MATH 313 HMWK 8

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March 23, 2018

Exercise 4.2.2

(a)

$$
\begin{aligned}
|f(x) - 9| &< 1 \\
|5x - 6 - 9| &< 1 \\
|5x - 15| &< 1 \\
5 \cdot |x - 3| &< 1 \\
|x - 3| &< \frac{1}{5}
\end{aligned}
$$

The largest possible δ-neighborhood is $V_{1/5}(3)$.

(b)

$$
\begin{aligned}
|f(x) - 2| &< 1 \\
\left|\sqrt{x} - 2\right| &< 1 \\
-1 < \sqrt{x} - 2 &< 1 \\
1 < \sqrt{x} &< 3 \\
1 < x &< 9 \\
-4 < x - 5 &< 4 \\
|x - 5| &< 4
\end{aligned}
$$

The largest δ-neighborhood is $V_4(5)$.

(c) If x is less than 3, then $[[x]]$ will be 2, which is 1 away from 3, so it will be out of the ϵ-neighborhood. Thus, our x must lie in a δ-neighborhood whose infimum is greater than 3. Thus, the largest δ-neighborhood is $V_{\pi-3}(\pi)$.

(d) If x is less than 3, then $[[x]]$ will be 2, which is more than 0.01 away from 3, so it will be out of the ϵ-neighborhood. Thus, our x must lie in a δ-neighborhood whose infimum is greater than 3. Thus, the largest δ-neighborhood is $V_{\pi-3}(\pi)$, so it is the same as the last problem.

Exercise 4.2.3

(a) Let $x_n = 1 - 1/n$, $y_n = 1 + 1/n$, and $z_n = 1 - (1/2)^n$.

(b)

$$\begin{aligned} x_n &= 1 - \frac{1}{n} \\ &= \frac{n}{n} - \frac{1}{n} \\ &= \frac{n-1}{n} \end{aligned}$$

By Thomae's function, $t(x_n) = 1/n$, since $n-1$ and n are relatively prime.

$$\begin{aligned} y_n &= 1 + \frac{1}{n} \\ &= \frac{n}{n} + \frac{1}{n} \\ &= \frac{n+1}{n} \end{aligned}$$

By Thomae's function, $t(y_n) = 1/n$, since $n+1$ and n are relatively prime.

$$\begin{aligned} z_n &= 1 - \frac{1}{2^n} \\ &= \frac{2^n}{2^n} - \frac{1}{2^n} \\ &= \frac{2^n - 1}{2^n} \end{aligned}$$

By Thomae's function, $t(z_n) = 1/2^n$, since $2^n - 1$ and 2^n are relatively prime. It should be evident that $\lim t(x_n) = 0$, $\lim t(y_n) = 0$, and $\lim t(x_n) = 0$.

(c) As evidenced by the last problem, it seems that $\lim_{t \to 1} t(x) = 0$. We consider the set $\{x \in \mathbb{R} : t(x) \geq \epsilon\}$. Such a set would contain rational numbers in the form of m/n, where $1/n \geq \epsilon$, with the exception of $x = 0$ and $\epsilon < 1$. The set contains isolated points, which are points that are not limit points of the set. Therefore, the set is not closed.

Exercise 4.2.6

(a) True. Making δ less means that values of $f(x)$ will lie in a smaller interval than before, and the smaller interval will still be in the ϵ-neighborhood.

(b) False. Consider a function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} x & x \neq 0 \\ 2 & x = 0 \end{cases}$$

It should be evident that $\lim_{x \to 0} f(x) = 0$, but $f(x) = 2$, even though $2 \in \mathbb{R}$.

(c) True.

$$\begin{aligned}
\lim_{x\to a} 3[f(x)-2]^2 &= 3\lim_{x\to a}[f(x)-2]^2 && \text{by (i) of Corollary 4.2.4}\\
&= 3\lim_{x\to a}[f(x)-2][f(x)-2]\\
&= 3\cdot\left(\lim_{x\to a} f(x)-2\right)\left(\lim_{x\to a} f(x)-2\right) && \text{by (iii) of Corollary 4.2.4}\\
&= 3\cdot(L-2)^2
\end{aligned}$$

(d) False. $\lim_{x\to\infty} 1/x = 0$, but $\lim_{x\to\infty} e^x/x \neq 0$.

Exercise 4.2.11 Say we are given some ϵ. By the definition of the limit, we know that

$$\begin{aligned}
\forall\epsilon \geq 0 \quad \exists\delta_1 : 0 < |x-c| < \delta_1 \implies |f(x)-L| < \epsilon\\
\forall\epsilon \geq 0 \quad \exists\delta_2 : 0 < |x-c| < \delta_2 \implies |h(x)-L| < \epsilon
\end{aligned}$$

We are given that $f(x) \leq g(x) \leq h(x)$. Then

$$\begin{gathered}
-\epsilon < f(x)-L < \epsilon\\
-\epsilon < g(x)-L < \epsilon\\
f(x) \leq g(x) \leq h(x)\\
f(x)-L \leq g(x)-L \leq h(x)-L\\
-\epsilon < f(x)-L \leq g(x)-L \leq h(x)-L < \epsilon\\
-\epsilon < g(x)-L < \epsilon\\
|g(x)-L| < \epsilon
\end{gathered}$$

And for our δ in the case of $g(x)$, we choose $\delta = \min\{\delta_1, \delta_2\}$