LECTURE 4

SOME BASIC MATHEMATICAL CONCEPTS

Mathematical induction

Example 1: Prove that the sum of integers 1 ... n is given by: n(n+1)/2.

Basis step:

Take n = 1.

Then the sum = 1, and 1(1+1)/2 = 1. The summation formula holds trivially.

Induction step:

Induction hypothesis: Assume that the result is true for all integers up to k.

That is, assume that: $\sum_{j=1,k} j = k(k+1)/2$.

We know that $\sum_{j=1,\mathbf{k+1}} j = \sum_{j=1,\mathbf{k}} j + (k+1)$.

Applying the induction hypothesis to the first term, this gives:

RHS = k(k+1)/2 + (k+1) = (k+1)(k/2+1) = (k+1)(k+2)/2

Induction step is proved. Therefore this is the required result.

Example 2: Prove that the sum of squares of integers 1^2, 2^2 ... n^2 is given by n(n+1)(2n+1)/6.

Example 3: Prove by mathematical induction that an undirected connected acyclic **graph** with n nodes has (n-1) edges.

[Note: Unless it is explicitly stated to be undirected, a graph will be assumed to be directed.]

Example 4: Prove by mathematical induction that a balanced, complete binary **tree** of height n has $2^{n+1} - 1$ nodes.

Mathematical induction is also used in proving the correctness of algorithms.

Quick review

1. When is a binary relation **transitive**, **reflexive**, **symmetric**?

2. What is a directed, acyclic graph (DAG)?

3. When is a binary relation also a **function**?

4. When is a function f: X → Y **one-to-one**, **onto**, **invertible**?

Selection from [CLRS]

A function f(n) is **monotonically increasing** if $m \le n$ implies that $f(m) \le f(n)$.

A function f(n) is **monotonically decreasing** if $m \le n$ implies that $f(m) \ge f(n)$.

A function f(n) is **strictly increasing** if $m < n$ implies that $f(m) < f(n)$.

A function f(n) is **strictly decreasing** if $m < n$ implies that $f(m) > f(n)$.

mod(a,n) = a – floor(a/n)*n

Note that integer arithmetic is in fact modulo arithmetic.

[BTW: Quick review of 2's complement number representation and floating point number representation.]

Polynomial p(n) of degree d: $p(n) = \sum_{j=0,d} a_j n^j$, asymptotically positive if $a_d > 0$.

A function f(n) is polynomially bounded if $f(n) = \Theta(n^k)$ for some constant k.

Miscellaneous

If $b^c = a$, then $c = \log_b a$ [Logarithm of a to base b]

$\log_b a = \log_c a / \log_c b$ [Change of base]

$\lg(n!) = \Theta(n \lg n)$

Infinite geometric series: For $0 \le r \le 1$, $1 + r + r^2 + r^3 ...... = 1/(1-r)$

Fibonacci numbers (or **series**)

$F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k \ge 2$.