MATH 313 HMWK 9

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Exercise 4.4.7 For any ϵ given, we simply choose $\delta = \epsilon \cdot \left|\sqrt{x} + \sqrt{x'}\right|$. Then

$$\begin{aligned}
|x - x'| &< \epsilon \cdot \left|\sqrt{x} + \sqrt{x'}\right| \\
\frac{|x - x'|}{\left|\sqrt{x} + \sqrt{x'}\right|} &< \epsilon \\
\left|\sqrt{x} - \sqrt{x'}\right| &< \epsilon
\end{aligned}$$

Exercise 4.4.9

(a) For any ϵ given, we simply choose $\delta = \epsilon/M$. Then

$$\begin{aligned}
|x - y| &< \frac{\epsilon}{M} \\
M \cdot |x - y| &< \epsilon
\end{aligned}$$

We know that

$$\begin{aligned}
\left|\frac{f(x) - f(y)}{x - y}\right| &\leq M \\
|f(x) - f(y)| &\leq M \cdot |x - y|
\end{aligned}$$

so

$$\begin{aligned}
|f(x) - f(y)| &\leq M \cdot |x - y| < \epsilon \\
|f(x) - f(y)| &< \epsilon
\end{aligned}$$

(b) False. The function $\sqrt{x}$ is a counterexample. It is continuous on the compact set $[0, 1]$, yet it is not a Lipschitz function. For any bound M given, we can choose $x = 1/(M + 1)^2$ and $y = 0$ that do not satisfy the requirement of a Lipschitz function, as seen below.

$$\begin{aligned}
\left|\frac{f(x) - f(y)}{x - y}\right| &= \frac{\sqrt{\dfrac{1}{(M+1)^2}} - \sqrt{0}}{\dfrac{1}{(M+1)^2} - 0} \\
&= \frac{1}{|M + 1|} \cdot (M + 1)^2 \\
&= M + 1 > M
\end{aligned}$$

Exercise 4.4.13 For any $\delta > 0$, we know that there exists $N \in \mathbb{N}$ such that for any $m \geq N$ and $n \geq N$, $|x_n - x_m| < \delta$ for any Cauchy sequence $x_n \subset A$. By the definition of a uniformly continuous function, this implies that $|f(x_n) - f(x_m)| < \epsilon$, which meets the definition of a Cauchy sequence.

Exercise 4.5.2

(a) Let us define the function $f(x) = x^3 - 9x$ on the open interval $(-3, 3)$. Then it will have range $[-6\sqrt{3}, 6\sqrt{3}]$, which is a closed interval. It should be readily apparent that for any $x \in (-3, 3)$, $\lim_{x \to c} f(x) = f(c)$, so f is continuous.

(b) Impossible. Let f be a function defined on the closed interval $[a, b]$ with range equal to the open interval (c, d). Let us assume that f is a continuous function. Then for all $V_\epsilon(a)$, there exists a $V_\delta(c)$ with the property that $xLetf(x) = 1/(1 - x^2$ be defined on the interval $(-1, 1)$.

(d) Impossible.

Exercise 4.5.8 We proceed by contradiction; we assume that f^{-1} is actually not continuous. Then by **Corollary 4.3.3**, there exists a sequence (x_n) in the domain of f^{-1} and a number c in the domain of f^{-1}, with $(x_n) \to c$ but $f^{-1}(x_n)$ does not converge to $f^{-1}(c)$. By **Theorem 4.3.2**, however, a characteristic of our continuous function f is that $f^{-1}(x_n) \to f^{-1}(c) \implies (x_n) \to c$. Since $f^{-1}(x_n) \to f^{-1}(c)$ is false by **Corollary 4.3.3**, but $f^{-1}(x_n)$ is true, our implication is false; however, we know that f is continuous. Thus, we must conclude that f^{-1} is also continuous.