	0.8885	1.58	0.8914	1.78	0.9262	1.98	0.9917
1.2	0.9182	1.4	0.8873	1.6	0.8935	1.8	0.9314	2.0	1.0

Table 5.3: Particular values of error function along with sine and cosine integrals.

x	erf(x)	Si(x)	ci(x)	x	erf(x)	Si(x)	ci(x)
0.0	0.0	0.0	$-\infty$	2.0	0.9953	1.6054	0.423
0.2	0.2227	0.1996	-1.0422	2.2	0.9981	1.6876	0.3751
0.4	0.4284	0.3965	-0.3788	2.4	0.9993	1.7525	0.3173
0.6	0.6039	0.5881	-0.0223	2.6	0.9998	1.8004	0.2533
0.8	0.7421	0.7721	0.1983	2.8	0.9999	1.8321	0.1865
1.0	0.8427	0.9461	0.3374	3.0	1.0	1.8487	0.1196
1.2	0.9103	1.108	0.4205	3.2	1.0	1.8514	0.0553
1.4	0.9523	1.2562	0.462	3.4	1.0	1.8419	-0.0045
1.6	0.9763	1.3892	0.4717	3.6	1.0	1.8219	-0.058
1.8	0.9891	1.5058	0.4568	3.8	1.0	1.7934	-0.1038
2.0	0.9953	1.6054	0.423	4.0	1.0	1.7582	-0.141

Table 5.4: Values of z for given values of the distribution function Φ(z) with $\Phi(-z) = 1 - \Phi(z)$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.01	0.504	0.51	0.695	1.01	0.844	1.51	0.934	2.01	0.978	2.51	0.994
0.02	0.508	0.52	0.698	1.02	0.846	1.52	0.936	2.02	0.978	2.52	0.994
0.03	0.512	0.53	0.702	1.03	0.848	1.53	0.937	2.03	0.979	2.53	0.994
0.04	0.516	0.54	0.705	1.04	0.851	1.54	0.938	2.04	0.979	2.54	0.994
0.05	0.52	0.55	0.709	1.05	0.853	1.55	0.939	2.05	0.98	2.55	0.995
0.06	0.524	0.56	0.712	1.06	0.855	1.56	0.941	2.06	0.98	2.56	0.995
0.07	0.528	0.57	0.716	1.07	0.858	1.57	0.942	2.07	0.981	2.57	0.995
0.08	0.532	0.58	0.719	1.08	0.86	1.58	0.943	2.08	0.981	2.58	0.995
0.09	0.536	0.59	0.722	1.09	0.862	1.59	0.944	2.09	0.982	2.59	0.995
0.1	0.54	0.6	0.726	1.1	0.864	1.6	0.945	2.1	0.982	2.6	0.995
0.11	0.544	0.61	0.729	1.11	0.867	1.61	0.946	2.11	0.983	2.61	0.995
0.12	0.548	0.62	0.732	1.12	0.869	1.62	0.947	2.12	0.983	2.62	0.996
0.13	0.552	0.63	0.736	1.13	0.871	1.63	0.948	2.13	0.983	2.63	0.996

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Table 5.4: Values of z for given values of the distribution function Φ(z) with $\Phi(-z) = 1-\Phi(z)$. (Continued)

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.14	0.556	0.64	0.739	1.14	0.873	1.64	0.949	2.14	0.984	2.64	0.996
0.15	0.56	0.65	0.742	1.15	0.875	1.65	0.951	2.15	0.984	2.65	0.996
0.16	0.564	0.66	0.745	1.16	0.877	1.66	0.952	2.16	0.985	2.66	0.996
0.17	0.567	0.67	0.749	1.17	0.879	1.67	0.953	2.17	0.985	2.67	0.996
0.18	0.571	0.68	0.752	1.18	0.881	1.68	0.954	2.18	0.985	2.68	0.996
0.19	0.575	0.69	0.755	1.19	0.883	1.69	0.954	2.19	0.986	2.69	0.996
0.2	0.579	0.7	0.758	1.2	0.885	1.7	0.955	2.2	0.986	2.7	0.997
0.21	0.583	0.71	0.761	1.21	0.887	1.71	0.956	2.21	0.986	2.71	0.997
0.22	0.587	0.72	0.764	1.22	0.889	1.72	0.957	2.22	0.987	2.72	0.997
0.23	0.591	0.73	0.767	1.23	0.891	1.73	0.958	2.23	0.987	2.73	0.997
0.24	0.595	0.74	0.77	1.24	0.893	1.74	0.959	2.24	0.987	2.74	0.997
0.25	0.599	0.75	0.773	1.25	0.894	1.75	0.96	2.25	0.988	2.75	0.997
0.26	0.603	0.76	0.776	1.26	0.896	1.76	0.961	2.26	0.988	2.76	0.997
0.27	0.606	0.77	0.779	1.27	0.898	1.77	0.962	2.27	0.988	2.77	0.997
0.28	0.61	0.78	0.782	1.28	0.9	1.78	0.962	2.28	0.989	2.78	0.997
0.29	0.614	0.79	0.785	1.29	0.901	1.79	0.963	2.29	0.989	2.79	0.997
0.3	0.618	0.8	0.788	1.3	0.903	1.8	0.964	2.3	0.989	2.8	0.997
0.31	0.622	0.81	0.791	1.31	0.905	1.81	0.965	2.31	0.99	2.81	0.998
0.32	0.626	0.82	0.794	1.32	0.907	1.82	0.966	2.32	0.99	2.82	0.998
0.33	0.629	0.83	0.797	1.33	0.908	1.83	0.966	2.33	0.99	2.83	0.998
0.34	0.633	0.84	0.8	1.34	0.91	1.84	0.967	2.34	0.99	2.84	0.998
0.35	0.637	0.85	0.802	1.35	0.911	1.85	0.968	2.35	0.991	2.85	0.998
0.36	0.641	0.86	0.805	1.36	0.913	1.86	0.969	2.36	0.991	2.86	0.998
0.37	0.644	0.87	0.808	1.37	0.915	1.87	0.969	2.37	0.991	2.87	0.998
0.38	0.648	0.88	0.811	1.38	0.916	1.88	0.97	2.38	0.991	2.88	0.998
0.39	0.652	0.89	0.813	1.39	0.918	1.89	0.971	2.39	0.992	2.89	0.998
0.4	0.655	0.9	0.816	1.4	0.919	1.9	0.971	2.4	0.992	2.9	0.998

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Table 5.4: Values of z for given values of the distribution function Φ(z) with $\Phi(-z) = 1 - \Phi(z)$. (Continued)

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.41	0.659	0.91	0.819	1.41	0.921	1.91	0.972	2.41	0.992	2.91	0.998
0.42	0.663	0.92	0.821	1.42	0.922	1.92	0.973	2.42	0.992	2.92	0.998
0.43	0.666	0.93	0.824	1.43	0.924	1.93	0.973	2.43	0.992	2.93	0.998
0.44	0.67	0.94	0.826	1.44	0.925	1.94	0.974	2.44	0.993	2.94	0.998
0.45	0.674	0.95	0.829	1.45	0.926	1.95	0.974	2.45	0.993	2.95	0.998
0.46	0.677	0.96	0.831	1.46	0.928	1.96	0.975	2.46	0.993	2.96	0.998
0.47	0.681	0.97	0.834	1.47	0.929	1.97	0.976	2.47	0.993	2.97	0.999
0.48	0.684	0.98	0.836	1.48	0.931	1.98	0.976	2.48	0.993	2.98	0.999
0.49	0.688	0.99	0.839	1.49	0.932	1.99	0.977	2.49	0.994	2.99	0.999
0.5	0.691	1.0	0.841	1.5	0.933	2.0	0.977	2.5	0.994	3.0	0.999

Table 5.5: Values of z for given values of the distribution function F(z) with $m = 1 - 10$.

	Degrees of Freedom									
$F(z)$	1	2	3	4	5	6	7	8	9	10
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.9	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.4	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81
0.975	12.71	4.3	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.82	6.96	4.54	3.75	3.36	3.14	3.0	2.9	2.82	2.76
0.995	63.66	9.92	5.84	4.6	4.03	3.71	3.5	3.36	3.25	3.17
0.995	63.66	9.92	5.84	4.6	4.03	3.71	3.5	3.36	3.25	3.17
0.999	318.31	22.33	10.21	7.17	5.89	5.21	4.79	4.5	4.3	4.14

Table 5.6: Values of z for given values of the distribution function F(z) with $m = 11 - 20$.

	Degrees of Freedom									
$F(z)$	11	12	13	14	15	16	17	18	19	20
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.54	0.54	0.54	0.54	0.54	0.54	0.53	0.53	0.53	0.53
0.8	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
0.9	1.36	1.36	1.35	1.35	1.34	1.34	1.33	1.33	1.33	1.33
0.95	1.8	1.78	1.77	1.76	1.75	1.75	1.74	1.73	1.73	1.72
0.975	2.2	2.18	2.16	2.14	2.13	2.12	2.11	2.1	2.09	2.09
0.99	2.72	2.68	2.65	2.62	2.6	2.58	2.57	2.55	2.54	2.53
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.9	2.88	2.86	2.85
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.9	2.88	2.86	2.85
0.999	4.02	3.93	3.85	3.79	3.73	3.69	3.65	3.61	3.58	3.55

Table 5.7: Values of z for given values of the distribution function F(z) with $m = 21 - 30$.

	Degrees of Freedom (m)									
$F(z)$	21	22	23	24	25	26	27	28	29	30
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53
0.8	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.85	0.85	0.85
0.9	1.32	1.32	1.32	1.32	1.32	1.31	1.31	1.31	1.31	1.31
0.95	1.72	1.72	1.71	1.71	1.71	1.71	1.7	1.7	1.7	1.7
0.975	2.08	2.07	2.07	2.06	2.06	2.06	2.05	2.05	2.05	2.04
0.99	2.52	2.51	2.5	2.49	2.49	2.48	2.47	2.47	2.46	2.46
0.995	2.83	2.82	2.81	2.8	2.79	2.78	2.77	2.76	2.76	2.75
0.995	2.83	2.82	2.81	2.8	2.79	2.78	2.77	2.76	2.76	2.75
0.999	3.53	3.5	3.48	3.47	3.45	3.43	3.42	3.41	3.4	3.39

Table 5.8: Values of z for given values of the distribution function F(z) with $m = 1 - 10$.

	Degrees of Freedom (m)									
$F(z)$	1	2	3	4	5	6	7	8	9	10
0.005	0.0	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16
0.01	0.0	0.02	0.11	0.3	0.55	0.87	1.24	1.65	2.09	2.56
0.025	0.0	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.7	3.25
0.05	0.0	0.1	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94
0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48
0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21
0.995	7.88	10.6	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19

Table 5.9: Values of z for given values of the distribution function F(z) with $m = 11 - 20$.

	Degrees of Freedom (m)									
$F(z)$	11	12	13	14	15	16	17	18	19	20
0.005	2.6	3.07	3.57	4.07	4.6	5.14	5.7	6.26	6.84	7.43
0.01	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26
0.025	3.82	4.4	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59
0.05	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85
0.95	19.68	21.03	22.36	23.68	25.0	26.3	27.59	28.87	30.14	31.41
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.99	24.72	26.22	27.69	29.14	30.58	32.0	33.41	34.81	36.19	37.57
0.995	26.76	28.3	29.82	31.32	32.8	34.27	35.72	37.16	38.58	40.0

Table 5.10: Values of z for given values of the distribution function F(z) with $m = 21 - 30$.

	Degrees of Freedom (m)									
$F(z)$	21	22	23	24	25	26	27	28	29	30
0.005	8.03	8.64	9.26	9.89	10.52	11.16	11.81	12.46	13.12	13.79

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Table 5.10: Values of z for given values of the distribution function F(z) with $m = 21 - 30$. (Continued)

	Degrees of Freedom (m)									
$F(z)$	21	22	23	24	25	26	27	28	29	30
0.01	8.9	9.54	10.2	10.86	11.52	12.2	12.88	13.56	14.26	14.95
0.025	10.28	10.98	11.69	12.4	13.12	13.84	14.57	15.31	16.05	16.79
0.05	11.59	12.34	13.09	13.85	14.61	15.38	16.15	16.93	17.71	18.49
0.95	32.67	33.92	35.17	36.42	37.65	38.89	40.11	41.34	42.56	43.77
0.975	35.48	36.78	38.08	39.36	40.65	41.92	43.19	44.46	45.72	46.98
0.99	38.93	40.29	41.64	42.98	44.31	45.64	46.96	48.28	49.59	50.89
0.995	41.4	42.8	44.18	45.56	46.93	48.29	49.64	50.99	52.34	53.67

Table 5.11: Values of z for which the distribution function F(z) has the value 0.95

	Degrees of Freedom (m)								
n	1	2	3	4	5	6	7	8	9
1	161.45	199.5	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.0	19.16	19.25	19.3	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.0
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.1
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.5	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.1	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.2	3.09	3.01	2.95	2.9
12	4.75	3.89	3.49	3.26	3.11	3.0	2.91	2.85	2.8
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.6	3.74	3.34	3.11	2.96	2.85	2.76	2.7	2.65
15	4.54	3.68	3.29	3.06	2.9	2.79	2.71	2.64	2.59

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Table 5.11: Values of z for which the distribution function F(z) has the value 0.95 (Continued)

	Degrees of Freedom (m)								
n	1	2	3	4	5	6	7	8	9
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.2	2.96	2.81	2.7	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.9	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.1	2.87	2.71	2.6	2.51	2.45	2.39
22	4.3	3.44	3.05	2.82	2.66	2.55	2.46	2.4	2.34
24	4.26	3.4	3.01	2.78	2.62	2.51	2.42	2.36	2.3
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
28	4.2	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
32	4.15	3.29	2.9	2.67	2.51	2.4	2.31	2.24	2.19
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15
38	4.1	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
50	4.03	3.18	2.79	2.56	2.4	2.29	2.2	2.13	2.07
60	4.0	3.15	2.76	2.53	2.37	2.25	2.17	2.1	2.04
70	3.98	3.13	2.74	2.5	2.35	2.23	2.14	2.07	2.02
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.0
90	3.95	3.1	2.71	2.47	2.32	2.2	2.11	2.04	1.99
100	3.94	3.09	2.7	2.46	2.31	2.19	2.1	2.03	1.97
100	3.94	3.09	2.7	2.46	2.31	2.19	2.1	2.03	1.97
150	3.9	3.06	2.66	2.43	2.27	2.16	2.07	2.0	1.94
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
1000	3.85	3.0	2.61	2.38	2.22	2.11	2.02	1.95	1.89

Table 5.12: Values of z for which the distribution function F(z) has the value 0.95

	Degrees of Freedom (m)								
n	10	15	20	30	40	50	100	200	500
1	241.88	245.95	248.01	250.1	251.14	251.77	253.04	253.68	254.06
2	19.4	19.43	19.45	19.46	19.47	19.48	19.49	19.49	19.49
3	8.79	8.7	8.66	8.62	8.59	8.58	8.55	8.54	8.53
4	5.96	5.86	5.8	5.75	5.72	5.7	5.66	5.65	5.64
5	4.74	4.62	4.56	4.5	4.46	4.44	4.41	4.39	4.37
6	4.06	3.94	3.87	3.81	3.77	3.75	3.71	3.69	3.68
7	3.64	3.51	3.44	3.38	3.34	3.32	3.27	3.25	3.24
8	3.35	3.22	3.15	3.08	3.04	3.02	2.97	2.95	2.94
9	3.14	3.01	2.94	2.86	2.83	2.8	2.76	2.73	2.72
10	2.98	2.85	2.77	2.7	2.66	2.64	2.59	2.56	2.55
11	2.85	2.72	2.65	2.57	2.53	2.51	2.46	2.43	2.42
12	2.75	2.62	2.54	2.47	2.43	2.4	2.35	2.32	2.31
13	2.67	2.53	2.46	2.38	2.34	2.31	2.26	2.23	2.22
14	2.6	2.46	2.39	2.31	2.27	2.24	2.19	2.16	2.14
15	2.54	2.4	2.33	2.25	2.2	2.18	2.12	2.1	2.08
16	2.49	2.35	2.28	2.19	2.15	2.12	2.07	2.04	2.02
17	2.45	2.31	2.23	2.15	2.1	2.08	2.02	1.99	1.97
18	2.41	2.27	2.19	2.11	2.06	2.04	1.98	1.95	1.93
19	2.38	2.23	2.16	2.07	2.03	2.0	1.94	1.91	1.89
20	2.35	2.2	2.12	2.04	1.99	1.97	1.91	1.88	1.86
22	2.3	2.15	2.07	1.98	1.94	1.91	1.85	1.82	1.8
24	2.25	2.11	2.03	1.94	1.89	1.86	1.8	1.77	1.75
26	2.22	2.07	1.99	1.9	1.85	1.82	1.76	1.73	1.71
28	2.19	2.04	1.96	1.87	1.82	1.79	1.73	1.69	1.67
30	2.16	2.01	1.93	1.84	1.79	1.76	1.7	1.66	1.64
32	2.14	1.99	1.91	1.82	1.77	1.74	1.67	1.63	1.61

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Table 5.12: Values of z for which the distribution function F(z) has the value 0.95 (Continued)

	Degrees of Freedom (m)								
n	10	15	20	30	40	50	100	200	500
34	2.12	1.97	1.89	1.8	1.75	1.71	1.65	1.61	1.59
36	2.11	1.95	1.87	1.78	1.73	1.69	1.62	1.59	1.56
38	2.09	1.94	1.85	1.76	1.71	1.68	1.61	1.57	1.54
40	2.08	1.92	1.84	1.74	1.69	1.66	1.59	1.55	1.53
50	2.03	1.87	1.78	1.69	1.63	1.6	1.52	1.48	1.46
60	1.99	1.84	1.75	1.65	1.59	1.56	1.48	1.44	1.41
70	1.97	1.81	1.72	1.62	1.57	1.53	1.45	1.4	1.37
80	1.95	1.79	1.7	1.6	1.54	1.51	1.43	1.38	1.35
90	1.94	1.78	1.69	1.59	1.53	1.49	1.41	1.36	1.33
100	1.93	1.77	1.68	1.57	1.52	1.48	1.39	1.34	1.31
100	1.93	1.77	1.68	1.57	1.52	1.48	1.39	1.34	1.31
150	1.89	1.73	1.64	1.54	1.48	1.44	1.34	1.29	1.25
200	1.88	1.72	1.62	1.52	1.46	1.41	1.32	1.26	1.22
1000	1.84	1.68	1.58	1.47	1.41	1.36	1.26	1.19	1.13

Table 5.13: Values of z for which the distribution function F(z) has the value 0.99

	Degrees of Freedom (m)								
n	1	2	3	4	5	6	7	8	9
1	4052.18	4999.5	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47
2	98.5	99.0	99.17	99.25	99.3	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.2	18.0	16.69	15.98	15.52	15.21	14.98	14.8	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.1	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91

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Table 5.13: Values of z for which the distribution function F(z) has the value 0.99 (Continued)

	Degrees of Freedom (m)								
n	1	2	3	4	5	6	7	8	9
9	10.56	8.02	6.99	6.42	6.06	5.8	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.2	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.5	4.39
13	9.07	6.7	5.74	5.21	4.86	4.62	4.44	4.3	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.0	3.89
16	8.53	6.23	5.29	4.77	4.44	4.2	4.03	3.89	3.78
17	8.4	6.11	5.18	4.67	4.34	4.1	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.6
19	8.18	5.93	5.01	4.5	4.17	3.94	3.77	3.63	3.52
20	8.1	5.85	4.94	4.43	4.1	3.87	3.7	3.56	3.46
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
24	7.82	5.61	4.72	4.22	3.9	3.67	3.5	3.36	3.26
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
30	7.56	5.39	4.51	4.02	3.7	3.47	3.3	3.17	3.07
32	7.5	5.34	4.46	3.97	3.65	3.43	3.26	3.13	3.02
34	7.44	5.29	4.42	3.93	3.61	3.39	3.22	3.09	2.98
36	7.4	5.25	4.38	3.89	3.57	3.35	3.18	3.05	2.95
38	7.35	5.21	4.34	3.86	3.54	3.32	3.15	3.02	2.92
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
50	7.17	5.06	4.2	3.72	3.41	3.19	3.02	2.89	2.78
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
70	7.01	4.92	4.07	3.6	3.29	3.07	2.91	2.78	2.67
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64

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Table 5.13: Values of z for which the distribution function F(z) has the value 0.99 (Continued)

	Degrees of Freedom (m)								
n	1	2	3	4	5	6	7	8	9
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61
100	6.9	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
100	6.9	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.6	2.5
1000	6.66	4.63	3.8	3.34	3.04	2.82	2.66	2.53	2.43

Table 5.14: Values of z for which the distribution function F(z) has the value 0.99

	Degrees of Freedom (m)								
n	10	15	20	30	40	50	100	200	500
1	6055.85	6157.28	6208.73	6260.65	6286.78	6302.52	6334.11	6349.97	6359.5
2	99.4	99.43	99.45	99.47	99.47	99.48	99.49	99.49	99.5
3	27.23	26.87	26.69	26.5	26.41	26.35	26.24	26.18	26.15
4	14.55	14.2	14.02	13.84	13.75	13.69	13.58	13.52	13.49
5	10.05	9.72	9.55	9.38	9.29	9.24	9.13	9.08	9.04
6	7.87	7.56	7.4	7.23	7.14	7.09	6.99	6.93	6.9
7	6.62	6.31	6.16	5.99	5.91	5.86	5.75	5.7	5.67
8	5.81	5.52	5.36	5.2	5.12	5.07	4.96	4.91	4.88
9	5.26	4.96	4.81	4.65	4.57	4.52	4.41	4.36	4.33
10	4.85	4.56	4.41	4.25	4.17	4.12	4.01	3.96	3.93
11	4.54	4.25	4.1	3.94	3.86	3.81	3.71	3.66	3.62
12	4.3	4.01	3.86	3.7	3.62	3.57	3.47	3.41	3.38
13	4.1	3.82	3.66	3.51	3.43	3.38	3.27	3.22	3.19
14	3.94	3.66	3.51	3.35	3.27	3.22	3.11	3.06	3.03
15	3.8	3.52	3.37	3.21	3.13	3.08	2.98	2.92	2.89
16	3.69	3.41	3.26	3.1	3.02	2.97	2.86	2.81	2.78

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Table 5.14: Values of z for which the distribution function F(z) has the value 0.99 (Continued)

	Degrees of Freedom (m)								
n	10	15	20	30	40	50	100	200	500
17	3.59	3.31	3.16	3.0	2.92	2.87	2.76	2.71	2.68
18	3.51	3.23	3.08	2.92	2.84	2.78	2.68	2.62	2.59
19	3.43	3.15	3.0	2.84	2.76	2.71	2.6	2.55	2.51
20	3.37	3.09	2.94	2.78	2.69	2.64	2.54	2.48	2.44
22	3.26	2.98	2.83	2.67	2.58	2.53	2.42	2.36	2.33
24	3.17	2.89	2.74	2.58	2.49	2.44	2.33	2.27	2.24
26	3.09	2.81	2.66	2.5	2.42	2.36	2.25	2.19	2.16
28	3.03	2.75	2.6	2.44	2.35	2.3	2.19	2.13	2.09
30	2.98	2.7	2.55	2.39	2.3	2.25	2.13	2.07	2.03
32	2.93	2.65	2.5	2.34	2.25	2.2	2.08	2.02	1.98
34	2.89	2.61	2.46	2.3	2.21	2.16	2.04	1.98	1.94
36	2.86	2.58	2.43	2.26	2.18	2.12	2.0	1.94	1.9
38	2.83	2.55	2.4	2.23	2.14	2.09	1.97	1.9	1.86
40	2.8	2.52	2.37	2.2	2.11	2.06	1.94	1.87	1.83
50	2.7	2.42	2.27	2.1	2.01	1.95	1.82	1.76	1.71
60	2.63	2.35	2.2	2.03	1.94	1.88	1.75	1.68	1.63
70	2.59	2.31	2.15	1.98	1.89	1.83	1.7	1.62	1.57
80	2.55	2.27	2.12	1.94	1.85	1.79	1.65	1.58	1.53
90	2.52	2.24	2.09	1.92	1.82	1.76	1.62	1.55	1.49
100	2.5	2.22	2.07	1.89	1.8	1.74	1.6	1.52	1.47
100	2.5	2.22	2.07	1.89	1.8	1.74	1.6	1.52	1.47
150	2.44	2.16	2.0	1.83	1.73	1.66	1.52	1.43	1.38
200	2.41	2.13	1.97	1.79	1.69	1.63	1.48	1.39	1.33
1000	2.34	2.06	1.9	1.72	1.61	1.54	1.38	1.28	1.19