1 Irrationality of $\sqrt{2}$

Theorem 1. *There is no irrational number whose square is 2.*

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. We proceed by contradiction and assume that there is a rational number whose square is 2. We also assume that this rational number is completely reduced. Then

$$\begin{aligned}
\left(\frac{p}{q}\right)^2 &= 2 \\
p^2 &= 2q^2 && \text{So } p \text{ must be an even number. Let } p = 2r \text{ with } r \in \mathbb{Z} \\
(2r)^2 &= 2q^2 \\
4r^2 &= 2q^2 \\
2r^2 &= q^2 && \text{WHAT!? So then } q \text{ is also even!? But } p/q \text{ was supposed to be completely reduced!}
\end{aligned}$$

As we can see, p/q was supposed to be completely reduced, yet we showed that it was not. Therefore, $\sqrt{2}$ cannot be irrational.

□