

Machine Learning Worksheet 4

Thomas Blocher – MatrNr. 03624034

Raphael Dümig – MatrNr. 03623199

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Problem 1

Problem 2

- $C = \{0, 1\}$: the tossed coin shows head or tail respectively

$$C \sim \text{Ber}\left(\frac{1}{2}\right)$$

- $T = \{0, 1, 2\}$: the number of two coin tosses that resulted in *tail*

$$\Rightarrow T \sim \text{Bin}\left(2, \frac{1}{2}\right)$$

- $R = \{0, 1, 2, 3\}$: number of red balls drawn from the box

$$R \sim \text{Bin}\left(3, \frac{t}{2}\right)$$

$$P(T = t) = \binom{2}{t} \cdot \left(\frac{1}{2}\right)^t \cdot \left(\frac{1}{2}\right)^{2-t} = \frac{1}{4} \cdot \binom{2}{t}$$

$$P(R = r | T = t) = \binom{3}{r} \cdot \left(\frac{t}{2}\right)^r \cdot \left(1 - \frac{t}{2}\right)^{3-r}$$

$$P(T | R) = \frac{P(R | T) \cdot P(T)}{P(R)}$$

$$\begin{aligned}
\Rightarrow P(T=2|R=3) &= \frac{P(R=3|T=2) \cdot P(T=2)}{P(R=3)} \\
&= P(R=3|T=2) \cdot \frac{P(T=2)}{P(R=3)} \\
&= P(R=3|T=2) \cdot \frac{P(T=2)}{\sum_{t \in T} P(R=3|T=t) \cdot P(T=t)} \\
&= 1 \cdot \frac{\frac{1}{4}}{0 \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4}} \\
&= \frac{\frac{1}{4}}{\frac{5}{16}} = \frac{4}{5}
\end{aligned}$$

Problem 3

$$P(X=x|\theta) = \theta^x(1-\theta)^{1-x}$$

goal: determine θ_{MLE}

$$\begin{aligned}
\theta_{MLE} &= \arg \max_{\theta} P(X=x|\theta) \\
\Rightarrow 0 &= \frac{dP}{d\theta}(\theta_{MLE}) \\
\Rightarrow 0 &= \frac{d \log P}{d\theta}(\theta_{MLE})
\end{aligned}$$

$$\begin{aligned}
\log P(X=x|\theta) &= \log \theta^x(1-\theta)^{1-x} \\
&= x \cdot \log \theta + (1-x) \cdot \log(1-\theta)
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{d}{d\theta}(x \cdot \log \theta + (1-x) \cdot \log(1-\theta)) \\
&= x \cdot \frac{1}{\theta} + (1-x) \cdot \frac{1}{1-\theta} \cdot (-1) \\
&= \frac{x}{\theta} - \frac{1-x}{1-\theta} \\
\Leftrightarrow \frac{1-x}{1-\theta} &= \frac{x}{\theta} \\
\Leftrightarrow \frac{1-x}{x} &= \frac{1-\theta}{\theta} \\
\Leftrightarrow \frac{1}{x} - 1 &= \frac{1}{\theta} - 1 \\
\Leftrightarrow \frac{1}{x} &= \frac{1}{\theta} \\
\Leftrightarrow \theta &= x \\
\Rightarrow \theta_{MLE} &= x
\end{aligned}$$

Problem 4

Show: *TODO: Show $\theta_{MLH} = \frac{|X=1|}{|X=1|+|X=0|}$ referece to prob 3

Formalized Problem:

$$mean(P(\theta|D)) = \lambda * mean(p(\theta)) + (1-\lambda) * \theta_{MLH}$$

$$mean(Beta(a + |X=1|, b + |X=0|)) = \lambda * mean(Beta(a, b)) + (1-\lambda) * \theta_{MLH}$$

$$\frac{a+|X=1|}{a+|X=1|+b+|X=0|} = \lambda * \frac{a}{a+b} + (1-\lambda) * \frac{|X=1|}{|X=1|+|X=0|}$$

With: $0 \leq \lambda \leq 1$

$$\begin{aligned}
\text{Solution: } \frac{a+|X=1|}{a+|X=1|+b+|X=0|} &= \\
\frac{a}{a+|X=1|+b+|X=0|} + \frac{|X=1|}{a+|X=1|+b+|X=0|} &= \\
\frac{a}{a+b+|X=1|+|X=0|} + \frac{|X=1|}{|X=1|+|X=0|+a+b} &= \\
\frac{a}{a+b*(1+\frac{|X=1|+|X=0|}{a+b})} + \frac{|X=1|}{|X=1|+|X=0|*(1+\frac{a+b}{|X=1|+|X=0|})} &= \\
\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \frac{1}{(1+\frac{a+b}{|X=1|+|X=0|})} &= \\
\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \frac{|X=1|+|X=0|}{(|X=1|+|X=0|+(a+b))} &= \\
\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \left(\frac{|X=1|+|X=0|+a+b}{(|X=1|+|X=0|+(a+b))} - \frac{a+b}{(|X=1|+|X=0|+(a+b))} \right) &= \\
\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \left(1 - \frac{a+b}{(|X=1|+|X=0|+(a+b))} \right) &= \\
\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \left(1 - \frac{1}{(\frac{|X=1|+|X=0|}{a+b} + 1)} \right) &=
\end{aligned}$$

insert:

$$\begin{aligned}\lambda &= \frac{1}{1 + \frac{|X=1| + |X=0|}{a+b}} \\ &= \lambda * \frac{a}{a+b} + (1 - \lambda) * \frac{|X=1|}{|X=1| + |X=0|}\end{aligned}$$

qed.

Problem 5

$$\begin{aligned}P(\lambda|x) &= \frac{P(X|\lambda)*P(\lambda)}{P(X)} \\ P(\lambda|x) &= \frac{P(X|\lambda)*P(\lambda)}{\int_0^\infty (P(X|\lambda)*P(\lambda))d\lambda}\end{aligned}$$

Step 1:

$$\begin{aligned}\int_0^\infty P(\lambda)D\lambda &= 1 \\ \int_0^\infty \text{Gamma}(a, b)D\lambda &= 1 \\ \int_0^\infty \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda}D\lambda &= 1 \\ \frac{b^a}{\Gamma(a)} * \int_0^\infty \lambda^{a-1} * e^{-b\lambda}D\lambda &= 1 \\ \int_0^\infty \lambda^{a-1} * e^{-b\lambda}D\lambda &= \frac{\Gamma(a)}{b^a}\end{aligned}$$

Step2:

$$\begin{aligned}P(X) &= \int_0^\infty (P(X|\lambda) * P(\lambda))d\lambda \\ \int_0^\infty (P(X|\lambda) * \text{Gamma}(a, b))d\lambda &= \\ \int_0^\infty \frac{e^{-\lambda} * \lambda^x}{x!} * \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda}d\lambda &= \\ \frac{b^a}{\Gamma(a)*x!} \int_0^\infty e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda}d\lambda &= \\ \frac{b^a}{\Gamma(a)*x!} \int_0^\infty \lambda^{x+a-1} * e^{-(b+1)\lambda}d\lambda &= \end{aligned}$$

$$\frac{b^a}{\Gamma(a)*x!} \int_0^\infty \lambda^{x+a-1} * e^{-(b+1)\lambda}d\lambda =$$

Applying Step1:

$$\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}$$

Putting it together:

$$P(\lambda|x) = \frac{P(X|\lambda)*P(\lambda)}{P(X)} =$$

$$\frac{\frac{e^{-\lambda} * \lambda^x}{x!} * \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}} =$$

$$\frac{\frac{b^a}{\Gamma(a)*x!} * e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}} =$$

$$\frac{e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda}}{\frac{\Gamma(x+a)}{(b+1)^{x+a}}} =$$

$$e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$$

$$\lambda^{a+x-1} * e^{-(b+1)\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$$

$$\text{Gamma}(a + x, b + 1)$$

Calculating Map:

$$\frac{d}{d\lambda} P(\lambda|x) = 0$$

$$\begin{aligned} \frac{d}{d\lambda} \ln(P(\lambda|x)) &= 0 \quad \frac{d}{d\lambda} \ln(P(\lambda|x)) = \frac{d}{d\lambda} \ln\left(\frac{(b+1)^{x+a}}{\Gamma(x+a)} * \lambda^{a+x-1} * e^{-(b+1)\lambda}\right) = \\ \frac{d}{d\lambda} \ln\left(\frac{(b+1)^{x+a}}{\Gamma(x+a)}\right) + \ln(\lambda^{a+x-1}) + \ln(e^{-(b+1)\lambda}) &= \\ \frac{d}{d\lambda} (a + x - 1) * \ln(\lambda) + -(b + 1)\lambda &= \\ (a + x - 1) * \frac{1}{\lambda} + -(b + 1) \end{aligned}$$

$$(a + x - 1) * \frac{1}{\lambda} + -(b + 1) = 0$$

$$(a + x - 1) * \frac{1}{\lambda} = (b + 1)$$

$$(a + x - 1) = (b + 1)\lambda$$

$$\lambda = \frac{a+x-1}{b+1}$$

$$\Theta_{MAP} = \frac{a+x-1}{b+1}$$