Chapter 1

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1. (a) We assume that $\sqrt{3}$ is rational. By definition, it can be written in the form p/q, where p and q are natural numbers. Let p/q be fully reduced.

$$\begin{aligned}
\sqrt{3} &= \frac{p}{q} \\
\sqrt{3} \cdot q &= p \\
\left(\sqrt{3} \cdot q\right)^2 &= p^2 \\
3q^2 &= p^2 && p \text{ must contain the factor 3. Let } p = 3k, k \in \mathbb{N} \\
3p^2 &= (3k)^2 \\
3p^2 &= 9k^2 \\
p^2 &= 3k^2
\end{aligned}$$

Since $3 \mid q^2$, then $3 \mid q$. Also, $3 \mid p$. This means that p/q can be reduced, but this contradicts our original assumption that p/q was fully reduced. Therefore, $\sqrt{3}$ is not rational. A similar argument does work to show that $\sqrt{6}$ is irrational.

(b)

$$\begin{aligned}
\sqrt{4} &= \frac{p}{q} \\
\sqrt{4} \cdot q &= p \\
\left(\sqrt{4} \cdot q\right)^2 &= p^2 \\
4q^2 &= p^2 && p \text{ can contain the factor 2.}
\end{aligned}$$

The proof breaks down when we try to claim that $p = 4k$, where $k \in \mathbb{N}$. It might be possible that $p = 2k$ since 4 is a perfect square.

2. We assume that r is rational. Then, it can be written as p/q, where p and q are natural numbers.

$$\begin{aligned}
2^r &= 3 \\
2^{p/q} &= 3 \\
\left(2^{p/q}\right)^q &= 3^q \\
2^p &= 3^q
\end{aligned}$$

Since p and q are natural numbers, then 2^p must contain a factor of 3 and 3^q must contain a factor of 2. However, 2^p does not contain 3 and 3^q does not contain 2, so we can conclude that r is not a rational number.

3. (a) false. When

$$\begin{aligned}
&A_1 = \mathbb{N} = \{1, 2, 3, \dots\} \\
&A_2 = \{2, 3, 4, \dots\} \\
&A_3 = \{3, 4, 5, \dots\} \\
&\dots \\
&A_n = \{n, n+1, n+2, \dots\}
\end{aligned}$$

Then
$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$
, which is clearly not infinite.

   (b) true

   (c) false. Let $A = \{1\}$, $B = \emptyset$, and $C = \{1, 2\}$. Then

$$\begin{aligned}
&A \cap (B \cup C) = \{1\} \\
&(A \cap B) \cup C = \{1, 2\} \\
&\{1\} \neq \{1, 2\}
\end{aligned}$$

   (d) true

   (e) true

4. Let A_1 contain 1 and every multiple of 2. Then, let A_2 contain every multiple of 3 not divisible by 2. Let A_3 contain every multiple of 5 not divisible by 2 or 3. Let A_k contain every multiple of the kth prime number not divisible by the previous $k-1$ prime numbers. Since there are an infinite number of prime numbers, there are an infinite number of sets. Also, there are an infinite number of multiples of prime numbers, so each set will have an infinite number of elements. None of the infinites sets will intersect since all of their elements are relatively prime to each other.

5. (a) If $x \in (A \cup B)^c$, then $x \notin (A \cup B)$. This means that $x \notin A$ or $x \notin B$, which is what $A^c \cup B^c$ means.

   (b) If $x \in A^c \cap B^c$, then $x \notin A$ or $x \notin B$. This means that $x \notin (A \cup B)$, which means $x \in (A \cup B)^c$.

   (c) Since $(A \cup B)^c \subset A^c \cup B^c$ and $A^c \cup B^c \subset (A \cup B)^c$, it is implied that $(A \cup B)^c = A^c \cap B^c$.

6. (a) We consider the case when $a > 0$ and $b > 0$. Then $a + b > 0$, so $|a + b| = a + b$. Since $a > 0$, then $|a| = a$. This is similarly true for

b. So $|a| + |b| = a + b$. By the transitive property, $|a + b| = |a| + |b|$, which verifies the triangle inequality for all $a > 0$ and $b > 0$.

We consider the case when $a < 0$ and $b < 0$. Then $a + b < 0$. Thus, $|a + b| = -(a + b)$. Since $a < 0$, $|a| = -a$. This is similarly true for b. So $|a| + |b| = -a + (-b) = -(a + b)$. By the transitive property, $|a + b| = |a| + |b|$, verifying the triangle inequality for $a < 0$ and $b < 0$.

(b) For all numbers, it is true that the square of the absolute value of the number is greater or equal than the square of the number. Therefore

$$\begin{aligned} a^2 + 2ab + b^2 &\leq |a|^2 + 2|a||b| + |b|^2 \\ (a + b)^2 &\leq (|a| + |b|)^2 \end{aligned}$$

(c)

$$\begin{aligned} |a - b| &= |a - c + c - d + d - b| \\ &\leq |a - c| + |c - d + d - b| && \text{By Triangle Inequality} \\ &\leq |a - c| + |c - d| + |d - b| && \text{By Triangle Inequality} \end{aligned}$$

(d)

$$\begin{aligned} ||a| - |b|| &= ||a - b + b| - |b|| && \text{By identity property} \\ &\leq ||a - b| + |b| - |b|| && \text{By triangle inequality} \\ &= |a - b| \end{aligned}$$

7. (a) Since $A = [0, 2]$ and $B = [1, 4]$, then $f(A) = [0, 4]$ and $f(B) = [1, 16]$. $A \cap B = [1, 2]$ and $A \cup B = [0, 4]$, so $f\,(A \cap B) = [1, 4]$ and $f\,(A \cup B) = [0, 16]$. $f(A) \cup f(B) = [0, 16]$, so $f\,(A \cup B) = f(A) \cup f(B)$. Likewise, $f(A) \cap f(B) = [1, 4]$, so $f\,(A \cap B) = f(A) \cap f(B)$.

(b) Let $A = [0, 2]$ and $B = [-3, 0]$. Then $A \cap B = [0, 0]$, $f(A) = [0, 4]$, and $f(B) = [0, 9]$. However, $f(A \cap B) = [0, 0]$ and $f(A) \cap f(B) = [0, 4]$. Evidently, $f(A \cap B) \neq f(A) \cap f(B)$.

(c) In the case of $g(A \cap B)$, it is true by definition that $\forall g(x) \in g(A \cap B).\, x \in A \cap B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. This implies that $g(x) \in g(A)$ and $g(x) \in g(B)$ by definition. This is exactly what $g(A) \cap g(B)$ is.

(d) We conjecture that for any arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cup B) \subset g(A) \cup g(B)$.

$\forall g(x) \in g(A \cup B).\, x \in A \cup B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. In the case where $x \in A$ but $x \notin B$, this implies that $g(x) \in A$, which means that $g(x) \in g(A \cup B)$. In the case where $x \in B$ but $x \notin A$, $g(x) \in B$, which means that $g(x) \in g(A \cup B)$. This is evidently also true when $x \in A$ and $x \in B$.

8. (a) $f(x) = 2x$ is an example of such a function. Let $a_1 \in A$ and $a_2 \in A$, with $a_1 \neq a_2$. Then $2a_1 \neq 2a_2$. However, if we choose $3 \in \mathbb{N}$, there is no natural number $a \in \mathbb{N}$ such that $f(a) = 3$.

   (b) Let $f(x) = |x-2| + 1$. Then for any $b \in \mathbb{N}$, it is always true that $a = b+1$ is in $\mathbb{N}$ and $f(a) = b$. However, $f(1) = 2$ and $f(3) = 2$ even though $1 \neq 3$.

   (c)

$$f(x) = \begin{cases} \frac{x-1}{2} \text{ if } x \text{ is odd} \\ -\frac{x}{2} \text{ if } x \text{ is even} \end{cases}$$

9. (a) $f^{-1}(A) = \{-2, 2\}$ and $f^{-1}(B) = \{-1, 1\}$. $f^{-1}(A \cap B) = [-1, 1]$ and $f^{-1}(A) \cap f^{-1}(B) = \{-1, 1\}$, so $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. $f^{-1}(A \cup B) = [-2, 2]$ and $f^{-1}(A) \cup f^{-1}(B) = [-2, 2]$, so $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

   (b) $x \in g^{-1}(A \cap B)$ implies that $g(x) \in A \cap B$, so $g(x) \in A$ and $g(x) \in B$. From this, we can conclude that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, which is equivalent to $g^{-1}(A) \cap g^{-1}(B)$. Thus, we have shown that $x \in g^{-1}(A \cap B) \implies x \in g^{-1}(A) \cap g^{-1}(B)$, meaning $g^{-1}(A \cap B) \subset g^{-1}(A) \cap g^{-1}(B)$. We must show that $g^{-1}(A) \cap g^{-1}(B) \subset g^{-1}(A \cap B)$ If $x \in g^{-1}(A) \cap g^{-1}(B)$, then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. This implies that $g(x) \in A$ and $g(x) \in B$, meaning that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$. Thus, we have shown that $g^{-1}(A) \cap g^{-1}(B) \subset g^{-1}(A \cap B)$.

   Thus, since $g^{-1}(A) \cap g^{-1}(B) \subset g^{-1}(A \cap B)$ and $g^{-1}(A) \cap g^{-1}(B) \subset g^{-1}(A \cap B)$, we can conclude that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.

   $x \in g^{-1}(A \cup B)$ implies that $g(x) \in A \cup B$, meaning that $g(x) \in A$ or $g(x) \in B$. This is equivalent to $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, so $x \in g^{-1}(A) \cup g^{-1}(B)$. This, we have shown that $g^{-1}(A \cup B) \subset g^{-1} \cup g^{-1}(B)$.

   $x \in g^{-1}(A) \cup g^{-1}(B)$ means that $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. This implies that $g(x) \in A$ or $g(x) \in B$, which is equivalent to $g(x) \in (A \cup B)$. From this, we conclude that $x \in g^{-1}(A \cup B)$. Thus, we have shown that $g^{-1}(A) \cup g^{-1}(B) \subset g^{-1}(A \cup B)$.

   Since $g^{-1}(A) \cup g^{-1}(B) \subset g^{-1}(A \cup B)$ and $g^{-1}(A \cup B) \subset g^{-1} \cup g^{-1}(B)$, we can conclude that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$.

10. (a) false. Let $a = 1$, $b = -2$, and $\epsilon = 8$. $1 < -2 + 8$, yet $1 \not< -2$.

    (b) false. Let $a = 1$, $b = -2$, and $\epsilon = 8$. $1 < -2 + 8$, yet $1 \not< -2$.

    (c) false. Let $a = 1$, $b = -2$, and $\epsilon = 8$. $1 < -2 + 8$, yet $1 \nleq -2$.

11. (a) There exists real number satisfying $a < b$, such that for all $n \in \mathbb{N}$ such that $a + 1/n > b$.

    (b) For all real numbers $x > 0$ there exists $n \in \mathbb{N}$ such that $x > 1/n$.

(c) There exists two real distinct real numbers for which there is no rational number between them.

12. (a) Base Case: $y_1 = 6 > -6$.

Inductive Step: First, we assume that $y_n > 6$. Now we must show that if $y_k > -6$, then $y_{k+1} > 6$.

$$\begin{aligned} y_k &> -6 \\ 2y_k &> -12 \\ 2y_k - 6 &> -18 \\ \frac{2y_k - 6}{3} &> -6 \\ y_{k+1} &> -6 \end{aligned}$$

(b) Base case:

$$\begin{aligned} y_1 &\overset{?}{>} y_2 \\ 6 &\overset{?}{>} \frac{2 \cdot 6 - 6}{3} \\ 6 &\overset{?}{>} \frac{6}{3} \\ 6 &> 2 \end{aligned}$$

Inductive Step: We assume that $y_n > y_{n+1}$. Now, we must show that if $y_k > y_{k+1}$, then $y_{k+1} > y_{k+2}$.

$$\begin{aligned} y_k &> y_{k+1} \\ 2y_k &> 2y_{k+1} \\ 2y_k - 6 &> 2y_{k+1} - 6 \\ \frac{2y_k - 6}{3} &> \frac{2y_{k+1} - 6}{3} \\ y_{k+1} &> y_{k+2} \end{aligned}$$