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Inference from Inadequate and Inaccurate Data, II*

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Abstract. Having measured D numerical properties of a physical object E which requires many more than D parameters for its complete specification, an observer seeks to estimate P other numerical properties of E. A previous paper discussed how he can proceed when E is adequately described by one member m_E of a Hilbert space $\mathfrak{M}$ of possible models of E and when all the observed and sought-for properties of E are Fréchet differentiable functionals on $\mathfrak{M}$. The present paper describes a technique often available for reducing to this tractable case problems in which the functionals are discontinuous and are defined only on parts of $\mathfrak{M}$, and $\mathfrak{M}$ is an arbitrary real linear space. Applications include geophysical inverse problems and numerical differentiation, integration, interpolation, and extrapolation.

1. **Introduction.** The language introduced in the first paper[1] of this series is used, except that now $\mathfrak{M}$, the space of models, will be an arbitrary real linear space. The data functionals $(g_1, \ldots, g_D)$ and the prediction functionals $(\tilde{g}_1, \ldots, \tilde{g}_P)$ are now defined only on subsets $\mathfrak{M}_1, \ldots, \mathfrak{M}_D, \tilde{\mathfrak{M}}_1, \ldots, \tilde{\mathfrak{M}}_P$ of $\mathfrak{M}$ and, if $\mathfrak{M}$ has a topology, may be discontinuous. For the moment the functionals need not be linear. I denote $\mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_D \cap \tilde{\mathfrak{M}}_1 \cap \ldots \cap \tilde{\mathfrak{M}}_P$ by $\mathfrak{M}$ $(\mathfrak{N}^*)$ and by $\mathfrak{N}^*$ I mean asp$\{g_1, \ldots, g_D, \tilde{g}_1, \ldots, \tilde{g}_P\}$, the set of all functionals on $\mathfrak{M}(\mathfrak{N}^*)$ which are real linear combinations of $g_1, \ldots, g_D, \tilde{g}_1, \ldots, \tilde{g}_P$. Even if these functionals are nonlinear, $\mathfrak{N}^*$ is a real linear space.

If $\mathbf{R}$ denotes the real line and $g : \mathfrak{B} \rightarrow \mathbf{R}$ is any functional (real valued function) on the arbitrary set $\mathfrak{B}$, I will write $[g, v]$ for the real number $g(v)$ which the functional g assigns to the member v of $\mathfrak{B}$. This number depends linearly on g but need not be linear in v.

A real pre-Hilbert space is a real linear space $\mathfrak{H}$ on which is defined a bilinear, positive-definite, real inner product. The real Hilbert space obtained by completing $\mathfrak{H}$ I write as $c\mathfrak{H}$. If h_1 and h_2 are in $c\mathfrak{H}$, I write their inner product as $\langle h_1, h_2\rangle$ and define $||h_1|| = \langle h_1, h_1\rangle^{1/2}$.

A functional $g^*:\mathfrak{H} \rightarrow \mathbf{R}$ on a pre-Hilbert space $\mathfrak{H}$ is "Cauchy continuous" if for every Cauchy sequence $\{h_1, h_2, \ldots\}$ in $\mathfrak{H}$, $\{[g^*, h_1], [g^*, h_2], \ldots\}$ is a Cauchy sequence in $\mathbf{R}$. If g^* is Cauchy continuous on $\mathfrak{H}$, it can be extended in exactly one way to a continuous functional $g^*:c\mathfrak{H} \rightarrow \mathbf{R}$. Conversely, if $g^*:c\mathfrak{H} \rightarrow \mathbf{R}$ is continuous then its restriction to $\mathfrak{H}$ is Cauchy continuous.

2. **Quellings.** Suppose $\mathfrak{M}$ and $\mathfrak{N}^*$ are as in the *Introduction*, $\mathfrak{H}$ is a pre-Hilbert space, and $Q:\mathfrak{H} \rightarrow \mathfrak{M}(\mathfrak{N}^*)$ is a possibly nonlinear mapping. Then if g is in $\mathfrak{N}^*$, a functional $g^*:\mathfrak{H} \rightarrow \mathbf{R}$ is defined by requiring that for each h in $\mathfrak{H}$

$$[g^*, h] = [g, Qh]. \tag{1}$$

I will call Q "$\mathfrak{N}^*$-Cauchy continuous" if for each g in $\mathfrak{N}^*$ the functional $g^*:\mathfrak{H} \rightarrow \mathbf{R}$ defined by equation (1) is Cauchy continuous.

If $Q:\mathfrak{H} \rightarrow \mathfrak{M}(\mathfrak{N}^*)$ is $\mathfrak{N}^*$-Cauchy continuous and g is in $\mathfrak{N}^*$ then the functional g^* defined by equation (1) has a unique continuous extension to all of $c\mathfrak{H}$, which will be called Q^*g. Thus $Q^*g:c\mathfrak{H} \rightarrow \mathbf{R}$ and $Q^*:\mathfrak{N}^* \rightarrow C(c\mathfrak{H})$ where $C(c\mathfrak{H})$ is the linear space of continuous functionals on $c\mathfrak{H}$. The mapping Q^* is linear, whether g and Q are or not. By the definition of Q^*, if g is any member of $\mathfrak{N}^*$ and h is any member of h then $[Q^*g, h] = [g, Qh]$.

"Aquelling" of $(\mathfrak{N}^*,\mathfrak{M})$ is defined to be an ordered pair $(Q,\mathfrak{H})$ with $\mathfrak{H}$ a real pre-Hilbert space and $Q:\mathfrak{H} \rightarrow \mathfrak{M}(\mathfrak{N}^*)$ an $\mathfrak{N}^*$-Cauchy continuous injection (Q is an injection if $Qh_1 = Qh_2$ implies $h_1 = h_2$). Then Q has an inverse mapping, Q^{-1}: $Q(\mathfrak{H}) \rightarrow \mathfrak{H}$, and if g is any functional in $\mathfrak{N}^*$ and m is any model in $Q(\mathfrak{H})$, $[g, m] = [Q^*g, Q^{-1}m]$. If Q is a linear mapping, $(Q,\mathfrak{H})$ is a linear quelling of $(\mathfrak{N}^*,\mathfrak{M})$.

If $(Q,\mathfrak{H})$ is any quelling of $(\mathfrak{N}^*,\mathfrak{M})$ for which m_E is in $Q(\mathfrak{H})$, we can replace $\mathfrak{M}$ by $\mathfrak{H}$, replace m_E by $Q^{-1}m_E$, replace any g in $\mathfrak{N}^*$ by Q^*g, and thus reduce the problem of prediction to prediction on a Hilbert space, the problem treated in the first paper.[1] If we regard the members of $c\mathfrak{H}$ as the possible models for E, the data and predictions now depend continuously on the model. Lack of such continuity for models in $\mathfrak{M}$ is an inaptness of $\mathfrak{M}$ to the problem.

If the nonlinear functionals $\{Q^*g_1, \ldots, Q^*g_D, Q^*\tilde{g}_1, \ldots, Q^*\tilde{g}_P\}$ are Fréchet differentiable on $c\mathfrak{H}$, *Section 6* of the first paper reduces the prediction problem locally to a linear problem. Therefore in the rest of the present paper it will be assumed that $\{g_1, \ldots, g_D, \tilde{g}_1, \ldots, \tilde{g}_P\}$ and hence all members of $\mathfrak{N}^*$ are linear functionals on $\mathfrak{M}(\mathfrak{N}^*)$, and we shall consider only linear quellings of $(\mathfrak{N}^*,\mathfrak{M})$.

When all members of $\mathfrak{N}^*$ are linear and $\mathfrak{H}$ is a pre-Hilbert space, a linear injection $Q:\mathfrak{H} \rightarrow \mathfrak{M}(\mathfrak{N}^*)$ is a quelling of $(\mathfrak{N}^*,\mathfrak{M})$ if and only if for each g in $\mathfrak{N}^*$ the linear functional $g^*:\mathfrak{H} \rightarrow \mathbf{R}$ defined by equation (1) is bounded. But if g^* is linear and bounded on $\mathfrak{H}$ then Q^*g, the extension of g^* to $c\mathfrak{H}$, is linear and bounded on $c\mathfrak{H}$. Then there is a unique member h^* of $c\mathfrak{H}$ such that for all h in $c\mathfrak{H}$, $[Q^*g, h] = \langle h^*, h\rangle$. Following usual practice, we shall identify Q^*g with h^*. Thus when the data functionals, the prediction functionals and the quelling $(Q,\mathfrak{H})$ are linear, Q^*g is defined for any g in $\mathfrak{N}^*$ as that member of $c\mathfrak{H}$ such that for all h in $\mathfrak{H}$, $[g, Qh] = \langle Q^*g, h\rangle$. It follows that if m is any model in $Q(\mathfrak{H})$ and g is any functional in $\mathfrak{N}^*$ then

$$[g, m] = \langle Q^*g, Q^{-1}m\rangle \tag{2}$$

Now Paper I of this series[1] gives the limitations on the predictions, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_P$, which are imposed by the data, $\gamma_1, \ldots, \gamma_D$, and the additional hypothesis that m_E satisfies

$$\|Q^{-1}m\| \leq M \tag{3}$$

or its probabilistic analog for some selected bound M. Quelling will usually be applied when some members of $\mathfrak{R}^*$ are discontinuous, so if $\mathfrak{M}$ is a function space, Q^* and hence Q will be, loosely speaking, smoothing operators, and Q^{-1} will "roughen." Then equation (3) will be a demand on the smoothness of m_E as well as on its general size.

3. **Extending Linear Quellings.** A linear quelling $(Q,\mathfrak{H})$ of $(\mathfrak{R}^*,\mathfrak{M})$ is useless if m_E is not in $Q(\mathfrak{H})$. The present section gives one way to extend $Q(\mathfrak{H})$.

If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are pre-Hilbert spaces, $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ denotes the pre-Hilbert space consisting of all ordered pairs (h_1, h_2) with h_1 in $\mathfrak{H}_1$ and h_2 in $\mathfrak{H}_2$; the linear combination $\alpha(h_1, h_2) + \alpha'(h_1', h_2')$ is defined to be $(\alpha h_1 + \alpha' h_1', \alpha h_2 + \alpha' h_2')$, and the inner product $\langle (h_1, h_2), (h_1', h_2')\rangle$ is defined to be $\langle h_1, h_1'\rangle + \langle h_2, h_2'\rangle$. The real line $\mathbf{R}$ is a real Hilbert space whose inner product is ordinary multiplication, and $\mathbf{R}^N$ is the Hilbert space $\mathbf{R} \oplus \mathbf{R} \oplus \ldots \oplus \mathbf{R}$ (N terms).

If $(Q_1,\mathfrak{H}_1)$ and $(Q_2,\mathfrak{H}_2)$ are linear quellings of $(\mathfrak{R}^*,\mathfrak{M})$, we can define a linear mapping $Q:\mathfrak{H}_1 \oplus \mathfrak{H}_2 \rightarrow \mathfrak{M}(\mathfrak{R}^*)$ as $Q(h_1, h_2) = Q_1h_1 + Q_2h_2$. This mapping Q, which we write $Q_1 \oplus Q_2$, is $\mathfrak{R}^*$-Cauchy continuous; and $Q(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ is $Q_1(\mathfrak{H}_1) + Q_2(\mathfrak{H}_2)$, the linear space of all those members of $\mathfrak{M}(\mathfrak{R}^*)$ which can be written $m_1 + m_2$ with m_1 in $Q_1(\mathfrak{H}_1)$ and m_2 in $Q_2(\mathfrak{H}_2)$. Therefore it may often happen that m_E is in $Q(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and in neither $Q_1(\mathfrak{H}_1)$ nor $Q_2(\mathfrak{H}_2)$. However, $Q_1 \oplus Q_2$ is an injection if and only if $Q_1(\mathfrak{H}_1) \cap Q_2(\mathfrak{H}_2) = \{0\}$; if $Q_1(\mathfrak{H}_1)$ and $Q_2(\mathfrak{H}_2)$ share a nonzero element $m = Q_1h_1 = Q_2h_2$, then $Q(0, h_2) = Q(h_1, 0) = m$, and $(0, h_2) \neq (h_1, 0)$. When $Q_1(\mathfrak{H}_1) \cap Q_2(\mathfrak{H}_2) = \{0\}$, we will call $(Q_1,\mathfrak{H}_1)$ and $(Q_2,\mathfrak{H}_2)$ orthogonal linear quellings of $(\mathfrak{R}^*,\mathfrak{M})$. The pair $(Q_1 \oplus Q_2, \mathfrak{H}_1 \oplus \mathfrak{H}_2)$ is a linear quelling of $(\mathfrak{R}^*,\mathfrak{M})$ if and only if $(Q_1,\mathfrak{H}_1)$ and $(Q_2,\mathfrak{H}_2)$ are orthogonal linear quellings of $(\mathfrak{R}^*,\mathfrak{M})$. (If $Q_1^{-1}(Q_1(\mathfrak{H}_1) \cap Q_2(\mathfrak{H}_2))$ is complete, a similar construction not described here gives a linear quelling $(Q,\mathfrak{H})$ with $Q(\mathfrak{H}) = Q_1(\mathfrak{H}_1) + Q_2(\mathfrak{H}_2)$.)

As an example, suppose that $(Q,\mathfrak{H})$ is a linear quelling of $(\mathfrak{R}^*,\mathfrak{M})$, that m_E is not in $Q(\mathfrak{H})$, but that there are finitely many models $\{m_1', \ldots, m_K'\}$ in $\mathfrak{M}(R^*)$ such that m_E is in $Q(\mathfrak{H}) + \text{asp}\,\{m_1', \ldots, m_K'\}$. If $\{m_1', \ldots, m_K'\}$ is linearly dependent modulo $Q(\mathfrak{H})$ (i.e., if there are real numbers $b_1, \ldots, b_K$ not all zero such that $b_1m_1' + \ldots + b_Km_K'$ is in $Q(\mathfrak{H})$) then $\{m_1', \ldots, m_K'\}$ contains a proper subset $\{m_1, \ldots, m_J\}$ which is linearly independent modulo $Q(\mathfrak{H})$ and such that $Q(\mathfrak{H}) + \text{asp}\{m_1', \ldots, m_K'\} = Q(\mathfrak{H}) + \text{asp}\{m_1, \ldots, m_J\}$. We work with $\{m_1, \ldots, m_J\}$. Its linear independence modulo $Q(\mathfrak{H})$ means $Q(\mathfrak{H}) \cap \text{asp}\{m_1, \ldots, m_J\} = \{0\}$. Define $Q':\mathbf{R}^J \rightarrow \mathfrak{M}(\mathfrak{R}^*)$ as $Q'(b_1, \ldots, b_J) = \sum_{j=1}^{J} b_jm_j$. Then since $\{m_1, \ldots, m_J\}$ is linearly independent, $(Q',\mathbf{R}^J)$ is a linear quelling of $(\mathfrak{R}^*,\mathfrak{M})$. And since $Q(\mathfrak{H}) \cap Q'(\mathbf{R}^J) = \{0\}$, $(Q \oplus Q', \mathfrak{H} \oplus \mathbf{R}^J)$ is also a linear quelling of $(\mathfrak{R}^*,\mathfrak{M})$. Write $\hat{Q} = Q \oplus Q'$, $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbf{R}^J$. If $\hat{h}$ is in $\hat{\mathfrak{H}}$, then $\hat{h} = (h, b_1, \ldots, b_J)$ with h in $\mathfrak{H}$ and $b_1, \ldots, b_J$ in $\mathbf{R}$, and $Qh = Qh + \sum_{j=1}^{J} b_jm_j$, so now there is an $\hat{h}_E$ in $\hat{\mathfrak{H}}$ with $\hat{Q}\hat{h}_E = m_E$.

4. **Quelling Derivatives of the Dirac Delta Function.** To illustrate the construction and extension of quellings, a single prediction problem will be treated by three different quellings, corresponding with three different ways to make a square-integrable function out of $\delta^{(J-1)}(x)$, the $(J-1)$'st derivative of the Dirac delta distribution. Each of these three quellings will make Paper I applicable to the prediction problem if the observer accepts a bound, equation (3), on the measure of size and smoothness of m_E appropriate to the given quelling.

The prediction problem is this: $\mathfrak{M} = C[0, 1]$, the linear space of all real-valued continuous functions on $[0, 1]$, the closed unit interval. The data functionals $g_1, \ldots, g_D$ have kernels $G_1, \ldots, G_D$ which are themselves in $C[0, 1]$, so for any m in $\mathfrak{M}$ and, for $i = 1, \ldots, D$, $[g_i, m] = \int_0^1 G_i(x)m(x)dx$. Only one "prediction," $\tilde{\gamma} = [\tilde{g}, m_E]$, is wanted, so $P = 1$. The prediction functional $\tilde{g}$ is defined by $[\tilde{g}, m] = m^{(J-1)}(x_o)$, the $(J-1)$'st derivative of m at a specified point x_o in $[0, 1]$. In the notation of distribution theory, $[\tilde{g}, m] = \int_0^1 \tilde{G}(x)m(x)dx$ with $\tilde{G}(x) = (-1)^{J-1}\delta^{(J-1)}(x - x_o)$. The space $\mathfrak{M}(\mathfrak{N}^*)$ consists of those real functions which are continuous on $[0, 1]$ and have a $(J-1)$'st derivative at x_o. The functionals g in $\mathfrak{N}^*$ can be expressed as $[g, m] = \int_0^1 G(x)m(x)dx$ where the kernel G is any linear combination of $G_1, \ldots, G_D$ and $\tilde{G}$.

In the first two methods of quelling $(\mathfrak{N}^*, \mathfrak{M})$ we take $\mathfrak{H} = C[0, 1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Then $c\mathfrak{H}$ is $L_2(0, 1)$, the Hilbert space of square integrable functions on $(0, 1)$.

Method 1. Quelling by multiplication: In this quelling, if h is in $\mathfrak{H}$ define Qh as the function $(x - x_o)^J h(x)$. Then $Q:\mathfrak{H} \to \mathfrak{M}(\mathfrak{N}^*)$ is a linear injection. If g is in $\mathfrak{N}^*$, the kernel of g can be written $G = a_o\tilde{G} + G'$ where $G' = \sum_{i=1}^{D} a_i G_i$ is in $C[0, 1]$. If h is in $\mathfrak{H}$ then $[g, Qh] = \int_0^1 G'(x)(x - x_o)^J h(x)dx$ because $(x - x_o)^J\delta^{(J-1)}(x - x_o) = 0$. Therefore, by Schwarz's inequality, $[g, Qh]^2 \leq ||h||^2\int_0^1 G'(x)^2(x - x_o)^{2J}dx$. Then g^* defined by equation (1) is a bounded linear functional on $\mathfrak{H}$. Consequently Q is $\mathfrak{N}^*$-Cauchy continuous, so $(Q,\mathfrak{H})$ is a quelling of $(\mathfrak{N}^*,\mathfrak{M})$. Clearly Q^*g is the function $(x - x_o)^J G(x) = (x - x_o)^J G'(x)$ in $C[0, 1]$.

If m is in $Q(\mathfrak{H})$ then $m(x_o) = m^{(1)}(x_o) = \ldots = m^{(J-1)}(x_o) = 0$. If m_E were in $Q(\mathfrak{H})$ the prediction problem would be trivial, so we must extend the quelling. We choose the J functions $m_j(x) = (x - x_o)^{j-1}$, $j = 1, 2, \ldots, J$. They are linearly independent modulo $Q(\mathfrak{H})$ so we take $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbf{R}^J$ and if $\hat{h} = (h, b_1, \ldots, b_J)$ is in $\hat{\mathfrak{H}}$ we take $\hat{Q}\hat{h} = (x - x_o)^J h(x) + \sum_{j=1}^{J} b_j(x - x_o)^{j-1}$. Then $\hat{Q}(\hat{\mathfrak{H}})$ consists of those functions m which have $J - 1$ derivatives at x_o and such that $h(x) = (x - x_o)^{-J}[m(x) - \sum_{j=1}^{J}(x - x_o)^{j-1}m^{(j-1)}(x_o)/(j - 1)!]$ is continuous in $[0, 1]$. Moreover, if m is in $\hat{Q}(\hat{\mathfrak{H}})$,

$$||\hat{Q}^{-1}m||^2 = \int_0^1 h(x)^2dx + \sum_{j=1}^{J} [m^{(j-1)}(x_o)/(j - 1)!]^2 \tag{4}$$

If the observer will accept an *a priori* bound, equation (3), for $\|\hat{Q}^{-1}m_E\|$ as defined by equation (4), he can use Paper I of the series to estimate $\tilde{\gamma} = m^{(J-1)}(x_o)$ from $\gamma_i = \int_0^1 G_i(x)m_E(x)dx, i = 1, \ldots, D$, and to assess the error of his estimate.

To discuss functions of depth defined in both the earth's core and mantle, we may want to permit m_E to be continuous except for a jump discontinuity of unknown amplitude $\Delta m_E = m_E(a+) - m_E(a-)$ at a known position $x = a$ in the open unit interval. As long as $x_o \neq a$ we can modify $\mathfrak{M}$ and $\mathfrak{H}$ to consist of such piecewise continuous functions. However, for a fixed m with $\Delta m \neq 0$, as x_o approaches a the right side of equation (4) approaches $+\infty$, so for any fixed M in equation (3) if we let x_o approach a we force Δm to zero. The *a priori* bounds M acceptable in equation (3) when $\Delta m \neq 0$ and x_o is close to a are too large to constrain usefully the prediction $\tilde{\gamma}$. The way out of this difficulty is a different extention of $(Q,\mathfrak{H})$, which we describe only for $\tilde{G}(x) = \delta(x - x_o)$. Let $H(x)$ be the Heaviside step function and let $H_a(x) = H(x - a)$. Let $\mathfrak{M}$ be all functions $m(x)$ continuous on $0 \leq x \leq a$ and $a < x \leq 1$ for which the limit $m(a+)$ exists, and for any such m define $\mathfrak{C}m$, its continuous part, as $\mathfrak{C}m = m - (\Delta m)H_a$ where $\Delta m = m(a+) - m(a)$. Take $\mathfrak{H}$ and Q as before, but set $m_1(x) = 1$, and $m_2(x) = H_a(x)$. Then $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbf{R}^2$, and if $\hat{h} = (h, b_1, b_2)$ is in $\hat{\mathfrak{H}}$ then $\hat{Q}\hat{h} = (x - x_o)h(x) + b_1 + b_2H_a(x)$. Therefore $Q(\hat{\mathfrak{H}})$ consists of those functions m in $\mathfrak{M}$ such that $[(\mathfrak{C}m)(x) - (\mathfrak{C}m)(x_o)]/(x - x_o)$ is in $C[0, 1]$, and for any such m

$$\|\hat{Q}^{-1}m\|^2 = \int_0^1 \left[\frac{(\mathfrak{C})m(x) - (\mathfrak{C}m)(x_o)}{x - x_o}\right]^2 dx + [(\mathfrak{C}m)(x_o)]^2 + [\Delta m]^2.$$

When $\Delta m \neq 0$, this expression need not approach $+\infty$ as x_o approaches a. In fact now the bound, equation (3), is simply an upper limit to the sizes of Δm and $\mathfrak{C}m$ and the slopes of the secants to $\mathfrak{C}m$ at $x = a$.

Method 2. Quelling by integration: Another way to make $\delta^{(J-1)}(x - x_0)$ square integrable is to integrate it J times from 0 to x. We take $\mathfrak{M} = \mathfrak{H} = C[0, 1]$. The identity

$$(J - 1)! \int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{J-1}} d\xi_J h(\xi_J) = \int_0^x d\xi(x - \xi)^{J-1}h(\xi)$$

permits us to define, for any h in $\mathfrak{H}$, $Qh = \int_0^x d\xi(x - \xi)^{J-1}h(\xi)$. Then $Q:\mathfrak{H} \to \mathfrak{M}(\mathfrak{N}^*)$ is a linear injection and if g is any linear combination of $g_1, \ldots, g_D$ then g^* defined by equation (1) is a bounded linear functional on $\mathfrak{H}$. The crucial observation is that $\tilde{g}^*$ is also a bounded linear functional on $\mathfrak{H}$. In fact, $[\tilde{g}^*, h] = [g, Qh]$ is obtained by $J - 1$ differentiations of Qh at x_o and is $(J - 1)! \int_0^{x_0} h(\xi)d\xi$. By Schwarz's inequality, $|[\tilde{g}^*, h]| \leq (J - 1)!\|h\|$. It follows that $(Q,\mathfrak{H})$ is a quelling of $(\mathfrak{N}^*,\mathfrak{M})$. If g in $\mathfrak{N}^*$ has kernel G then J integrations by parts show that $Q^*g = \int_x^1 d\xi(\xi - x)^{J-1}G(\xi)$. As expected, this is always a square integrable function because $Q^*\tilde{g} = (-1)^{J-1}(J - 1)!\, H(x_o - x)$.

If m is in $Q(\mathfrak{H})$, then $m(0) = m^{(1)}(0) = \ldots = m^{(J-1)}(0) = 0$; the quelling $(Q,\mathfrak{H})$ must be extended to eliminate these restrictions. We take $m_j(x) = x^{j-1}$, $j = 1, \ldots, J$. Then $\{m_1, \ldots, m_J\}$ are linearly independent modulo $Q(\mathfrak{H})$ so we define $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbf{R}^J$ and if $\hat{h} = (h, b_1, \ldots, b_J)$ is in $\hat{\mathfrak{H}}$ we define $Q\hat{h} = \sum_{j=1}^{J} b_j x^{j-1} + \int_0^x d\xi(x - \xi)^{J-1}h(\xi)$. Then m is in $\hat{Q}(\hat{\mathfrak{H}})$ if and only if $m^{(J)}$ is in $C[0, 1]$; and then

$$||\hat{Q}^{-1}m||^2 = \int_0^1 \left[\frac{m^{(J)}(x)}{(J-1)!}\right]^2 dx + \sum_{j=1}^{J} \left[\frac{m^{j-1}(0)}{(j-1)!}\right]^2. \tag{5}$$

As with Method 1, a different extension of $(Q,\mathfrak{H})$ is required if m_E is only piecewise continuously differentiable on $[0, 1]$.

Method 3. Quelling by convolution: Another way to make $\delta^{(J-1)}(x - x_0)$ square integrable is to convolute it with a function s whose $(J - 1)$'st derivative is square integrable. For simplicity we assume now that $\mathfrak{M}$ consists of all functions continuous, bounded and absolutely integrable on the whole real line. We take $\mathfrak{H} = \mathfrak{M}$ and on $\mathfrak{H}$ define the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$. For any f in $\mathfrak{H}$ we define the Fourier transform $\mathfrak{F}f$ by $\mathfrak{F}f(k) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi ikx)dx$, and if f and g are in $\mathfrak{H}$ we define $(f*g)(x) = \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi$. Then $\mathfrak{F}(f*g) = (\mathfrak{F}f)(\mathfrak{F}g)$. We assume that the kernels $G_1(x), \ldots, G_D(x)$ for $g_1, \ldots, g_D$ are in $\mathfrak{M}$.

Let s be a function in $\mathfrak{M}$ whose $(J - 1)$'st derivative is also in $\mathfrak{M}$ and whose Fourier transform $\mathfrak{F}s$ never vanishes. If h is in $\mathfrak{H}$, let $Qh = s*h$. Then $\mathfrak{F}(Qh) = (\mathfrak{F}s)(\mathfrak{F}h)$ so h is uniquely determined by Qh and $Q:\mathfrak{H} \rightarrow \mathfrak{M}(\mathfrak{N}^*)$ is a linear injection. Furthermore, for $i = 1, \ldots, D$, g_i^* as defined by equation (1) is known[3] to be a bounded linear functional on $\mathfrak{H}$. Finally, $[\bar{g}^*, h] = \int_{-\infty}^{\infty} s^{(J-1)}(x - \xi)h(\xi)d\xi$ so by Schwarz's inequality $\bar{g}^*$ is also a bounded linear functional on $\mathfrak{H}$. Therefore Q is $\mathfrak{N}^*$-Cauchy continuous and $(Q,\mathfrak{H})$ is a quelling of $(\mathfrak{N}^*,\mathfrak{M})$. Let $s_-(x) = s(-x)$. Then if g in $\mathfrak{N}^*$ has kernel G, a change of order of integration in $[g, Qh]$ shows that $Q^*g = s_-*G$.

By the Plancherel theorem,[3] if m is in $Q(\mathfrak{H})$ then

$$||Q^{-1}m||^2 = \int_{-\infty}^{\infty} (\mathfrak{F}m)^2(\mathfrak{F}s)^{-2}dk \tag{6}$$

so with convolution quelling the *a priori* bound, equation (3), is a restriction on the amount of "energy" (squared amplitude) in m's Fourier transform at high wave numbers.

Convolution quelling can be applied to the unit interval by using a function s which is $(J - 1)$ times continuously differentiable on $\mathbf{R}$ and periodic with unit period. Then $Qm = s*m$ is defined as $\int_0^1 s(x - \xi)m(\xi)d\xi$ and Fourier coefficients replace the Fourier transform. In this case $Q(\mathfrak{H})$ consists of functions m which are $(J - 1)$ times continuously differentiable on $[0, 1]$ and satisfy $m^{(j)}(0) = m^{(j)}(1), j = 0, 1, \ldots, J - 1$, so the quelling $(Q,\mathfrak{H})$ must be extended to be useful. One extension is obtained by taking $m_j(x) = x^j, j = 1, \ldots, J$.

5. **Other Prediction Problems.** If $P \geq 1$ and $\{g_1, \ldots, g_D, \bar{g}_1, \ldots, \bar{g}_P\}$ involve $\delta^{(J_1-1)}(x - x_1), \ldots, \delta^{(J_t-1)}(x - x_t)$ then Quelling Method 1 in Section 4 will fail, but Methods 2 and 3 work if $J \geq \max\{J_1, \ldots, J_t\}$. Method 1 can be repaired by defining $Qh = (x - x_1)^{J_1} \ldots (x - x_t)^{J_t}h(x)$.

When $G_i = \delta(x - x_i), i = 1, \ldots, D$ in Section 4, then that section gives three different ways of estimating $[\tilde{g}, m]$ from $m(x_1), \ldots, m(x_D)$. If $\tilde{G}(x) = 1$, $[\tilde{g}, m] = \int_0^1 m(x)dx$, and if $\tilde{G}(x) = (-1)^{J-1}\delta^{(J-1)}(x - x_o)$ then $[\tilde{g}, m] = m^{(J-1)}(x_o)$, so we have three different methods of numerical integration, differentiation, extrapolation, and interpolation. All three quellings on [0, 1] must be extended to be useful, and with Method 1 when $G_i = \delta(x - x_i)$, $i = 1, \ldots, D$, the extension requires D functions $m_1, \ldots, m_D$ and $m^{(J-1)}(x_o)$ or $\int_0^1 m(x)dx$ is estimated simply by fitting a linear combination of these D functions to the D data points. Even in Method 1, however, a bound on the error in this estimate is obtained which is applicable to a larger class of functions m than that usually considered in numerical analysis, and Methods 2 and 3 appear to be new ways to perform numerical integration, differentiation, interpolation and extrapolation.

All three quelling methods can be generalized to models m which are functions defined on subsets of $\mathbf{R}^N$. Method 1 needs no modification, nor does Method 3 for functions defined on all of $\mathbf{R}^N$ or on the surface of the unit sphere in $\mathbf{R}^N$. The generalization of Method 2 takes Q^{-1} to be a differential operator. These generalizations will be discussed in detail elsewhere.

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