Problem 2.3.4

(a)

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)}\\ &= \frac{1}{1}\\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n+4-4}{a_n}\right)\\ &= \lim\left(\frac{a_n^2+4a_n}{a_n}\right)\\ &= \lim(a_n+4)\\ &= \lim(a_n)+4\\ &= 0+4\\ &= 4\end{aligned}$$

(c)

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\cdot\frac{a_n}{a_n}\right)\\ &= \lim\left(\frac{2+3a_n}{1+5a_n}\right)\\ &= \left(\frac{\lim(2+3a_n)}{\lim(1+5a_n}\right)\\ &= 2\end{aligned}$$

Problem 2.3.9

(a) If (a_n) is bounded, then there exists a number $M > 0$ such that $|a_n| \leq M$. If $\lim b_n = 0$, then for all $\epsilon_0 > 0$, there exists a $N_0 \in \mathbb{N}$ such that whenever $n \geq N_0$ it follows that $|b_n - 0| < \epsilon_0$. Then

$$\begin{aligned}|b_n - 0| &< \epsilon_0\\ |b_n| &< \epsilon_0\\ |a_n|\cdot|b_n| &< M\cdot\epsilon_0\\ |a_n\cdot b_n| &< M\cdot\epsilon_0\\ |a_n\cdot b_n - 0| &< M\cdot\epsilon_0\end{aligned}$$

Whatever ϵ is chosen, we can find a N for ϵ for the sequence (a_nb_n) satisfying the definition of convergence in the following way. Divide the ϵ by the upper bound M to get a ϵ_0 and find a N_0 for that ϵ for the case of b_n. Then, we can use the same N_0 as N for $a_n b_n$. Thus, $\lim(a_n b_n) = 0$ since a N can be found for any ϵ that satisfies the condition of convergence. We are not allowed to use the Algebraic Limit Theorem because it is not known whether a_n is convergent.

Problem 2.3.11

Problem 2.4.3 We first show that the limit of the sequence exists using the monotone convergence theorem. We proceed by induction to show that the sequence is monotone; that is, $\forall n x_{n+1} > x_n$. We consider the base case.

$$\sqrt{5+\sqrt{5}} \overset{?}{\geq} \sqrt{5}$$
$$5+\sqrt{5} > 5$$

$$\begin{aligned} x &= \sqrt{2+x} \\ x^2 &= 2+x \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \text{ or } 2 \end{aligned}$$

Problem 2.4.5

(a) We proceed by induction. For the base case, $x_1^2 = 2^2 = 4 \geq 2$. Now for the inductive step. We must now show that if $x_k^2 \geq 2$, then $x_{k+1}^2 \geq 2$. So

$$\begin{aligned} x_k^2 &\geq 2 \\ x_k^2 - 2 &\geq 0 \\ \left(x_k^2 - 2\right)^2 &\geq 0 \\ x_k^4 - 4x_k^2 + 4 &\geq 0 \\ x_k^2 - 4 + \frac{4}{x_k^2} &\geq 0 \\ x_k^2 + 4 + \frac{4}{x_k^2} &\geq 8 \\ \frac{1}{4}\left(x_k^2 + 4 + \frac{4}{x_k^2}\right) &\geq 2 \\ \frac{1}{4}\left(x_k + \frac{2}{x_k}\right)^2 &\geq 2 \\ x_{k+1}^2 &\geq 2 \end{aligned}$$

Thus, we know that $x_n^2 \geq 2$. Now, we have to show that $x_n - x_{n+1} \geq 0$. So

$$\begin{aligned}
x_n^2 &\geq 2 \\
x_n^2 - 2 &\geq 0 \\
\frac{1}{2x_n} \cdot \left(x_n^2 - 2\right) &\geq \frac{1}{2x_n} \cdot 0 \\
\frac{1}{2}x_n - \frac{1}{x_n} \geq 0 \\
\frac{1}{2}x_n + \frac{1}{2}x_n - \frac{1}{2}x_n - \frac{1}{x_n} &\geq 0 \\
x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) &\geq 0 \\
x_n - x_{n+1} &\geq 0
\end{aligned}$$

(b) Let $x_1 = c$ and define

$$x_{n+1} = \frac{x_n + \frac{x_n}{c}}{2}$$

Problem 2.4.8

(a) A explicit formula would be $1 - \frac{1}{2^n}$. The sequence converges to 1.

(b) A explicit formula would be $\frac{n}{n+1}$. The sequence converges to 1.

(c) A explicit formula would be $\log(n+1)$.