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Numerical Applications of a Formalism for Geophysical Inverse Problems

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Summary

A gross datum of the Earth is a single measurable number describing some property of the whole Earth, such as mass, moment of inertia, or the frequency of oscillation of some identified elastic-gravitational normal mode. We prove that the collection of Earth models which yield the physically observed values of any independent set of gross Earth data is either empty or infinite dimensional. We exploit this very high degree of non-uniqueness in real geophysical inverse problems to generate computer programs which iteratively produce Earth models to fit given gross Earth data and satisfy other criteria. We describe techniques for exploring the collection of all Earth models which fit given gross Earth data. Finally, we apply the theory to the normal modes of elastic-gravitational oscillation of the Earth.

1. Introduction

Any single number which describes some property of the whole Earth will be called a gross datum of the Earth. Examples of gross data are the Earth's mass, its moments of inertia, its Love numbers, the frequencies of oscillation of its elastic-gravitational normal modes, the quality factors (Q's) of those normal modes, the rotational splitting parameters of the normal modes, the travel time of S or P waves from a particular source to a particular receiver, and the coefficients in the spherical harmonic expansion of the Earth's external gravitational potential.

Human limitations are such that at any given epoch only a finite number of gross Earth data will have been measured. This paper is a discussion of the extent to which these finitely many gross data can be used to determine the Earth's internal structure. The general procedure to be described is applicable to any finite set of gross Earth data which vary continuously in response to variations in the Earth's internal structure. For concreteness and because of the authors' present interests, detailed calculations will be given only for the mass, the moment of inertia, and the frequencies of the elastic-gravitational normal modes.

It will be assumed that the Earth is invariant under all rigid rotations about its centre, that the angular velocity of steady rotation of the Earth is zero, and that the static stress field and dynamic stress–strain relation at every point in the Earth are isotropic. Each of these assumptions is false, but probably to an extent which only slightly perturbs the theoretical calculations of normal modes (Toksöz & Ben-Menahem 1963, Backus & Gilbert 1961, Hast 1958). The problem of determining the deep asphericities and anisotropies from measured gross Earth data is an inverse problem of the general sort to be discussed in the rest of this paper, but we shall not consider the details here.

The problem we shall consider is the following: suppose a non-rotating, spherical, isotropic Earth of radius a has density $\rho(r)$, bulk modulus $\kappa(r)$, and shear modulus $\mu(r)$, all functions only of r, the radial distance from the centre. Suppose a finite number J of gross Earth data $E_1, E_2, \ldots, E_J$ have been measured, and that these data depend only on the functions ρ, κ, μ. Given the observed values of $E_1, \ldots, E_J$, what can be said about the unknown functions ρ, κ, μ?

In carrying out the quantitative discussion, we shall use the radius of the Earth as the unit of length (6·371 megametres), the Earth's mean density $\bar{\rho}$ as a unit of density (5·517 g/cm^3), and $2\pi(\pi\bar{\rho}G)^{-\frac{1}{2}}$ as a unit of time (5844 seconds). Here G is Newton's universal constant of gravitation. Then the unit of velocity is 1·090 km/s and the unit of pressure or elastic modulus is 65·57 kilobars.

2. The nature of the inverse problem

An ordered triple of functions, $\mathbf{m} = (\rho, \kappa, \mu)$, defined on $0 \leqslant r \leqslant 1$, will be called an Earth model. For the moment we put no restrictions whatever on the functions ρ, κ, μ except that they be real-valued, piecewise continuous functions of r, the radial distance from the centre of the Earth. If $\mathbf{m} = (\rho, \kappa, \mu)$ and $\mathbf{m}' = (\rho', \kappa', \mu')$ are two Earth models and b and b' are two real numbers, we shall use the symbol $b\mathbf{m} + b'\mathbf{m}'$ to denote the Earth model $(b\rho + b'\rho', b\kappa + b'\kappa', b\mu + b'\mu')$. With linear combinations of Earth models being thus defined, Earth models can be thought of as points in an infinite-dimensional linear space $\mathfrak{M}$, the space of all conceivable Earth models. On $\mathfrak{M}$ we introduce an inner product as follows:

$$(\mathbf{m}, \mathbf{m}') = \int_0^1 dr\, r^2 [\rho(r)\rho'(r) + \kappa(r)\kappa'(r) + \mu(r)\mu'(r)], \tag{1}$$

and a norm $\|\mathbf{m}\| = (\mathbf{m}, \mathbf{m})^{\frac{1}{2}}$. With these definitions, $\mathfrak{M}$ becomes an inner product space, which can be completed to a Hilbert space (Riesz & Nagy 1955). In (1) three positive weighting functions which vary with r may be introduced if some particular shell of the Earth is to be emphasized. We use unit weights so as to emphasize all volumes of the Earth equally.

A gross datum of the Earth, such as the squared eigenfrequency of the normal mode ${}_0S_2$, is a real number E which can be calculated for each Earth model $\mathbf{m}$ in $\mathfrak{M}$. A gross datum can be thought of as a rule which assigns to every point $\mathbf{m}$ in $\mathfrak{M}$ (that is, to every Earth model) a real number $E(\mathbf{m})$. *In fine*, a gross Earth datum is a real-valued function on the space $\mathfrak{M}$. Since $\mathfrak{M}$ is itself a function space, real-valued functions on $\mathfrak{M}$ are commonly called functionals. In general, a gross Earth datum will be a non-linear functional on $\mathfrak{M}$. However, the mass and moment of inertia are clearly linear functionals on $\mathfrak{M}$: the mass is

$$E_1 = 4\pi \int_0^1 dr\, r^2 \rho(r)$$

and the moment of inertia is

$$E_2 = \frac{8\pi}{3} \int_0^1 dr\, r^4 \rho(r).$$

If measurement shows that the value of a particular gross Earth datum E for the real Earth is E^0, then we know that the point $\mathbf{m}_e$ in $\mathfrak{M}$ which describes the real Earth lies on the hypersurface in $\mathfrak{M}$ defined by the equation

$$E(\mathbf{m}) = E^0. \tag{2}$$

If E is a linear functional, the hypersurface (2) will be a hyperplane; this hyperplane will contain $\mathfrak{M}$'s origin [the Earth model (0, 0, 0)] if and only if $E^0=0$. If E is non-linear, in general (2) will be a curved surface. If we have measured J gross Earth data for the real Earth and found their values to be $E_j^0, j=1, \ldots, J$, then we know that the point $\mathbf{m}_e$ which describes the real Earth lies on the intersection in $\mathfrak{M}$ of the J hypersurfaces

$$E_j(\mathbf{m})=E_j^0, \quad j=1, \ldots, J. \tag{3}$$

From the measurements $E_1^0, \ldots, E_J^0$ we can deduce nothing whatever about the Earth, except what follows from the J equations (3).

The inverse problem for the gross Earth data $E_1, \ldots, E_J$ consists in trying to understand what is the totality of Earth models which lie on all the J hypersurfaces (3). If we are fortunate or shrewd in our choice of which J gross data to measure, then all the different Earth models which satisfy (3) may share some common property. For example, they may all have a low-velocity zone in the upper mantle; or they may all become essentially the same when we take running averages of their ρ, κ and μ over some fixed depth interval H. In the first example, we can definitely assert that the Earth has a low-velocity zone in the upper mantle. In the second example, we can claim to know ρ, κ and μ as functions of radius r, except for unresolved details whose vertical length scale is H or less.

Of course, if $E_1^0, \ldots, E_J^0$ are real measurements they will contain errors of observation (such as incorrect identification of a mode) and will be perturbed by the rotation, asphericity, and anisotropy of the Earth. Therefore in principle it is possible that the set of Earth models which satisfy equations (3) will be empty, that is, that no spherical Earth model $\mathbf{m}$ is capable of producing the observed values of the gross Earth data $E_1, \ldots, E_J$. As far as we know, all of the theoretical work so far published on the inverse normal mode problem has been addressed to the question of whether equations (3) have even one solution $\mathbf{m}$ when $E_1^0, \ldots, E_J^0$ are the squared raw frequencies obtained from the real Earth and uncorrected for rotation, asphericity, and anisotropy. As yet no exact solutions have been published. The published Earth models have maximum errors between 0·5 and 1·5% for the normal modes with periods longer than about 300 s (Pekeris & Jarosch 1958, Alterman *et al.* 1959, Sato *et al.* 1960, Pekeris *et al.* 1961a, Bolt & Dorman 1961, MacDonald & Ness 1961, Alsop 1963a, Landisman *et al.* 1965). Slichter (1967) believes that many of the longer periods of spheroidal modes are now measured with an error no larger than 0·1%.

3. Non-uniqueness in the inverse problem

The linear space $\mathfrak{M}$ of all conceivable Earth models $\mathbf{m}$ is infinite dimensional, and equations (3) place only finitely many restrictions on $\mathbf{m}$. Intuitively it is reasonable that if equations (3) have any solution at all, they will have an infinite-dimensional (usually curved) manifold of solutions. That this intuitive argument needs some examination is indicated by the following example: for any $\mathbf{m}=(\rho, \kappa, \mu)$ define

$$E(\mathbf{m})=4+\int_0^1 dr\, r^2[(\rho-3)^2+(\kappa-2)^2+(\mu-1)^2]. \tag{4}$$

Then the equation $E(\mathbf{m})=4$ imposes only one condition on the infinite-dimensional space of $\mathbf{m}$'s, and yet there is exactly one $\mathbf{m}$ which satisfies that condition, namely $\mathbf{m}=(\rho, \kappa, \mu)$ with $\rho(r)=3$, $\kappa(r)=2$, $\mu(r)=1$ for $0 \leqslant r \leqslant 1$. To suppress pathologies like (4) we appeal to Fréchet differentiability. A functional E on the inner-product space $\mathfrak{M}$ is Fréchet-differentiable at the point $\mathbf{m}$ in $\mathfrak{M}$ if there exists a member $\mathscr{M}$ of $\mathfrak{M}$, determined by E and $\mathbf{m}$, such that for any member $\delta\mathbf{m}$ of $\mathfrak{M}$

$$E(\mathbf{m}+\delta\mathbf{m})=E(\mathbf{m})+(\mathscr{M}, \delta\mathbf{m})+\varepsilon(\delta\mathbf{m}), \tag{5}$$

where $\varepsilon(\delta\mathbf{m})||\delta\mathbf{m}||^{-1}$ approaches zero uniformly as $\delta\mathbf{m}$ approaches zero. We shall call $\mathcal{M}$ the differential kernel of E at $\mathbf{m}$. In our space $\mathfrak{M}$ of function triples, Fréchet differentiability of E would require the existence of a function triple $\mathcal{M}=(P, K, M)$ such that if $\delta\mathbf{m}=(\delta\rho, \delta\kappa, \delta\mu)$ then correct to first order in the arbitrary small perturbations $\delta\rho$, $\delta\kappa$, $\delta\mu$ we have

$$E(\mathbf{m}+\delta\mathbf{m})=E(\mathbf{m})+\int_0^1 dr\, r^2[P\delta\rho+K\delta\kappa+M\delta\mu]. \tag{6}$$

Clearly mass is a Fréchet-differentiable gross Earth datum with $\mathcal{M}=(4\pi, 0, 0)$ and moment of inertia is a Fréchet-differentiable gross Earth datum with $\mathcal{M}=(8\pi r^2/3, 0, 0)$. The travel time for P or S waves from any particular source to any particular receiver is a Fréchet-differentiable gross Earth datum, as is evident from the expression for the travel time as an integral along the path of propagation. (The kernel $\mathcal{M}$ in this case is singular, being integrable but not square integrable; this requires some inessential changes in the formalism.) It is shown in Appendix B that the squared frequency of any normal mode of elastic-gravitational oscillation is a Fréchet-differentiable gross Earth datum. In that appendix, the kernel functions $P(r)$, $K(r)$, $M(r)$ in (6) are given explicitly as expressions quadratic in the eigenfunctions for the normal mode. Rotational splitting parameters are also Fréchet-differentiable gross Earth data, as is evident from the explicit expressions which have been derived for them (Backus & Gilbert 1961, Pekeris *et al.* 1961b, MacDonald & Ness 1961). Finally, as shown by Anderson & Archambeau (1964), quality factors Q expressing the damping of the normal modes are Fréchet-differentiable gross Earth data if the bulk and shear moduli are permitted to be complex and frequency-dependent. In short, all the gross Earth data which have so far received geophysical attention are Fréchet-differentiable. From a practical point of view the restriction to Fréchet-differentiable gross data is not severe.

Now suppose that $\mathbf{m}_0$ is a solution of (3) and that the gross Earth data $E_1, \ldots, E_J$ are Fréchet-differentiable at and near $\mathbf{m}_0$, with differential kernels $\mathcal{M}_1^{\,0}, \ldots, \mathcal{M}_J^{\,0}$ at $\mathbf{m}_0$. Suppose further that $\mathcal{M}_1^{\,0}, \ldots, \mathcal{M}_J^{\,0}$ are linearly independent. Then (Halmos 1958) we can find J linear combinations of $\mathcal{M}_1^{\,0}, \ldots, \mathcal{M}_J^{\,0}$, say $\mathbf{m}_1, \ldots, \mathbf{m}_J$, such that

$$(\mathcal{M}_j^{\,0}, \mathbf{m}_k)=\delta_{jk} \tag{7}$$

when $j, k=1, \ldots, J$. Here δ_{jk} is the Kronecker delta. Since $\mathfrak{M}$ is infinite-dimensional, for any finite K larger than J we can find $K-J$ Earth models $\mathbf{m}_k$, with $k = J+1, \ldots, K$, such that the K Earth models $\mathbf{m}_1, \ldots, \mathbf{m}_k$ are linearly independent. Let $\alpha_1, \ldots, \alpha_K$ be arbitrary real numbers and define

$$\mathbf{m}=\mathbf{m}_0+\sum_{k=1}^{K}\alpha_k\,\mathbf{m}_k. \tag{8}$$

We shall show that within the K-parameter family of Earth models (8) there is contained a family with $K-J$ independent parameters (or dimensions) which satisfies (3) exactly when the parameters are sufficiently small. The proof is quite simple. When $\mathbf{m}$ is given by (8), equations (3) are J equations for the K unknowns $\alpha_1, \ldots, \alpha_K$. When $\alpha_1=\ldots=\alpha_K=0$, then for any j between 1 and J and any k between 1 and K we have

$$\frac{\partial E_j}{\partial\alpha_k}=(\mathcal{M}_j^{\,0}, \mathbf{m}_k). \tag{9}$$

Thus, according to (7), the $J\times J$ matrix $\partial E_j/\partial\alpha_k$, with $j, k=1, \ldots, J$, is non-singular (it is, in fact, the $J\times J$ identity matrix) when $\alpha_1=\ldots=\alpha_K=0$. It follows from a standard theorem in implicit function theory (Graves 1946) that for any sufficiently small $\alpha_{J+1}, \ldots, \alpha_K$ there is exactly one set of values of $\alpha_1, \ldots, \alpha_J$ such that the $\mathbf{m}$ defined by (8) satisfies (3) exactly. Therefore $\alpha_{J+1}, \ldots, \alpha_K$ are the $K-J$ independent parameters of a $K-J$ dimensional family of solutions (8) of (3). Since K can be arbitrarily large, we are justified in asserting that the totality of solutions of (3) near any solution $\mathbf{m}_0$ is infinite-dimensional if the differential kernels $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$ at $\mathbf{m}_0$ are linearly independent.

Without this hypothesis of linear independence serious complications ensue which we shall not examine, but which can lead to failure of the argument. In the numerical examples to be discussed in this paper we have computed $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$ explicitly and verified their linear independence by Gram-Schmidt orthogonalizing them. Physically, when $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$ are linearly dependent the gross Earth data $E_1, \ldots, E_J$ are not independent, at least to first order in variations from the Earth model $\mathbf{m}_0$. Such a special situation is possible, but rare. The more usual situation is that $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$ are almost but not quite linearly dependent. That is, the determinant of the $J \times J$ matrix $(\mathscr{M}_j{}^0, \mathscr{M}_k{}^0)$ is not zero but is very much smaller than the product of its diagonal elements. This approximate linear dependence calls for great care in numerical calculations and error estimates, but does not affect the validity of the foregoing proof of non-uniqueness.

Geometrically, the linear dependence of $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$ means that one of the J hypersurfaces (3) is tangent at $\mathbf{m}_0$ to some linear combination of the others. Examination of the case $J=2$ makes clear that this tangency permits the set of solutions of (3) to be infinite dimensional or finite dimensional or to consist of $\mathbf{m}_0$ alone.

One physical reason for the infinite-dimensional non-uniqueness of the inverse problem (3) becomes clear on examining (6). Only finitely many gross Earth data $E_1{}^0, \ldots, E_J{}^0$ have been measured. There is a length H such that none of the kernels P_j, K_j, M_j, with $j=1, \ldots, J$, changes appreciably in a length H. Then if to ρ, κ, μ we add perturbations $\delta\rho$, $\delta\kappa$, $\delta\mu$ whose Fourier series contain no wavelengths greater than H, the perturbations produced in $E_1, \ldots, E_J$ by $\delta\rho$, $\delta\kappa$, $\delta\mu$ will be extremely small. With only finitely many gross data we cannot expect to resolve details of arbitrarily small vertical scale; our vertical resolution is finite. This remark is sufficiently tautological to be without much geophysical interest. A geophysically more interesting question is whether there is any other source of the non-uniqueness in (3) besides the finite resolving power inherent in a finite set of gross Earth data. We shall see that in general there is.

Suppose that for the J gross Earth data $E_1, \ldots, E_J$ we have found an exact solution $\mathbf{m}_0$ of (3). Suppose that $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$, the differential kernels of $E_1, \ldots, E_J$ at $\mathbf{m}_0$, are linearly independent. Simply to know that there are other exact solutions of (3) is unsatisfying. We would like to know what they look like and how they differ from $\mathbf{m}_0$. One approach to this problem is as follows: Let $\mathbf{n}$ be any Earth model orthogonal to all of $\mathscr{M}_1{}^0, \ldots, \mathscr{M}_J{}^0$; that is, for any j among $1, \ldots, J$, suppose

$$(\mathscr{M}_j{}^0, \mathbf{n}) = 0. \tag{10}$$

Then according to (5) if β is any small number the Earth model $\mathbf{m}_0 + \beta\mathbf{n}$ is a solution of (3), at least to first order in β. It is not generally an exact solution unless $\beta = 0$.

We can find an exact solution which is nearly $\mathbf{m}_0 + \beta\mathbf{n}$ when β is small. To start the procedure, let $\mathbf{m}_1{}^0, \ldots, \mathbf{m}_J{}^0$ be chosen to satisfy (7). We have seen that for any sufficiently small β there is exactly one solution of (3) which has the form

$$\mathbf{m} = \mathbf{m}_0 + \sum_{k=1}^{J} \alpha_k \mathbf{m}_k{}^0 + \beta\mathbf{n}. \tag{11}$$

Since this solution is determined by β, the coefficients α_j are functions of β, as is the solution $\mathbf{m}$ itself. For any β, let $\mathbf{m}(\beta)$ be the solution of (3) which has the form (11).

Let $\mathscr{M}_j(\beta)$ be the differential kernel of E_j at $\mathbf{m}(\beta)$, so $\mathscr{M}_j(0)=\mathscr{M}_j^0$. If we change β to $\beta+\delta\beta$ we will change $\mathbf{m}$ to $\mathbf{m}+\delta\mathbf{m}$ where, from (11),

$$\delta\mathbf{m}=\sum_{k=1}^{J}\mathbf{m}_k{}^0\delta\alpha_k+\mathbf{n}\delta\beta \tag{12}$$

and $\delta\alpha_k=(d\alpha_k/d\beta)\delta\beta+O[(\delta\beta)^2]$. Then $E_j(\mathbf{m})$ will change to $E_j(\mathbf{m}+\delta\mathbf{m})$ where, according to (5),

$$E_j(\mathbf{m}+\delta\mathbf{m})=E_j(\mathbf{m})+(\mathscr{M}_j(\beta),\delta\mathbf{m})+\varepsilon\|\delta\mathbf{m}\|. \tag{13}$$

If both $\mathbf{m}$ and $\mathbf{m}+\delta\mathbf{m}$ are exact solutions of (3), then from (12) and (13) we have

$$\sum_{k=1}^{J}(\mathscr{M}_j(\beta),\mathbf{m}_k{}^0)\delta\alpha_k+(\mathscr{M}_j(\beta),\mathbf{n})\delta\beta+\varepsilon\|\delta\mathbf{m}\|=0.$$

Dividing by $\delta\beta$ and letting $\delta\beta$ approach zero, we have

$$\sum_{k=1}^{J}(\mathscr{M}_j(\beta),\mathbf{m}_k{}^0)\frac{d\alpha_k}{d\beta}=-(\mathscr{M}_j(\beta),\mathbf{n}), \tag{14}$$

with $j=1,\ldots,J$.

Assume that near $\mathbf{m}_0$ the differential kernels $\mathscr{M}_j$ depend continuously on $\mathbf{m}$. Then the $J\times J$ matrix $(\mathscr{M}_j(\beta),\mathbf{m}_k{}^0)$ depends continuously on β. Since it is the identity matrix when $\beta=0$, it is non-singular for all sufficiently small β. Therefore for all sufficiently small β, equations (14) can be solved for $d\alpha_k/d\beta$. The numerical procedure for constructing the one-parameter family $\mathbf{m}(\beta)$ of exact solutions of (3) is now evident. We start with $\mathbf{m}(0)=\mathbf{m}_0$, and $\alpha_1=\ldots=\alpha_J=0$ in (11). Using $\mathscr{M}_j(0)$ we find $d\alpha_k/d\beta$ at $\beta=0$ from (14), and then take a small step $\Delta\beta$ in β. With $\Delta\alpha_j=\Delta\beta(d\alpha_j/d\beta)$ we construct $\mathbf{m}(\Delta\beta)$ from (11). For $\mathbf{m}(\Delta\beta)$ we find the differential kernels $\mathscr{M}_j(\Delta\beta)$ as described in Appendix B. Then we use (14) to obtain $d\alpha_k/d\beta$ at $\Delta\beta$, and proceed to another step in β. Notice that the curve $\mathbf{m}(\beta)$ thus constructed in the space $\mathfrak{M}$ does have the form $\mathbf{m}_0+\beta\mathbf{n}+O(\beta^2)$ when β is small, because, according to (10), (11), and (14), at $\beta=0$ we have $\alpha_1=\ldots=\alpha_J=d\alpha_1/d\beta=\ldots=d\alpha_J/d\beta=0$.

The result of the foregoing discussion can be summarized as follows: let $\mathscr{M}_1{}^0,\ldots,\mathscr{M}_J{}^0$ be the differential kernels of $E_1,\ldots,E_J$ at $\mathbf{m}_0$, assumed linearly independent. Let $\mathbf{n}$ be any Earth model orthogonal to all of $\mathscr{M}_1{}^0,\ldots,\mathscr{M}_J{}^0$ in the sense of (10). Then there is a unique one-parameter family $\mathbf{m}(\beta)$ of Earth models (a unique curve in the space $\mathfrak{M}$) which starts at $\mathbf{m}_0$ when $\beta=0$ and leaves $\mathbf{m}_0$ in the direction described by $\mathbf{n}$.

The key relations in these remarks are (10). Any model near $\mathbf{m}_0$ which satisfies (3) exactly has the form $\mathbf{m}_0+\mathbf{n}+$terms of second order in $\mathbf{n}$, where $\mathbf{n}$ is an Earth model satisfying (10). Conversely, any Earth model $\mathbf{n}$ satisfying (10) generates a one-parameter family of exact solutions of (3) which, for small β, have the form $\mathbf{m}_0+\beta\mathbf{n}+O(\beta^2)$.

4. Resolving power in the inverse problem

It is now possible to give a quantitative discussion of the question of resolving power. We define the mean vertical wave number k_f of a function $f(r)$ by the equation

$$k_f{}^2=\frac{\int_0^1 dr\,r^2(df/dr)^2}{\int_0^1 dr\,r^2f^2}. \tag{15}$$

(A definition in terms of the Bessel, Legendre, Chebyshev or Fourier expansion coefficients of f involves less computing in what follows. We use (15) because of its formal simplicity and intuitive appeal.) Then the resolving power with which a gross Earth datum E looks at the density ρ is simply the mean vertical wave number of $P(r)$ in the differential kernel of E at $\mathbf{m}_0$. This idea could be elaborated by considering mean wave numbers in different regions, but we content ourselves here with the coarser discussion.

If we assume that the P and S wave velocities, $v_P(r)$ and $v_S(r)$, are known exactly, so that the inverse problem is to find the density ρ from (3), then we must recall that $\kappa=\rho(v_P{}^2-\frac{4}{3}v_S{}^2)$ and $\mu=\rho v_S{}^2$, so that, from (6),

$$\delta E_j=\int_0^1 dr\, r^2\, \tilde{P}_j \delta\rho,$$

where

$$\tilde{P}_j=P_j+K_j(v_P{}^2-\tfrac{4}{3}v_S{}^2)+M_j v_S{}^2. \tag{16}$$

Then the resolving power of a gross Earth datum E is the mean vertical wave number of $\tilde{P}$.

In the rest of this section, we will in fact assume that v_P and v_S are known, in order to simplify the discussion. The required modifications for the more general problem will be apparent. We point out that all the results about non-uniqueness proved hitherto in this paper apply *in toto* to the case where v_P and v_S are known and only ρ is sought.

We have defined the resolving power of a single gross Earth datum. Now we must define the resolving power of a finite set of gross Earth data $E_1, \ldots, E_J$ at a point $\mathbf{m}_0$ in $\mathfrak{M}$ where their differential kernels are $\tilde{P}_1, \ldots, \tilde{P}_J$ [see (16)]. One definition which at first looks reasonable is as follows: if we have available measurements of $E_1, \ldots, E_J$, then we have available a measurement of the gross Earth parameter

$$E=\sum_{j=1}^{J}\alpha_j E_j,$$

where $\alpha_1, \ldots, \alpha_J$ are any constants whatever. The differential kernel of E at $\mathbf{m}_0$ is

$$\tilde{P}=\sum_{j=1}^{J}\alpha_j \tilde{P}_j. \tag{17}$$

If $k_{\tilde{P}}$ is the mean wave number of P then this resolving power is available to us from our data. The best resolving power available from all our data is the largest value that $k_{\tilde{P}}$ can take as $\alpha_1, \ldots, \alpha_J$ in (17) take all their possible values. This maximum we will call $K(E_1, \ldots, E_J)$. It seems at first to be a reasonable definition of the resolving power of our data. For normal modes of oscillation it corresponds to the assertion that we can expect to see unambiguous detail in ρ whose vertical wavelength is approximately the shortest wavelength of any of the squared wave functions or linear combinations of squared wave functions whose eigenfrequencies we have measured. It is easy to compute $K^2(E_1, \ldots, E_J)$, since it is λ, the largest root of the equation

$$\det(A_{ij}-\lambda B_{ij})=0$$

where

$$A_{ij}=\int_0^1 dr\, r^2 \frac{d\tilde{P}_i}{dr}\frac{d\tilde{P}_j}{dr}$$

and

$$B_{ij}=\int_0^1 dr\, r^2\, \tilde{P}_i \tilde{P}_j.$$

Furthermore, lower bounds for $K(E_1, \ldots, E_J)$ are easily obtained. The mean wave number of any particular linear combination

$$\sum_{j=1}^{J} \alpha_j \tilde{P}_j$$

is such a lower bound. Estimates of such lower bounds can be obtained by Gram-Schmidt orthogonalizing the sequence $\tilde{P}_1, \ldots, \tilde{P}_J$ relative to the inner product (1) and inspecting the wavelengths of the oscillations in the resulting functions. In other words, we expect detail in ρ to be meaningful if its vertical scale is not smaller than the shortest wavelength available in the Gram–Schmidt orthogonalization of $\tilde{P}_1, \ldots, \tilde{P}_J$.

In general this expectation will be disappointed. A little consideration shows that the real resolving power of our data is obtained from a second definition. Suppose we have a density $\rho_0(r)$ which satisfies (3) exactly. Among all perturbations $\delta\rho$ which can be added to ρ_0 without affecting the validity of (3), which is that one, $f(r) = \delta\rho$, whose mean vertical wave number k_f is least (whose mean wavelength is longest)? Since, to first order in β, $\rho_0 + \beta f$ satisfies (3), we cannot tell from our data how much of f is present in the true ρ. And among all such undetectable perturbations $\delta\rho$ in ρ_0, f has the longest mean wavelength. Clearly we want to know what f is, and what k_f is. Equally clearly, the real resolving power of our data is not $K(E_1, \ldots, E_J)$ but k_f.

The demand that $\rho_0 + \beta f$ satisfy (3) to first order in β is simply the demand that for $j = 1, \ldots, J$ we have

$$\int_0^1 dr\, r^2 \tilde{P}_j f = 0. \tag{18}$$

Thus the problem is to find that f which minimizes the k_f of (15) subject to the constraints (18). This is a classical variational problem. The method of Lagrange multipliers shows that there are Lagrange multipliers λ, $\alpha_1, \ldots, \alpha_J$ such that

$$r^{-2} \frac{d}{dr} r^2 \frac{d}{dr} f + \lambda f = \sum_{j=1}^{J} \alpha_j \tilde{P}_j \tag{19}$$

while f is finite at $r = 0$, and

$$\frac{df}{dr} = 0 \tag{20}$$

at $r = 1$. For any fixed λ let f_j be chosen so as to be finite at $r = 0$, to satisfy (20) at $r = 1$, and to satisfy

$$r^{-2} \frac{d}{dr} r^2 \frac{d}{dr} f_j + \lambda f_j = \tilde{P}_j$$

in $0 \leqslant r \leqslant 1$. Then the solution of (19) and (20) is

$$f = \sum_{j=1}^{J} \gamma_j f_j$$

where the constants γ_j are determined by the requirements

$$\int_0^1 dr\, r^2 f^2 = 1 \tag{21}$$

and, for $k = 1, \ldots, J$,

$$\sum_{j=1}^{J} \gamma_j \int_0^1 dr\, r^2 f_j \tilde{P}_k = 0. \tag{22}$$

The existence of a nonzero solution $\gamma_1, \ldots, \gamma_J$ of (22) is a condition on λ, namely

$$\det\left(\int_0^1 dr\, r^2 f_j \tilde{P}_k\right) = 0.$$

The smallest λ which satisfies this condition is the desired minimum k_f^2, as can be seen on multiplying (19) by $r^2 f$, integrating from 0 to 1, and applying (15), (18) and (21).

We will see from a numerical example later in this paper that k_f can indeed be considerably smaller than $K(E_1, \ldots, E_J)$. That is, the real resolving power of a finite set of normal modes can be considerably poorer than the resolving power estimated from the vertical wavelengths present in the eigenfunctions of the normal modes. In the light of the foregoing discussion, this result is not surprising. Speaking very loosely, what we know from $E_1^0, \ldots, E_J^0$ are the coefficients in the expansion

$$\rho_0 = \sum_{j=1}^{\infty} \alpha_j \tilde{P}_j(r),$$

where $\tilde{P}_1, \ldots, \tilde{P}_J, \tilde{P}_{J+1}, \ldots$ is some sort of a complete set on $0 \leqslant r \leqslant 1$ whose first members are $\tilde{P}_1, \ldots, \tilde{P}_J$. Among the remaining functions $\tilde{P}_{J+1}, \ldots$ may be some whose vertical wavelength is longer than the shortest of the wavelengths of $\tilde{P}_1, \ldots, \tilde{P}_J$. For example, the whole set might be $\sin n\pi r$, $n = 1, 2, \ldots$, and $\tilde{P}_j$ might be $\sin \pi(j+1) r$ for $j = 1, \ldots, J$. The procedure described above for calculating k_f and f is essentially a constructive procedure for finding the lowest such 'gap' among the vertical wave numbers made available by $\tilde{P}_1, \ldots, \tilde{P}_J$. The next gap can be computed by adjoining f to the set $\tilde{P}_1, \ldots, \tilde{P}_J$ and calculating a new function g which minimizes k_g^2 subject to (18) and the new constraint $\int_0^1 dr\, r^2 fg = 0$.

Some of the modifications made possible in the foregoing discussion by the presence of discontinuities in ρ, κ and μ will be discussed elsewhere.

5. Finding solutions to the inverse problem

Since the set of solutions $\mathbf{m}$ of (3) is usually infinite dimensional, any practical numerical technique for finding one such solution must somehow remove this serious ambiguity. When there are ν functions of radius to be determined and J Earth data available, previous systematic procedures for dealing with the ambiguity have involved subdividing a part or all of the Earth radially into J/ν or fewer shells in which the functions are assumed constant or linear (Dorman & Ewing 1962, Anderson & Archambeau 1964, Anderson 1967, Landisman *et al.* 1965, Verreault 1965b).

The question arises whether it is not possible to make more explicit use of the very large extent of the non-uniqueness in the inverse problem. For example, the ideas in Section 4 might be pursued. Given P and S velocity distributions $v_P(r)$ and $v_S(r)$, we might seek the smoothest $\rho(r)$ which solves equations (3). That is, we might seek to minimize

$$k_\rho^2 = \frac{\int_0^1 dr\, r^2 (d\rho/dr)^2}{\int_0^1 dr\, r^2 \rho^2} \tag{23}$$

subject to the constraints (3) on ρ. All the structural detail in such a smoothest solution would be real, in the sense that, roughly speaking, the shortest vertical wavelength present in such a solution would be longer than the longest vertical wavelength present in any acceptable perturbation to the solution. An iterative numerical technique can be set up, using Lagrange multipliers and the differential kernels of $E_1, \ldots, E_J$, which produces this smoothest solution of (3). The technique is essentially the same as another which we shall describe in detail later in this section, so we postpone the details.

Another use to which the non-uniqueness of the inverse problem can be put is to test the validity of guesses about the internal structure of the Earth. Suppose again that $v_P(r)$ and $v_S(r)$ are known, and that a guess $\rho_G(r)$ is put forth concerning the internal density of the Earth. The model $\mathbf{m}_G = (\rho_G, \kappa_G, \mu_G)$ with $\kappa_G = \rho_G(v_P{}^2 - \frac{4}{3}v_S{}^2)$ and $\mu_G = \rho_G v_S{}^2$ may not solve (3). Yet we can always ask, what is the model $\mathbf{m}$ closest to $\mathbf{m}_G$ in the least squares sense which does solve (3). That is, we can seek to minimize

$$||\mathbf{m} - \mathbf{m}_G||^2 = \int_0^1 dr\, r^2 [(\rho - \rho_G)^2 + (\kappa - \kappa_G)^2 + (\mu - \mu_G)^2] \tag{24}$$

subject to the constraints (3). In general this problem will have a unique solution $\mathbf{m}$, and if $\mathbf{m}_G$ is at all close to $\mathbf{m}$ then $\mathbf{m}$ can be found numerically by a simple iterative procedure. This particular application of the non-uniqueness of the solutions of (3) produces a simple, systematic scheme for generating exact solutions of (3). Every reasonable guess will generate such a solution. The scheme is very close to the numerical procedure we have used to generate solutions of (3), so we describe it in some detail.

Suppose, for generality, that we are trying to obtain ρ, κ, μ from $E_1{}^0, \ldots, E_J{}^0$. When $v_P(r)$ and $v_S(r)$ are already known the required modifications in our procedure will be obvious, and we have in fact used both procedures. If $\mathbf{m}^{(n)}$ is a good approximation to an exact solution $\mathbf{m}$ of (3) we may write, to first order in $\mathbf{m} - \mathbf{m}^{(n)}$,

$$E_j(\mathbf{m}) = E_j(\mathbf{m}^{(n)}) + (\mathscr{M}_j{}^{(n)}, \mathbf{m} - \mathbf{m}^{(n)}) \tag{25}$$

for $j = 1, \ldots, J$. Here $\mathscr{M}_j{}^{(n)}$ is the differential kernel of E_j at $\mathbf{m}^{(n)}$. Then to first order in $\mathbf{m} - \mathbf{m}^{(n)}$ we have, from (3) and (25),

$$(\mathscr{M}_j{}^{(n)}, \mathbf{m} - \mathbf{m}^{(n)}) = E_j{}^0 - E_j(\mathbf{m}^{(n)}) \tag{26}$$

for $j = 1, \ldots, J$. Thus we seek to minimize (24) subject to the constraints (26). Introducing Lagrange multipliers α_j, we conclude that

$$\mathbf{m} - \mathbf{m}_G = \sum_{k=1}^{J} \alpha_k \mathscr{M}_k{}^{(n)}, \tag{27}$$

and that the α_k are determined from (26) as the solution of the linear system

$$\sum_{k=1}^{J} \alpha_k (\mathscr{M}_j{}^{(n)}, \mathscr{M}_k{}^{(n)}) = E_j{}^0 - E_j(\mathbf{m}^{(n)}) - (\mathscr{M}_j{}^{(n)}, \mathbf{m}_G - \mathbf{m}^{(n)}) \tag{28}$$

for $j = 1, \ldots, J$. When we have solved this system we define

$$\mathbf{m}^{(n+1)} = \mathbf{m}_G + \sum_{k=1}^{J} \alpha_k \mathscr{M}_k{}^{(n)}.$$

If (25) were exact, we would have $E_j(\mathbf{m}^{(n+1)}) = E_j{}^0$, and $\mathbf{m}^{(n+1)}$ would be the exact solution to our problem. Because of the non-linearity of (3), usually we will have only

$$|E_j(\mathbf{m}^{(n+1)}) - E_j{}^0| \ll |E_j(\mathbf{m}^{(n)}) - E_j{}^0|,$$

so that $\mathbf{m}^{(n+1)}$ will be more nearly a solution of (3) than is $\mathbf{m}^{(n)}$. It seems likely that fairly weak hypotheses on the continuity of the differential kernels of $E_1, \ldots, E_J$ as functions of $\mathbf{m}$ will suffice to ensure that if $\mathbf{m}_G$ is sufficiently close to the solution $\mathbf{m}$ of our problem and if we take $\mathbf{m}^{(0)}=\mathbf{m}_G$, then as n approaches infinity $\mathbf{m}^{(n)}$ converges to $\mathbf{m}$.

The numerical procedure we have actually used to generate solutions of (3) is like the foregoing except that at the nth stage we try to minimize $\mathbf{m}-\mathbf{m}^{(n)}$ rather than $\mathbf{m}-\mathbf{m}_G$, subject to the constraints (26). In all of the numerical work to be reported here we have assumed that the Earth had a radius of 6371 km, consisted of a fluid core of radius 3473 km and outside the core a solid mantle. We have described any Earth model numerically by giving the values of ρ, κ and μ at 66 core points, 98 mantle points, and 4 crust points spaced for Gauss–Legendre integration, and we have obtained intermediate values of ρ, κ, and μ by linear interpolation. Thus we have in fact confined our attention to a 438 dimensional subspace of $\mathfrak{M}$. Since we always have $J \ll 438$, this limitation affects none of the foregoing arguments. The problem attempted has been to find ρ and κ in the core and ρ, κ, and μ in the mantle from (3).

By introducing into the inner products (1) and (24) and the differential kernels $\mathscr{M}$ terms involving the depths of the discontinuities in ρ and κ, it is possible to let the depths of the discontinuities as well as their magnitudes be determined by the data in (3). The necessary formalism is presented in Appendix C. In this paper the depths of the discontinuities were assumed to be known exactly, and only their magnitudes were determined by (3). In a subsequent paper we will report on calculations in which a solid inner core was used and the depths as well as the magnitudes of the discontinuities were assumed at the outset to be unknown.

There is, of course, no reason in principle why one must postulate discontinuities in trying to solve (3). However, when all the observational data in (3) are at very low frequencies and when $\mathbf{m}_G$ is continuous it seems most unlikely that among the infinitely many solutions of (3) the iteration procedure will select a discontinuous one. High-frequency data assure us that the real Earth has at least one discontinuity on the scale on which we work here. Ideally we would like to know the whole manifold (3), but limitations of time and ingenuity make it possible for us to explore only a small part of it. We have elected to try to arrange that the small part we do explore is near the real Earth, so we have explicitly introduced a discontinuity at the core–mantle boundary in our starting models $\mathbf{m}_G$.

6. Numerical experiments with raw data

In a first application of the foregoing procedure we took as gross Earth data E_j^0 the Earth's mass (Gutenberg 1959), the Earth's moment of inertia (Jeffreys 1963, King-Hele *et al.* 1964), and the squared frequencies observed by Slichter (1966) for ${}_0S_0$, ${}_0S_2$, ${}_0S_3, \ldots, {}_0S_7$. Here ${}_nS_l$ denotes the $(n+1)$th term in a list of all spheroidal modes of angular order l, arranged in order of increasing frequency. Similarly ${}_nT_l$ denotes a toroidal mode. For $\mathbf{m}_G$ we used Gutenberg's v_P and v_S as reported by Bullard (1957), smoothed so as to have no low velocity zone, and Bullen's density model A, slightly modified to agree with recent satellite measurements of the moment of inertia. This $\mathbf{m}_G$ will be called the smoothed Gutenberg model in what follows. We performed seven successively more demanding calculations: in the first, only mass, moment, and ${}_0S_0$ were used; in the lth we used mass, moment, ${}_0S_0$, ${}_0S_2, \ldots, {}_0S_l$. All of ρ, κ, μ were regarded as unknown. In each of the seven calculations only one iteration was performed. The quality of the Earth model $\mathbf{m}$ so produced was measured by its r.m.s. error $\varepsilon_{\mathbf{m}}$, defined by

$$(J-2)\,\varepsilon_{\mathbf{m}}^2 = \sum_{j=3}^{J} (1 - E_j(\mathbf{m})/E_j^0)^2.$$

Since E_1^0 and E_2^0, the mass and moment, were always fitted exactly, we did not include them in our measurement of error. Table 1 gives the r.m.s. errors ε_0 and ε_1

Table 1

The r.m.s. errors ε_0 and ε_1 for the starting model (smoothed Gutenberg) and the first iterate, using as gross data the mass, moment, and Slichter's observed raw frequencies for ${}_0S_0$, ${}_0S_2$, ${}_0S_3$, …, ${}_0S_l$

l	0	2	3	4	5	6	7
$\varepsilon_0 \times 10^2$	1·1	1·0	1·0	1·0	0·9	1·0	1·0
$\varepsilon_1 \times 10^4$	2·6	1·7	3·1	17·0	25·0	320·0	—

for $\mathbf{m}_G$ and the first iterate in each of the seven calculations. At $l=7$ the first iterate for the density was negative and the computer program was unable to produce eigenfrequencies. That is, the procedure described in Section 5 was unable to produce a model to fit Slichter's data for ${}_0S_0$, ${}_0S_2$, ${}_0S_3$, …, ${}_0S_7$.

In a second experiment we used as Earth data mass, moment, and Slichter's (1966) observations of ${}_0S_0$, ${}_0S_{2q}$, $q = 2, 3, \ldots, 9$. We used only even angular orders so as to reach moderately high frequencies without excessive computer time, and we omitted ${}_0S_2$ as the mode most heavily contaminated by the ellipticity and rotation of the Earth. Again all of ρ, κ, μ were regarded as unknown. In the core we took for $\mathbf{m}_G$ the smoothed Gutenberg model already mentioned, while in the mantle we took velocity and density distributions of the form $A+Br^2$, with A and B chosen so that the mass and moment of inertia of the whole Earth were correct, while the corresponding moments of v_P and v_S agreed with those for the Gutenberg velocities. This $\mathbf{m}_G$ we will call 'quadratic Gutenberg'. Taking this $\mathbf{m}_G$ as a zeroth iterate we had, for the r.m.s. error ε_n at the nth stage of iteration the result given in Table 2. The r.m.s. error had dropped by the third iteration to a value which was below the estimated errors of observation (Slichter 1966) but it remained essentially unchanged through six further iterations. We regard this as an unsuccessful outcome.

Table 2

The r.m.s. error ε_n of the n-th iterate, starting with the quadratic Gutenberg model and using as gross data the mass, moment, and Slichter's observed raw frequencies for ${}_0S_0$, ${}_0S_4$, ${}_0S_6$, ${}_0S_8$, …, ${}_0S_{18}$

n	0	1	2	3	4	5	6	7	8	9
$\varepsilon_n \times 10^4$	200	8	11	5	4	4	4	6	3	4

The negative results of these two experiments on the real raw data imply one or more of the following conclusions:

(i) there is an undiscovered conceptual or numerical error in our iteration procedure;

(ii) the models were too far from the real Earth to permit the linearization (26);

(iii) the raw observational data cannot be obtained exactly from any spherical, isotropic, non-rotating Earth model of radius 6371 km with a fluid core of radius 3473 km inside a solid mantle.

The last possibility might be the result of contamination by asphericity, anisotropy and rotation, or the misidentification of one or more modes. It seems unlikely that any inner core mode was included in our list of observations, because their surface amplitudes are small (Alsop 1963b); but if an inner core mode was mistaken for a mantle mode, we might expect difficulty from our failure to include an inner core in our models.

7. Numerical experiments with artificial data

In the hope of understanding our lack of success with real data, we replaced the real gross Earth data by artificial gross data generated from a particular model Earth which we shall call Model I. It has the correct mass and moment, fits 56 of Slichter's (1966) spheroidal frequencies, including ${}_0S_0, {}_0S_2, {}_0S_3, \ldots, {}_0S_{10}$, with a maximum error of 1·3%, and fits the lowest ten ${}_0T$ frequencies (Benioff *et al.* 1961) with a maximum error of 0·9%. For all these 66 modes the r.m.s. error of Model I is

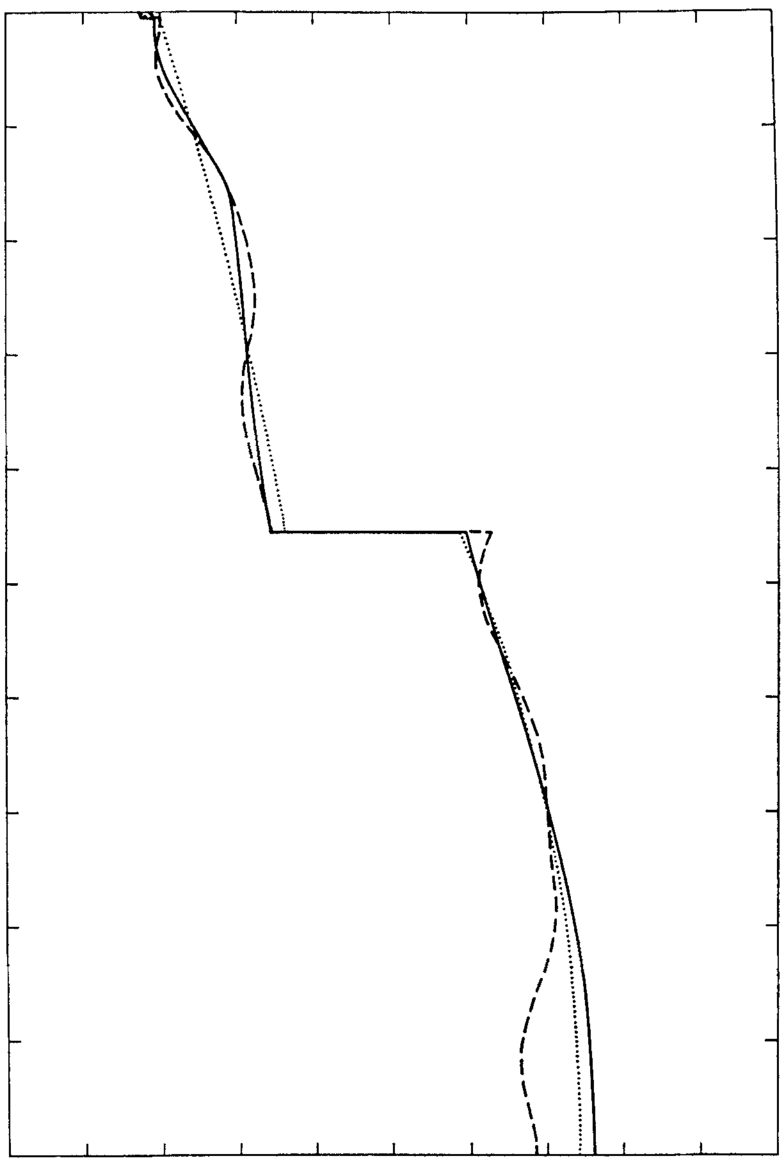

FIG. 1. The dimensionless density $\rho(r)$ for Model I (solid line), the initial guess (dotted line), and the thirteenth iterate (dashed line), using 20 gross Earth data calculated from Model I, and permitting ρ, κ and μ to vary during iteration. The centre of the Earth is at bottom, the surface at top. Values of ρ range from 0 at left to 3 dimensionless units at right.

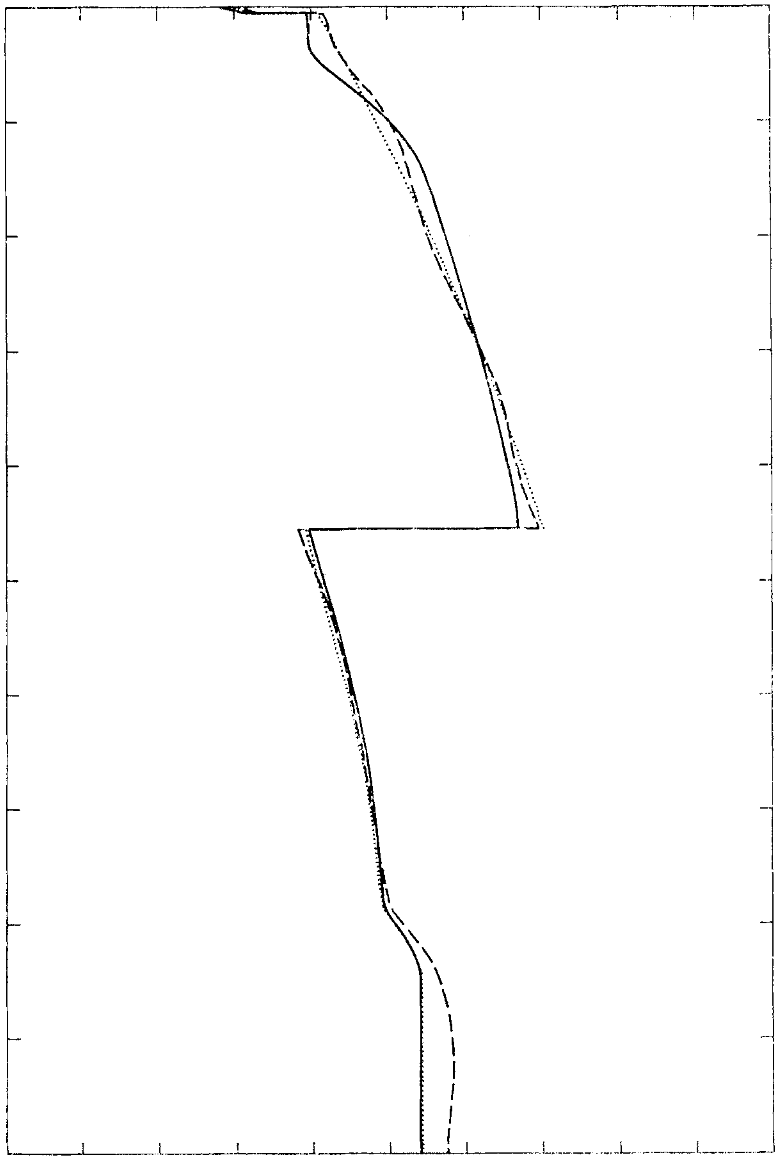

FIG. 2. The dimensionless compressional velocities $v_P(r)$ for the same numerical experiment as in Fig. 1. Full scale for v_P is 6π dimensionless units.

0·3%. We took as our 'observed' gross Earth data, E_j^0, the mass, moment, and various calculated eigenfrequencies of Model I. In a first experiment we used ${}_0S_0$, ${}_0S_2$, ${}_0S_3, \ldots, {}_0S_{18}$ and took for $\mathbf{m}_G$, our starting model, the smoothed Gutenberg model. All of ρ, κ, μ were allowed to vary. One iteration reduced the r.m.s. error from $1{\cdot}2 \times 10^{-2}$ to $1{\cdot}9 \times 10^{-4}$.

In a second experiment we used the same gross Earth data and took $\mathbf{m}_G$ to be the quadratic Gutenberg model, a poorer starting point. All of ρ, κ, μ were allowed to vary. The r.m.s. error ε_n after n iterations is given in Table 3.

In a third experiment we used as gross Earth data twenty numbers, the mass, the moment, and the squared eigenfrequencies, computed from Model I, of the eighteen modes ${}_0S_0$, ${}_1S_0$, ${}_2S_0$, ${}_1S_1$, ${}_2S_1, \ldots, {}_7S_1$, ${}_0S_2$, ${}_1S_2, \ldots, {}_7S_2$. The starting model $\mathbf{m}_G$ was

Table 3

The r.m.s. error ε_n of the nth iterate, starting with the quadratic Gutenberg model and using as gross data the mass, moment, and frequencies calculated from Model I for ${}_0S_0$, ${}_0S_2$, ${}_0S_3$, ..., ${}_0S_{18}$

n	0	1	2	3
$\varepsilon_n \times 10^4$	200	50	16	0·6

the quadratic Gutenberg model. All of ρ, κ, μ were allowed to vary. The r.m.s. error ε_n after n iterations decreased steadily from $1{\cdot}75 \times 10^{-2}$ for ε_0 to $1{\cdot}1 \times 10^{-8}$ for ε_{13}. Fig. 1 shows $\rho(r)$ for Model I, the starting model, and the thirteenth iterate. Fig. 2 shows $v_P(r)$ for these same three models, and Fig. 3 shows $v_S(r)$ for them. Model I

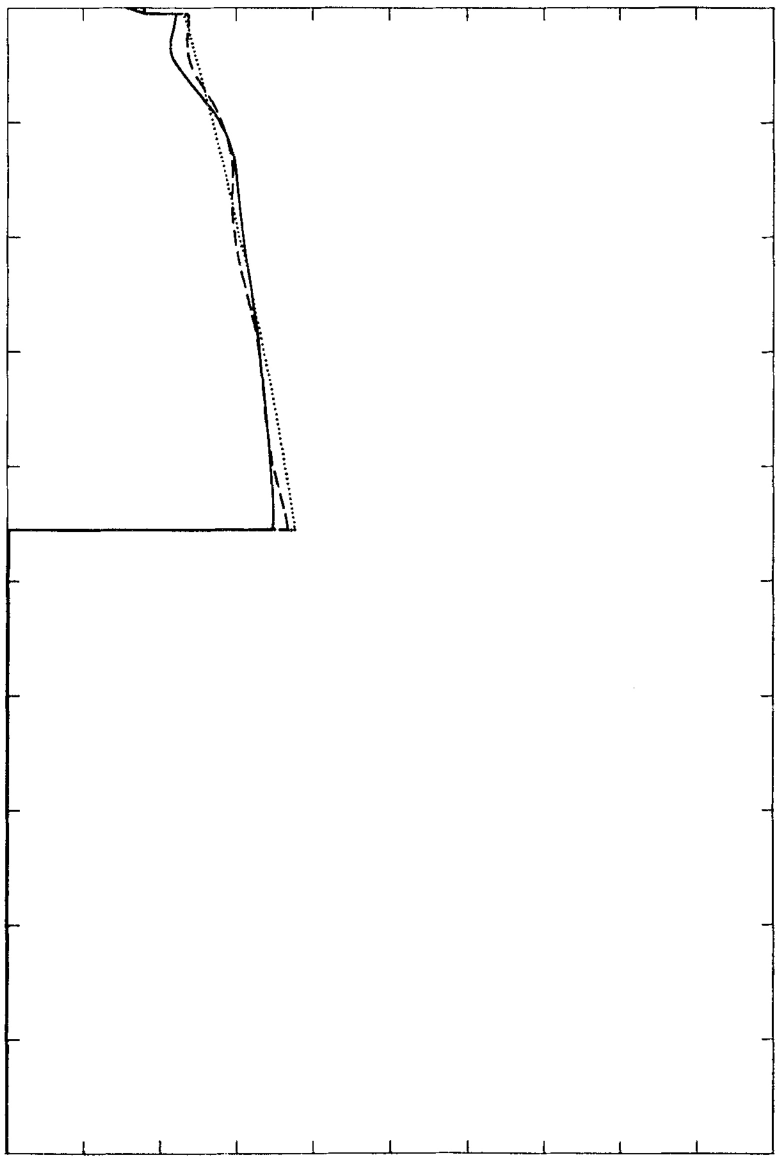

FIG. 3. The dimensionless shear velocities $v_S(r)$ for the same numerical experiment as in Fig. 1. Full scale for v_S is 6π dimensionless units.

and the thirteenth iterate have the same gross Earth data $E_1{}^0, \ldots, E_{20}{}^0$ to eight significant figures.

In a fourth experiment we used the same gross Earth data, but we permitted only ρ to vary during our attempts to fit the data. We required that the starting model, $\mathbf{m}_G$, and all subsequent iterates have the same $v_P(r)$ and $v_S(r)$ as Model I, the source of the 'observed' gross Earth data being used. The initial guessed density was obtained essentially by demanding that the Brunt–Vaisala frequency vanish in a fluid core and solid mantle and that the mass and moment of inertia be correct. The r.m.s. error ε_n after n iterations decreased steadily from $\varepsilon_0 = 0{\cdot}9 \times 10^{-2}$ to $\varepsilon_6 = 1{\cdot}6 \times 10^{-5}$, and then steadily but more slowly to $\varepsilon_{11} = 3{\cdot}6 \times 10^{-6}$. In Fig. 4 are shown the densities for model I and the 11th iterate.

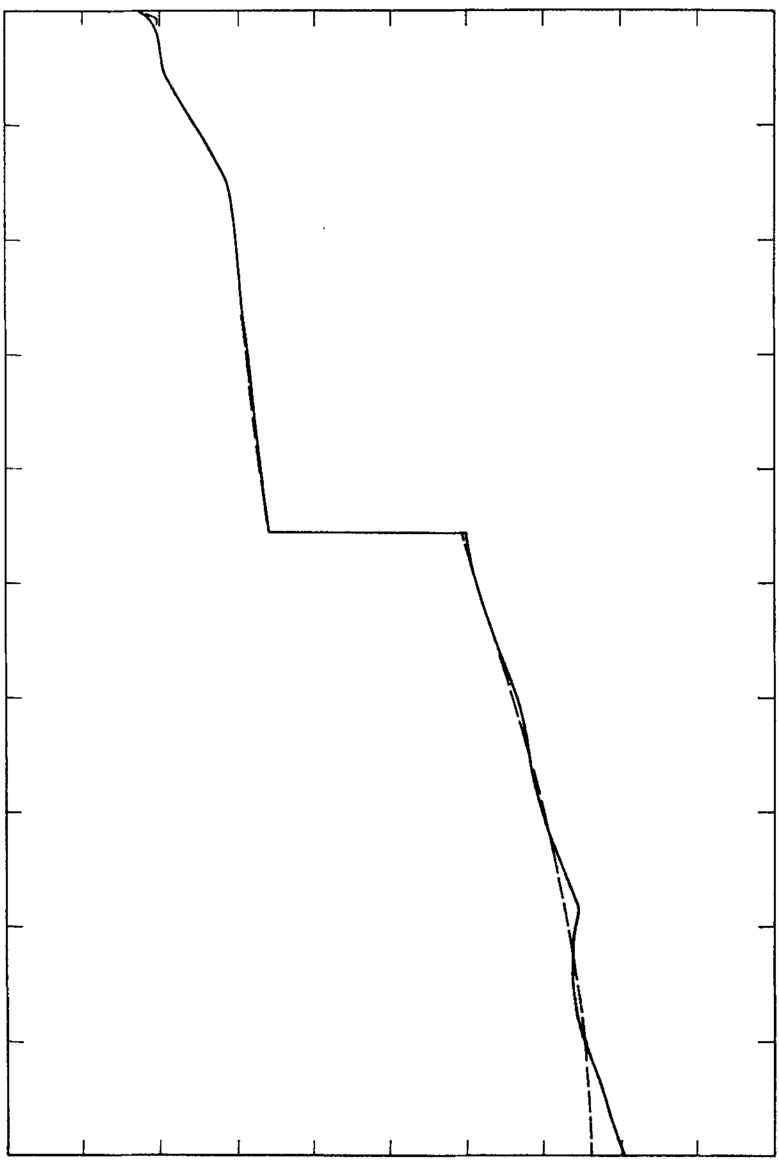

FIG. 4. The dimensionless density $\rho(r)$ for Model I (dashed line) and the eleventh iterate (solid line) using the same gross data and the same units on horizontal and vertical axes as in Fig. 1. Only ρ was varied during iteration.

8. Conclusions

We regard the experiments with artificial Earth data as successful. Since they were reasonable imitations of the unsuccessful experiments performed with the real data, there is some suggestion that our difficulties with the real data lay not in the inversion technique nor the initial guess, but in contamination of the real data by rotation, asphericity, anisotropy, mode misidentification and other effects. The evidence is inadequate to justify a firm conclusion, but we intend to keep the suggestion in mind as a guide to future work. In a subsequent paper we propose to correct the real data as far as possible for rotation and the known asphericities, to introduce an inner core and variable core boundaries, and to learn what we can about the possibility of mode misidentification. In this latter connection, a world-wide array of long-period seismometers (tiltmeters, gravimeters and strain gauges) would be extremely helpful, although some mode-identifying criteria can also be obtained by observing correlations among components of strain, gravity and tilt at a single station (Gilbert & Backus 1965, Smith 1966). Admittedly a world-wide array of strain-gauges, tiltmeters and gravimeters would be expensive to establish and maintain, but it might be a good investment in earthquake prediction. Press (1965) has shown that the Alaskan earthquake of 1964 produced a permanent strain change of 10^{-8} at Hawaii which could be explained approximately by treating the stressed source region before the earthquake as a dislocation loop. Such a dislocation would strain the whole Earth and redistribute its mass, so it ought to produce over the whole Earth changes in gravity of about one part in 10^8 as well as the observed changes in tilt and strain. A world-wide network of long-period seismometers appears to be capable of detecting and locating such localized accumulations of large stress as occurred prior to the Alaskan earthquake. It is to be hoped that if such a network is established, the data it produces will be made available for work on normal modes.

Our third experiment with artificial gross Earth data from Model I leads to an interesting conclusion about resolving power. Model I and the thirteenth iterate had the same values for mass, moment, and the eigenfrequencies of ${}_0S_0$, ${}_1S_0$, ${}_2S_0$, ${}_1S_1$, ${}_2S_1$, ${}_3S_1, \ldots, {}_7S_1$, ${}_0S_2$, ${}_1S_2, \ldots, {}_7S_2$, correct to eight significant figures. The twenty gross Earth data used to find the thirteenth iterate included frequencies of modes with high radial orders. It is widely believed that if such data can be observed they will give unambiguous results about the deep interior of the Earth. The Gram–Schmidt orthogonalization of the sequence of differential kernels $P_1, P_2, \ldots, P_{20}$ produces functions the nineteenth of which is shown in Fig. 5. In the mantle this kernel has wavelengths shorter than 900 km. Our naive expectations about resolving power [as measured by $K(E_1, \ldots, E_J)$] might lead us to expect that the data permitted us to determine the density structure except for details with wavelengths shorter than 900 km. In fact the density difference between Model I and the thirteenth iterate, as shown in Fig. 1, has wavelengths of 1800 km in the mantle.

By contrast with the third experiment, the fourth numerical experiment with artificial gross Earth data produced a very good fit between the eleventh iterate and the Model I which was the source of the data. This fit is seen in Fig. 4. In the third experiment all of ρ, κ, μ were to be determined from (3), while in the fourth experiment v_P and v_S were assumed known and only ρ was to be determined from (3). The same gross Earth data were used in both experiments. Is it possible that those gross data are sufficient to determine ρ uniquely (except for small-scale ambiguities due to lack of resolving power) if v_P and v_S are given, but are not adequate to determine all three of ρ, v_P and v_S? We have no idea whether this suggestion is correct, but a very naive argument may be advanced in its favour. Borg (1945) has shown that when a Sturm–Liouville problem on a finite interval is reduced to its canonical form as a Schrödinger equation for a wave function $\psi(x)$ on the interval $0 \leqslant x \leqslant 1$ then the spectrum of eigenvalues does not determine the potential $V(x)$ uniquely. However, if for a single potential $V(x)$ the eigenvalues are known for two different pairs of boundary conditions, say

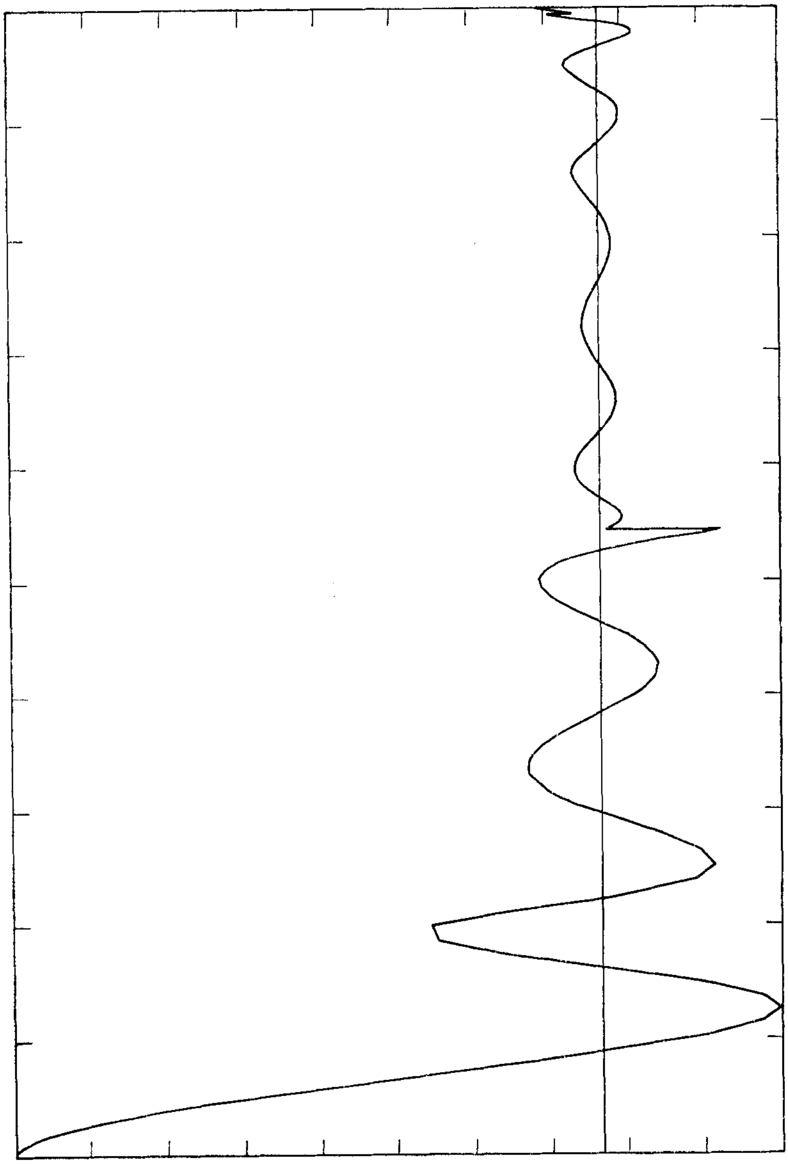

FIG. 5. The 19th member of the Gram-Schmidt orthogonalized sequence obtained from the differential kernels $P_1, .., P_{20}$ appropriate to Fig. 1. The centre of the earth is at bottom, the surface at top. The horizontal scale is arbitrary.

$\psi(0)=\psi(1)=0$ and $\psi(0)=\partial_x\psi(1)=0$, then V is uniquely determined. Suppose that the mass, moment of inertia, and shear velocity in the mantle of a spherical isotropic Earth are given. Suppose that for some fixed l the eigenfrequencies of ${}_nT_l$ are known for all n. As shown in Appendix E, Borg's result implies that these data do not determine the density, but that if in addition the eigenfrequencies of ${}_nT_l$ for all n are known when the surface is rigidly fixed, then the density must be one of two uniquely determined functions. Borg's result suggests the following conjecture: suppose that for a spherical Earth v_S is a known function of r and for two different angular orders l and l' the eigenfrequencies of $_nT_l$ and $_nT_{l'}$ are known for all radial orders n. Then the density is determined if the total mass and moment of inertia of the mantle are known. This conjecture suggests a further conjecture: that if we seek uniquely to determine v of the functions ρ, κ, μ (with $v = 1, 2, 3$) from the mass, moment, and eigenfrequencies of normal modes, we need $2v$ different infinite sequences of eigenfrequencies, each sequence consisting either of all toroidal modes of a given angular order or all spheroidal modes of a given angular order. By thus piling conjecture on conjecture we produce a conjectured explanation for the different outcomes of our third and fourth experiments with artificial data. In the fourth experiment we had the beginnings of two infinite sequences of the required type, namely $_nS_l$ with $l=1, 2$ and $n=0, 1, 2, \ldots, 7$ (and the brief beginnings of a third sequence with $l=0, n=0, 1, 2$). We were trying to determine only one function, the density, and the conjecture suggests this is possible, at least within the limits of resolution of the data. In the third experiment we had exactly the same three sequences of data, but we were trying to determine three functions. The conjecture suggests that to do this we would need at least six sequences of the form $_0S_l$, $_1S_l$, $_2S_l, \ldots$ or $_0T_l$, $_1T_l, \ldots$. We emphasize that we have no reason to accept the conjectures just described except the very naive counting of variables based on Borg's result. Our only reasons for mentioning the conjectures are that they do explain the different outcomes of experiments three and four, and that they may be suggestive to investigators proposing to pursue the theoretical inverse problem.

This work was supported by the National Science Foundation under NSF Grant GP-4096.

Appendix A

Rayleigh's principle and the differential kernels for normal modes of oscillation of the Earth

Rayleigh's principle for the eigenfrequencies of the normal modes of an arbitrary conservative system in a stationary equilibrium configuration has been written in several different forms for application to the Earth (Pekeris & Jarosch 1958, Jobert 1961). The form which is most convenient for our purposes is

$$\omega^2 \int_V dv\{\rho s^2\} = \int_V dv\{\kappa\tilde{K} + \mu\tilde{M} + \rho s_i s_j \partial_i \partial_j \phi_0 + \rho \partial_j \phi_0 (s_i \partial_i s_j - s_j \partial_i s_i)\} + \int_E dv\{(4\pi G)^{-1} |\nabla\phi_1|^2 + 2\rho s_i \partial_i \phi_1\}. \tag{29}$$

In equation (29), ρ is the density and ϕ_0 the gravitational potential in an elastic, self-gravitating body at rest in hydrostatic equilibrium and occupying the bounded region V. All of space is E. At any point $\mathbf{r}$ within the body, κ and μ are the bulk and shear moduli appropriate to the hydrostatically compressed state of the material at $\mathbf{r}$ (Rayleigh 1906). Newton's universal constant of gravitation is G, and ∂_i denotes partial differentiation with respect to x_i, the ith Cartesian component of $\mathbf{r}$. The vector $\mathbf{s}$ is a possible displacement field for an elastic deformation of the body, s_i being the ith Cartesian component of s; and ϕ_1 is a possible disturbance in the gravitational potential at points fixed in space (not moving with the body). The quantities $\frac{1}{2}\kappa\tilde{K}$ and $\frac{1}{2}\mu\tilde{M}$ are the energy densities of elastic compression and shear produced by $\mathbf{s}$, correct to second order in s. Therefore

$$\tilde{K} = (\partial_i s_i)^2, \tag{30}$$

and
$$\tilde{M}=2\Delta_{ij}\Delta_{ij} \tag{31}$$
where Δ_{ij} is the strain deviator:
$$\Delta_{ij}=\tfrac{1}{2}(\partial_i s_j+\partial_j s_i)-\tfrac{1}{3}(\partial_k s_k)\delta_{ij}. \tag{32}$$

The content of equation (29) is twofold. First, if an elastic-gravitational normal mode has displacement field $e^{i\omega t}\mathbf{s}(\mathbf{r})$ and gravitational potential perturbation field $e^{i\omega t}\phi_1(\mathbf{r})$ then ω, $\mathbf{s}$, and ϕ_1 are related by (29). Second, if $\mathbf{s}$ is permitted to be an arbitrary vector field defined in V and ϕ_1 is an arbitrary scalar field defined in E [not necessarily the potential produced by the density disturbance $\rho_1=-\nabla.(\rho\mathbf{s})$] and if (29) is regarded as defining a functional ω^2 of $\mathbf{s}$ and ϕ_1, then that functional is stationary to first order in arbitrary independent small variations of $\mathbf{s}$ and ϕ_1 if and only if $\mathbf{s}$ and ϕ_1 are the displacement and gravitational perturbation fields of a normal mode of oscillation whose angular frequency is ω.

The derivation of (29) from Rayleigh's principle consists in equating the kinetic energy $\frac{\omega^2}{2}\int_V dv\{\rho\mathbf{s}^2\}$ to the total potential energy of the disturbance $\mathbf{s}$, ϕ_1. This potential energy includes elastic energy, work done against hydrostatic pressure, gravitational energy, and the self-energy of the deviation of ϕ_1 from the gravitational potential produced by the density perturbation $\rho_1=-\nabla.(\rho\mathbf{s})$. If ψ is that deviation in ϕ_1, its self-energy is $(8\pi G)^{-1}\int_E dv|\nabla\psi|^2$.

Rayleigh (1877) points out that the small change $\delta\omega$ produced in any eigenfrequency ω by small perturbations $\delta\kappa$, $\delta\mu$, $\delta\rho$, $\delta\phi_0$ in the static body can be calculated directly from (29) if the eigenfunctions $\mathbf{s}$ and ϕ_1 for the unperturbed body are known. The perturbations $\delta\kappa$, $\delta\mu$, $\delta\rho$, $\delta\phi_0$ will produce perturbations $\delta\omega$, $\delta\mathbf{s}$, $\delta\phi_1$ in the normal mode ω, $\mathbf{s}$, ϕ_1. The first order relation among all these perturbations is computed by taking first-order variations in (29). The terms containing $\delta\mathbf{s}$ and $\delta\phi_1$ will sum to zero on account of the stationary character of ω relative to any small changes in the eigenfunctions. Therefore

$$\begin{aligned}(\delta\omega^2)\int_V dv\{\rho\mathbf{s}^2\}+\omega^2\int_V dv\{\delta\rho\mathbf{s}^2\}\\ =\int_V dv\{\delta\kappa\tilde{K}+\delta\mu\tilde{M}+\delta\rho s_i s_j\partial_i\partial_j\phi_0+\rho s_i s_j\partial_i\partial_j\delta\phi_0\\ +\delta\rho\partial_j\phi_0(s_i\partial_i s_j-s_j\partial_i s_i)+\rho\partial_j\delta\phi_0(s_i\partial_i s_j-s_j\partial_i s_i)+2\delta\rho s_i\partial_i\phi_1\}.\end{aligned} \tag{33}$$

The foregoing applications of Rayleigh's remarks require no restrictions on the shape of V or the symmetries (if any) of κ, μ, ρ and ϕ_0, nor do they demand that ϕ_0 be entirely due to ρ. Rayleigh's suggestion for using his variational principle to simplify first-order perturbation theory has been a classical technique in mechanics for almost a century, and in quantum mechanics for at least one-third of a century. Its first appearance in seismology seems to be an application to dispersion relations for Love waves in a half-space in the absence of gravitation (Meissner 1926, Stoneley 1926). Subsequent applications are described by Jeffreys (1935, 1961) and Takeuchi *et al.* (1964).

Appendix B

Rayleigh's principle for spherical bodies

For a self-gravitating body, $\delta\rho$ and $\delta\phi_0$ are not independent. This makes calculation of $\delta\omega$ from (33) cumbersome. We prefer to simplify (29) at the outset by making explicit use of spherical symmetry and the absence of gravitational fields other than that due to the body itself. We assume that V is a sphere of radius a whose centre is the origin of a system of Cartesian co-ordinates x_1, x_2, x_3. We assume that ρ, κ, μ depend only on r ($r^2 = x_1^2 + x_2^2 + x_3^2$) and vanish when $r > a$. We assume that ϕ_0 is entirely due to ρ; then

$$\partial_r^2 \phi_0 + 2r^{-1} \partial_r \phi_0 = 4\pi G\rho \tag{34}$$

and ϕ_0 and $\partial_r \phi_0$ are continuous at $r = a$.

Since ϕ_0 depends only on r, we have

$$\partial_j \phi_0 = r^{-1} x_j \partial_r \phi_0 \tag{35}$$

and

$$\partial_i \partial_j \phi_0 = r^{-3}(r^2 \delta_{ij} - x_i x_j) \partial_r \phi_0 + r^{-2}(x_i x_j) \partial_r^2 \phi_0.$$

Then according to (34),

$$\partial_i \partial_j \phi_0 = r^{-3}(r^2 \delta_{ij} - 3x_i x_j) \partial_r \phi_0 + 4\pi G\rho r^{-2}(x_i x_j). \tag{36}$$

If we denote by s_r the r component of $\mathbf{s}$, then (35) and (36) reduce (29) to

$$\omega^2 \int_V dv \{\rho \mathbf{s}^2\} = \int_V dv \{\kappa \tilde{K} + \mu \tilde{M} + 4\pi G\rho^2 s_r^2 + \rho \Lambda \partial_r \phi_0\}$$
$$+ \int_E dv \{(4\pi G)^{-1} |\nabla \phi_1|^2 + 2\rho \mathbf{s} . \nabla \phi_1\}, \tag{37}$$

where

$$\Lambda = (\mathbf{s} . \nabla) s_r - s_r (\nabla . \mathbf{s}) - 2r^{-1} s_r^2. \tag{38}$$

If $\mathbf{s}$ and ϕ_1 are the eigenfunctions of a normal mode with angular frequency ω, then Rayleigh's argument which led to (33) can be used to calculate from (37) the first-order perturbation $\delta\omega^2$ produced in ω^2 by small perturbations $\delta\kappa$, $\delta\mu$, $\delta\rho$, $\delta\phi_0$ in the equilibrium configuration. The result is

$$(\delta\omega^2) \int_V dv \{\rho \mathbf{s}^2\} + \omega^2 \int_V dv \{\delta\rho \mathbf{s}^2\} = \int_V dv \{\delta\kappa \tilde{K} + \delta\mu \tilde{M}$$
$$+ 8\pi G\rho \delta\rho s_r^2 + (\delta\rho \partial_r \phi_0 + \rho \partial_r \delta\phi_0) \Lambda + 2\delta\rho \mathbf{s} . \nabla \phi_1\}. \tag{39}$$

Note that

$$\partial_r \delta\phi_0 = 4\pi G r^{-2} \int_0^r dr' [r'^2 \delta\rho(r')].$$

If we define

$$S(r) = \int_r^a \rho \Lambda \, dr$$

then

$$\int_V dv \{\rho \Lambda \partial_r \delta\phi_0\} = -4\pi G \int_{S_1} dA \int_0^a dr (\partial_r S) \int_0^r dr' [r'^2 \delta\rho(r')],$$

where dA is the element of area on S_1, the surface of the unit sphere. Integrating by parts with respect to r, we obtain

$$\int_V dv \{\rho \Lambda \partial_r \delta\phi_0\} = 4\pi G \int_V dv \{\delta\rho S\}.$$

Therefore we can write (39) in the form

$$(\delta\omega^2)\int_V dv\{\rho \mathbf{s}^2\} = \int_V dv\{\tilde{K}\delta\kappa + \tilde{M}\delta\mu + \tilde{R}\delta\rho\} \tag{40}$$

where

$$\tilde{R} = 8\pi G\rho s_r^2 - \omega^2 \mathbf{s}^2 + \Lambda\partial_r\phi_0 + 4\pi G\int_r^a \rho\Lambda\, dr + 2\mathbf{s}.\nabla\phi_1. \tag{41}$$

The kernels $\tilde{K}$, $\tilde{M}$, $\tilde{R}$ can be reduced to a form more amenable to numerical calculation by noting (Backus 1967) that for any vector field s defined in $0 \leqslant r \leqslant a$ there are unique scalar fields U, V, W such that V and W average to zero on every spherical surface concentric with the origin, and

$$\mathbf{s} = U\hat{\mathbf{r}} + \nabla_1 V - \hat{\mathbf{r}}\, X\, \nabla_1 W. \tag{42}$$

$\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\lambda}}$ are unit vectors in the directions of increasing radius r, colatitude θ, and longitude λ, while

$$\nabla_1 = \hat{\boldsymbol{\theta}}\partial_\theta + \operatorname{cosec}\theta\,\hat{\boldsymbol{\lambda}}\partial_\lambda. \tag{43}$$

In a spheroidal normal mode (Hoskins 1910) W vanishes and U, V and ϕ_1 are all products of the form $U(r)\, Y_l^m(\theta,\lambda)$, $V(r)\, Y_l^m(\theta,\lambda)$, $\phi_1(r)\, Y_l^m(\theta,\lambda)$, where Y_l^m is a normalized surface spherical harmonic of angular order l and azimuthal order m. In a toroidal normal mode (Alterman *et al.* 1959) U, V and ϕ_1 vanish, while W has the form $W(r)\, Y_l^m(\theta,\lambda)$. There are no other normal modes (Backus 1967).

To apply (42) to (40) we introduce a shorthand notation. If f and g are two functions of position we say $f \equiv g$ when $f-g$ averages to zero on every sphere concentric with the origin. Since $\delta\kappa$, $\delta\mu$ and $\delta\rho$ depend only on r, the validity of (40) is unaffected if we replace $\tilde{K}$, $\tilde{M}$ and $\tilde{R}$ by functions K', M' and R', as long as $K' \equiv \tilde{K}$, $M' \equiv \tilde{M}$ and $R' \equiv \tilde{R}$. We note that for any functions f and g we have

$$\nabla_1 f.\nabla_1 g \equiv -f\nabla_1^2 g, \tag{44}$$

where $\nabla_1^2 = \nabla_1.\nabla_1 = \operatorname{cosec}\theta\partial_\theta \sin\theta\partial_\theta + \operatorname{cosec}^2\theta\partial_\lambda^2$, so that $\nabla_1^2\, Y_l^m = -l(l+1)\, Y_l^m$.

We restrict attention to a normal mode (either spheroidal or toroidal) of angular order l. We define

$$F = r^{-1}[2U - l(l+1)\, V]. \tag{45}$$

Then (Backus 1967) some applications of (42) and (44) show that $K' \equiv \tilde{K}$, $M' \equiv \tilde{M}$, and $R' \equiv \tilde{R}$, where

$$K' = (\partial_r U + F)^2, \tag{46}$$

$$M' = \tfrac{1}{3}(2\partial_r U - F)^2 + r^{-2}l(l+1)[(r\partial_r V - V + U)^2 + (r\partial_r W - W)^2] + r^{-2}(l-1)l(l+1)(l+2)[V^2 + W^2], \tag{47}$$

and

$$R' = -\omega^2[U^2 + l(l+1)(V^2+W^2)] + 2U(\partial_r\phi_1 + 4\pi\rho G U - F\partial_r\phi_0) + 2r^{-1}l(l+1)\, V\phi_1 - 8\pi G\int_r^a dr(\rho U F). \tag{48}$$

It follows that

$$(\delta\omega^2)\int_V dv\{\rho\mathbf{s}^2\} = \int_V dv\{K'\delta\kappa + M'\delta\mu + R'\delta\rho\}. \tag{49}$$

Appendix C

Locations of discontinuities

In all of the foregoing discussion it has been assumed that the radii of the discontinuities in ρ, κ and μ are known exactly. Since this is not the case for the Earth, it is of some interest to include the locations of discontinuities as unknowns in the inversion procedure described in Section 5. We need to know how varying the location of a jump discontinuity in ρ, κ or μ affects the frequencies of the normal modes. In what follows, if a function $f(r)$ has a jump discontinuity at $r=b$, we will denote by $[f]_-^+$ the limit of $[f(b+\varepsilon)-f(b-\varepsilon)]$ as ε approaches 0 through positive values.

Suppose that ω is the angular frequency of a normal mode whose eigenfunctions $\mathbf{s}$ and ϕ_1 are known, and that ρ, κ and μ have jump discontinuities at $r=b$. Suppose we perturb the Earth model (the equilibrium configuration) by raising the level of the discontinuity to $b+h$ but leaving ρ, κ and μ otherwise unaffected. We seek the resulting change $\delta\omega^2$, correct to first order in h. To this order $\delta\omega^2$ is unaffected by the manner in which ρ, κ and μ are defined in the gap $b \leqslant r \leqslant b+h$, as long as they are continuous in $b \leqslant r < b+h$.

If K', M' and R' were continuous, $\delta\omega^2$ could be computed directly from (49) by replacing $\delta\kappa$ by $-[\kappa]_-^+ h\delta(r-b)$ and similarly for μ and ρ, $\delta(r-b)$ being the Dirac delta function. Since K', M' and R' may be discontinuous at $r=b$, we must start from (37).

As in the deduction of (33) from (29), Rayleigh's (1877) argument applied to (37) in the present situation implies

$$(\delta\omega^2)\int_V dv\{\rho\mathbf{s}^2\} = h\int_{S_b} dA[\omega^2\rho\mathbf{s}^2-(4\pi G)^{-1}|\nabla\phi_1|^2-2\rho\mathbf{s}.\nabla\phi_1$$
$$-K'\kappa-M'\mu-4\pi G\rho s_r^2-\rho\Lambda\partial_r\phi_0]_-^+ + \int_V dv(\rho\Lambda\partial_r\delta\phi_0). \quad (50)$$

In (50) the surface integral is over the surface S_b defined by $r=b$, and the volume integral arises because moving the discontinuity in ρ from $r=b$ to $r=b+h$ perturbs the equilibrium potential ϕ_0 in the region $b \leqslant r \leqslant a$ by the amount

$$\partial_r\delta\phi_0 = -4\pi G r^{-2} h[\rho]_-^+.$$

If we define

$$g_1 = \partial_r\phi_1 + 4\pi G\rho U \quad (51)$$

then g_1, U, ϕ_0, $\partial_r\phi_0$, ϕ_1 and $\nabla_1\phi_1$ are continuous at $r=b$ while V, W, $\partial_r U$, $\partial_r V$, $\partial_r W$ and $\partial_r\phi_1$ may be discontinuous. Calculations like those in Appendix B, including appropriate integrations by parts, lead finally to the formula

$$(\delta\omega^2)\int_V dv\{\rho\mathbf{s}^2\} = -h\int_{S_b} dA[K'\kappa+M'\mu+R'\rho-2\rho U g_1]_-^+. \quad (52)$$

Appendix D

Finding the zeros of the secular equation from a variational principle with boundary term

Verreault (1965a) has pointed out that a generalization of the variational principle for torsional modes to include a boundary term provides a rapidly convergent iterative scheme for numerical calculation of the torsional eigenfrequencies of any spherical

Earth model. We have extended Verreault's result to spheroidal modes and have used it in our calculations of all eigenfrequencies ω. We find that if an initial estimate of ω was accurate to one part in 10^2, three integrations of the radial ordinary differential equations (Alterman *et al.* 1959) usually reduce the error to one part in 10^8.

To obtain the desired generalization of the variational principle we assume that for some value of ω, not necessarily an eigenfrequency, we have a vector field $\mathbf{s}(\mathbf{r})$ and a scalar field $\phi_1(\mathbf{r})$ which satisfy the momentum and Poisson equations inside a model Earth V, but do not necessarily satisfy the appropriate boundary conditions on ∂V, the surface of V. If we define

$$E_{ij} = (\kappa - \tfrac{2}{3}\mu)(\nabla . \mathbf{s})\delta_{ij} + \mu(\partial_i s_j + \partial_j s_i),$$

$$\rho_1 = -\partial_i(\rho s_i),$$

then the equations in question are

$$\omega^2 \rho s_i = -\partial_j E_{ij} + \partial_i(\rho s_j \partial_j \phi_0) + \rho_1 \partial_i \phi_0 + \rho_0 \partial_i \phi_1, \tag{53}$$

$$0 = \rho_1 - (4\pi G)^{-1} \nabla^2 \phi_1. \tag{54}$$

We multiply (53) by s_i, (54) by ϕ_1, add the two, and integrate over V. With the help of Gauss's theorem we obtain

$$\omega^2 \int_V dv\{\rho \mathbf{s}^2\} = \int_V dv\{\kappa \tilde{K} + \mu \tilde{M} + \rho \partial_j \phi_0 (s_i \partial_i s_j - s_j \partial_i s_i)$$
$$+ \rho s_i s_j \partial_i \partial_j \phi_0 + \rho s_i \partial_i \phi_1\} - \int_{\partial V} dA\{s_i E_{ij} n_j\}, \tag{55}$$

where n_j are the Cartesian components of the unit outward normal to ∂V.

Now we assume further that $\nabla^2 \phi_1 = 0$ in $E - V$ (outside V) and that ϕ_1 is continuous at ∂V. Then

$$0 = \int_V dv\{\rho s_i \partial_i \phi_1\} + (4\pi G)^{-1} \int_E dv |\nabla \phi_1|^2 + \int_{\partial V} dA\, \phi_1 n_i [(4\pi G)^{-1} \partial_i \phi_1 + \rho s_i]_-^+. \tag{56}$$

Adding (55) and (56) gives

$$\omega^2 \mathscr{T} = \mathscr{V} - \mathscr{B}, \tag{57}$$

where

$$\mathscr{T} = \int_V dv\{\rho \mathbf{s}^2\},$$

$$\mathscr{V} = \int_V dv\{\kappa \tilde{K} + \mu \tilde{M} + \rho \partial_j \phi_0 (s_i \partial_i s_j - s_j \partial_i s_i)$$
$$+ \rho s_i s_j \partial_i \partial_j \phi_0 + 2\rho s_i \partial_i \phi_1\} + (4\pi G)^{-1} \int_E dv |\nabla \phi_1|^2,$$

and

$$\mathscr{B} = \int_{\partial V} dA\, n_i \{E_{ij} s_j - \phi_1 [(4\pi G)^{-1} \partial_i \phi_1 + \rho s_i]_-^+\}.$$

In case $\mathbf{s}$, ϕ_1 is an eigenfunction pair, the boundary conditions imply $\mathscr{B} = 0$, so (57) reduces to (29).

To deal with a spherical Earth model, we denote by $\hat{\mathbf{x}}_i$ the unit vector in the direction of x_i, so that the radial stress field is $\mathbf{T} = r^{-1} x_i E_{ij} \hat{\mathbf{x}}_j$. Then we write $\mathbf{s}$ in the form (42) and $\mathbf{T}$ in the form

$$\mathbf{T} = \hat{\mathbf{r}} P + \nabla_1 Q - \hat{\mathbf{r}} \times \nabla_1 R.$$

We confine attention to normal modes of a fixed angular order l, and introduce the ellipsis $U=U(r)\,Y_l^m(\theta,\lambda)$, and similarly for V, W, P, Q, R, ϕ_1, and g_1. Here Y_l^m is a surface spherical harmonic normalized so that

$$\int_0^{2\pi} d\lambda \int_0^{\pi} d\theta \sin\theta (Y_l^m)^2 = 1.$$

Then if $r=a$ is the surface ∂V, we have

$$\mathscr{T} = 4\pi \int_0^a dr\, r^2 \rho(r) \{U^2(r) + l(l+1)[V^2(r) + W^2(r)]\} \tag{58}$$

and

$$\mathscr{B} = 4\pi a^2 \{U(a)\,P(a) + l(l+1)[V(a)\,Q(a) + R(a)\,W(a)] \\ + (4\pi G)^{-1} \phi_1(a)[g_1(a) + (l+1)\,a^{-1}\,\phi_1(a)]\}, \tag{59}$$

where g_1 is given by (51).

If ω, $\mathbf{s}$, ϕ_1 are close to some normal mode $\tilde{\omega}$, $\tilde{\mathbf{s}}$, $\tilde{\phi}_1$, then because (29) is a variational principle we have

$$\tilde{\omega}^2 = \mathscr{V}\mathscr{T}^{-1} + \varepsilon,$$

where ε is a term of the order of $|\mathbf{s}-\tilde{\mathbf{s}}|^2$ and $|\phi_1 - \tilde{\phi}_1|^2$, and $\mathscr{V}$ and $\mathscr{T}$ are calculated from $\mathbf{s}$ and ϕ_1 rather than $\tilde{\mathbf{s}}$ and $\tilde{\phi}_1$. Then, from (57),

$$\tilde{\omega}^2 = \omega^2 + \mathscr{B}\mathscr{T}^{-1} + \varepsilon. \tag{60}$$

When the equations of motion have been integrated from $r=0$ to $r=a$ for some value of ω near an eigenfrequency $\tilde{\omega}$, then equation (60) without ε can be used to improve the estimate of $\tilde{\omega}$.

If the normal mode under study is torsional or radial (spheroidal with $l=0$) then the closeness of ω to $\tilde{\omega}$ and the fact that $\mathbf{s}$ and ϕ_1 are regular at $r=0$ and solve the equations of motion suffice to ensure that $\mathbf{s}$ and ϕ_1 are near $\tilde{\mathbf{s}}$ and $\tilde{\phi}_1$, so that (60) is a good estimate. If, however, the normal mode under study is spheroidal with $l \geqslant 1$, the pairs $\mathbf{s}$, ϕ_1 which are regular at $r=0$ and solve the equations of motion constitute a three-dimensional linear space. The numerical integration (Alterman *et al.* 1959) produces three linearly independent six-by-one row vectors

$$(f_{i1}(r), f_{i2}(r), f_{i3}(r), f_{i4}(r), f_{i5}(r), f_{i6}(r)) \\ = (P^{(i)}(r),\, Q^{(i)}(r),\, h^{(i)}(r),\, U^{(i)}(r),\, V^{(i)}(r),\, \phi_1{}^{(i)}(r))$$

where $h(r) = g_1(r) + (l+1)\,r^{-1}\,\phi_1(r)$ and $i=1, 2, 3$. The problem is to find a linear combination of these three row vectors which is nearly the row eigenvector corresponding to $\tilde{\omega}$. If ω were an eigenvalue, the 3×3 matrix $f_{ij}(a)$, with $i, j = 1, 2, 3$, would be singular, and its adjugate $F_{ij}(a)$ [the transpose of the 3×3 matrix of cofactors of $f_{ij}(a)$] would be of rank 1. The eigenvector would be

$$f_j(r) = F_{ki}(a) f_{ij}(r), \tag{61}$$

where $j=1, \ldots, 6$, and k is any one of 1, 2 or 3. When ω is not an eigenvalue we define f_j and hence $\mathbf{s}$ and ϕ_1 via (61). For numerical stability we choose k so as to maximize the largest element of $F_{ki}(a)$, $i=1, 2, 3$. Then when $\omega=\tilde{\omega}$, $\mathbf{s}$ and ϕ_1 become $\tilde{\mathbf{s}}$ and $\tilde{\phi}_1$, so by continuity $\mathbf{s}$ and ϕ_1 will be nearly $\tilde{\mathbf{s}}$ and $\tilde{\phi}_1$ when ω is near $\tilde{\omega}$.

Appendix E

Determination of the density from the torsional eigenfrequencies for rigid and free surfaces

In this appendix we show that the density $\rho(r)$ in the mantle of a spherical isotropic Earth model is one of a pair of functions uniquely determined by the following data:

(i) the inner and outer radii of the mantle, b and a;

(ii) the total mass and moment of inertia of the mantle;

(iii) the shear velocity $v_S(r)$ for $b \leqslant r \leqslant a$;

(iv) the values of all the torsional eigenfrequencies for some fixed angular order l when the boundaries of the mantle at $r=b$ and $r=a$ are free;

(v) the values of all the torsional eigenfrequencies for the same fixed angular order l as in (iv) when the boundary of the mantle at $r=b$ is free while that at $r=a$ is rigid.

If we write the displacement as $\mathbf{s} = -\mathbf{r} \times \nabla_1 [w(r)\, Y_l^m(\theta, \lambda)]$ then the equations of motion (Alterman *et al.* 1959) are

$$\frac{d}{dr}\left(r^4 \mu \frac{dw}{dr}\right) + [\omega^2 \rho r^4 - (l+2)(l-1)\, \mu r^2]\, w = 0 \tag{62}$$

in $b \leqslant r \leqslant a$, while the boundary conditions are $w=0$ at a rigid boundary and $dw/dr=0$ at a free boundary. Equation (62) can be reduced to Schrödinger's equation,

$$\frac{d^2u}{dt^2} + [\omega^2 - V(t)]\, u = 0. \tag{63}$$

The new independent variable t, the new dependent variable u, and the 'potential' V are obtained from the following substitutions (Courant & Hilbert 1953):

$$t(r) = \int_b^r v_S^{-1}(r)\, dr; \tag{64}$$

$$f(t) = r^2 \rho^{\frac{1}{2}} v_S^{\frac{1}{2}}; \tag{65}$$

$$u(t) = fw; \tag{66}$$

$$V(t) = \frac{1}{f}\frac{d^2 f}{dt^2} + \frac{(l+2)(l-1)\, v_S^2}{r^2}. \tag{67}$$

The boundary conditions become

$$\frac{du}{dt} = \left(\frac{d \ln f}{dt}\right) u \tag{68}$$

at a free boundary and

$$u = 0 \tag{69}$$

at a rigid boundary. The boundaries are now at $t(b)=0$ and at $t(a)$.

If the 'free surface' torsional eigenfrequencies are known, the eigenvalues ω^2 of (63) are known when the boundary condition is (68) at both ends of the t interval. If the 'rigid surface' torsional eigenfrequencies are also known, the eigenvalues ω^2 of (63) are known when the boundary condition is (68) at $t(b)$ and (69) at $t(a)$. According to Borg (1945) these two spectra suffice to determine $V(t)$ uniquely on the interval

$t(b) \leqslant t \leqslant t(a)$. But from (64) r and v_S are known as functions of t, so (67) can be integrated for f. Since (67) is linear in f, we may write the general solution as $A_1 f_1 + A_2 f_2$ where f_1 and f_2 are solutions chosen so that

$$4\pi \int_b^a dr\, r^{-2} v_S{}^{-1} f_i f_j = \delta_{ij}.$$

Then from (65)

$$\rho = r^{-4} v_S{}^{-1} (A_1 f_1 + A_2 f_2)^2, \tag{70}$$

so the mass and moment of inertia are respectively

$$M = A_1{}^2 + A_2{}^2, \tag{71}$$

$$I = \frac{8\pi}{3} \sum_{i=1}^{2} \sum_{j=1}^{2} A_i A_j \int_b^a dr\, v_S{}^{-1} f_i f_j. \tag{72}$$

We assume that the observed spectra come from an Earth model, so the circle (71) and the ellipse (72) intersect. Except in the singular event of tangency, there will be four points of intersection, (A_1, A_2), (A_1', A_2'), $(-A_1, -A_2)$, and $(-A_1', -A_2')$. Of these, only two will produce different functions ρ, so there will be exactly two densities (70) satisfying the given data (i), ..., (v).

Appendix F

Relative energies for some of the normal modes of Model I

Using (37) we have computed the relative energies of compression, shear, and gravitational effects for 144 spheroidal normal modes of Model I. Normalizing $\mathbf{s}$ for each mode so that

$$\omega^2 \int_V dv \{\rho \mathbf{s}^2\} = 1 \tag{73}$$

we have the relative compressional elastic energy, E_c, shear elastic energy, E_s, and energy due to gravitational effects, E_g;

$$\begin{aligned} E_c &= \int_V dv \{\kappa \tilde{K}\} \\ E_s &= \int_V dv \{\mu \tilde{M}\} \\ E_g &= 1 - E_c - E_s. \end{aligned} \tag{74}$$

Both E_c and E_s are necessarily positive but E_g may be either positive or negative. Numerical results for 17 low frequency modes are given in Table 4. The mode ${}_0S_0$ has much less shear elastic energy than any other mode; a fact that correlates interestingly with the observed high Q of ${}_0S_0$. If dissipation is roughly proportional to E_s then a Q of 350 for ${}_0S_2$ implies a Q of 6800 for ${}_0S_0$.

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Table 4

Relative energies for some of the spheroidal modes of Model I. f(c/h) is the frequency in cycles/hour

n	l	f(c/h)	E_c	E_s	E_g
0	0	2·932	1·304	0·028	−0·332
1	0	5·961	0·946	0·159	−0·105
2	0	9·100	0·870	0·172	−0·042
1	1	1·460	0·295	0·689	0·016
2	1	3·400	0·831	0·306	−0·137
0	2	1·115	0·115	0·546	0·339
1	2	2·452	0·196	0·791	0·013
2	2	3·946	0·552	0·491	−0·043
0	3	1·685	0·166	0·642	0·192
1	3	3·383	0·116	0·838	0·046
2	3	4·499	0·437	0·573	−0·010
0	4	2·327	0·179	0·716	0·105
1	4	4·221	0·056	0·875	0·069
2	4	4·984	0·396	0·609	−0·005
0	5	3·022	0·186	0·757	0·057
1	5	4·931	0·031	0·888	0·081
2	5	5·473	0·362	0·643	−0·005

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