Machine Learning Worksheet 4

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Problem 1

Since we throw the coin until one Head occurce E(H) =1

The number of Tails: expected:
$E(T) = \sum_{n=0}^{\inf}(\frac{1}{2}^n * n * \frac{1}{2})$
$= \frac{1}{2} * \sum_{n=0}^{\inf}(\frac{1}{2}^n * n) = \frac{1}{2} * 2 = 1$

Problem 2

random variables:

- U: the selected urn

- B: the number of black balls drawn from an urn

$$U \sim \mathrm{U}(-1, 10)$$
$$B \sim \mathrm{Bin}\left(10, 10 \cdot \frac{u}{10}\right) = \mathrm{Bin}(10, u)$$
$$\Rightarrow P(U) = \frac{1}{11}$$
$$\Rightarrow P(B|U) = \binom{10}{b} \cdot \left(\frac{u}{10}\right)^b \cdot \left(1 - \frac{u}{10}\right)^{10-b}$$

$$P(U|B) = \frac{P(B|U) \cdot P(U)}{P(B)}$$

$$P(B) = \sum_{u' \in U} P(B|U = u') \cdot P(U = u')$$

$$= \sum_{u' \in U} \binom{10}{b} \cdot \left(\frac{u'}{10}\right)^b \cdot \left(1 - \frac{u'}{10}\right)^{10-b} \cdot \frac{1}{11}$$

$$= \frac{1}{11} \cdot \binom{10}{b} \cdot \sum_{u' \in U} \left(\frac{u'}{10}\right)^b \cdot \left(1 - \frac{u'}{10}\right)^{10-b}$$

$$= \frac{1}{11} \cdot \binom{10}{b} \cdot \sum_{u'=0}^{10} \left(\frac{u'}{10}\right)^b \cdot \left(1 - \frac{u'}{10}\right)^{10-b}$$

$$P(U = u|B = b) = \frac{P(B = b|U = u) \cdot P(U = u)}{P(B = b)}$$

$$= \frac{\binom{10}{b} \cdot \left(\frac{u}{10}\right)^b \cdot \left(1 - \frac{u}{10}\right)^{10-b} \cdot \frac{1}{11}}{\frac{1}{11} \cdot \binom{10}{b} \cdot \sum\limits_{u'=0}^{10} \left(\frac{u'}{10}\right)^b \cdot \left(1 - \frac{u'}{10}\right)^{10-b}}$$

$$= \frac{\left(\frac{u}{10}\right)^b \cdot \left(1 - \frac{u}{10}\right)^{10-b}}{\sum\limits_{u'=0}^{10} \left(\frac{u'}{10}\right)^b \cdot \left(1 - \frac{u'}{10}\right)^{10-b}}$$

$$P(U = u|B = 3) = \frac{\left(\frac{u}{10}\right)^3 \cdot \left(1 - \frac{u}{10}\right)^7}{\sum\limits_{u'=0}^{10} \left(\frac{u'}{10}\right)^3 \cdot \left(1 - \frac{u'}{10}\right)^7}$$

$$\Rightarrow P(U=0|B=3) = 0$$
$$\Rightarrow P(U=1|B=3) = \frac{1594323}{25277575}$$
$$\Rightarrow P(U=2|B=3) = \frac{16777216}{75832725}$$
$$\Rightarrow P(U=3|B=3) = \frac{7411887}{25277575}$$
$$\Rightarrow P(U=4|B=3) = \frac{5971968}{25277575}$$
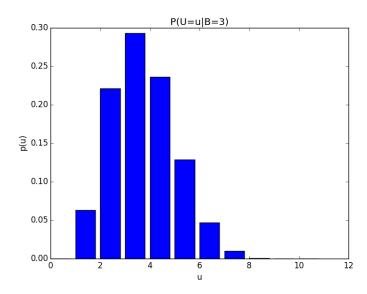$$\Rightarrow P(U=5|B=3) = \frac{390625}{3033309}$$
$$\Rightarrow P(U=6|B=3) = \frac{1179648}{25277575}$$
$$\Rightarrow P(U=7|B=3) = \frac{250047}{25277575}$$
$$\Rightarrow P(U=8|B=3) = \frac{65536}{75832725}$$
$$\Rightarrow P(U=9|B=3) = \frac{243}{25277575}$$
$$\Rightarrow P(U=10|B=3) = 0$$

(result calculated using *ipython* interactive console – log attached to mail)

- $X = 1$: next ball drawn from one of the urns is black

- $X = 0$: next ball drawn from one of the urns is white

$$X \sim \mathrm{Ber}\left(\frac{u}{10}\right)$$

$$P(X = 1) = \sum_{u=0}^{10} P(X = 1 | U = u) \cdot P(U = u)$$
$$= \frac{673471}{2022206}$$
$$\approx 0.33304$$

(result calculated using *ipython* interactive console – log attached to mail)

Problem 3

$$P(X = x | \theta) = \theta^x (1 - \theta)^{1-x}$$

goal: determine θ_{MLE}

$$\theta_{MLE} = \arg\max_\theta P(X = x | \theta)$$
$$\Rightarrow 0 = \frac{dP}{d\theta}(\theta_{MLE})$$
$$\Rightarrow 0 = \frac{d \log P}{d\theta}(\theta_{MLE})$$

$$\log P(X = x | \theta) = \log \theta^x (1 - \theta)^{1-x}$$
$$= x \cdot \log \theta + (1 - x) \cdot \log (1 - \theta)$$

$$0 = \frac{d}{d\theta}(x \cdot \log \theta + (1 - x) \cdot \log (1 - \theta))$$
$$= x \cdot \frac{1}{\theta} + (1 - x) \cdot \frac{1}{1 - \theta} \cdot (-1)$$
$$= \frac{x}{\theta} - \frac{1 - x}{1 - \theta}$$
$$\Leftrightarrow \frac{1 - x}{1 - \theta} = \frac{x}{\theta}$$
$$\Leftrightarrow \frac{1 - x}{x} = \frac{1 - \theta}{\theta}$$
$$\Leftrightarrow \frac{1}{x} - 1 = \frac{1}{\theta} - 1$$
$$\Leftrightarrow \frac{1}{x} = \frac{1}{\theta}$$
$$\Leftrightarrow \theta = x$$
$$\Rightarrow \theta_{MLE} = x$$

Problem 4

Show:

$$\theta_{MLE} = \arg\max_\theta P(X = x|\theta)$$

$$\Rightarrow 0 = \frac{\mathrm{d}P}{\mathrm{d}\theta}(\theta_{MLE})$$

$$\Rightarrow 0 = \frac{\mathrm{d}\log P}{\mathrm{d}\theta}(\theta_{MLE})$$

$$\log P(X = m|\theta) = \log \binom{N}{m}\theta^m(1-\theta)^{n-m}$$

$$= \log \binom{N}{m} + m \cdot \log\theta + (N-m) \cdot \log(1-\theta)$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\log\binom{N}{m} + m \cdot \log\theta + (N-m) \cdot \log(1-\theta)\right)$$

$$= m \cdot \frac{1}{\theta} + (N-m) \cdot \frac{1}{1-\theta} \cdot (-1)$$

$$= \frac{m}{\theta} - \frac{N-m}{1-\theta}$$

$$\Leftrightarrow \frac{N-m}{1-\theta} = \frac{m}{\theta}$$

$$\Leftrightarrow \frac{N-m}{m} = \frac{1-\theta}{\theta}$$

$$\Leftrightarrow \frac{N-m}{m} = \frac{1}{\theta} - 1$$

$$\Leftrightarrow \frac{N-m}{m} + 1 = \frac{1}{\theta}$$

$$\Leftrightarrow \frac{N-m+m}{m} = \frac{1}{\theta}$$

$$\Leftrightarrow \theta = \frac{m}{N}$$

$$\Rightarrow \theta_{MLE} = \frac{|X=1|}{|X=1| + |X=0|}$$

Formalized Problem:

$mean(P(\theta|D)) = \lambda * mean(p(\theta)) + (1-\lambda) * \theta_{MLH}$

$mean(Beta(a + |X=1|, b + |X=0|)) = \lambda * mean(Beta(a,b)) + (1-\lambda) * \theta_{MLH}$

$\frac{a+|X=1|}{a+|X=1|+b+|X=0|} = \lambda * \frac{a}{a+b} + (1-\lambda) * \frac{|X=1|}{|X=1|+|X=0|}$

With: $0 \leq \lambda \leq 1$

Solution:

$$\frac{a + |X = 1|}{a + |X = 1| + b + |X = 0|}$$

$$= \frac{a}{a + |X = 1| + b + |X = 0|} + \frac{|X = 1|}{a + |X = 1| + b + |X = 0|}$$

$$= \frac{a}{a + b + |X = 1| + |X = 0|} + \frac{|X = 1|}{|X = 1| + |X = 0| + a + b}$$

$$= \frac{a}{a + b * (1 + \frac{|X=1|+|X=0|}{a+b})} + \frac{|X = 1|}{|X = 1| + |X = 0| * (1 + \frac{a+b}{|X=1|+|X=0|})}$$

$$= \frac{a}{a + b} * \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}} + \frac{|X = 1|}{|X = 1| + |X = 0|} * \frac{1}{(1 + \frac{a+b}{|X=1|+|X=0|})}$$

$$= \frac{a}{a + b} * \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}} + \frac{|X = 1|}{|X = 1| + |X = 0|} * \frac{|X = 1| + |X = 0|}{(|X = 1| + |X = 0| + (a + b))}$$

$$= \frac{a}{a + b} * \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}} + \frac{|X = 1|}{|X = 1| + |X = 0|} * \left(\frac{|X = 1| + |X = 0| + a + b}{(|X = 1| + |X = 0|) + (a + b)}\right)$$

$$- \frac{a + b}{(|X = 1| + |X = 0| + (a + b))}$$

$$= \frac{a}{a + b} * \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}} + \frac{|X = 1|}{|X = 1| + |X = 0|} * \left(1 - \frac{a + b}{(|X = 1| + |X = 0| + (a + b))}\right)$$

$$= \frac{a}{a + b} * \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}} + \frac{|X = 1|}{|X = 1| + |X = 0|} * \left(1 - \frac{1}{(\frac{|X=1|+|X=0|}{a+b} + 1)}\right)$$

Insert:

$$\lambda = \frac{1}{1 + \frac{|X=1|+|X=0|}{a+b}}$$
$$mean(P(\theta|D)) = \lambda * \frac{a}{a+b} + (1 - \lambda) * \frac{|X=1|}{|X=1|+|X=0|}$$

qed.

Problem 5

$$P(\lambda|x) = \frac{P(X|\lambda) * P(\lambda)}{P(X)}$$
$$P(\lambda|x) = \frac{P(X|\lambda) * P(\lambda)}{\int_0^\infty (P(X|\lambda) * P(\lambda)) d\lambda}$$

Step 1:

$\int_0^\infty P(\lambda)d\lambda = 1$
$\int_0^\infty Gamma(a,b)d\lambda = 1$
$\int_0^\infty \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda}d\lambda = 1$
$\frac{b^a}{\Gamma(a)} * \int_0^\infty \lambda^{a-1} * e^{-b\lambda}d\lambda = 1$
$\int_0^\infty \lambda^{a-1} * e^{-b\lambda}d\lambda = \frac{\Gamma(a)}{b^a}$

Step2:

$P(X) = \int_0^\infty (P(X|\lambda) * P(\lambda))d\lambda$
$\int_0^\infty (P(X|\lambda) * Gamma(a,b))d\lambda =$
$\int_0^\infty \frac{e^{-\lambda}*\lambda^x}{x!} * \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda})d\lambda =$
$\frac{b^a}{\Gamma(a)*x!} \int_0^\infty e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda})d\lambda =$

$\frac{b^a}{\Gamma(a)*x!} \int_0^\infty \lambda^{x+a-1} * e^{-(b+1)\lambda})d\lambda =$

Applying Step1:
$\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}$

Putting it together:
$P(\lambda|x) = \frac{P(X|\lambda)*P(\lambda)}{P(X)} =$

$\frac{\frac{e^{-\lambda}*\lambda^x}{x!} * \frac{b^a}{\Gamma(a)}*\lambda^{a-1}*e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}} =$

$\frac{\frac{b^a}{\Gamma(a)*x!}*e^{-\lambda}*\lambda^x*\lambda^{a-1}*e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}}$

$\frac{e^{-\lambda}*\lambda^x*\lambda^{a-1}*e^{-b\lambda}}{\frac{\Gamma(x+a)}{(b+1)^{x+a}}}$

$e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$

$\lambda^{a+x-1} * e^{-(b+1)\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$

$Gamma(a+x, b+1)$

Calculating Map:
$\frac{d}{d\lambda}P(\lambda|x) = 0$
$\frac{d}{d\lambda}ln(P(\lambda|x)) = 0$
$\frac{d}{d\lambda}ln(P(\lambda|x)) = \frac{d}{d\lambda}ln(\frac{(b+1)^{x+a}}{\Gamma(x+a)} * \lambda^{a+x-1} * e^{-(b+1)\lambda}) =$
$\frac{d}{d\lambda}ln(\frac{(b+1)^{x+a}}{\Gamma(x+a)}) + ln(\lambda^{a+x-1}) + ln(e^{-(b+1)\lambda}) =$
$\frac{d}{d\lambda}(a+x-1) * ln(\lambda) + -(b+1)\lambda =$
$(a+x-1) * \frac{1}{\lambda} + -(b+1)$

$(a+x-1) * \frac{1}{\lambda} + -(b+1) = 0$
$(a+x-1) * \frac{1}{\lambda} = (b+1)$
$(a+x-1) = (b+1)\lambda$

$$\lambda = \frac{a+x+1}{b+1}$$
$$\Theta_{MAP} = \frac{a+x+1}{b+1}$$