MATH 313 HMWK 5

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Exercise 2.6.2

(a) The sequence $a_n = \sin n/n$ is a Cauchy sequence that is not monotone. $-1 < \sin n < 1$, so $-1/n < \sin n/n < 1/n$. By the Squeeze theorem, a_n converges to 0 since both $1/n$ and $-1/n$ converge to zero. By the Cauchy Criterion, a_n must also be a Cauchy sequence. It is easily seen that a_n is not monotone, as $a_1 > 0$, $a_4 < 0$, and $a_7 > 0$.

(b) Impossible. By Lemma 2.6.3, all Cauchy sequences are bounded, so any subsequences of Cauchy sequences will also be bounded.

(c) Impossible

(d) The sequence

$$a_n = e^n \sin\left(\frac{\pi}{2}n\right)$$

is such a sequence.

Exercise 2.6.5 (i) is false and (ii) is true. For (ii), we are given that (x_n) and (y_n) are psudo-Cauchy and we want to show that for any ϵ given, there exists N such that for all $n \geq N$, $|(x_{n+1} + y_{n+1}) - (x_n + y_n)| < \epsilon$. Since it is given that (x_n) and (y_n) are psuedo-Cauchy, we know that there exists N_1 and N_2 that satisfy the definition of psudo-Cauchiness for any real number given. We choose the error for both x_n and y_n to be $\epsilon/2$. Then we observe that

$$\begin{aligned}
|x_{n+1} - x_n| &< \frac{\epsilon}{2} \\
|y_{n+1} - y_n| &< \frac{\epsilon}{2} \\
|x_{n+1} - x_n| + |y_{n+1} - y_n| &< \epsilon \\
|(a_{n+1} - a_n) + (b_{n+1} - b_n)| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| &< \epsilon \qquad \text{By triangle inequality} \\
|(a_{n+1} - a_n) + (b_{n+1} - b_n)| &< \epsilon
\end{aligned}$$

For choosing N for $\epsilon > 0$, just let $N = \max\{N_1, N_2\}$.

Exercise 2.7.1

(a)

(b) It is easily seen that

$$s_2 \leq s_4 \leq s_6 \leq \ldots \leq s_{2k} \ldots \leq \ldots s_{2k-1} \leq \ldots \leq s_5 \leq s_3 \leq s_1$$

So let $I_n = [s_{2k}, s_{2k-1}]$.

Exercise 2.7.4

1. Let $x_n = 1/n$ and $y_n = 1/n$. Then $\sum x_n y_n$ converges by Corollary 2.4.7.

2. By Corollary 2.4.7, $a_k = \sum_{i=1}^{k} 1/i^2$ converges. Since

$$\sum_{k=1}^{n} \frac{\sin k}{k^2} \leq \sum_{k=1}^{n} 1/k^2$$

for all n, then $sum_{n=1}^{\infty} 1/n^2$ converges as well.

3. Impossible by the Caucy Criterion.

4. Impossible. Consider the series $a_n = \sum_{i=1}^{n}(-1)^i 1/n$. By the alternating series test, $(a_n$ converges. By Theorem 2.5.2, any subsequences of (a_n) also converge. Consider the subsequence of (a_n) $y_n = x_{2n}$ and $z_n = x_{2n-1}$. It is easily seen that

$$y_n \leq \sum (-1)^n x_n \leq z_n.$$