MATH 313 HMWK 11

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Exercise 5.3.8 We want to show that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

exists. Let us define functions $m(x) = f(x) - f(0)$ and $n(x) = x$ with m and n continuous on the same interval that f is. Then $m(0) = f(0) - f(0) = 0$ and $n(0) = 0$. It is evident that

$$f'(0) = \lim_{x \to 0} \frac{m(x)}{n(x)}$$

and

$$\begin{aligned} \lim_{x \to 0} \frac{m'(x)}{n'(x)} &= \lim_{x \to 0} \frac{f'(0)}{1} \\ &= \lim_{x \to 0} f'(0) \\ &= L && \text{this is given} \end{aligned}$$

By L'Hospital's Rule,

$$\lim_{x \to 0} \frac{m(x)}{n(x)} = L$$

and since

$$f'(0) = \lim_{x \to 0} \frac{m(x)}{n(x)}$$

$f'(0) = L$.

Exercise 5.3.11

(a) Say that a sequence $x_n \to a$, with $x_n > 0$. Then

$$\begin{aligned} \frac{f(x_n)}{g(x_n)} &= \frac{f(x_n) - 0}{g(x_n) - 0} \\ &= \frac{f(x_n) - f(a)}{g(x_n) - g(a)} \\ &= \frac{f'(c_n)}{g'(c_n)} \\ &\to L && \text{by definition of limit} \end{aligned}$$

(b) No, it does not necessarily follow. Let

$$f(x) = \frac{1}{6}x^5 \quad \text{and} \quad g(x) = \frac{1}{2}x^2 - 3x$$

Then

$$\begin{aligned}
\lim_{x \to 3} \frac{f(x)}{g(x)} &= \lim_{x \to 3} \frac{\frac{1}{6}x^5}{\frac{1}{2}x^2 - 3x} \\
&= \frac{\frac{243}{6}}{-\frac{9}{2}} \\
&= \frac{243}{6} \cdot -\frac{2}{9} \\
&= -9
\end{aligned}$$

However,

$$\begin{aligned}
\lim_{x \to 3} \frac{f'(x)}{g'(x)} &= \lim_{x \to 3} \frac{x^5}{x-3} \\
&= \infty
\end{aligned}$$

Exercise 6.2.1

(a)

$$\begin{aligned}
\lim_{n \to \infty} g_n(x) &= \lim_{n \to \infty} \frac{nx}{1 + nx^2} \\
&= \lim_{n \to \infty} \frac{nx}{n \cdot \left(\frac{1}{n} + x^2\right)} \\
&= \lim_{n \to \infty} \frac{x}{\frac{1}{n} + x^2} \\
&= \frac{1}{x}
\end{aligned}$$

(b) Yes. For any $c \in (0, 1)$, given any arbitrary $\epsilon > 0$, we can just choose $\delta = |c| \cdot \epsilon$. Then, since

$$\left| \frac{x - c}{x} \right| < |x - c|$$

then

$$\left| \frac{c - x}{cx} \right| < \epsilon$$

(c) Yes. See above.

(d) Yes. See above.