MATH 313 HMWK 6

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1. Since any real number squared is greater than zero, then for any two real number x and y, $0 \leq \left(x^2 - y^2\right)^2$. Then

$$
\begin{aligned}
0 &\leq \left(x^2 - y^2\right)^2 \\
0 &\leq \frac{\left(x^2 - y^2\right)^2}{4} \\
0 &\leq \frac{x^4 - 2x^2y^2 + y^2}{4} \\
\frac{4x^2y^2}{4} &\leq \frac{x^4 + 2x^2y^2 + y^2}{4} \\
x^2y^2 &\leq \frac{x^4 + 2x^2y^2 + y^2}{4} \\
\sqrt{x^2y^2} &\leq \sqrt{\frac{x^4 + 2x^2y^2 + y^2}{4}} \\
|xy| &\leq \frac{x^2 + y^2}{2}
\end{aligned}
$$

Now we can apply this to our proof. So

$$
\begin{aligned}
|a_n b_n| &\leq \frac{a_n^2 + b_n^2}{2} \\
\sum_{n=0}^{\infty} |a_n b_n| &\leq \sum_{n=0}^{\infty} \frac{a_n^2 + b_n^2}{2} \\
\sum_{n=0}^{\infty} |a_n b_n| &\leq \frac{1}{2} \sum_{n=0}^{\infty} a_n^2 + b_n^2 \\
\sum_{n=0}^{\infty} |a_n b_n| &\leq \frac{1}{2} \left(\sum_{n=0}^{\infty} a_n^2 + \sum_{n=0}^{\infty} b_n^2 \right)
\end{aligned}
$$

Therefore, $\sum_{n=0}^{n=\infty} |a_n \cdot b_n|$ absolutely converge if $\sum_{n=0} a_n^2 < +\infty$ and $\sum_{n=0} b_n^2 < +\infty$

2. (a) We can use the Ratio Test. So

$$\lim \left| \frac{\frac{(n+1)^{n+1}}{2^{n+1} \cdot ((n+1)!)^2}}{\frac{n^n}{2^n \cdot (n!)^2}} \right| = \lim \left| \frac{(n+1)^{n+1}}{2^{n+1} \cdot ((n+1)!)^2} \cdot \frac{2^n \cdot (n!)^2}{n^n} \right|$$

$$= \lim \left| \frac{(n+1)^{n+1}}{2 \cdot (n+1)^2 \cdot n^n} \right|$$

$$= \lim \left| \frac{(n+1)^{n-1}}{2 \cdot n^n} \right|$$

$$= 0$$

Since the ratio is 0, which is less than 1, the sum converges. Since n is a natural number, and there are no operations in the sum that could result in a negative number, the sum is absolutely convergent as well.

(b) It can be shown that the sequence $a_n = 1/(8n \cdot \log n)$ is greater than $1/(n^{1/n} \cdot (n+1) \cdot \log(5n))$ for $n \geq 3$. Since a_n diverges, we know that by the Comparison Test that this sum must diverge as well.

(c) Let $a_n = 1/\log(2n+1)$. Then this sum can be written as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(2n+1)} = \sum_{n=1}^{\infty} (-1)^n a_n$$

We can apply the Alternating Series test. First, we show that $a_n \geq a_{n+1}$. We consider the base case.

$$a_1 \overset{?}{\geq} a_2$$

$$\frac{1}{\log(2 \cdot 1 + 1)} \overset{?}{\geq} \frac{1}{\log(2 \cdot 2 + 1)}$$

$$\frac{1}{\log(3)} \overset{?}{\geq} \frac{1}{\log(5)}$$

$$\log(3) \overset{?}{\leq} \log(5)$$

$$3 \leq 5 \qquad \text{since log is an increasing function}$$

Now that we have established the base case, we must show that

$a_k \geq a_{k+1}$ implies $a_{k+1} \geq a_{k+2}$. So

$$\begin{aligned}
\frac{1}{\log(2 \cdot k + 1)} &\geq \frac{1}{\log(2 \cdot (k+1) + 1)} \\
\log(2k+1) &\leq \log(2k+3) \\
2k+1 &\leq 2k+3 \\
2k+3 &\leq 2k+5 \\
2(k+1)+1 &\leq 2(k+2)+2 \\
\log(2(k+1)+1) &\leq \log(2(k+2)+2) \\
\frac{1}{\log(2(k+1)+1)} &\leq \frac{1}{\log(2(k+2)+2)}
\end{aligned}$$

We show that (a_n) converges to zero. For any arbitrary $\epsilon > 0$, we pick $N \geq \frac{e^{1/\epsilon}-1}{2}$. Now we must show that for any $n \geq N$, $|a_n - 0| < \epsilon$. If $n \geq N$, then

$$\begin{aligned}
n &\geq \frac{e^{1/\epsilon}-1}{2} \\
2n &\geq e^{1/\epsilon} - 1 \\
2n+1 &\geq e^{1/\epsilon} \\
\log(2n+1) &\geq \frac{1}{\epsilon} \\
\frac{1}{\log(2n+1)} &\leq \epsilon \\
\left| \frac{1}{\log(2n+1)} - 0 \right| &\leq \epsilon
\end{aligned}$$

Therefore, by the Alternating Series Test, this sum must converge.

3. (a) We can use the ratio test to find out for which values of x this series is convergent. So

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-2x)^{n+1}}{n+1} \cdot \frac{n}{(-2x)^n} \right| \\
&= \left| \frac{n}{-2x \cdot (n+1)} \right|
\end{aligned}$$

Now we have to find which values of x make the sequence converge to some value less than 1. A little more work shows

$$\lim \left| \frac{n}{-2x \cdot (n+1)} \right| = \frac{1}{2|x|} \cdot \lim \left(\frac{n}{n+1} \right)$$

It is trivial to show that $\lim n/(n+1) = 1$. So if the series is to converge, then

$$\frac{1}{2|x|} < 1$$
$$|x| > \frac{1}{2}$$

So x has to be greater than $1/2$ or less than $-1/2$

(b) We can use the Ratio test again for this problem. So

$$\begin{aligned}
\lim \left| \frac{\frac{x^{n+1}}{3^{n+1} - 2^{n+1}}}{\frac{x^n}{3^n - 2^n}} \right| &= \lim \left| \frac{x^{n+1}}{3^{n+1} - 2^{n+1}} \cdot \frac{3^n - 2^n}{x^n} \right| \\
&= \lim \left| x \cdot \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}} \right| \\
&= |x| \cdot \lim \left| \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}} \right| \\
&= |x| \cdot \lim \left| \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \right| \\
&= |x| \cdot \lim \left| \frac{1 - \left(\frac{2}{3}\right)^n}{3 - 2 \cdot \left(\frac{2}{3}\right)^n} \right| \\
&= |x| \cdot \lim \left| \frac{1 - 0}{3 - 2 \cdot 0} \right| && \text{by } \textbf{Example 2.5.3} \\
&= |x| \cdot \frac{1}{3}
\end{aligned}$$

Therefore, $x < 3$ or $x > -3$.

(c) We apply the Ratio Test once again.

$$\begin{aligned}
\lim \left| \frac{\frac{x^{2(n+1)}}{(n+1)^3}}{\frac{x^{2n}}{n^3}} \right| &= \lim \left| \frac{x^{2(n+1)}}{(n+1)^3} \cdot \frac{x^{2n}}{n^3} \right| \\
&= x^2 \cdot \lim \left| \frac{n^3}{(n+1)^3} \right| \\
&= x^2
\end{aligned}$$

Therefore, $x < 1$ and $x > -1$.

4. (a) Since $\sum_{n=1}^{\infty} a_n$ is a positive convergent series, it must be bounded. So

there exists some M such that

$$\sum_{n=1}^{\infty} a_n \leq M$$

So

$$\sum_{n=1}^{\infty} a_n \leq M$$
$$\left(\sum_{n=1}^{\infty} a_n\right)^2 \leq M^2$$
$$\sum_{n=1}^{\infty} a_n^2 \leq \left(\sum_{n=1}^{\infty} a_n\right)^2 \leq M^2$$

Therefore, $\sum_{n=1}^{\infty} a_n^2$ is bounded. We also know that it is monotone since a_n^2 will always be a positive number or 0. By the Monotone Convergence Theorem, $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Let $a_n = (-1)^{n+1} \cdot 1/\sqrt{n}$. Then $\sum_{n=1}^{\infty} a_n$ converges since $a_{n+1} < a_n$ and a_n converges to zero. However, $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} 1/n$ does not converge, which was shown in **Exercise 2.4.5**