MATH 313 HMWK 2

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1. 3.8.
   i. Infima: 0, Suprema: 1
   ii. Infima: -1, Suprema: 1
   iii. Infima: $\frac{1}{4}$, Suprema: $\frac{1}{3}$
   iv. Infima: 0, Suprema: 1

   4.1.
   i. Let $a = \frac{m}{n}$ and $b = \frac{p}{q}$. Then

$$\begin{aligned} ab &= \frac{m}{n} \cdot \frac{p}{q} \\ &= \frac{mp}{nq} \end{aligned}$$

$$\begin{aligned} a + b &= \frac{m}{n} + \frac{p}{q} \\ &= \frac{mq}{qn} + \frac{pn}{qn} \\ &= \frac{mq + pn}{qn} \end{aligned}$$

   Since ab and $a + b$ can be written as fractions, by definition, they must be rational. Therefore, the set of rational numbers are closed under addition and multiplication.

   ii. Let us assume that $a + t \in \mathbb{Q}$. Then we should be able write $a + t$ as p/q, where p and q are integers. The same goes for a. Then

$$\begin{aligned} \frac{m}{n} + t &= \frac{p}{q} \\ t &= \frac{p}{q} - \frac{m}{n} \\ &= \frac{pn}{qn} - \frac{mq}{qn} \\ &= \frac{pn - mq}{qn} \end{aligned}$$

This, however, contradicts our given statement that t is irrational. Therefore, $a + t$ cannot be rational, so $a + t \in \mathbb{I}$. As for showing that $at \in \mathbb{I}$, we proceed in a similar fashion: assume that $at \in \mathbb{Q}$. Then

$$\begin{aligned} \frac{m}{n} \cdot t &= \frac{p}{q} \\ t &= \frac{p}{q} \cdot \frac{m}{n} \\ &= \frac{pm}{qn} \end{aligned}$$

Once again, we reach a contradiction. Since t is irrational, it cannot be written as a fraction, so therefore, $at \in \mathbb{I}$.

iii. If $s, t \in \mathbb{I}$, then $s + t \notin I$ and $st \notin I$. Let $s = -\sqrt{2}$. Let $t = \sqrt{2}$, which we proved to irrational in the first chapter. Then $s + t = 0$, which is a rational number.
Let $s = \sqrt{2}$ and $t = \sqrt{2}$. Then $st = \sqrt{2} \cdot \sqrt{2} = 2$. 2 is a rational number, even though $\sqrt{2}$ is not rational.

4.2. We approach this problem in a similar fashion as **Theorem 1.4.5**. First, we assume that $s \neq \sup A$. In order for this to be true, there must exist a real number $a \in A$ such that $a > s$. In search of a real number greater than s, let $a = s+r$, where r is a real number. However, we can do a little better than that. By the second part of the **Archimedean Property**, we can choose a rational number $1/n$ such that $1/n < r$. So we let $a = s + \frac{1}{n}$, which will be less than the quantity $s + r$. However, it is given that $s + \frac{1}{n}$ is an upper bound for A for all $n \in \mathbb{N}$! Thus, we now know that s must be an upper bound for A.

However, in order for $s = \sup A$, for any upper bound b of A, $s \geq b$. We assume that the opposite is true, that there exists b such that $s < b$. In search of such a b, we might set $b = s-t$, where $t \in \mathbb{R}$. We can do better than that, since by the **Archimedean Property** (again), we can choose a rational number $1/n$ such that $1/n < t$. So we set $b = s - \frac{1}{n}$ in hope of finding a b that is less than s. As given by the problem though, $b - \frac{1}{n}$ is not an upper bound for A! Thus, there does not exist an upper bound b such that $s < b$, so we can safely conclude that $s \geq b$.

Thus, we have proven the two requirements for $s = \sup A$.

4.6. i. Not dense in $\mathbb{R}$. Since $\mathbb{Q} \subset \mathbb{R}$, any rational number is also a real number. Let $a = 1/100$ and $b = 0$. Assume that there exists p and q such that $0 < p/q > 1/100$ with $q \leq 10$, $p \in \mathbb{Z}$

and $q \in \mathbb{N}$. Then

$$\begin{aligned}
0 &< \frac{p}{q} < \frac{1}{100} \\
100q \cdot 0 &< 100q \cdot \frac{p}{q} < 100q \cdot \frac{1}{100} \\
0 &< 100p < q \\
0 &< 100p < q \leq 10 \\
0 &< 100p \leq 10 \\
0 &< p \leq \frac{1}{10}
\end{aligned}$$

However, this contradicts the fact that $p \in \mathbb{Z}$ since there are no integers greater than zero but less than 1/10. Therefore, the set is not dense in $\mathbb{R}$.

ii. Dense. If q was just a natural number, then we would already be done since we proved that the rational numbers were dense in the reals. Even though q must be a power of 2, we can still proceed in a similar fashion if we show that for any real number $x \in \mathbb{R}$, there exists $q : q = 2^k$ for some $k \in \mathbb{N}$. such that $q > x$. In this case, let $T = q : q = 2^k$ for some $k \in \mathbb{N}$. For contradiction, we assume that T is bounded above. By **AoC**, T should have a least upper bound, so we can set $\alpha = \sup T$. If we consider $\alpha - 2^p$ for some p

5.4. i. It is known that the function $f(x) = \frac{x}{(x-1)(x+1)}$ is a bijection from the interval (-1,1) to the real numbers. We get a similar bijective function if we translate $f(x)$ to halfway the interval (a, b) and then increase the width of the translated $f(x)$ by a factor of $(b-a)/2$. Recall that the function

$$f(x) = \frac{x}{1-x^2}$$

is a bijection $(-1, 1) \to \mathbb{R}$.

$$g : (a, b) \to (-1, 1) g(x) = \lambda x + \tau g(b) = 1 \lambda a + \tau = -1 \lambda b + \tau = 1 \lambda (b-a) = 2 \lambda = \frac{2}{b-a} \tau =$$

ii. By the Archimedean property, we can choose $b \in \mathbb{N}$ such that $b > a$. Let $f(x) = \frac{x^2-b^2}{x-a}$ with domain (a, ∞). If $f(x)$ is a bijection from $(a, \infty$ to the reals, then (a, ∞) $\mathbb{R}$. To show that $f(x)$ is a bijection, we show that $f(x)$ is an injection and surjection first. For the case of surjection, we must show that for every $r \in \mathbb{R}$, there exists $x \in (a, \infty)$ such that $f(x) = r$.

$$\begin{aligned}
\frac{x^2 - b^2}{x - a} &= r \\
x^2 - b^2 &= r \cdot (x - a) \\
x^2 - b^2 &= rx - ra \\
x^2 - rx - b^2 + ra &= 0 \\
x &= \frac{r + \sqrt{r^2 - 4\left(ra - b^2\right)}}{2}
\end{aligned}$$

Thus, for every $r \in \mathbb{R}$, we have a corresponding x such that $f(x) = r$, showing that $f(x)$ is a surjection.
Now, we must show that f is surjective or that $f(x_1 = f(x_2) \implies x_1 = x_2$.

$$\begin{aligned}
f\left(x_1\right) &= f\left(x_2\right) \\
\frac{x_1^2 - b^2}{x_1 - a} &= \frac{x_2^2 - b^2}{x_2 - a} \\
\left(x_1^2 - b^2\right)\left(x_2 - a\right) &= \left(x_2^2 - b^2\right)\left(x_1 - a\right) \\
x_2 x_1^2 - a x_1^2 - b^2 x_2 + a b^2 &= x_1 x_2^2 - a x_2^2 - b^2 x_1 + a b^2 \\
x_2 x_1^2 - x_1 x_2^2 + a x_2^2 - a x_1^2 - b^2 x_2 + b^2 x_1 &= 0 \\
x_2 x_1 \cdot \left(x_1 - x_2\right) + a\left(x_2 - x_1\right)\left(x_2 + x_1\right) + b^2(x_1 - x_2) &= 0 \\
\left(x_1 - x_2\right) \cdot \left[x_1 x_2 + b^2 - a x_1 - a x_2\right] &= 0
\end{aligned}$$

The only way for the last line to be true is if $x_1 = x_2$, so $f(x)$ is also injective. Therefore, f is bijective and $(a, \infty) \sim \mathbb{R}$.

5.5. Since $C \subset [0, 1]$, then we can say that is bounded below by zero. By the **AoC** for the lower bound, we know that there exists m such that $m = \inf C$. Set $a = m$