Tutorial Book

M.Sc Higher Mathematics II

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List of Questions

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Theory of Probability

[Q1] Assembling Computers

Sam is going to assemble a computer by himself. He has the choice of chips from **??** brands, a hard drive from **??**, memory from **??**, and an accessory bundle from **??** local stores.
How many different ways can Sam order the parts?

(**Answer:** **??**)

[A1]

As $n_1 = \textbf{??}$, $n_2 = \textbf{??}$, $n_3 = \textbf{??}$, and $n_4 = \textbf{??}$ there are:

$$n_1 \times n_2 \times n_3 \times n_4 = \textbf{??} \times \textbf{??} \times \textbf{??} \times \textbf{??} = \textbf{??} \quad \blacksquare$$

[Q2] Four Digits

How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

(**Answer:** 156)

[A2]

Since the number must be even, we have only $n_1 = 3$ choices for the units position. However, for a four-digit number the thousands position **cannot** be 0. Hence, we consider the units position in two (2) parts: 0 or not 0. If the units position is 0 (i.e., $n_1 = 1$), we have $n_2 = 5$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. Therefore, in this case we have a total of

$$n_1 n_2 n_3 n_4 = 1 \times 5 \times 4 \times 3 = 60,$$

even four-digit numbers. On the other hand, if the units position is not 0 (i.e., $n_1 = 2$), we have $n_2 = 4$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position.

In this situation, there are a total of

$$n_1 n_2 n_3 n_4 = 2 \times 4 \times 4 \times 3 = 96,$$

even four-digit numbers. Since the above two cases are mutually exclusive, the total number of even four-digit numbers can be calculated as 60 + 96 = 156 ■

[Q3] Choosing the President in the Club

A president and a treasurer are to be chosen from a student club consisting of **??** people. How many different choices of officers are possible if:

i. there are no restrictions;
ii. A will serve only if he is president;
iii. B and C will serve together or not at all;
iv. D and E will not serve together?

(**Answer:** i. **??**, ii. **??**, iii., 2258, iv. 2448)

[A3]

i. The total number of choices of officers, without any restrictions, is

$$\frac{\mathbf{??}!}{\mathbf{??}!} = \mathbf{??} \quad ■$$

ii. Since A will serve only if they are president, we have two (2) situations here:

   a) A is selected as the president, which gives **??** possible outcomes for the treasurer's position
   b) Officers are selected from the remaining **??** people **without** A, which has the number of choices:

$$\frac{\mathbf{??}!}{\mathbf{??}!} = \mathbf{??}$$

   Therefore, the total number of choices is **??** + **??** = **??** ■

iii. The number of selections when B and C serve together is **??**. The number of selections when both B and C are not chosen is

$$\frac{\mathbf{??}!}{\mathbf{??}!} = \mathbf{??}$$

   Therefore, the total number of choices in this situation is **??** + **??** = **??**.

iv. The number of selections when D serves as an officer but not E is (2)(48) = 96, where 2 is the number of positions D can take and 48 is the number of selections of the other officer from the remaining people in the club except E. The number of selections when E serves as an officer but not D is also (2)(48) = 96. The number of selections when both D and E are not chosen is 48P2 = 2256. Therefore, the total number of choices is (2)(96) + 2256 = 2448. This problem also has another short solution: Since D and E can only serve together in 2 ways, the answer is 2450 2 = 2448.

[Q4] The Lineup

In a rugby training session, the defensive coordinator needs to have ten (10) players standing in a row. Among these ten (10) players, there are one (1) freshman, two (2) sophomores, four (4) juniors, and three (3) seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

(**Answer:** 12 600)

[A4]

We find that the total number of arrangements is:

$$\frac{10!}{1!2!3!4!} = 12\,600 \quad \blacksquare$$

[Q5] Arranging Accommodation

In how many ways can **??** graduate students be assigned to one triple and two double hotel rooms during a conference?

(**Answer: ??**)

[A5]

The total number of possible partitions would be

$$\binom{\textbf{??}}{\textbf{??}, \textbf{??}, \textbf{??}} = \frac{\textbf{??}!}{\textbf{??}!\ \textbf{??}!\ \textbf{??}!} = \textbf{??} \quad \blacksquare$$

[Q6] Buying Games

A young boy asks his mother to get 5 games from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

(**Answer:** 1200)

[A6]

The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = 120.$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = 10.$$

Using the multiplication rule with $n_1 = 120$ and $n_2 = 10$, we have

$$120 \times 10 = 1200 \quad \text{ways} \quad \blacksquare$$

[Q7] Letter Soup

How many different letter arrangements can be made from the letters in the word STATISTICS ?

(**Answer: ??**)

[A7]

We can use the classification idea we discussed:

$$\binom{10}{3,\ 3,\ 2,\ 1,\ 1} = \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!} = 50400 \quad \blacksquare$$

Here we have 10 total letters, with 2 letters (S, T) appearing 3 times each, letter I appearing twice, and letters A and C appearing once each.

[Q8] Flip of a Coin

A coin is tossed twice. What is the probability that at least 1 head occurs?

(**Answer:** $^{3}/_{4}$)

[A8]

The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of p to each sample point. Then:

$$4p = 1, \qquad \text{or} \qquad p = 1/4.$$

If A represents the event of **at least 1 head occurring**, then:

$$A = \{HH, HT, TH\} \qquad \text{and} \qquad P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \quad \blacksquare$$

[Q9] That's not Fair!

A die is loaded in a way that an even number is **twice as likely** to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

(**Answer:** $^{4}/_{9}$)

[A9]

The sample space is:

$$S = \{1, 2, 3, 4, 5, 6\} .$$

We assign a probability of p to each odd number and a probability of $2p$ to each even number. Since the sum of the probabilities must be 1, we have

$$9p = 1 \qquad \text{and} \qquad p = 1/9.$$

Therefore, probabilities of $^1/_9$ and $^2/_9$ are assigned to each odd and even number, respectively. Therefore,

$$E = \{1, 2, 3\} \qquad \text{and} \qquad P(E) = {}^1/_9 + {}^2/_9 + {}^1/_9 = {}^4/_9 \quad \blacksquare$$

[Q10] Who Wants to Help

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students.

If a person is **randomly** selected by the instructor to answer a question, find the probability that the student chosen is:

a. an industrial engineering major,

b. a civil engineering or an electrical engineering major.

(**Answer:** a.$^{25}/_{53}$, b. $^{18}/_{53}$)

[A10]

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively.

The total number of students in the class is 53, all of whom are equally likely to be selected.

a. Since 25 of the 53 students are majoring in industrial engineering, the probability of event I, selecting an industrial engineering major at random, is:

$$P(I) = \frac{25}{53}$$

b. Since 18 of the 53 students are civil or electrical engineering majors, it follows that:

$$P(C \cup E) = \frac{10}{53} + \frac{8}{53} = \frac{18}{53} \quad \blacksquare$$

[Q11] I Fold

In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

(**Answer:** 0.9×10^{-5})

[A11]

The number of ways of being dealt 2 aces from 4 cards is 4 = 4! =6, 2 2! 2! and the number of ways of being dealt 3 jacks from 4 cards is 4 = 4! =4. 3 3! 1! By the multiplication rule (Rule 2.1), there are n = (6)(4) = 24 hands with 2 aces and 3 jacks. The total number of 5-card poker hands, all of which are equally likely, is N = 52 = 52! = 2,598,960. 5 5! 47! Therefore, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is P(C) = 24 = 0.9 Œ 105.

[Q12] Any Colour, as Long as it is Black

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colours?

(**Answer:** 0.68)

[A12]

Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is:

$$P(G \cap W \cap R \cap B) = P(G) + P(W) + P(R) + P(B) = 0.09 + 0.15 + 0.21 + 0.23 = 0.68 \quad \blacksquare$$

[Q13] Check your Engine Light

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

(**Answer:** 0.69)

[A13]

Let E be the event that at least 5 cars are serviced. Now:

$$P(E) = 1 - P\left(E^{c}\right).$$

Since;

$$P(E) = 0.12 + 0.19 = 0.31,$$

it follows that

$$P(E) = 1 - 0.31 = 0.69 \quad \blacksquare$$

[Q14] Got to Catch My Flight

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane:

a. arrives on time, given that it departed on time, and

b. departed on time, given that it has arrived on time.

(**Answer:** a. 0.94, b. 0.95)

[A14]

(a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time. Solution : Using Definition 2.10, we have the following. (a) The probability that a plane arrives on time, given that it departed on time, is P(A|D)= P(DA) = 0.78 =0.94. P (D) 0.83 (b) The probability that a plane departed on time, given that it has arrived on time, is P(D|A)= P(DA) = 0.78 =0.95.

[Q15] The Calculation Looms

The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two (2) ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

(**Answer:** 0.08)

[A15]

Consider the events L: length defective, T: texture defective. Given that the strip is length defective, the probability that this strip is texture defective is given by P(T|L) = P(T L) = 0.008 = 0.08. P (L) 0.1 Thus, knowing the conditional probability provides considerably more information than merely knowing P(T).

[Q16] Fuse In, Fuse Out

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

(**Answer:** $^{1}/_{19}$)

[A16]

We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective; then we interpret A B as the event that A occurs and then B occurs after A has occurred. The probability of first removing a defective fuse is 1/4; then the probability of removing a second defective fuse from the remaining 4 is 4/19. Hence, P(AB)=1 4=1.

[Q17] White or Black

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

(**Answer:** $^{38}/_{63}$)

[A17]

Let B1, B2, and W1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events B1 B2 and W1 B2. The various possibilities and their probabilities are illustrated in Figure 2.8. Now P[(B1 B2) or (W1 B2)] = P(B1 B2) + P(W1 B2) = P (B1)P (B2|B1) + P (W1)P (B2|W1) = 3 6 + 4 5 =38. 7 9 7 9 63

[Q18] Achtung!

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

(**Answer:** 0.9016)

[A18]

Let A and B represent the respective events that the fire engine and the ambulance are available. Then P (A B) = P (A)P (B) = (0.98)(0.92) = 0.9016.

[Q19] It's your Turn

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

(**Answer:** $^{8}/_{5525}$)

[A19]

First we define the events A1: the first card is a red ace, A2: the second card is a 10 or a jack, A3: the third card is greater than 3 but less than 7. Now P(A1)= 2 , P(A2|A1)= 8 , P(A3|A1 A2)= 12, 52 51 50 and hence, by Theorem 2.12, P(A1 A2 A3)=P(A1)P(A2|A1)P(A3|A1 A2) = 2 8 12 = 8 . 52 51 50 5525

[Q20] This Car went to Vegas and Back

If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

(**Answer:** $f(x) = \frac{1}{16}\binom{4}{x}$, for $x = 0, 1, 2, 3, 4.$)

[A20]

Since the probability of selling an automobile with side airbags is 0.5, the 24 = 16 f (x)points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16. To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other. This can be
f(x)= 1 4 , forx=0,1,2,3,4. 16 x

[Q21] Let me Get my Pen

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

(**Answer:** 1.7)

[A21]

Let X represent the number of good components in the sample. The probability distribution of X is 43 x 3x f(x)= , x=0,1,2,3. 7 3 Simple calculations yield f(0) = 1/35, f(1) = 12/35, f(2) = 18/35, and f(3) = 4/35. Therefore,
=E(X)=(0) 1 +(1) 12 +(2) 18 +(3) 4 = 12 =1.7. 35 35 35 35 7

2

Statistical Methods

[Q22] Maximum Likelihood of Poisson Distribution

Consider a Poisson distribution with probability mass function

$$f(x|\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0, 1, 2, \ldots.$$

Suppose that a random sample $x_1, x_2, \ldots, x_n$ is taken from the distribution. What is the maximum likelihood estimate of μ?

[A22]

The likelihood function is

$$L(x_1, x_2, \ldots, x_n; \mu) = \prod_{i=1}^{n} f(x_i|\mu) = \frac{e^{-n\mu} \sum\limits_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i!}.$$

Now consider

$$\ln L(x_1, x_2, \ldots, x_n; \mu) = -n\mu + \sum_{i=1}^{n} x_i \ln \mu - \ln \prod_{i=1}^{n} x_i!$$

$$\frac{\partial \ln L(x_1, x_2, \ldots, x_n; \mu)}{\partial \mu} = -n + \sum_{i=1}^{n} \frac{x_i}{\mu}.$$

Solving for $\hat{\mu}$, the maximum likelihood estimator, involves setting the derivative to zero and solving for the parameter. Thus,

$$\hat{\mu} = \sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}.$$

The second derivative of the log-likelihood function is negative, which implies that the solution above indeed is a maximum. Since μ is the mean of the Poisson distribution (Chapter 5), the sample average would certainly seem like a reasonable estimator.

[Q23] Maximum Likelihood of Normal Distribution

Consider a random sample $x_1, x_2, \ldots, x_n$ from a normal distribution $N(\mu, \sigma)$. Find the maximum likelihood estimators for μ and σ^2.

[A23]

The likelihood function for the normal distribution is

$$L(x_1, x_2, \dots, x_n; \mu, \sigma^2) = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2\right].$$

Taking logarithms gives us

$$\ln L(x_1, x_2, \dots, x_n; \mu, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2.$$

Hence,

$$\frac{\partial \ln L}{\partial \mu} = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma^2}\right)$$

. and

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Setting both derivatives equal to 0, we obtain

$$\sum_{i=1}^{n} x_i - n\mu = 0 \quad \text{and} \quad n\sigma^2 = \sum_{i=1}^{n}(x_i - \mu)^2.$$

Thus, the maximum likelihood estimator of μ is given by

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x},$$

. which is a pleasing result since $\bar{x}$ has played such an important role in this chapter as a point estimate of μ. On the other hand, the maximum likelihood estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Checking the second-order partial derivative matrix confirms that the solution results in a maximum of the likelihood function.

[Q24] Rats in a Ship

Suppose 10 rats are used in a biomedical study where they are injected with cancer cells and then given a cancer drug that is designed to increase their survival rate. The survival times, in months, are 14, 17, 27, 18, 12, 8, 22, 13, 19, and 12. Assume that the exponential distribution applies. Give a maximum likelihood estimate of the mean survival time.

[A24]

From Chapter 6, we know that the probability density function for the exponential random variable X is

$$f(x,\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, the log-likelihood function for the data, given $n = 10$, is

$$\ln L(x_1, x_2, \dots, x_{10}; \beta) = -10 \ln \beta - \frac{1}{\beta} \sum_{i=1}^{10} x_i.$$

Setting

$$\frac{\partial \ln L}{\partial \beta} = -\frac{10}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^{10} x_i = 0$$

implies that

$$\hat{\beta} = \frac{1}{10} \sum_{i=1}^{10} x_i = \bar{x} = 16.2.$$

Evaluating the second derivative of the log-likelihood function at the value $\hat{\beta}$ above yields a negative value. As a result, the estimator of the parameter β, the population mean, is the sample average $\bar{x}$.

[Q25] Sampling the Population

It is known that a sample consisting of the values 12, 11.2, 13.5, 12.3, 13.8, and 11.9 comes from a population with the density function

$$f(x;\theta) = \begin{cases} \frac{\theta}{\theta^{x+1}}, & x > 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$. Find the maximum likelihood estimate of θ.

[A25]

The likelihood function of n observations from this population can be written as

$$L(x_1, x_2, \dots, x_{10}; \theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^{\theta+1}} = \frac{\theta^n}{(\prod_{i=1}^{n} x_i)^{\theta+1}},$$

.

which implies that

$$\ln L(x_1, x_2, \dots, x_{10}; \theta) = n \ln(\theta) - (\theta + 1) \sum_{i=1}^{n} \ln(x_i).$$

Setting $0 = \frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum\limits_{i=1}^{n} \ln(x_i)$ results in ...

$$\begin{aligned} \dot{\theta} &= \frac{n}{\sum\limits_{i=1}^{n} \ln(x_i)} \\ &= \frac{6}{\ln(12) + \ln(11.2) + \ln(13.5) + \ln(12.3) + \ln(13.8) + \ln(11.9)} = 0.3970. \end{aligned}$$

Since the second derivative of L is $-n/\theta^2$, which is always negative, the likelihood function does achieve its maximum value at $\dot{\theta}$.