MATH 313 HMWK 4

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Problem 2.3.4

(a)

$$\begin{aligned}
\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n+4-4}{a_n}\right) \\
&= \lim\left(\frac{a_n^2+4a_n}{a_n}\right) \\
&= \lim(a_n+4) \\
&= \lim(a_n)+4 \\
&= 0+4 \\
&= 4
\end{aligned}$$

(c)

$$\begin{aligned}
\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\cdot\frac{a_n}{a_n}\right) \\
&= \lim\left(\frac{2+3a_n}{1+5a_n}\right) \\
&= \left(\frac{\lim(2+3a_n)}{\lim(1+5a_n}\right) \\
&= 2
\end{aligned}$$

Problem 2.3.9

(a) If (a_n) is bounded, then there exists a number $M > 0$ such that $|a_n| \leq M$. If $\lim b_n = 0$, then for all $\epsilon_0 > 0$, there exists a $N_0 \in \mathbb{N}$ such that whenever $n \geq N_0$ it follows that $|b_n - 0| < \epsilon_0$. Then

$$\begin{aligned} |b_n - 0| &< \epsilon_0 \\ |b_n| &< \epsilon_0 \\ |a_n| \cdot |b_n| &< M \cdot \epsilon_0 \\ |a_n \cdot b_n| &< M \cdot \epsilon_0 \\ |a_n \cdot b_n - 0| &< M \cdot \epsilon_0 \end{aligned}$$

Whatever ϵ is chosen, we can find a N for ϵ for the sequence $(a_n b_n)$ satisfying the definition of convergence in the following way. Divide the ϵ by the upper bound M to get a ϵ_0 and find a N_0 for that ϵ for the case of b_n. Then, we can use the same N_0 as N for $a_n b_n$. Thus, $\lim(a_n b_n) = 0$ since a N can be found for any ϵ that satisfies the condition of convergence. We are not allowed to use the Algebraic Limit Theorem because it is not known whether a_n is convergent.

Problem 2.3.11 Seeing that x_n is convergent, we know by **Theorem 2.5.2** that x_n is also bounded. So there exists a M such that for all x_n, $x_n \leq M$. Even though M may be equal to some x_n, we can just choose a M slightly greater than the previous one. We can then add up all of the elements in the sequence from x_1 to x_n, and they will all be less than $n \cdot M$.

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &< nM \\ x_1 + x_2 + \ldots + x_n - nL &< nM - nL \\ \frac{x_1 + x_2 + \ldots + x_n}{n} - L &< M - L \\ |y_n - L| &< M - L \end{aligned}$$

Then, we just use $M - L$ as ϵ.

Problem 2.4.3 We first show that the limit of the sequence exists using the monotone convergence theorem. We proceed by induction to show that the sequence is monotone; that is, $\forall n x_{n+1} > x_n$. We consider the base case.

$$\begin{aligned} \sqrt{2 + \sqrt{2}} &\overset{?}{\geq} \sqrt{2} \\ 2 + \sqrt{2} &> 2 \end{aligned}$$

We now proceed to the inductive step. We must show that if $x_{k+1} \geq x_k$ for some particular k, then it will also be true for $x_{k+2} \geq x_{k+1}$. So

$$\begin{aligned} x_{k+1} &> x_k \\ 2 + x_{k+1} &> 2 + x_k \\ \sqrt{2 + x_{k+1}} &> \sqrt{2 + x_k} \\ x_{k+1} &> x_k \end{aligned}$$

Now that we have show that (x_n is monotone, we must now show that it is bounded. We show that $x_k < 16$. We first consider the base case. $x_1 = \sqrt{2}$ is indeed less than 16. We can now proceed to the inductive step, where we show that if $x_k < 16$, then $x_{k+1} < 16$. We first assume to the contrary that $x_{k+1} \geq 16$. Then

$$\begin{aligned} x_{k+1} &\geq 16 \\ \sqrt{2+x_k} &\geq 16 \\ 2+x_k &\geq 256 \\ x_k \geq 254 \end{aligned}$$

However, this contradicts our original assumption that $x_k < 16$. Thus, we deduce that x_n converges to some limit L. Now we define a sequence $y_n = x_{n+1}$. y_n, which is a subsequence of x_n, converges to the same limit that y_n does. So

$$\begin{aligned} x_n &\to y \\ 2+x_n &\to 2+y \\ \sqrt{5+x_n} &\to \sqrt{2+y} \\ y_n &\to \sqrt{2+y} \end{aligned}$$

Then

$$\begin{aligned} y &= \sqrt{2+y} \\ y^2 &= 2+y \\ y^2 - y - 2 &= 0 \\ (y+1)(y-2) &= 0 \\ y &= 2 \end{aligned}$$

Problem 2.4.5

(a) We proceed by induction. For the base case, $x_1^2 = 2^2 = 4 \geq 2$. Now for the inductive step. We must now show that if $x_k^2 \geq 2$, then $x_{k+1}^2 \geq 2$. So

$$\begin{aligned}
x_k^2 &\geq 2 \\
x_k^2 - 2 &\geq 0 \\
\left(x_k^2 - 2\right)^2 &\geq 0 \\
x_k^4 - 4x_k^2 + 4 &\geq 0 \\
x_k^2 - 4 + \frac{4}{x_k^2} &\geq 0 \\
x_k^2 + 4 + \frac{4}{x_k^2} &\geq 8 \\
\frac{1}{4}\left(x_k^2 + 4 + \frac{4}{x_k^2}\right) &\geq 2 \\
\frac{1}{4}\left(x_k + \frac{2}{x_k}\right)^2 &\geq 2 \\
x_{k+1}^2 &\geq 2
\end{aligned}$$

Thus, we know that $x_n^2 \geq 2$. Now, we have to show that $x_n - x_{n+1} \geq 0$. So

$$\begin{aligned}
x_n^2 &\geq 2 \\
x_n^2 - 2 &\geq 0 \\
\frac{1}{2x_n} \cdot \left(x_n^2 - 2\right) &\geq \frac{1}{2x_n} \cdot 0 \\
\frac{1}{2}x_n - \frac{1}{x_n} &\geq 0 \\
\frac{1}{2}x_n + \frac{1}{2}x_n - \frac{1}{2}x_n - \frac{1}{x_n} &\geq 0 \\
x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) &\geq 0 \\
x_n - x_{n+1} &\geq 0
\end{aligned}$$

Now we have to how that $\lim x_n = \sqrt{2}$. We proceed in a fashion similar to the last problem. We let $y_n = x_{n+1}$, which will converge to the same limit y as x_n. Then

$$\begin{aligned}
y &= \frac{1}{2}\left(y + \frac{2}{y}\right) \\
2y &= y + \frac{2}{y} \\
2y^2 &= y^2 + 2 \\
y^2 &= 2 \\
y &= \sqrt{2}
\end{aligned}$$

(b) Let $x_1 = c$ and define

$$x_{n+1} = \frac{x_n + \frac{x_n}{c}}{2}$$

Problem 2.4.8

(a) A explicit formula would be $1 - \frac{1}{2^n}$. The sequence converges to 1.

(b) A explicit formula would be $\frac{n}{n+1}$. The sequence converges to 1.

(c) A explicit formula would be $\log(n+1)$.

Problem 2.5.1

(a) Impossible. According to the Bolzano-Weierstrass Theorem, every bounded sequence contains a convergent subsequence. Since the sequence has a bounded subsequence, the bounded subsequence contains a subsubsequence that converges which is itself a subsequence of the original sequence.

(b) Define
$$x_n = \frac{1}{2} + \frac{1}{2}(-1)^k \frac{k}{k+1}$$

(c) In Theorem 1.5.6, we saw that the rational numbers were countable, meaning that there exists a bijection from the natural numbers to the rational numbers. Such a bijection would contain a subsequence that converges to all numbers in the sequence $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

(d) Impossible. Such a sequence would contain a subsequence that converges to zero, since the sequence $x_n = 1/n$ itself converges to zero, which is a point outside of the set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

Exercise 2.5.2

(a) True. Define a proper subsequence (y_n) of (x_n) such that $y_n = x_{n+1}$. It can be easily seen that if y_n converges, then so will x_n

(b) True. Consider the contrapositive of the statement, if (x_n) converges, then x_n contains a convergent subsequence. Since x_n is a subsequence of itself, then x_n will always have a convergent subsequence. Since the contrapositive of the statement is true, the statement is true.

(c) True.

(d) True.