Machine Learning Worksheet 4

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November 3, 2015

Problem 1

Problem 2

Problem 3

Problem 4

Show: *TODO: Show $\theta_{MLH} = \frac{|X=1|}{|X=1|+|X=0|}$ referece to prob 3

Formalized Problem:
$mean(P(\theta|D)) = \lambda * mean(p(\theta)) + (1-\lambda) * \theta_{MLH}$
$mean(Beta(a+|X=1|, b+|X=0|)) = \lambda * mean(Beta(a,b)) + (1-\lambda) * \theta_{MLH}$
$\frac{a+|X=1|}{a+|X=1|+b+|X=0|} = \lambda * \frac{a}{a+b} + (1-\lambda) * \frac{|X=1|}{|X=1|+|X=0|}$

With: $0 \leq \lambda \leq 1$

Solution: $\frac{a+|X=1|}{a+|X=1|+b+|X=0|} =$
$\frac{a}{a+|X=1|+b+|X=0|} + \frac{|X=1|}{a+|X=1|+b+|X=0|} =$
$\frac{a}{a+b+|X=1|+|X=0|} + \frac{|X=1|}{|X=1|+|X=0|+a+b} =$
$\frac{a}{a+b*(1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|*(1+\frac{a+b}{|X=1|+|X=0|}} =$
$\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \frac{1}{(1+\frac{a+b}{|X=1|+|X=0|}} =$
$\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * \frac{|X=1|+|X=0|}{(|X=1|+|X=0|+(a+b)} =$
$\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * (\frac{|X=1|+|X=0|+a+b}{(|X=1|+|X=0|+(a+b)} - \frac{a+b}{(|X=1|+|X=0|+(a+b))} =$
$\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * (1 - \frac{a+b}{(|X=1|+|X=0|+(a+b))}) =$
$\frac{a}{a+b} * \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}} + \frac{|X=1|}{|X=1|+|X=0|} * (1 - \frac{1}{(\frac{|X=1|+|X=0|}{a+b}+1)}) =$

insert:
$\lambda = \frac{1}{1+\frac{|X=1|+|X=0|}{a+b}}$
$= \lambda * \frac{a}{a+b} + (1-\lambda) * \frac{|X=1|}{|X=1|+|X=0|}$

qed.

Problem 5

$P(\lambda|x) = \frac{P(X|\lambda)*P(\lambda)}{P(X)}$
$P(\lambda|x) = \frac{P(X|\lambda)*P(\lambda)}{\int_0^\infty (P(X|\lambda)*P(\lambda))d\lambda}$

Step 1:

$\int_0^\infty P(\lambda)D\lambda = 1$
$\int_0^\infty Gamma(a,b)D\lambda = 1$
$\int_0^\infty \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda}D\lambda = 1$
$\frac{b^a}{\Gamma(a)} * \int_0^\infty \lambda^{a-1} * e^{-b\lambda}D\lambda = 1$
$\int_0^\infty \lambda^{a-1} * e^{-b\lambda}D\lambda = \frac{\Gamma(a)}{b^a}$

Step2:

$P(X) = \int_0^\infty (P(X|\lambda) * P(\lambda))d\lambda$
$\int_0^\infty (P(X|\lambda) * Gamma(a,b))d\lambda =$
$\int_0^\infty \frac{e^{-\lambda}*\lambda^x}{x!} * \frac{b^a}{\Gamma(a)} * \lambda^{a-1} * e^{-b\lambda})d\lambda =$
$\frac{b^a}{\Gamma(a)*x!} \int_0^\infty e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda})d\lambda =$

$\frac{b^a}{\Gamma(a)*x!} \int_0^\infty \lambda^{x+a-1} * e^{-(b+1)\lambda})d\lambda =$
Applying Step1:
$\frac{b^a}{\Gamma(a)*x!} * \frac{\Gamma(x+a)}{(b+1)^{x+a}}$
Putting it together:
$P(\lambda|x) = \frac{P(X|\lambda)*P(\lambda)}{P(X)} =$
$\frac{\frac{e^{-\lambda}*\lambda^x}{x!}*\frac{b^a}{\Gamma(a)}*\lambda^{a-1}*e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!}*\frac{\Gamma(x+a)}{(b+1)^{x+a}}} =$
$\frac{\frac{b^a}{\Gamma(a)*x!}*e^{-\lambda}*\lambda^x*\lambda^{a-1}*e^{-b\lambda}}{\frac{b^a}{\Gamma(a)*x!}*\frac{\Gamma(x+a)}{(b+1)^{x+a}}}$
$\frac{e^{-\lambda}*\lambda^x*\lambda^{a-1}*e^{-b\lambda}}{\frac{\Gamma(x+a)}{(b+1)^{x+a}}}$
$e^{-\lambda} * \lambda^x * \lambda^{a-1} * e^{-b\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$
$\lambda^{a+x-1} * e^{-(b+1)\lambda} * \frac{(b+1)^{x+a}}{\Gamma(x+a)} =$

$Gamma(a+x, b+1)$

Calculating Map:

$\frac{d}{d\lambda}P(\lambda|x) = 0$

$\frac{d}{d\lambda}ln(P(\lambda|x)) = 0$ $\frac{d}{d\lambda}ln(P(\lambda|x)) = \frac{d}{d\lambda}ln(\frac{(b+1)^{x+a}}{\Gamma(x+a)} * \lambda^{a+x-1} * e^{-(b+1)\lambda}) =$
$\frac{d}{d\lambda}ln(\frac{(b+1)^{x+a}}{\Gamma(x+a)}) + ln(\lambda^{a+x-1}) + ln(e^{-(b+1)\lambda}) =$
$\frac{d}{d\lambda}(a+x-1) * ln(\lambda) + -(b+1)\lambda =$
$(a+x-1) * \frac{1}{\lambda} + -(b+1)$

$(a+x-1) * \frac{1}{\lambda} + -(b+1) = 0$
$(a+x-1) * \frac{1}{\lambda} = (b+1)$
$(a+x-1) = (b+1)\lambda$
$\lambda = \frac{a+x+1}{b+1}$
$\Theta_{MAP} = \frac{a+x+1}{b+1}$