Random Approximation of the Sierpinski Gasket

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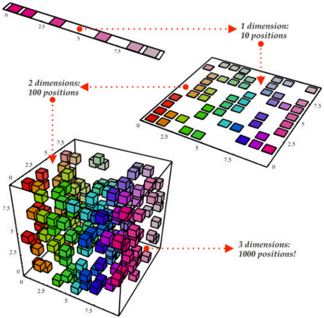
Non-Linear Dimension Reduction

- Non-Linear Dimension Reduction (NLDR) is an important technique used to reduce a high-dimensional data set into a lower dimension manifold.
- Using NLDR, algorithms can be run on the reduced data set much faster, while still preserving the information contained within.
- Laplacian Eigenmaps are a popular technique for NLDR that preserves the geometric structure of the data

Rough NLDR

- While data is usually assumed to lie on a manifold, there is no reason to assume this is always true
- Real world data sets, especially in ML, may lie in a rough space without the manifold structure
- Using Laplacian eigenmaps, we can do NLDR on rough surfaces, such as fractals [1]



The Resistance Metric

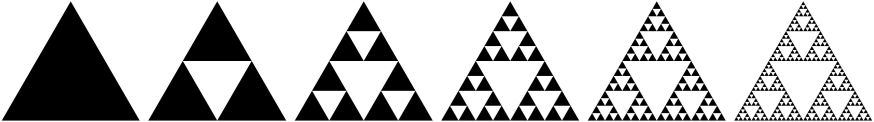
- Rather than find eigenmaps directly, we investigate the resistance metric on graph approximations of the fractal
- The resistance metric can be used to find Green's functions using [2]

$$g_x(y,z) = \frac{R(x,y) + R(x,z) - R(y,z)}{2}.$$

- Green's functions can be used to find Laplacian eigenmaps

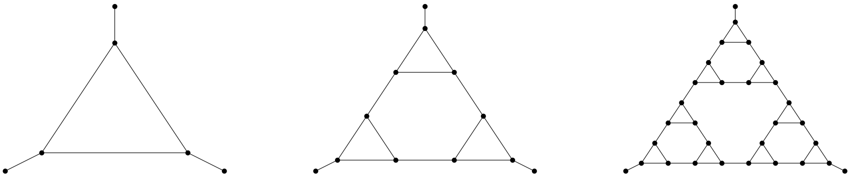
Sierpinski Gasket

- The Sierpinski Gasket is a commonly studied fractal
- It can be constructed by from successive graph approximations, G_n
- We will study the graph Laplacian on SG by investigating the resistance metric

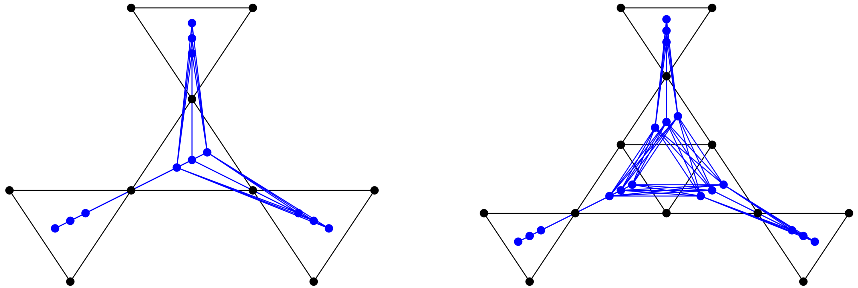


Amalgamation Process

- We scatter points randomly on the nth level of the SG
- We connect points in adjacent cells, forming the nth level Hanoi graph H_n. [3]
- In order to simplify, we perform an "amalgamation" process where n edges between multiple vertices becomes a single edge with weight n



Amalgamation Process



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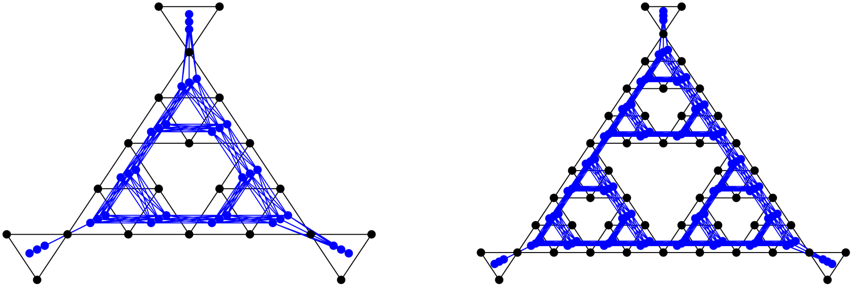


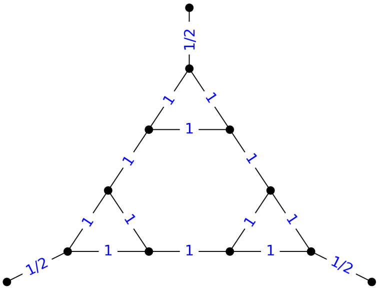
Electrical Network Interpretation

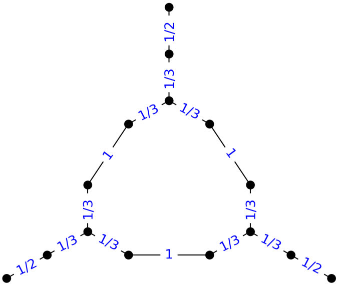
- We can interpret a graph with edge weights as an electrical network, where the weights are conductances c_{xy}, and have resistances of $r_{xy} = 1/c_{xy}$. [4]
- Physically, this can be realized by creating a network where each vertex of the graph is a node, and edges connecting nodes are resistors with conductance given by the edge weight.

The Resistance Metric

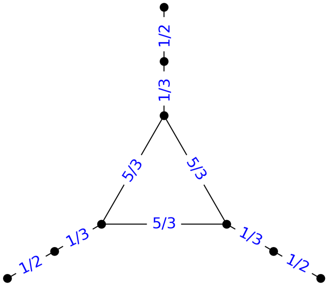
- This lets us define a metric, the resistance metric or effective resistance. [5]
- The effective resistance between x and y, $R(x, y)$ can be physically interpreted as measuring the current flow between the two points.
- Mathematically, this can be done by reducing the graph to just the two points x and y using energy-minimizing rules.

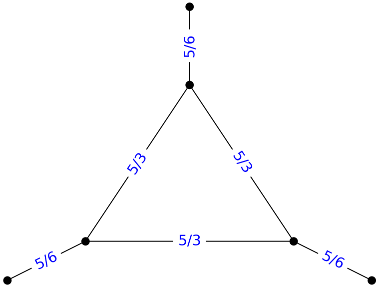
Reduction





Reduction





Normalizing Resistance

- We can ask what edge weights should be assigned to the edge weights of H_n to ensure that between two tail points p and q, $R_n(p, q) = 1$
- In the deterministic case, we find that reducing from H_n to H_{n-1} puts a factor of 5/3 on every edge weight
- Thus, we find that to get $R_n(p, q) = 1$, we should have $r_{xy} = (\frac{3}{5})^n$, except for the tail weights, which should be $\frac{1}{2}(\frac{3}{5})^n$

Random Model

- What about the probabilistic case, motivated by the construction of a graph Laplacian on random points?
- We must first choose how the points are distributed. We choose to distribute the points according to identical, independent Poisson processes in each cell
- However, to ensure no cell is empty, and thus the correct Hanoi graph is always formed, we modify this slightly to have $1 + X$ points in each cell, where X is a Poisson random variable

Random Model

Conjecture

Forming H_n through the random process described earlier, as $n \to \infty$,

$$\mathbb{E}(R_n(p,q)) = (\frac{5}{3})^n \mathbb{E}(r_{xy})$$

for tail points p and q

Conjecture

Forming H_n through the random process described earlier, as $n \to \infty$,

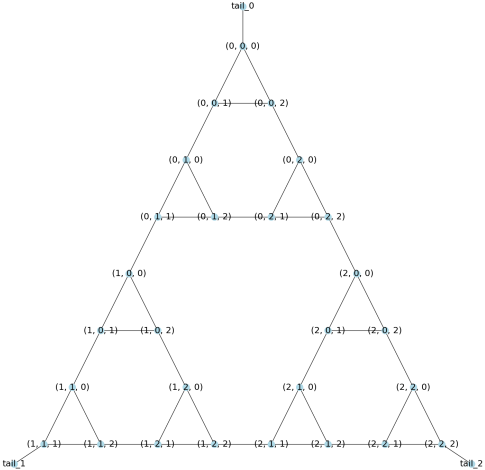
$$\mathbb{V}(R_n(x,y)) = 0$$

for any two points x and y

Graph Creation

To investigate these conjectures numerically, we first must create H_n: vertex naming convention.

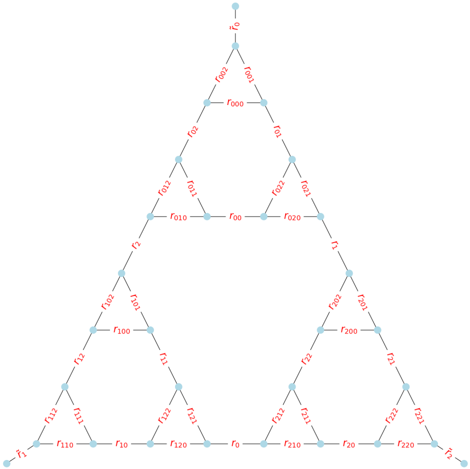
- Each vertex is labeled with a tuple of length n whose elements are either $0, 1$, or 2.
- The i-th entry denotes which component of the H_{n-i} subgraph the node is in.



Graph Creation

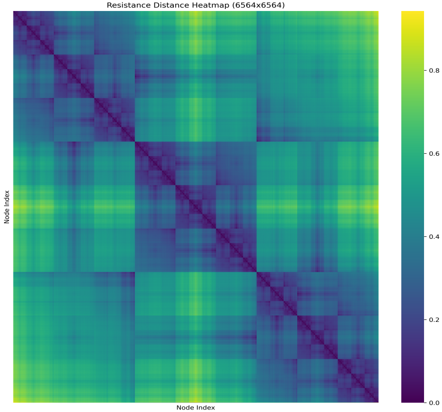
Edge naming convention: two slight complications.

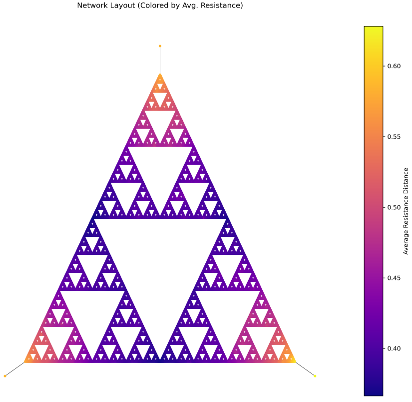
- Delta edges exist within the smallest H_1 triangles and have a subscript of length n.
- Bridge edges connect larger subgraphs. An edge that bridges two H_k subgraphs has a subscript of length $n - k$.



Computing R_n, $\mathbb{E}(R_n)$ and $\mathbb{V}(R_n)$

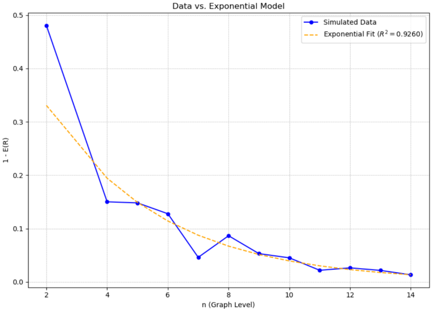
- With our graph created and edge weights randomly assigned via the rule $r_{xy} = [(1+X)(1+Y)]^{-1}$ where $X, Y \overset{\text{i.i.d.}}{\sim} \text{Pois}(\lambda)$ and $\lambda(n)$ fulfills $\mathbb{E}(r_{xy}) = (3/5)^n$, we can compute the effective resistance matrix R [5].
- The following plots show the behavior of our metric for $n = 8$.

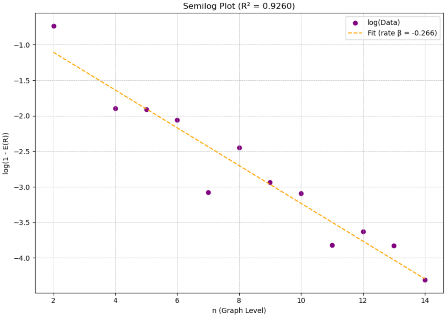




Computing R_n, $\mathbb{E}(R_n)$ and $\mathbb{V}(R_n)$

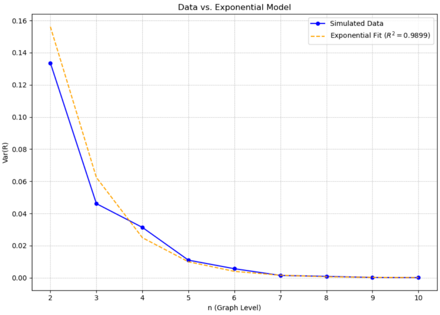
- With our resistance metric we investigate the scaling of $\mathbb{E}(R_n(p,q))$ where p, q are tail nodes.
- Our conjecture states $\mathbb{E}(R_n(p,q)) \to 1$, so we plot $1 - \mathbb{E}(R_n(p,q))$ as n increases and find the following.

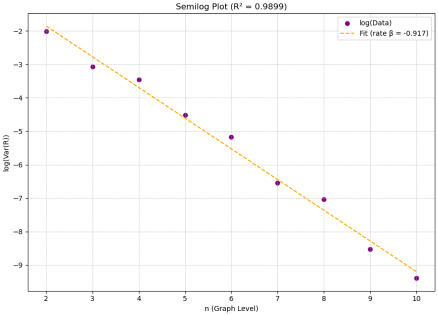




Computing R_n, $\mathbb{E}(R_n)$ and $\mathbb{V}(R_n)$

- Similar analysis on $\mathbb{V}(R_n(p,q))$ demonstrates stronger exponential decay.
- These two results indicate that the system is stabilizing as n increases, but by what mechanism?





- The stabilizing mechanism is the **graph reduction** process.
- We aim to define this process as a composition of mappings from edges on H_{n+1} to edges on H_n.
- Let $\varphi_n^j : \mathbb{R}^7 \to \mathbb{R}$ be the reduction map from H_{n+1} to H_n for a single edge j.

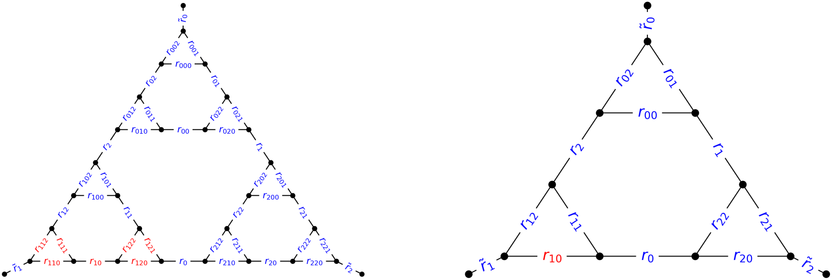


Figure: The 7 input edges to $\varphi_2^{r_{10}}$.

The Reduction Mappings: Formal Definitions

To formalize the graph reduction mappings, let $\mathbf{r}_k \in \mathbb{R}^{|E_k|}$ be the vector of edge resistances on the graph H_k.

Definition (Local Map (φ_n^j))

The local map $\varphi_n^j : \mathbb{R}^7 \to \mathbb{R}$ computes the new resistance r_j' for a single edge $j \in E_n$.

$$r_j' = r_j + \frac{r_a r_b}{r_a + r_b + r_c} + \frac{r_d r_e}{r_d + r_e + r_f}$$

Here, r_j is the resistance of the corresponding edge on H_{n+1}, and $\{r_a, r_b, r_c\}$ and $\{r_d, r_e, r_f\}$ are the resistances of the edges that form the two delta networks that j connects on H_{n+1}.

The Reduction Mappings: Formal Definitions

To formalize the graph reduction mappings, let $\mathbf{r}_k \in \mathbb{R}^{|E_k|}$ be the vector of edge resistances on the graph H_k.

Definition (One-Step Map (φ_n))

The one-step map $\varphi_n : \mathbb{R}^{|E_{n+1}|} \to \mathbb{R}^{|E_n|}$ reduces the entire graph by one level, where

$$\mathbf{r}_n = \varphi_n(\mathbf{r}_{n+1}).$$

Its components are computed using the local maps φ_n^j for all $j \in E_n$.

Definition (Full Reduction Map (Φ_n))

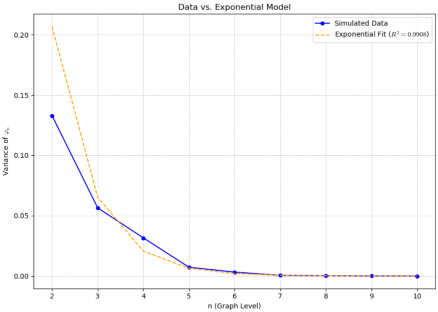
The full map $\Phi_n : \mathbb{R}^{|E_n|} \to \mathbb{R}^{|E_1|}$ is the composition of all one-step maps from level n down to level 1.

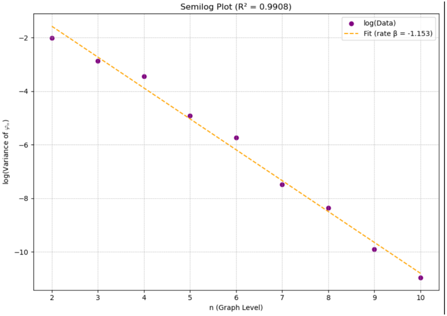
$$\Phi_n = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1}$$

Numerical Analysis of φ_n^j

Having defined the local map φ_n^j as the core mechanism, we now test its key property: does it effectively reduce variance?

- Our analysis confirms strong exponential decay for the variance of the local map, $\mathbb{V}(\varphi_n^j)$, with a decay rate of $\beta \approx -1.15$.
- This provides a solid foundation for the convergence of the full map Φ_n.





Expected Value and Variance of r_{xy}

- We will now go over some results involving the expected value and variance of the resistance metric
- These proofs rely on relating the expected value and variance of the final level of reduction to the expected value and variance of the distribution of edge weights r_{xy}
- Thus, we need to find $\mathbb{E}(r_{xy})$ and $\mathbb{V}(r_{xy})$ and confirm their asymptotic behavior.

Expected Value

Because X and Y are independent and identically distributed, we can say that $\mathbb{E}(r_{xy}) = \mathbb{E}(\frac{1}{1+X} \cdot \frac{1}{1+Y}) = [\mathbb{E}(\frac{1}{1+X})]^2$. Using the fact that X is Poisson distributed with expectation λ, we can write that

$$\mathbb{E}(\frac{1}{1+X}) = \sum_{n=0}^{\infty} \frac{1}{1+n} \cdot \frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+1)!}$$

$$= e^{-\lambda} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} = \frac{e^{-\lambda}}{\lambda}(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} - 1) = \frac{1}{\lambda}(1 - e^{-\lambda}).$$

So we get that $\mathbb{E}(r_{xy}) = (\frac{1}{\lambda}[1 - e^{-\lambda}])^2$, and when forming H_n, the expectation of the Poisson distribution should fulfill the equation

$$\left[\frac{1}{\lambda}(1 - e^{-\lambda})\right]^2 = \left(\frac{3}{5}\right)^n$$

Variance

For variance, we know that $\mathbb{V}(r_{xy}) = \mathbb{E}(r_{xy}^2) - [\mathbb{E}(r_{xy})]^2$. We already know the second term, so we will focus on the first.

$$\mathbb{E}(r_{xy}^2) = \mathbb{E}(\frac{1}{(1+X)^2}\frac{1}{(1+Y)^2}) = (\mathbb{E}\frac{1}{(1+X)^2})^2.$$

$$\mathbb{E}(\frac{1}{(1+X)^2}) = \sum_{n=0}^{\infty}\frac{1}{(1+n)^2}\frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\int_0^{\lambda}\frac{e^x-1}{x}dx.$$

So

$$\mathbb{V}(r_{xy}) = (e^{-\lambda}\int_0^{\lambda}\frac{e^x-1}{x}dx)^2 - (\frac{1}{\lambda}[1-e^{-\lambda}])^4.$$

Variance Bound Proof

We found a relation between $\mathbb{E}(R_n)$ and $\mathbb{E}(R_{n-1})$ involving a factor of $5/3$, similar to the deterministic case.

Theorem

For any effective resistance vector r composed of $(r^{(0)}, r^{(1)}, r^{(2)})$ we have a bound

$$||\mathbb{E}(r) - \frac{5}{3}\mathbb{E}(r^{(0)})||^2 \leq C\mathbb{V}(r^{(0)})$$

Where $(r^{(0)}, r^{(1)}, r^{(2)})$ are composed of individual edge resistances in their cluster.

Proof Sketch

- Take second degree Taylor Error
- Linearize around the vector $(\mathbb{E}(r^{(0)}), \mathbb{E}(r^{(1)}), \mathbb{E}(r^{(2)}))$
- With projective space normalize gradient term around vector $(1, 1, ..., \frac{1}{2}, \frac{1}{2})$ and Hessian around fixed ζ
- Use holders inequality to bound hessian term
- Take expectation which kills linear terms

Variance Bound Corollary

Corollary

Between 2 tail points p and q,

$$||\mathbb{E}(R_n(p,q)) - (\frac{5}{3})^n \mathbb{E}(R_1(p,q))||^2 \leq C \sum_{k=0}^{n-1} (\frac{5}{3})^{n-k} \mathbb{V}(R_{k+1})$$

Variance going to 0

Theorem

For the reduction map $\Phi_n : H_{n+1} \to H_1$ we have that the

$$\mathbb{V}(\Phi_n) \to 0$$

Note: This guarantees that variances of the reduction mappings converge to 0, but does not provide a rate of convergence.

Proof Sketch

- Breakdown Φ_n as a composition of $\varphi_1 \circ \varphi_2 \circ ... \circ \varphi_{n-1}$ and rewrite $\Phi_n = \varphi_1$ with suppressed arguments
- Use bound of

$$\mathbb{V}(\varphi_1^j) \leq 12 \cdot \sum_{j=1}^{k} \mathbb{E}\left(\varphi_2^j\right)^2$$

  where $k = N(n) = \frac{3^{n+1}+3}{2}$
- Each map can be bounded with a similar operation and a constant of 12 will be pulled out for each level
- The φ_1^j composes the φ_1 map
- The composition $\mathbb{V}(\Phi_n) \to 0$

Closing Thoughts and Next Steps

- We have an alternative proof which provides a bound on the rate of $\mathbb{V}(\Phi_n) \to 0$, giving a rate of roughly -0.07 exponentially
- We are seeking a proof that aligns with the numerical rate, which is around -0.95.
- Next steps on this project could include investigating Green's functions using this metric, and using that to find eigenmaps

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