Quantum Spin Hall Effect

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Overview

Haldane's graphene model

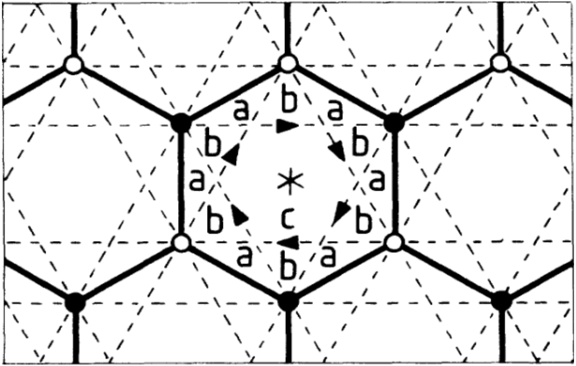

Figure: The honeycomb model showing nearest neighbor hopping (solid lines) and second neighbor hopping (dash lines), two sublatices are charaterized by open and soilid points.

Hamiltonian

In 1988, Duncan Haldane proposed the scenario for the quantum Hall state which absence the magnetic field. In his paper, he add a periodic staggered local magnetic-flux density **B(r)** in the $\hat{z}$ direction normal to the 2D plane, but with *zerototalflux* through the unit cell.

$$
\begin{aligned}
H = & [-t \sum_{\langle i,j \rangle} a_i^\dagger b_j + \frac{M}{2} \sum_{i=1} (a_i^\dagger a_i - b_i^\dagger b_i)] + h.c. \\
& + t_2 \sum_{\langle\langle i,j \rangle\rangle} [e^{i\varphi} a_i^\dagger a_j + e^{-i\varphi} b_i^\dagger b_j]
\end{aligned}
$$

where the t and t_2 are hopping amplitude, the φ is the phase caused by the total fluxes threading through the second hopping. The third term means that second neighbor hopping term,

the conductance of this system

The conclusion of this paper is, while the system is in presence of magnetic flux, the Chern number is not zero which shows the topological property which same as the quatun Hall effect system. Haldane calculate the Hall conductane when system break the time reversal symmetry which is

$$\sigma_H = \frac{\partial \rho}{\partial B_z}\bigg|_{\mu} = \frac{\nu e^2}{h}\bigg|_{\nu=1}$$

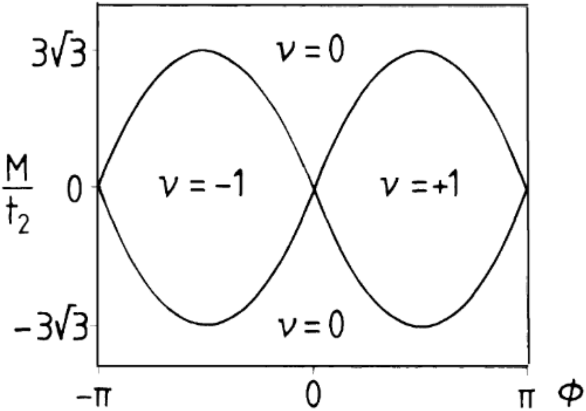

Figure: By tuning the parameter in the Hamiltonian the system would show the zero-field quatum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu \frac{e^2}{h}$)

The proposal of QSHE

In 2005, Kane and Mele proposed the idea which considers the strong spin orbital interaction in the sigle layer graphene. They found that spin and charge current can be transported in the gapless edge states. The Hamiltonian in this model whcih can be written as

$$\mathcal{H}_0 = -i\hbar v_F \psi^\dagger(\sigma_x \tau_z \partial_x + \sigma_y \partial_y)\psi.$$

The SO interaction allows for a new terms which doesn't break the time reversal symmetry.

$$\mathcal{H}_{SO} = \Delta_{so}\psi^\dagger \sigma_z \tau_z s_z \psi.$$

Here s_z is Pauli matrix representing the electron's spin,which respects all of the symmetry of graphene
$\sigma_z = \pm 1$ describing states on the $A(B)$ sublattice and $\tau_z = \pm 1$ describing states at the $K(K')$ points.

They consider the Rashba term which the mirror symmetry is broken.

$$\mathcal{H}_R = \lambda_R \psi^\dagger (\sigma_x \tau_z s_y - \sigma_y s_x)\psi.$$

If Rashba term is zero, there leads to an energy gap with $E(\mathrm{q}) = \pm\sqrt{(\hbar \mathrm{v_F q})^2 + \Delta_{\mathrm{so}}^2}$. For $0 < \lambda_R < \Delta_{so}$ the energy gap $2(\Delta_{so} - \lambda_R)$ remains finite. For $\lambda_R > \Delta_{so}$ the gap closes, and the electronic structure is that of a zero gap semiconductor with quadradically dispersing bands.

- $\sigma_z \tau_z s_z$ is different from the gap that would be generated by the staggered sublattice potentials, σ_z or $\sigma_z s_z$.
- The spin dependent Hamiltonian which violet the time reversal symmetry are equivalent the Haldane's model with spinless electrons which periodic magectic field with no net flux.
- Spin would gives rise a gap where has a opposite signs at K and K' points.

In this system we can computed the quantized Hall conductance which can be $\sigma_{xy} = \pm e^2/h$. Since the SO interaction would cause the opposite sign of gaps for opposite spins which induce the opposite current for the opposite spins. Spin current is $\mathrm{J}_s = (\hbar/2e)(\mathrm{J}_\uparrow - \mathrm{J}_\downarrow)$ characterized by a quantized spin Hall conductivity

$$\sigma^s_{xy} = \frac{e}{2\pi}.$$

Band structure after adding Rashba term

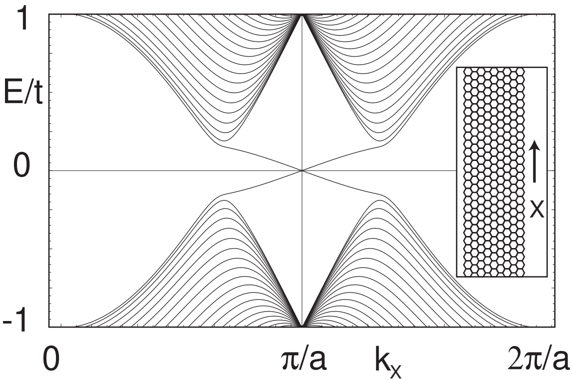

Figure: In this figure we would see that there is crossing at π/a which is robust protected by the time revesal symmetry follows the Kramer theorem. We can image that the QSHE can be caused by two copies of Haldane's model which preserves the time reveseral symmetry which is broken in Haldane's.

The QSHE of Bernevig and Zhang

Bernevig and Zhang proposed that the SO is not easy to realized in term of electric field, but there is another way which the shear strain gradients can play a similar role.
For the purpose, we use replace the electric field by using the shear strain which is,

$$\epsilon_{xy} \leftrightarrow E_z; \quad \epsilon_{xz} \leftrightarrow E_y; \quad \epsilon_{yz} \leftrightarrow E_x$$

After this replacement our Hamiltonian can be written as,

$$H = \frac{p^2}{2m} + Btr\epsilon + \frac{1}{2}\frac{C_3}{\hbar}[(\epsilon_{xy}p_y - \epsilon_{xz}p_z)\sigma_x \\ +(\epsilon_{zy}p_z - \epsilon_{xy}p_x)\sigma_y + (\epsilon_{zx}p_x - \epsilon_{yz}p_y)\sigma_z]$$

For GaAs, the constant $\frac{C_3}{\hbar} = 8 \times 10^5 m/s$.

Bervenig and Zhang consider that system in quantum well in xy plane which is parabolic and Hamiltonian with the strain fonfiguration can becomes:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}\frac{C_3}{\hbar} g(yp_x - xp_y)\sigma_z + D(x^2 + y^2)$$

Then make the change of variable, our Hamiltonian can be $H = (1/2m)(\vec{p} - e\vec{A}\sigma_z)^2$ with $\vec{A} = (mC_3 g/2\hbar e)(y, -x, 0)$. This Hamiltonian is equivalent the charge in the uniform magnetic field which two different spin experience the opposite directions of magnetic field.

Hamiltonian

In this Hamiltonian the s_z is a good quantum number, therefore our Hamiltonian can be written as:

$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix}$$

$$H_{\downarrow,\uparrow} = \sqrt{\frac{D}{2m}}[p_x^2 + p_y^2 + x^2 + y^2 \pm R(xp_y - yp_x)]$$

Then choosing the $z = x + iy$, we obtain two sets of raising and lowering operators:

$$a = \partial_{z^\star} + \frac{z}{2}, \quad a^\dagger = -\partial_z + \frac{z^\star}{2}$$
$$b = \partial_z + \frac{z^\star}{2}, \quad b^\dagger = -\partial_{z^\star} + \frac{z}{2}$$

After introducing the raising and lowering operators, our Hamiltonain can be:

$$H_{\downarrow,\uparrow} = 2\sqrt{\frac{D}{2m}}\left[(1 \mp \frac{R}{2})aa^\dagger + (1 \pm \frac{R}{2})bb^\dagger + 1\right]$$

The eigenstates of this system are harmonic oscillators $|m,n\rangle = (a^\dagger)^m (b^\dagger)^n |0,0\rangle$ of energy is

$$E_{m,n}^{\downarrow,\uparrow} = \frac{1}{2}\sqrt{\frac{D}{2m}}\left[(1 \mp \frac{R}{2})m + (1 \pm \frac{R}{2})n + 1\right]$$

Following discussion we focus on $R = 2$ ($R = \frac{1}{2}\frac{C_3}{\hbar}\sqrt{\frac{2m}{D}}g$) where there is no addtional static potential within Landau level

- For the spin up electron, the vicinity of $R \approx 2$ is characterized by the Hamiltonian $H_{\uparrow} = (1/2)(C_3/\hbar)g(2aa^{\dagger} + 1)$ with the LLL wave function $\phi_n^{\uparrow}(z) = \frac{z^n}{\sqrt{\pi n!}} \exp\left(\frac{-zz^{\star}}{2}\right)$. These up spin electrons are the chiral, and their charge conductance is quantized in units of e^2/h.
- The spin down is same as spin up just replce $aa^{\dagger}$ by $bb^{\dagger}$. These electrons are also chiral, but conductance is opposite of sign of spin up one.

Figure describes by Hamiltonian

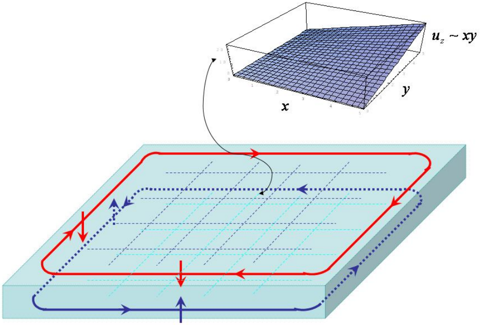

Figure: Spin up and down electrons have opposite chirality as they feel the opposite spin-orbit coupling force. Total charge conductance vanishes but spin conductance is quantized. The inset shows the lattice displacement leading to the strain configuration.

In the previous slide's figure the totoal charge conductance is zero for the whole system. But, the time reversal smmetry reserves the direction of the "effective" magnetic, and interchanges the laers at the same time. This means that the spin Hall conductance is remain finite which is quantized in units of $2\frac{e^2}{h}\frac{\hbar}{2e} = 2\frac{e}{4\pi}$.

Realization of the strain gradients

- The strain tensor is related to displacement of lattice atoms from ther equilibrium position u_i in the familiar way $\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, as we know that the strain gradients should be $\epsilon_{zx} = gy$ and $\epsilon_{yx} = gx$ in this model. If we want to satify the condition that we should choose oue displacement should have the form which is $\vec{u} = (0, 0, 2gxy)$.
- This can be realize by pulverizing GaAs on the a subtrate in MBE at the rate which is a function of the position which vary as $xy \sim r^2 \sin(2\phi)$, where r is the distance from one corners of sample.

- With the different strain architectures, we can careate the Landua gauge Hamiltonian and indeed other gauges.
- Landau gauge can be create by growing the quantum well in the [110] direcion. The spin orbit part Hamiltonian can be $\frac{C_3}{\hbar}\epsilon_{xy}(p_x\sigma_y - p_y\sigma_x)$. If make some coordinate transformation then our Hamiltonian could be written as:

$$H = \frac{p^2}{2m} + \frac{C_3}{\hbar} g y' p_{x'} \sigma_{z'} + D y^2$$

  This kind of method is easily realized by experiment.
- In conclusion they created the effective quantum spin Hall effect Hamiltonian by suing the gradient of the strain field, rather than the magnetic field, which does not violet the time reversal symmetry.