Exam Higher Mathematics II Final

Neighbours

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Lecturer: Daniel T. McGuiness, Ph.D

SEMESTER: SS 2025
DATE: 10.06.2025
TIME: 09:00 - 11:15

First and Last Name

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Student Registration Number

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Grading Scheme	$\geq 90\%$	1
	$\leq 80\%$ and $\geq 90\%$	2
	$\leq 70\%$ and $\geq 80\%$	3
	$\leq 60\%$ and $\geq 70\%$	4
	$\leq 60\%$	5

Result:

___/ max. 100 points

Grade:

Student Cohort	MA-MECH-24-VZ
Study Programme	M.Sc Smart Technologies
Permitted Tools	One two-sided hand-written A4 paper and a calculator are allowed.

Important Notes

Unnecessary Items

Place all items not relevant to the test (including mobile phones, smartwatches, etc.) out of your reach.

Identification (ID)

Lay your student ID or an official ID visibly on the table in front of you.

Examination Sheets

Use only the provided examination sheets and label each sheet with your name and your student registration number. The sheets be labelled on the front. Do not tear up the examination sheets.

Writing materials

Do not use a pencil or red pen and write legibly.

Good Luck!

Please read the following instructions carefully.

- You have **135 minutes** to complete this exam. This question booklet contains 5 question(s), 9 pages (including the cover) for the total of 100 points.
- Check to see if any pages are missing.
- All the questions are **compulsory** and all the notations used in the questions have their usual meaning taught at the lectures and done in practice.
- **Read the instructions for individual questions carefully** before answering the questions.

Question	Maximum Point	Received Point
Laplace Equation	30	
Measure the Distance Correctly	15	
Filtering Out the Harmonics	15	
Fourier Series Expansion	20	
Triangular Initial Temperature	20	
Sum	100	

[Q1] Laplace Equation 30

Solve Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{where} \qquad 0 < x < a \quad \text{and} \quad 0 < y < b$$

for a rectangular plate subject to the given initial and boundary conditions:

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = f(x), \quad u(x, b) = g(x).$$

[A1] Laplace Equation 30

Since the boundary conditions at $y = 0$ and $y = b$ are functions of x we choose to separate Laplace's equation as:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

so that:

$$X'' + \lambda^2 X = 0, \quad Y'' - \lambda^2 Y = 0,$$

Which are 2$^{\text{nd}}$ order ODE. Based on their order the following solutions are valid.

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x, \qquad \text{and} \qquad Y(y) = c_3 \cosh \lambda y + c_2 \sinh \lambda y.$$

Now, using the given conditions, $X(0) = 0$ gives $c_1 = 0$ and $X(a) = 0$ implies $\sin \lambda a = 0$ or $\lambda = \pi/a$ for $n = 1, 2, 3, \ldots$. Therefore:

$$u_n(x,\, y) = XY = \left(A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y\right) \sin \frac{n\pi}{a} x$$

and

$$u(x,\, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y\right) \sin \frac{n\pi}{a} x$$

We then need to solve it for the initial conditions. At $y = 0$ we have:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x$$

and consequently

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

At $y = b$ we have:

$$g(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{a} b + B_n \sinh \frac{n\pi}{b} a\right) \sin \frac{n\pi}{a} x$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{a} b + B_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx$$

We can now solve for B_n:

$$\begin{aligned} B_n \sinh \frac{n\pi}{a} b &= \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b \\ B_n &= \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b\right) \end{aligned}$$

Which finally forms the solution to the boundary value problem ■

[Q2] Measure the Distance Correctly 15

During the design process of a robot you are tasked with finding a range sensor with a specific resolution. A key point which was requested from you is the sensor variance must lie within $\sigma^2 = 10$. After being sent 15 samples (n) from a company which claim their sensor variance $\sigma^2 = 10$, you measure their variance as $s^2 = 13$. Based on this information, and assuming the measurements are governed by normal distribution, you come to the conclusion that the actual variance of the sensors are actually $\sigma^2 = 20$. Assuming a significance level of $\alpha = 5\%$, please decide whether the manufacturer's claim is True or Not.

[A2] Measure the Distance Correctly 15

We are given a significance level of $\alpha = 5\%$. If the hypothesis is true:

$$Y = (n-1)\frac{S^2}{\sigma_0^2} = 14\frac{S^2}{10} = 1.4S^2$$

Has a chi-square distribution with $n-1 = 14$d.f.. Therefore we can define:

$$P(Y > c) = \alpha = 0.05$$

And using Table 3, we obtain $c = 23.68$. This is the critical value of Y. Therefore to $S^2 = \sigma_0^2 Y/(n-1) = 0.714Y$ there corresponds the critical value:

$$c^* = 0{,}714 \times 23{,}68 = 16.91.$$

Since $s^2 < c^*$ we accept the hypothesis.

[Q3] Filtering Out the Harmonics 15

During the manufacturing process of Printed Circuit Boards (PCBs) for use in Radio-Frequency communication systems, you are required to use a specialised inductance to filter out harmonics which should at least should be $200\,\mathrm{mH}$, preferably more. You contact a company which sends you 10 samples (n), and after doing measurements on them you realise the sample mean ($\overline{x}$) to be $195\,\mathrm{mH}$ with a standard deviation of $s = 8$. Given a significance level $\alpha = 1\%$, please decide whether this company's inductance's would fit to your application.

[A3] Filtering Out the Harmonics 15

We assume that the measurements are normally distributed. Then:

$$T = \frac{\overline{X} - \mu_0}{S\sqrt{n}},$$

with μ_0 having a t-distribution with $n-1$ degrees of freedom. Also

$$X = 195 \qquad \text{and} \qquad s = 8.$$

With a significance level of $\alpha = 0.01$ we look at Table 5 to find our critical value as ± 2.82. We can assume a symmetry as the data is assumed to be normally distributed. If our value is lower than this critical value we must reject the hypothesis.
Calculating our t value we get:

$$T = \frac{\overline{X} - \mu_0}{S\sqrt{n}} = -1.98$$

Given our value is within the boundary set by c, we accept this hypothesis.

[Q4] Fourier Series Expansion 20

While working on analysing an electrical signal, you come across a saw-tooth waveform. Mathematically this waveform is represented as:

$$f(x) = x + 2\pi \qquad \text{if} \qquad -2\pi < x < 2\pi \quad \text{and} \quad f(x + 4\pi) = f(x)$$

Using this information, and assuming this wave-form is only defined in $[-2\pi, 2\pi]$ please approximate it using the Fourier series.

NOTE: The integration of parts may be needed. The method of integration by parts is as follows:

$$\begin{aligned}\int_a^b u(x)v'(x)\,dx &= \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx \\ &= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.\end{aligned}$$

[A4] Fourier Series Expansion 20

We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = 2\pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is 2π. Therefore, the Fourier coefficients a_n, b_n are those of f_1, except for a_n, which is 2π.
Given f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and:

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f_1(x)\sin\frac{n\pi x}{2\pi}\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin\frac{nx}{2}\,dx.$$

Integrating by parts, we obtain

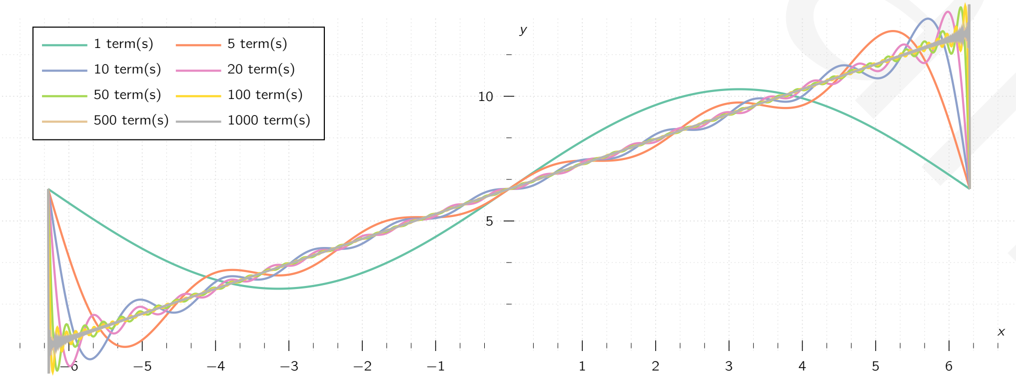
$$b_n = \frac{1}{\pi}\left[\frac{-2x\cos\dfrac{nx}{2}}{n}\Bigg|_0^{2\pi} + \cancel{\frac{2}{n}\int_0^{2\pi}\cos\frac{nx}{2}\,dx}^{\,0}\right] = -\frac{4}{n}\cos n\pi$$

Therefore:

$$b_1 = 4, b_2 = -\frac{4}{2}, b_3 = -\frac{4}{3}, \cdots$$

and the Fourier series of $f(x)$ is:

$$f(x) = 2\pi + 4\left(\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - \cdots\right) \quad \blacksquare$$



[Q5] Triangular Initial Temperature 20

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is:

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < L/2, \\ L - x & \text{if} \quad L/2 < x < L, \end{cases}$$

[A5] Triangular Initial Temperature 20

From

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

we get

$$B_n = \frac{2}{L} \left(\int_0^{L/2} x \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} (L - x) \sin \frac{n\pi x}{L} \, dx \right).$$

Integration gives $B_n = 0$ if n is even,

$$B_n = \frac{4L}{n^2 \pi^2} \quad (n = 1, 5, 9, \cdots) \qquad \text{and} \qquad B_n = -\frac{4L}{n^2 \pi^2} \quad (n = 3, 7, 11, \cdots).$$

Therefore the solution is:

$$u(x, t) = \frac{4L}{\pi^2} \left\{ \sin \frac{\pi x}{L} \exp \left[- \left(\frac{c\pi}{L} \right)^2 t \right] - \frac{1}{9} \sin \frac{3\pi x}{L} \exp \left[- \left(\frac{3c\pi}{L} \right)^2 t \right] + - \cdots \right\}. \quad \blacksquare$$

Supplemental Information

Table 1: Solutions to particular forms of ODEs.

ODE	Solution
$X'' + k^2 X = 0$	$X(x) = c_1 \cos kx + c_2 \sin kx$
$X'' - k^2 X = 0$	$X(x) = c_1 \cosh kx + c_2 \sinh kx$
$X'' = 0$	$X(x) = c_1 + c_2 x$
$X' + k^2 X = 0$	$X(x) = c_1 e^{k^2 x}$
$X' - k^2 Y = 0$	$X(x) = c_1 e^{-k^2 x}$
$X' = 0$	$X(x) = c_1$

Table 2: The Chi-square distribution values of z for given values of the distribution function F(z) with $m = 1 - 10$.

	Degrees of Freedom (m)									
$F(z)$	1	2	3	4	5	6	7	8	9	10
0.005	0.0	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16
0.01	0.0	0.02	0.11	0.3	0.55	0.87	1.24	1.65	2.09	2.56
0.025	0.0	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.7	3.25
0.05	0.0	0.1	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94
0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48
0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21
0.995	7.88	10.6	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19

Table 3: The Chi-square distribution values of z for given values of the distribution function F(z) with $m = 11 - 20$.

	Degrees of Freedom (m)									
$F(z)$	11	12	13	14	15	16	17	18	19	20
0.005	2.6	3.07	3.57	4.07	4.6	5.14	5.7	6.26	6.84	7.43
0.01	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26
0.025	3.82	4.4	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59
0.05	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85

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Table 3: The Chi-square distribution values of z for given values of the distribution function F(z) with $m = 11 - 20$. (Continued)

	Degrees of Freedom (m)									
$F(z)$	11	12	13	14	15	16	17	18	19	20
0.95	19.68	21.03	22.36	23.68	25.0	26.3	27.59	28.87	30.14	31.41
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.99	24.72	26.22	27.69	29.14	30.58	32.0	33.41	34.81	36.19	37.57
0.995	26.76	28.3	29.82	31.32	32.8	34.27	35.72	37.16	38.58	40.0

Table 4: The Chi-square distribution values of z for given values of the distribution function F(z) with $m = 21 - 30$.

	Degrees of Freedom (m)									
$F(z)$	21	22	23	24	25	26	27	28	29	30
0.005	8.03	8.64	9.26	9.89	10.52	11.16	11.81	12.46	13.12	13.79
0.01	8.9	9.54	10.2	10.86	11.52	12.2	12.88	13.56	14.26	14.95
0.025	10.28	10.98	11.69	12.4	13.12	13.84	14.57	15.31	16.05	16.79
0.05	11.59	12.34	13.09	13.85	14.61	15.38	16.15	16.93	17.71	18.49
0.95	32.67	33.92	35.17	36.42	37.65	38.89	40.11	41.34	42.56	43.77
0.975	35.48	36.78	38.08	39.36	40.65	41.92	43.19	44.46	45.72	46.98
0.99	38.93	40.29	41.64	42.98	44.31	45.64	46.96	48.28	49.59	50.89
0.995	41.4	42.8	44.18	45.56	46.93	48.29	49.64	50.99	52.34	53.67

Table 5: The Student-t distribution values of z for given values of the distribution function F(z) with $m = 1 - 10$.

	Degrees of Freedom									
$F(z)$	1	2	3	4	5	6	7	8	9	10
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.9	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.4	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81

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Table 5: The Student-t distribution values of z for given values of the distribution function F(z) with $m = 1 - 10$. (Continued)

	Degrees of Freedom									
$F(z)$	1	2	3	4	5	6	7	8	9	10
0.975	12.71	4.3	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.82	6.96	4.54	3.75	3.36	3.14	3.0	2.9	2.82	2.76
0.995	63.66	9.92	5.84	4.6	4.03	3.71	3.5	3.36	3.25	3.17
0.995	63.66	9.92	5.84	4.6	4.03	3.71	3.5	3.36	3.25	3.17
0.999	318.31	22.33	10.21	7.17	5.89	5.21	4.79	4.5	4.3	4.14

Table 6: The Student-t distribution values of z for given values of the distribution function F(z) with $m = 11 - 20$.

	Degrees of Freedom									
$F(z)$	11	12	13	14	15	16	17	18	19	20
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.54	0.54	0.54	0.54	0.54	0.54	0.53	0.53	0.53	0.53
0.8	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
0.9	1.36	1.36	1.35	1.35	1.34	1.34	1.33	1.33	1.33	1.33
0.95	1.8	1.78	1.77	1.76	1.75	1.75	1.74	1.73	1.73	1.72
0.975	2.2	2.18	2.16	2.14	2.13	2.12	2.11	2.1	2.09	2.09
0.99	2.72	2.68	2.65	2.62	2.6	2.58	2.57	2.55	2.54	2.53
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.9	2.88	2.86	2.85
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.9	2.88	2.86	2.85
0.999	4.02	3.93	3.85	3.79	3.73	3.69	3.65	3.61	3.58	3.55