MATH 313 HMWK 7

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Exercise 3.2.2

(a) The limit points for A are 1 and -1. To see why, simply let $a_n = A_{2n}$ and $b_n = A_{2n-1}$, which are sequences contained in A. Then $a_n = 1 + 1/n$ and $b_n = -1 + 2/(2n-1)$. We can use the Algebraic limit theorem to show that a_n converges to 1:

$$\begin{aligned}
\lim\left(1+\frac{1}{n}\right) &= \lim(1) + \lim\left(\frac{1}{n}\right) && \text{by (ii) of Theorem 2.3.3 (Algebraic Limit Theorem)}\\
&= 1 + 0\\
&= 1
\end{aligned}$$

We can show that b_n converges to -1 through the definition of the limit. For any arbitrary $\epsilon > 0$, choose $N > 1/\epsilon + 1/2$. Then for all $n \geq N$,

$$\begin{aligned}
n &> \frac{1}{\epsilon} + \frac{1}{2}\\
2n &> \frac{2}{\epsilon} + 1\\
2n - 1 &> \frac{2}{\epsilon}\\
\left|\frac{1}{2n-1}\right| &< \frac{\epsilon}{2} && \text{since } n \text{ and } \epsilon \text{ are both positive}\\
\left|\frac{2}{2n-1}\right| &< \epsilon\\
\left|-1 + \frac{2}{2n-1} - (-1)\right| &< \epsilon\\
|b_n - (-1)| &< \epsilon
\end{aligned}$$

Therefore, b_n converges to -1, so -1 is a limit point of A. The limit points for B are all real numbers between 1 and 0 inclusive. By **Theorem 1.4.3**, for every real number r such that $0 < r < 1$, there exists a rational number within $V_\epsilon(r)$.

(b) Set A is neither open nor closed. We choose the element $2 = (-1)^2 + (2/2)$ from the set A, which we must show does not contain any ϵ-neighborhoods that are a subset of A. For 2 to contain a ϵ-neighborhood, there must be an element $a \in A$ such that $a > 2$. However, it is not to hard to see that there is no such a. Consider the sequences we defined before, a_n and b_n. If we defined two sets to contain the elements of a_n and b_n, and took the union of those sets, we would get A. So if we can show that a_n does not contain any elements greater than 2 and b_n does not contain any elements greater than 2, then we can conclude that 2 does not have any ϵ-neighborhoods that a subset of A. So we assume to the contrary that a_n does contain an element greater than 2. Then for some n, $1 + 1/n > 2$. However,

$$\begin{aligned} 1 + \frac{1}{n} &> 2 \\ \frac{1}{n} &> 1 \\ n &< 1 \end{aligned}$$

However, n must be a natural number, so it cannot be less than 1. Now we assume that b_n contains some element greater than 2. Then there exists some n such that $-1 + 2/(2n - 1) > 2$. However,

$$\begin{aligned} -1 + \frac{2}{2n-1} &> 2 \\ \frac{2}{2n-1} &> 3 \\ \frac{2n-1}{2} &< \frac{1}{3} \\ 2n - 1 &< \frac{2}{3} \\ 2n &< \frac{5}{3} \\ n &< \frac{5}{6} \end{aligned}$$

As stated before, n must be a natural number, so n cannot be less than $5/6$. Thus, b_n does not contain an element greater than 2. So, if neither a_n and b_n does not contain an element greater than 2, then A does not contain an element greater than 2; therefore, there is no ϵ neighborhood for 2 that is a subset of A and A is not open.

To show that A is closed, we just have to show that there is a limit point that A does not contain. We consider -1. In order for A to contain -1, there must be an element in A for which $-1 = (-1)^n + 2/n$. However,

$$\begin{aligned} (-1)^n + \frac{2}{n} &= -1 \\ \frac{2}{n} &= -1 - (-1)^n \\ \frac{2}{n} &= -2 \text{ or } 0 \end{aligned}$$

Then n be equal to -1. However, n cannot be equal to -1, since n must be a natural number. And $2/n$ is never equal to 0. Therefore, the set A is closed.

The set B is neither open nor closed. Any element $b \in B$ will have a ϵ-neighborhood containing a real number not part of the rational numbers, so for any $V_\epsilon(b)$, $V_\epsilon(b) \not\subset B$. Therefore, B is not closed. B is also not closed, since it does not contain its limit point 1.

(c) A does contain isolated points. Simply choose $a \in A$ such that $a \neq 1$ and $a \neq -1$. B, on the other hand, does not contain any isolated points. Since the limit points of B are all reals between 0 and 1 inclusive, B contains all of its limit points.

Exercise 3.2.3

1. Open but not closed. By **Theorem 3.2.5**, e is a limit point of $\mathbb{Q}$ since the sequence $a_n = (1 + (1/n))^n$ is contained in $\mathbb{Q}$ due to the fact that rational numbers are closed under multiplication and addition. a_n converges to e which is an irrational number.

2. Closed but not open. Choose any element $n \in \mathbb{N}$. Then any $V_\epsilon(n)$ will contain a number that is not in $\mathbb{N}$.

3. Open but not closed. 0 is most certainly a limit point of $\{x \in \mathbb{R} : x \neq 0\}$, but is not in the set.

4. Not open and not closed. 1 is in the set, yet there is no ϵ-neighborhood around 1 that is a subset of the set, since the ϵ-neighborhood that will contain a real number greater than 1 but less than $5/4$. Such a number will not be in the set, so for any ϵ, the ϵ-neighborhood will not be a subset of the set. $\pi^2/6$ is a limit point of the set, yet the set does not contain $\pi^2/6$.

5. Not open but closed. 1 is in the set, yet there is no ϵ-neighborhood around 1 that is a subset of the set, since the ϵ-neighborhood that will contain a real number greater than 1 but less than $3/2$. Such a number will not be in the set.

Exercise 3.2.6

1. True. For any point in

2. True. Since all closed intervals are closed sets, the Nested Interval Property still holds.

3. True

Exercise 3.3.1 By **Theorem 3.3.4**, any compact set will be closed and bounded. By the Axiom of Completeness, any set that is bounded and is a subset of the real numbers will have a least upper bound and a least upper bound.

Exercise 3.3.2

(a) Not compact. Let $a_n = n$, which we know to not contain any convergent subsequences.

(b) Compact.

(c)

(d) Not compact. We showed that this set was not closed since it does not contain the limit point $\pi^2/6$.

(e) Compact

Exercise 3.3.5

(a) True. We consider n compact sets. By **Theorem 3.3.4**, any compact set is closed and bounded. By **Theorem 3.2.14**, the intersection of our compact sets will be closed. Since compact sets are bounded, there must exist $M_1, M_2, \ldots M_n$ that are bounds on our n compact sets. For the intersection, we simply select bound $M = \max\{M_1, M_2, \ldots M_n\}$. Since the intersection is bounded and closed, then it must be compact.

(b) True. We consider n compact sets. By **Theorem 3.3.4**, any compact set is closed and bounded. By **Theorem 3.2.14**, the union of our compact sets will be closed. Since compact sets are bounded, there must exist $M_1, M_2, \ldots M_n$ that are bounds on our n compact sets. For the intersection, we simply select bound $M = \max\{M_1, M_2, \ldots M_n\}$. Since the union is bounded and closed, then it must be compact.

(c) False. Let $A = (0, 1]$ and $K = [0, 1]$. Then $A \cap K = (0, 1]$, which is not compact since $A \cap K$ is not closed.

(d) True by Nested Interval property.