MATH 313 HMWK 10

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Exercise 5.2.3

(a)

$$\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{h(c)-h(x)}{c-x} \\
&= \lim_{x\to c} \frac{\frac{1}{c}-\frac{1}{x}}{c-x} \\
&= \lim_{x\to c} \frac{\frac{x-c}{cx}}{c-x} \\
&= \lim_{x\to c} -\frac{c-x}{cx} \cdot \frac{1}{c-x} \\
&= \lim_{x\to c} \frac{1}{cx} \\
&= \frac{1}{c^2}
\end{aligned}$$

(b)

$$\begin{aligned}
(f/g)'(c) &= (f \cdot (1/g))\,(c) \\
&= f'(c) \cdot \frac{1}{g(c)} + f(c) \cdot (1/g)'\,(c) \\
&= \frac{f'(c)}{g(c)} - \frac{f(c) \cdot g'(c)}{[g(c)]^2} \\
&= \frac{f'(c) \cdot g(c)}{[g(c)]^2} - \frac{f(c) \cdot g'(c)}{[g(c)]^2} \\
&= \frac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{[g(c)]^2}
\end{aligned}$$

(c)

$$\begin{aligned}
(f/g)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(c)}{g(c)}}{x-c} \\
&= \lim_{x\to c} \frac{\dfrac{f(x)\cdot g(c)}{g(x)\cdot g(c)} - \dfrac{f(c)\cdot g(x)}{g(c)\cdot g(x)}}{x-c} \\
&= \lim_{x\to c} \frac{1}{x-c}\cdot\frac{1}{g(x)\cdot g(c)}\cdot\left(f(x)\cdot g(c) - f(c)\cdot g(x)\right) \\
&= \lim_{x\to c} \frac{1}{x-c}\cdot\frac{1}{g(x)\cdot g(c)}\cdot\left(f(x)\cdot g(c) + f(c)\cdot g(c) - f(c)\cdot g(c) - f(c)\cdot g(x)\right) \\
&= \lim_{x\to c} \frac{1}{g(x)\cdot g(c)}\cdot\left(\frac{f(x)\cdot g(c) - f(c)\cdot g(c)}{x-c} - \frac{g(x)\cdot f(c) - g(c)\cdot f(c)}{x-c}\right) \\
&= \lim_{x\to c} \frac{g(c)\cdot f'(c) - f(c)\cdot g'(c)}{[g(c)]^2}
\end{aligned}$$

Exercise 5.2.7

(a)

$$\begin{aligned}
g_a'(0) &= \lim_{x\to 0} \frac{g_a(x) - g_a(0)}{x-0} \\
&= \lim_{x\to 0} \frac{x^a \sin(1/x) - 0}{x-0} \\
&= \lim_{x\to 0} x^{a-1}\sin(1/x)
\end{aligned}$$

In this situation, if $a = 0$ or $a = 1$, the limit will not exist. We choose $a = 2$.

Exercise 5.3.2 We take the contrapositve of Rolle's theorem. If for all points c in our interval A $f'(c) \neq 0$, then for any two distinct points a and b in our interval, $f(a) \neq f(b)$. Since $a \neq b \implies f(a) \neq f(b)$, then f is one-to-one on A. The function $f(x) = x^3$ is one-to-one on the interval $[-1, 1]$, yet $f'(0) = 0$.

Exercise 5.3.3

Exercise 5.3.4

(a) Let $f(x_n) \to L$. Then by the definition of the limit for a sequence, $\forall \epsilon > 0 \exists \delta \forall n \geq N \quad |f(x_n - L)| < \epsilon$. Since $f(x_n) = 0$ for all $n \geq N$, then we are left with $|-L| < \epsilon$. The only way that $|-L| < \epsilon$ is that $L = 0$. So $f(x_n) \to 0 = f(0)$ by the characterizations of continuity (f must be continuous on the interval since it is differentiable).

When we evaluate $f'(0)$, we get

$$\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x-0} \\
&= \lim_{x\to 0} \frac{f(x)}{x} && \text{since } f(0) = 0
\end{aligned}$$

Now we must show that $\lim_{x\to 0} f(x)/x = 0$. We can do this by showing that for all $\epsilon > 0$, there exists δ such that $|x - 0| < \delta \implies |f(x)/x - 0| < \epsilon$.