Introduction to the Fourier Series

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This paper gives an introduction to the Fourier Series that is suitable for students with an understanding of Calculus. The emphasis is on introducing useful terminology and providing a conceptual level of understanding of Fourier analysis without getting too hung up on details of mathematical rigor.

How to Approach this Document. To be an electrical engineer you need to understand Fourier series very well, which means that you should be able to derive every result and work every example given in this paper. I recommend that you try. Read this paper, then put it aside and try to do every derivation and example. If you can do that, then you have a good understanding of Fourier series. In particular, I recommend that you:

1. Apply (10)-(12) to (4) to prove that they are correct.

2. Calculate each term in the Fourier series of (13).

3. Apply (69) to (68) to prove that it is correct.

4. Calculate Fourier series of (73).

5. Show that each of the properties given in Section 6 are correct.

Contents

1 Introduction

Consider the situation where you are working with a voltage waveform that you know is purely sinusoidal with a known frequency f, but for which you do not yet know its amplitude and phase. Such a waveform is shown in Figure 1. A waveform of this form can be represented with the following equation: l

$$v(t) = A\cos(2\pi ft + \phi). \tag{1}$$

In this equation, only the values of A and ϕ are unknown. Once we specify those values, we have fully specified the waveform. Thus, we now have two ways to specify such a waveform: we can either give v, a function that maps time into voltage, or we can sim-

FIGURE 1   *A voltage waveform to consider. The vertical axis represents voltage and the horizontal axis represents time.*
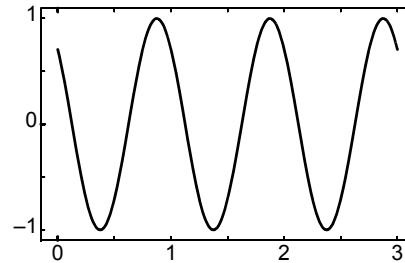
ply give A and ϕ. Given our assumption that the waveform must be sinusoidal with frequency f, both specify the same waveform, but there are significant benefits to just using A and ϕ. First, specifying two numbers is simpler than specifying an entire function. Second, the numbers can offer insight into the nature of the waveform that is difficult to get from the waveform directly. For example, in this case A is the amplitude of the waveform and ϕ is its phase.

The function v is called a time-domain representation of the waveform because it is a function that specifies the waveform and whose domain is time (meaning that it maps time into voltage).

The alternate representation of v can be denoted (A, ϕ). By themselves these numbers are not sufficient to represent the waveform, but when combined with (1) they provide an alternate way of specifying a signal. Notice that with (A, ϕ) we are specifying a waveform that varies with time, but that time is not included in (A, ϕ). Rather time comes in through our waveform model, given by (1). Thus, this alternative representation of the waveform consists of two pieces, a model of our waveform, in this case given by (1), and a set of coefficients that fit into that model, in this case A and ϕ. It is often the case that we must work with many waveforms that all fit the same waveform model. For example, a linear amplifier will produce a sinusoid at its output when driven with a sinusoid at its input, and they will both have the same frequency. In this case you can distinguish between the waveforms using only their coefficients with the knowledge that they all assume the same model. For the amplifier (A_{in}, ϕ_{in}) represents the waveform at the input and (A_{out}, ϕ_{out}) represents the one at the output. The gain of the amplifier is A_{out}/A_{in}.

With Fourier series we will be generalizing this concept. Use of Fourier series allows us to provide an alternative representations for not just a purely sinusoidal waveforms, but for any periodic waveform with a given period, but it still involves a waveform model (the Fourier series) and a set of coefficients (the Fourier coefficients).

2 The Fourier Series

The basic idea of the Fourier series is that any periodic waveform can be represented with a sum of harmonically related sinusoids. Let's break this statement down. First, a **waveform** is a function of time, such as the one shown in Figure 1. A waveform is peri-

odic if it repeats itself identically after a period of time. Let the period be denoted T. Then mathematically, a T-periodic waveform v satisfies

$$\boxed{v(t) \ = \ v(t+T)} \text{ — } \textit{\textbf{a periodic waveform with period T}} \qquad (2)$$

for all t. To make things simpler, let's further assume that v is a continuous function of time. The Fourier series of this waveform can be written as

$$v(t) \ = \ a_0 + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + b_2 \sin\frac{4\pi t}{T} + a_3 \cos\frac{6\pi t}{T} + \dots, \qquad (3)$$

which is written more succinctly as

$$\boxed{v(t) \ = \ \sum_{k=0}^{\infty} a_k \cos\frac{2\pi kt}{T} + b_k \sin\frac{2\pi kt}{T}} \text{ — } \textit{\textbf{trigonometric form of the Fourier series}} \qquad (4)$$

where b_0 is assumed to be 0. This is an infinite series, and in order for it to converge $|a_k|$ and $|b_k|$ both must go to 0 as k goes to infinity. In other words there must be some value of K beyond which the contributions of a_k and b_k become negligible for $k > K$. This implies that if the sum only includes a finite number of terms it becomes an approximation for v and the accuracy of the approximation improves as K increases.

$$v(t) \cong \sum_{k=0}^{K} a_k \cos\frac{2\pi kt}{T} + b_k \sin\frac{2\pi kt}{T} \qquad (5)$$

for some K. This point is illustrated in Figure 2, which shows a square wave being approximated by a finite Fourier series. The sum is shown for the cases in which K equals 1, 3, 5, and 7. You can see that the series more closely approximates the function as K increases. This process is illustrated for a sawtooth waveform on the Wikipedia page for Fourier Series (*en.wikipedia.org/wiki/Fourier_series*). I encourage you to take a look at it.

The fundamental frequency f_0 is defined as the reciprocal of the period,

$$\boxed{f_0 \ = \ \frac{1}{T}} \text{ — } \textit{\textbf{the fundamental frequency}} \qquad (6)$$

This allows us to rewrite the Fourier series slightly as:

$$v(t) \ = \ \sum_{k=0}^{\infty} a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t . \qquad (7)$$

You will notice that the sinusoids that make up the sum are all at frequencies that are integer multiples of the fundamental frequency. These frequencies are referred to as **harmonics** of the fundamental frequency. Now you can see that the Fourier series, which is the right side of (7), is a sum of harmonically related sinusoids that represents a periodic waveform v, which is the left side of (7).

It is possible to condense (7) a bit more by rewriting it in terms of angular frequencies. Let

FIGURE 2   *A square wave being approximated by a finite Fourier series with an increasing number of terms.*
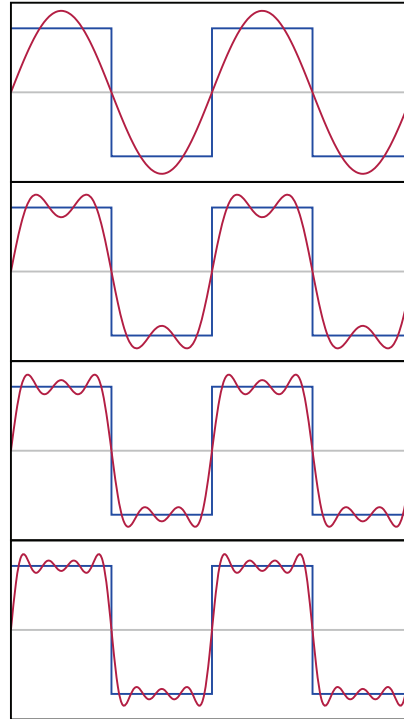
$$\boxed{\omega_0 \ = \ 2\pi f_0} \text{ — \textbf{\textit{the angular fundamental frequency}}} \tag{8}$$

Then

$$v(t) \ = \ \sum_{k=0}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t . \tag{9}$$

The coefficients a_k for $k = 0$ to ∞ and b_k for $k = 1$ to ∞ (we define b_0 to be 0) are referred to as the **Fourier coefficients** of v.

The waveform v can be represented with its Fourier coefficients, but the sequence of Fourier coefficients is not a waveform, because it is not a function of time. A more general term is needed. A **signal** is a physical quantity or effect, such as voltage, current, or an electromagnetic field strength, that can be varied in such a way as to convey information. Then a waveform, such as v, is often referred to as the **time-domain representation** of the signal because it represents the signal as a function of time. The Fourier coefficients are generally referred to as the **frequency-domain representation** of the signal as they vary as a function of the harmonic number k, which corresponds to a frequency, kf_0. These two are both representations of the same underlying signal, and so are equivalent and interchangeable. Both representations are very useful and engineers become quite adept at mentally converting from one domain to the other. The reason why is that certain concepts are more easily grasped in one domain than the other, some operations are

3 The Fourier Coefficients

Given a *T*-periodic signal *v*, it is possible to compute the Fourier coefficients of (4) with the following equations:

$$a_0 = \frac{1}{T}\int_0^T v(t)dt \quad — \textit{trigonometric form of the Fourier coefficients} \tag{10}$$

$$a_k = \frac{2}{T}\int_0^T v(t)\cos\frac{2\pi kt}{T}dt, \, k > 0, \tag{11}$$

$$b_k = \frac{2}{T}\int_0^T v(t)\sin\frac{2\pi kt}{T}dt, \, k > 0. \tag{12}$$

a_0 is referred to as the **DC component** of the signal, and a_k and b_k are the Fourier coefficients for the kth harmonic. Notice that a_0 is nothing more than the average of the function computed over one period (the interval $[0,T]$). Furthermore, notice that a_k and b_k are nothing more than the average of the function over one period after it has been multiplied by either $\cos(2\pi kt/T)$ and $\sin(2\pi kt/T)$.

3.1 Example

Calculate the Fourier coefficients of

$$v(t) = \alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t . \tag{13}$$

This is actually written in the form of a Fourier series, so we can determine the Fourier coefficients by inspection. In particular, $a_0 = \alpha$, $a_1 = \gamma$, $b_1 = \sigma$, with all other Fourier coefficients being identically equal to zero. But we will go through the process of computing them using (10)-(12) to familiarize ourselves with the process.

Let's start by calculating the DC component:

$$a_0 = \frac{1}{T}\int_0^T v(t)dt , \tag{14}$$

$$a_0 = \frac{1}{T}\int_0^T (\alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t)dt , \tag{15}$$

$$a_0 = \frac{\alpha}{T}\int_0^T dt + \frac{\gamma}{T}\int_0^T \cos 2\pi f_0 t dt + \frac{\sigma}{T}\int_0^T \sin 2\pi f_0 t dt . \tag{16}$$

Notice that the first term integrates 1 from 0 to *T*, getting *T*, and then multiplies by α over *T*, giving α. The second and third terms integrate a cosine and sine function over one complete period, which gives 0. Thus, $a_0 = \alpha$, as expected.

Let's examined what happened here. The signal consists of three components, a DC component and two components at the fundamental frequency (cosine and sine). This is shown in Figure 3. When computing a_0 the DC component is extracted from the composite signal by computing the average over exactly one period. The other components of the signal are at the fundamental frequency and so would be ignored because we integrate over one full cycle of these components and they are symmetric about zero over one period and so average to zero.

FIGURE 3   *The components that we integrate over when computing a_0.*
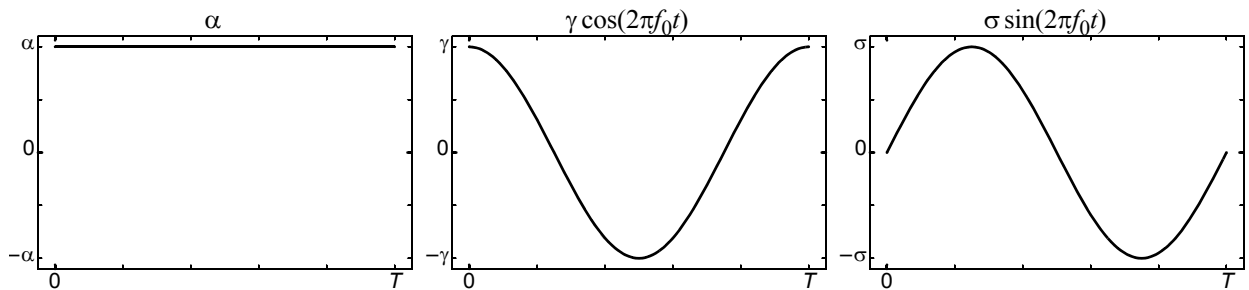
Now let's compute a_1.

$$a_1 = \frac{2}{T}\int_0^T v(t)\cos\frac{2\pi t}{K}dt,$$
(17)

$$a_1 = \frac{2}{T}\int_0^T (\alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t)\cos\frac{2\pi t}{T}dt.$$
(18)

Recall that $f_0 = 1/T$,

$$a_1 = \frac{2\alpha}{T}\int_0^T \cos\frac{2\pi t}{T}dt + \frac{2\gamma}{T}\int_0^T \cos^2\frac{2\pi t}{T}dt + \frac{2\sigma}{T}\int_0^T \sin\frac{2\pi t}{T}\cos\frac{2\pi t}{T}dt.$$
(19)

The arguments to the integrals in each of the three terms is shown in Figure 4.

FIGURE 4   *The components that we integrate over when computing a_1.*

e use the trigonometric identity $\cos^2 x = (1 + \cos 2x)/2$ to simplify the second term and $2\sin x\cos y = \sin(x + y) + \sin(x - y)$ to simplify the third.

$$a_1 = \frac{2\alpha}{T}\int_0^T \cos\frac{2\pi t}{T}dt + \frac{\gamma}{T}\int_0^T \left(1 + \cos\frac{4\pi t}{T}\right)dt + \frac{\sigma}{T}\int_0^T \sin\frac{4\pi t}{T}dt \,. \tag{20}$$

Here, the second term evaluates to 1 while the first, third and fourth terms integrate a cosine and sine function over either one or two full periods and so are 0. Thus, $a_1 = \gamma$. as expected.

Now lets compute a_k for k other than 0 or 1.

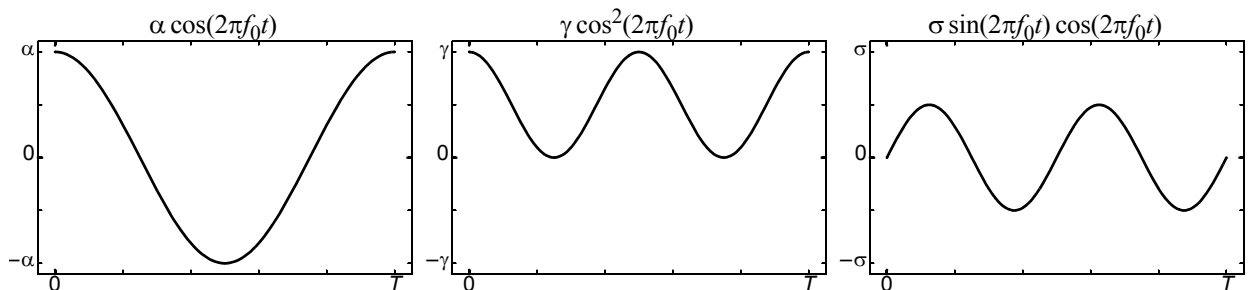$$a_k = \frac{2}{T}\int_0^T v(t)\cos\frac{2\pi kt}{T}dt \,, \tag{21}$$

$$a_k = \frac{2}{T}\int_0^T (\alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t)\cos\frac{2\pi kt}{T}dt \,. \tag{22}$$

Recall that $f_0 = 1/T$,

$$a_k = \frac{2}{T}\int_0^T \left(\alpha\cos\frac{2\pi kt}{T} + \gamma\cos\frac{2\pi t}{T}\cos\frac{2\pi kt}{T} + \sigma\sin\frac{2\pi t}{T}\cos\frac{2\pi kt}{T}\right)dt \,. \tag{23}$$

Using the trigonometric identity $2\cos x\cos y = \cos(x+y) + \cos(x-y)$ and $2\sin x\cos y = \sin(x+y) + \sin(x-y)$, we get

$$a_k = \frac{2\alpha}{T}\int_0^T \cos\frac{2\pi kt}{T}dt \tag{24}$$

$$+ \frac{\gamma}{T}\int_0^T \left(\cos\frac{2\pi(k+1)t}{T} + \cos\frac{2\pi(k-1)t}{T}\right)dt$$

$$+ \frac{\sigma}{T}\int_0^T \left(\sin\frac{2\pi(k+1)t}{T} + \sin\frac{2\pi(k-1)t}{T}\right)dt \,.$$

Remember that k is greater than one. Each of these three terms all become 0 because we are integrating one or more full periods of a cosine function. Thus, $a_k = 0$, for $k > 1$, as expected.

Now compute b_1.

$$b_1 = \frac{2}{T}\int_0^T v(t)\sin\frac{2\pi t}{T}dt \,, \tag{25}$$
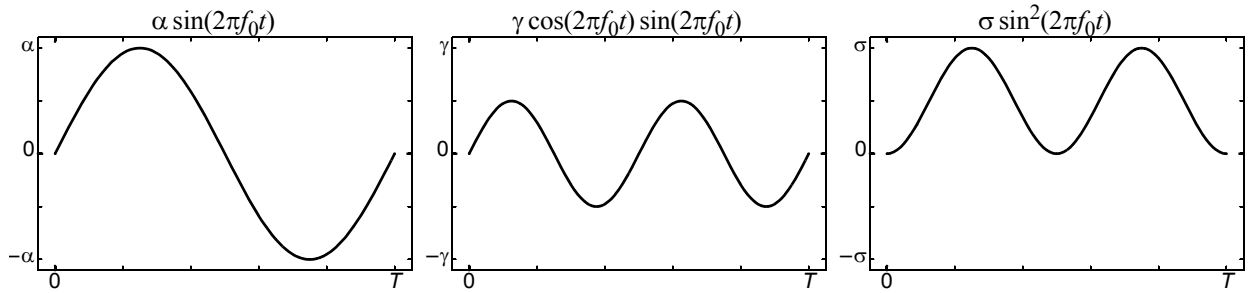
$$b_1 = \frac{2}{T}\int_0^T (\alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t)\sin\frac{2\pi t}{T}dt \,, \tag{26}$$

Recall that $f_0 = 1/T$,

$$b_1 = \frac{2}{T}\int_0^T \left(\alpha\sin\frac{2\pi kt}{T} + \gamma\cos\frac{2\pi t}{T}\sin\frac{2\pi t}{T} + \sigma\sin^2\frac{2\pi t}{T}\right)dt \,. \tag{27}$$

The three arguments to the integral is shown in Figure 5.

FIGURE 5 *The components that we integrate over when computing b_1.*

$\alpha \sin(2\pi f_0 t)$ | $\gamma \cos(2\pi f_0 t)\sin(2\pi f_0 t)$ | $\sigma \sin^2(2\pi f_0 t)$

Using the trigonometric identity $2\cos x \sin y = \sin(x+y) - \sin(x-y)$ to simply the second term and $\sin^2 x = (1 - \cos 2x)/2$ to simplify the third, we get

$$b_1 = \frac{1}{T}\int_0^T \left(2\alpha\sin\frac{2\pi kt}{T} + \gamma\cos\frac{4\pi t}{T} + \sigma - \sigma\cos\frac{4\pi t}{T}\right) dt. \tag{28}$$

All of the terms integrate to zero except the third term, which integrates to σ, and so $b_1 = \sigma$, as expected.

Finally, now compute b_k for $k > 1$.

$$b_k = \frac{2}{T}\int_0^T v(t)\sin\frac{2\pi kt}{T} dt, \tag{29}$$

$$b_k = \frac{2}{T}\int_0^T (\alpha + \gamma\cos 2\pi f_0 t + \sigma\sin 2\pi f_0 t)\sin\frac{2\pi kt}{T} dt, \tag{30}$$

$$b_k = \frac{2}{T}\int_0^T \left(\alpha\sin\frac{2\pi kt}{T} + \gamma\cos\frac{2\pi t}{T}\sin\frac{2\pi kt}{T} + \sigma\sin\frac{2\pi t}{T}\sin\frac{2\pi kt}{T}\right) dt. \tag{31}$$

Using the trigonometric identities $2\cos x \sin y = \sin(x+y) - \sin(x-y)$ and $2\sin x \sin y = \cos(x-y) - \cos(x+y)$, we get

$$b_k = -\frac{2\alpha}{T}\int_0^T \sin\frac{2\pi kt}{T} dt \tag{32}$$

$$+ \frac{\gamma}{T}\int_0^T \left(\sin\frac{2\pi(k+1)t}{T} - \sin\frac{2\pi(k-1)t}{T}\right) dt$$

$$+ \frac{\sigma}{T}\int_0^T \left(\cos\frac{2\pi(k-1)t}{T} - \cos\frac{2\pi(k+1)t}{T}\right) dt.$$

All three terms will be 0 because we are integrating one or more full cycles of a sine function. Thus, $b_k = 0$ for $k > 1$, as expected.

3.1.1 Comments

It may seem to you that it is unnecessarily repetitive to go through the calculation of each coefficient in so much detail, and if so I apologize. However I think it is very

important that you fully understand these calculations and be very comfortable with them. In fact, I encourage you to repeat these calculations on your own.

These ideas will come up repeatedly when studying both Fourier analysis and communication systems. The basic idea being employed when calculating the Fourier coefficients stems from the observation that if you multiply a signal by a sinusoid of frequency f_0, then you will translate all of the frequency components in that signal both up and down by f_0. For example, if our signal is

$$v_{in}(t) = V\cos 2\pi f_1 t \tag{33}$$

and we multiply this by a cosine wave with frequency f_0, we end up with

$$\begin{aligned} v_{out}(t) &= v_{in}(t)\cos(2\pi f_0 t) \\ &= V\cos(2\pi f_1 t)\cos(2\pi f_0 t) \\ &= V(\cos(2\pi(f_0 + f_1)t) + \cos(2\pi(f_0 - f_1)t)) \end{aligned} \tag{34}$$

Generally we are only interested in one of these two terms, and we use some kind of filtering to eliminate the other. When computing Fourier coefficients, the integration operator acts as a low pass filter eliminating all but the DC component, which would be the $f_0 - f_1$ term if $f_0 = f_1$.

This idea is used in wireless communication systems to translate an information signal such as v_{in} up to the carrier frequency $(f_0 + f_1)$ for transmission on the wireless channel and then translating it back down to an intermediate frequency $(f_0 - f_1)$ where it is easy to process.

3.2   Orthogonal Decomposition

If you review the above example you will see that for each coefficient, if we wanted a_k, the coefficient of the kth harmonic cosine, we multiplied the signal by $\cos(2\pi k f_0 t)$, which translates the component of interest to DC, where it is extracted while discarding all other terms by integrating over exactly one cycle of the fundamental frequency. Similarly, if we are interested in b_k, the coefficient of the kth harmonic sine, we multiply the signal by $\sin(2\pi k f_0 t)$, which translates the component of interest to DC, where it is extracted while discarding all other terms by integrating over exactly one cycle of the fundamental frequency.

This works because $\cos(2\pi k f_0 t)$ and $\sin(2\pi k f_0 t)$ are ***orthogonal functions***, meaning that following orthogonality conditions are satisfied by these functions:

$$\frac{1}{T}\int_0^T \cos(2\pi k f_0 t)\cos(2\pi l f_0 t)dt = 0 \text{ if } k \neq l, \tag{35}$$

$$\frac{1}{T}\int_0^T \cos(2\pi k f_0 t)\cos(2\pi l f_0 t)dt = \begin{cases} 1 & \text{if } k = 0 \\ 1/2 & \text{otherwise} \end{cases} \text{ if } k = l, \tag{36}$$

$$\frac{1}{T}\int_0^T \sin(2\pi k f_0 t)\sin(2\pi l f_0 t)dt = 0 \text{ if } k \neq l, \tag{37}$$

$$\frac{1}{T}\int_0^T \sin(2\pi kf_0 t)\sin(2\pi lf_0 t)dt = \begin{cases} 1 & \text{if } k = 0 \\ 1/2 & \text{otherwise} \end{cases} \quad \text{if } k = l, \text{ and} \tag{38}$$

$$\frac{1}{T}\int_0^T \cos(2\pi kf_0 t)\sin(2\pi lf_0 t)dt = 0 \text{ for all } k, l. \tag{39}$$

Thus, if we multiply a T-periodic signal by either $\cos(2\pi kf_0 t)$ or $\sin(2\pi kf_0 t)$ and integrate over one cycle, we isolate that component in the signal. Contributions from all other components drop out of the result. This is shown in the next section.

Orthogonality implies that if there is a $\cos(2\pi kf_0 t)$ component in a signal, it can only be represented by a_k in the Fourier series. In particular, no other combination of coefficients could be used to represent that component. The same is true for a $\sin(2\pi kf_0 t)$ component and b_k. Thus, orthogonality implies that the Fourier series of a waveform is unique.

3.3 Confirming the Fourier Coefficient Formulas

Assume v is a signal of the form

$$v(t) = \sum_{l=0}^{\infty} a_l \cos\frac{2\pi lt}{T} + b_l \sin\frac{2\pi lt}{T}. \tag{40}$$

To confirm (10), apply it to this signal and show that the result is a_0.

$$a_0 = \frac{1}{T}\int_0^T v(t)dt \tag{41}$$

$$a_0 = \frac{1}{T}\int_0^T \left(\sum_{l=0}^{\infty} a_l \cos\frac{2\pi lt}{T} + b_l \sin\frac{2\pi lt}{T} \right) dt \tag{42}$$

$$a_0 = \frac{1}{T}\int_0^T a_0 dt + \frac{1}{T}\int_0^T \left(\sum_{l=1}^{\infty} a_l \cos\frac{2\pi lt}{T} + b_l \sin\frac{2\pi lt}{T} \right) dt \tag{43}$$

Clearly everything in the second term goes to zero because we are integrating sine and cosine functions over exactly one or more periods. And so,

$$a_0 = \frac{1}{T}\int_0^T a_0 dt + 0 = a_0, \tag{44}$$

which confirms (10).

Now confirm (11) by applying it to v.

$$a_k = \frac{2}{T}\int_0^T v(t)\cos\frac{2\pi kt}{T}dt, \tag{45}$$

$$a_k = \frac{2}{T}\int_0^T \left(\sum_{l=0}^{\infty} a_l \cos\frac{2\pi l t}{T} + b_l \sin\frac{2\pi l t}{T} \right) \cos\frac{2\pi k t}{T} dt, \tag{46}$$

Apply the appropriate trigonometric identities.

$$a_k = \sum_{l=0}^{\infty} \frac{a_l}{T}\int_0^T \left(\cos\frac{2\pi(k+l)t}{T} + \cos\frac{2\pi(k-l)t}{T} \right) dt \tag{47}$$

$$+ \sum_{l=0}^{\infty} \frac{b_l}{T}\int_0^T \left(\sin\frac{2\pi(k+l)t}{T} + \sin\frac{2\pi(k-l)t}{T} \right) dt$$

In the first integral, the first term is always 0 because we are integrating over one or more complete cycles of a cosine wave (because $k > 1$). For similar reasons the second term in the first integral will also equal 0 for all $l \neq k$. The same is true for the second integral, except the second term in the second integral is also 0 when $l = k$ because the argument of the sine function will identically equal 0. Thus, (47) simplifies to

$$a_k = \frac{a_k}{T}\int_0^T \cos(0)dt = \frac{a_k}{T}\int_0^T dt = a_k, \tag{48}$$

which confirms (11).

Finally, we confirm (12) by applying it to v.

$$b_k = \frac{2}{T}\int_0^T v(t)\sin\frac{2\pi k t}{T} dt, \tag{49}$$

$$b_k = \frac{2}{T}\int_0^T \left(\sum_{l=0}^{\infty} a_l \cos\frac{2\pi l t}{T} + b_l \sin\frac{2\pi l t}{T} \right) \sin\frac{2\pi k t}{T} dt. \tag{50}$$

Apply the appropriate trigonometric identities.

$$b_k = \sum_{l=0}^{\infty} \frac{a_l}{T}\int_0^T \left(\sin\frac{4\pi(k+l)t}{T} - \sin\frac{4\pi(k-l)t}{T} \right) dt \tag{51}$$

$$+ \sum_{l=0}^{\infty} \frac{b_l}{T}\int_0^T \left(\cos\frac{4\pi(k+l)t}{T} + \cos\frac{4\pi(k-l)t}{T} \right) dt$$

In the second integral, the first term is always 0 because we are integrating over one or more complete cycles of a cosine wave (because $k > 1$). For similar reasons the second term in the second integral will also equal 0 for all $l \neq k$. The same is true for the first integral, except the second term in the first integral is also 0 when $l = k$ because the argument of the sine function will identically equal 0. Thus, (51) simplifies to

$$b_k = \frac{b_k}{T}\int_0^T \cos(0)dt = \frac{b_k}{T}\int_0^T dt = b_k,\tag{52}$$

which confirms (12).

4 Exponential Form of the Fourier Series

The Fourier series given in (4) is referred to as the **trigonometric form**. It is also referred to as the **single-sided form** because the Fourier coefficients all have non-negative indices (they are all on one side of zero). An alternate, often simpler form is the **exponential form**, also known as the **double-sided form** because the Fourier coefficients have both positive and negative indices. The exponential form uses complex numbers and is notationally simpler because you can use one complex coefficient to play the role of the two coefficients required per harmonic in the trigonometric form.

The exponential form of the Fourier series uses Euler's formula,

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j\sin k\omega_0 t\tag{53}$$

where $j = \sqrt{-1}$. Now, a more general Fourier series is

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.\tag{54}$$

This is more general in that it allows v to be complex, which is often not needed. If we constrain the Fourier coefficients so that

$$c_k = c_{-k}{}^*,\tag{55}$$

then v will be a real function. What this means is that c_k must be the complex conjugate of c_{-k}, or that the real parts of c_k and c_{-k} must be the same, but the imaginary parts must have opposite signs.

The Fourier coefficients are then given by:

$$c_k = \frac{1}{T}\int v(t)e^{-jk\omega_0 t}dt.\tag{56}$$

Assuming a real signal v, the various Fourier coefficients are related in the following manner:

$$a_k = c_k + c_{-k} \text{ for } k = 0, 1, 2, \ldots\tag{57}$$

$$b_k = j(c_k - c_{-k}) \text{ for } k = 1, 2, \ldots\tag{58}$$

and

$$c_k = \begin{cases} (a_k - jb_k)/2 & \text{for } k > 0 \\ a_0 & \text{for } k = 0 \\ (a_{-k} + jb_{-k}) & \text{for } k < 0 \end{cases} \quad . \tag{59}$$

Using c_k to represent the Fourier coefficients is cumbersome in practice because we often are working with several signals at once. The engineering convention is to use lower case letters to represent signals in the time domain and upper case letters to represent signals in the frequency domain. So for example, v might represent a signal as a function of time over all time, and $v(t)$ represents one point on that signal, whereas V represents that signal as a function of frequency (it holds all of the Fourier coefficients) and V_k represents one particular Fourier coefficient. This special relationship between v and V is sometimes denoted using the notation:

$$v \leftrightarrow V \tag{60}$$

which means that V is the sequence of Fourier coefficients for v, and you can convert from one form to the other using the updated versions of (54) and (56),

$$\boxed{v(t) = \sum_{k = -\infty}^{\infty} V_k e^{jk\omega_0 t}} \; \text{— \textbf{\textit{exponential form of the Fourier series.}}} \tag{61}$$

$$\boxed{V_k = \frac{1}{T}\int_0^T v(t) e^{-jk\omega_0 t} dt} \; \text{— \textbf{\textit{exponential form Fourier coefficients.}}} \tag{62}$$

As mentioned previously, a signal in the time domain, such a v, is referred to as a ***waveform***. A signal in the frequency domain, such as V, is referred to as a ***spectrum***.

When applied for all k, (62) converts a waveform into a spectrum. Convenient shorthand for this transformation is

$$V = \mathcal{F}(v) . \tag{63}$$

Similarly, applying (61) over all t converts a spectrum to a waveform and can be conveniently denoted

$$v = \mathcal{F}^{-1}(V) . \tag{64}$$

Thus $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be used to transform the signal back and forth between the two domains, frequency and time. Note that while $\mathcal{F}$ represents a Fourier transform, it is not *the* Fourier Transform (see Section 6).

4.1 Phasors

A signal that takes the form

$$x(t) = Xe^{j\omega_0 t} \tag{65}$$

is referred to being a signal in phasor form, with X being a **phasor**. The name *phasor* is used because of its similarity to *vector*. Phasors, like vectors, have a magnitude and phase.

$$|X| = \sqrt{\Re(X)^2 + \Im(X)^2} \tag{66}$$

$$\angle X = \pm\tan^{-1}\frac{\Im(X)}{\Re(X)} \,\dagger \tag{67}$$

where $\Re(X)$ is the real part of X, and $\Im(X)$ is the imaginary part. Thus, the complex exponential form of the Fourier series is a sum of phasors at harmonic frequencies.

4.2 Pulse Train Example

Define $\Pi(t, \tau, T)$ το be a T-periodic function that has a value of 1 if $|t| < \tau/2$ and 0 otherwise on the interval $-T/2$ to $T/2$. This function produces a pulse train as illustrated in Figure 6. If $t = T/2$, then the pulse train is a square wave.

FIGURE 6   *Pulse train waveform.*
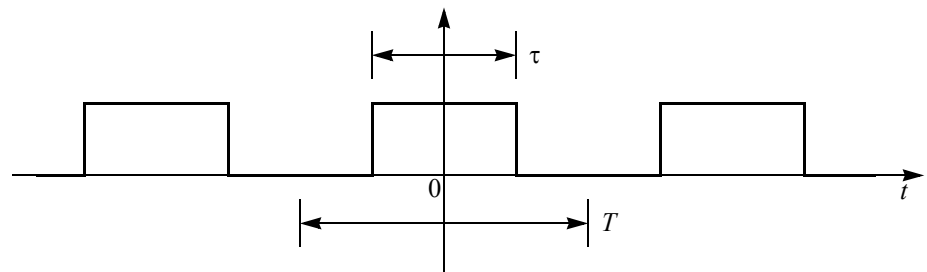
Calculate the Fourier series for

$$v(t) = \Pi(t, \tau, T) \tag{68}$$

The frequency domain representation of this signal is computed with (62) (we have shifted the limits of integration over by a half a cycle, which we are free to do because v is periodic.

$$V_k = \frac{1}{T}\int_{-T/2}^{T/2} v(t)e^{-jk\omega_0 t}dt . \tag{69}$$

$$V_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-jk\omega_0 t}dt . \tag{70}$$

$$V_k = \frac{-1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-\tau/2}^{\tau/2} . \tag{71}$$

---

†. $\pm\tan^{-1}(y/x)$ resolves to two different values, and you must choose the correct one based on the quadrant of (x, y). In programming languages such as C or Python, use the atan2(y, x) function to get the right value.

$$V_k = \frac{-1}{jk\omega_0 T}\left(e^{-\frac{jk\omega_0\tau}{2}} - e^{\frac{jk\omega_0\tau}{2}}\right). \tag{72}$$

$$V_k = \frac{2}{k\omega_0 T}\left(\frac{e^{\frac{jk\omega_0\tau}{2}} - e^{-\frac{jk\omega_0\tau}{2}}}{2j}\right). \tag{73}$$

The term on the right is the complex exponential form of the sine function, and can be derived from Euler's formula.

$$V_k = \frac{2}{k\omega_0 T}\sin\frac{k\omega_0\tau}{2}. \tag{74}$$

This has the form

$$V_k = \frac{\tau}{T}\frac{\sin x}{x} \text{ where } x = \frac{k\omega_0\tau}{2}. \tag{75}$$

The $\sin(x)/x$ function comes up a lot and so it is important to be familiar with it. It is shown in (7). This function is well defined for $x = 0$, where its value is 1, as can be shown using l'Hôpital's rule.

FIGURE 7 *The* $\sin(x)/x$ *function.*
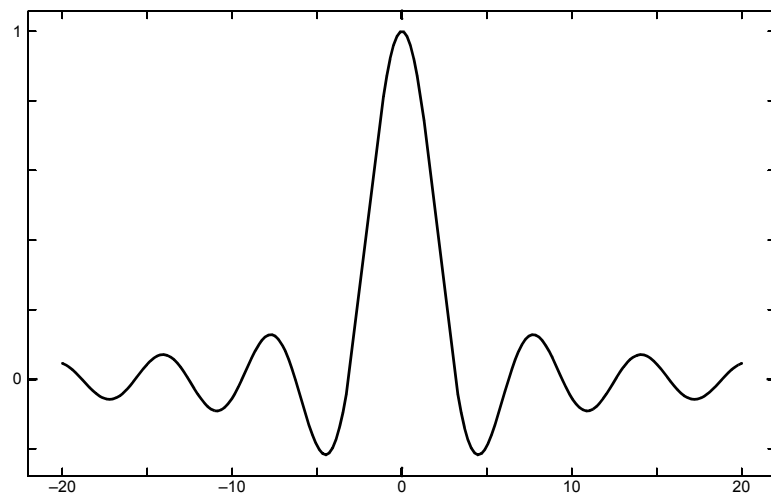
The spectrum V is shown for various values of duty cycle, where the duty cycle is defined as the percentage of the time that v is non-zero. So a 50% duty cycle is one where $\tau = T/2$. Notice as the as the pulse width becomes smaller (the duty cycle decreases) the bandwidth of the spectrum increases.

5 Power and Parseval's Theorem

Electrical energy is the product of voltage and current integrated over time,

FIGURE 8

| 50% Duty Cycle | 25% Duty Cycle | 12.5% Duty Cycle |
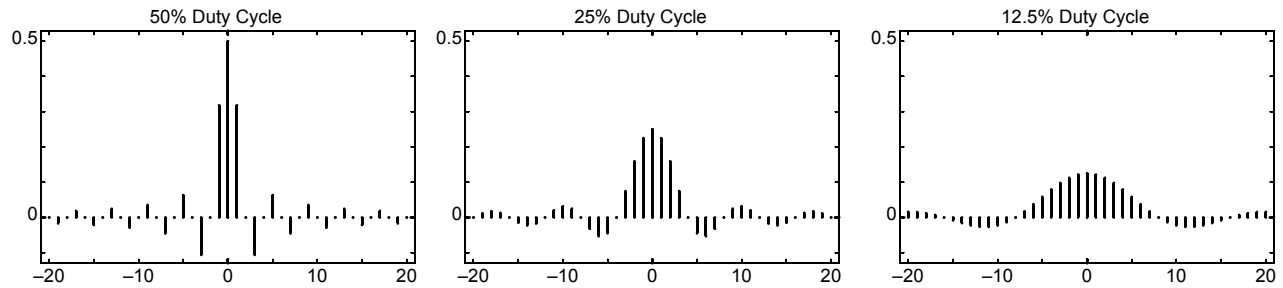
$$E = \int_{-\infty}^{\infty} v(t)i(t)^* dt \,.^{\dagger} \tag{76}$$

Periodic signals persist forever, and the energy associated with them is either zero or infinite, so energy is not a very interesting quantity. More interesting for periodic signals is power, which equals energy per unit time,

$$P = \frac{1}{T}\int_{0}^{T} v(t)i(t)^* dt \tag{77}$$

To compute power we need to know both the voltage and the current. However, it is often interesting to consider power even if you know only one of these quantities, say the voltage v. Knowing voltage is not enough to tell us the current because we do not know the circuit to which the voltage is applied. In these situations, we often just assume that the voltage is applied to a resistor R. This assumption is a good approximation for many common and important situations, and when it is not, knowing what the power would be with a pure resistive load is often a good starting point in predicting the power when the load is not a resistor. This assumption allows us to make statements about power and power-like quantities that are very useful. In this case, the current is $i(t) = v(t)/R$, and so the instantaneous power becomes

$$P(t) = \frac{|v(t)|^2}{R} \,. \tag{78}$$

For simplicity, we further assume that R = 1. While this is clearly artificial, making this assumption allows us to compute a "normalized power", that can be easily scaled to the true power once R is known.

The power of a periodic signal (this abstract normalized power that we just introduced) represented in the time domain is computed using

$$P = \frac{1}{T}\int_{0}^{T} |v(t)|^2 dt \,. \tag{79}$$

The power of a periodic signal represented in the frequency domain is computed using

---

†. Generally our waveforms v and i would be real, in which case you would ignore the complex conjugate operator.

$$P = \sum_{k=-\infty}^{\infty} |V_k|^2 . \tag{80}$$

Parseval's theorem states that these two are the same,

$$\frac{1}{T}\int_0^T |(v(t))|^2 dt = \sum_{k=-\infty}^{\infty} |V_k|^2 , \tag{81}$$

as you would expect.

6 Properties of the Fourier Series

Many useful properties of the Fourier series are presented in this section and summarized in Table 1 on page 22.

Let $\mathcal{F}(x)$ denote a transformation of a T-periodic waveform x into its sequence of Fourier coefficients X by repeated application of (62),

$$\mathcal{F}(x) = X. \tag{82}$$

6.1 Linearity

$\mathcal{F}$ is a linear transformation and so superposition holds. In other words, assume that a and b are simple real numbers, that x and y are T-periodic functions, and that

$$z(t) = ax(t) + by(t) \tag{83}$$

If $\mathcal{F}(x) = X$, $\mathcal{F}(y) = Y$, and $\mathcal{F}(z) = Z$, then

$$Z_k = aX_k + bY_k \text{ for all } k. \tag{84}$$

6.2 Symmetry

6.2.1 Even Waveforms

A waveform x is ***even*** if

$$x(t) = x(-t). \tag{85}$$

If x is also a T-periodic function and $\mathcal{F}(x) = X$, then

$$\Im(X) = 0 \tag{86}$$

for all k.

6.2.2 Odd Waveforms

A waveform x is ***odd*** if

$$x(t) = -x(-t). \tag{87}$$

If x is also a T-periodic function and $\mathcal{F}(x) = X$, then

$$\Re(X) = 0 \tag{88}$$

for all k.

6.2.3 Real Waveforms

If x is a real-valued T-periodic function and $\mathcal{F}(x) = X$, then

$$X_k = X_{-k}{}^* \tag{89}$$

for all k. In other words, X_k equals the complex conjugate of X_{-k}. As a result, the real part of X is even and the imaginary part is odd.

6.3 Time Shift

Let x be a T-periodic function and

$$y(t) = x(t - t_0) \tag{90}$$

for all t. If $\mathcal{F}(x) = X$ and $\mathcal{F}(y) = Y$, then

$$Y_k = e^{-jk\omega_0 t_0} X_k \tag{91}$$

for all k.

6.4 Time Reversal

Let x be a T-periodic function and

$$y(t) = x(-t) \tag{92}$$

for all t. If $\mathcal{F}(x) = X$ and $\mathcal{F}(y) = Y$, then

$$Y_k = X_{-k} \tag{93}$$

for all k.

6.5 Frequency Translation

Let x be a T-periodic function and

$$y(t) = x(t)e^{jl\omega_0 t} \tag{94}$$

for all t. If $\mathcal{F}(x) = X$ and $\mathcal{F}(y) = Y$, then

$$Y_k = X_{k-l} \tag{95}$$

for all k.

Notice that $e^{jl\omega_0 t}$ is complex, so that y would be a complex function even if x is real.

6.6 Modulation

Modulation that results in a real-valued waveform could be accomplished by multiplying either by $\cos(l\omega_0 t)$ or $\sin(l\omega_0 t)$.

Let x be a T-periodic function. If $y(t) = x(t)\cos(l\omega_0 t)$, then

$$Y_k = \frac{X_{k-l} + X_{k+l}}{2}. \tag{96}$$

If $y(t) = x(t)\sin(l\omega_0 t)$, then

$$Y_k = \frac{X_{k-l} - X_{k+l}}{2j}. \tag{97}$$

6.7 Differentiation

Let x be a T-periodic function and

$$y(t) = \frac{dx(t)}{dt} \tag{98}$$

for all t. If $\mathcal{F}(x) = X$ and $\mathcal{F}(y) = Y$, then

$$Y_k = jk\omega_0 X_k \tag{99}$$

for all k.

6.8 Integration

Let x be a T-periodic function and

$$y(t) = \int_{-\infty}^{t} x(t)dt \tag{100}$$

for all t. If $\mathcal{F}(x) = X$ and $\mathcal{F}(y) = Y$, then

$$Y_k = \frac{1}{jk\omega_0} X_k \tag{101}$$

for all k. In this case X_0 must be zero, otherwise the integral would not be a periodic function and its Fourier series would not exist.

6.9 Multiplication

Let x and y be T-periodic functions and let

$$z(t) = x(t)y(t) \tag{102}$$

for all t. If $\mathcal{F}(x) = X$, $\mathcal{F}(y) = Y$, and $\mathcal{F}(z) = Z$, then

$$Z_k = \sum_{l=-\infty}^{\infty} X_l Y_{k-l} \tag{103}$$

for all k. That is to say that multiplication in the time-domain is equivalent to convolution in the frequency domain.

6.10 Convolution

Let x and y be T-periodic functions and let

$$z(t) = \frac{1}{T}\int_0^T x(t-\tau)y(\tau)d\tau = \frac{1}{T}\int_0^T x(\tau)y(t-\tau)d\tau \tag{104}$$

for all t. If $\mathcal{F}(x) = X$, $\mathcal{F}(y) = Y$, and $\mathcal{F}(z) = Z$, then

$$Z_k = X_k Y_k \tag{105}$$

for all k. That is to say that convolution in the time-domain is equivalent to multiplication in the frequency domain.

7 Conclusion

The Fourier series as presented here maps a periodic waveform defined on continuous time into a discrete spectrum. This is not the only possibility. Other forms of Fourier analysis are also in common use. I will only mention them so you are aware of their existence.

Discrete Fourier Transform. The discrete Fourier transform is akin to the Fourier series except that the waveform is a periodic sequence. In other words, the waveform is defined only at discrete evenly spaced points in time. Since there will only be a finite number of points in the sequence before it repeats, the resulting spectrum will only have a finite number of unique harmonics. As such, like the waveform, the spectrum is also periodic. The number of unique Fourier coefficients in the spectrum sequence is exactly the same as the number of unique time values in the waveform sequence.

The Fast Fourier Transform (FFT) is a highly efficient numerical algorithm for computing discrete Fourier transforms.

Fourier Transform. The Fourier transform maps a finite energy waveform into a continuous spectrum (defined for all values of frequencies, not just discrete evenly spaced values as is the case for the Fourier series). In this case, the waveforms are not periodic. Rather the finite energy requirement means the waveforms are finite in extent.

Fourier Analysis. The basic concepts of the various forms of Fourier analysis are all the same. Once you understand the Fourier series, you have most of what you need to understand all forms of Fourier analysis.

7.1 If You Have Questions

If you have questions about what you have just read, feel free to post them on the *Forum* section of *The Designer's Guide Community* website. Do so by going to *www.designers-guide.org/Forum*. For more in depth questions, feel free to contact me in my role as a consultant at *ken@designers-guide.com*.

TABLE 1 *Summary of Fourier series properties.*

Name	Time Domain	Frequency Domain
Transformation	$v(t) = \displaystyle\sum_{k=-\infty}^{\infty} V_k e^{jk\omega_0 t}$	$V_k = \dfrac{1}{T}\displaystyle\int_0^T v(t) e^{-jk\omega_0 t} dt$
Superposition	$ax(t) + by(t)$	$aX_k + bY_k$
Symmetry	x is an even function	$\Im(X_k) = 0$
	x is an odd function	$\Re(X_k) = 0$
	x is a real function	$X_k = X_{-k}{}^*$
Time shift	$y(t) = x(t - t_0)$	$Y_k = e^{-jk\omega_0 t_0} X_k$
Time reversal	$y(t) = x(-t)$	$Y_k = X_{-k}$
Frequency translation	$y(t) = x(t)e^{jl\omega_0 t}$	$Y_k = X_{k-l}$
Modulation	$y(t) = x(t)\cos(l\omega_0 t)$	$Y_k = \dfrac{X_{k-l} + X_{k+l}}{2}$
	$y(t) = x(t)\sin(l\omega_0 t)$	$Y_k = \dfrac{X_{k-l} - X_{k+l}}{2j}$
Differentiation	$y(t) = \dfrac{dx(t)}{dt}$	$Y_k = jk\omega_0 X_k$
Integration	$y(t) = \displaystyle\int_{-\infty}^t x(t)dt$	$Y_k = \dfrac{1}{jk\omega_0}X_k$
Multiplication	$z(t) = x(t)y(t)$	$Z_k = \displaystyle\sum_{l=-\infty}^{\infty} X_l Y_{k-l}$
Convolution	$z(t) = \dfrac{1}{T}\displaystyle\int_0^T x(t-\tau)y(\tau)d\tau$	$Z_k = X_k Y_k$
Power	$P = \dfrac{1}{T}\displaystyle\int_0^T |(v(t))|^2 dt$	$P = \displaystyle\sum_{k=-\infty}^{\infty} |V_k|^2$

Postscript

I would like to thank Sohaib Afridi for pointing out errors in equations (36), (38), and (47) and providing the corrections.

Appendix A Proofs

The following are short, not terribly formal, proofs for the properties described in Section 6.

A.1 Linearity

Prove linearity using superposition.

$$z(t) = ax(t) + by(t) \tag{106}$$

$$Z_k = \frac{1}{T}\int_0^T (ax(t) + by(t))e^{-jk\omega_0 t}dt \tag{107}$$

$$Z_k = \frac{a}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt + \frac{b}{T}\int_0^T y(t)e^{-jk\omega_0 t}dt \tag{108}$$

$$Z_k = aX_k + bY_k \text{ for all } k. \tag{109}$$

A.2 Symmetry

A.2.1 Even Waveforms

$$X_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \tag{110}$$

By Euler's formula, (53),

$$X_k = \frac{1}{T}\int_0^T x(t)(\cos(k\omega_0 t) - j\sin(k\omega_0 t))dt \tag{111}$$

Separating into real and imaginary components and shifting the limits of integration:

$$\Re(X_k) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt, \; \Im(X_k) = \frac{-1}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt \tag{112}$$

Since sin() is an odd function, if x is even, then the imaginary part of X_k ($\Im(X_k)$) must be zero.

A.2.2 Odd Waveforms

From the above, since cos() is an even function, if x is odd, then the real part of X_k ($\Re(X_k)$) must be zero.

A.2.3 Real Waveforms

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}. \tag{113}$$

By Euler's formula, (53),

$$x(t) = \sum_{k=-\infty}^{\infty} X_k(\cos(k\omega_0 t) + j\sin(k\omega_0 t)). \tag{114}$$

Separate X_k into real and imaginary parts,

$$x(t) = \sum_{k=-\infty}^{\infty} (\Re(X_k) + j\Im(X_k))(\cos(k\omega_0 t) + j\sin(k\omega_0 t)). \tag{115}$$

Expand the complex multiplication,

$$x(t) = \sum_{k=-\infty}^{\infty} \Re(X_k)\cos(k\omega_0 t) - \Im(X_k)\sin(k\omega_0 t) \tag{116}$$
$$+ j\sum_{k=-\infty}^{\infty} \Re(X_k)\sin(k\omega_0 t) + \Im(X_k)\cos(k\omega_0 t).$$

Reformulate the summation so index variable k is only positive.

$$x(t) = \Re(X_0) \tag{117}$$
$$+ \sum_{k=1}^{\infty} (\Re(X_k) + \Re(X_{-k}))\cos(k\omega_0 t) - (\Im(X_k) - \Im(X_{-k}))\sin(k\omega_0 t)$$
$$+ j\sum_{k=1}^{\infty} (\Re(X_k) - \Re(X_{-k}))\sin(k\omega_0 t) + (\Im(X_k) + \Im(X_{-k}))\cos(k\omega_0 t)$$

Now, for x to be real the third term must be zero, so

$$\Re(X_k) = \Re(X_{-k}), \ \Im(X_k) = -\Im(X_{-k}), \text{ and} \tag{118}$$

$$X_k = X_{-k}^*. \tag{119}$$

A.3 Time Shift

$$y(t) = x(t - t_0) \tag{120}$$

$$Y_k = \frac{1}{T}\int_0^T x(t - t_0)e^{-jk\omega_0 t}dt \tag{121}$$

Let $\tau = t - t_0$.

$$Y_k = \frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0(\tau + t_0)}d\tau \tag{122}$$

$$Y_k = e^{-jk\omega_0 t_0}\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0\tau}d\tau \tag{123}$$

$$Y_k = e^{-jk\omega_0 t_0}X_k \tag{124}$$

A.4 Time Reversal

$$y(t) = x(-t) \tag{125}$$

$$Y_k = \frac{1}{T}\int_0^T x(-t)e^{-jk\omega_0 t}dt \tag{126}$$

Let $\tau = -t$.

$$Y_k = \frac{1}{T}\int_0^T x(\tau)e^{jk\omega_0\tau}d\tau \tag{127}$$

$$Y_k = X_{-k} \tag{128}$$

A.5 Frequency Translation

$$y(t) = x(t)e^{jl\omega_0 t} \tag{129}$$

$$Y_k = \frac{1}{T}\int_0^T x(t)e^{jl\omega_0 t}e^{-jk\omega_0 t}dt \tag{130}$$

$$Y_k = \frac{1}{T}\int_0^T x(t)e^{-j(k-l)\omega_0 t}dt \tag{131}$$

$$Y_k = X_{k-l} \tag{132}$$

A.6 Modulation

The complex exponential form of the cosine function is given by:

$$\cos(l\omega_0 t) = \frac{e^{jl\omega_0 t} + e^{-jl\omega_0 t}}{2}. \tag{133}$$

Using the frequency translation property (95) twice shows that if
$y(t) = x(t)\cos(l\omega_0 t)$, then

$$Y_k = \frac{X_{k-l} + X_{k+l}}{2}. \tag{134}$$

Similarly, the complex exponential form of the sine function is given by:

$$\sin(l\omega_0 t) = \frac{e^{jl\omega_0 t} - e^{-jl\omega_0 t}}{2j}. \tag{135}$$

Using the frequency translation property (95) twice shows that if $y(t) = x(t)\sin(l\omega_0 t)$, then

$$Y_k = \frac{X_{k-l} - X_{k+l}}{2j}. \tag{136}$$

A.7 Differentiation

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \tag{137}$$

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt} \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \tag{138}$$

$$y(t) = \sum_{k=-\infty}^{\infty} X_k \frac{d}{dt} e^{jk\omega_0 t} \tag{139}$$

$$y(t) = \sum_{k=-\infty}^{\infty} jk\omega_0 X_k e^{jk\omega_0 t} \tag{140}$$

$$Y_k = jk\omega_0 X_k \tag{141}$$

A.8 Integration

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \tag{142}$$

$$y(t) = \int_{-\infty}^{t} x(t)dt = \int_{-\infty}^{t} \left(\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \right) dt \tag{143}$$

$$y(t) = \sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{t} e^{jk\omega_0 t} dt \tag{144}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{X_k}{jk\omega_0} e^{jk\omega_0 t} \tag{145}$$

$$Y_k = \frac{X_k}{jk\omega_0} \tag{146}$$

A.9 Multiplication

$$z(t) = x(t)y(t) \tag{147}$$

$$Z_k = \frac{1}{T}\int_0^T x(t)y(t)e^{-jk\omega_0 t}dt \tag{148}$$

$$Z_k = \frac{1}{T}\int_0^T \left(\sum_{l=-\infty}^{\infty} X_l e^{jl\omega_0 t} \right) y(t)e^{-jk\omega_0 t}dt \tag{149}$$

$$Z_k = \sum_{l=-\infty}^{\infty} \frac{X_l}{T}\int_0^T y(t)e^{-j(k-l)\omega_0 t}dt \tag{150}$$

$$Z_k = \sum_{l=-\infty}^{\infty} X_l Y_{k-l} \tag{151}$$

A.10 Convolution

$$z(t) = \frac{1}{T}\int_0^T x(t-\tau)y(\tau)d\tau . \tag{152}$$

$$Z_k = \frac{1}{T}\int_0^T \left(\frac{1}{T}\int_0^T x(t-\tau)y(\tau)d\tau \right) e^{-jk\omega_0 t}dt . \tag{153}$$

Swap the order of the integrations,

$$Z_k = \frac{1}{T}\int_0^T y(\tau)\left(\frac{1}{T}\int_0^T x(t-\tau)e^{-jk\omega_0 t}dt \right)d\tau . \tag{154}$$

Let $h = t - \tau$, then $dh = dt$,

$$Z_k = \frac{1}{T}\int_0^T y(\tau)\left(\frac{1}{T}\int_0^T x(h)e^{-jk\omega_0(h+\tau)}dh \right)d\tau . \tag{155}$$

Finally, separate the integrations,

$$Z_k = \left(\frac{1}{T}\int_0^T y(\tau)e^{-jk\omega_0 \tau}d\tau \right)\left(\frac{1}{T}\int_0^T x(h)e^{-jk\omega_0 h}dh \right), \tag{156}$$

$$Z_k = Y_k X_k . \tag{157}$$

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