MATH 313 HMWK 3

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Exercise 2.2.1 An example of a vercongent sequence would be $x_n = \sin x$. Let $\epsilon = \frac{101}{100}$ and let $x = 0$. Then x_n converges to 0, since $|\sin x - 0| < \frac{101}{100}$. Since x_n does not converge to any number, it is a vercongent sequence that is also divergent. What a vercongent sequence describes is a sequence for whom all values lie in a certain bound, namely $(x - \epsilon, x + \epsilon)$.

Excercise 2.2.2

(a) Let $\epsilon > 0$ be arbitrary. Now choose N such that

$$N > \frac{3}{25\epsilon} - \frac{4}{5}$$

To verify that our choice of N is appropiate, let $n \in \mathbb{N}$ satisfy $n \geq N$. Then, $n \geq N$ implies that

$$\begin{aligned}
n &> \frac{3}{25\epsilon} - \frac{4}{5} \\
5n &> \frac{3}{5\epsilon} - 4 \\
5n + 4 &> \frac{3}{5\epsilon} \\
\epsilon &> \frac{3}{5 \cdot (5n+4)} \\
\epsilon &> \frac{2 \cdot (5n+4) - 5 \cdot (2n+1)}{5 \cdot (5n+4)} \\
\epsilon &> \frac{2 \cdot (5n+4)}{5 \cdot (5n+4)} - \frac{5 \cdot (2n+1)}{5 \cdot (5n+4)} \\
\epsilon &> \frac{2}{5} - \frac{2n+1}{5n+4} \\
\epsilon &> \left| \frac{2}{5} - \frac{2n+1}{5n+4} \right| \\
\left| \frac{2}{5} - \frac{2n+1}{5n+4} \right| &< \epsilon \\
\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| &< \epsilon
\end{aligned}$$

(b) Let $\epsilon > 0$ be arbitrary and let $a_n = 2n^2/\left(n^3+3\right)$. Choose $N \in \mathbb{N}$ with $N > \frac{4}{\epsilon^2}$. Now, we must show that for all $n \in \mathbb{N}$, $n \geq N$ satisfies $|a_n - L| < \epsilon$. If $n \geq N$, then

$$\begin{aligned} n &> \frac{4}{\epsilon^2} \\ \sqrt{n} &> \frac{2}{\epsilon} \\ \frac{2}{\sqrt{n}} &< \epsilon \end{aligned}$$

If we can show that $2/\sqrt{n} > \frac{2n^2}{n^3+3}$, then we are done. So,

$$\begin{aligned} \frac{2n^2}{n^3+3} &< \frac{2}{\sqrt{n}} \\ \frac{1}{n^3+3} &< \frac{1}{\sqrt{n^5}} \\ n^3+3 &> \sqrt{n^5} \end{aligned}$$

The last line is a true statement, so we are done.

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon^3}$. To verify that the choice of N was appropiate, let $n \in \mathbb{N}$ satisfy $n \geq N$. Then

$$\begin{aligned} n &\geq \frac{1}{\epsilon^3} \\ \frac{1}{n} &\leq \epsilon^3 \\ \sqrt[3]{\frac{1}{n}} &\leq \sqrt[3]{\epsilon^3} \\ \frac{1}{\sqrt[3]{n}} &\leq \epsilon \\ \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} &\leq \sin\left(n^2\right) \cdot \epsilon \end{aligned}$$

However, $-1 \leq \sin x \leq 1$, so

$$\left| \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} - 0 \right| \leq \epsilon$$

Exercise 2.3.1

(a) If $x_n \to 0$, then for any $\epsilon_0 > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n \geq \mathbb{N}_{\nvdash}$, $|x_n - 0| < \epsilon_0$. Since $x_n \geq 0$, then

$$\begin{aligned} x_n &< \epsilon_0 \\ \sqrt{x_n} &< \sqrt{\epsilon_0} \\ \sqrt{x_n} - 0 &< \sqrt{\epsilon_0} \\ |\sqrt{x_n} - 0| &< \sqrt{\epsilon_0} \end{aligned}$$

Now, we can just set any ϵ given to us to $\sqrt{\epsilon_0}$. To supply a given N for an ϵ, we can use the N_0 supplied to us by ϵ_0.

(b) If $x_n \to x$, then for any $\epsilon_0 > 0$, there exists $N_0 \in \mathbb{N}\not{}$ such that for all $n \geq \mathbb{N}$, $|x_n - x| < \epsilon_0$. Since $x_n \geq 0$, so is x. Then

$$\begin{aligned} x_n - x &< \epsilon_0 \\ x_n &< \epsilon_0 + x \\ \sqrt{x_n} &< \sqrt{\epsilon_0 + x} \\ \sqrt{x_n} - \sqrt{x} &< \sqrt{\epsilon_0 + x} - \sqrt{x} \\ \left|\sqrt{x_n} - \sqrt{x}\right| &< \sqrt{\epsilon_0 + x} - \sqrt{x} \end{aligned}$$

We can use the same strategy as last time. We let any ϵ given to us be equal to $\sqrt{\epsilon_0 + x} - \sqrt{x}$ and solve for that particular ϵ_0. Then, we can find a N_0 that satisfies the definition of convergence for x_n. We set $N = N_0$ and we have an algorithm for finding a N for any given ϵ.

Exercise 2.3.5 If x_n and y_n are both convergent, then there must exist a $N_1, N_2 \in \mathbb{N}$ such that for all $n_1 \geq N_2$ and for all $n_2 \geq N_2$, $|a_n - L| < \epsilon_1$ and $|b_n - L| < \epsilon_2$ for any $\epsilon > 0$ since a_n and b_n both converge to the same limit. In the case of z_n, we choose $N = \max(N_1, N_2)$. Then for all $n \geq N$, we know that $|z_n - L| < \epsilon$. Thus z_n must converge.

Now we must prove the converse. We start with the fact that z_n is convergent. Then by definition, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ greater than or equal to N, $z_n - L < \epsilon$, where L is the limit that z_n converges to. If we split z_n into x_n and y_n, we can use the same N when given an ϵ for x_n or y_n.

Exercise 2.3.7

(a) Let $x_n = n$ and $y_n = -n$. Both x_n and y_n diverge. However, $x_n + y_n$ converge to zero.

(b) Impossible by Algebraic Limit Theorem ii.

(c) Let $b_n = \frac{1}{n}$ with $b_n \neq 0$ for all $n \in \mathbb{N}$. However, $(1/b_n)$ diverges.

(d) Impossible. By Theorem 2.3.2, a_n must be a divergent sequence, so then $(a_n - b_n$ is a sequence that diverges.

(e) Let $a_n = \frac{1}{n}$ and $b_n = n$. a_n converges to 0 and b_n diverges. (a_nb_n) converges to 1.