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GAUSSIAN MEASURES ON LINEAR SPACES

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Introduction

Gaussian distributions play a fundamental role in all natural sciences. The growing importance of infinite-dimensional Gaussian distributions has become clear over the past decades.

The aim of this survey is to present a systematic exposition of the theory of Gaussian measures on general locally convex spaces. Various results on this topic are scattered in a very extensive literature. There are several books devoted to some special aspects of the theory of Gaussian measures such as [420, 395, 478, 199, 407, 429, 289, 230, 118, 480, 80, 359, 304, 114]; however, only [420, 395] and [289] (written more than 20 years ago) present a general theory. The locally convex space setting provides a natural framework for most of the results in this theory. It should be noted that overall this generality does not bring new technical difficulties; moreover, it makes basic constructions more natural and clearer. There are very few cases where the proofs in the Hilbert space setting are really simpler. In such cases we shall mention these shorter proofs.

There are several main directions in the study of Gaussian measures. The first one might be called a "linear-topological theory." We discuss the most important results in this direction in Chapter 1 and Chapter 5. Among them: classical theorems on equivalence/singularity, zero–one laws, the reproducing kernel Hilbert spaces, measurable linear functionals and operators, and topological properties of supports.

The second direction is connected with convexity and isoperimetry. In Chapter 2 we discuss several fundamental inequalities and estimates, such as exponential integrability, tail behavior, and measures of small balls. This direction is closely related to the theory of Gaussian processes, in particular, to the study of the sample path properties. However, we do not discuss these important applications here. Finally, we make several introductory remarks about the central limit theorem and related questions. A good account of the whole direction is given in a recent monograph by Ledoux and Talagrand [304].

The third direction we discuss might be called a "nonlinear theory" or "analysis on Wiener spaces." This is a very actively developing part of infinite-dimensional stochastic analysis. Nonlinear transformations of Gaussian measures, including the study of absolute continuity, and the Malliavin calculus are the heart of this direction. These problems are briefly discussed in Chapter 4. They will be the subject of a separate survey of the author. Some basic information about Sobolev classes over Gaussian measures is presented in Chapter 3, where, in addition, we discuss Gaussian capacities.

In Chapter 5 we introduce Wiener processes in infinite dimensions and study several related problems.

This survey is based on the lectures of the author at Moscow State University. Functional–analytic aspects are emphasized, though we do not avoid probabilistic constructions.

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Chapter 0
Background

A. Hilbert–Schmidt and Nuclear Operators on Hilbert Spaces.

In this section, we recall some well-known facts about linear operators on Hilbert spaces. The proofs can be found in [289, 368].

Let H be a separable Hilbert space. Denote by $\mathcal{L}(H)$ the space of all bounded linear operators on H equipped with the operator norm $\|A\|_{\mathcal{L}(H)}$.

A.1 Definition. An operator $A \in \mathcal{L}(H)$ is said to be a *Hilbert-Schmidt operator* if the series

$$\sum_{n=1}^{\infty} |Ae_n|^2$$

converges for some orthogonal basis $\{e_n\}$ in H.

In this case, the series above converges for every orthogonal basis in H and its sum does not depend on $\{e_n\}$.

A symmetric operator $A \in \mathcal{L}(H)$ is Hilbert–Schmidt if and only if there is an orthogonal basis $\{e_n\}$ in H such that $Ae_n = a_n e_n$ and

$$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

Let $A = U|A|$ be the polar decomposition of the operator $A \in \mathcal{L}(H)$. Then A is a Hilbert–Schmidt operator if and only if $|A|$ is also.

The class of all Hilbert–Schmidt operators on H is denoted by $\mathcal{L}_{(2)}(H)$ or by $\mathcal{H}(H)$.

The second notation will be convenient in Chapter 3, where we discuss Sobolev classes. The following more general object will also be useful there.

A.2 Definition. Let H and E be separable Hilbert spaces. A continuous linear mapping $A: H^n \to E$ is called a *Hilbert–Schmidt n-linear map* on H, if for some orthogonal basis $\{e_n\}$ in H, one has

$$\sum_{i_1,\ldots,i_n=1}^{\infty} \left|A(e_{i_1},\ldots,e_{i_n})\right|_E^2 < \infty.$$

In this case, the sum above is finite for any orthogonal basis in H and does not depend on a particular choice of a basis.

$\mathcal{H}_n(H,E)$ stands for the space of all n-linear Hilbert–Schmidt mappings between H and E. If $E = H$ we use the notation $\mathcal{H}_n$.

Note that $\mathcal{H}_n(H,E)$ with the inner product

$$(A,B)_{\mathcal{H}_n} = \sum_{i_1,\ldots,i_n=1}^{\infty} \left(A(e_{i_1},\ldots,e_{i_n}), B(e_{i_1},\ldots,e_{i_n})\right)_E$$

is a Hilbert space. In particular, the space of all Hilbert–Schmidt operators on H with the inner product

$$(A,B)_{\mathcal{H}} = \sum_{n=1}^{\infty} (Ae_n, Be_n)_H$$

is a Hilbert space.

A.3 Definition. An operator $A \in \mathcal{L}(H)$ is called nuclear (or a trace class operator) if for any orthogonal basis $\{e_n\}$ in H the series

$$\sum_{n=1}^{\infty}(Ae_n, e_n)_H$$

converges.

In this case, the sum above does not depend on a particular choice of $\{e_n\}$. A symmetric operator $A \in \mathcal{L}(H)$ is nuclear if and only if there is an orthogonal basis $\{e_n\}$ in H such that

$$Ae_n = a_n e_n, \quad \sum_{n=1}^{\infty}|a_n| < \infty.$$

$\mathcal{L}_{(1)}(H)$ is the class of all nuclear operators in H.

Clearly, $\mathcal{L}_{(1)}(H) \subset \mathcal{L}_{(2)}(H)$.

A.4 Lemma. *Let H, E, X be Hilbert spaces.*

(i) *If $A \in \mathcal{L}_{(2)}(H,K)$, $B \in \mathcal{L}(K,X)$ or $A \in \mathcal{L}(H,K)$, $B \in \mathcal{L}_{(2)}(K,X)$, then $AB \in \mathcal{L}_{(2)}(H,X)$.*

(ii) *If A, $B \in \mathcal{L}_{(2)}(H)$, then $AB \in \mathcal{L}_{(1)}(H)$, $A^* \in \mathcal{L}_{(2)}(H)$, and $||A||_{\mathcal{H}} = ||A^*||_{\mathcal{H}}$.*

(iii) *If $A \in \mathcal{L}_{(2)}(H,K)$ and if $B \in \mathcal{L}(H,K)$ is such that $B(H) \subset A(H)$, then $B \in \mathcal{L}_{(2)}(H,K)$.*

(iv) *Let H be a Hilbert space and let $A:\ H \to C[0,1]$ be a continuous linear operator. Then composing this operator with the natural embedding $C[0,1] \to L^2[0,1]$ one gets a Hilbert–Schmidt operator $A:\ H \to L^2[0,1]$.*

Proof. The first two assertions can be found in [289]. To prove (iii) note that, assuming both spaces H and K to be infinite dimensional, one can find an injective operator $C \in \mathcal{L}_{(2)}(H,K)$ such that $A(H) = C(H)$. In this case put $D = C^{-1}B$. By the closed graph theorem D is continuous. Hence $B = CD \in \mathcal{L}_{(2)}(H,K)$. Finally, (iv) is proved in [368].

We shall say that an operator $D \in \mathcal{L}(H)$ is *diagonal* if there is an orthogonal basis $\{e_n\}$ in H such that $De_n = d_n e_n$.

A.5 Lemma. *For any invertible operator $A \in \mathcal{L}(H)$ there exist an orthogonal operator U, a symmetric Hilbert–Schmidt operator S, and a bounded diagonal operator D such that $I + S$ and D are invertible and*

$$A = U(I+S)D.$$

Proof. See [420].

A.6 Lemma. *Let E, H be Hilbert spaces, $A \in \mathcal{L}(E,H)$. Assume that H is separable and that A is injective. Then E is separable.*

Proof. Note that $A^*(H^*)$ is dense in E^* by the injectivity of A. Hence E^* is separable, which implies the separability of E.

B. Locally Convex Spaces.

In this section, we recall some standard facts from the theory of locally convex spaces. Those readers who prefer to deal with Gaussian measures on Banach spaces can omit this section without any loss.

Recall that a real vector space X is called a locally convex space if there is a family of seminorms $\mathcal{P} = (p_\alpha)_{\alpha\in A}$ on X which separates points (that is, for any nonzero element $x \in X$ there an index $\alpha \in A$ such that $p_\alpha(x) > 0$). The topology on X generated by this family $\mathcal{P}$ consists of all open sets which are unions of the basic neighborhoods of the form

$$U_{\alpha_1,\ldots,\alpha_n,\varepsilon_1,\ldots,\varepsilon_n}(a) = \{x:\ p_{\alpha_i}(x-a) < \varepsilon_i,\ i=1,\ldots,n\},\ \alpha_i \in A,\ a \in X.$$

Certainly, a family of seminorms defining the topology of a locally convex space is not unique.

A normed space is a particular case of a locally convex space (in this case the family $\mathcal{P}$ contains only one element).

The topological dual (the space of all continuous linear functionals) to a locally convex space X is denoted by X^*.

A typical example of a locally convex space arising in the theory of random processes is the space R^T of all real functions on a nonempty set T endowed with the topology of the pointwise convergence. In other words, the topology is defined by the family of seminorms $p_t(x) = |x(t)|,\ t \in T$. The dual to R^T coincides with the linear span of the functionals $x \mapsto x(t),\ t \in T$. The space R^T is called the product of T copies of R^1. In particular, if T is the set of natural numbers, then the corresponding space is denoted by R^∞. This space, consisting of all real sequences, is very important for the theory of Gaussian measures.

A set A in a locally convex space is said to be *symmetric* if $A = -A$. A convex set A is said to be *absolutely convex* (or convex, balanced) if $\lambda A \subset A$ for any scalar λ such that $|\lambda| \leq 1$. Clearly, this is the same as saying that A is convex and symmetric. By the absolutely convex (closed) hull of a set A we mean the minimal absolutely convex (closed) set containing A.

A continuous linear map P from a locally convex space X to R^n is called a finite-dimensional projection. Clearly, such a map can be written as

$$Px = f_1(x)e_1 + \cdots + f_n(x)e_n,$$

where $f_i \in X^*$ and $e_1, \ldots, e_n$ is a basis in R^n.

A function f on a locally convex space X is said to be cylindrical if there are a projection $P : X \to R^n$ and a Borel function ϕ on R^n such that $f(x) = \phi(Px)$.

If E is a linear space and F is some linear space of linear functionals on E separating points, then $\sigma(E, F)$ denotes the weakest locally convex topology on E such that all the elements of F are continuous. This is the topology of the pointwise convergence on F. The corresponding family of seminorms defining the topology is given by

$$p_f(x) = \big|f(x)\big|,\ f \in F.$$

Typical examples are the weak topology $\sigma(X, X^*)$ on a locally convex space X and the weak-$*$ topology $\sigma(X^*, X)$ on its dual. An important property of the topology $\sigma(E, F)$ is that the dual to $(E, \sigma(E, F))$ coincides (as a linear space) with F. In particular, any linear functional F on the space X^* with the topology $\sigma(X^*, X)$ has the form $F(f) = f(a)$ for some $a \in X$.

The *Mackey topology* $\tau_M(X, X^*)$ on a locally convex space X is the topology of uniform convergence on all absolutely convex $\sigma(X^*, X)$-compact subsets of X^*. The corresponding family of seminorms is described by the formula

$$p_Q(x) = \sup_{f \in Q}\big|f(x)\big|, \quad Q \in \mathcal{Q},$$

where $\mathcal{Q}$ stands for the collection of all absolutely convex $\sigma(X^*, X)$-compact subsets of X^*. This is the strongest locally convex topology on X such that the dual remains X^* (see [405]).

In a similar way one defines the Mackey topology $\tau_M(X^*, X)$ on X^* by means of the seminorms

$$p_K(f) = \sup_{x \in K}\big|f(x)\big|, \quad K \in \mathcal{K},$$

where $\mathcal{K}$ stands for the collection of all absolutely convex $\sigma(X, X^*)$-compact subsets of X. According to a classical result (see [405]), any linear functional F on the space X^* which is continuous in the topology $\tau_M(X^*, X)$ has the form $F(f) = f(a)$ for some $a \in X$.

A net $\{x_\lambda\}$ in a locally convex space X is called fundamental (or a Cauchy net) if it is fundamental with respect to any seminorm p_α from some family which defines the topology of X. We say that X is complete if any fundamental net in X is converging. We say that X is sequentially complete if this holds for any countable fundamental sequence in X.

Recall that a topological space T is called metrizable if the topology of T is generated by a metric on T.

A complete metrizable locally convex space is called a *Fréchet space*.

Typical examples are:

(1) all Banach spaces,

(2) the countable product of lines R^∞,
(3) the Schwartz space $S(R^n)$ of rapidly decreasing infinitely differentiable functions on R^n,
(4) the space $C_b^\infty(U)$ of functions possessing bounded derivatives in a domain U, equipped with the seminorms

$$p_n(f) = \sup_U |f^{(n)}(x)|,$$

(5) the subspace $\mathcal{D}_m(R^n)$ in $C_b^\infty(R^n)$, formed by all the functions with supports in the ball of radius m centered at the origin.

Note that the spaces mentioned in (2)–(5) are nonnormable. Important examples of spaces that do not belong to the class of Fréchet spaces are: infinite-dimensional normed spaces with the weak topology, dual spaces to nonnormable Fréchet spaces, the Schwartz space $\mathcal{D}(R^n)$ of smooth compactly supported functions on R^n with the topology of the inductive limit of the spaces $\mathcal{D}_m(R^n)$, and the space of distributions $\mathcal{D}'(R^n)$.

Recall that any locally convex space X is completely regular (see [143], p. 32). In particular (see [143], p. 19), for any compact set $K \subset X$ and any open set U containing K, there is a continuous function $f : X \to [0, 1]$, such that $f = 1$ on K and $f = 0$ outside of U. Another useful property of such spaces is that any continuous function on K admits a continuous extension to X with the same sup-norm.

B.1 Lemma. *Let K be a metrizable compact in a locally convex space X. Then its absolutely convex closed hull $\widetilde{K}$ is metrizable. $\widetilde{K}$ is compact provided X is sequentially complete.*

Proof. See [389] or [59]. A short proof is given in [52].

Now we describe one technical construction from the theory of locally convex spaces which has a lot of applications in infinite-dimensional analysis. Let X be a locally convex space, $B \subset X$ a bounded absolutely convex sequentially closed set. Denote by E_B the linear span of B. The gauge function p_B of B, defined by

$$p_B(x) = \inf\{t > 0 \colon\ x \in tB\},$$

is a norm on E_B. The natural embedding of $(E_B,\ p_B)$ into X is continuous. If, in addition, B is sequentially complete, then E_B is a Banach space (see [143], Lemma 6.5.2). In particular, this is the case if B is the absolutely convex closed hull of a compact subset in a sequentially complete locally convex space.

C. σ-fields in Locally Convex Spaces.

Recall that for any collection of functions G on a set X there is the smallest σ-field $\mathcal{E}(X, G)$ with respect to which all functions from G are measurable. This σ-field is generated by all the sets $\{x \in X : g < c\},\ g \in G,\ c \in R^1$.

In this section we discuss three important σ-fields in locally convex spaces which arise in connection with Gaussian measures. Let X be a locally convex space.

A set $C \subset X$ is said to be a cylindrical set in X (or a cylinder in X) if $C = P^{-1}(B)$, where $P : X \to R^n$ is a finite-dimensional projection and $B \in \mathcal{B}(R^n)$. Clearly, the collection of all cylindrical subsets of a locally convex space X is an algebra, which is denoted by the symbol $\mathcal{R}(X)$.

Denote by $C(X)$ the space of all continuous functions on X and by $C_b(X)$ its subspace consisting of bounded functions.

$\sigma(X)$ is the σ-field generated by X^*;
$\mathcal{B}_0(X)$ is the σ-field generated by $C(X)$;
$\mathcal{B}(X)$ is the Borel σ-field of X, generated by all open sets.

Note that $\sigma(X)$ is exactly the σ-field generated by all cylindrical sets in X.

Clearly, $\sigma(X) \subset \mathcal{B}_0(X) \subset \mathcal{B}(X)$. It is easy to see that $\mathcal{B}_0(X) = \mathcal{B}(X)$ for any metric space X.

C.1 Proposition. *The equality $\sigma(X) = \mathcal{B}_0(X) = \mathcal{B}(X)$ holds in any of the following cases:*
(i) *X is a Polish space (that is, separable complete metrizable);*
(ii) *X is one of the spaces $\mathcal{D}(R^n)$, $\mathcal{D}'(R^n)$, $S'(R^n)$.*

However, in some important cases which we shall encouter below the equality above fails.

C.2 Example. *Let T be an uncountable set and let $X = R^T$. Then $\sigma(X) = \mathcal{B}_0(X)$ is smaller than $\mathcal{B}(X)$.*

Proof. The equality $\sigma(R^T) = \mathcal{B}_0(R^T)$ follows from the result below. It is easy to see that any set $A \in \sigma(X)$ has the form $A = \{x : (x(t_i)) \in B\}$ for some sequence $\{t_i\} \subset T$ and some $B \in \mathcal{B}(R^\infty)$. In particular, any single point set does not belong to $\sigma(R^T)$.

C.3 Example. *Let X be a nonseparable Banach space. Then $\sigma(X)$ is smaller than $\mathcal{B}_0(X) = \mathcal{B}(X)$.*

Proof. See [480], Proposition I.2.5.

C.4 Theorem. *Let X be a locally convex space equipped with the weak topology $\sigma(X, X^*)$. Then $\sigma(X) = \mathcal{B}_0((X, \sigma(X, X^*)))$.*

Proof. See [142].

C.5 Remark. Let X be a Polish space or let X be a σ-compact space. If a family G of continuous functions separates points in X, then $\mathcal{B}_0(X) = \mathcal{E}(X, G)$ (see [480], Theorem I.1.2).

D. Regularity Properties of Measures on Locally Convex Spaces.

In this section, we recall some basic notions and results about the regularity properties of measures on topological spaces. We shall consider only nonnegative measures. The proofs of the results below and further references can be found in [407] and [480].

Let μ be a measure on a σ-field $\mathcal{M}$. We say that a set A is μ-measurable if it is measurable with respect to the Lebesgue extension of μ, that is, it belongs to the Lebesgue completion of $\mathcal{M}$.

Let X be a topological space.

D.1 Definition.

(i) A countably additive measure on $\mathcal{B}(X)$ is called a Borel measure on X.
(ii) A countably additive measure on $\mathcal{B}_0(X)$ is called a Baire measure on X.
(iii) A Borel measure μ on X is said to be a Radon measure if for any $B \in \mathcal{B}(X)$ and any $\varepsilon > 0$ there is a compact set $K \subset B$ such that

$$\mu(B \backslash K) < \varepsilon.$$

The following classical result is very important.

D.2 Theorem. *Let X be a Polish space. Then any Borel measure on X is Radon.*

This theorem fails for general topological spaces. However, it can be extended to a wide class of spaces, including such spaces as $\mathcal{D}(R^n)$, $\mathcal{D}'(R^n)$, and $S'(R^n)$.

D.3 Definition. Let $\mathcal{A}$ be a σ-field (or algebra) of subsets of a topological space X.

(i) A measure μ on $\mathcal{A}$ is called tight if for any $\varepsilon > 0$ there is a compact set K in X such that $\mu(A) < \varepsilon$ for any $A \in \mathcal{A}$ disjoint with K.
(ii) A measure μ on $\mathcal{A}$ is said to be regular if

$$\mu(A) = \sup\{\mu(F) : \ F \subset A, \ F \in \mathcal{A} \text{ is closed}\} \quad \forall A \in \mathcal{A}.$$

It is known (see [480], Lemma I.3.1) that for any collection G of continuous functions on a topological space X, any measure on the σ-field $\mathcal{E}(X, G)$ is regular. In particular, all Baire measures and all measures on $\sigma(X)$ are regular.

Clearly, any Radon measure is tight. However, a tight measure need not be Radon (see [480] for the Dieudonné example of a Borel measure on a compact space which is not a Radon measure). The following very important result gives a partial compensation.

D.4 Theorem. *Let $\mathcal{A}$ be an algebra of subsets of a topological space X containing a basis of the topology. If a measure μ on $\mathcal{A}$ is regular and tight, then it admits a unique extension to a Radon measure on X. In particular, any Baire measure μ on a completely regular space X which is tight admits a unique extension to a Radon measure.*

D.5 Corollary. *Let X be a locally convex space and let μ be a tight measure on $\sigma(X)$. Then μ has a unique extension to a Radon measure on X.*

It should be noted that the Lebesgue extension may be not sufficient to give the extension which is guaranteed by the theorem above.

D.6 Example. *Let $X = R^T$, where T is an uncountable set and let μ be the Dirac measure at zero, considered as a Baire measure. Clearly, this is a tight measure, which obviously admits an extension to a Radon measure on X as the Dirac measure at zero. However, the point $x = 0$ is not measurable with respect to the Lebesgue extension of the measure μ considered on $\mathcal{B}_0(X)$! Indeed, $\mu^*(X\backslash\{0\}) = 1$, since the whole space is the only Baire set which contains $X\backslash\{0\}$.*

On the other hand, the next result shows that any Radon measure on a locally convex space can be specified by its values on $\sigma(X)$.

D.7 Proposition. *Let μ be a Radon measure on a locally convex space X. Then for any μ-measurable set A there is a set $B \in \sigma(X)$ such that*

$$\mu(A\Delta B) = 0.$$

Moreover, if $G \subset X^*$ is an arbitrary linear subspace separating points of X, then such a set B can be chosen in $\mathcal{E}(X, G)$.

D.8 Corollary. *Let μ be a Radon measure on a locally convex space X. Then the class of all bounded cylindrical functions on X is dense in $L^p(\mu)$ for any $p > 0$. The same is true for the linear space T formed by the functions $\exp(if)$, $f \in X^*$. Moreover, this assertion is valid if one replaces X^* by any linear subspace $G \subset X^*$, which separates points of X.*

D.9 Definition. A Borel measure μ on a topological space X is said to be τ-additive if for any increasing net of open sets $(U_\lambda)_{\lambda\in\Lambda}$ one has

$$\mu\Big(\bigcup_{\lambda\in\Lambda} U_\lambda\Big) = \lim \mu(U_\lambda).$$

It is known (see [480]) that any Radon measure is τ-additive. Below we shall encounter an example of a Gaussian τ-additive Borel measure which is not tight (hence is not Radon).

D.10 Definition. Let μ be a Borel measure on a topological space X. We say that a closed set $S_\mu \subset X$ is the topological support of μ if $\mu(X\backslash S_\mu) = 0$ and there is no smaller closed set with this property.

D.11 Proposition. *Any τ-additive (in particular, any Radon) measure μ on a topological space X has the support which can be defined by the formula*

$$S_\mu = \bigcap \{S \subset X : \ S \text{ is closed and } \mu(X\backslash S) = 0\}.$$

As we shall see below, the support of a Radon Gaussian measure is an affine subspace.

Let μ be a measure on a measurable space $(X, \mathcal{B})$, Y a space equipped with a σ-field $\mathcal{E}$, and let $f: X \to (Y, \mathcal{E})$ be a μ-measurable map. The measure

$$\mu \circ f^{-1} : \ A \mapsto \mu(f^{-1}(A))$$

on $\mathcal{E}$ is called the image of the measure μ under the map f.

The proof of the following important result can be found in [407].

D.12 Theorem. *Let μ be a Radon measure on a Hausdorff topological space X. Then for any Borel set B in a separable complete metric space M and any continuous mapping $f: \ B \to X$ the set $f(B)$ is μ-measurable.*

Recall that sets of the form indicated in Theorem D.12 are called *Souslin sets.* It is known that every Borel measure on a Souslin space is Radon (see [407]).

A very important object connected with a measure on a locally convex space is its Fourier transform.

D.13 Definition. Let X be a locally convex space and let μ be a measure on $\sigma(X)$. The Fourier transform $\widetilde{\mu}$ of μ is defined by the formula:

$$\widetilde{\mu}: \ X^* \to C, \quad \widetilde{\mu}(f) = \int_X \exp\big(if(x)\big)\mu(dx). \tag{D.1}$$

It is easy to see that two measures on $\sigma(X)$ with equal Fourier transforms coincide. According to Corollary D.8 the same is true for any Radon measures.

One might ask under which conditions is a function $\phi : X^* \to C$ the Fourier transform of a (Radon) measure on X. In the case R^n the classical Bochner theorem says that if and only if ϕ is continuous and positive-definite (see [480]). In general this is false in infinite dimensions (for example, as we shall see, no Borel measure on an infinite-dimensional Hilbert space has the function $\exp(-(x,x))$ for its Fourier transform). Sazonov's theorem [402] asserts that a function ϕ on a Hilbert space X is the Fourier transform of a Borel measure on X if and only if it is positive-definite and continuous in the topology generated by all seminorms of the form $x \mapsto \|Tx\|$, where T is a Hilbert–Schmidt operator on X. If X is the dual to a barrelled nuclear space Y, then the same is true for the Mackey topology of X. This is Minlos's theorem [341]. The role of Hilbert–Schmidt operators in both theorems was clarified by Kolmogorov [273]. Additional references may be found in [343, 434, 480]. For practical purposes, it is important that analogs of Bochner's theorem hold true in such spaces as R^∞, $\mathcal{S}(R^n)$, $\mathcal{S}'(R^n)$, $\mathcal{D}(R^n)$, and $\mathcal{D}'(R^n)$.

Note that if μ and ν are two measures defined on $\sigma(X)$ in a locally convex space X, then their product $\mu \times \nu$ is a measure on $\sigma(X \times X)$. However, the product of two Borel measures may be defined on a σ-field smaller than $\mathcal{B}(X \times X)$. But, as it follows from Theorem D.4, if μ and ν are Radon measures, then their product $\mu \times \nu$ admits a unique extension to a Radon measure on $X \times X$. By a product of Radon measures we always mean this extension.

D.14 Definition. Let μ and ν be Radon measures on a locally convex space X. Their convolution $\mu * \nu$ is defined as the image of the measure $\mu \times \nu$ on the space $X \times X$ under the mapping $X \times X \to X$, $(x,y) \mapsto x+y$.

D.15 Theorem. *Let μ and ν be Radon measures on a locally convex space X. Then for any Borel set $B \subset X$ the function $x \mapsto \mu(B-x)$ is ν-measurable and*

$$\mu * \nu(B) = \int_X \mu(B-x)\nu(dx).$$

In addition, $\mu * \nu = \nu * \mu$ and $\widetilde{\mu * \nu} = \widetilde{\mu}\widetilde{\nu}$.

In this article, we do not discuss cylindrical measures (not necessarily σ-additive); however it should be noted that the corresponding notion is useful even for the study of Radon measures.

D.16 Definition. Let X be a locally convex space. An additive set function m on the algebra of cylindrical sets $\mathcal{R}(X)$ is said to be a cylindrical measure on X if for any finite-dimensional projection P the set function $m \circ P^{-1}$ is a Borel measure on R^n.

The Fourier transform of a cylindrical measure m is defined by

$$\widetilde{m}(f) = \int_{R^1} \exp(it)\, m \circ f^{-1}(dt).$$

D.17 Proposition. *Let μ and λ be two Radon probability measures on a locally convex space X. Assume that there exists a positive-definite function $\phi : X^* \to C$, such that*

$$\widetilde{\lambda} = \phi\widetilde{\mu}.$$

Then there exists a Radon probability measure ν on X such that $\widetilde{\nu} = \phi$. In addition, $\lambda = \nu * \mu$.

Proof. Note that the function ϕ is continuous at the origin on finite-dimensional subspaces. By the Bochner theorem ϕ is the Fourier transform of a cylindrical measure ν on X. We shall prove that ν is tight: For any $\varepsilon > 0$ there is a compact set K such that $\nu(C) < \varepsilon$ for every cylindrical set $C \subset X$ disjoint with K. Let V be a compact set such that

$$\mu(X \backslash V) + \lambda(X \backslash V) < \varepsilon/2.$$

Let $K = V - V$. Assume that $C \in \mathcal{R}(X)$ is disjoint with K. By definition $C = P^{-1}(B)$, where $B \in \mathcal{B}(R^n)$ and $P : X \to R^n$ is a continuous linear mapping. Then $P(K) \in \mathcal{B}(R^n)$ and $S = P^{-1}(P(K)) \in \mathcal{R}(X)$. It follows from the equality $\widetilde{\lambda} = \widetilde{\nu}\widetilde{\mu}$ that for any cylindrical set E one has

$$\lambda(E) = \int_X \nu(E-x)\mu(dx).$$

Therefore

$$1-\varepsilon/2<\lambda(S)=\int\limits_X \nu(S-x)\mu(dx)\le\int\limits_S \nu(S-x)\mu(dx)+\varepsilon/2\le\nu(S-S)+\varepsilon/2.$$

Hence $\nu(S-S)>1-\varepsilon$ and $\nu(X\backslash(S-S))<\varepsilon$. Note that $C\subset X\backslash(S-S)$. Indeed, if $x\in S-S$, then $x=x_1-x_2,\ x_1,\ x_2\in S$. So $x=v_1-v_2+k$, where $v_1,\ v_2\in V,\ k\in Ker(P)$. Now, if $x\in C$, then $v_1-v_2=x-k\in C$. This is a contradiction, since $v_1-v_2\in K$. Note that ν is automatically countably additive on $\mathcal{R}(X)$, since it is tight. According to Theorem D.4, ν admits a Radon extension to X.

The following result was proved in [79] for Banach spaces (earlier somewhat weaker results were found in [284] and [398]). We present here a shorter proof suggested in [41] and based on the method of [79].

D.18 Theorem. *Let μ be a Radon measure on a Fréchet space X. Then there exists a separable reflexive Banach space B such that:* (1) *B is embedded in X by a compact linear operator*; (2) $\mu(X\backslash B)=0$.

Proof. For any n there is a compact set K_n with $\mu(X\backslash K_n)<1/n$. We can take $c_n>0$ such that the set c_nK_n is contained in the ball (with respect to a metric generating the topology of X) of radius $1/n$ around the origin. The set $\bigcup\limits_{n=1}^{\infty}c_nK_n$ has compact closure K. Indeed, for any sequence $\{x_n\}$ in this set either an infinite subsequence is contained in one of the c_nK_n's or the whole sequence is converging to zero. According to Lemma 9.6.4 in [143] there is an absolutely convex compact set A such that K is a compact as a subset of the Banach space E_A. Hence the closure B_1 of the linear span of K in E_A is a separable Banach space of full measure. Now we can repeat the same trick with the measure μ considered on B_1. This is possible since all Borel subsets of B_1 are μ-measurable by virtue of Theorem D.12 and Theorem D.2. Thus, there is a separable Banach space B_2, continuously embedded in B_1, such that the unit ball of B_2 is compact in B_1 and $\mu(X\backslash B_2)=0$. According to the lemma on p. 124 of [127] there exists a third Banach space B_3, which is reflexive, continuously embedded in B_1 and in addition the unit ball of B_2 is bounded in B_3. Let B be the closure of B_2 in B_3. Clearly, B is a reflexive Banach space, has full measure, and its unit ball is compact in B_1, and hence in X.

D.19 Definition. We say that a sequence of Baire measures μ_n on a completely regular topological space converges weakly to a Baire measure μ if

$$\lim_{n\to\infty}\int\limits_X f(x)\mu_n(dx)=\int\limits_X f(x)\mu(dx)\quad\forall f\in C_b(X).$$

In the same way one defines weak convergence for nets of measures.

D.20 Theorem.

(i) *Let X be a locally convex space (or, more generally, a completely regular topological space), and let $\mathcal{M}$ be a family of Radon probability measures on X such that for any $\varepsilon>0$ there is a compact set K with*

$$\mu(X\backslash K)<\varepsilon\quad\forall\mu\in\mathcal{M}.$$

Then $\mathcal{M}$ is relatively compact in the space of Radon measures with the weak topology. If compact sets in X are metrizable (e.g., X is metrizable), then $\mathcal{M}$ is sequentially relatively compact, that is, any sequence in $\mathcal{M}$ contains a weakly converging subsequence.

(ii) *If X is a Polish space, then the condition above is also necessary for the relative weak compactness.*

(iii) *Let X be a complete barrelled nuclear locally convex space and let $\mathcal{M}$ be a family of Radon probability measures on X^* such that their Fourier transforms are equicontinuous on X. Then $\mathcal{M}$ is relatively compact in the weak topology.*

Finally, let us make a remark about random vectors and random processes. Let $(\Omega,\mathcal{F},P)$ be a probability space, and X a locally convex space. A measurable map $\xi:\Omega\to(X,\sigma(X))$ is called a *random vector* in X. The measure $P_\xi(C)=P(\xi^{-1}(C))$ is called the distribution (the law) of ξ. Clearly, each probability measure on $\sigma(X)$ can be obtained in this form (with the identical map $\xi(x)=x$). If we have a family of probability measures μ_n on X, then there is a family of independent random vectors ξ_n on one and the same probability space Ω such that $P_{\xi_n} = \mu_n$ (take $\Omega = \prod_n X_n$, $X_n = X$, $P = \otimes\mu_n$, $\xi_n(\omega) = \omega_n$).

Recall that a random process $\xi = (\xi_t,\ t \in T)$ is just a collection of random variables on some probability space $(\Omega, \mathcal{F}, P)$. Then for any Borel set $B \in \mathcal{B}(R^n)$ and all $t_1, \dots, t_n \in T$ the set

$$C_{t_1,\dots,t_n,B} = \left\{\omega:\ (\xi_{t_1}(\omega), \dots, \xi_{t_n}(\omega)) \in B\right\}$$

is in $\mathcal{F}$. Thus, we can define a measure on $\mathcal{R}(R^T)$ by

$$\mu^\xi(C_{t_1,\dots,t_n,B}) = P\big((\xi_{t_1}, \dots, \xi_{t_n}) \in B\big).$$

This measure is automatically countably additive and hence is uniquely extended to a countably additive measure on $\sigma(R^T)$. Its extension is denoted by the same symbol μ^ξ and is called the distribution of ξ in the function space (or the measure generated by ξ). Conversely, any probability measure μ on $\sigma(R^T)$ is the distribution of the random process $\xi_t(\omega) = \omega(t)$, if we take $\Omega = R^T$, $P = \mu$.

Note that for any finite collection $t_1, \dots, t_n \in T$ the formula above defines the probability measure $P_{t_1,\dots,t_n}$ on R^n that is called the finite-dimensional distribution of ξ. Clearly, the image of $P_{t_1,\dots,t_n,t_{n+1}}$ under the natural projection $R^{n+1} \to R^n$ coincides with $P_{t_1,\dots,t_n}$. The celebrated Kolmogorov's theorem [271] asserts that, conversely, if for any (not arranged) collection $t_1, \dots, t_n \in T$ there is a probability measure $P_{t_1,\dots,t_n}$ on R^n such that the above-mentioned property holds, then there exists a probability measure P whose finite-dimensional projections are the $P_{t_1,\dots,t_n}$'s.

Chapter 1
Linear-Topological Properties of Gaussian Measures

1. Gaussian Measures: Definitions and Examples.

1.1 Definition. A probability measure γ on a line is called Gaussian if it is either a Dirac measure δ_a or has density

$$p(\cdot, a, \sigma) : t \mapsto \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(t-a)^2}{2\sigma}\right)$$

with respect to the Lebesgue measure. In the latter case the measure γ is called nondegenerate.

The parameters a and σ are called the mean and dispersion (covariance) respectively. The measure with the density $p(\cdot, 0, 1)$ is called standard. By a Gaussian random variable one means a random variable with Gaussian distribution. Clearly, any Gaussian variable of this type can be expressed as $\sigma^{1/2}\xi + a$, where ξ is a random variable with the standard Gaussian distribution.

Put

$$\Phi(t) = \int_{-\infty}^{t} p(s, 0, 1)ds.$$

Denote by Ψ the inverse function to Φ.

A useful property of Gaussian random variables is that two orthogonal jointly Gaussian random variables are independent. Moreover, if random variables $\xi_1, \dots, \xi_n$ have joint centered Gaussian distribution and ξ_1 is orthogonal to $\xi_2, \dots, \xi_n$, then ξ_1 is independent of the σ-field generated by $\xi_2, \dots, \xi_n$. Note that a random vector $\xi = (\xi_1, \xi_2)$ whose coordinates ξ_1 and ξ_2 are Gaussian random variables need not be Gaussian.

The following classical result plays a very important role in the theory of Gaussian measures. The proof can be found in [320].

1.2 Theorem. *Let ξ_n be a sequence of independent centered Gaussian random variables with dispersions σ_n. Then the following conditions are equivalent:*

(1) *the series $\sum_{n=1}^{\infty} \xi_n$ converges a.s.;*
(2) *the series in* (1) *converges in probability;*
(3) *the series in* (1) *converges in L^2;*
4) $\sum_{n=1}^{\infty} \sigma_n < \infty$.

1.3 Definition. A probability measure γ on a locally convex space X is called Gaussian if it is defined on the σ-field $\sigma(X)$ generated by X^*, and for any $f \in X^*$ the induced measure $\gamma \circ f^{-1}$ on the line is Gaussian.

A random vector is called Gaussian if it induces a Gaussian measure.

Note that sometimes the following more general definition is used (cf. [161]). Recall that a linear space E with a σ-field $\mathcal{E}$ is said to be a measurable linear space if the operations $X \times X \to X$, $(x, y) \mapsto x + y$, and $R^1 \times X \to X$, $(t, x) \mapsto tx$, are measurable with respect to the σ-field $\mathcal{E}$ on E and the Borel σ-field on the line.

1.4 Definition. Let $(E, \mathcal{E})$ be a measurable linear space. A measurable map ξ with values in E is called a Gaussian random vector in E if for any pair (ξ_1, ξ_2) of independent copies of ξ and for any real ϕ the mappings

$$\sin\phi\xi_1 + \cos\phi\xi_2, \qquad \cos\phi\xi_1 - \sin\phi\xi_2$$

are independent copies of ξ.

1.5 Lemma. *If $\mathcal{E} = \sigma(X)$, then both definitions are equivalent.*

The second definition may be useful for studying random elements with values in topological vector spaces with trivial duals (for example, in L^p with $0 < p < 1$ or in the space of measurable functions on $[0, 1]$ equipped with the topology of the convergence in measure). It should be noted that a locally convex space (or even a Banach space) with the Borel σ-algebra need not be a measurable linear space (see [480]).

One can check (see [119]) that a random vector ξ is Gaussian if and only if for any random vector η, which is independent of ξ and has the same distribution, the random vectors $\xi - \eta$ and $\xi + \eta$ are independent.

There is a useful characterization of the Gaussian property by means of triple convolutions. We introduce the following notations. Let X be a linear space. For any $\phi \in R^1$ set

$$\begin{aligned}
\alpha &= 1/3 + 2\cos\varphi/3,\\
\beta &= 1/3 - \cos\varphi/3 - 3^{-1/2}\sin\varphi,\\
\gamma &= 1/3 - \cos\varphi/3 + 3^{-1/2}\sin\varphi.
\end{aligned}$$

Note that $\alpha + \beta + \gamma = 1$ and the matrix

$$T_\varphi = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}$$

is orthogonal. Using this matrix the operator $T_\phi : X^3 \to X^3$ can be defined by $(x, y, z) \mapsto T_\phi(x, y, z)$.

1.6 Theorem. *Let X be a locally convex space.*

(i) *Let μ be a Gaussian measure on X. Then the measure*

$$\mu^3 = \mu \times \mu \times \mu$$

is invariant under T_φ for any φ.

(ii) *Let μ be a probability measure on $\sigma(X)$ such that μ^3 is invariant under T_φ for any φ such that $\varphi \neq 2k\pi/3$. Then μ is a Gaussian measure.*

Proof. See [455].

1.7 Definition. Let X be a locally convex space and let μ be a measure on $\sigma(X)$ such that $X^* \subset L^2(\mu)$. The element a_μ in the algebraic dual $(X^*)'$ to X^*, defined by the formula

$$a_\mu(f) = \int\limits_X f(x)\mu(dx), \tag{1.1}$$

is called the mean of μ. The operator $R_\mu : X^* \to (X^*)'$, defined by the formula

$$R_\mu(f)(g) = \int_X [f(x) - a_\mu(f)]\,[g(x) - a_\mu(g)]\,\mu(dx), \tag{1.2}$$

is called the covariance operator of μ.

As we shall see below, $a_\mu \in X$ and $R_\mu(X^*) \subset X$ for any Radon Gaussian measure μ.

Let γ be a Gaussian measure on a locally convex space X.

X^*_γ will stand for the closure of the set $\{f - a_\gamma(f),\ f \in X^*\}$ in $L^2(\gamma)$. This space is called the *reproducing kernel Hilbert space* (RKHS).

We set

$$H(\gamma) := R_\gamma(X^*_\gamma), \qquad \|h\|_\gamma = \sup\{f(h) : f \in X^*, R_\gamma(f)(f) \le 1\},$$

if $R_\gamma(X^*_\gamma) \subset X$. In this case we call $H(\gamma)$ the *Cameron–Martin space.*

Note that $a_\gamma(g)$ is defined by (1.1) for any $g \in X^*_\gamma$. In a similar way the operator R_γ can be extended by (1.2) to the Hilbert space X^*_γ.

1.8 Theorem. *A probability measure γ on a locally convex space X is Gaussian if and only if its Fourier transform has the following form:*

$$\widetilde{\gamma}(f) = \exp\Bigl(iL(f) - \frac{1}{2}B(f,f)\Bigr), \tag{1.3}$$

where L is a linear function on X^* and B is a symmetric bilinear function on $X^* \times X^*$ such that $B(f,f) \ge 0$. In addition, γ is symmetric precisely when $L = 0$.

Proof. See [480].

Clearly, in (1.3) one has $B(f,f) = R_\gamma(f)(f)$. To simplify notations we put

$$\sigma(f) := R_\gamma(f)(f).$$

1.9 Example. *A measure γ on R^n is Gaussian exactly when its Fourier transform is given by*

$$\widetilde{\gamma}(y) = \exp\bigl(i(y,a) - (Kx,x)/2\bigr),$$

where a is a vector and K is a nonnegative matrix. The measure γ has a density if and only if K is nondegenerate.

Clearly, Gaussian measures on R^n can be described as the images of the standard Gaussian measure on R^n (which is the product of n standard Gaussian measures on lines) under the affine maps $x \mapsto \sqrt{K}x + a$.

1.10 Theorem. *Let γ be a Gaussian measure on a separable Hilbert space X and let X^* be identified with X by the Riesz representation. Then there exist a vector $a \in X$ and a symmetric nonnegative nuclear operator K such that the Fourier transform of γ equals*

$$x \mapsto \exp\bigl(i(a,x) - (Kx,x)/2\bigr). \tag{1.4}$$

Conversely, for any pair (a, K) as above the function (1.4) is the Fourier transform of a Gaussian measure on X. In this case a is the mean of γ and K is its covariance operator.

Proof. By definition, the Fourier transform of γ is given by (1.1), where $x \mapsto m(x)$ is linear and $x \mapsto B(x,x)$ is a quadratic form. By the Lebesgue theorem $\widetilde{\gamma}$ is continuous. This implies the continuity of both functions m and B. Therefore, there are a vector a and a bounded symmetric nonnegative operator K such that

$$m(x) = (a,x), \ \ B(x,x) = (Kx,x).$$

Clearly, K is a compact operator. Indeed, if $x_n \to 0$ weakly, then $\widetilde{\gamma}(x_n) \to 0$ by the Lebesgue theorem. Hence $\|K^{1/2}x_n\| \to 0$. Now there are three possibilities to see that K is nuclear. The first one is to prove that

$$\int_X (x,x)\gamma(dx) < \infty \tag{1.5}$$

(see Chapter 2). Then, assuming that $a = 0$, for any orthonormal basis $\{e_n\}$ in X one has

$$\sum_{n=1}^{\infty}(Ke_n, e_n) = \sum_{n=1}^{\infty}\int_X (x,e_n)^2\gamma(dx) = \int_X (x,x)\gamma(dx) < \infty.$$

The second possibility is to use Theorem 1.2. Indeed, we may assume that $a = 0$ (shifting the measure). Since K is compact and symmetric we can take the orthonormal basis $\{e_n\}$ consisting of the eigenvectors of K. Then the orthogonal centered Gaussian random variables (x, e_n) on (X, γ) are independent and

$$\sum_{n=1}^{\infty}(x,e_n)^2 < \infty$$

for any x. This implies (1.5). Finally, the third possibility is to apply Sazonov's theorem mentioned in Section D.

Note that in representation (1.4) the operator K coincides with R_γ. Hence, the closure of $X = X^*$ in $L^2(\gamma)$ (in the case $a = 0$) is the completion of X with respect to the norm $\|\sqrt{K}x\|_X$. Thus, if $\{e_n\}$ is an orthonormal basis in X formed by the eigenvectors of K corresponding to the eigenvalues k_n, then the completion above is identified with the weighted Hilbert space of sequences $\{x_n : \sum_n k_n x_n^2 < \infty\}$. The operator K extends naturally to this completion X_γ and $H(\gamma) = K(X_\gamma)$. However, if we consider K only on the initial space X, then

$$H(\gamma) = \sqrt{K}(X).$$

In the natural coordinates associated with $\{e_n\}$, formula (1.4) reads as

$$\widetilde{\gamma}(y) = \exp\Bigl(i\sum_n a_n y_n - \sum_n k_n y_n^2/2\Bigr).$$

It follows from the result above that if $\dim X = \infty$, then there is no countably additive measure on X with the Fourier transform $\phi(x) = \exp(-\|x\|^2/2)$.

1.11 Example. *Let γ_n be Gaussian measures on R^1. Then the measure*

$$\gamma = \prod_{n=1}^{\infty}\gamma_n$$

is Radon and Gaussian on R^∞.

Proof. Note that γ is defined on the σ-field generated by the coordinate functions $x = (x_n) \mapsto x_n$, which coincides with the Borel σ-field of R^∞ by Proposition 0.C.1. Since R^∞ is a Polish space, γ is a Radon measure. Any continuous linear functional f on R^∞ is a finite linear combination of the coordinate functions. Therefore, for some n the induced measure $\gamma \circ f^{-1}$ coincides with the measure

$$\Bigl(\prod_{i=1}^{n}\gamma_i\Bigr) \circ f_n^{-1},$$

where $f_n((x_1,\dots,x_n)) = f((x_1,\dots,x_n,0,0,\dots))$.

We shall see below that this apparently special example provides, in fact, a representation of a general Radon Gaussian measure.

1.12 Example. *Let γ be the countable product of centered Gaussian measures on R^1 with covariances σ_n. Then γ is a Radon Gaussian measure on R^∞. In addition, γ is a Radon Gaussian measure on any weighted Hilbert space*

$$E_\alpha = \left\{ x \in R^\infty : \ \|x\|_\alpha^2 = \sum_{n=1}^{\infty} \alpha_n^2 x_n^2 < \infty \right\},$$

where $\sum_{n=1}^{\infty} \alpha_n^2 \sigma_n < \infty$.

Proof. It suffices to note that $\gamma(E_\alpha) = 1$.

The notion of Gaussian measure is closely related to the notion of Gaussian random process. The latter is defined as a family of random variables $(\xi_t)_{t\in T}$ such that their linear combinations are Gaussian. Indeed, the measure induced by such a process on the space of trajectories R^T with the pointwise convergence topology is Gaussian. Some aspects of this relation will be briefly discussed below.

1.13 Example.

(i) *Let $\xi_t, t \in T$, be a Gaussian random process on a set T. This means that for any $t_1, \ldots, t_n \in T$ the random vector $(\xi_{t_1}, \ldots, \xi_{t_n})$ has Gaussian distribution. Then the measure μ^ξ induced by ξ on the space of functions R^T with the topology of the pointwise convergence is Gaussian.*

(ii) *Let T be a nonempty set and let $X = R^T$. Then any function F of the form* (1.3) *is the Fourier transform of some Gaussian measure on X.*

Proof. It suffices to note that any continuous linear functional on R^T can be written as $x \mapsto c_1 x(t_1) + \cdots + c_n x(t_n)$ for a suitable choice of c_i and t_i (see [405]). To get (ii) it suffices to apply Kolmogorov's theorem on the existence of the measure with given finite-dimensional projections (see [199]).

We have seen in Theorem 1.10 that the analogue of this statement fails for infinite-dimensional Hilbert spaces. However, there is a class of infinite-dimensional locally convex spaces X such that any nonnegative continuous quadratic form on the dual space is the covariance of a Radon Gaussian measure on X. This class contains, for example, the duals to complete nuclear barrelled locally convex spaces.

1.14 Example. *In the situation of Example* 1.13 (ii) *let* $T = [0, 1]$, $L = 0$, *and*

$$B(\delta_a, \delta_b) = \min(a, b),$$

where $\delta_a(x) = x(a)$ and B is extended by linearity to linear combinations of such functionals. It is easy to check that $B(l, l) \geq 0$. Then the corresponding measure P^W on the path space is called a Wiener measure. For this measure P^W the following estimate holds:

$$\int_X |x(t) - x(s)|^4 P^W(dx) \leq C|t - s|^3.$$

Hence, by the classical Kolmogorov theorem (*see* [199]) *there exists a Radon measure P_1^W on the space of continuous functions $C[0, 1]$, such that its extension to the cylindrical σ-field $\sigma(X)$ coincides with P^W.*

Proof. Note that linear combinations of Dirac's functionals δ_a are dense in $C[0, 1]^*$ with the weak-$*$ topology, where the dual to $C[0, 1]$ is identified with the space of all signed Borel measures λ on $[0, 1]$. Moreover, any element in $C[0, 1]$ is the limit in the weak-$*$ topology of a sequence of linear combinations of Dirac's functionals. Hence, the Fourier transform of P_1^W has the following representation:

$$\widetilde{P_1^W}(\lambda) = \exp\Bigl(-\frac{1}{2}\int_0^1\int_0^1 \min(s, t)\lambda(ds)\lambda(dt)\Bigr).$$

Thus, P_1^W is a Gaussian measure.

Note that for any cylindrical set $C = \{x \in C[0, 1] : \ (x(t_1), \ldots, x(t_n)) \in B\}$, $B \in \mathcal{B}(R^n)$, one has

$$P^W(C) = c_n \int_B \exp\left(-\frac{1}{2}\left[\frac{u_1^2}{t_1} + \frac{(u_2 - u_1)^2}{t_2 - t_1} + \cdots + \frac{(u_n - u_{n-1})^2}{t_n - t_{n-1}}\right]\right) du_1 \ldots du_n,$$

$$c_n = [(2\pi)^n t_1(t_2 - t_1)\dots(t_n - t_{n-1})]^{-1/2}.$$

Let us find the eigenvectors and eigenvalues of the operator R_W with the kernel $\min(t,s)$ on $L^2[0,1]$ which is the covariance operator of the Wiener measure P^W regarded on $L^2[0,1]$. Since this is a symmetric compact operator its eigenvectors form an orthonormal basis in $L^2[0,1]$. We have for almost all t

$$\int_0^1 \min(t,s)f_\lambda(s)ds = \lambda f_\lambda(t).$$

Hence almost everywhere

$$\int_0^t sf_\lambda(s)ds + \int_t^1 f_\lambda(s)ds = \lambda f_\lambda(t).$$

Clearly, in the case $\lambda \neq 0$ f_λ is absolutely continuous. Differentiating the equality above we get a.e.

$$\int_t^1 f_\lambda(s)ds = \lambda f'_\lambda(t).$$

Hence f'_λ is absolutely continuous and differentiating the last equality we arrive at the differential equation

$$\lambda f''_\lambda(t) = -f_\lambda(t),\ f_\lambda(0) = f'_\lambda(0) = 0.$$

Therefore, we get the countable collection

$$f_n(t) = \sqrt{2}\sin\frac{t}{\sqrt{\lambda_n}},\ \lambda_n = \left(\pi\left(n - \frac{1}{2}\right)\right)^{-2}.$$

In a similar way one proves that $\lambda = 0$ is not an eigenvalue.

In particular, this calculation shows that the embedding of the space $H(P^W) = \sqrt{R_W}(L^2[0,1])$ into $L^2[0,1]$ is not a nuclear operator (but certainly it is a Hilbert–Schmidt operator). One can check that $H(P^W)$ coincides with the Sobolev space $W_0^{1,2}$ of functions f on $[0,1]$ such that f is absolutely continuous, $f' \in L^2[0,1]$ and $f(0) = 0$.

In Example 1.13 (ii) the measure μ^ξ may fail to have a Radon extension to R^T. Now we shall examine an example of a Gaussian measure which is not tight (and, therefore, does not admit Radon extensions).

1.15 Example. *Let γ on $R^c = R^{[0,1]}$ be the product of continuum standard one-dimensional Gaussian measures (this corresponds to independent random variables ξ_t above). Then for any compact subset K of R^c one has $\gamma^*(K) = 0$.*

Proof. Note that nonvoid compact subsets of R^c do not belong to $\sigma(R^c)$. Indeed, each compact set is contained in a product of the continuum number of segments $[-C_t, C_t]$. One can find a number N such that for an infinite number of indices t the inequality $C_t \leq N$ holds. Thus, the initial compact is covered by the set $\Pi_1 \times \Pi_2$, where Π_1 is the countable product of copies of the segment $[-N,N]$ and Π_2 is the product of the lines corresponding to the remaining coordinates. Clearly, $\Pi_1 \times \Pi_2$ belongs to $\sigma(X)$ and has measure zero.

1.16 Proposition. *Let γ be a Gaussian measure on a locally convex space X, $g \in X^*_\gamma$. Then:*

(i) *the measure $\gamma \circ g^{-1}$ on the line is Gaussian with mean $a_\gamma(g)$ and dispersion $\sigma(g) = R_\gamma(g)(g)$;*

(ii) *the measure ν on X defined by the density*

$$\rho(x) = \exp\left(g(x) - a_\gamma(g) - \frac{1}{2}R_\gamma(g)(g)\right)$$

with respect to the measure γ is Gaussian with the Fourier transform

$$\widetilde{\nu}(f) = \exp\left(-iR_\gamma(g)(f) + ia_\gamma(f) - \frac{1}{2}R_\gamma(f)(f)\right). \tag{1.6}$$

Proof. Let $\{f_n\}$ be a sequence of elements of X^* which converges to g in $L^2(\gamma)$. Then it converges to g in measure and we have

$$\int_X \exp\big(itg(x)\big)\gamma(dx) = \lim_{n\to\infty}\int_X \exp\big(itf_n(x)\big)\gamma(dx) = \lim_{n\to\infty}\exp\left[ita_\gamma(f_n) - \frac{1}{2}t^2\sigma(f_n)\right].$$

Therefore, there exist the limits

$$m = \lim_{n\to\infty} a_\gamma(f_n),$$
$$d = \lim_{n\to\infty} \sigma(f_n).$$

This means that the measure $\gamma\circ g^{-1}$ is Gaussian.

It follows from the above that the function $\exp|g|$ is γ-integrable, so the function ρ defines a finite Radon measure ν. Let $f \in X^*$. Put

$$k = \exp\left[-a_\gamma(g) - \frac{1}{2}\sigma(g)\right].$$

Consider the following function of real argument z:

$$\phi(z) = k\int_X \exp\big[i(f(x) - zg(x)\big]\gamma(dx).$$

We have by assertion (i):

$$\begin{aligned}\varphi(z) &= k\exp\left(ia_\gamma(f-zg) - \frac{1}{2}R_\gamma(f-zg)(f-zg)\right)\\ &= k\exp\left[ia_\gamma(f) - iza_\gamma(g) - \frac{1}{2}\sigma(f) - \frac{1}{2}z^2\sigma(g) + zR_\gamma(f)(g)\right].\end{aligned}$$

It remains to note that ϕ admits a holomorphic extension to the complex plane. By virtue of the Lebesgue theorem $\phi(z) \to \widetilde{\nu}(f)$ as $z \to i$. On the other hand, the limit of the right-hand side of the relationship above coincides with the right-hand side of (1.6).

1.17 Corollary. *Let γ be a Gaussian measure on a locally convex space X. Then for any $h \in X$ such that $h = R_\gamma(g)$, where $g \in X^*_\gamma$, the measures γ and γ_h are equivalent and the corresponding Radon–Nikodym density is given by expression* (1.6). *In addition,*

$$1 - \exp(-\|h\|_\gamma^2/8) \le \|\gamma_h - \gamma\| \le \|h\|_\gamma. \tag{1.7}$$

Proof. It is easy to check that the Fourier transform of the measure ν with density ρ given by (1.6) coincides with the Fourier transform of the measure γ_h. Now one can prove estimate (1.7) in the same way as estimate (2.1) below.

1.18 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X. Then its mean is an element of X and $R_\gamma(X^*) \subset X$. Moreover, $R_\gamma(X^*_\gamma) \subset X$.*

Proof. There are several ways of proving this result. For example, if compact sets in X have compact absolutely convex closed hulls (say, if X is complete), it is possible to check that for any $g \in X^*_\gamma$ the functionals a_γ and $R_\gamma(g)$ are continuous in the Mackey topology $\tau_M(X^*, X)$ (this is almost immediate). This gives the assertion in the case where X is complete (see Section B above). However, the reduction of the general case to the case of a complete space requires some additional constructions which are almost the same as in the proof below, which works for any space.

We shall use the symmetrization γ^s of the measure γ which is defined by the formula

$$\gamma^s(A) = \gamma * \gamma_1(2^{-1/2}A), \text{ where } \gamma_1(A) = \gamma(-A).$$

The Fourier transform of γ^s is given by

$$\widetilde{\gamma^s}(f) = \exp\bigl(-R_\gamma(f)(f)/2\bigr) = \bigl|\widetilde{\gamma}(f)\bigr|.$$

Therefore, γ^s is a Radon Gaussian measure with zero mean. According to Proposition 0.D.17 the function

$$\phi(f) = \exp\bigl(ia_\gamma(f)\bigr)$$

is the Fourier transform of a Radon probability measure ν on X. Clearly, ν is a Gaussian measure such that for any $f \in X^*$ the measure $\nu \circ f^{-1}$ is a Dirac measure. This implies that the support of ν is a singleton a and ν itself is a Dirac measure. Indeed, the support S_ν of ν exists by Proposition 0.D.11. If a and b are two different points of S_ν, then by the Hahn–Banach theorem there is a functional $f \in X^*$ such that $f(a) > 0$ and $f(b) < 0$. By the definition of support, $\nu(f \geq 0) < 1$ and $\nu(f \leq 0) < 1$. This is a contradiction, since one of these two sets should have full measure, $\nu \circ f^{-1}$ being a Dirac measure. Thus, a_γ is in X.

To prove the second assertion consider the measure λ defined in Proposition 1.16 (ii). By virtue of Proposition 1.16, λ is a Gaussian measure with mean $a_\gamma - R_\gamma(g)$ (certainly, it is a Radon measure). Now we apply the first assertion.

Note that the covariance function $R_\gamma(f)(f)$ of a Gaussian measure is automatically sequentially continuous on X^* with the weak-$*$ topology (by virtue of the Lebesgue theorem). However, even if γ is a Radon measure, it need not be continuous. For example, if X is equipped with the weak topology, then the Fourier transform of γ is continuous in the weak-$*$ topology if and only if the measure γ is concentrated on a finite-dimensional subspace (since any weak-$*$ continuous seminorm is finite-dimensional). On the other hand, as has been pointed out in the proof above, the Fourier transform of γ is continuous on X^* with the Mackey topology.

There are examples (see [455]) showing that Theorem 1.18 may fail for non-Radon measures. However, this theorem holds for measures on the space $X = R^T$ for any set T. Indeed, the dual to this space is the direct sum of lines; hence X coincides with the algebraic dual to X^*. In this connection, we mention the following nice result obtained by Talagrand [454].

1.19 Theorem. *Any Gaussian measure on R^T is τ-additive.*

This property may be useful since any Gaussian measure on $\sigma(X)$ can be extended to a Gaussian measure on the space R^T with $T = X^*$ (X embeds into R^T by considering the elements of X as functions on X^*). The extension is given by the formula

$$\gamma\Bigl(\widetilde{x} \in R^T : (\widetilde{x}(t_1), \ldots, \widetilde{x}(t_n)) \in B\Bigr) = \gamma\Bigl(x \in X : (t_1(x), \ldots, t_n(x)) \in B\Bigr), \quad t_i \in X^*, \ B \in \mathcal{B}(R^n).$$

Below we shall make use of the following important result, which can be deduced from Proposition 0.D.17 and Theorem 2.5 below.

1.20 Theorem. *Let X be a locally convex space, $G \subset X^*$ a linear subspace separating points in X, and let V and W be two nonnegative quadratic forms defined on G. Assume that the function $g \mapsto \exp\bigl(-V(g)\bigr)$ on G coincides with the Fourier transform of a Radon Gaussian measure γ on X and that*

$$W(g) \leq V(g) \quad \forall g \in G.$$

Then there is a Radon Gaussian measure λ on X, whose Fourier transform coincides on G with the function $g \mapsto \exp\bigl(-W(g)\bigr)$. In addition, $\lambda(C) \geq \gamma(C)$ for any closed absolutely convex set C.

1.21 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X. For any bounded γ-measurable function and any $p \in [1, \infty)$ the mapping*

$$\bigl(H(\gamma),\ \|\cdot\|_\gamma\bigr) \to L^p(\gamma),\ h \mapsto f(\cdot + h),$$

is continuous.

Proof. This result can be easily deduced from Corollary 1.17 taking into account that $H(\gamma) \subset X$. See [63] for the details.

1.22 Corollary. *Let γ be a Radon Gaussian measure on a locally convex space and let A be a set of positive γ measure. Then there exists $c > 0$ such that $cU_H \subset A - A$, where U_H is the closed unit ball in the Hilbert space $H(\gamma)$.*

Proof. The map $h \mapsto \gamma\big((A+h)\cap A\big)$ is continuous by Theorem 1.21 and positive at the origin.

Gaussian measures constitute a subclass of the class of stable measures.

1.23 Definition. A probability measure μ on a locally convex space X, defined on $\sigma(X)$, is called stable of order $\alpha \in (0,2]$ if it coincides with the distribution of a random vector ξ with the following property: for each n there exists a vector $a_n \in X$, such that for any independent copies $\xi_1, \ldots, \xi_n$ of ξ the random vector

$$(\xi_1 + \cdots + \xi_n)/n^{-1/\alpha} - a_n$$

has the same distribution μ.

A measure μ is said to be strictly convex of order α if the condition above holds for $a_n = 0$. Note that a measure is Gaussian if and only if it is stable of order 2. It should be noted that Dirac measures are the only measures stable of order bigger than 2. A detailed discussion of one-dimensional stable distributions can be found in [505]. A typical example of a nongaussian stable measure is the measure on the line with the Fourier transform $\exp(-|y|^\alpha)$, where $\alpha \in (0,2)$. It is known (see [452]) that any symmetric stable measure is a mixture of some Gaussian measures. Some classes of mixtures of Gaussian measures are considered in [443, 480].

1.24 Remark. Unlike Gaussian measures, stable distributions of order less than 2 cannot be characterized by the stability of the one-dimensional projections. There exists an example [200] of a probability measure on the plane which is not stable, but whose one-dimensional projections are stable of some order α. On the other hand [139, 200], a measure μ is stable provided all of its two-dimensional projections are stable of order α.

2. Gaussian Measures on $\mathbf{R}^n$.

2.1 Proposition. *Let γ be a Gaussian measure on R^n. Then, for any $h \in R_\gamma(R^n)$, one has*

$$1 - \exp(-\|h\|_\gamma^2/8) \le \|\gamma - \gamma_h\| \le 2\sqrt{1 - \exp(-\|h\|_\gamma^2/4)}, \tag{2.1}$$

where

$$\|h\|_\gamma = \sup\{|(v,h)| : R_\gamma(v)(v) \le 1\}.$$

Proof. Clearly, it suffices to prove (2.1) for centered measures. Let γ be the standard Gaussian measure on R^n with density p. We have

$$\begin{aligned}
\|\gamma_h - \gamma\| &= \int_{R^n} |p(x-h) - p(x)|\,dx \\
&\le \Big(\int_{R^n} (\sqrt{p(x-h)} - \sqrt{p(x)})^2 dx\Big)^{1/2} \Big(\int_{R^n} (\sqrt{p(x-h)} + \sqrt{p(x)})^2 dx\Big)^{1/2} \\
&\le \Big(2 - 2\int_{R^n} \sqrt{p(x-h)p(x)}\,dx\Big)^{1/2} \Big(2 + 2\int_{R^n} \sqrt{p(x-h)p(x)}\,dx\Big)^{1/2} \\
&= 2\sqrt{1 - \Big(\int_{R^n} \sqrt{p(x-h)p(x)}\,dx\Big)^2}.
\end{aligned}$$

On the other hand,

$$\left|1-\int_{R^n}\sqrt{p(x-h)p(x)}dx\right| = \left|\int_{R^n}\sqrt{p(x)}\left[\sqrt{p(x)}-\sqrt{p(x-h)}\right]dx\right|$$
$$\leq \int_{R^n}\frac{\sqrt{p(x)}}{\sqrt{p(x)}+\sqrt{p(x-h)}}|p(x)-p(x-h)|dx$$
$$\leq \int_{R^n}|p(x)-p(x-h)|dx.$$

Note that

$$\int_{R^n}\sqrt{p(x)p(x-h)}dx = \exp\Big(-\sum_{i=1}^n h_i^2/8\Big).$$

In addition, $R_\gamma(v)=v$, hence

$$\|h\|_\gamma^2 = \sum_{i=1}^n h_i^2.$$

Thus, we arrive at (2.1) in our special case. Now it remains to note that any centered Gaussian measure γ on R^n is the image of the standard Gaussian measure μ under some linear map T, and

$$\gamma_{Th}-\gamma = (\mu_h-\mu)\circ T^{-1},\ \|Th\|_\gamma = \|h\|_\mu.$$

Recall that Ψ is the inverse function to the standard Gaussian distribution function.

2.2 Theorem. *Let γ be a centered Gaussian measure on R^n. Then for any Borel sets A and B and any $\lambda\in[0,1]$, one has*

$$\gamma(\lambda A+(1-\lambda)B)\geq\gamma(A)^\lambda\gamma(B)^{1-\lambda}. \tag{2.2}$$

If A and B are convex, then

$$\Psi\big(\gamma(\lambda A+(1-\lambda)B)\big)\geq\lambda\Psi\big(\gamma(A)\big)+(1-\lambda)\Psi\big(\gamma(B)\big). \tag{2.3}$$

In particular, for any symmetric convex set A and any vector a,

$$\gamma(A+a)\leq\gamma(A). \tag{2.4}$$

2.3 Lemma. *Let μ and ν be two symmetric Gaussian measures on R^n whose covariance matrices have the same eigenvectors and the eigenvalues (α_i) and (β_i), respectively. Assume that $\alpha_i\leq\beta_i$ for all i. Then, for any convex Borel set $V\subset R^n$ symmetric about the origin, one has $\mu(V)\geq\nu(V)$.*

Proof. Let γ be the centered Gaussian measure on R^n with covariance having eigenvalues $(\beta_i-\alpha_i)$ and the same eigenvectors as R_μ and R_ν. Note that $\nu=\mu*\gamma$. Therefore,

$$\nu(V)=\int_{R^n}\mu(V-x)\gamma(dx)\leq\mu(V),$$

since $\mu(V-x)\leq\mu(V)$ for any $x\in R^n$ by Theorem 2.2.

2.4 Corollary. *Let $\xi_1,\dots\xi_n$ be random variables whose joint distribution is the centered Gaussian measure with the covariance matrix A. Assume that $A\geq dI$. Then the following estimate holds:*

$$P\big(\omega:\max_{i\leq n}|\xi_i(\omega)|\leq 1\big)\leq\Big(\int_{-1}^{1}p(x,0,d)dx\Big)^n. \tag{2.5}$$

Proof. Denote by ν the distribution of $(\xi_1,\dots,\xi_n)$ in R^n. The probability on the left-hand side of (2.5) coincides with $\nu(V)$, where V is the unit cube in R^n. According to the lemma above, $\nu(V)\leq\mu(V)$, where μ is the Gaussian measure with covariance dI. This implies (2.5).

The following result, obtained by Anderson [10], generalizes Lemma 2.3.

2.5 Theorem. *Let μ and ν be two centered Gaussian measures on R^n. Then the following statements are equivalent:*

(i) $\int (y,x)^2\mu(dx) \geq \int (y,x)^2\nu(dx) \quad \forall y \in R^n$.

(ii) *there is a centered Gaussian measure σ such that $\mu = \nu * \sigma$.*

(iii) *$\mu(A) \leq \nu(A)$ for any convex symmetric set A.*

2.6 Corollary. *Let C be an absolutely convex set in R^n, γ_n the standard Gaussian measure on R^n, T an operator on R^n with $\|T\| \leq 1$. Then*

$$\gamma_n(T(C)) \leq \gamma_n(C).$$

Proof. It suffices to consider the case where T is invertible. In this case we put $\mu(A) = \gamma_n(T(A))$, $\nu = \gamma_n$. Since $(T^{-1}y, T^{-1}y) \geq (y,y)$, we may apply Theorem 2.5.

2.7 Corollary. *Let C and γ_n be as in Corollary 2.6 and let C be closed. Then for any linear subspace $L \subset R^n$ one has:*

$$\gamma_n(C) \leq \gamma_n((C \cap L) + L^{\perp}). \tag{2.6}$$

Proof. Indeed, the right-hand side of (2.6) is the measure of C with respect to the image of γ_n under the orthogonal projection on L. Clearly, the covariance of this image is majorized by that of γ_n.

Now we shall discuss estimates of distributions of nonlinear functions on a Gauss space. Let $f : R^n \to R^k$ be a measurable function which is locally absolutely continuous on every line. Then almost everywhere there exist partial derivatives $\partial_v f$, which we denote below by $f'(x)(v)$.

2.8 Theorem. *Let ξ and η be two independent Gaussian vectors in R^n with one and the same distribution γ. Then, for any convex function $F : R^k \to R^1$, we have*

$$EF(f(\xi) - Ef(\xi)) \leq EF\Big(\frac{\pi}{2}f'(\xi)(\eta)\Big). \tag{2.7}$$

In particular, if γ is the standard Gaussian measure, then

$$\int F\Big(f(x) - \int f\gamma\Big)\gamma(dx) \leq \iint F\Big(\frac{\pi}{2}f'(x)(y)\Big)\gamma(dx)\gamma(dy). \tag{2.8}$$

Proof. The same proof as in [369] works. Indeed, put $\xi(\theta) = \xi\sin\theta + \eta\cos\theta$ for $\theta \in [0, 2\pi]$. Then $\xi'(\theta) := \partial_\theta \xi(\theta) = \xi\cos\theta - \eta\sin\theta$. By our assumption we have with probability one

$$f(\xi) - f(\eta) = \int_0^{\pi/2} \partial_\theta f(\xi(\theta))d\theta = \int_0^{\pi/2} f'(\xi(\theta))(\xi'(\theta))\, d\theta.$$

By the convexity of F we get:

$$EF(f(\xi) - f(\eta)) \leq \frac{2}{\pi}\int_0^{\pi/2} EF(\frac{\pi}{2}f'(\xi(\theta))(\xi'(\theta)))\, d\theta.$$

By the Gaussian property the random vector $(\xi(\theta), \xi'(\theta))$ has the same distribution as (ξ, η). In particular, for all θ

$$EF\Big(\frac{\pi}{2}f'(\xi(\theta))\Big) = EF\Big(\frac{\pi}{2}f'(\xi)(\eta)\Big).$$

Therefore, we arrive at (2.7).

2.9 Corollary. *If γ is centered, $k = 1$, and $F(x) = |x|^r$, then we have*

$$E\big|f(\xi) - Ef(\xi)\big|^r \le E\left|\frac{\pi}{2}f'(\xi)(\eta)\right|^r = M_r(\pi/2)^r E\big[\sigma_\xi(f')^{r/2}\big], \tag{2.9}$$

where $M_r = \int_{-\infty}^{+\infty} |t|^r p(0,1,t)dt$. In particular, if γ is the standard Gaussian measure,

$$\int\left|f(x) - \int f\gamma\right|^r \gamma(dx) \le M_r(\pi/2)^r \int ||\nabla f(x)||^r \gamma(dx). \tag{2.10}$$

Proof. It suffices to note that for any linear functional l the random variable $l(\eta)$ is Gaussian with variance $\sigma_\xi(l)$.

2.10 Corollary.

(i) *Assume that $Ef(\xi) = 0$ and that $E||\nabla f||^r = N_r$, $r \in N$. Then*

$$E|f|^r \le (\pi/2)^r \sqrt{r!} N_r. \tag{2.11}$$

(ii) *If $E\exp\big(C||\nabla f||^2\big) < \infty$ for all C, then*

$$E\exp\big(C|f|\big) < \infty \quad \forall C.$$

Proof. Indeed, it follows from the condition that

$$\sum_{n=1}^{\infty}(4C^2)^n E||\nabla f||^{2n}/n! < \infty.$$

Hence there is K such that $E||\nabla f||^{2n} \le Kn!(4C^2)^{-n}$. According to Corollary 2.9, for even $n = 2m$ this implies

$$E|Cf|^n/n! \le C^{2m}(\pi/2)^{2m}((2m)!)^{-1/2} Km!(4C^2)^{-m} \le K(\pi/4)^{2m},$$

since $M_{2m} \le \sqrt{(2m)!}$. A similar estimate holds for odd n. Thus, $E\exp\big(C|f|\big) < \infty$.

2.11 Remark. The estimates above can be improved a little if one uses the following trick (see [369]). Let W_t be Brownian motion in R^n and let P_t be the associated semigroup. Then, using the Ito formula, one can check the following identity:

$$f(W_1) - Ef(W_1) = \int_0^1 \nabla(P_{1-t}f)(W_t)dW_t.$$

Finally, note that for many pairs of convex sets A, B in R^n the following inequality holds: $\gamma_n(A \cap B) \ge \gamma_n(A)\gamma_n(B)$. For example, this is true if A and B are ellipsoids or if A is any convex set and B is a half-space (see [120, 67]). It seems to be open whether this is true for all pairs of convex sets.

3. Zero–One Laws and the Equivalence–Singularity Dichotomy.

3.1 Theorem. *Let γ be a Gaussian measure on a locally convex space X such that $R_\gamma(X^*) \subset X$. Assume that a γ-measurable set A satisfies the condition*

$$\gamma(A + h) = \gamma(A) \quad \forall h \in R_\gamma(X^*).$$

Then either $\gamma(A) = 1$ or $\gamma(A) = 0$.

Proof. To simplify notations we shall assume that $a_\gamma = 0$. For any vectors

$$h_1 = R_\gamma(l_1), \dots, h_n = R_\gamma(l_n) \text{ with } l_i \in X^*$$

the function

$$F(t_1,\dots,t_n) = \gamma(A + t_1h_1 + \cdots + t_nh_n)$$
$$= \int\limits_A \exp\Big(\sum_{i=1}^n t_il_i(x) - \|\sum_{i=1}^n t_ih_i\|_\gamma^2/2\Big)\gamma(dx)$$

is constant. Therefore,

$$\frac{\partial^n F}{\partial t_1\dots\partial t_n}(0,\dots,0) = 0.$$

Hence

$$\int\limits_A l_1(x)\cdots l_n(x)\gamma(dx) = 0.$$

Thus, the indicator function I_A of A is orthogonal to all polynomials $l_1\cdots l_n$, $l_i \in X^*$, $n \geq 1$. This means that I_A is a constant. Clearly, this may be only 0 or 1.

It is clear that it suffices to claim the condition of Theorem 3.1 to hold for all vectors from $R_\gamma(Y)$, where Y is dense in X^* with the L^2-norm. Below we get a countable version of this result.

3.2 Corollary. *Let γ be a Gaussian measure on $X = R^T$ and let $A \in \sigma(R^T)$ be a set such that*

$$\gamma(A\backslash(A+h)) = 0 \quad \forall h \in R_\gamma(X).$$

Then either $\gamma(A) = 1$ or $\gamma(A) = 0$.

It follows from Theorem 3.1 that for any measurable linear subspace L of a locally convex space with a Gaussian measure γ the following alternative holds: either $\gamma(L) = 0$ or $\gamma(L) = 1$. Below we deduce the same result from the membership of γ in the class of stable measures. It is worth mentioning that the zero–one law holds for convex measures (see [61]). Moreover, it holds for additive subgroups of X. This observation (included in the theorem below) is due to Kallianpur [254]. First results on the zero–one laws for Gaussian measures were obtained, in fact, by Kolmogorov [271] in his classical theorem and by Cameron and Graves (see [90]) in the case of Wiener measure.

3.3 Theorem.

(i) *Let μ be a stable measure on a locally convex space X and let L be an affine subspace of X, which is an element of $\sigma(X)$. Then either $\mu(L) = 1$ or $\mu(L) = 0$. If μ is, in addition, a Radon measure, then the same holds for any μ-measurable affine subspace.*

(ii) *Let γ be a Radon Gaussian measure on X, G a measurable additive subgroup of X. Then either $\gamma(G) = 0$ or $\gamma(G) = 1$.*

Proof. We shall prove only assertion (i). Let L be a linear subspace. Assume that $\mu(L) > 0$ and μ is strictly convex. Choose numbers q and p from (0,1) so that q is rational and

$$p^\alpha + q^\alpha = 1,\; H_p(x) = px.$$

Let x be outside of L and $\mu(L - x) > 0$. Then

$$\mu(L - qx) = \int\limits_X \mu\circ H_q^{-1}(L - qx - y)\mu\circ H_p^{-1}(dy)$$
$$= \int\limits_X \mu(L - x - q^{-1}py)\mu(dy) \geq \int\limits_L \mu(L - x - q^{-1}py)\mu(dy) \geq \mu(L-x)\mu(L) > 0.$$

Thus, for different rational q's the sets $L - qx$ are disjoint and have positive measures separated from zero, which is impossible. This contradiction shows that $\mu(L - x) = 0$ for any x outside of L. Substituting $x = 0$ into the formula above we get

$$\mu(L) = \int\limits_X \mu(L - q^{-1}py)\mu(dy) = \int\limits_L \mu(L - q^{-1}py)\mu(dy) = \mu(L)^2.$$

Therefore, we come to the equality $\mu(L) = 1$.

Now let μ be an arbitrary stable measure. Consider its symmetrization ν:

$$\nu(B) = \int\limits_X \mu(B + x)\mu(dx).$$

It is easy to check that ν is strictly stable. An obvious estimate $\nu(L) \geq \mu(L)^2$ implies $\mu(L) > 0$ in the case $\nu(L) > 0$. If $\mu(L) > 0$, then the same estimate and the assertion for strictly stable measures give the equality $\nu(L) = 1$. Therefore, for μ-a.a. x the equality

$$\mu(L + x) = 1$$

holds. Hence, an element with this property can be found in the set L of positive measure. Thus, $\mu(L) = 1$. The theorem is proved for linear subspaces belonging to $\sigma(X)$. Consequently, it holds for affine subspaces since a shift of a stable measure is a stable measure.

To achieve the proof it remains to note that if μ is a Radon measure and if a μ-measurable linear subspace L has positive measure, then it contains a Borel linear subspace of positive measure, which by the above has full measure. Indeed, let K be a compact subset of L of positive measure. Then its linear span is Borel since it can be written as $\bigcup\limits_{n,m} nQ_m$, where Q_m is the image of the compact set $[-1,1]^m \times K^m$ under the continuous map

$$(t_1, \ldots, t_m, k_1, \ldots k_m) \mapsto \sum_{i=1}^m t_i k_i.$$

The theorem is proved.

3.4 Remark. (a) Let μ be a strictly stable measure and let L be a μ-measurable linear subspace. Then either $\mu(L) = 0$ or $\mu(L) = 1$.

(b) If a measure μ is stable of some rational order α and a set $L \subset \sigma(X)$ is linear over the field of rational numbers, then either $\mu(L) = 0$ or $\mu(L) = 1$.

3.5 Theorem. *Let γ be a Gaussian measure on a locally convex space X. Let $h \in X$ be such that*

$$\|h\|_\gamma = \sup\{f(h) :\ R_\gamma(f)(f) \leq 1\} = \infty.$$

Then the measures γ_h and γ are mutually singular.

Proof. Note that for any finite-dimensional projection P one has

$$\|\gamma_h - \gamma\| \geq \|(\gamma \circ P^{-1})_{Ph} - \gamma \circ P^{-1}\|,$$

$$\|h\|_\gamma = \sup\{\|Ph\|_{\gamma \circ P^{-1}},\ P \in \mathcal{P}(X)\},$$

where $\mathcal{P}(X)$ stands for the collection of all finite-dimensional projections in X. Applying Proposition 2.1 we get $\|\gamma_h - \gamma\| = 2$. This is equivalent to the mutual singularity of these two measures.

3.6 Lemma. *Let γ be a Gaussian measure on a locally convex space X and let $A \in \sigma(X)$ be a set of positive measure. If the measure $\nu = I_A\mu/\mu(A)$ is Gaussian, then $\mu(A) = 1$. The same is true for Borel sets if μ is τ-smooth.*

Proof. As mentioned above, we can assume that $X = R^I$. Let $h \in H(\mu)$. For sufficiently small t we have, according to Theorem 1.21,

$$\mu\big(A \cap (A + th)\big) > 0.$$

Hence the measures ν and ν_{th} are not mutually singular. Therefore, $h \in H(\nu)$ and so $\nu_{th} \ll \nu$. This means that $\mu\big(A \backslash (A + th)\big) = 0$. Applying Corollary 3.2 we get our assertion.

The following classical result was found by several authors (Hajek [222, 223] and Feldman [156] were the first) in different degrees of generality (see [395]). The elegant proof presented below was suggested by Talagrand [455].

3.7 Theorem. *Any two Gaussian measures are either equivalent or mutually singular.*

Proof. Let μ and ν be two Gaussian measures. Then $\mu = \mu_1 + \mu_2$, where μ_2 is mutually singular with ν, $\mu_1 \ll \nu$ and $\mu_1 = \mu|_A$ for some $A \in \sigma(X)$. Then $\mu_1^3 \ll \nu^3$ and $(\mu^3 - \mu_1^3)$ is mutually singular with ν^3. Note that T_ϕ preserves both properties and

$$T_\phi(\mu^3) = \mu^3,\ T_\phi(\nu^3) = \nu^3.$$

Therefore,

$$T_\phi(\mu_1^3) = \mu_1^3.$$

If $\mu(A) > 0$, then by Theorem 1.6 $\mu_1/\mu(A)$ is a Gaussian measure. By virtue of Lemma 3.6 this measure coincides with μ, which implies $\mu \ll \nu$. If $\mu(A) = 0$, then μ and ν are mutually singular. Finally, note that if μ and ν are not mutually singular, then by the same argument $\nu \ll \mu$ and these measures are equivalent.

In the finite-dimensional case, any two Gaussian measures with full support are equivalent. In infinite dimensions the situation is absolutely different.

3.8 Example. *Let γ be a centered Gaussian measure on a locally convex space X. If $dim(X_\gamma^*) = \infty$, then for any reals r and q with $|r| \neq |q|$ the measures $\gamma^r(A) = \gamma(rA)$ and $\gamma^q(A) = \gamma(qA)$ are mutually singular.*

Proof. Let $\{\xi_n\}$ be an orthonormal sequence in X_γ^*. Set

$$S_n = \frac{\sum_{i=1}^{n} \xi_i^2}{n}.$$

By the Gaussian property ξ_n are independent. According to Lemma 3.6 above, $S_n \to r^{-2}$ γ^r-a.s. and $S_n \to q^{-2}$ γ^q-a.s. Hence these two measures are mutually singular.

The proof of the following classical result can be found in [320].

3.9 Lemma. *Let ξ_n be a sequence of independent centered Gaussian random variables with dispersion d. Then almost surely*

$$S_n = \frac{\sum_{i=1}^{n} \xi_i^2}{n} \to d.$$

Keeping the notation of Example 3.8, put

$$E_c = \left\{x :\ \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \xi_i^2(x)}{n} = c\right\},\ c \in R^1.$$

Since the set E_1 has full measure, E_c is γ-negligible if $|c| \neq 1$. Therefore, the map $x \mapsto cx$ sends some measurable sets to nonmeasurable sets. A similar effect is exhibited by the following surprising example, discovered by Cameron and Martin [89].

3.10 Example. *Let $f:\ (0,\infty) \to (0,1)$ be an arbitrary function. Then there exists a γ-measurable set $E \subset X$ such that cE is measurable for every $c > 0$ and*

$$\gamma(cE) = f(c) \quad \forall c > 0.$$

Proof. Let $r:\ R^1 \to R^1$ be an arbitrary function. Put

$$A_t = \{x \in X :\ \xi_1(x) > r(t)\}.$$

Clearly, $sA_t = A_{sr(t)}$. Let

$$E = \bigcup_{t\in R^1} (A_t \cap E_t).$$

Note that

$$cE = \bigcup_t cA_t \cap cE_t = \bigcup_t A_{cr(t)} E_{c^2 t}.$$

Hence cE is the union of the measurable set $A_{cr(1/c^2)} \cap E_1$ and a subset of $X \backslash E_1$ which has measure zero. Thus, cE is γ-measurable and

$$\gamma(cE) = \frac{1}{\sqrt{2\pi}} \int_{cr(1/c^2)}^{\infty} \exp(-s^2/2) ds.$$

It remains to choose r in such a way that $cr(1/c^2) = f(c)$. Clearly, this is possible.

The following important theorem of Ito and Nisio (in a more general form proved in [247]) is connected with the zero–one law. A detailed proof and related discussion may be found in [247, 80].

3.11 Theorem. *Let T be a separable metric space and let ξ_t be a centered separable Gaussian random process with continuous covariance function. Then there exists a deterministic function α on T such that with probability one for all t*

$$\alpha(t) = \lim_{\varepsilon \to 0} \sup\{|\xi_u(\omega) - \xi_v(\omega)|,\ u, v \in K(t, \varepsilon)\},$$

where $K(t, r)$ stands for the ball of radius r around t in T.

4. Measurable Linear Functionals and the Cameron–Martin Subspace.

4.1 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X. Then the Hilbert space X^*_γ is separable (hence so is $H(\gamma)$).*

Proof. We may assume that γ is centered. Let K be a compact set of positive γ-measure. Note that $X^* = \cup_n nK^0$, where

$$K^0 = \Big\{ f \in X^* : \sup_{x \in K} |f(x)| \le 1 \Big\}.$$

Thus, it suffices to prove the separability of K^0 in $L^2(\gamma)$. By induction we can find a sequence $\{f_n\} \subset K^0$, such that letting

$$X_n = \operatorname{span}(f_1, \dots, f_n),\ X_0 = 0,$$

one has

$$\operatorname{dist}(f_n, X_{n-1}) \ge \operatorname{dist}(K^0, X_{n-1})/2 = d_n,$$

where "dist" denotes the distance in $L^2(\gamma)$. Since the finite-dimensional spaces X_n are separable, the separability of K^0 follows from the relationship

$$\lim_{n \to \infty} d_n = 0,$$

which we shall prove. Assume that $\inf_n d_n > 0$. Then for some $d > 0$

$$\operatorname{dist}(f_n, X_{n-1})^2 \ge d \quad \forall n.$$

This estimate means that the covariance matrix A_n of the distribution of the Gaussian random vector $(f_1, \dots, f_n)$ satisfies the condition $A_n \ge dI$. By virtue of Corollary 2.4 we get the following estimate:

$$\gamma\big(x : \sup_{i \le n} |f_i(x)| \le 1\big) \le \Big(\int_{-1}^{1} p(x, 0, d) dx \Big)^n,$$

which tends to zero as n goes to infinity. Therefore,

$$\gamma\Big(x : \sup_n |f_n(x)| \le 1\Big) = 0.$$

This contradicts the condition $\gamma(K) > 0$, since for any $x \in K$ and all n one has $|f_n(x)| \leq 1$ by the inclusion $f_n \in K^0$.

The preceding theorem implies, in particular, the existence of a countable basis in X_γ^*, which consists of continuous linear functionals f_n. Since the linear span of functions $\exp(if)$, $f \in X^*$, is dense in $L^2(\gamma)$ by Corollary 0.D.8, we get that $L^2(\gamma)$ is separable. In addition, it follows from Proposition 0.D.7 that any γ-measurable set coincides with some $\sigma(\{f_n\})$-measurable set up to a set of measure zero. Below we get a stronger result, but even now we are able to obtain the following countable version of Theorem 3.1 (we do not include its proof since it is similar to that of Theorem 3.1).

4.2 Theorem. *Let γ be a centered Radon Gaussian measure on a locally convex space $X, \{e_n\}$ an orthonormal basis in $H(\gamma)$, B a γ-measurable set. If $B + e_n = B$ for any n, then $\gamma(B) = 0$ or $\gamma(B) = 1$.*

4.3 Theorem. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Then its Cameron–Martin subspace $H(\gamma)$ coincides with the intersection of all linear subspaces of γ-full measure.*

Proof. Let L be a linear subspace of full measure, $h \in H(\gamma)$. Then $\gamma(L - h) = 1$ since the measures γ and γ_h are equivalent by Proposition 1.16. Therefore, $h \in L$. Conversely, assume that h does not belong to $H(\gamma)$. Hence

$$\sup\{f(h) : \ f \in X^*, \ R_\gamma(f)(f) \leq 1\} = \infty.$$

We can choose elements $f_n \in X^*$ such that $R_\gamma(f_n)(f_n) = 1$ and $f_n(h) > n$. Since

$$\sum_{n=1}^{\infty} n^{-2} \int_X |f_n(x)|\gamma(dx) < \infty,$$

we obtain a linear space

$$L = \left\{ x \in X : \ \text{the series } \sum_{n=1}^{\infty} n^{-2} f_n(x) \text{ converges} \right\}$$

of full measure. By construction, h does not belong to L.

4.4 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X and let G be an additive γ-measurable subgroup of X with $\gamma(G) > 0$. Then $2a_\gamma \in G$ and $H(\gamma) \subset G$.*

Proof. According to Theorem 3.3, $\gamma(G) = 1$. Let $\gamma_0 = \gamma_{-a_\gamma}$. Then

$$\gamma_0(-a_\gamma + G) = 1 = \gamma(-(-a_\gamma + G)) = 1.$$

Therefore, the set $(-a_\gamma + G) \cap (a_\gamma + G)$ is nonempty, which gives the first assertion. To prove the second one, note that for any $h \in H(\gamma)$ the set $h - a_\gamma + G$ has full measure; hence the set $(h - a_\gamma + G) \cap (-a_\gamma + G)$ is nonempty.

4.5 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X and let q be a γ-measurable seminorm on X. Then the restriction of q to the Hilbert space $H(\gamma)$ is continuous.*

Proof. Take n such that the set $V_n = \{x : \ q(x) \leq n\}$ has positive measure. By Corollary 1.22 for a sufficiently small positive number c the set $V_n - V_n$ contains a ball of radius c in $H(\gamma)$. Hence q is bounded on the unit ball of $H(\gamma)$ which is equivalent to the continuity of q.

4.6 Corollary. *Let γ be a Radon Gaussian measure on a locally convex space X and let l be a γ-measurable linear function on X. Then the restriction of l to the Hilbert space $H(\gamma)$ is continuous.*

4.7 Definition. Let γ be a Gaussian Radon measure on a locally convex space X. A function f on X is called a measurable linear functional on (X, γ) if there exists a linear subspace L of full γ-measure and a measurable linear function f_0 on L such that $f = f_0$ γ-a.e.

In a similar way one defines a measurable linear mapping F on X with values in a locally convex space Y as a measurable map which admits a version F_0 that is linear on a linear subspace L of full measure.

Note that one can always redefine a γ-measurable linear map $F : X \to Y$ in such a way that F_0 becomes linear on the whole space X. Indeed, using a Hamel basis in X one extends F_0 to a linear map on X. Clearly, all these extensions are γ-equivalent functions.

Consider the following instructive example. Let γ be the countable product of standard Gaussian measures on the line, $X = R^\infty$. Then for any $(c_n) \in l^2$ the series $\sum_n c_n x_n$ converges γ-a.e. and hence defines a measurable linear functional on X. However, only for finite sequences do we get continuous functionals.

4.8 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X. The following conditions are equivalent:*

(1) *f is a measurable linear functional;*

(2) *$f \in X^*_\gamma$;*

(3) *there exists a sequence $\{f_n\} \subset X^*$ converging to f in measure.*

Proof. Clearly, (2) implies (3). Let $\{f_n\} \subset X^*$ be converging to f in measure. There exists a subsequence of $\{f_n\}$ (denoted by the same symbol) which converges to f γ-a.e. Let

$$L = \left\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\right\}.$$

Then L is a linear subspace of full γ-measure. We can define f_0 by

$$f_0(x) = \lim_{n\to\infty} f_n(x)$$

on X. Hence (3) implies (1). Assume now that f is a measurable linear functional on (X, γ). We shall assume from the very beginning that $f = f_0$, where f_0 is from the definition above.

Choose an orthonormal basis $\{g_n\}$ in X^*_γ such that $\{g_n\} \subset X^*$. Let

$$e_n = R_\gamma g_n.$$

As we know, $\{e_n\}$ is an orthonormal basis in $H(\gamma)$. By Theorem 4.5 the restriction of f to $H(\gamma)$ is continuous. Hence

$$\sum_{n=1}^{\infty} f(e_n)^2 < \infty.$$

Therefore, we can consider the functional

$$g = \sum_{n=1}^{\infty} f(e_n) g_n.$$

For any n one has $f(e_n) = g(e_n)$, since $g_n(e_k) = \delta_{nk}$. Hence $f(x + e_n) - g(x + e_n) = f(x) - g(x)$ γ-a.e. By virtue of Theorem 3.1 $f - g$ is a constant (γ-a.e.) Thus, $f = g$ γ-a.e.

4.9 Corollary. *Any γ-measurable linear functional f on X is a centered Gaussian random variable with dispersion $\|f\|^2_{L^2(\gamma)}$.*

4.10 Corollary. *Let γ be a Gaussian Radon measure on a locally convex space X, X_0 a Souslin locally convex space continuously embedded in X and having positive measure. Then the restriction of γ to X_0 is a Gaussian Radon measure on X_0.*

Proof. By virtue of Theorem 0.D.12 all Borel subsets of X_0 are γ-measurable and the restriction is a Radon measure. By the zero–one law, $\gamma(X_0) = 1$. If $f \in X_0^*$, then f is a measurable linear functional on X. Now we apply the result above.

4.11 Corollary. *Let γ be a centered Gaussian Radon measure on a locally convex space X, $\{\xi_n\}$ an orthonormal basis in $X^*\gamma$, $l \in X^*_\gamma$. Then there is a sequence $(c_n) \in l^2$ such that*

$$l(x) = \sum_{n=1}^{\infty} c_n \xi_n(x) \quad \gamma\text{-a.e.}$$

4.12 Theorem. *Any two measurable linear functionals f and g on a locally convex space X with a centered Gaussian measure γ either differ almost everywhere or coincide almost everywhere. The latter occurs if and only if $f = g$ on $H(\gamma)$.*

Proof. This follows from the zero–one law and the results above.

4.13 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X. For any continuous linear functional l on the Hilbert space $H(\gamma)$ there is a measurable linear functional l_0 on X which coincides with l on $H(\gamma)$. In addition, $\|l_0\|_2 = \|l\|$.*

Proof. Let $\{\xi_n\}$ be an orthonormal basis in $H(\gamma)$. Then the vectors $e_n = R_\gamma \xi_n$ form an orthonormal basis in the Hilbert space $H(\gamma)$. Put $c_n = l(e_n)$. Then $(c_n) \in l^2$. Hence we have the measurable linear functional

$$l_0 = \sum_{n=1}^{\infty} c_n \xi_n.$$

Since $\xi_n(e_k) = \delta_{nk}$, we have $l_0(e_n) = c_n = l(e_n)$.

The results of this section show that the space $H(\gamma)$ is naturally isomorphic to the dual of X^*_γ. The map R_γ gives this isomorphism.

4.14 Theorem. *Let μ and ν be two centered Radon Gaussian measures on a locally convex space X. Then:*

(i) *The following conditions are equivalent:* (a) $H(\nu) \subset H(\mu)$; (b) $\nu(L) = 1$ *for any ν-measurable linear subspace of full μ-measure.*

(ii) *For any Hilbert space E continuously embedded in $H(\mu)$ there is a centered Radon Gaussian measure λ on X such that $H(\lambda) = E$.*

Proof. (i) Assume that $H(\nu) \subset H(\mu)$. By the closed graph theorem there is a constant C such that for any $h \in H(\nu)$ one has the following estimate:

$$|h|_{H(\mu)} \le C|h|_{H(\nu)}.$$

Replacing the measure μ by its homothetic image, we can always assume that $C = 1$. Let L be a ν-measurable linear subspace with $\mu(L) = 1$. We can consider μ as a Radon measure on L with the induced topology. The Fourier transform of μ at a point $f \in L^*$ equals $\exp(-M(f)^2/2)$, where $M(f) = \sup\{f(h) : |h|_{H(\mu)} \le 1\}$. Consider the following function on L^*:

$$\phi(f) = \exp\bigl(-m(f)^2/2\bigr), \quad m(f) = \sup\bigl\{f(h) : |h|_{H(\nu)} \le 1\bigr\}.$$

Since $m(f) \le M(f)$ we can apply Theorem 1.20, which says that there is a Radon Gaussian measure λ on L with the Fourier transform ϕ. Regarded as a measure on X this measure λ has the same Fourier transform as ν. Hence $\lambda = \nu$ on X. In particular, $\nu(L) = 1$. Conversely, let the condition (b) hold. If there is an element $h \in H(\nu)$ which does not belong to $H(\mu)$, then according to Theorem 4.2 one can find a Borel linear subspace L of full μ-measure which does not contain h. By Theorem 4.3 $\nu(L) = 0$, which is a contradiction.

To prove (ii) note that the same argument as above can be applied to E instead of $H(\nu)$ (with $L = X$).

Now we shall consider measurable linear functionals on the classical Wiener space $(C[0,1], P^W)$. Recall that for any function $\phi \in L^2[0,1]$ Wiener's stochastic integral

$$J\phi = \int_0^1 \phi(t)\,dw_t$$

with respect to the Wiener process w_t is defined as follows. If ϕ is a step function of the form

$$\phi(t) = \sum_{i=1}^{n} c_i I_{[t_i, t_{i+1}]}, \quad t_1 = 0 \le t_2 \le \cdots \le t_{n+1} = 1,$$

we put

$$J\phi = \int_0^1 \phi(t)\,dw_t := \sum_{i=1}^{n} c_i (w_{t_{i+1}} - w_{t_i}).$$

Clearly, $J\phi$ is a Gaussian random variable with zero mean and covariance

$$\sum_{i=1}^{n} c_i^2(t_{i+1} - t_i) = \|\phi\|_{L^2}^2.$$

Therefore, J admits the unique extension to an isometry from $L^2[0,1]$ into the subspace in $L^2(P^W)$ formed by centered Gaussian random variables. Note that the stochastic integral above cannot be defined as a Stieltjes integral since the path w_t almost surely has unbounded variation. However, if we take $(C[0,1], P^W)$ as a probability space for w_t, then $J\phi$ is a measurable linear functional. Indeed, the functionals

$$l_n : \ w \mapsto \sum_{i=1}^{n} c_i(w_{t_{i+1}} - w_{t_i})$$

are continuous on $C[0,1]$ and $J\phi$ may be obtained as their limit.

Conversely, let l be a P^W-measurable linear functional. Put $h = R_W l$. As we have seen above, $h \in W_0^{1,2}$, so $\phi = h'$ is in $L^2[0,1]$. In order to prove that $l = J\phi$ it suffices to check that both functionals coincide on $W_0^{1,2}$. For any $\psi \in W_0^{1,2}$ we have

$$l(\psi) = (R_W l, \psi)_H = (h, \psi)_H = \int_0^1 h'(t)\psi'(t)dt = \int_0^1 \phi(t)\psi'(t)dt.$$

Note that the right-hand side of the last equality coincides with $\int_0^1 \phi(t)d\psi(t) = J\phi$. To see this let us note that it is easy to deduce from the definition of J that there is an increasing sequence of partitions of $[0,1]$ by the points $t_1^n, \dots, t_{N_n}^n$, such that the functions $\sum_1^{N_n} \phi(t_k^n) I_k$, $I_k = [t_{k+1}^n, t_k^n]$, converge to ϕ in $L^2[0,1]$ and

$$J\phi(w) = \lim_{n\to\infty} \sum_{k=1}^{N_n} \phi(t_{t_k}^n)\left[w(t_{k+1}^n) - w(t_k^n)\right]$$

P^W-a.e. Since the domain of the convergence is linear it contains $W_0^{1,2}$. Obviously, on $W_0^{1,2}$ the limit equals $\int_0^1 \phi(t)w'(t)dt$ by virtue of our choice of t_k^n.

Note that in the case of the classical Wiener measure, an analogue of Theorem 4.7 was obtained in [90]. The corresponding result for general Gaussian measures (within the framework of Gaussian processes) is contained in [261], Theorem 2.3. Analogous statements for product-measures can be found in [231, 433].

Some additional properties of measurable linear functionals and operators will be discussed in the next section after we prove a fundamental theorem on the existence of Souslin supports.

5. Supports of Gaussian Measures and Measurable Linear Maps.

The following very important result was found by Tsirelson [463]. Shorter proofs, based on the same idea, were suggested in [399, 455].

5.1 Theorem. *Let γ be a Gaussian measure on a locally convex space X. Then there exists a sequence of metrizable compacts K_n such that*

$$\gamma(\cup K_n) = 1.$$

Proof. Let $\varepsilon > 0$. We shall prove that there exists a compact set K, such that

$$\gamma(K) > 1 - \varepsilon,$$

and the topology of X restricted to K is metrizable. The main idea of the construction is as follows. We shall find an absorbing set $T \subset X^*$ and equip it with some separable metric d. Then we shall find a compact set $K \subset X$ with $\gamma(K) > 1 - \varepsilon$, such that for any $x \in K$ the function

$$F_x : g \mapsto g(x)$$

on (T, d) is continuous.

Assume that this is done. In this case the initial topology of X is metrizable on K. Indeed, the initial topology on K coincides with the weak topology $\sigma(X, X^*)$, since K is compact. We shall prove that under our conditions the weak topology on K has the countable base formed by all the sets

$$U = \{x \in K : (g_1(x), \dots, g_n(x)) \in V\}, \ V \in \mathcal{V},$$

where $g_i \in Q$, Q is a fixed countable set which is dense in (T, d), and $\mathcal{V}$ is the collection of all cubes in R^n with rational vertices. Since T is absorbing, we have a basis of the weak topology formed by all the sets

$$W = \{x : (f_1(x), \dots, f_n(x)) \in B\},$$

where B is open in R^n and $f_i \in T$. In addition, any function $f \in T$ on K is the pointwise limit of a sequence of elements $f_n \in Q$. Indeed, let $\{f_n\} \subset Q$ converge to f in the metric d. By the choice of T for any $x \in K$ the functions $g \mapsto g(x)$ are continuous on (T, d). Therefore, $f_n(x) \to f(x)$ for any $x \in K$. Hence, K is a Hausdorff compact in the topology generated by the sets U. Thus, W contains a set U of the type above.

Now we shall choose T. This will be the set

$$T = \{l \in X^* : \sup_S |l(x)| \leq 1\},$$

where S is an arbitrary compact subset of X of positive measure (such a set exists since γ is a Radon measure). In other words, T is the polar set of S. Consider the metric r on T defined by the formula

$$r(f, g) = \Big(\int_X \big(f(x) - g(x)\big)^2 \gamma(dx) \Big)^{1/2}.$$

Let

$$S^* = \{x \in X : \sup_T |l(x)| \leq 1\}.$$

Thus, S^* is the bipolar of S and coincides with the closed absolutely convex hull of S (see [405]). Denote by E the linear subspace $\cup_n n S^*$. E contains S and by virtue of the zero–one law has full measure. Consider the random process $\xi(t, x) = t(x), t \in T, x \in E$, where E is equipped with the measure γ. Clearly, ξ is a Gaussian process with bounded trajectories. According to Theorem 4.1 the space (T, r) is separable. By a standard argument (see [199]) the process ξ on T admits a separable modification.

By the Ito–Nisio theorem there is a function α on T such that the set

$$Z = \Big\{x \in E : \limsup_{y \to x} \xi(t, y) - \liminf_{y \to x} \xi(t, y) = \alpha(t) \text{ for all } t \in T\Big\}$$

has γ-measure 1. Now we define a new metric d on T by the formula

$$d(f, g) = r(f, g) + |\alpha(f) - \alpha(g)|.$$

Note that (T, d) is separable. Indeed, take a countable subset of (T, r) such that the set of pairs $(q, \alpha(q)), q \in Q$, is dense in the graph of the function α on (T, r). This is possible since the space $(T, r) \times R^1$ is separable. Clearly, Q is dense in (T, d). It remains to note that for any $z \in Z$ the function $t \mapsto t(z) = \xi(t, z)$ is continuous on (T, d). Therefore, we can take any compact set $K \subset Z$ with $\gamma(K) > 1 - \varepsilon$. The theorem is proved.

One might get the impression that the previous theorem enables us to reduce the study of general Radon Gaussian measures to that of measures on a metrizable locally convex space. However, the following example shows that this is not true.

5.2 Example. *Let X be the space l^∞ equipped with the topology $\sigma(l^\infty, l^1)$ and let γ be the restriction to l^∞ of the countable product of the one-dimensional centered Gaussian measures with the dispersions $\sigma_n = \log(n+1)$. Then γ is a Radon Gaussian measure on X, but $\gamma(E) = 0$ for any separable Fréchet space E continuously embedded in X.*

Proof. The proof is based on the following two facts, which can be verified directly:

$$\prod_{n=1}^{\infty} \int_{-1/2}^{1/2} g(\sigma_n t)dt/\sigma_n = 0,$$

but for sufficiently large R

$$\prod_{n=1}^{\infty} \int_{-R}^{R} g(\sigma_n t)dt/\sigma_n > 0.$$

By the Banach–Alaoglou theorem the sets

$$K_R = \{x \in l^\infty : \|x\|_\infty \le R\}$$

are compact in $(l^\infty, \sigma(l^\infty, l^1))$. The measure γ defined on R^∞ is concentrated on their union. By virtue of Corollary 0.D.5, the restriction of γ to X endowed with the σ-field generated by the coordinate functions has a unique extension to a Radon measure on $(l^\infty, \sigma(l^\infty, l^1))$. It is easy to check that this extension is again a Gaussian measure. Assume now that E is a separable Fréchet space which is continuously embedded in X (or, in a more general way, let E be an ultrabornological space continuously embedded in X such that γ is a Radon measure on E).

In this case there is an open neighborhood of zero U in E which is contained in $K_{1/2}$ and hence has γ-measure zero. Applying Theorem 5.6 below, we see that γ is vanishing on E.

5.3 Theorem. *Let γ be a centered Radon Gaussian measure on a locally convex space $X, H = H(\gamma), \{e_n\}$ an orthonormal basis in $H, \{\xi_n\}$ a sequence of independent standard Gaussian variables and let $A \in \mathcal{L}(H)$. Then the series*

$$\sum_{n=1}^{\infty} \xi_n A e_n$$

a.s. converges in X. The distribution of its sum is a Radon Gaussian measure with the covariance $R(g,g) = (A^* j_H(g), A^* j_H(g))_H$. In particular, if $A = I$ and ξ_n are the measurable linear functionals associated with e_n, then

$$x = \sum_{n=1}^{\infty} \xi_n(x) e_n \quad \gamma\text{-a.e.}$$

Proof. It is possible to reduce this theorem to the case $A = I$ considered by Sato [399], introducing a new Hilbert space $E = A(H)$ with the induced norm (which in the case of injective A is given by $||h||_E = ||A^{-1}h||_H$). However, we suggest the following direct argument. By virtue of Theorem 1.20, there exists a Radon Gaussian measure λ on X with the Fourier transform $g \mapsto \exp(-(A^* j_H(g), A^* j_H(g))_H/2)$. Let S be a random vector with distribution λ (defined on the same probability space as $\{\xi_n\}$). For any element $g \in X^*$ the series $\sum_n \xi_n g(Ae_n)$ converges a.s. to $g(S)$. Indeed,

$$\sum_{n=1}^{\infty} g(Ae_n)^2 = \sum_{n=1}^{\infty} (j_H(g), Ae_n)_H^2 = \sum_{n=1}^{\infty} (A^* j_H(g), e_n)_H^2 = (A^* j_H(g), A^* j_H(g))_H.$$

Note that it suffices to prove the convergence of our series in the completion of X, since S with probability one is in X. Thus, we may assume from the very beginning that X is complete. Then there is an absolutely convex metrizable compact set K with $\lambda(K) = P(S \in K) > 0$. Hence $\lambda(nK) = P(S \in nK) \to 1$. Note that the initial topology coincides with the weak one on K due to its compactness. Therefore, by the metrizability of K, there is a countable family of continuous linear functionals $\{g_i\}$ generating the topology of K and consequently of every nK. Unfortunately, this family need not generate the topology of the union of the nK's (for example, in l^2 with the weak topology the coordinate functions generate the topology of every ball, but not of the whole space). Thus, we need to prove convergence in X, given that with probability one $g_i(S_n) \to g(S)$ for every i. The shortest way to do this is to prove that with probability one the sequence S_j remains within one of the nK's. Denoting by q the gauge function of K, it suffices to check that $\sup_n q(S_j) < \infty$ a.s. To this end, note that the sequence of finite-dimensional vectors S_j forms a martingale (with respect to the sequence of σ-fields σ_j generated by $\xi_1, \ldots, \xi_j$). Then the sequence $q(S_j)$ is a submartingale. The covariances of the S_n are given by $\left(P_n A^* j_H(g), P_n A^* j_H(g)\right)_H$, where P_n are the orthogonal projections in H on the span of $e_1, \ldots, e_n$. These covariances are majorized by that of S and according to Theorem 2.5 for every symmetric convex set C we have $P(S_n \in C) \geq P(S \in C)$. This implies that $Eq(S_n) \leq Eq(S)$, which is finite by Fernique's theorem. By a classical result (see [320], Section 32.2) we get the statement above. It should be noted that if we apply a Banach-space-valued version of the martingale convergence theorem (see [480], Exercise 4b, p. 297), then we get almost sure convergence of S_n with respect to the norm q.

Note that in the case where ξ is a centered Gaussian random vector in a separable Banach space X, one can find [297] the representation above with the additional property that

$$\sum_{n=1}^{\infty} ||e_n||_X^2 < \infty.$$

5.4 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X. Then the topological support of γ coincides with the affine subspace $a_\gamma + \overline{H}(\gamma)$, where $\overline{H}(\gamma)$ stands for the closure in X. In particular, the support of γ is separable.*

Proof. Without any loss of generality, we may assume that γ is centered. Let L be the closure of $H(\gamma)$. Clearly, $\gamma(L) = 1$, since by Theorem 5.3, γ-almost each x is the limit of linear combinations of $e_n \in H(\gamma)$. Thus, the topological support S_γ is contained in L. Assume that there is $x \in L$ which does not belong to S_γ. Hence x is contained in some open set V of zero measure. Then V contains an element $h \in H(\gamma)$. Therefore, the open set $W = V - h$ also has measure zero and, in addition, it contains the origin. It follows from the existence of the topological support that the union of all sets $W + h$, $h \in H(\gamma)$, has measure zero as well. It remains to note that this union covers L, which is a contradiction with $\gamma(L) = 1$.

The following technical result is an important corollary of Theorem 5.3.

5.5 Theorem. *Let γ be a centered Radon Gaussian measure on a locally convex space X, $\{\xi_n\}$ an orthonormal sequence in X_γ^*, $e_n = R_\gamma(\xi_n)$. Then, for any function $f \in L^p(\gamma)$, the function*

$$f_n(x) = \int_X f(P_n x + S_n y)\gamma(dy),$$

where

$$P_n x = \sum_{i=1}^{n} \xi_i(x) e_i, \qquad S_n y = \sum_{i=n+1}^{\infty} \xi_i(y) e_i,$$

can be taken as a version of the conditional expectation of f with respect to the σ-field generated by $\xi_1, \ldots, \xi_n$. In particular, $\{f_n\} \to f$ in $L^p(\gamma)$.

Proof. For any bounded Borel function g of the form $g(x) = g(P_n x)$, we have by Theorem 5.3

$$\int f(x) g(P_n x)\gamma(dx) = \int f\Big(\sum_{n=1}^{\infty} x_n e_n\Big) g\Big(\sum_{i=1}^{n} x_i e_i\Big)\mu(dx),$$

where μ is the countable product of the standard Gaussian measures γ_i on the line. In a similar way

$$\int f_n(x) g(P_n x)\gamma(dx) = \int\int f\Big(\sum_{i=1}^{n} x_i e_i + \sum_{i>n} x_i e_i\Big) g\Big(\sum_{i=1}^{n} x_i e_i\Big)\mu(dx)\mu(dy).$$

Now it remains to note that μ coincides with the product of two measures $\mu_n = \prod_{i=1}^{n} \gamma_i$ and $\mu^n = \prod_{i=n+1}^{\infty} \gamma_i$. Therefore, by Fubini's theorem the integrals in the right-hand sides of the two equalities above coincide. Thus,

$$\int f(x)g(x)\gamma(dx) = \int f_n(x)g(x)\gamma(dx),$$

which shows that f_n coincides with the conditional expectation.

Now we shall apply Theorem 5.3 to investigate the structure of measurable linear mappings on a locally convex space X with a centered Radon Gaussian measure γ.

5.6 Theorem.

(i) *Let F be a γ-measurable linear map on X, $\lambda = \gamma \circ F^{-1}$. If λ is a Radon measure on X (which is always the case when X is a Souslin space), then denoting by F_0 a linear version of F one has*

$$H(\lambda) = F_0(H(\gamma)).$$

In addition, the map $F_0 : H(\gamma) \to H(\lambda)$ is a continuous operator between these Hilbert spaces.

(ii) *Let μ be an arbitrary centered Radon Gaussian measure on X. If $dim H(\gamma) = \infty$, then there exists a γ-measurable linear map F on X such that $\mu = \gamma \circ F^{-1}$.*

(iii) *Let $A \in \mathcal{L}(H(\gamma))$. Then A admits an extension to a γ-measurable linear map on X. Any two such extensions coincide γ-a.e. In addition, the image of γ under this extension is a Radon Gaussian measure on X.*

Proof. (i) Clearly, λ is a Gaussian measure. It is known (see [407]) that it is automatically a Radon measure, provided X is a Souslin space. By Theorem 4.7 and Proposition 1.16 it is a centered Gaussian measure.

The equality

$$H(\lambda) = F_0(H(\gamma))$$

follows from the characterization of $H(\gamma)$ as the collection of all vectors of quasiinvariance. This can also be seen from the fact that a vector h belongs to $H(\lambda)$ precisely when

$$\sup\{f(h) : f \in X^*, \ \|f\|_{L^2(\lambda)} \le 1\} < \infty,$$

which is equivalent to

$$\sup\{f(h) : f \in X^*, \ \|f \circ F_0\|_{L^2(\gamma)} \le 1\} < \infty.$$

The continuity of the map $F_0 : H(\gamma) \to H(\lambda)$ follows from the continuity of $F_0 : H(\gamma) \to X$ and the closed graph theorem, which can be applied since $H(\lambda)$ is continuously embedded in X.

(ii) This is a direct corollary of Theorem 5.3.

(iii) Let $H = H(\gamma)$, $A \in \mathcal{L}(H)$. Let $\{f_n\}$ be an orthonormal basis in X^*_γ such that $f_n \in X^*$. Then the vectors

$$e_n = R_\gamma f_n$$

form an orthonormal basis in $H(\gamma)$ and $f_n(e_k) = \int f_n(x) f_k(x)\gamma(dx) = \delta_{nk}$. According to Theorem 5.3, the series

$$\sum_{n=1}^{\infty} f_n(x) A e_n$$

converges γ-a.e.. Denote its sum by $A_0 x$. Clearly, A_0 is a γ-measurable linear map. For $h \in H$ one has

$$A_0 h = \sum_{n=1}^{\infty} f_n(h) A e_n = \sum_{n=1}^{\infty} (e_n, h)_H A e_n = Ah.$$

The uniqueness of A_0 follows from Theorem 4.11.

Put $E = A(H)$. With the graph norm, E becomes a continuously embedded Hilbert space. According to Theorem 1.20 there is a Radon Gaussian measure λ on X such that $E = H(\lambda)$. It is easy to check that λ coincides with the image of γ under A_0.

Let us make several additional remarks about measurable linear functionals. First of all, note that by virtue of Theorem 4.1 and Lusin's theorem, for any measurable linear functional f and any positive ε, we can find a metrizable compact set with measure greater than $1-\varepsilon$ on which f is continuous. Simple observations lead to the following result.

5.7 Proposition. *Let γ be a centered Gaussian measure on a locally convex space X and let $f \in X^*_\gamma$. Then one can find a properly linear version of f (denoted by the same symbol) with the following properties:*

(i) *There exists a Borel Souslin linear subspace $X_0 \in X$ of full γ-measure, such that f is Borel measurable on X_0.*

(ii) *If X is sequentially complete, then there is a metrizable absolutely convex compact set K such that f is continuous on nK for every n and $\gamma(nK) \to 1$.*

Proof. Take a sequence $\{f_n\} \in X^*$ convergent to f γ-a.e. By virtue of the classic Egorov theorem and Theorem 5.1, there is a metrizable compact S of positive measure on which $f_n \to f$ uniformly. Let K be the closed absolutely convex hull of S. It follows from the Hahn–Banach theorem that for any $l \in X^*$ we have $\sup_K |l(x)| = \sup_S |l(x)|$. Hence, the sequence $\{f_n\}$ converges uniformly on K. Letting $f^*(x) = \lim_{n\to\infty} f_n(x)$ on the set L, where the limit exists (L is linear and has full measure), we get a measurable linear functional which coincides with f on a set of positive measure, hence almost everywhere. Now f^* can be extended to X by linearity using any Hamel basis in X. Since the f_n's are continuous, L is a Borel set (moreover, $L \in \mathcal{B}_0(X)$) and f^* is a Borel measurable on L (it even belongs to the first Baire class). To prove (i), note that the linear span X_0 of S belongs to L and coincides with the union of sets

$$D_{n,m} = \{t_1 s_1 + \cdots + t_n s_n \mid |t_i| \le m,\ k_i \in S\}.$$

Clearly, these sets are metrizable compacts, since they are the continuous images of metrizable compact sets $[-m, m]^n \times S^n$. X_0 has full measure by the zero–one law. To prove (ii) it remains to note that $\gamma(nK) \to 1$ and that nK are metrizable compacts provided X is sequentially complete (see Lemma 0.B.1).

5.8 Remark. (a) Even if X is a separable Hilbert space, a γ-measurable linear functional f may fail to have a Borel linear version on the whole space. Indeed, any Borel linear function on a Fréchet space is known to be continuous. Moreover, as shown in [63], if a linear functional on a Fréchet (or on a barrelled space) is measurable with respect to every Gaussian Radon measure on X, then it is continuous. Other related results may be found in [455].

(b) A linear functional f on a locally convex space X with a Radon measure μ is called Lusin if for any positive ε there is a convex compact set K_ε on which f is continuous. In fact, this is equivalent to f being the limit of a sequence of continuous linear functionals f_n converging in measure (see [472]). If any μ-measurable linear functional is Lusin, then μ is said to have the Riesz property (see [472]). Thus, the Riesz property is a consequence of Theorem 4.7. However, there are examples (see [261, 472]) of symmetric measures μ on a separable Hilbert space and μ-measurable linear functionals which are not Lusin.

5.9 Proposition.

(i) *Let γ be a Radon centered Gaussian measure on a locally convex space X, H its Cameron–Martin space, and let $A : X \to H$ be a γ-measurable linear map. Then $A|_H \in \mathcal{L}_{(2)}(H)$ and its Hilbert-Schmidt norm equals $\int_X ||Ax||_H^2 \gamma(dx)$.*

(ii) *If, in addition, X is a Banach space, $A \in \mathcal{L}(X)$, and $A(X) \subset j_H(X^*)$ (see Chapter 5, Section 1), then $A|_H \in \mathcal{L}_{(1)}(H)$.*

Proof. We already know that $C = A|_H \in \mathcal{L}(H)$. By virtue of Fernique's theorem, $||Ax||_H$ is exponentially integrable. According to Lemma 0.A.4 (ii) it suffices to check that $C^* \in \mathcal{L}_{(2)}(H)$ and to get the corresponding equality. Let $\{e_n\}$ be an orthogonal basis in H. Then

$$\int_X ||Ax||_H^2 \gamma(dx) = \int_X \sum_{n=1}^{\infty} (Ax, e_n)_H^2 \gamma(dx).$$

It remains to note that

$$\int_X (Ax, e_n)_H^2 \gamma(dx) = \|C^* e_n\|_H^2.$$

Indeed, $(Ax, e_n)_H$ is a γ-measurable functional. Its restriction to H coincides with the functional $h \mapsto (Ch, e_n)_H = (h, C^* e_n)_H$ whose norm is $\|C^* e_n\|_H$. The proof of (ii) is given in [289], Theorem I.4.6.

Finally, we shall make several remarks on linear supports of Gaussian measures. The terminology here differs from that of Section D in Chapter 0, where topological supports were discussed. We say that a measure μ on a locally convex space X has a Banach (or Hilbert) support if there is a continuously embedded Banach (respectively, separable Hilbert) space $B \subset X$ of full μ-measure. Typically, a support in this sense is not a topological support, because B need not be closed in X. For example, Theorem 0.D.18 says that any Radon measure on a Fréchet space has a reflexive Banach support. One might wish to go further and ask whether there is a Hilbert support. The answer is negative. Moreover, if any Borel measure on a separable Banach (or Fréchet) space X has a Hilbert support, then X is linearly homeomorphic to l^2 (see [342, 267]). However, the situation is different if we consider only Gaussian measures.

Recall that a Banach space X is said to have cotype 2 if $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$ for any sequence $\{x_n\} \subset X$ such that the series $\sum\limits_{n}^{\infty} \varepsilon_n x_n$ converges a.s., where $\{\varepsilon_n\}$ is a sequence of independent random variables with $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$.

5.10 Theorem. *Let γ be a Radon Gaussian measure on a Banach space X. If X has cotype 2, then γ has a Hilbert support. In particular, this holds for $X = l^p$, $1 < p \le 2$.*

The first example of a Gaussian measure on a separable Banach space without Hilbert supports was constructed by Dudley [135]. Later it was proved in [221] and [435] ([221] also mentions an unpublished proof by Kwapien) that the classical Wiener measure on $C[0,1]$ has no Hilbert support. Below we give a very short proof of this fact.

5.11 Example. *Let μ be the classical Wiener measure on $C[0,1]$. Then $\mu(E) = 0$ for any Hilbert space E continuously embedded in $C[0,1]$.*

Proof. First of all, note that E is separable (see Lemma 0.A.6). Therefore, any Borel set in E (with its Hilbert norm) is μ-measurable. Hence the restriction of μ to E is a Borel measure on E. If $\mu(E) > 0$, then $\mu(E) = 1$ by the zero–one law and μ on E is a Gaussian measure. The Cameron–Martin space $H(\mu)$ does not change if we consider μ on E. Thus, the embedding $H(\mu) \to E$ is a Hilbert–Schmidt operator. By Lemma 0.A.4 the embedding $E \to L^2[0,1]$ is also a Hilbert–Schmidt map. Hence the embedding $H(\mu) \to L^2[0,1]$ is a nuclear operator. This is a contradiction, because we know that the covariance operator of the Wiener measure has the eigenvalues $c(n - 1/2)^{-2}$.

Sample paths of the Wiener process have very interesting properties. There is an extensive literature on this subject (see, e.g. [227, 309]). Here we only mention that with probability one the path w_t has no points of differentiability, has unbounded variation on every interval, and does not belong to the Hölder class $H^{1/2}$, but for any positive $\alpha < 1/2$ is in the Hölder class H^α. In this connection let us mention the following surprising smoothing property of Brownian paths (the proof is left as an exercise). Let f be an arbitrary square-integrable Borel function on the line and let

$$F(x) = \int_0^1 f(x + w_t)\,dt.$$

Then F is absolutely continuous for almost all paths!

6. Ornstein–Uhlenbeck Semigroup and Wiener–Ito Expansions.

Let γ be a centered Gaussian measure on a locally convex space X. The Ornstein–Uhlenbeck semigroup T_t is defined on $L^p(\gamma)$ by the formula

$$T_t f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}\,y)\,\gamma(dy). \tag{6.1}$$

Certainly, one has to check that this semigroup is well defined. This can be done as follows. It follows from the definition that the measure γ is the image of the measure $\gamma \times \gamma$ on $X \times X$ under the mapping

$$(x,y) \mapsto e^{-t}x + \sqrt{1-e^{-2t}}y.$$

Hence for any $f \in L^p(\gamma)$ one has

$$\int\limits_X |f(x)|^p \gamma(dx) = \int\limits_X \int\limits_X \left|f(e^{-t}x + \sqrt{1-e^{-2t}}y)\right|^p \gamma(dx)\gamma(dy). \tag{6.2}$$

Applying the Fubini theorem and the Hölder inequality to (6.2) we get the inclusion $T_t f \in L^p(\gamma)$. Moreover,

$$\|T_t f\|_p \le \|f\|_p.$$

Some additional considerations lead to the following result.

6.1 Theorem. *T_t is a strongly continuous contraction semigroup on $L^p(\gamma)$ for any $p > 1$.*

The Ornstein–Uhlenbeck semigroup plays an important role in the theory of Gaussian measures. Its properties were investigated in many papers (see [336–338, 173], where additional references can be found). More general Ornstein–Uhlenbeck semigroups are studied in [52, 54, 178].

Now we shall derive another representation of the Ornstein–Uhlenbeck semigroup using the so-called Wiener–Ito expansions. From now on we assume that γ is a Radon measure.

Let H_n be the nth Hermite polynomial normalized in such a way that $\|H_n\|_{L^2(p)} = \frac{1}{n!}$, where p is the standard Gaussian density on the line. An important property of these functions is that $H_n' = H_{n-1}$.

Let I be the family of finite sequences of nonnegative integers $k = (k_1, k_2, \dots)$ such that $k_i = 0$ except for a finite number of indices. Let $\{l_n\}$ be an orthonormal basis in X_γ^*. Then the functions

$$H_k(x) = \prod_{i=1}^{\infty} H_{k_i}(l_i(x)), \ k \in I,$$

form an orthonormal basis in $L^2(\gamma)$.

For any $f \in L^2(\gamma)$ denote by Q_n the orthogonal projection of f onto the closed subspace E_n generated by the polynomials H_k with $|k| \le n$, where $|k| = \sum k_i$. Put $f_n = Q_n - Q_{n-1}$, $n = 1, 2, \dots$, $f_0 = Q_0$. Thus, we get the following orthogonal expansion:

$$f = \sum_n f_n. \tag{6.3}$$

The following result is a direct consequence of its finite-dimensional version.

6.2 Theorem. *For any $t \ge 0$ and any $f \in L^2(\gamma)$ one has*

$$T_t f = \sum_n e^{-nt} f_n.$$

Let L be the generator of T_t on $L^2(\gamma)$. Clearly,

$$Lf = -\sum_n n f_n,$$

with domain $\{f: \sum\limits_n n^2 \|f_n\|_2^2 < \infty\}$.

Sometimes expansion (6.3) is called the Wiener chaos decomposition. It is natural to regard the functions f_n as power polynomials. In a more general way, a function f is said to be a polynomial of degree $\le n$ if it belongs to E_n. The minimal n with this property is said to be the degree of f. Finally, f is called homogeneous if $f \in E_n$ and $f \perp E_{n-1}$.

It is easy to see that the limit in L^2 of the sequence of polynomials of degree $\le n$ is again a polynomial of degree $\le n$. However, it can happen that homogeneous polynomials converge to a nonhomogeneous limit.

6.3 Example. *Let γ be the countable product of the standard Gaussian measures on the line, $S_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$. Clearly, the functions S_n on R^∞ are continuous quadratic forms. It is known (see* [320]*) that $S_n \to 1$ in $L^2(\gamma)$. Thus, the limit is not a quadratic form.*

Motivated by the results above on measurable linear functionals, one might suggest more algebraic definition of measurable polynomials.

Following [432] we shall inductively define measurable polylinear forms on (X^n, γ^n), where $\gamma^n = \gamma \times \cdots \times \gamma$.

6.4 Definition. For $n = 1$ a measurable polylinear functional is a measurable linear functional. Suppose for $k = n - 1$ the definition of measurable polylinear functional on X^k has already been given. By a measurable polylinear functional on X^n we shall mean any measurable function $Q : X^n \to R^1$, defined γ^n-a.e. such that

(i) If $j \le n$ and for some $x^j \in X$ the function

$$(x^1, \ldots, x^k) \mapsto Q(x^1, \ldots, x^j, \ldots, x^n)$$

is defined a.e. on X^k, then it is a polylinear measurable functional on X^k.

(ii) If $j \le n$ and for some element $(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots x^n)$ the function

$$x \to Q(x^1, \ldots, x^{j-1}, x, x^{j+1}, \ldots, x^n)$$

is γ-a.e. defined, then it is a measurable linear functional.

A measurable polylinear functional Q on X^n is called symmetric if for any transformation T_{ij} which changes the ith and the jth components of $(x^1, \ldots, x^n)$ the functions Q and $Q \circ T_{ij}$ coincide γ^n-a.e.

6.5 Definition. We say that a measurable a.e. defined function P on X is a power functional of order n if there is a measurable polylinear functional Q on X^n such that the domain of definition of the function $x \mapsto Q(x, \ldots, x)$ coincides with the domain of definition of P and on this domain both functions coincide.

The following results were obtained in [432] for more general product measures, but we formulate them for Gaussian measures. We do not give the proofs since these assertions can be easily deduced by induction from the results above on linear functionals. Let $\{e_n\}$ be an orthonormal basis in $H(\gamma)$, $x_i = l_i(x)$.

6.6 Theorem.

(i) *Any two measurable polylinear functionals on X^n either differ almost everywhere or coincide almost everywhere. The latter occurs if and only if the functionals coincide on $H(\gamma)^n = H(\gamma^n)$.*

(ii) *If Q is a measurable polylinear functional on X^n, then $Q \in L^2(\gamma^n)$ and*

$$\|Q\|_2^2 = \sum_{i_1, \ldots, i_n = 1}^{\infty} Q(e_{i_1}, \ldots, e_{i_n})^2.$$

In addition,

$$Q(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n} Q(e_{i_1}^1, \ldots, e_{i_n}^n) x_{i_1}^1 \ldots x_{i_n}^n \quad \gamma^n\text{-a.e.}$$

6.7 Theorem. *Let Q be a symmetric measurable bilinear functional on X^2. Then:*

(i) *there is a γ-equivalent symmetric bilinear functional V such that its domain of definition coincides with the region of convergence of the series*

$$\sum_{k=1}^{\infty} \lambda_k \left(\sum_{n=1}^{\infty} u_{kn} x_n^1\right) \left(\sum_{n=1}^{\infty} u_{kn} x_n^2\right),$$

where (u_{kn}) is some orthogonal matrix and $(\lambda_k) \in l^2$, and on this domain V coincides with the series above.

(ii) *The domain of definition of V contains a linear subspace of full measure if and only if $\sum_n |\lambda_n| < \infty$.*

6.8 Theorem. *For any measurable polylinear functional Q there exists a symmetric measurable polylinear functional V such that $Q = V$ γ^n-a.e. and the function $x \mapsto V(x, \dots, x)$ is defined γ-a.e. and is γ-measurable.*

It should be noted that even if the initial function Q is symmetric the map $x \mapsto V(x, \dots, x)$ need not be defined a.e.

6.9 Example. *Let $X = R^\infty$, and γ be the countable product of the standard Gaussian measures on the line. Then the function*

$$Q(x,y) = \sum_{n=1}^{\infty} \frac{1}{n} x_n y_n$$

is a measurable bilinear functional, but $Q(x,x) = \infty$ γ-a.e.

Proof. Almost sure convergence of the series above follows from general results on the convergence of series of independent random variables, since

$$E(x_n y_n / n) = 0, \;\; E[(x_n y_n/n)^2] = n^{-2}.$$

Note that the series $\sum_{n=1}^{\infty} n^{-2} x_n^2$ converges γ-a.e. (see Theorem 1.2). For such values of x the map $y \mapsto Q(x,y)$ is a measurable linear functional. Conversely, if for some x the function $Q(x,y)$ is defined for a.e. y, then it is a measurable linear functional in y and necessarily $\sum_{n=1}^{\infty} n^{-2} x_n^2 < \infty$. Clearly, Q is symmetric. However, $Q(x,x) = \infty$ a.e. since $\sum_{n=1}^{\infty} n^{-1} = \infty$.

It should be noted, however, that it is not clear whether the function Q above is equivalent to a function which is bilinear (in the usual algebraic sense) on the whole space $X \times Y$. A related question, raised by H. V. Weizsäcker, concerns the possibility of finding a bilinear function on $C[0,1] \times C[0,1]$, which coincides $P^W \times P^W$-a.e. with the stochastic integral $\int_0^1 x(t) dy(t)$.

6.10 Theorem. *Let F be a measurable power functional on X of order n. Then:*

(i) *it is defined on $H(\gamma)$;*

(ii) *there is a symmetric measurable polylinear functional Q on X^n such that $F(x) = Q(x, \dots, x)$ γ-a.e.*

6.11 Theorem. *If two measurable power functionals coincide on the linear span of $\{e_n\}$, then their difference is almost everywhere constant.*

Measurable polylinear functionals on the classical Wiener space can be described as multiple stochastic integrals (see [432]). An investigation of multiple stochastic integral as polylinear forms was carried out in [354, 448].

7. Abstract Wiener Spaces.

In this section, we discuss the notion of an abstract Wiener space, introduced by L. Gross [215]. The starting point is a separable Hilbert space H with the canonical cylindrical Gaussian measure ν whose Fourier transform is $x \mapsto \exp\bigl(-(x,x)_H/2\bigr)$. We already know that this measure is not countably additive. The main idea of Gross's method is to find a suitable extension of H on which the measure ν becomes countably additive. Having said this, we must be precise in which sense we talk about "extensions," since a measure which is not countably additive on a smaller space neither is on a bigger one. A correct interpretation may be obtained in terms of linear images of cylindrical measures. Recall that if ν is a cylindrical measure on a locally convex space E and $F: E \to X$ is a continuous linear mapping with values in a locally convex space X, then $\nu \circ F^{-1}: C \mapsto \nu\bigl(F^{-1}(C)\bigr)$ is a cylindrical measure on X. Now let E be continuously embedded in X and let ν be a cylindrical measure on E. Denote by j the injection $E \to X$. We say that ν is countably additive on X if the measure $\nu \circ j^{-1}$ is countably additive.

7.1 Definition. Let H be a separable Hilbert space. A seminorm q on H is said to be measurable in the sense of Gross if for any $\varepsilon > 0$ there is an orthogonal projector $P_\varepsilon \in \mathcal{P}(X)$ such that

$$\nu\bigl(x \in H: \; q(Px) > \varepsilon\bigr) < \varepsilon \quad \forall P \in \mathcal{P}(X), \; P \perp P_\varepsilon, \tag{7.1}$$

where ν is the canonical cylindrical Gaussian measure on H.

Note that the set on the left-hand side of (7.1) is ν-measurable, since q is continuous on finite-dimensional subspaces, since it is a seminorm. The following lemma shows that, in fact, q is continuous on the whole space H.

7.2 Lemma. *Any seminorm measurable in the sense of Gross is continuous.*

Proof. It suffices to prove that q is bounded on the unit ball U of H. Assume that $\sup_{h\in U} q(h) = \infty$. It is easy to see that in this case one can find an orthonormal sequence $\{h_n\} \subset U$ such that $q(h_n) = C_n \to \infty$. Let $P_n x = (x, h_n)h_n$. Then we have

$$\nu\big(x\colon\ q(P_n x) > \varepsilon\big) = 1 - \int_{-\varepsilon/C_n}^{\varepsilon/C_n} p(t)dt,$$

where p is the standard Gaussian density on the line. Moreover, given $P_\varepsilon \in \mathcal{P}(X)$ one can find a sequence such that $P_n \perp P_\varepsilon$. Clearly, the right-hand side does not tend to zero as $n \to \infty$.

7.3 Definition. A triple (i, H, B) is said to be an abstract Wiener space if B is a separable Banach space, H is a separable Hilbert space, $i : H \to B$ is a continuous linear injection with dense range, and the norm of B is measurable on H in the sense of Gross.

7.4 Theorem. *Let (i, H, B) be an abstract Wiener space. Then the canonical cylindrical Gaussian measure ν on H is countably additive on B.*

Proof. By definition, we can find an increasing sequence of finite-dimensional projections $P_n \to I$ and a sequence $c_n \to 0$ such that

$$\nu\big(x:\ q(P_m) > c_n\big) < c_n \quad \forall m \ge n. \tag{7.2}$$

Let $\{e_n\}$ be an orthonormal basis in H chosen in $\cup_n P_n(H)$. Consider the Hilbert space E obtained as the completion of H with respect to the norm

$$x \mapsto \sqrt{\sum_n n^{-2}(x, e_n)_H^2}.$$

Denote by j the injection $H \to E$. Applying Theorem 1.10 we see that $\nu_1 = \nu \circ j^{-1}$ is countably additive on E. We get the sequence of B-valued random vectors on (E, ν_1):

$$S_n(x) = \sum_{k=1}^{n} k(x, e_k)_E e_k.$$

It follows from (7.2) that the sequence S_n is fundamental in ν_1-measure. Since B is complete, this sequence converges in measure to a random vector S in B (the proof of this fact is the same as for real random variables). Now it is easy to check that the distribution of S coincides with the measure $\nu \circ i^{-1}$ on B.

7.5 Theorem. *Let γ be a centered Gaussian measure on a separable Banach space X, $H = H(\gamma)$. Then (i, H, X) is an abstract Wiener space.*

Proof. Let $\{\xi_n\}$ be an orthogonal basis in X^*_γ, $\xi_n \in X^*$, $e_n = R_\gamma(\xi_n)$, $P_n = \sum_{i=1}^{n} \xi_i(x)e_i$, $\mu = \nu \circ i^{-1}$.

Note that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \gamma\big(x \in X:\ \|x - P_n x\|_X > \varepsilon\big) = 0.$$

Indeed, we know that $P_n x \to x$ in X γ-a.e. Hence there is N such that

$$\gamma\big(x \in X:\ \|P_n x - P_m x\|_X > \varepsilon\big) < \varepsilon \quad \forall n, m \ge N.$$

By definition

$$\gamma\big(x \in X:\ \|P_n x - P_m x\|_X > \varepsilon\big) = \nu\big(x \in H:\ \|P_n x - P_m x\|_X > \varepsilon\big).$$

For any orthogonal projection $P \in \mathcal{P}(H)$ with $P \perp P_N$ we have

$$\nu\big(x \in H : \ \|Px\|_X > \varepsilon\big) = \lim_{k\to\infty} \nu\big(x \in H : \ \|(P_{N+k} - P_N)Px\|_X > \varepsilon\big).$$

Finally, by virtue of Corollary 2.7 we get

$$\nu\big(x \in H : \ \|(P_{N+k} - P_N)Px\|_X > \varepsilon\big) \le \nu\big(x \in H : \ \|(P_{N+k} - P_N)x\|_X > \varepsilon\big) \le \varepsilon.$$

7.6 Concluding remarks. 1. There are several different ways of proving the existence of a Wiener measure. Bachelier [17] was the first to consider Brownian motion from the mathematical point of view. The first rigorous construction, suggested by Wiener [492, 493, 494], was based on the Fourier series with random coefficients. As noticed in [106], an easier proof of this way is obtained if one uses Haar functions. The second proof (presented above) is based on two Kolmogorov's theorems. The third possibility is provided by Gross's abstract Wiener spaces. Finally, there are proofs (see [368]) exploiting the concept of the so-called absolutely summing operators (which, in fact, is related to the previous possibility). Of course, there exist modifications of these approaches (e.g., [420]).

2. Various results related to the discussion above can be found in the list of references. In particular, zero–one laws, equivalence/singularity dichotomies, and absolute continuity are discussed in [18, 73, 81, 82, 97, 98, 139, 140, 160, 163, 198, 231, 245, 248, 249, 250, 251, 257, 258, 306, 307, 357, 372, 373, 390, 391, 394, 400, 414, 481, 482, 483, 497];

classical Wiener measure, cylindrical Gaussian measures, Gaussian measures in function spaces and abstract Wiener spaces: [19, 20, 21, 24, 93, 94, 95, 99, 100, 138, 150, 157, 159, 217, 229, 280, 397, 404, 415, 450, 500];

general theory: [63, 64, 78, 99, 128, 129, 165, 178, 182, 183, 184, 186, 187, 196, 199, 212, 241, 246, 247, 253, 275, 281, 282, 283, 284, 289, 300, 311, 343, 346, 349, 380, 381, 382, 395, 398, 401, 409, 410, 411, 412, 420, 429, 443, 454, 456, 478, 479, 480, 484, 485, 486, 498];

applications in statistics: [235, 236, 237, 255, 327, 370, 465].

Infinite-dimensional Gaussian distributions play an important role in financial mathematics (see [423]).

3. There are several very important directions, closely connected with the material of Chapter 1 and Chapter 2, but not discussed in this survey.

The first one concerns various notions of radonifying operators. Let X and Y be two locally convex spaces with some classes of cylindrical measures $\mathcal{K}_X$ and $\mathcal{K}_Y$. A continuous linear operator $T : X \to Y$ is said to be $(\mathcal{K}_X, \mathcal{K}_Y)$-radonifying if $\nu \circ T^{-1} \in \mathcal{K}_Y$ for any $\nu \in \mathcal{K}_X$. For example, if $\mathcal{K}_X$ consists of all Gaussian cylindrical measures on X with continuous Fourier transforms and $\mathcal{K}_Y$ is the class of all Radon measures on Y, then we come to the notion of γ-radonifying operators. In the case where X and Y are Hilbert spaces, these are exactly Hilbert–Schmidt operators. For further references, see [316, 332, 343, 369, 404, 408, 480, 496].

The second direction deals with the characterization of quadratic forms on X^* which are Gaussian covariances. In the case of Hilbert spaces this class coincides with the family of nonnegative forms generated by nuclear operators, hence, coincides with the class of covariances of all measures μ with $\int \|x\|^2 \mu(dx) < \infty$. As is already known, the situation is different in general Banach spaces. However, as follows from Minlos's theorem, mentioned in Section 0.D, if X is a dual to a nuclear barrelled space Y, then for any nonnegative continuous quadratic form V on $X^* = Y$ the function $\exp(-V)$ is the Fourier transform of a Radon Gaussian measure on X. See [103, 343, 381, 382, 480] for further references.

Finally, the third direction, probably the most important, is the study of the sample path properties of Gaussian processes. Many results and constructions in abstract theory originate in this field. Various geometric characteristics generated by the covariance functions of Gaussian processes (such as metric entropy, GB- and GC-sets, etc.) have deep connections with supports of Gaussian measures. Many beautiful results on this subject can be found in [2, 135, 136, 137, 161, 304, 443, 458].

4. There exist analogs of Gaussian measures on more general spaces (e.g., on groups, on spaces over p-adic fields, in noncommutative analysis). Some related references can be found in [108, 224, 269, 270, 279, 313, 339, 462].

Connections with the Feynman integral are discussed in [118, 195, 334, 437].

For applications in mathematical physics see [202, 424, 425].

Chapter 2.
Convexity, Inequalities, and Applications

1. Convexity of Gaussian Measures.

Let

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-x^2/2)dx. \tag{1.1}$$

Convexity of Gaussian measures plays an important role in diverse applications. Typically, quantitative theorems of this kind assert that for a certain function $\phi(\cdot,\cdot,\cdot)$ the inequality

$$\gamma(\lambda A + (1-\lambda)B) \geq \phi(\lambda, \gamma(A), \gamma(B))$$

holds for all A and B from a certain class of sets and for any $\lambda \in [0,1]$. The next two theorems follow from Theorem 1.2.2 and Theorem 1.2.5.

1.1 Theorem. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Then for any Borel sets A and B and any $\lambda \in [0,1]$ one has*

$$\Phi^{-1}\big(\gamma_*(\lambda A + (1-\lambda)B)\big) \geq \lambda\Phi^{-1}\big(\gamma(A)\big) + (1-\lambda)\Phi^{-1}\big(\gamma(B)\big), \tag{1.2}$$

$$\gamma_*\big(\lambda A + (1-\lambda)B\big) \geq \gamma(A)^{\lambda}\gamma(B)^{1-\lambda}. \tag{1.3}$$

In particular, for any symmetric convex measurable set A and any vector a,

$$\gamma(A+a) \leq \gamma(A). \tag{1.4}$$

Note that if X is a separable Fréchet space then the sum of two Borel sets is Souslin, hence measurable.

1.2 Theorem. *Let μ and ν be two centered Radon Gaussian measures on a locally convex space X. Then the following statements are equivalent:*

(i) $\int l(x)^2\mu(dx) \geq \int l(x)^2\nu(dx) \quad \forall l \in X^*$.

(ii) *There is a centered Radon Gaussian measure σ such that $\mu = \nu * \sigma$.*

(iii) $\mu(A) \leq \nu(A)$ *for any convex symmetric Borel set A.*

1.3 Lemma. *Let γ be a Radon Gaussian measure on a locally convex space X, Q a closed ball in the Hilbert space $H(\gamma)$. Then for any Borel set $B \subset X$ the set $B+Q$ is γ-measurable.*

Proof. We know that there is a metrizable compact set K of positive γ-measure. Let T_n be the cube in R^n defined by the condition $|x_i| \leq n$, $i = 1,\ldots,n$. Denote by K_n the image of the metrizable compact set $T_n \times K^n$ under the mapping

$$\big((t_1,\ldots,t_n),(k_1,\ldots,k_n)\big) \mapsto k_1t_1 + \cdots + k_nt_n.$$

Then K_n is a metrizable compact set as well. Therefore, the set $(B \cap K_n) + Q$ is a Souslin subset of X as the continuous image of the Borel set $(B \cap K_n) \times Q$ in the complete separable metric space $K_n \times Q$ (see [407]). By virtue of Theorem 0.D.12, for any n the set $(B \cap K_n) + Q$ is γ-measurable. The union $L = \cup_n K_n$ is a linear subspace. In addition, $\gamma(L) > 0$. By the zero–one law $\gamma(L) = 1$. In particular, L contains $H(\gamma)$. Now it remains to note that, letting $C = \cup_n((B \cap K_n) + Q)$, we get

$$(B+Q)\backslash C \subset (B\backslash L) + L \subset X\backslash L.$$

Thus, $\gamma\big((B+Q)\backslash C\big) = 0$.

1.4 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X, U_H the closed unit ball in $H(\gamma)$, A a γ-measurable set. Then*

$$\gamma(A + tU_H) \geq \Phi(a+t) \quad \forall t \geq 0,$$

where a is chosen so that $\Phi(a) = \gamma(A)$.

1.5 Corollary. *For every positive α there exist $r_0(\alpha) \geq 0$ and a real number $c(\alpha)$, such that for all $r \geq r_0(\alpha)$*

$$\gamma(A + rU_H) \geq 1 - \exp\left(-\frac{r^2}{2} + c(\alpha)r\right), \tag{1.5}$$

provided $\gamma(A) = \alpha > 0$.

Many other useful inequalities may be found in [61–70, 144–148, 304].

2. Exponential Integrability.

For a measurable function f on a measurable space (X, μ) we denote by $M(f)$ a median of f, i.e., $M(f)$ is a number such that

$$\mu\big(x : f(x) \geq M(f)\big) \geq 1/2 \text{ and } \mu\big(x : f(x) \leq M(f)\big) \geq 1/2.$$

Clearly, zero is a median of the function $f - M(f)$. Note that a median always exists (see [320], Section 18), but need not be unique. If the distribution function of f is continuous, then we can take for a median any solution t of the equation $\mu(x : f(x) \leq t) = 1/2$. In the general case, one can take $M(f) = \sup\{t : \mu(x : f(x) < t) < 1/2\}$. Note that if f is a continuous function on R^1 with a Gaussian measure, then $M(f)$ is unique.

If f is integrable we put

$$E_\mu(f) = \int_X f(x)\mu(dx).$$

2.1 Lemma. *Let μ be a Gaussian Radon measure on a locally convex space X and let f be a μ-measurable function such that for μ-a.e. x the function $t \mapsto f(x + th)$ is continuous for any $H(\mu)$. Then $M(f)$ is unique.*

Proof. Let $\{e_n\}$ be an orthonormal basis in $H(\mu)$. Assume that there are two different medians m and M. This means that $\mu(f^{-1}(m, M)) = 0$. Hence for μ-a.e. x we get for all n and all $t \in R^1$

$$f(x + te_n) \leq m \text{ or } f(x + te_n) \geq M.$$

Therefore, two sets of x's arise, $A(m)$ and $A(M)$, each of which is invariant under all translations along $\{e_n\}$. Hence one of these two sets is of full measure and the other is of zero measure. This would be impossible if both m and M were medians.

The proof of the following theorem can be found in [304]. Another approach is discussed below.

2.2 Theorem. *Let γ_n be the standard Gaussian measure on R^n and let $f : R^n \to R^1$ be a Lipschitzian function with*

$$|f(x) - f(y)| \leq L\|x - y\| \quad \forall x, y \in R^n.$$

Then the following estimates hold for all $R > 0$:

$$\gamma_n\big(x : |f(x) - M(f)| > R\big) \leq 2\Psi(R/L) \leq \exp(-2^{-1}L^{-2}R^2).$$

$$\gamma_n\big(x\colon |f(x)| > R\big) \leq 2\Psi\big((R - M(f))/L\big) \leq \exp\big(-2^{-1}L^{-2}(R - M(f))^2\big).$$

$$\gamma_n\big(x : |f(x) - E_\mu(f)| > R\big) \leq 2\exp\big(-2^{-1}L^{-2}R^2\big).$$

For vector-valued maps there are similar estimates.

2.3 Theorem. *Let γ be a centered Gaussian measure on a locally convex space X. Assume that a γ-measurable function f satisfies the condition*

$$|f(x + h) - f(x)| \leq L\|h\|_H \quad \forall h \in H \text{ for } \gamma\text{-a.e. } x.$$

Then the following estimates hold for any $R \geq 0$:

$$\gamma\big(x : |f(x) - M(f)| > R\big) \leq 2\Psi(R/L) \leq \exp(-2^{-1}L^{-2}R^2). \tag{2.1}$$

$$\gamma\big(x : |f(x)| > R\big) \leq 2\Psi(R/L) \leq \exp\big(-2^{-1}L^{-2}(R - M(f))^2\big). \tag{2.2}$$

$$\gamma\big(x : |f(x)| > R\big) \leq 2\exp\big(-(R - E_\gamma(f))^2/(2L^2)\big), \tag{2.3}$$

where $E_\gamma(f) = \int\limits_X f(x)\gamma(dx)$.

Proof. We shall use the following simple observation. Let f_n be functions satisfying the estimates

$$|f_n(x+h) - f_n(x)| \le L\|h\|_H \ \forall h \in H \ \text{ for } \gamma\text{-a.e. } x,$$

$$\gamma\big(x : |f_n(x) - M(f_n)| > R\big) \le 2\Psi(R/L) \le \exp(-2^{-1}L^{-2}R^2). \tag{2.4}$$

Assume that the sequence $\{f_n\}$ converges in measure to f. Then (2.1)-(2.3) hold. Indeed, by Lemma 2.1 the functions f_n possess the unique medians $M(f_n)$. In addition,

$$\lim_{n\to\infty} M(f_n) = M(f). \tag{2.5}$$

The relationship (2.5) can be easily checked directly (or see [320], Section 18.1, Corollary 1). Thus, it suffices to consider the case where $M(f) = M(f_n) = 0$ for all n. In this case, we get for any $n > n(\varepsilon)$

$$\gamma\big(x : |f(x)| > R\big) \le \gamma\big(x : |f_n(x)| > R - \varepsilon\big) + \varepsilon \le 2\Psi\big((R-\varepsilon)/L\big) + \varepsilon.$$

Since ε is arbitrary we arrive at the desired estimates.

It remains to note that by Theorem 1.5.7 we can find a sequence of functions f_n such that: (1) $f_n(x) = F_n(g_1(x), \dots, g_n(x))$, where F_n are Lipschitzian functions on R^n with the Lipschitz constant L; (2) $f_n \to f$ in measure. According to Theorem 2.2 and the arguments above we get (2.1)–(2.3).

2.4 Example. Let f be a γ-measurable seminorm on X. Then f satisfies the condition in Theorem 2.3 with some L. Indeed, according to Theorem 1.4.4, f is continuous on $H(\gamma)$. Hence

$$|f(x+h) - f(x)| \le f(h) \le C\|h\|_H \quad \forall h \in H(\gamma).$$

This leads to the following celebrated Fernique's result [158] (related results were obtained in [428, 298]).

2.5 Corollary. *There is $c > 0$ such that*

$$\int\limits_X \exp\big(cf(x)^2\big)\gamma(dx) < \infty.$$

A closer look enables us to get the estimates with some universal constants.

2.6 Proposition. *Let γ be a centered Gaussian measure on a locally convex space X and let D be a γ-measurable disc in X (that is, a bounded absolutely convex closed set). Assume that $\gamma(D) \ge d > 1/2$. Then*

$$\gamma(X \backslash tD) \le \exp\big(-M(d)t^2\big), \qquad M(d) = \frac{1}{24}\log\frac{d}{d-1}.$$

2.7 Remark. The idea to apply the finite-dimensional estimates (2.3) to the infinite-dimensional case was used in [369, 304, 295, 473] to prove the exponential integrability of seminorms. In [473] the estimate (2.3) was obtained under the a priori assumption that f is square-integrable. Therefore, Fernique's theorem was not a direct corollary of [473]. The proof above is a modification of the argument from [473]. It should be noted that this result can be easily derived from the following nice result due to Ehrhard [144].

For a measurable function f on (X, γ) denote by f^* the function on the line with the standard Gaussian measure γ_0, such that f^* is nondecreasing and equimeasurable with f. Such a function exists:

$$f^*(t) = \inf\big\{s\colon\ \gamma_0((-\infty, t]) \le \gamma(f \le s)\big\}.$$

2.8 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X. Assume that a measurable function f on X satisfies the condition*

$$|f(x+h) - f(x)| \le L\|h\|_H \quad \forall h \in H(\gamma) \ a.e.$$

Then f^* is Lipschitzian with constant L. If f is convex, then f^* is convex as well.

2.9 Theorem. *Let γ_n be a tight sequence of centered Gaussian Radon measures on a locally convex space X. Then for any continuous seminorm q on X there is a constant $c > 0$ such that*

$$\limsup_n \int_X \exp\bigl(cq(x)^2\bigr)\gamma_n(dx) < \infty.$$

If, in addition, the sequence $\{\gamma_n\}$ converges weakly to a measure γ, then for any $\delta < c$

$$\lim_n \int_X \exp\bigl(\delta q(x)^2\bigr)\gamma_n(dx) = \int_X \exp\bigl(\delta q(x)^2\bigr)\gamma(dx).$$

In particular, for any positive r

$$\lim_n \int_X q(x)^r \gamma_n(dx) = \int_X q(x)^r \gamma(dx).$$

Proof. Let Q be a compact set such that $\gamma_n(Q) > 1/2$ for all n. Denote by K the closed absolutely convex hull of K. Clearly,

$$\gamma_n(K) > 1/2 \quad \forall n.$$

Since q is a continuous seminorm we have

$$\sup_K q(x) = S < \infty.$$

Therefore, $q(x) \le S g_K(x)$ for any x in the linear span of K, which has full measure by the zero–one law. According to Proposition 2.6, there is a positive constant k such that

$$\limsup_n \int_X \exp\bigl(k g_K(x)^2\bigr)\gamma_n(dx) < \infty.$$

This implies the first assertion. Two other assertions follow from the first one and the general properties of weak convergence (see [480]).

Related results, in particular, estimates of the tail probabilities of the suprema of Gaussian processes, may be found in [62, 63, 66, 67, 68, 69, 70, 72, 80, 101, 120, 144, 145, 146, 147, 148, 149, 166, 230, 233, 238, 302, 303, 310, 315, 312, 388, 444, 464, 471]. For more information on exponential integrability see [12, 188, 480, 503]. Some additional results on exponential integrability will be presented in the next chapter in the section devoted to logarithmic inequalities.

3. Onsager–Machlup Functionals.

Let μ be a measure on a metric space X and let $K(x,\varepsilon)$ be the closed ball of radius ε centered at x. In various applications one has to study the existence of the following limit:

$$\lim_{\varepsilon\to 0} \frac{\mu\bigl(K(a,\varepsilon)\bigr)}{\mu\bigl(K(b,\varepsilon)\bigr)} = I(a,b).$$

The functional $\mathcal{F}(a,b) = \log I(a,b)$ (defined for those pairs (a,b), for which the limit exists) is called the Onsager–Machlup functional (we set $\log(\infty) = \infty, \log(0) = -\infty$). If the measure μ is defined by a continuous density p on R^n, the limit above equals $p(a)/p(b)$. If we only assume that p is integrable, then the same holds for $\mu\times\mu$-almost all pairs (a,b). The situation is more complicated in infinite-dimensions due to the absence of analogs of Lebesgue measure. Here the Onsager–Machlup functional is not defined for all (or almost all) arguments. In the pioneering papers by Onsager and Machlup [355], the measure μ was generated by a diffusion process ξ_t (with the constant diffusion term). These papers became a starting point of intensive investigations (see [239, 96, 495]). Many results obtained in this direction are based on a preliminary study of the Gaussian case. For this reason, we shall discuss here only Gaussian measures (however, some additional information can be found in the remarks below).

Let γ be a centered Gaussian Radon measure on a locally convex space X and let q be a γ-measurable seminorm on X. Then the problem is: does the limit

$$\lim_{\varepsilon\to 0}\frac{\gamma(V_\varepsilon+h)}{\gamma(V_\varepsilon)},\quad V_\varepsilon=\{x: q(x)\le\varepsilon\}, \tag{3.1}$$

exist? The first general results in this direction were obtained in [355, 451, 185] for the Wiener measure γ on $C[0,1]$ with the sup-norm q and for Gaussian measures on a Hilbert space with the usual norm q. A stronger result was obtained in [65]. Let V be a bounded symmetric convex γ-measurable set such that $\gamma(\varepsilon V)>0$ for all $\varepsilon>0$. Then, according to [65], for any $h\in H(\gamma)$

$$\lim_{\varepsilon\to 0}\frac{\gamma(V_\varepsilon+h)}{\gamma(V_\varepsilon)}=\exp(-|h|^2_{H(\gamma)}/2),\quad V_\varepsilon=\varepsilon V. \tag{3.2}$$

Later Hoffmann-Jorgensen, Shepp, and Dudley [232] proved the same if a Gaussian measure γ on R^∞ is the countable product of measures on lines and the γ-measurable norm q satisfies the following condition:

$$q(x)=\sup q(x_1,\dots,x_n,0,0,\dots). \tag{3.3}$$

Shepp and Zeitouni [416] obtained (3.2) in the case where γ is the Wiener measure on $C[0,1]$ and q is a norm satisfying the following conditions:

(1) $\|\phi\|_{L^1}\le Cq(\phi)$ for all $\phi\in C[0,1]$;

(2) $\lim\limits_{\varepsilon\to 0}E(\exp(cw_1)|q\le\varepsilon)=1\ \forall c$, where w_t is the Wiener process.

In this section, we improve all these results. In particular, it will be proved that for any γ-measurable seminorm q and any sequence ε_n tending to zero one can find a subsequence $\varepsilon_{n(k)}$ such that

$$\lim_{k\to\infty}\frac{\gamma(V_{\varepsilon_{n(k)}}+h)}{\gamma(V_{\varepsilon_{n(k)}})}$$

exists. The relationship (3.2) holds provided q is a continuous seminorm such that q on $H(\gamma)$ is a norm, or $\{q<1\}$ is bounded. Another sufficient condition is that

$$q(x)=\sup_n|f_n(x)|,\quad f_n\in X^*.$$

For a very large class of seminorms the value of the limit in (3.1) can be calculated. In order to do this we shall introduce some additional characteristics of q. Finally, note that it follows from the results below that any of the two conditions above used by Shepp and Zeitouni can be omitted.

Let us assume that $\gamma(\varepsilon V)>0$ for all $\varepsilon>0$ (otherwise we may assign the value 1 to (3.1)).

Now we shall introduce some additional notations:

$$J_\varepsilon(f)=\frac{\int_{V_\varepsilon}\exp(f(x))\gamma(dx)}{\gamma(V_\varepsilon)},\quad V_\varepsilon=\{q\le\varepsilon\}.$$

$$E_q=\bigl\{f\in X^*_\gamma:\ \lim_{\varepsilon\to 0}J_\varepsilon(f)\ \text{ exists}\bigr\}.$$

$$F_q=\bigl\{f\in X^*_\gamma:\ \lim_{\varepsilon\to 0}J_\varepsilon(f)=1\bigr\}. \tag{3.4}$$

Recall that if q is a γ-measurable seminorm on X, then $q|_{H(\gamma)}$ is continuous in the natural norm of $H(\gamma)$. Let

$$Z=\{a\in H(\gamma):\ q(a)=0\},$$

$$Z^0=\bigl\{f\in X^*_\gamma:\ |f(z)|\le 1\ \forall z\in Z\bigr\}.$$

Note that Z^0 is a closed linear subspace in the Hilbert space X^*_γ.

π_q is the orthogonal projection on Z^0 in X^*_γ,

$$\delta_q=Id-\pi_q.$$

A trivial example of an element from F_q is given by f satisfying $f\le Cq$ for some C.

The following simple lemma plays an essential role below.

3.1. Lemma. *F_q is a closed linear subspace in X_γ^*. E_q is closed in X_γ^*.*

Proof. We shall use the following well-known estimate:

$$1 \leq \frac{\int\limits_{V_\varepsilon} \exp(f(x))\gamma(dx)}{\gamma(V_\varepsilon)} \leq \exp(\|f\|_2^2/2). \tag{3.5}$$

Indeed, by the convexity,

$$1 = \exp\Big(\int\limits_{V_\varepsilon} f(x)\gamma(dx)/\gamma(V_\varepsilon)\Big) \leq \frac{\int\limits_{V_\varepsilon} \exp(f(x))\gamma(dx)}{\gamma(V_\varepsilon)}. \tag{3.6}$$

On the other hand, according to Theorem 3.1,

$$\gamma(V_\varepsilon + h) \leq \gamma(V_\varepsilon)$$

for any h. For $h \in H(\gamma)$ this implies

$$\int\limits_{V_\varepsilon} \exp(-2^{-1}\|h\|_\gamma^2 + f_h(x))\gamma(dx) \leq \gamma(V_\varepsilon),$$

since f_h is an element in X_γ^*, associated with h by $h = R_\gamma(f_h)$. Since any $f \in X_\gamma^*$ corresponds to some h with $\|f\|_2 = \|h\|_\gamma$ we get (3.5). By (3.5) the functions $J_\varepsilon : X_\gamma^* \to R$ are locally uniformly Lipschitzian. Indeed, in view of (3.5)

$$\begin{aligned}\big|J_\varepsilon(f+g) - J_\varepsilon(f)\big| &\leq \sqrt{J_\varepsilon(2f)}\sqrt{J_\varepsilon(2g) - 2J_\varepsilon(g) + 1}\\ &\leq \exp(\|f\|_2^2)\sqrt{\exp(2\|g\|_2^2) - 2\exp(\|g\|_2^2/2) + 1}\\ &\leq \sqrt{2}\exp(\|f\|_2^2)\exp(\|g\|_2^2)\|g\|_2\end{aligned}$$

since

$$2e^{2t} - 2e^{t/2} + 1 \leq 2e^{2t}t$$

for $t \geq 0$. This uniform Lipschitz property implies that E_q and F_q are closed in X_γ^*.

Now we prove that F_q is linear. Let $f \in F_q$. For $\lambda \in (0,1)$ we have by (3.5) and the Hölder inequality

$$1 \leq J_\varepsilon(\lambda f) \leq J_\varepsilon(f)^\lambda.$$

Hence $\lambda f \in F_q$. Set

$$T_\varepsilon(z) = J_\varepsilon(zf).$$

The functions T_ε are holomorphic and satisfy the estimate:

$$|T_\varepsilon(z)| \leq J_\varepsilon(\mathrm{Im}(z)f) \leq \exp(\mathrm{Im}(z)^2\|f\|_2^2/2)\,. \tag{3.7}$$

Since

$$\lim_{\varepsilon\to 0} T_\varepsilon(z) = 1$$

for $z \in (0,1)$, by a classical theorem these functions are converging to 1 uniformly on bounded subsets in the complex plane and their derivatives are converging to zero locally uniformly. Thus $\lambda f \in F_q$ for all λ. Let $g \in F_q$. By (3.5)

$$1 \leq J_\varepsilon(f+g) \leq \sqrt{J_\varepsilon(2f)J_\varepsilon(2g)} \to 1 \quad \text{as } \varepsilon \to 0.$$

Hence $f + g \in F_q$.

3.2 Corollary. *For any sequence $\varepsilon_n \to 0$ there exists a subsequence $\delta_i = \varepsilon_{n_i}$ such that*

$$\lim_{i\to\infty} J_{\delta_i}(f)$$

exists for all $f \in X^*_\gamma$.

Proof. Choose a dense countable subset $\{f_j\} \subset X^*_\gamma$. For each j, since the sequence $J_{\varepsilon_n}(f_j)$ is bounded, by the standard diagonal procedure we can choose a subsequence ε_{n_i} ensuring the convergence at all points f_j. It follows from the above that then the limit exists at all points.

3.3 Corollary. *If F_q contains a dense subset, then $F_q = X^*_\gamma$. The same is true for E_q. In particular, this is the case if F_q or E_q contains X^*.*

3.4 Corollary. *If the set $\{q \leq 1\}$ is bounded or q is a continuous norm, then* (3.4) *holds for all measurable linear functionals.*

Proof. The first assertion follows from Corollary 3.3. To prove the second assertion note that by the Hahn–Banach theorem the family of continuous linear functionals on X that are majorized by q separates points in $H(\gamma)$ and hence is dense in X^*_γ. For such functionals (3.4) holds in a trivial way.

3.5 Remark. We do not know whether the equality $E_q = X^*_\gamma$ always holds.

Now we give conditions sufficient for the equalities $F_q = X^*_\gamma$ or $E_q = X^*_\gamma$ and making it possible to calculate the exact values of the corresponding limits.

Denote by P_q the orthogonal projection on F_q in X^*_γ,

$$\Delta_q = Id - P_q.$$

3.6 Proposition. *Assume that q is measurable with respect to $\sigma(F_q)$ (or is equivalent to a $\sigma(F_q)$-measurable function). Then*

$$\lim_{\varepsilon\to 0} J_\varepsilon(f) = \exp(\|\Delta_q f\|_2^2/2). \tag{3.8}$$

Proof. For $f = P_q f + \Delta_q f$ we have

$$J_\varepsilon(f) = J_\varepsilon(P_q f) \int\limits_X \exp(\Delta_q f)\gamma,$$

since $\Delta_q f$ and $\exp(P_q f) I_{V_\varepsilon}$ are independent random variables. The latter follows from the fact that q is equivalent to a $\sigma(F_q)$-measurable function and, consequently, is equivalent to a Borel function of some countable collection $\{f_j\}$ of elements from F_q, while $\Delta_q f$ is independent of each sequence of such elements, since it is orthogonal to all of them. By the direct calculation for the Gaussian random variable $\Delta_q f$, we obtain

$$\int\limits_X \exp(\Delta_q f(x))\gamma(dx) = \exp\Big[\int\limits_X (\Delta_q f(x))^2 \gamma(dx)/2\Big] = \exp(\|\Delta_q f\|_2^2/2).$$

Let us introduce some additional notions. We shall say that a seminorm q is of the sup-type if there exists a sequence $f_n \in X^*_\gamma$ and a linear subspace L with $\gamma(L) = 1$ such that

$$q(x) = \sup_n |f_n(x)|, \quad x \in L;$$

it will be called of the limsup-type if

$$q(x) = \limsup_n |f_n(x)|.$$

3.7 Example.

1. *These types are stable under countable operations of the corresponding character (sup or limsup).*
2. *Any continuous seminorm is of the sup-type.*
3. *Seminorms on R^∞ of the form*

$$q(x) = \sup_n q(x_1, \dots, x_n, 0, 0, \dots),$$

considered in [232], are of the sup-type (this follows from 1 and 2).

4. *The seminorm*

$$q(x) = \limsup_n |(n \log n)^{-1/2} x_n|$$

on the space R^∞ equipped with the product γ of standard one-dimensional Gaussian distributions is of the limsup-type but not of the sup-type, since it is finite on a linear subspace of the full γ-measure, but vanishes on $l^2 = H(\gamma)$.

5. *Seminorms of the sup-type are measurable with respect to the Lebesgue completion of $\sigma(F_q)$.*

Proof. Only the second statement needs a proof. Let L be the closure of $H(\gamma)$ in X. According to Theorem 1.5.6, $\gamma(L) = 1$ and $H(\gamma)$ is dense in each open part of L. Take a dense countable subset $\{a_j\}$ in the separable Hilbert space $H(\gamma)$. By the Hahn–Banach theorem and the compactness of closed balls of $H(\gamma)$ in X for each j and each closed ball $K(a_j, r)$ in this Hilbert space, such that r is rational and $K(a_j, r)$ does not intersect $Q = \{q \leq 1\}$, there exists $f_{j,r} \in X^*$ such that $f_{j,r} \leq 1$ on Q and $f_{j,r} \geq 1$ on $K(a_j, r)$. Hence, there exists a countable family $f_n \in X^*$ such that $|f_n| \leq q$ and

$$q(a) = \sup |f_n(a)| \quad \forall a \in H(\gamma)$$

(recall that $Q \cap H(\gamma)$ is closed in $H(\gamma)$). Now we obtain that

$$q(x) = \sup |f_n(x)| \quad \forall x \in L.$$

Indeed, for such x and each $\delta > 0$ the open set $x + \{q < \delta\}$ contains some $a \in H(\gamma)$, from which

$$q(x) < q(a) + \delta = \sup_n |f_n(a)| + \delta \leq \sup_n |f_n(x)| + \sup_n |f_n(a - x)| + \delta \leq$$

$$\leq \sup_n |f_n(x)| + q(a - x) + \delta \leq \sup_n |f_n(x)| + 2\delta.$$

This gives the estimate

$$q(x) \leq \sup_n |f_n(x)| \quad \forall x \in L.$$

The inverse estimate holds by our choice of $\{f_n\}$.

3.8 Lemma. $F_q \subset Z^0$. *If q is of the sup-type, then $F_q = Z^0$.*

Proof. Assume that $f \in F_q$ and $q(a) = 0$. Then $V_\varepsilon + a = V_\varepsilon$. Hence

$$J_\varepsilon(f) = \frac{\int\limits_{V_\varepsilon + a} \exp(f(x))\gamma(dx)}{\gamma(V_\varepsilon)} = \frac{\int\limits_{V_\varepsilon} \exp(f(x + a)) \exp(-\|a\|_\gamma^2/2) \exp(g_a(x))\gamma(dx)}{\gamma(V_\varepsilon)},$$

where $g_a \in X_\gamma^*$ is the functional corresponding to a. For any $r > 1$ and $s = (1 - 1/r)^{-1}$ the right-hand side of this equality is majorized by

$$\exp(f(a)) \exp(-\|a\|_\gamma^2/2) (J_\varepsilon(sf))^{1/s} (J_\varepsilon(rg_a))^{1/r}$$
$$\leq \exp(f(a)) \exp(-\|a\|_\gamma^2/2) (J_\varepsilon(sf))^{1/s} \exp(r\|a\|_\gamma^2/2),$$

which tends to $\exp(f(a))\exp((r-1)\|a\|_\gamma^2/2)$ as $\varepsilon \to 0$. This implies the estimate

$$1 = \lim_{\varepsilon\to 0} J_\varepsilon(f) \le \exp(f(a)).$$

The same is true for $-f$. Hence $f(a) = 0$ (otherwise one of the two numbers $\exp(f(a))$ or $\exp(-f(a))$ would be less than 1). If q is of the sup-type and f_n are the corresponding functionals, then clearly $f_n \in F_q$ and $Z = \bigcap_n \mathrm{Ker}(f_n)$. Functionals orthogonal to Z belong to the closure of the linear span of $\{f_n\}$ and hence to F_q.

3.9 Corollary. *$F_q = Z^0$ provided q is continuous or $\{q < 1\}$ is bounded.*

3.10 Remark. Note that the second assertion of this lemma is not true for norms of the limsup-type since they can vanish identically on $H(\gamma)$.

The next theorem summarizes our results.

3.11 Theorem. *Let q be a γ-measurable seminorm on a locally convex space X.*

(i) *Then there exists a sequence $\varepsilon_n \to 0$ such that*

$$\lim_{\varepsilon_n\to 0} J_{\varepsilon_n}(f)$$

exists for all $f \in X_\gamma^*$.

(ii) *Assume that q is measurable with respect to the completion of $\sigma(F_q)$. Then for all $f \in X_\gamma^*$*

$$\lim_{\varepsilon\to 0} J_\varepsilon(f) = \exp(\|\Delta_q f\|_2^2/2).$$

(iii) *If q is of the sup-type, then*

$$\lim_{\varepsilon\to 0} J_\varepsilon(f) = \exp(\|\Delta_q f\|_2^2/2) = \exp(\|\delta_q f\|_2^2/2).$$

3.12 Corollary. *Any of the following conditions is sufficient for the equality $F_q = X_\gamma^*$:*

(a) *F_q is dense in X_γ^*;*
(b) *q is of the sup-type and $q\,|_{H(\gamma)}$ is a norm;*
(c) *q is continuous and $q\,|_{H(\gamma)}$ is a norm;*
(d) *$\{q < 1\}$ is bounded.*

3.13 Examples.

1. *Let γ be the Wiener measure on $C[0,1]$ with a γ-measurable seminorm q. If*

$$\lim_{\varepsilon\to 0} E\big[\exp w(t) \mid q \le \varepsilon\big] = 1 \quad \forall t \in [0,1],$$

then

$$\lim_{\varepsilon\to 0} E\left[\exp\left(\int_0^1 \phi(t)dw_t\right) \,\middle|\, q \le \varepsilon\right] = 1 \quad \forall \phi \in L^2[0,1]. \tag{3.9}$$

Thus, the condition $\|x\|_{L^1} \le Cq(x)$ from [416] can be omitted.

2. *On the other hand if $\|x\|_{L^1} \le Cq(x)$, then (3.9) holds without additional conditions and*

$$\lim_{\varepsilon\to 0} E\big[\exp w(t) \mid q \le \varepsilon\big] = 1$$

holds automatically (and thus can be omitted in [416]).

3.14 Remark. For related results see [23, 65, 207, 208, 209, 285, 286, 314, 317]. Similar results can be proved for some measures equivalent to Gaussian measures, for example, for the distributions of diffusion processes with constant diffusion coefficients (see [96, 124, 239, 495, 499, 504]). But, as noticed in [141], for a diffusion ξ_t with a nonconstant diffusion coefficient σ the Onsager–Machlup function does not exist if one chooses the usual sup-norm on the path space X. In [141] the described phenomenon was explained by the fact that the measures μ^ξ and $\mu^{\xi+th}$, where $h \neq 0$ is smooth, are not equivalent. Later it was realized that it is more natural here to choose a nonlinear Riemannian metric on the state space of ξ_t in which σ gives the Laplace–Beltrami operator and thus to introduce a new metric on X. This can also be understood as follows.

3.15 Example. *Let γ be a centered Gaussian measure on $X, F : X \to X$ a diffeomorphism, $\mu = \gamma \circ F^{-1}, G = F^{-1}$. Then*

$$\frac{\mu(\varepsilon U + h)}{\mu(\varepsilon U)} = \frac{\gamma(G(\varepsilon U + h))}{\gamma(G(\varepsilon U))}, \quad G(\varepsilon U + h) = G(h) + \varepsilon G'(h)U + \cdots,$$

and if $G'(h) \neq G'(0)$ one cannot hope to get a reasonable limit. But if one chooses a metric depending on a point it becomes possible to achieve a suitable scaling to reduce the problem to the Gaussian case. It would be interesting to develop this idea.

Now we shall briefly mention some results on the existence of the Onsager–Machlup functionals for non Gaussian measures.

First of all note, that the Onsager–Machlup functional exists for the diffusion process ξ_t in a Riemannian manifold M generated by the operator $\Delta_M/2 + b$, where Δ_M is the Laplace–Beltrami operator and b is a smooth vector field on M. A discussion of this case and further references can be found in [239].

In [328, 495] the existence of the Onsager–Machlup functional is investigated for the solution of the stochastic partial differential equation

$$Pu + F(u) = n$$

in a bounded domain $D \subset R^d$ with zero boundary condition, where P is an elliptic operator of order $2k$, F is smooth (in a suitable sense), and n is a white noise process. The idea of the proof is to reduce the problem to case of an equivalent Gaussian measure.

Another interesting case was studied in [96].

The study of Onsager–Machlup functionals is connected with the differentiation problems considered in [379, 460]. Another interesting direction of investigation is to study possible links between the Onsager–Machlup functionals, the Cameron–Martin formula, and generalized Girsanov's transformations.

3.16 Remark. It should be mentioned that in [376] there is an example of a centered Gaussian measure γ on a separable Hilbert space X and a set A with $\gamma(A) > 0$, such that

$$\lim_{r\to 0} \frac{\gamma(A \cap B(x,r))}{\gamma(B(x,r))} = 0 \quad \gamma\text{-a.e.},$$

where $B(x,r)$ is a closed ball of radius r centered at x. In [377] an example is given of a function $f \in L^1(\gamma)$ such that

$$\lim_{r\to 0} \inf\left\{\gamma(B(x,r))^{-1} \int\limits_{B(x,r)} f(y)\gamma(dy) : \ x \in X, \ 0 < s < r\right\} = +\infty.$$

On the other hand, the following result is proved in [379]. Let γ be the countable product of standard Gaussian measures on the line, regarded as a measure on the weighted Hilbert space X of the sequences (x_n) such that $\sum_n c_n x_n^2 < \infty$, where $0 < c_{k+1}/c_k \leq q, \ q < 1$. Then for every $f \in L^1(\gamma)$

$$\gamma(B(x,r))^{-1} \int\limits_{B(x,r)} f(y)\gamma(dy) \to f \quad \text{in } \gamma\text{-measure as } r \to 0.$$

4. Large Deviations and Applications.

The principles of large deviations for Gaussian measures started with Shilder's theorem [406] for Wiener measure. The extension to arbitrary separable Banach spaces is due to Donsker and Varadhan [131–133]. Borell [63] considered the case of a general locally convex space. An attempt to find a measure-theoretic version of this result was made by Ben Arous and Ledoux [31]. The exposition in this section follows [31].

Let γ be a centered Radon Gaussian measure on a locally convex space X, $\gamma^\varepsilon(A) = \gamma(A/\varepsilon)$, and U_H be the closed unit ball in the Hilbert space $H = H(\gamma)$.

For any two sets $A, \ B \subset X$ put

$$A \ominus B = \{x \in A : \ x + B \in A\}.$$

Denote by $\mathcal{BAL}$ the class of all Borel subsets V of X such that

$$\liminf_{\varepsilon\to 0} \gamma^{\varepsilon}(V) > 0.$$

Note that if $V \in \mathcal{BAL}$, then $\lambda V \in \mathcal{BAL}$ for every $\lambda > 0$. Any Borel set V such that $\lambda V \subset V$ and $\gamma(V) > 0$ belongs to $\mathcal{BAL}$.

Set

$$I(A) = \frac{1}{2} \inf_{x\in A} I(x)^2, \qquad I(x) = \|x\|_H,$$

where $\|x\|_H = \infty$ if x is not in H.

Introduce the following functionals on $\mathcal{B}(X)$:

$$r(A) = \sup\{r \geq 0 :\ \exists V \in \mathcal{BAL},\ (rU_H + V) \cap A = \varnothing\}$$

($r(A) = 0$ if for no r does such a set V exist) and

$$s(A) = \inf\{s \geq 0 :\ \exists V \in \mathcal{BAL},\ (A \ominus V) \cap sU_H \text{ is not empty}\}$$

($s(A) = +\infty$ if for no s does such a set V exist).

The classical large deviation principle says (see [16, 126]) that for every Borel set $A \in X$ one has

$$-I(A^{\circ}) \leq \liminf_{\varepsilon\to 0} \varepsilon^2 \log \gamma^{\varepsilon}(A) \leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log \gamma^{\varepsilon}(A) \leq -I(\overline{A}),$$

where A° is the interior of A in X and $\overline{A}$ is its closure.

The role of the topology in this statement was studied in [31], where the following results were obtained. We have to stress that, although the space X was assumed to be a separable Banach in [31], the same proofs apply to general locally convex spaces (to illustrate this circumstance we reproduce some proofs from [31]). The connection of the material of this section with the main subject of the present chapter comes from estimate (1.3), which plays an important role below.

4.1 Lemma. *For any subset A of X,*

$$\frac{1}{2} r(A)^2 \geq I(A) \textit{ for any closed } A \subset X$$

and

$$\frac{1}{2} s(A)^2 \leq I(A) \textit{ for any open } A \subset X.$$

Proof. Assume that A is closed and that $I(A) > 0$. Let r be such that $0 < r < I(A)$. It follows from the definition of $I(A)$ that $\sqrt{2r}U_H \cap A = \varnothing$. Since $\sqrt{2r}U_H$ is compact and A is closed, there is a convex circled neighborhood of the origin V such that $(\sqrt{2r}U_H + V) \cap A$ is still empty. Clearly, $V \in \mathcal{BAL}$. Therefore, $r(A) \geq \sqrt{2r}$. This implies the first assertion. Now let A be open and such that $I(A) < \infty$. Then we can find an element $h \in A \cap H$. There is an open absolutely convex open set V such that $V \subset A$, which means that $(A \ominus V) \cap \|h\|_H U_H$ is nonempty. Hence $s(A) \leq \|h\|_H$, from which the second assertion follows.

4.2 Theorem. *For every Borel set A in X,*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \gamma^{\varepsilon}(A) \leq -\frac{1}{2} r(A)^2$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \gamma^{\varepsilon}(A) \geq -\frac{1}{2} s(A)^2.$$

Proof. Let $r \geq 0$ be such that $(rU_H + V) \cap A$ is empty for some $V \in \mathcal{BAL}$. Then

$$\gamma^\varepsilon(A) \leq 1 - \gamma(\varepsilon^{-1} r U_H + \varepsilon^{-1} V).$$

Since $V \in \mathcal{BAL}$, there is $\alpha > 0$ such that $\gamma(\varepsilon^{-1}V) \geq \alpha$ for all sufficiently small positive ε. Hence, by virtue of (1.3), for some real number $c(\alpha)$ depending on α only, and all sufficiently small positive numbers ε one has

$$\gamma^\varepsilon(A) \leq \exp\left(-\frac{r^2}{2\varepsilon^2} + c(\alpha)\frac{r}{\varepsilon}\right),$$

from which (i) immediately follows since r can be chosen arbitrarily less than $r(A)$.

To prove (ii) let us take a positive number s such that $(A \ominus V) \cap sU_H$ is nonempty for some V in $\mathcal{BAL}$. Therefore, there exists $h \in sU_H$ such that $V + h \subset A$. Hence, for every $\varepsilon > 0$,

$$\gamma^\varepsilon(A) \geq \gamma^\varepsilon(V + h).$$

Denoting by g_h the measurable linear functional associated with h, we get, by the Cameron–Martin formula,

$$\gamma^\varepsilon(V + h) = \exp\left(-\frac{\|h\|_H^2}{2\varepsilon^2}\right) \int_{V/\varepsilon} \exp(-g_h(x)/\varepsilon)\gamma(dx).$$

Since $V \in \mathcal{BAL}$, there is α such that $\gamma(V/\varepsilon) \geq \alpha > 0$ for every ε small enough. By Jensen's inequality

$$\int_{V/\varepsilon} \exp(-g_h(x)/\varepsilon)\gamma(dx) \geq \gamma(V/\varepsilon) \exp\left[-\int_{V/\varepsilon} \frac{g_h(x)}{\varepsilon}\gamma(dx)/\gamma(V/\varepsilon)\right].$$

By the Cauchy inequality

$$\int_{V/\varepsilon} g_h(x)\gamma(dx) \leq \|h\|_H.$$

Thus, for every $\varepsilon > 0$ small enough

$$\gamma^\varepsilon(A) \geq \alpha \exp\left(-\frac{\|h\|_H^2}{2\varepsilon^2} - \frac{\|h\|_H}{\alpha\varepsilon}\right).$$

This leads to the estimate

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log \gamma^\varepsilon(A) \geq -\frac{\|h\|_H^2}{2} \geq -\frac{s^2}{2}.$$

Since s may be chosen arbitrarily less than $s(A)$ the claim (ii) follows.

4.3 Example. *Let X be a separable Banach space with norm $\|\cdot\|$ and let σ be the minimal positive number such that $U_H \subset \{x \in X : \|x\| \leq \sigma\}$ (this is the norm of the natural embedding $H \to X$). Then for the both sets $A = \{x : \|x\| \geq 1\}$ and $A = \{x : \|x\| > 1\}$ one has $I(A) = \sigma^2/2$. Hence*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \gamma\left(x : \|x\| \geq \frac{1}{\varepsilon}\right) = -\frac{1}{2\sigma^2}.$$

The constructions described above may be applied to Laplace approximations (see also [30]).

4.4 Theorem. *Let $F : X \to \overline{R}$ be a γ-measurable function. If F is bounded below, then for any Borel subset A of X*

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log\left(\int_A \exp\left(-\frac{1}{\varepsilon^2}F(\varepsilon x)\right)\gamma(dx)\right) \leq -\inf_{t \in R}\left(t + \frac{1}{2}r(A \cap \{F \leq t\})^2\right).$$

If F is bounded above, then for any Borel subset A of X,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log\left(\int_A \exp\left(-\frac{1}{\varepsilon^2}F(\varepsilon x)\right)\gamma(dx)\right) \geq -\inf_{t \in R}\left(t + \frac{1}{2}s(A \cap \{F \leq t\})^2\right).$$

In the following, $\inf\limits_{x \in A}(F(x) + I(x))$ is meant as the infinum over all $x \in A$ such that either $F(x) \neq -\infty$ or $I(x) \neq +\infty$.

4.5 Theorem. *Let $F: X \to \overline{R}$ be γ-measurable.*

(i) *Assume that F is lower semi-continuous on H and that for every $r > 0$ and every $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \gamma^\varepsilon \left(x : \sup_{\|h\|_H \le r} |F(h) - F(x+h)| > \delta \right) < 1.$$

Then, for every closed set A in X,

$$\inf_{x \in A} (F(x) + I(x)) \le \inf_{t \in R} \left(t + \frac{1}{2} r(A \cap \{F \le t\})^2 \right).$$

(ii) *Assume that for every h in H and every $\delta > 0$*

$$\limsup_{\varepsilon \to 0} \gamma^\varepsilon (x : |F(x+h) - F(h)| > \delta) < 1.$$

Then, for every open set A in X,

$$\inf_{x \in A} (F(x) + I(x)) \ge \inf_{t \in R} \left(t + \frac{1}{2} s(A \cap \{F \le t\})^2 \right).$$

4.6 Corollary. *Let $F: X \to R$ be bounded and such that for every $r > 0$ and every $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \gamma^\varepsilon \left(x : \sup_{\|h\|_H \le r} |F(h) - F(x+h)| > \delta \right) < 1.$$

Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \left(\int_X \exp \left(-\frac{1}{\varepsilon^2} F(\varepsilon x) \right) \gamma(dx) \right) = - \inf_{x \in X} (F(x) + I(x))$$

Let M be a Polish space with metric d and let $F: X \to M$ be a Borel map. Put $\nu^\varepsilon = \gamma^\varepsilon \circ F^{-1}$. The rate function for this family is defined by

$$J(y) = \inf\{I(x) : F(x) = y\}, \ y \in M.$$

For any subset A of M put

$$J(A) = \inf_{y \in A} J(y).$$

4.7 Lemma. *For any Borel set $A \subset M$ one has*

$$J(A) \le \frac{1}{2} r(F^{-1}(A))^2 \text{ if } A \text{ is closed,}$$

$$J(A) \ge \frac{1}{2} s(F^{-1}(A))^2 \text{ if } A \text{ is open.}$$

4.8 Theorem.

(i) *Assume that F is continuous on H and that for every $r > 0$ and every $\delta > 0$*

$$\limsup_{\varepsilon \to 0} \gamma^\varepsilon \left(x : \sup_{\|h\|_H \le r} d(F(h), F(x+h)) > \delta \right) < 1.$$

Then, for every closed subset A of M,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \nu^\varepsilon(A) \le -J(A).$$

(ii) *Assume that for every $h \in H$ and every $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \gamma^\varepsilon (x : d(F(h), F(x+h)) > \delta) < 1.$$

Then, for every open subset A of M,

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log \nu^\varepsilon(A) \ge -J(A).$$

It is well known that the large deviation estimates yield Strassen's functional law of the iterated logarithm for Brownian motion (see, e.g., [126]). One can sharpen Strassen's result with the measure theoretic estimates presented above (see [31] for details).

5. Additional Remarks.

Gaussian measures on infinite-dimensional spaces play a vital role in the limit theorems for infinite-dimensional random elements. Further information about this important field can be found in [12, 33, 137, 230, 301, 304, 359, 386, 480]. We shall make several remarks connected with the central limit theorem.

Let X be a locally convex space and let X_n be a sequence of X-valued independent centered random vectors with one and the same Radon distribution μ. Set

$$S_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Note that the distribution of X_n coincides with the measure μ^{*n}, defined by

$$\mu^{*n}(A) = (\mu * \ldots \mu)(n^{-1/2}A),$$

where the convolution is n-fold.

The central limit problem studies the following two questions:

1) Does the sequence of random vectors S_n converge (in a suitable sense)?

2) If it converges to some random element Y, then what is the rate of convergence on a certain class of sets ? More precisely, let $\mathcal{M}$ be a fixed class of subsets of X (say, some class of balls in a Banach space). Then the problem is to estimate the quantity

$$\Delta_n(\mathcal{M}) = \sup_{M \in \mathcal{M}} |P(S_n \in M) - P(Y \in M)|.$$

A typical problem of this kind is to estimate

$$\Delta_n(f, r) = \big|P(f(S_n) < r) - P(f(Y) < r)\big|,$$

where f is a function on X (typically, a norm or a smooth function).

In order to formulate some related results, we introduce the following notions.

In this section we consider only Radon probability measures μ which satisfy the following condition:

$$\int_X l(x)^2 \mu(dx) < \infty \quad \forall l \in X^*. \tag{5.1}$$

In this case we say that the μ has weak second moment.

5.1 Definition. We say that a probability measure μ with mean m on a locally convex space X is pregaussian, if condition (5.1) holds and there exists a Gaussian measure γ with mean m on X such that

$$\int_X f(x)g(x)\mu(dx) = \int_X f(x)g(x)\gamma(dx) \quad \forall f, g \in X^*.$$

In Example 5.9 below we shall encounter a probability measure which has bounded support in c_0, but is not pregaussian.

5.2 Definition. We say that a probability measure μ with zero mean on a locally convex space X satisfies the central limit theorem (CLT), if condition (5.1) holds and the sequence $\{\mu^{*n}\}$ is tight. A probability measure μ with mean m is said to satisfy the CLT if the measure μ_{-m} with zero mean satisfies the CLT.

5.3 Lemma. *Let μ be a probability measure with zero mean on a locally convex space X. If the sequence $\{\mu^{*n}\}$ is tight, then it converges weakly to some centered Radon Gaussian measure γ. In addition, μ is pregaussian.*

Proof. According to the Prokhorov theorem 0.D.20 the sequence $\{\mu^{*n}\}$ is relatively compact in the weak topology. Hence there is a Radon probability measure γ which is its weak limit point. Note that γ is a Gaussian measure. Indeed, for any $l \in X^*$ the measure $\gamma \circ l^{-1}$ is a weak limit point of the sequence of measures $\mu^{*n} \circ l^{-1}$, which converges weakly to the centered Gaussian measure with dispersion $\int l(x)^2 \mu(dx)$ by virtue of the one-dimensional central limit theorem. This argument also proves the uniqueness of the limit point. Hence $\{\mu^{*n}\}$ converges weakly to γ.

We say that a measure μ on a locally convex space X has strong second moment if for any continuous seminorm q on X one has

$$\int_X q(x)^2 \mu(dx) < \infty. \tag{5.2}$$

5.4 Definition. A locally convex space X is said to have the central limit theorem property (the CLT property) if any measure μ on X which satisfies condition (5.2) satisfies the CLT. X is said to have the strict CLT property if the CLT holds for any probability measure μ on X which has mean zero and satisfies (5.1).

If $X = R^n$, then any measure satisfying condition (5.1) satisfies the CLT. Certainly, such a measure satisfies condition (5.2) (moreover, (5.2) follows from the tightness of the sequence $\{\mu^{*n}\}$). The situation is different in infinite-dimensions. For the proofs of the following four statements, see [304, 480].

5.5 Proposition.

(i) *A Banach space X has the strict CLT property if and only if* $\dim X < \infty$.

(ii) *The space $C[0,1]$ does not have the CLT property. Moreover, there exists a pregaussian measure with bounded support in $C[0,1]$ that does not satisfy the CLT.*

(iii) *There is a measure with bounded support in $C[0,1]$ which is not pregaussian.*

(iv) *There is a measure on $C[0,1]$ which satisfies the CLT, but*

$$\int_X \|x\|^2 \mu(dx) = \infty.$$

5.6 Proposition. *Any Hilbert space has the CLT property.*

5.7 Theorem. *Let X be a Banach space. The following conditions are equivalent:*

(i) *X has cotype 2.*

(ii) *Any pregaussian measure on X satisfies the CLT.*

(iii) *For any pregaussian measure μ*

$$\int_X \|x\|^2 \mu(dx) < \infty. \tag{5.3}$$

(iv) *If a measure μ satisfies the CLT, then* (5.3) *holds.*

5.8 Theorem. *A Banach space X has cotype 2 if and only if the CLT holds for any measure which satisfies condition* (5.2) *and has mean zero.*

It is worth mentioning that in standard examples of measures on Banach spaces which are not pregaussian these measures have the form

$$\mu = \sum_{n=1}^{\infty} c_n \delta_{a_n},\ c_n > 0,\ a_n \in X.$$

In these examples it is easy to find a pregaussian "part" ν of μ. As the following example shows, there are measures with compact supports without pregaussian "parts."

5.9 Example. *Let X be a separable Banach space which contains a closed linear subspace linearly homeomorphic to the space c_0. Then there is a probability measure μ on X with compact support such that μ is mutually singular with any pregaussian measure on X. In particular, this holds for $X = C[0,1]$.*

Proof. Let $X = c_0$. Take an infinitely differentiable probability density p with support in the segment $[-1,1]$ such that

$$I = \int_{-1}^{1} \frac{p'(x)^2}{p(x)} dx < \infty.$$

Put $p_n(t) = c_n^{-1} p(t/c_n)$, where $c_n = \sqrt{\log(n+1)}$. Let μ be the product of the measures μ_n with the densities p_n. Clearly, $\mu(c_0) = 1$. In addition, the support of μ is contained in the compact set $\{(x_n) : |x_n| \leq c_n^{-1}\}$. The proof of the orthogonality of μ to all pregaussian measures is based on the following three observations:

1. For any vector h in the set

$$H = \left\{(h_n) : \ \|h\|_H^2 = \sum_{n=1}^{\infty} c_n^2 h_n^2 < \infty\right\}$$

one has

$$\lim_{t\to 0} \|\mu_{th} - \mu\| = 0. \tag{5.4}$$

2. If $\nu \ll \mu$, then (5.4) holds for ν.
3. If γ is a Gaussian measure on X such that $\nu \ll \mu$ and $\nu \ll \gamma$, then $H = H(\gamma)$.

Technical details of the proof may be found in [50].

Now we shall discuss some properties of locally convex spaces with the strict CLT property.

5.10 Proposition. *Let X be the strict inductive limit of a sequence of locally convex spaces X_n. If μ is a probability measure on X such that $X^* \subset L^p(\mu)$ for some $p \geq 1$, then there is n such that*

$$\mu(X_n) = 1 \text{ and } X^* \subset L^p(\mu).$$

Proof. Assuming the converse, one gets a contradiction using the Hahn–Banach theorem (see [41]). Note that the strictness of the limit is essential (see Example 2 in [41]).

5.11 Theorem. *The strict CLT property is inherited by closed subspaces and is preserved in the formation of strict inductive limits, countable products, and direct sums.*

Proof. Let μ be a probability measure with zero mean and weak second moment on a countable product $\prod_{n=1}^{\infty} X_n$ where X_n is a locally convex space with the CLT property. Denote by μ_n the image of μ under natural projection onto X_n. Note that the image of μ^{*k} under the same projection coincides with μ_n^{*k}. For any $\varepsilon > 0$ and any n there is a compact set $K_n \in X_n$ such that

$$\mu_n^{*k}(X_n \backslash K_n) < \varepsilon/2^n \quad \forall k.$$

Then $K = \prod_{n=1}^{\infty} K_n$ is compact in X and

$$\mu^{*k}(X \backslash K) \leq \varepsilon.$$

Indeed, letting $Q_n = X_1 \times \cdots \times X_{n-1} \times K_n \times X_{n+1} \times \ldots$, for any n one has

$$\mu^{*k}(X\backslash K) \leq \sum_{n=1}^{\infty} \mu^{*k}(X\backslash Q_n) \leq \sum_{n=1}^{\infty} \varepsilon/2^n \leq \varepsilon.$$

In particular, we get our assertion for finite products, hence for finite direct sums. Applying Proposition 5.10 we get this assertion for strict inductive limits and direct sums.

5.12 Corollary. *The strict CLT property is retained in the formation of countable projective limits.*

For uncountable products, Theorem 5.11 does not hold. Indeed, let μ be the product of an uncountable number of copies of the measure ν on the line which assigns 1/2 to the points -1 and 1. Clearly, μ admits a Radon extension to the Borel σ-field of the corresponding product of lines. It is easy to see that the only candidate for a weak limit point of the sequence $\{\mu^{*n}\}$ is the product of the standard Gaussian measures on the line, which is not a tight measure, as we already know.

5.13 Theorem. *Let X be the dual space to a complete nuclear barrelled locally convex space Y. Then X equipped with the strong topology has the strict CLT property.*

Proof. Let μ be a probability measure on X with zero mean and weak second moment. For any n one has

$$\left|1 - \int_X \exp(il(x))\mu^{*n}(dx)\right| \leq \int_X l(x)^2\mu(dx).$$

From this estimate one can deduce the tightness of $\{\mu_n\}$ (see Theorem 0.D.20 or [41]).

Note that the space X itself need not have the CLT property. We have already encountered such an example: an uncountable product of lines (this is a complete nuclear barrelled space).

5.14 Corollary. *Let X be the dual space to a nuclear Fréchet space. Then X has the strict CLT property.*

5.15 Corollary. *The following spaces have the strict CLT property: $\mathcal{D}[-n,n]$, $\mathcal{D}[-n,n]^*$, $\mathcal{D}(R^k)$, $\mathcal{D}(R^k)^*$, $\mathcal{S}(R^k)$, $\mathcal{S}(R^k)^*$, $\mathcal{E}(R^k)$, $\mathcal{E}(R^k)^*$, R^∞.*

Proof. Follows from Theorem 5.11 and Theorem 5.13.

5.16 Theorem. *Let X be the inductive limit of an increasing sequence of locally convex spaces X_n such that for any n the natural embedding of X_n in X_{n+1} is compact. If a probability measure μ on X satisfies the CLT, then there is n such that*

$$\mu(X_n) = 1 \text{ and } X_n^* \subset L^2(\mu).$$

Proof. See [41].

5.17 Corollary. *Under the conditions of Theorem* 5.16 *for any measure μ on X which satisfies the CLT there is a separable Banach space B, compactly embedded in X, such that $\mu(B) = 1$.*

Proof. See [41].

5.18 Corollary. *If X is the dual to a nuclear Fréchet space, then for any measure μ on X with weak second moment there is a separable Hilbert space H, compactly embedded in X, such that $\mu(H) = 1$.*

5.19 Corollary. *The assertion of Corollary 5.18 holds for the following spaces: $\mathcal{D}[-n,n]^*$, $\mathcal{S}(R^k)^*$, $\mathcal{E}(R^k)^*$.*

Additional information can be found in the references above. The central limit theorem is just one problem in the large area of limit theorems for infinite-dimensional random variables. A lot of interesting material, including convergence of sums of independent random vectors, is presented in the above quoted references. One of the related questions is the Log law. In the simplest formulation it says that if $\{\xi_n\}$ are independent random vectors in a separable Banach space X with one and the same centered Gaussian distribution γ, then with probability one the sequence $\sum_1^n \xi_i/\sqrt{2n\log\log n}$ clusters (in the norm of X) at any point of the unit ball U_H of the Hilbert space $H(\gamma)$ (see [305]).

Chapter 3
Sobolev Classes over Gaussian Measures

1. Integration-by-Parts Formulas.

In this section, we discuss integration-by-parts formulas for Gaussian measures. It would be natural to discuss this topic within the framework of the theory of differentiable measures, suggested by Fomin [181] and developed by many authors (see the recent survey [50]). However, in order to keep our exposition within reasonable limits, we just mention several basic notions. We start with the differentiability of functions.

Different types of differentiability can be described by the following scheme of differentiability with respect to a class of sets $\mathcal{M}$. Let X, Y be locally convex spaces and let $\mathcal{M}$ be some class of nonvoid subsets in X.

1.1 Definition. A map $F: X \to Y$ is said to be differentiable with respect to $\mathcal{M}$ at a point x if there exists a continuous linear map from X to Y, denoted by $DF(x)$, such that uniformly in h from each fixed $M \in \mathcal{M}$

$$\lim_{t\to 0} \frac{F(x+th)-F(x)}{t} = DF(x)h.$$

1.2 Example. 1. Taking the collection of all finite subsets for $\mathcal{M}$ we get the Gâteaux differentiability. 2. Choosing for $\mathcal{M}$ all compact subsets we come to the Hadamard differentiability (in the case of normed spaces). 3. Finally, if X, Y are normed spaces and $\mathcal{M}$ consists of all bounded sets, we get the definition of the Fréchet differentiability.

It is clear that in finite-dimensional spaces the Hadamard definition is equivalent to the Fréchet one and is stronger than the Gâteaux differentiability. An easy but useful fact is that for locally Lipschitzian mappings in normed spaces Gâteaux and Hadamard differentiabilities coincide (see [107]).

In infinite-dimensional Banach spaces Fréchet differentiability is strictly stronger than Hadamard differentiability. For example, the function

$$f:\ L^1[0,1] \to R^1, \quad f(x) = \int_0^1 \sin x(t)dt,$$

is Hadamard differentiable but not Fréchet differentiable. The same holds for the map

$$F:\ L^2[0,1] \to L^2[0,1], \quad F(x)(t) = \sin x(t).$$

If E is a linear subspace in X, then one can define differentiability along E (in the corresponding sense) at a point x as differentiability at $h = 0$ of the map $h \mapsto F(x+h)$ from E to Y in this sense. The derivative along E is denoted by $D_E F$. When E is one-dimensional this gives the usual partial derivative $\partial_h F$.

1.3 Definition. A Radon measure μ on a locally convex space X is said to be differentiable along a vector $h \in X$ in the sense of Fomin if for any Borel set A there exists the limit

$$\lim_{t\to 0} \frac{\mu(A+th)-\mu(A)}{t}. \tag{1.1}$$

In this case there is a signed Radon measure ν, absolutely continuous with respect to μ, such that the limit above equals $\nu(A)$ for all Borel A (see [15, 118]). This measure is denoted by the symbol $d_h\mu$ and is called the derivative of μ along h. The Radon–Nikodym density $\beta_h(\mu)$ of $d_h\mu$ with respect to μ is called the logarithmic derivative of μ along h.

By induction one defines higher-order differentiability.

1.4 Example. Let μ be a measure on the line with density p. Assume that p is an absolutely continuous function with $p' \in L^1(R^1)$. Then μ is differentiable along any $h \in R^1$ and $\beta_1(\mu) = p'/p$.

In infinite-dimensions a nonzero measure cannot be differentiable along all directions.

1.5 Proposition. *Let γ be a Radon Gaussian measure on a locally convex space X. Then its reproducing kernel Hilbert space coincides with the collection of all vectors of differentiability. In addition, γ is infinitely differentiable along $H(\gamma)$. If γ is centered and if $h = R_\gamma g$, $g \in X_\gamma^*$, then*

$$\beta_h(\gamma)(x) = -g(x).$$

This result can be easily derived from the formula for the density of the translate of a Gaussian measure. See [50] for details. Moreover, a closer look shows [6, 32] that γ is analytic along directions from the Cameron–Martin space, that is, for any $h \in H(\gamma)$ all the functions $t \mapsto \gamma(A+th)$ admit holomorphic extensions. Such a property holds [40] for any measure μ which is stable of index $\alpha > 1$ and differentiable along h.

1.6 Example. Let γ be a centered Gaussian measure on $X = l^2$ which coincides with the countable product of Gaussian measures on the line with covariances σ_n such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then for any

$$h \in H(\gamma) = \left\{ h \in l^2 : \sum_{n=1}^{\infty} \sigma_n^{-1} h_n^2 < \infty \right\}$$

one has

$$\beta_h(\gamma)(x) = -\sum_{n=1}^{\infty} \sigma_n^{-1} h_n x_n.$$

Note that $\partial_k \beta_h = -(k, h)_{H(\gamma)}$ for any $k \in H(\gamma)$. By induction one can prove that for any r there exists a polynomial E_r on R^r such that

$$d_{h_1} \dots d_{h_r} \gamma = E_r(g_1, \dots, g_r)\gamma, \; h_i = R_\gamma g_i, \; g_i \in X_\gamma^*.$$

Recall that a normed space E is said to have the Radon–Nikodym property if any absolutely continuous map $f : [0,1] \to E$ is differentiable a.e. This is equivalent to many other properties, such as almost everywhere differentiability of all Lipschitzian functions with values in E (see [127, 360]). It is known that all reflexive Banach spaces and all separable dual spaces possess the Radon–Nikodym property. In particular, any Hilbert space has this property.

1.7 Theorem. *Let γ be a Radon Gaussian measure on a locally convex space X, $h \in H(\gamma)$. Assume that $F : X \to E$ is a measurable mapping with values in a separable Banach space E with the Radon–Nikodym property such that:*

(i) *For γ-a.e. x the mapping $t \mapsto F(x + th)$ is either absolutely continuous or everywhere differentiable;*

(ii) *the maps $\partial_h F$ and $\beta_h(\gamma)F$ are γ-integrable.*

Then the following formula holds:

$$\int_X \partial_h F(x)\gamma(dx) = -\int_X \beta_h(\gamma)(x)F(x)\gamma(dx). \tag{1.2}$$

Stronger results may be found in [48, 50, 265]. Various useful integration-by-parts formulas are obtained in [15, 116, 117].

One might ask to what extent are Gaussian measures determined by their logarithmic derivatives. There is an example [47, 55] of two different centered Gaussian measures on a separable Hilbert space X which have continuous and equal logarithmic derivatives along vectors from a dense linear subspace $L \subset X$. On the other hand, it is known [46] that if γ is a centered Radon Gaussian measure on a locally convex space X and μ is a Radon probability measure on X such that γ and μ have continuous and equal logarithmic derivatives β_{a_n} along some vectors a_n, then $\mu = \gamma$ provided the sequence $\{\beta_{a_n}\}$ separates the points in X. Related results may be found in [55, 393].

The notion of logarithmic derivative is closely related to the stochastic integral (see [47, 115, 116, 117, 192, 350, 352, 430]).

2. Sobolev Classes over Gaussian Measures.

Denote by $\mathcal{F}C^\infty$ the class of all functions f on the locally convex space X of the form

$$f(x) = \phi(l_1(x), \dots, l_n(x)), \quad l_i \in X^*, \; \phi \in C_b^\infty(R^n).$$

For any $p \geq 1$ and any $r \in N$ the Sobolev norm $\|\cdot\|_{p,r}$ is defined by the following formula:

$$\|\phi\|_{p,r} = \sum_{k \leq r} \left(\int [\sum_{i_1, \dots, i_k \geq 1} (\partial_{i_1} \partial_{i_k} \phi(x))^2]^{p/2} \gamma(dx) \right)^{1/p} \tag{2.1}$$

Denote by $W^{p,r}$ the completion of $\mathcal{F}C^\infty$ with respect to the norm $\|\cdot\|_{p,r}$. Note that the same norm may be defined as

$$\|\phi\|_{p,r} = \sum_{k\le r} \|D_H^k \phi\|_{L^p(X,\mathcal{H}_k)}.$$

A function f on X is said to be a finite-dimensional polynomial on X if there exists a polynomial Q on R^n and functionals $f_1, \dots, f_n \in X^*$ such that

$$f(x) = Q(f_1(x), \dots, f_n(x)).$$

Denote by $\mathcal{P}ol(X)$ the class of all finite-dimensional polynomials on X.

2.1 Lemma. *$W^{p,r}$ coincides with the completion of $\mathcal{P}ol(X)$ with respect to the norm $\|\cdot\|_{p,r}$.*

Proof. This follows from the classical theory of Sobolev spaces in finite dimensions.

In a similar way, one defines the Sobolev spaces $W^{p,r}(X,E)$ of mappings with values in a Hilbert space E. The corresponding norms are denoted by $\|\cdot\|_{p,r,E}$.

There are several equivalent definitions of the Sobolev spaces over a Wiener space. The next section is devoted to one of them.

A. The Ornstein–Uhlenbeck semigroup characterization.

Recall that the Ornstein–Uhlenbeck semigroup T_t is defined by the formula

$$T_t f(x) = \int_X f(e^{-t}x + (1-e^{-2t})^{1/2}y)\gamma(dy).$$

Let

$$V_r f = \Gamma(r/2)^{-1} \int_0^\infty t^{r/2-1} e^{-t} T_t f\, dt.$$

One can check that for any $p > 1$ V_r is a bounded linear operator on $L^p(\gamma)$. Hence the space

$$H^{p,r} := V_r(L^p(\gamma)), \qquad \|f\|^*_{p,r} = \|V_r^{-1} f\|_{L^p},$$

is complete. Let

$$H^{+\infty} = \bigcap_{p,r>1} H^{p,r}.$$

The proof of the following result can be found in [336–338].

2.2 Theorem. *For any $r \in N$ and any $p > 1$ the norms $\|\cdot\|_{p,r}$ and $\|\cdot\|^*_{p,r}$ are equivalent.*

The same norms can be introduced by means of the Ornstein–Uhlenbeck operator L. Recall (see Section 1.6) that for any $f \in \mathcal{P}l\updownarrow(X)$ with the Wiener–Ito–Wick decomposition

$$f = \sum_n f_n$$

one has

$$Lf = \sum_n (-n) f_n.$$

For any real r let

$$(I-L)^r f = \sum_n (1+n)^r f_n \quad \forall f \in \mathcal{POL}(X).$$

Then

$$\|f\|^*_{p,r} = \|(I-L)^{r/2} f\|_{L^p}.$$

3.1 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X, $H = H(\gamma)$, and let C be a compact linear operator on H. Then for any $c > 0$ the set*

$$F = \left\{ f \in W^{1,2}(X,\gamma)\colon\ D_H f(x) \in \mathrm{Range}(C) \ \ \gamma\text{-a.e. and } \|f\|_{L^2(\gamma)} + \|C^{-1} D_H f\|_{L^2(\gamma,H)} \le c \right\}$$

is relatively compact in $L^2(\gamma)$.

3.2 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X. A set $F \subset W^{2,\infty}(X,\gamma)$ is relatively compact in $W^{2,\infty}(X,\gamma)$ endowed with its natural topology of a Fréchet space if and only if it is bounded in $L^2(\gamma)$ and for any $n \ge 1$ there exists a self-adjoint compact operator C_n on $\mathcal{H}_n$ such that*

$$\forall f \in F \ D^n_H f(x) \in \mathrm{Range}(C_n) \ \ \gamma\text{-a.e. and } \sup_{f\in F} \|C_n^{-1} D^n_H f\|_{L^2(\gamma,\mathcal{H}_n)} < \infty.$$

3.3 Theorem. *Let γ be a centered Gaussian Radon measure on a locally convex space X. A set $F \subset W^{\infty}(X,\gamma)$ is relatively compact in $W^{\infty}(X,\gamma)$ endowed with its natural topology of a Fréchet space if and only if the following two conditions are satisfied:*

(i) $\sup_{f\in F} \|f\|_{p,r} < \infty \quad \forall p, r$;

(ii) *for any $n \ge 1$ there exists a self-adjoint compact operator C_n on $\mathcal{H}_n$ such that*

$$\forall f \in F \ D^n f \in C_n(\mathcal{H}_n) \ \ \gamma\text{-a.e. and } \sup_{n} \|C^{-1} D^n f\|_{L^2(\gamma,\mathcal{H}_n)} < \infty.$$

3.4 Corollary. *Let X be a Hilbert space. The set of continuously Fréchet differentiable functions f on X such that*

$$\|f\|_{L^2} + \|D_X f\|_{L^2(X,X)} \le C$$

is relatively compact in $L^2(\gamma)$.

4. Gaussian Capacities.

In this section we define Gaussian capacities on locally convex spaces and discuss their basic properties such as invariance under embeddings and the existence of Souslin supports.

Using capacities one can get some classical results (limit theorems, zero–one laws, etc.) in a refined form, since capacities provide a finer characterization of the smallness of sets.

4.1 Definition. Let X be a locally convex space with a centered Gaussian Radon measure γ, $r > 0, p \ge 1$. Let U be an open subset of X. We define the capacity $C_{p,r}$ on U by

$$C_{p,r}(U) = \inf\{\|f\|^p_{p,r} \mid f \in H^{p,r},\ f \ge 1 \ \text{ a.e. on } \ U\}.$$

For an arbitrary set $A \subset X$ we define $C_{p,r}(A)$ by

$$C_{p,r}(A) = \inf\{C_{p,r}(U) \mid A \subset U,\ U \text{ is open}\}.$$

4.2 Definition. A set D is called *slim* if $C_{p,r}(D) = 0$ for all p, r.

A property is said to hold quasi-surely if it holds outside of a slim set.

Certainly, any slim set has measure zero, but the converse is not true.

4.3 Theorem. *For any open set U there is a unique element $\pi_U \in H^{p,r}$ such that $\pi_U \ge 1$ γ-a.e. on U and*

$$\|\pi_U\|_{p,r} = C_{p,r}(U).$$

In addition, π_U admits a representation $\pi_U = V_r f$ for some nonnegative function $f \in L^p(\gamma)$.

Proof. Since $H^{p,r}$ is isometric to $L^p(\gamma)$, it has the Banach–Saks property: any bounded sequence $\{\phi_n\} \subset H^{p,r}$ contains a subsequence $\{\psi_n\}$ such that the sequence

$$S_n = \frac{\psi_1 + \cdots + \psi_n}{n}$$

converges in $H^{p,r}$ (see [127]). We apply this property to a sequence $\{\phi_n\}$ chosen so that $\phi_n \geq 1$ γ-a.e. on U and

$$\lim_{n\to\infty} \|\phi_n\|_{p,r}^p = C_{p,r}(U).$$

Let π_U be the limit of $\{S_n\}$ in $H^{p,r}$. Clearly, $S_n \geq 1$ γ-a.e. on U and $\|S_n\|_{p,r}^p \to C_{p,r}(U)$. Hence $\pi_U \geq 1$ γ-a.e. on U and $\|\pi_U\|_{p,r}^p = C_{p,r}(U)$. To prove the uniqueness we shall use another property of L^p-spaces: the uniform convexity (see [127]). This means that for any positive ε and M there is a positive number δ such that the conditions $\|u\| \leq M$, $\|v\| \leq M$, $\|u-v\| > \varepsilon$ imply $\|u+v\| \leq 2M - \delta$. If q is another function with the properties indicated in the theorem and $\varepsilon > 0$ is chosen so that

$$\varepsilon < \|\pi_U - q\|_{p,r},$$

then $\phi = (\pi_U + q)/2 \geq 1$ γ-a.e. on U and $\|\phi\|_{p,r}^p < C_{p,r}(U)$, which is a contradiction.

Finally, note that if $\pi_U = V_r f$, $f \in L^p(\gamma)$, then $V_r f \leq V_r f^+$ and

$$\|f\|_{p,r} = \|f^+\|_{p,r}.$$

Therefore, $f = f^+$ by the uniqueness.

4.4 Proposition. *The capacities $C_{p,r}$ have the following properties:*

(i) $\gamma(A) \leq C_{p,r}(A)$;
(ii) $C_{p,r}(A) \leq C_{p,d}(A)$ *for* $d \geq r$;
(iii) $C_{p,r}(A) \subset C_{p,r}(B)$ *if* $A \subset B$;
(iv) $C_{p,r}(\cup_n A_n) \leq \sum_n C_{p,r}(A_n)$,

Proof. (i)-(iii) are trivial. To prove the last assertion note that it is sufficient to assume that the sets A_n are open and that the series of their capacities converges. Let

$$\pi_{A_n} = V_r f_n,\ f_n \in L^p(\gamma),\ f_n \geq 0,\ f = \sup_n f_n.$$

Since $f^p \leq \sum_n f_n^p$ and

$$\sum_{n=1}^{\infty} \|f_n\|_p^p = \sum_{n=1}^{\infty} C_{p,r}(A_n) < \infty,$$

we have that $f \in L^p(\gamma)$ and

$$\|V_r f\|_{r,p}^p \leq \sum_{n=1}^{\infty} C_{p,r}(A_n).$$

On the other hand,

$$C_{p,r}\left(\bigcup_n A_n\right) \leq \|V_r f\|_{r,p}^p$$

since $V_r f \geq 1$ γ-a.e. on $\cup_n A_n$ by virtue of the estimate $V_r f \geq V_r f_n = \pi_{A_n}$.

A function f on (X, γ) is said to be quasi-continuous if for any $\varepsilon > 0$ there is an open set U with $C_{p,r}(U) < \varepsilon$ such that f is continuous on $X \backslash U$.

4.5 Theorem.

(i) *If f is quasi-continuous and $f \geq 0$ a.e. on an open set U, then $f \geq 0$ (p,r)-quasi-everywhere on U (that is, on the complement to a set of $C_{p,r}$-capacity zero).*
(ii) *For any $f \in H^{p,r}$ there is a quasi-continuous modification $\widetilde{f}$ such that*

$$C_{p,r}(|\widetilde{f}| > R) \leq R^{-p} \|f\|_{p,r}^p \quad \forall R > 0.$$

(iii) *If a sequence $\{f_n\}$ converges to f in $H^{p,r}$, then a subsequence of $\{f_n\}$ converges quasi-everywhere to a quasi-continuous modification of f.*

(iv) *For any set B there is a unique element*

$$u_B \in F_B = \{u \in H^{p,r}:\ \widetilde{u} \le 1 \text{ quasi-everywhere on } B\}$$

with minimal norm. In addition, u_B is nonnegative and

$$C_{p,r}(B) = \|u_B\|^p_{p,r}.$$

Recall that the map $J = T_{t_0},\ t_0 = \log 2/2$, was defined in Section 1.6 by

$$Jf(x) = \int\limits_X f\left(\frac{x+y}{\sqrt{2}}\right)\gamma(dy)\ \forall f \in L^p(\gamma).$$

The proof of the following result is similar to that of [189], where the case of a metric space was considered.

4.6 Theorem. *For any increasing sequence of sets B_n one has*

$$\lim_{n\to\infty} C_{p,r}(B_n) = C_{p,r}\left(\bigcup_n B_n\right).$$

The next result, obtained in [51] (see also [52]), gives a full answer to the problem posed by K. Ito and P. Malliavin (see [191]). In less general situations this assertion was proved in [291, 447, 7, 8, 175].

4.7 Theorem. *For any $\varepsilon > 0$ there exists a metrizable compact K_ε such that*

$$C_{p,r}(X\backslash K_\varepsilon) < \varepsilon. \tag{4.1}$$

In addition,

$$C_{p,r}(B) = \sup\{C_{p,r}(K);\ K \subset B \text{ is a metrizable compact}\} \tag{4.2}$$

for any Borel set B (the same holds for any Souslin set B).

4.8 Corollary. *Let E be a γ-measurable linear subspace in X with $\gamma(E) > 0$. Then $C_{p,r}(X\backslash E) = 0$.*

4.9 Corollary. *Let $T:\ X \to Y$ be an injective continuous linear map, where Y is a locally convex space. Denote by $C^T_{p,r}$ the capacities associated with the measure $\nu = \gamma\circ T^{-1}$. Then for any set Borel $B \subset Y$ one has*

$$C^T_{p,r}(B) = C_{p,r}(T^{-1}(B)).$$

4.10 Corollary. *Let Y be a locally convex space, $T:\ X \to Y$ a γ-measurable linear map. If the measure $\nu = \gamma\circ T^{-1}$ is Radon on Y, then T is quasi-surely continuous and the assertion of Corollary 4.9 is true.*

4.11 Corollary. *A compact set K in Theorem 4.6 can be chosen symmetric and convex provided X is sequentially complete.*

4.12 Corollary. *There exists a constant $k(p,r) > 0$ such that*

$$C_{p,r}(Jf > R) \le k(p,r)R^{-p}\|f\|^p_{L^p} \quad \forall f \in L^p(\gamma). \tag{4.3}$$

Proof of Theorem 4.7. Step 1. We shall consider first the case where X is quasi-complete. According to Theorem 1.5.1 there is a metrizable compact set S such that $\gamma(S) > 1/2$. Let S^* be the convex symmetric hull of S, K the closure of S^*. By Lemma 0.B.1, K is a metrizable compact. Denote by g the gauge functional of K, defined on the linear span E of K. Put $g = \infty$ on $X\backslash E$. Since $\gamma(E) > 0$ we get $\gamma(E) = 1$ by the zero–one law. By virtue of the Fernique theorem (see Chapter 2), $g \in \cap L^p$.

Take a function $\psi \in C_b^\infty(R^1)$ such that $0 \le \psi \le 1$, $\psi = 0$ on $[-\infty, 0]$, $\psi = 1$ on $[1, +\infty]$. Let

$$f_n(x) = \psi(Jg(x) - n).$$

Denote by K_R the closure of the set $\{Jg \leq R\}$. We have

$$\left|\sqrt{2}Jg(x) - g(x)\right| \leq \int\limits_X g(y)\gamma(dy) = d,$$

from which

$$\{Jg \leq R\} \subset \{g \leq \sqrt{2}R + d\} \subset mK \text{ if } m > 2R + d.$$

Hence K_R is a metrizable compact.

Following [447], we shall prove that there is a constant $c(p,r)$ such that

$$\|f_n\|_{p,r} \leq c(p,r)\gamma(Jg > n) \quad \forall n.$$

Note that

$$Df_n = \psi'(Jg - n)D_H(Jg),$$

$$D_H^2 f_n = \psi''(Jg - n)D_H(Jg) \otimes D_H(Jg) + \psi'(Jg - n)D_H^2(Jg),$$

$$D_H^r f_n = \sum_{k \leq r} c_k \psi^{(k)}(Jg - n)G_k,$$

where G_k are some mappings which do not depend on n and possess finite L^p-norms. Now it suffices to note that $\psi^{(k)}(Jg(x) - n) = 0$ if $Jg(x) \leq n$ and $|\psi^{(k)}(Jg - n)|, k = 1, \ldots r$, are uniformly bounded. The estimate above gives the following relationship:

$$\lim_{n \to \infty} C_{p,r}(X \backslash K_n) = 0.$$

Indeed, if $n > 2d$, one has

$$C_{p,r}(X \backslash K_{2n}) \leq C_{p,r}(Jg > 2n) \leq C_{p,r}(g > \sqrt{2}\,2n - d) \leq C_{p,r}(g > 2n) \leq \|f_n\|_{p,r}^p,$$

since the set $\{g > 2n\} = X \backslash (2nK)$ is open and $f_n = 1$ on this set by virtue of the estimate $Jg \geq \sqrt{2}(g + d)$. It remains to note that

$$\gamma(Jg > n) \leq \gamma(g > \sqrt{2}n - d) \to 0.$$

Step 2. Now we are going to establish (4.3). To this end denote by $\mathcal{E}$ the class of all Borel functions $f \in L^p(\gamma)$, satisfying (4.3). This class contains $\mathcal{FC}^\infty$. Indeed, if $f \in \mathcal{E}$, then the function Jf is continuous, belongs to $W^\infty(X,\gamma)$ and $Jf/R > 1$ on the open set $\{Jf > R\}$. $\mathcal{E}$ admits monotone limits. Indeed, if $\{f_n\} \subset \mathcal{E}$ is increasing to a Borel function $f \in L^p(\gamma)$, then by the Lebesgue theorem

$$Jf(x) = \lim_{n \to \infty} Jf_n(x)$$

for each x. Therefore, the family of sets $\{Jf_n > R\}$ is increasing to the set $\{Jf > R\}$. By virtue of Theorem 4.6

$$\lim_{n \to \infty} C_{p,r}(Jf_n > R) = C_{p,r}(Jf > R).$$

Since $\|f_n\|_p \to \|f\|_p$ we get the inclusion $f \in \mathcal{E}$. If $\{f_n\} \subset \mathcal{E}$ is decreasing to a Borel function $f \in L^p(\gamma)$, then the sequence $\{Jf_n\}$ is decreasing to Jf and $\|f_n\|_p \to \|f\|_p$. Therefore,

$$C_{p,r}(Jf > R) \leq C_{p,r}(Jf_n > R) \leq k(p,r)R^{-p}\|f_n\|_p^p \to k(p,r)R^{-p}\|f\|_p^p.$$

Finally, let $\{f_n\} \subset \mathcal{E}$ be a sequence which converges to f uniformly. Then for any positive ε and sufficiently large n we have

$$|Jf(x) - Jf_n(x)| < \varepsilon.$$

Therefore,

$$C_{p,r}(Jf>R)\le C_{p,r}(Jf_n>R-\varepsilon)\le k(p,r)(R-\varepsilon)^{-p}\|f_n\|_p^p\to k(p,r)(R-\varepsilon)^{-p}\|f\|_p^p.$$

Hence (4.3) holds for f. According to the monotone classes theorem (see [335], Remark after Theorem 20, Chap. 1), $\mathcal{E}$ contains all Borel functions $f\in L^p(\gamma)$ which are measurable with respect to the σ-field $\sigma(X)$.

Now let f be an arbitrary bounded Borel function, $C=\sup|f(x)|$. For any $\varepsilon>0$ we can choose a metrizable symmetric convex compact set K with

$$C_{p,r}(X\backslash K)<\varepsilon,\quad \gamma(X\backslash K)<\varepsilon.$$

It follows from Remark 0.C.5 that there exists a $\sigma(X)$-measurable function g, which agrees with f on $2K$ and satisfies the estimate $\sup|g(x)|\le C$. Note that for any $x\in K$ one has

$$|Jf(x)-Jg(x)|\le\int_X\left|f(\sqrt{2}(x+y))-g(\sqrt{2}(x+y))\right|\gamma(dy)\le 2C\varepsilon,$$

since $f(\sqrt{2}(x+y))=g(\sqrt{2}(x+y))$ for any $y\in K$. Thus, we get

$$\begin{aligned}C_{p,r}(Jf>R)&\le C_{p,r}(Jg>R-2C\varepsilon)+\varepsilon\\&\le k(p,r)(R-2C\varepsilon)^{-p}\|g\|_p+\varepsilon\\&\le k(p,r)(R-2C\varepsilon)^{-p}(\|f\|_p+C\varepsilon)+\varepsilon.\end{aligned}$$

Since ε was arbitrary we get (4.3).

Now let $f\in L^p(\gamma)$ be an arbitrary Borel function. Since $Jf\le J|f|$ we may assume that f is nonnegative. Put $f_n=\min(f,n)$. The inclusion $f_n\in\mathcal{E}$ is proved already, so $f\in\mathcal{E}$.

Finally, note that for any $f\in L^p(\gamma)$ one can find a Borel function g such that $f(x)\le g(x)$ for any x and $g=f$ γ-a.e. Then $Jf(x)\le Jg(x)$ for any x. Therefore,

$$C_{p,r}(Jf>R)\le C_{p,r}(Jg>R)\le k(p,r)R^{-p}\|g\|_p^p=k(p,r)R^{-p}\|f\|_p^p.$$

Step 3. Now let X be an arbitrary locally convex space. Denote by Z its completion (see [143]). Consider our measure γ on Z. Let D be a compact subset of X of positive γ-measure and let E be the linear span of D. Then E is σ-compact, since

$$D=\bigcup_{k,n}D_{k,n}=\{t_1d_1+\cdots+t_nd_n|d_i\in D,|t_i|\le k\},$$

and $D_{k,n}$ are compact. The indicator function I_M of the set $M=Z\backslash E$ is Borel in Z and $\gamma(E)=1$ by the zero–one law. Clearly, $JI_M=I_M$. According to Step 2 we have

$$C_{p,r}(M)=C_{p,r}(I_M=1)=C_{p,r}(JI_M=1)\le 2k(p,r)\|I_M\|_p^p=0.$$

Hence for any $\varepsilon>0$ there exist a metrizable compact Q and an open set U in Z such that

$$M\subset U,\ C_{p,r}(U)<\varepsilon,\ C_{p,r}(Z\backslash Q)<\varepsilon.$$

Finally, take $K=Q\cap(Z\backslash U)$ and note that:

(i) $K\subset E\subset X$;

(ii) K is compact and metrizable, since it is a closed subset of Q;

(iii) $C_{p,r}(X\backslash K)\le C_{p,r}(Z\backslash K)<\varepsilon$. The theorem is proved.

4.13 Theorem. *Let X be a locally convex space with a centered Gaussian measure γ. Then there exists a γ-measurable linear map $j: X \to R^N$, where N is finite or $N = \infty$, such that j is injective on a linear subspace of full γ-measure, $\gamma \circ j^{-1}$ coincides with the product ν of N copies of the standard one-dimensional Gaussian measures, and for any Borel set B one has*

$$C^{\nu}_{p,r}(B) = C^{\gamma}_{p,r}(j^{-1}(B)).$$

In particular, all infinite-dimensional Gaussian spaces (with Radon measures) are linearly isomorphic in this sense.

Since the property of having capacity zero is stronger than that of having measure zero, the problem arises to characterize measures vanishing on slim sets. This problem is solved in [264] and [421] (see also [447] for a partial result). Measures ν vanishing on all sets of $C_{p,r}$-capacity zero can be described as those for which the functionals $\phi \mapsto \int \phi\nu$ are continuous on $H^{p,r}$.

It should be noted that it is possible to define more general Gaussian capacities associated with a semigroup of bounded operators $\{T_t\}_{t\geq 0}$ (see [52, 54, 175]).

Let us make several remarks concerning quasi-sure analysis (see [325]). Many classical results which deal with "almost sure properties" can be obtained in a sharper form as "quasi-sure properties." There are two main directions in which such refinements have been obtained:

A. Existence of maps with certain properties which hold quasi-surely (for example, modifications of functions in Sobolev classes);

B. Quasi-sure results connected with limit theorems. Below we describe some typical examples.

First of all, we recall that any assertion saying that something takes place on a linear subspace of full measure implies the corresponding quasi-sure analogue.

4.14 Example. *Let γ be a Gaussian measure on a locally convex space $X, \{e_n\}$ an orthogonal basis in $H = H(\gamma)$. Let*

$$S_n(x) = \sum_{i=1}^{n} \xi_i(x)e_i,$$

where ξ_i denotes the measurable linear functional associated with e_i. Then

$$\lim_{n\to\infty} S_n(x) = x \text{ quasi-surely.}$$

The situation becomes more difficult when the set on which the corresponding limit theorem holds is not linear. The simplest example is the law of iterated logarithms (see [263]). Further examples can be found in [325].

5. Additional Remarks.

A. Negligible sets.

We shall recall some concepts of a "zero-set" in infinite dimensions. Since here there is no reasonable substitute for the Lebesgue measure (as well as no preference in the choice of, say, some distinguished nondegenerate Gaussian measure among a continuum mutually singular ones), one has to define this concept in some intrinsic way not involving any fixed measure. One of the definitions is due to Christensen ([104, 105]), who called a Borel set A in a Banach space X universally zero if there exists nonzero Borel measure μ such that $\mu(A + x) = 0$ for all x. Another definition was introduced by Aronszajn [14], who introduced the following class $\mathcal{A}^{\mathcal{B}}$ of exceptional Borel sets.

For any vector e in a locally convex space X, denote by $\mathcal{A}^{\mathcal{B}}_e$ the class of all Borel sets A such that $\mathrm{mes}(t : x + te \in A) = 0$ for every $x \in X$, where "mes" is the Lebesgue measure. For any sequence $\{e_n\}$ in X, let $\mathcal{A}^{\mathcal{B}}\{e_n\}$ be the class of sets of the form $A = \cup_n A_n$, where $A_n \in \mathcal{A}^{\mathcal{B}}_{e_n}$ for all n.

5.1 Definition. A set A is said to be *exceptional* ($A \in \mathcal{A}^{\mathcal{B}}$) if it belongs to the class $\mathcal{A}^{\mathcal{B}}\{e_n\}$ for each sequence $\{e_n\}$ whose linear span is dense in X.

The corresponding class is smaller than that of Christensen, but both coincide with Lebesgue zero Borel sets in finite-dimensional spaces. Then Phelps [361] introduced the class $\mathcal{G}^{\mathcal{B}}$ of Gaussian null sets:

5.2 Definition. A Borel set A is called a *Gaussian null set* if it is a zero set for each nondegenerate (that is of the full support) Gaussian measure on X.

According to [361], $\mathcal{A}^{\mathcal{B}} \subset \mathcal{G}^{\mathcal{B}}$, but it remains open whether this inclusion is strict. Finally, the author [37] suggested the following definition.

5.3 Definition. A Borel set A is called *negligible* ($A \in \mathcal{P}^{\mathcal{B}}$) if it is a zero set for each measure which is differentiable along directions from a dense set.

The classes above can be extended in order to cover in finite-dimensional spaces all Lebesgue zero sets (not necessarily Borelian). To this end, let $\mathcal{L}$ be the class of all sets in X that are measurable with respect to every measure on X which is differentiable along directions from a dense subspace. Then the definitions above extend naturally to $\mathcal{L}$ (in particular, in the definitions of $\mathcal{A}_e$ and $\mathcal{A}\{e_n\}$ now sets from $\mathcal{L}$ are admissible). Let us denote the new classes we get by $\mathcal{A}$, $\mathcal{G}$, $\mathcal{P}$, respectively. Then we have [38, 39, 40]:

5.4 Theorem. *$\mathcal{A}^{\mathcal{B}} \subset \mathcal{G}^{\mathcal{B}} = \mathcal{P}^{\mathcal{B}}$ and $\mathcal{A} = \mathcal{G} = \mathcal{P}$.*

Note that, in particular, $\mathcal{G}^{\mathcal{B}} \subset \mathcal{A}$; however, we do not know whether the sets A_n in the corresponding decomposition can be chosen in $\mathcal{B}(X)$ (and not just in $\mathcal{L}$). One gets similar classes $\mathcal{A}^G$, $\mathcal{G}^G$, considering, instead of $\mathcal{L}$, the class $\mathcal{L}^G$ of all sets that are measurable with respect to each nondegenerate Gaussian measure on X. Then, by the same argument as in [39, 40], $\mathcal{A}^G = \mathcal{G}^G$. We have no examples distinguishing the classes $\mathcal{L}$ and $\mathcal{L}^G$ (or the classes $\mathcal{G}^G$ and $\mathcal{P}$).

The classes above are invariant under affine isomorphisms of X and enjoy many useful properties of the finite-dimensional Lebesgue zero sets (see, e.g., the next section). However, they are not stable under nonlinear diffeomorphisms. For example, if f is a real-analytic diffeomorphism of the line that is not affine, then the map $F : x \mapsto f \circ x$ on $C[0,1]$ is a diffeomorphism under which the image of some negligible set is not negligible (see [461]). It is still an open problem (posed in [502]) whether a Lipschitzian image of any Borel negligible set is zero in the sense of Christensen.

Basic facts about Rademacher's theorem in infinite dimensions.

The classical Rademacher theorem states that a Lipschitz map $F: R^n \to R^k$ is almost everywhere Fréchet differentiable. This result has no direct infinite-dimensional extension. The main reason is not the absence of infinite-dimensional analogs of the Lebesgue measure, but simply the existence of Lipschitz maps between Hilbert spaces without points of Fréchet differentiability. However, there are many papers devoted to various generalizations of Rademacher's theorem to the infinite-dimensional case. The point is that this theorem can be reformulated (in finite-dimensions) in many equivalent ways, admitting infinite-dimensional extensions. Here, following [57], we discuss one such possibility (see Theorem 5.8 below and [291, 153, 57]).

Although there are easy examples of Lipschitzian everywhere Gâteaux differentiable maps from a Hilbert space H to a (infinite-dimensional) Hilbert space E without points of Fréchet differentiability, it was open for a long time what happens if E is finite-dimensional. Only recently has Preiss [378] proved that each Lipschitzian real function on an Asplund space (in particular, on a Hilbert space) is Fréchet differentiable on a dense set. However, it is still open whether the same is true if $\dim(E) > 1$. But even for real functions Preiss's positive result does not give an extension of Rademacher's theorem since for each Borel measure μ on l^2 there exists a Lipschitzian convex function f which is not Fréchet differentiable μ-a.e. (see [378]).

The situation with Gâteaux differentiability is more favorable. For example, every locally Lipschitzian function on a separable normed space X is Gâteaux differentiable outside of some "exceptional" set for many possible choices of the notion of "exceptional" (for example, for the classes mentioned above, see, e.g., [14, 105, 361, 39]).

5.5 Theorem. *Let X be a separable normed space and let $F : X \to Y$ be a locally Lipschitzian map with values in a Banach space Y which has the Radon–Nikodym property. Then F is Gâteaux differentiable outside of some Borel negligible* (*and exceptional*) *set.*

In particular, this implies Hadamard differentiability and hence Fréchet differentiability along any compactly embedded normed space E. However, when admitting differentiability along a smaller subspace, it would be quite natural to also impose the Lipschitz condition only along this subspace. Such a point of view is highly consistent with basic constructions of the Malliavin calculus and the theory of differentiable measures. A typical result in this direction is due to Kusuoka [291]. If γ is a Gaussian measure on a separable Banach space X, H is the reproducing kernel space of γ (Hilbert space of differentiability), and

F is a measurable map of X to a reflexive Banach space Y, such that

$$\|F(x+h) - F(x)\|_Y \le C|h|_H \quad \forall x \in X,\ h \in H,$$

then F is γ-a.e. Gâteaux differentiable along H. For the case $Y = R^1$ it was noticed in [153] that it suffices to claim the condition above to hold for γ-a.a. x. All these investigations lead to the following natural question. Let F be a map from a space X to a normed space $Y \in RN$ measurable with respect to some measure μ on X and such that for μ-almost all x

$$\|F(x+h) - F(x)\|_Y \le C|h|_E \quad \forall h \in E,$$

where E is some normed space continuously embedded in X. Is F differentiable along E μ-a.a.? We prove below that under some natural conditions on μ the answer is positive for Gâteaux differentiability and negative for Fréchet differentiability.

5.6 Lemma. *Let $F:\ E \to Y$ be a map between normed spaces such that*

$$Z = \Big\{x:\ \|F(x+h) - F(x)\| \le C\|h\|\ \forall h \in E\Big\}$$

is dense. Then $Z = E$.

Proof. For each fixed $\delta > 0$ there exists a point $z \in Z$ with $\|(x+y)/2 - z\| < \delta$. Then

$$\|F(x) - F(y)\| \le \|F(x) - F(z)\| + \|F(z) - F(y)\| \le C\big(\|x-z\| + \|y-z\|\big) \le C(\|x-y\| + 2\delta).$$

Since δ was arbitrary the lemma follows.

5.7 Corollary. *Let $E = R^n$ and let Z be of full Lebesgue measure. Then $Z = R^n$.*

Denote by $Q(\mu)$ the collection of all vectors of quasi-invariance of a measure μ (that is, the vectors h such that μ_{th} is equivalent to μ for all t, where $\mu_h(A) = \mu(A-h)$).

5.8 Theorem. *Let μ be a Radon measure on a locally convex space X, $\{a_n\} \subset Q(\mu)$, E a normed space linearly embedded in X such that the linear span of $\{a_n\}$ is dense in E. If a measurable map F from X to a Banach space $Y \in RN$ satisfies the condition*

$$\|F(x+h) - F(x)\| \le C\|h\|_E, \quad \forall h \in E,\ \textit{for } \mu\textit{-a.a. } x, \tag{5.1}$$

then:

(1) *there exists a modification of F satisfying* (5.1) *for all x;*
(2) *μ-a.e. there exists Gâteaux derivative $D_E F$. In particular, the statement holds if μ is a Gaussian measure and $E = H(\mu)$.*

Proof. Let $E_n = \mathrm{span}\{a_1, \dots, a_n\}$ and let X_n be a closed subspace in X complementing E_n. It is known (see, e.g., [50]) that there exists a family of measures $\{\mu^y, y \in X_n\}$, on $E_n + y$, quasiinvariant along E_n, such that for all measurable B

$$\mu(B) = \int\limits_{X_n} \mu^y(B)\nu(dy),$$

where ν is the image of $|\mu|$ under the natural projection of X on X_n. We equip E_n with the Lebesgue measure λ induced from the algebraic isomorphism with a finite-dimensional space. Then all measures μ^y are equivalent to the corresponding translates of λ, and hence for a set A of full μ-measure we have that the sections $A \cap (E_n + x)$ are of full Lebesgue measure for μ-a.a. x. In particular, such sections are dense in $E_n + x$. This implies that for μ-a.a. x the sets $\{h \in E : x + h \in A\}$ are dense in E. Now, applying Lemma 5.6, we get the first assertion.

To prove the second one, denote by M_n the set of points where F is Gâteaux differentiable along E_n. Note that M_n are measurable. Indeed, the maps $x \mapsto F(x+h)$ are measurable; hence also $x \mapsto \partial_h F(x)$, since these maps exist a.e. Let $\{h_i\}$ be a countable dense subset of E_n. One can check that the set of points where all $\partial_{h_i} F(x)$ exist and are linear in h over the field of rational numbers is measurable and coincides with M_n. By our condition on Y and the classical Rademacher theorem, $M_n \cap (E_n + y)$ is of full Lebesgue measure for all y and thus M_n is of full μ-measure. Hence the same holds for $M = \cap M_n$. We claim that F is Gâteaux differentiable at each point x in M. Indeed, the linear map

$$D_E F(x) : h \mapsto \partial_h F(x)$$

is already defined on the span of $\{a_n\}$ and $\|D_E F(x)\| \leq C$. Hence this map admits a unique linear continuous extension on E. Let $h \in E$. Choose $h_n \to h$ in E, $h_n \in \mathrm{span}(a_i)$. By virtue of the estimate

$$\left\| \frac{F(x+th_n) - F(x)}{t} - \frac{F(x+th) - F(x))}{t} \right\| \leq C\|h - h_n\|,$$

the vectors

$$\frac{F(x+th_n) - F(x)}{t} - \partial_{h_n} F(x)$$

converge uniformly in t to the limit

$$\frac{F(x+th) - F(x)}{t} - \partial_h F(x).$$

Hence

$$\lim_{t \to 0} \frac{F(x+th) - F(x)}{t} - \partial_h F(x) = 0.$$

5.9 Corollary. *Under the conditions of Theorem 5.8 for every normed space B compactly embedded in E there exists the Fréchet derivative $D_B F$ μ-a.e.*

Proof. Note that F is Hadamard differentiable at all points of Gâteaux differentiability. This implies Fréchet differentiability along compactly embedded spaces.

The last corollary does not hold in general for the space E itself.

5.10 Example. *Let $X = R^\infty$, $\mu = \prod_{n=1}^{\infty} \mu_n$, where μ_n are identical standard Gaussian measures on the line, $E = l^2$, $F: X \to l^2$, $F(x) = (f_n(x_n))$, where f_n are 2^{1-n}-periodic functions on the line with*

$$f_n(t) = t \ \text{ if } \ t \in [0, 2^{-n}], \qquad f_n(t) = 2^{1-n} - t \ \text{ if } \ t \in [2^{-n},\ 2^{1-n}].$$

Then $E = H(\mu) = Q(\mu)$ and F is Lipschitzian along E, but at no point is Fréchet differentiable along E.

Proof. Assume that F is Fréchet E-differentiable at a point x. Then there exists a $\delta > 0$ such that

$$\left\| \frac{F(x+te_n) - F(x)}{t} - \partial_{e_n} F(x) \right\| < 1/2 \quad \forall n \in N, \quad \forall t \in (0, \delta].$$

Choose n with $2^{-n} < \delta/2$ and $x \in X$. Since $|f_n'(t)| = 1$ for $t \neq k2^{-n}$ we can assume that $f_n'(x_n) = 1$. Then we can find a $t \in (0, \delta]$ such that

$$\frac{f_n(x_n + t) - f_n(x_n)}{t} = 0$$

whence

$$\left\| \frac{F(x+te_n) - F(x)}{t} - D_E F(x) \right\| \geq 1,$$

which is a contradiction.

It is easy to modify this example to make F Gâteaux differentiable along E everywhere. Certainly, one can choose some Hilbert space X instead of R^∞.

In [57] one can find an example of a probability measure μ on R^∞ that is quasi-invariant along $E = l^2$, such that the function $f(x) = \sup_n |x_n|$ is μ-a.e. finite and Lipschitzian along E, but μ-a.e. fails to be Fréchet differentiable along E. It was conjectured in [153] that a similar example exists for a Gaussian measure. Though this conjecture seems to be very likely, we have no such examples.

5.11 Proposition. *Let γ be a centered Gaussian measure on a locally convex space X with the reproducing kernel H, F a γ-measurable map to a separable normed space Y satisfying* (5.1). *Then there exists a sequence of smooth cylindrical functions f_n satisfying* (5.1) *and such that*

$$\|D_H f_n\| \le C,$$

and, as $n \to \infty$,

$$\|f - f_n\|_2 + \|D_H f - D_H f_n\|_2 \to 0. \tag{5.2}$$

Conversely, (5.2) *implies* (5.1).

Proof. According to Theorem 2.2.3, $\|F\|$ is exponentially integrable and hence is in L^p for all $p \in [1, \infty)$. Thus the direct implication follows from Theorem 1.5.7. To prove the converse, note that if a sequence of uniformly Lipschitzian maps converges on a dense set then it converges pointwise everywhere. Now we can apply the arguments in the proof of Theorem 5.8 based on conditional measures.

Let us mention several open problems posed in [57] in connection with the results above.

1. Does there exist a Lipschitz function f on a separable Hilbert space H such that the set of points of Fréchet differentiability of f is negligible? Can such a set be empty for a map with values in R^2?

2. Let μ be a centered Gaussian measure on a separable Hilbert space X and let f be a Borel function on X, Lipschitzian along $H = H(\mu)$. Can it happen that the set of points of Fréchet differentiability of f along H is μ-zero ?

3. Let B be a Borel set. Define a function F on a locally convex (say, on a separable Hilbert) space X equipped with a Gaussian measure as follows:

$$F(x) = \inf\{\|h\|_H : h \in B - x\}$$

if the set of such h is not empty, and 0 otherwise. Clearly, F is Lipschitzian along H. What can be said about points of Fréchet differentiability of F along H ?

To conclude this chapter note that additional information may be found in [32, 34, 102, 171, 172, 173, 175, 189, 190, 191, 218, 219, 220, 228, 239, 260, 276, 277, 278, 288, 291, 299, 302, 322, 325, 449, 488, 489, 490].

Chapter 4
Nonlinear Transformations of Gaussian Measures

1. Absolutely Continuous Linear Transformations.

As is already known, even the simplest linear maps such as $x \mapsto 2x$ transform any Gaussian measure with infinite-dimensional support into a nonequivalent one. In this section, we shall study this question in more detail.

1.1 Theorem. *Let γ be a centered Gaussian measure on a locally convex space $X, H = H(\gamma)$ and let $T : X \mapsto X$ be a γ-measurable linear map such that $\gamma \circ T^{-1} = \gamma$. Then letting T_0 be a properly linear modification of T, one has $T_0 : H \to H$ is an orthogonal operator. Conversely, for any orthogonal operator $U : H \to H$ there is a γ-measurable properly linear map T, preserving γ, such that $T|_H = U$.*

Proof. According to Theorem 1.5.5, $T_0(H) = H$. For any $h \in H$ we have

$$\|h\|_H = \|g_h\|_{L^2(\gamma)}, \qquad h = R_\gamma g_h.$$

Since $g_{T_0 h} = g_h \circ T_0$ and T preserves γ, we get the orthogonality of T_0 on H. Conversely, an orthogonal map $U \in \mathcal{L}(H)$ can be extended to a γ-measurable map T (see Theorem 1.5.5). The orthogonality of U means that the induced measure $\gamma \circ T^{-1}$ has the same covariance as γ. Hence $\gamma \circ T^{-1} = \gamma$.

We shall say that an operator A on a Hilbert space H has property (E) (associated with the word "equivalence") if

(i) A is invertible;

(ii) $A^*A - I \in \mathcal{L}_{(2)}(H)$.

One can easily check (using the polar decomposition $A = U\sqrt{A^*A}$) that this is equivalent for A to have the form

$$A = U(I + K),$$

where U is an orthogonal operator, $K \in \mathcal{L}_{(2)}(H)$ is a symmetric operator and $I + K$ is invertible. Note that the composition of two operators with property (E) has this property as well.

1.2 Theorem. *Let γ be a centered Gaussian measure on a locally convex space X, $H = H(\gamma)$, and let $T: X \to X$ be a γ-measurable linear map such that the measure $\gamma \circ T^{-1}$ is equivalent to γ. Denote by T_0 a properly linear modification of T. Then T_0 maps H into itself and the operator $A = T_0|_H$ has property (E). Conversely, for any operator $A \in \mathcal{L}(H)$ with property (E) there is a γ-measurable properly linear map T, such that $T|_H = A$ and the measure $\gamma \circ T^{-1}$ is equivalent to γ.*

Proof. According to Theorem 1.1, T_0 maps H onto $H(\gamma \circ T^{-1}) = H(\gamma) = H$ by the equivalence of these two measures. Note that T_0 on H is injective. Indeed, let $T_0 h = 0$. This means that $g_h \circ T = 0$ γ-a.e. Since $\gamma \circ T^{-1}$ is equivalent to γ one gets $g_h = 0$ γ-a.e. and hence $h = 0$. Assume first that $A = I + S$, where S is a diagonal operator in some basis $\{e_n\}$ with the eigenvalues $\alpha_n \in [0, 1)$. Let us prove that $\gamma \circ A^{-1}$ is equivalent to γ if and only if $\sum_n \alpha_n^2 < \infty$. The most economical way to do this is to use the following Kakutani's theorem [252].

Let $\{\mu_n\}$ and $\{\nu_n\}$ be two sequences of probability measures on the line such that $\mu_n \sim \nu_n$. Set $\mu = \otimes\mu_n$ and $\nu = \otimes\nu_n$. Then the measures μ and ν are either equivalent or mutually singular. The latter is equivalent to the equality

$$\prod_{n=1}^{\infty} \int \sqrt{q_n(t)}\mu_n(dt) = 0,$$

where q_n is the Radon–Nikodym density of ν_n with respect to μ_n.

In our case we get the expression

$$\prod_{n=1}^{\infty} \sqrt{1 - \frac{\alpha_n^2}{2 + 2\alpha_n + \alpha_n^2}},$$

which is positive if and only if $\sum_n \alpha_n^2 < \infty$.

Certainly, it is possible to do the same thing in a more direct way. For example, let p_n be the standard Gaussian density on R^n, g_n the density of the image of the standard Gaussian measure on R^n under the map $(x_i) \mapsto ((1 + \alpha_i)x_i)$, ξ_n the γ measurable linear functional corresponding to e_n, $P_n(x) = \sum_{i=1}^n \xi_n(x)e_i$. Then, in the same way as in the proof of Proposition 1.2.1, we get the estimate

$$\|p_n - g_n\|_{L^1(R^n)} \geq 1 - \int \sqrt{p_n(x)g_n(x)}dx = 1 - \prod_{i=1}^{n} \sqrt{1 - \frac{\alpha_i^2}{2 + 2\alpha_i + \alpha_i^2}}.$$

Together with the equality

$$\|\gamma - \gamma \circ T^{-1}\| = \sup_n \|\gamma \circ P_n^{-1} - (\gamma \circ T^{-1}) \circ P_n^{-1}\| = \sup_n \|p_n - g_n\|_{L^1(R^n)},$$

this gives one implication. To get the other it suffices to note that by direct calculations

$$\sup_n \int \log\left(\frac{g_n}{p_n}\right) \frac{g_n}{p_n} p_n dx < \infty,$$

provided $\sum_n \alpha_i^2 < \infty$. Hence the sequence $g_n(P_n)/p_n(P_n)$ is uniformly integrable. Since it forms a martingale (with respect to the σ-fields σ_n generated by P_n), it converges in L^1 to some function f. Then $\int f\gamma = 1$, so it cannot happen that $\|p_n - g_n\|_{L^1(R^n)} = \|1 - g_n/p_n\|_{L^1(p_n)}$ converges to 1.

Note that the same argument shows that the γ-measurable extension of any operator A of the form described maps γ to an equivalent measure.

Now we return to the general case. By virtue of Lemma 0.A.5, we can write A in the following form:

$$A = U(I+S)D,$$

where U is an orthogonal operator, S is a symmetric Hilbert–Schmidt operator, D is a diagonal operator, and all the multipliers are invertible. Note that

$$(I+S)^{-1} = (I+S_1),$$

where S_1 is again a symmetric Hilbert-Schmidt operator (if $\{s_i\}$ are the eigenvalues of S, then S_1 has the eigenvalues $\sigma_i = -s_i(1+s_i)^{-1}$). The operator $U_1 = U^{-1}$ is orthogonal. According to Theorem 1.1 and the remark above, the operator $(I+S)^{-1}U_1$ generates a γ-measurable map T_1 which preserves γ-equivalence. It is easy to see that then the map $D = (I+S)^{-1}U_1A$ corresponds to the map T_1T, preserving equivalence to the measure γ. Thus, we come to the diagonal case, which has already been studied. Finally, if A on H has the property (E), then we can write $A = U(I+K)$, where U is orthogonal and K is a Hilbert–Schmidt operator. As explained above, both multipliers have extensions transforming γ into equivalent measures.

2. Absolute Continuity of Gaussian Measures under Nonlinear Transformations.

Let γ be a centered Gaussian measure on a locally convex space X, $H = H(\gamma)$. We shall study the image of γ under a measurable map $T: X \to X$ which has the following special form:

$$T(x) = x - F(x),$$

where $F: X \to H$. The results presented below were proved by Kusuoka [290] in the case of a separable Banach space X. Earlier, under additional assumptions, similar results were found by Gross, Kuo, Ramer, and other researchers. It follows from the formulations below and the results above on measurable linear mappings that these assertions are invariant under linear measurable isomorphisms. Thus, one can reduce the general case to that of a Gaussian measure on a separable Hilbert space.

In this section we denote the derivative along H by D. Recall that a map $F: X \to H$ is said to be an $\mathcal{H}-C^1$ map if the following holds:

(1) F is Fréchet differentiable along H at any point, that is, for any $x \in X$ there is an operator $DF(x) \in \mathcal{L}(H)$ such that

$$\left\|F(x+h) - F(x) - DF(x)(h)\right\|_H = o(\|h\|_H) \quad \text{as} \quad \|h\|_H \to 0;$$

(2) for any x $DF(x)$ is a Hilbert–Schmidt operator on H and the map

$$h \mapsto DF(x+h),\ H \to \mathcal{L}_{(2)}(H),$$

is continuous.

2.1 Theorem. *Assume that F satisfies the following conditions*:

(1) *$F: X \to H$ is an $\mathcal{H}-C^1$ map;*
(2) *$I_H - DF(x): H \to H$ is invertible for γ-a.e. x. Then the measure $\gamma \circ T^{-1}$ is absolutely continuous with respect to γ. Moreover, if B is a measurable subset of X such that condition* (2) *holds only on B, then the measure $\gamma_B \circ T^{-1}$ is absolutely continuous with respect to γ.*

Let F be an $\mathcal{H}-C^1$-map with values in H. As we know (see Chapter 1), any projector $P \in \mathcal{P}(H)$ can be uniquely extended to a measurable linear operator $\widetilde{P}: X \to H$. Put

$$L_PF(x) = (F(x), \widetilde{P}(x))_H - \mathrm{trace}_H(PDF(x)).$$

The following result is due to Ramer [383].

2.2 Theorem. *Assume that F is an $\mathcal{H}-C^1$ map such that*

$$I = \int\limits_X \Big[\big\|F(x)\big\|_H^2 + \big\|DF(x)\big\|_{\mathcal{L}_{(2)}(H)}^2\Big]\gamma(dx) < \infty.$$

Then for any sequence $\{P_n\}$ of finite-dimensional orthogonal projections on H which converges pointwise to the identity, the sequence $L_{P_n}F$ converges in measure to some function LF. In addition,

$$\int\limits_X \big\|LF(x)\big\|^2\gamma(dx) \le I.$$

2.3 Definition. For each $\mathcal{H}-C^1$ map F define

$$d_F(x) = \delta\big(I_H - DF(x)\big)\exp\big(LF(x) - \|F(x)\|_H^2/2\big),$$

where $\delta(A)$ stands for the Carleman–Fredholm determinant of an operator $A \in \mathcal{L}(H)$.

For $K \in \mathcal{L}_{(2)}(H)$, the Carleman–Fredholm determinant of $I_H - K$ is given by

$$\delta(I_H - K) = \prod_{n=1}^{\infty} \lambda_n \exp(1 - \lambda_n),$$

where the λ_n's are the eigenvalues of $I_H - K$ counted with their multiplicities. Note that if K is nuclear then

$$\delta(I_H - K) = \det(I_H - K)\exp(\operatorname{trace}(K)).$$

2.4 Theorem. *Assume that an $\mathcal{H}-C^1$-map F satisfies the following conditions:*

(1) *T is bijective;*

(2) *$I_H - DF(x) : H \to H$ is invertible for any $x \in X$. Then the measure $\gamma \circ T^{-1}$ is equivalent to γ and its Radon–Nikodym derivative equals $|d_F(x)|$.*

Let us mention here one related result, due to A. B. Cruseiro [109].

2.5 Theorem. *Let f be a Borel map on X (not necessarily continuous) with values in the Cameron–Martin subspace H and belonging to the Sobolev class $W^\infty(X, \gamma, H)$. Assume that*

$$\exp(\lambda\|f\|_H) + \exp(\lambda|\delta f|) + \exp(\lambda\|\nabla f\|_H) \in L^1(\gamma) \quad \forall\lambda.$$

Then there exists a family of transformations $U_t : X \to X$ such that

$$U_t(x) = x + \int\limits_0^t f(U_s(x))ds \quad \text{for all } t \text{ and a.e. } x.$$

In addition, for any t the image of the measure γ under the map U_t is equivalent to γ.

It should be pointed out here that the transformations considered above after Theorem 3.2.6 typically are not absolutely continuous. For example, let $g : R^1 \to R^1$ be a smooth function with bounded derivatives such that $0 < c_1 \le g' \le c_2 < \infty$. The map $G : C[0,1] \to C[0,1]$, $G(x)(t) = g(x(t))$, is a diffeomorphism (if we consider G on $L^2[0,1]$, then it is a W^∞-map). The image of the Wiener measure P^W under this map coincides with the measure induced by the solution of the stochastic equation

$$d\xi_t = a(\xi_t)dw_t + b(\xi_t)dt, \ \xi_0 = g(0),$$

where $a(s) = g'(g^{-1}(s))$, $b(s) = g''(g^{-1}(s)/2$. This follows from Ito's formula applied to the process $\xi_t = g(w_t)$ (see [318], p. 118, Theorem 4.4). If $a = \text{const}$, then according to the Girsanov–Skorohod theorem [201, 426], the measure μ^ξ is equivalent to the Gaussian measure generated by the process aw_t (that is, to the homothetic image of P^W). In particular, the measures μ^ξ and P^W are equivalent if $a = 1$. One can check that then μ^ξ is differentiable along all vectors in the Cameron–Martin space of P^W (see [371, 50]). It was conjectured by Pitcher [371] that this is no longer true for a nonconstant a. This conjecture was proved by the author [45]. Moreover, it was proved there that in this case the measure μ^ξ has no nonzero directions of continuity. Under the additional assumption that $a \neq$ const is real-analytic, a stronger result has been obtained in [461] that μ^ξ is concentrated on a set which is zero for any Gaussian measure (that is, on a Gaussian null set; see Section 3.5).

It is also worth mentioning that, unlike the linear case, a smooth transformation F which preserves a Gaussian measure γ need not be of the type indicated above. Consider the following example, suggested to us by M. Zakai. Let γ be the countable product of the standard Gaussian measures on lines, $X = R^\infty$, and let F be a transformation, which, for every odd n, is an orthogonal rotation in the plane (x_n, x_{n+1}), but with varying angle. Formally:

$$F(x)_n = x_n \cos\theta(x) + x_{n+1} \sin\theta(x), \qquad F(x)_{n+1} = -x_n \sin\theta(x) + x_{n+1} \cos\theta(x),$$

where $\theta(x)$ is a nonconstant function in $W^\infty(X, \gamma)$. Clearly, F preserves γ, but in general is not of the type described in Theorem 4.2.1.

2.6 Remark. The results presented in this section go back to the pioneering works [84, 330, 90, 86, 91], where consequently translations, linear mappings and then more general nonlinear transformations $T\omega = \omega + F(\omega)$ were investigated in the case of the classical Wiener space. One of the basic assumptions in this circle of problems is that F takes values in the Cameron–Martin space. Further progress is due to Girsanov and Skorohod. In their articles [201] and [426], which appeared in the same volume of "Teoria veroyatnostei i ee primenenia" in 1960 (submitted in 1958), they proved independently that the distributions of two diffusion processes with the same diffusion coefficient A are equivalent (in particular, if $A = I$, they are equivalent to the Wiener measure). In addition, Girsanov's paper contains a more general statement, which in the case $A = I$ says that the measure induced by the process

$$\xi_t = w_t + \int_0^t f(s, \omega)ds$$

is equivalent to the Wiener measure, provided $f(t, \omega)$ is adapted, square-integrable in each argument, and $E\Phi = 1$, where

$$\Phi = \exp\left(\int_0^1 f(s,\omega)dw_s - \frac{1}{2}\int_0^1 f(s,\omega)^2 ds\right).$$

It was shown in [201] that the condition $E\Phi = 1$ is fulfilled in many important cases. In some special cases this identity was obtained in [331], however, the problem of absolute continuity of measures was not considered there.

Starting with Gross's paper [213], many authors (see [22, 427, 428, 429 287, 150, 383, 290]) investigated related problems within the framework of abstract Wiener spaces (or Hilbert spaces with Gaussian measures), where the adaptivity condition is replaced by a certain smoothness of F. The papers [74–76] gave an impetus to active investigations of the Girsanov-type transformations without nonanticipativity conditions. Further references may be found in [77, 152, 153, 256, 294, 351, 476, 477, 506].

Interesting applications of the results above to constructions of quasi-invariant measures and diffusions on infinite-dimensional manifolds can be found in [134, 266, 326, 348, 413].

3. Finite-Dimensional Images of Gaussian Measures and Surface Measures.

One of the classical problems of stochastic analysis is the investigation of measures induced by functionals of random processes. In diverse applications (in mathematical statistics, to limit theorems in probability theory and mathematical physics) the following classes of (numerical or vector-valued) functionals arise most often: (1) solutions of stochastic differential equations at a fixed moment of time; (2) polynomial functionals, in particular, quadratic forms and multiple stochastic integrals; (3) functionals of supremum type, norms; (4) integral functionals (given by ordinary or stochastic integrals). Various results and further references relating to these classes may be found in [5, 44, 45, 50, 121, 193, 292, 293, 296, 345, 469]. It is more natural to discuss this range of problems in a more general framework in connection with the Malliavin calculus (the reader may consult [28, 35, 50, 239, 352, 487, 488]). This will be the subject of a forthcoming survey of this author. Here we restrict ourselves to introductory remarks.

Let γ be a centered Gaussian Radon measure on a locally convex space X and let $F: X \to R^n$ be a map which is sufficiently regular. What can be said about the induced measure $\mu = \gamma \circ F^{-1}$? Is it absolutely continuous with respect to the Lebesgue measure on R^n? Does it have a bounded density? Is it possible to choose a smooth version of this density? Clearly, we need some "nondegeneracy" conditions on F, otherwise μ may have atoms. For example, if $X = R^1$, $n = 1$, and F is smooth, then a necessary and sufficient condition for μ to be absolutely continuous is the equality $\gamma(x: F'(x) = 0) = 0$. Let us discuss the infinite-dimensional case. The proofs of the results presented below can be found in [50]. The statements about absolute continuity of the induced measures can be deduced from classical theorems in geometric measure theory and the following elementary lemma (see [121] for the proof).

3.1 Lemma. *Let $E \subset H(\gamma)$ be a finite-dimensional linear subspace, Y a closed linear subspace in X such that $X = Y \oplus E$, ν the image of γ under the natural projection $X \to Y$. Then for every $y \in Y$ there exists a nondegenerate Gaussian measure γ^y on $E \oplus y$ such that*

$$\gamma(B) = \int_Y \gamma^y(B)\nu(dy) \quad \forall B \in \mathcal{B}(X).$$

3.2 Theorem. *Let $\{a_i\}$ be an arbitrary sequence in $H(\gamma)$, $F: X \to R^m$ a γ-measurable map with the following property: for γ-a.e. x there exist vectors $v_1(x), \ldots, v_m(x)$ in $\{a_i\}$, such that the vectors*

$$\partial_{v_j} F(x) = \lim_{t \to 0} \frac{F(x + tv_j(x)) - F(x)}{t}, \ j = 1, \ldots, m,$$

exist and are linearly independent. Then the measure $\gamma \circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on R^m. In particular, this holds if F belongs to the Sobolev class $W^{1,1}$ and $D_H F(x)$ is surjective for almost all x.

It should be noted that unlike the finite-dimensional case, the condition in Theorem 3.2 is far from being necessary. There is an example [262] of a function $F: C[0,1] \to R^n$, where $C[0,1]$ is equipped with Wiener measure P^W, such that F is infinitely Fréchet differentiable, but the image under F of the restriction of P^W on the set of critical points $\{x: F'(x) = 0\}$ possesses smooth density. In particular, there is no infinite-dimensional analog of the Sard theorem in this sense (see, however, [197] and [308]).

3.3 Corollary. *Let X be a Banach space, $F: X \to R^m$ a locally Lipschitzian map such that the set of x's where the Gateaux derivative $F'(x)$ exists but is not surjective has measure zero. Then $\gamma \circ F^{-1}$ is absolutely continuous.*

3.4 Example. *Let $X = C([0,T], R^d)$ and let μ be the measure generated by the solution of the stochastic differential equation*

$$d\xi_t = A(\xi_t)dw_t + B(\xi_t)dt, \quad \xi_0 = x,$$

with Lipschitzian coefficients A and B. Assume that the operator $A(x)$ is invertible for every x and that $F: X \to R^m$ satisfies the condition of the previous corollary. Then the measure $\mu \circ F^{-1}$ is absolutely continuous.

The next result provides a version of Theorem 3.2 for weak derivatives.

3.5 Theorem. *Let $\{a_i\}$ be an arbitrary sequence in $H(\gamma)$. Assume that a function F on X satisfies the following conditions:*

(1) *F is the limit of an almost everywhere converging sequence of measurable functions F_n differentiable in the directions $\{a_i\}$ and with $\partial_{a_i} F_n$ integrable on every compact set.*

(2) *For every i there exists a measurable function g_i integrable on every compact set $K \subset X$ and such that as $n \to \infty$*

$$\int_K |\partial_{a_i} F(x) - g_i(x)| \gamma(dx) \to 0.$$

(3) $\gamma(x: \ g_i(x) = 0 \ \forall i) = 0$.
Then the measure $\gamma \circ F^{-1}$ is absolutely continuous.

3.6 Theorem. *Let $Q = \sum_{n=0}^{\infty} Q_n$, where Q_n are γ-measurable polynomials of degree less than or equal to n and the series converges in $L^2(\gamma)$. Assume that*

$$\sum_{n=0}^{\infty} |\lambda|^n \|Q_n\|_{L^2}^2 < \infty$$

for some $\lambda > 1$. Then either Q has absolutely continuous distribution or $Q = \mathrm{const}$ γ-almost everywhere.

We proceed now to smoothness results.

3.7 Theorem. *Let $F = (F_1, \dots, F_n): \ X \to R^n$ be a map such that $F_i \in W^\infty(X, \gamma)$, $i = 1, \dots, n$, and $\frac{1}{\Delta} \in \bigcap_{p>1} L^p(\gamma)$, where $\Delta = det((D_H f_i, D_H F_j)_H)$, $H = H(\gamma)$. Then the measure $\gamma \circ F^{-1}$ admits a density in the Schwartz class $S(R^n)$.*

It is worth mentioning that the support of the measure $\gamma \circ F^{-1}$ is always connected, provided $F = (F_1, \dots, F_n)$, where $F_i \in W^\infty(X, \gamma)$ (see [154]).

3.8 Corollary. *Let Q be a γ-measurable polynomial. Assume that for any n the space X can be decomposed in the direct topological sum of its closed linear subspaces $X_0, \dots, X_n$ in such a way that $Q(x) = Q_0(x) + \dots + Q_n(x)$, where Q_i are measurable polynomials depending only on the projections of x on X_0 and X_i. Assume, in addition, that there are vectors $h_1, \dots, h_n$, such that $degQ_i|_{R^1 h_i} \geq 1$. Then the distribution density of Q is in $S(R^1)$.*

We say that a quadratic form Q on a Hilbert space H is infinite-dimensional if $Q(x) = (Ax, x)$, where $A \in \mathcal{L}(H)$ is a symmetric operator with $\dim A(H) = \infty$.

3.9 Corollary. *Let Q be a γ-measurable quadratic form on X such that Q is infinite-dimensional on $H(\gamma)$. Then the measure $\gamma \circ Q^{-1}$ admits a density in the class $S(R^1)$. An analogous statement holds for the map $Q = (Q_1, \dots, Q_n): \ X \to R^n$, if Q_i are quadratic forms whose nontrivial linear combinations satisfy the condition above.*

3.10 Example.

(a) *Let $X = C[a, b]$, $F(x) = \int_a^b f(t, x(t))dt$, where*

$$f(t, z) = \sum_{j=0}^{d} a_j(t) z^j, \quad a_j \in L^1[a, b].$$

Assume that $H(\gamma)$ is dense in X and that $\|a_d\|_{L^1} \neq 0$. Then $\gamma \circ F^{-1}$ admits a density in $S(R^1)$.

(b) *Let μ be the same as in Example 3.4. Assume that A and B are in C_b^∞ and that A is uniformly nondegenerate. Let*

$$F(\xi_t) = \int_0^T f(t, \xi_t)\, dt,$$

where the function f has measurable derivatives $\partial_x^r f$, satisfying the estimates $\|\partial_x^r f(t, x)\| \leq c_r \exp(\|x\|^{\alpha_r})$, $\alpha_r < 2$. Assume, in addition, that f is real-analytic in a neighborhood of zero $V \subset R^{d+1}$ and that $\partial_x f$ is not identically zero in V. Then $\gamma \circ F^{-1}$ admits a density in $S(R^1)$.

Recall that the modulus of convexity of a norm q is defined by

$$\delta_q(\varepsilon) = \inf\left(1 - \frac{q(x+y)}{2}, \ q(x), q(y) \leq 1, \ q(x-y) \geq \varepsilon\right).$$

3.11 Theorem. *Let X be a Banach space such that its norm q has k Lipschitzian Fréchet derivatives on the unit sphere and satisfies the condition $\delta_q(\varepsilon) \geq C\varepsilon^\alpha$, where $C, \alpha > 0$. If $\dim H(\gamma) = \infty$, then the function $M: t \mapsto \gamma(x: q(x) < t)$ is k times differentiable and $M^{(k)}$ is absolutely continuous. In addition, the function $Q: x \mapsto \gamma(U + x)$, has k continuous Fréchet derivatives, where U is a ball in X. Moreover, the map $(t, x) \mapsto \gamma(tU + x)$ is in C^k. In particular, the condition above holds for the spaces L^{2n}.*

Much more is known about the distribution function of the norm in a Hilbert space with a Gaussian measure. See [233, 359, 385, 387, 471].

One needs much less to get the boundedness of the distribution density of a quadratic form.

3.12 Theorem. *Let Q be a γ-measurable quadratic form on X. A necessary and sufficient condition for the density of the distribution of Q to be bounded is the existence of a two-dimensional subspace L of $H(\gamma)$ on which the form Q is positive or negative definite (moreover, in this case, the density of the distribution of Q automatically has bounded variation).*

The next result was proved in [314]. A short proof was suggested in [43].

3.13 Theorem. *Let γ be a nondegenerate Gaussian measure on a Banach space X, $M(t, x) = \gamma(y : \|y - x\| \leq t$. Then the function M is locally Lipschitzian on $(0, \infty) \times X$ and is Hadamard differentiable.*

Note that M need not (in the case $X = C[0, 1]$ even cannot) be Fréchet differentiable, in particular, its Gateaux derivative need not be continuous. Indeed, according to [60, 491] there is no nontrivial Fréchet differentiable function on $C[0, 1]$. This implies that there is no nontrivial Fréchet differentiable function on $C[0, 1]$ which tends to zero as x goes to infinity.

Now let F be a measurable function on a Banach space X possessing almost everywhere Gateaux derivative F' that is locally integrable.

Put $S = \{x : F(x) = 0\}$.

3.14 Definition. A locally finite measure σ on S is called the *local surface measure* for γ if every point $s \in S$ has a neighborhood V such that the measures

$$\eta_t(B) = (2t)^{-1} \int\limits_{B \cap V \cap \{-t < F < t\}} \|F'(x)\| \gamma(dx)$$

converge weakly to $\sigma(\cdot \cap V)$ as $t \to 0$. We call σ the surface measure for γ and denote it by γ^S if it has bounded variation. If $\{S_n\}$ is a countable union of surfaces of the type considered, then we define the surface measure γ^S on $S = \cup_n S_n$ as the sum of the series $\sum_n \gamma^{S_n}$ if the latter converges weakly.

3.15 Proposition. *Assume that $H(\gamma)$ is dense in X, F has locally bounded Gateaux derivative not vanishing at any point and that the partial derivatives of F are continuous. Then the local surface measure on S exists.*

Various results on the existence of bounded surface measures may be found in [43, 45, 50] taking into account that results of this kind can be derived from statements on the boundedness of the distribution densities.

The next theorem connects surface integrals with integrals over the space. See [438, 206, 429, 289, 466, 467, 5] for related results.

3.16 Theorem. *Let V be an open set with the boundary S which locally has the form indicated in Proposition 3.15, and let n be the continuous normal covector to the surface S. Assume that $H(\gamma)$ is dense in X, γ^S is finite, $\{e_n\}$ is an orthonormal basis in $H(\gamma)$ and that for some function ϕ, differentiable along e_i, the series $G = \sum_{i=1}^\infty [\partial_{e_i}^2 \phi + \partial_{e_i} \phi \beta_{e_i}]$ converges in $L^1(\gamma)$ and the series $\partial_n \phi = \sum_{i=1}^\infty \partial_{e_i} \phi \langle n, e_i \rangle$ converges in $L^1(\gamma^S)$. Then the following formula holds:*

$$\int\limits_S \partial_n \phi(s) \gamma^S(ds) = \int\limits_V G(x) \gamma(dx).$$

3.17 Corollary. *Assume that the conditions of Proposition* 3.16 *are fulfilled for the surfaces* $V + ta$, $0 < t < r$, *where* $a \in X$ *is fixed. Then for such* t

$$\gamma(V+ta) - \gamma(V) = \int_0^t \int_{S+ra} \langle n(x), a\rangle \, \gamma^{S+ra}(dx)dr.$$

3.18 Remark. It is possible to define surface measures in another way (see [5]), replacing $\|F'\|$ by $\|D_H F\|_H$. In a similar way, surface measures on surfaces of codimension $n > 1$ can be defined [5] . This modification is especially useful for introducing surface measures on level surfaces of functions from Sobolev spaces over abstract Wiener spaces since such functions need not even be continuous. The construction is as follows. Let $F : X \to R^n$ satisfy the conditions of Theorem 3.7. Then by virtue of this theorem the measure $\gamma \circ F^{-1}$ has infinitely differentiable density k. In addition, on the sets $S_y = F^{-1}(y)$ there are conditional measures $\gamma(\cdot|y)$. Let us define the surface measure a^F on S by

$$\int_S f(x)a^F(dx) = k(0) \int_S f(x)\sqrt{\Delta(x)}\gamma(dx|0).$$

Then the following relationship holds:

$$\int_S f(x)a^F(dx) = \lim_{\varepsilon\to 0} \frac{1}{\mathrm{Vol}B(0,\varepsilon)} \int_X f(x)\sqrt{\Delta(x)} I_{|F|\le\varepsilon}(x)\gamma(dx),$$

where $\mathrm{Vol}B(0,\varepsilon)$ is the volume of the ball of radius ε in R^n.

For these surface measures there is the Stokes formula as well [5]. Geometrically the difference between these two types of surface measures is that in the first case (at least for sufficiently regular surfaces) the surface measure is obtained as the limit of the ratio of the measure of the ε-neighborhood of S, generated by the normal n, to ε. In the second case, the ε-neighborhood is constructed as $S + \varepsilon U_H$, where U_H is the unit ball in $H(\gamma)$. Obviously, the first approach corresponds to the geometry of the Banach space X, while the second one is connected only with the geometry of $H(\gamma)$ and does not depend at all on the geometry of X (in particular, this approach applies to locally convex spaces).

Nice differentiability properties of Gaussian measures can be used for constructing partitions of unity and smooth approximations consisting of functions infinitely differentiable along a dense subspace H. Some results in this directions are contained in [33, 43, 205, 216, 366].

For related results, see also [44, 71, 174, 176, 206, 323, 324, 345, 417, 418, 466, 467, 468, 469, 474, 475].

Chapter 5
Infinite-Dimensional Wiener Processes and Diffusions

1. Wiener Processes in Infinite-Dimensional Spaces.

Now we shall briefly discuss the notion of a Wiener process in infinite-dimensional locally convex spaces (see, e.g., [9, 114, 177, 289, 396]). Note that if W_t is a standard Wiener process in R^n, then for any unit vector $v \in R^n$ the process (v, W_t) is a one-dimensional Wiener process. Thus, one might try to define a Wiener process in a separable Hilbert space X as a continuous process W_t with values in X such that for any unit vector $v \in X$ the real process $(v, W_t)_X$ is Wiener. However, such a process does not exist if X is infinite dimensional. Indeed, let u and v be two orthogonal unit vectors in X. Then we have

$$E\Big[\Big(\frac{u+v}{\sqrt{2}}, W_t\Big)_X^2\Big] = t = \frac{1}{2}E\big[(u, W_t)_X^2\big] + \frac{1}{2}E\big[(v, W_t)_X^2\big].$$

Hence $E\big[(u, W_t)_X (v, W_t)_X\big] = 0$. Let $\{e_n\}$ be an orthonormal basis in X. Then the orthogonal Gaussian random variables $(e_n, W_t)_X$ are independent and $E\big[(e_n, W_t)_X^2\big] = t$. By a classical result, the series

$$(W_t, W_t)_X = \sum_{n=1}^{\infty} (e_n, W_t)_X^2$$

diverges a.s., which is a contradiction.

Nevertheless, the idea above can be implemented in the following way. Let X be a locally convex space and let H be a separable Hilbert space, which is continuously and densely embedded in X. For any $v \in X^*$ the functional $h \mapsto {}_{X^*}\langle v, h\rangle_X$ is continuous on H. Hence there is a vector $j(v) \in H$ such that for all $h \in H$

$$\big(j(v), h\big)_H = {}_{X^*}\langle v, h\rangle_X .$$

Thus, we get the embedding $j: X^* \to H$.

1.1 Definition. Let X and H be as above. A continuous stochastic process W_t in X is called a Wiener process associated with H, if for each $v \in X^*$ with $\|j(v)\|_H = 1$ the one-dimensional process ${}_{X^*}\langle v, W_t\rangle_X$ is Wiener.

Let $\mathcal{F}_t$ be an increasing sequence of σ-fields. A Wiener process W_t is said to be an $\mathcal{F}_t$-Wiener process if, for any $t,\ s > \tau$, $W_t - W_s$ is independent of $\mathcal{F}_\tau$ and W_t is $\mathcal{F}_t$-measurable.

Note that in the case $X = H = R^n$ we come to the usual definition. The definition above being given, the following two problems arise.

(a) Let X be a locally convex space. Can one find an H such that there exists an associated Wiener process in X?

(b) Let $H \subset X$ be fixed. Does there exist an associated Wiener process in X?

The answer to the first question is positive for many spaces.

1.2 Proposition. *Let X be a locally convex space. One can construct a Wiener process in X if and only if there exists a separable Hilbert space E which is continuously and densely embedded into X. If X is sequentially complete, then this is equivalent to the existence of a bounded sequence in X whose linear span is dense.*

Proof. The necessity of the condition above is trivial. Assume that X is a separable Hilbert space. Choose positive numbers t_n such that $\sum\limits_{n=1}^{\infty} t_n^2 < \infty$. Let

$$H = \left\{ h = \sum_{n=1}^{\infty} t_n x_n e_n \;\middle|\; \sum_{n=1}^{\infty} x_n^2 < \infty \right\}.$$

Then H is a separable Hilbert space with the norm $\|h\|_H = \big(\sum\limits_{n=1}^{\infty} x_n^2\big)^{1/2}$. Clearly, the natural embedding $H \to X$ is a Hilbert–Schmidt operator. Let $\{w_n(t)\}$ be a sequence of independent real Wiener processes. Then the series $\sum\limits_{n=1}^{\infty} t_n^2 w_n(t)^2$ converges a.s. Set

$$W_t = \sum_{n=1}^{\infty} t_n w_n(t) e_n .$$

One can check that the process W_t is continuous (see [114], Proposition 4.2). Let $v \in X$ and $v_n = (v, e_n)_X$. Then

$$j(v) = \sum_{n=1}^{\infty} t_n^2 v_n e_n .$$

If $\|j(v)\|_H = 1$, then $\sum\limits_{n=1}^{\infty} t_n^2 v_n^2 = 1$. Hence the process $(v, W_t)_X = \sum\limits_{n=1}^{\infty} t_n v_n w_n(t)$ is Wiener.

Let X be a locally convex space such that there exists a separable Hilbert space E which is continuously and densely embedded into X. We can take any Wiener process in E, associated with some Hilbert space H embedded in E, and consider it as an X-valued process. Clearly, it is continuous. Let $v \in X^*$ be such that $\|j(v)\|_H = 1$. Denote by u the restriction of v to E. We have

$${}_{E^*}\langle u, W_t\rangle_E = {}_{X^*}\langle v, W_t\rangle_X$$

and

$$\|j(u)\|_H = \sup\{(j(u),h)_H : \|h\|_H \le 1\} = \sup\{{}_{E^*}\langle u,h\rangle_E : \|h\|_H \le 1\} =$$

$$= \sup\{{}_{X^*}\langle v,h\rangle_X : \|h\|_H \le 1\} = \sup\{(j(v),h)_H : \|h\|_H \le 1\} = \|j(v)\|_H = 1.$$

Therefore, W_t is a Wiener process in X. To prove the last assertion, note that if $\{a_n\}$ is a bounded sequence in a sequentially complete locally convex space X, then, according to [50], Lemma 1.1, there is a separable Hilbert space containing $\{a_n\}$ and continuously embedded in X.

The problem (b) is more delicate. If (H, X) is an abstract Wiener space, then a Wiener process associated with H exists (see [289]). If X is a Hilbert space, then a necessary and sufficient condition is that the natural embedding $H \to X$ is a Hilbert–Schmidt operator. If X is a locally convex space with a centered Radon Gaussian measure γ such that $H = H(\gamma)$, then one can define a Wiener process in X by

$$W_t = \sum_{n=1}^{\infty} w_n(t)e_n,$$

where $\{w_n(t)\}$ is a sequence of independent standard one-dimensional Wiener processes and $\{e_n\}$ is an orthonormal basis in H. Thus, we return to the problem of the existence of a Radon Gaussian measure on X with the reproducing kernel Hilbert space H.

Having defined an infinite-dimensional Wiener process, one might ask questions about its realizations (the asymptotic behavior of the sample paths, various moduli of continuity, etc.). Further information on this subject can be found in [95, 167, 168, 340] and in the references therein.

2. Infinite-Dimensional Diffusions.

Let X be a locally convex space, $H \subset X$ a separable Hilbert space which is continuously and densely embedded into X, $B : X \to X$ a Borel mapping. Consider the following stochastic differential equation:

$$d\xi_t = dW_t + B(\xi_t)dt, \ \xi_0 = x. \tag{2.1}$$

By a solution we mean a stochastic process ξ_t in X such that there exists a filtration $\mathcal{F}_t$, an $\mathcal{F}_t$-Wiener process W_t in X associated with H, and

$$\xi_t = x + W_t + \int_0^t B(\xi_s)ds \quad \text{a. s.}$$

It should be noted that in infinite-dimensions (say, in infinite-dimensional Banach spaces) Eq. (2.1) need not be solvable even for a bounded continuous mapping B (see [58] for some examples). Sufficient conditions for the existence of a solution of (2.1) can be found in [6, 9, 114, 118, 123, 125, 243, 289, 396]. If X is a Banach space, then it suffices that B be Lipschitzian.

An interesting direction connected with Eq. (2.1) is the study of the transition probabilities and the invariant measures of the diffusion ξ_t. Let H be the Hilbert space associated with W_t. We shall consider the case where

$$B(x) = -\frac{1}{2}x + b(x), \quad v : X \to H.$$

Denote by γ the law of W_1. Clearly, this is a centered Gaussian measure with $H(\gamma) = H$.

The following result was obtained in [419].

2.1 Theorem. *Assume that* $\sup \|v(x)\|_H < \infty$. *Then the process* ξ_t *exists and possesses an invariant probability measure* μ *which is equivalent to* γ.

Any invariant measure of the diffusion above satisfies the following elliptic equation:

$$L^*\mu = 0\,.$$

Here L is defined on $\mathcal{FC}^\infty$ by

$$Lf(x) = \frac{1}{2}\Delta f(x) + {}_{X^*}\langle f'(x), B(x)\rangle_X, \tag{2.2}$$

where $B: X \to X$ is a μ-measurable map,

$$\Delta f(x) := \Delta_H f(x) := \sum_{n=1}^{\infty} \partial_{e_n}^2 f(x),$$

and $\{e_n\}$ is an orthonormal basis in H. Note that the sum above does not depend on the particular choice of an orthonormal basis in H.

If X is a Hilbert space and the embedding $H \to X$ is compact (which is always the case if $H = H(\gamma)$) we get the following picture: $H = T(X)$, where T is a nonnegative injective Hilbert–Schmidt operator with the eigenvectors h_n, forming an orthonormal basis in H, and the eigenvalues t_n. The vectors $e_n = t_n h_n$ form an orthonormal basis in H, and for any vector $v = \sum\limits_{n=1}^{\infty} v_n h_n \in H$,

$$(v, v)_H = \sum_{n=1}^{\infty} t_n^{-2} v_n^2.$$

With this notation,

$$Lf = \frac{1}{2}\sum_{n=1}^{\infty} t_n^2 \partial_{h_n}^2 f + \sum_{n=1}^{\infty} B_n \partial_{h_n} f,$$

where $B_n = (B, h_n)_X$.

We always assume that

$${}_{X^*}\langle l, B\rangle_X \in L^2(\mu) \quad \forall l \in X^*. \tag{2.3}$$

We understand the equation $L * \mu = 0$ in the following weak sense:

$$\int_X Lf(x)\mu(dx) = 0 \quad \forall f \in \mathcal{FC}^\infty. \tag{2.4}$$

In the case where $X = R^n$ and B is smooth, a classical result says that any solution of (3.3) is an absolutely continuous measure which has a smooth density with respect to the Lebesgue measure. An extension of this result to the case of a locally bounded (or locally integrable) coefficient B gives the existence of a density in some Sobolev space. However, in our more general setting the classical theory does not apply, since B is not assumed to be locally integrable and has sense only with respect to a solution. Nevertheless, some regularity holds even in this general setting. The following result is proved in [55, 56].

2.2 Proposition. *Let μ be a probability measure on R^n such that*

$$\int L\phi\mu = 0 \text{ for all } \phi \in C_b^\infty(R^n),$$

where $L\phi = \Delta\phi + (\nabla\phi, b)$ and b is a Borel vector field on R^n with $|b| \in L^2(\mu)$. Then:

(1) *μ has a density $p \in W^{1,1}(R^n)$, in particular, μ is differentiable along R^n;*

(2) *the following estimate holds:*

$$\int \frac{|\nabla p|^2}{p} dx \le \int |b|^2 \mu; \tag{2.5}$$

(3) *if b belongs to the closure of smooth gradients in $L^2(\mu, R^n)$, then $\nabla p/p = b$ a.e.*

2.3 Theorem. *Let X, H, γ be as above. Assume that a probability measure μ on X satisfies Eq.* (2.4), *where*

$$B(x) = -x + v(x), \ v: \ X \to H, \ \|v\|_{H} \in L^2(\mu).$$

Then:

(i) *μ is differentiable along all vectors $h \in H$;*

(ii) *$\beta_h(x) = -g_h(x) + (u(x), h)_H \quad \forall h \in H$, where $h = R_\gamma g_h$, $u: \ X \to H$ is the orthogonal projection of v on the closure of the set $\{D_H f| \ f \in \mathcal{F}C^\infty\}$ in the Hilbert space $L^2(\mu, H)$;*

(iii) *μ is absolutely continuous with respect to γ and its Radon–Nikodym derivative f admits the representation $f = F^2$, where F is in the Sobolev class $W^{1,2}(X, \gamma)$.*

Various results and further references on infinite-dimensional Ornstein–Uhlenbeck processes and linear stochastic equations can be found in [1, 11, 52, 54, 123, 168, 169, 170, 177, 240, 242, 274, 340, 392].

For additional information on regularity of the transition probabilities and invariant measures of infinite-dimensional diffusions see [29, 53, 114, 118, 179, 180, 194, 216, 268, 319, 344, 362, 363, 364, 365, 366, 367].

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