Quantum Spin Hall Effect

Chia-Wei Wei

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Abstract

The quantum Hall effect is a state with some interesting properties that the resistivity is quantized which is in unit of $\frac{e}{2\pi}$ by applying the external magnetic field. The spin Hall effect is an effect that intrinsic spin of electron would be split which also caused by external magnetic field. But, for quantum spin Hall effect it is contrast to the spin Hall effect with zero charge Hall conductance that the spin Hall conductance is quantized that $\sigma_{xy}^{spin} = 2$ in unit of $\frac{e}{4\pi}$ with the absence of the external magnetic field. The first proposal of existence of quantum spin Hall state[**?**] is developed be Charles Kane and Gene Mele who observe the spin up and down exhibits the chiral and anti-chiral integer quantum Hall effect in model of graphene. Then, B. Bernevig and S.C. Zhang[**?**] give the model for the quantum spin Hall effect. Here we want to show that how quantum spin Hall effect be constructed in their paper, the arxiv number of Bernevig and Zhang is arXiv:cond-mat/0504147.

Introduction

The Bloch Hamiltonian of graphene is a 2×2 matrix which can be expressed as

$$\mathcal{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and $\mathbf{h}(\mathbf{k}) = (h_x(\mathbf{k}), h_y(\mathbf{k}), 0)$. The combination of inversion ($\mathcal{P}$) and time reversal ($\mathcal{T}$) symmetry requires $h_z(\mathbf{k}) = 0$ because $\mathcal{P}$ takes $h_z(\mathbf{k})$ to $-h_z(-\mathbf{k})$, while $\mathcal{T}$ takes $h_z(\mathbf{k})$ to $+h_z(-\mathbf{k})$.

In 1988, Duncan Haldane[**?**] proposed the scenario for the quantum Hall state which absence the magnetic field. In his paper, he add a periodic staggered local magnetic-flux density $\boldsymbol{B}(\mathbf{r})$ in the $\hat{\mathbf{z}}$ direction normal to the 2D plane, but with *zero total flux* through the unit cell. The staggerd magnetic field break the time-reversal invariant, but there is no net magnetic field in this system.

$$H =[-t\sum_{\langle i,j\rangle} a_i^\dagger b_j + \frac{M}{2}\sum_{i=1}(a_i^\dagger a_i - b_i^\dagger b_i)] + h.c. + t_2 \sum_{\langle\langle i,j\rangle\rangle}[e^{i\varphi}a_i^\dagger a_j + e^{-i\varphi}b_i^\dagger b_j]$$

where the t and t_2 are hopping amplitude, the φ is the phase caused by the total fluxes threading through the second hopping. The third term means that second neighbor

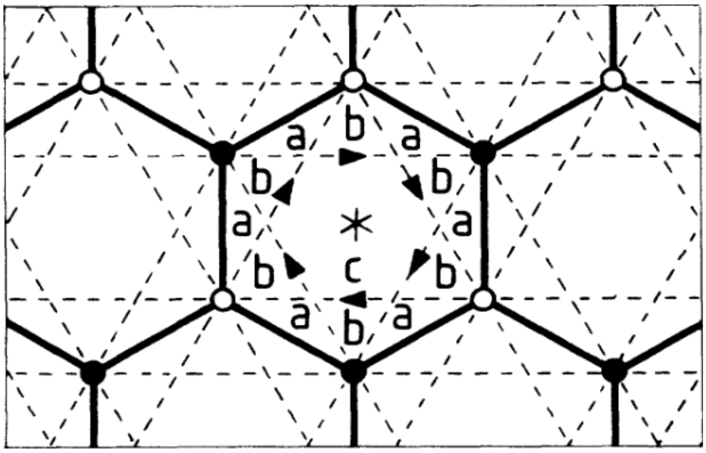

Figure 1: The honeycomb model showing nearest neighbor hopping (solid lines) and second neighbor hopping (dash lines), two sublatices are charaterized by open and soilid points.

hopping term, In his paper we will see that there is an additional phase which totol flux enclose the the second neighbor hopping path would also break the time-reversal symmetry. The coefficient M in second term represents the on-site chemical potential which break the inversion symmetry in the system. If we turn on the M term, we can find the gap of Dirac cone is opend. The Cherm number that we calculate is zero. But, if we consider the magnetic field is turn on, then we find there is emergent topological property which Hall conductance is quantized. In other word, there is non-zero Cherm number appear in this condition. From this point of view, we find that different symmetry breaking play the different role which open the band gap and generate the different topological results. The conclusion of this paper is, while the system is in presence of magnetic flux, the Chern number is not zero which shows the topological property which same as the quatun Hall effect system. Haldane calculate the Hall conductane when system break the time reversal symmetry which is

$$\sigma_H = \frac{\partial \rho}{\partial B_z}\bigg|_{\mu} = \frac{\nu e^2}{h}\bigg|_{\nu=1}$$

Haldane proposed the first quantum Hall system without adding the global magnetic field,that's why quantum anomaly Hall effect be proposed. After the several years, in 2005, Kane and Mele considerthe spin-orbital coupling base on the Haldan's model. They also found that there also open the gap at the Dirac point which posses the same non-trivial topological electric structure. But this kind of topological isulator preserve the time reversal symmetry which emerge the Quantum spin Hall effect rather than the Quantum Hall effect.

The effective Hamiltonian of graphene model at the Dirac points which is

$$\mathcal{H}_0 = -i\hbar v_F \psi^\dagger (\sigma_x \tau_z \partial_x + \sigma_y \partial_y)\psi.$$

Here $\vec{\sigma}$ and $\vec{\tau}$ are Pauli matrices with $\sigma_z = \pm 1$ describing states on the $A(B)$ sublattice and $\tau_z = \pm 1$ describing states at the $K(K')$ points. As we know that if we want to open the gap at the Dirac point, we should add the mass term in the Hamiltonian.

In their paper, they stated that we can allow the new term which preserves the time reversal invariant in this system. The spin-orbital interaction is allowed which can be described as

$$\mathcal{H}_{SO} = \Delta_{so}\psi^\dagger \sigma_z \tau_z s_z \psi.$$

where the s_z is Pauli matrix which represents the electron's spin. They also introduced Rashbe term which break the mirror symmetry

$$\mathcal{H}_R = \lambda_R \psi^\dagger (\sigma_x \tau_z s_y - \sigma_y s_x)\psi.$$

For $\lambda_R = 0$, Δ_{so} there would be opened the gap at the Dirac point. For $0 < \lambda_R < \Delta_{so}$, the energy remain the fintine until the λ_R is graeter than Δ_{so} whcih energy gap would be close with quafrafically dispersing bands.

Therefore, Kane and Mele found that if system remains the time reversal symmetry there would emerges the new topological property in this system. The effect of spin-orbital interaction is distinct from the ordinary effect caused by the mass term (σ_z or $\sigma_z s_z$). If we consider the $\sigma_z \tau_z s_z$, we will find that is produce the gaps with the *oppositesigns* at the K and K' points. This kind of special property prevents that we adiabatically smooth changing from the state generated by σ_z to the state of $\sigma_z \tau_z s_z$. It means that if we want to change the phase we should close the the gap between the process. If we consider one spin for $s_z = +1$ or $s_z = -1$, which violet the time reversal symmetry, is equivalent to the Haldane's model introdcing the periodic magenetic field with no net flux. If we calculate the Hall conductance, the result is same as the Haldane's model. So, this system can be seen as two copies of the Haldane's model but the total Hall conductance is zero for two opposite sign Hall conductance. Since the spin accumulate in the opposite gaps for the opposite spin, so electric field will induce the spin current which is $\mathbf{J}_s = (\hbar/2e)(\mathbf{J}_\uparrow - \mathbf{J}_\downarrow)$ characterized by a quantized spin Hall conductivity

$$\sigma^s_{xy} = \frac{e}{2\pi}.$$

As they stated there is come novel phase in this system, if we consider the Rashba term to break the mirror symmetry we would find the interesting property in the band structure.

The figure **??** shows that at k_x at π/a there is a degeneracy. Kane and Mele had proved that this kind of degeneracy is protected by the time reversal symmetry which follows the Kramer theorem. It stated that if system is in time reversal symmetry with half-integer total spin the ground state should be at least twofold degeneracy. This is a novel state of the matter which is distinct from the quatum Hall state.

Based on the QSHE model of Kane and Mele, Bernevig and Zhang proposed the another model that realized the QSHE which replaces the electric field in the spin-orbiral term by the stress strain tensor. However, this kinds of anology works in this model and they also explained that how to realize experimentally. In the QSHE system, we want to realized the spin-orbiral term which satisfies that $E\sigma_z(xp_y - yp_x)$, we should consider the particel confined in a two dimensional xy plane, and the electric

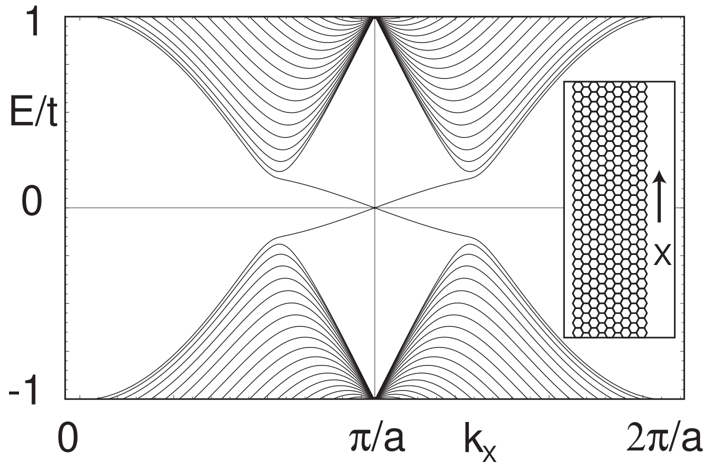

Figure 2: In this figure we would see that there is crossing at π/a which is robust protected by the time revesal symmetry follows the Kramer theorem.

field direction should in xy plane as well, the direction spin lies on the z direction. Besides, electric field is not a constant rather than varies with the xy plane. But, we know that this is difficult to realize experimentally. So, the scenario of stress strian tensor plays the important role in the Bernevig and Zhang's model. The following section, I will show that how deos it do with the strain tensor.

Method

In their paper, they use GaAs as the matrial to realize QSHE. They stated that GaAs which is Zinc-blende semiconductors has the point-group symmetry T_d. According to the this symmetry, the symmetry tensor tensor is $(xy + yx$, etc.$)$ transform in the same way as vector. The shear strain is symmetry tensor $\epsilon_{ij} = \epsilon_{ji}$ which is equivalent to the electric field term in the Hamiltonian:

$$\epsilon_{xy} \leftrightarrow E_z; \quad \epsilon_{xz} \leftrightarrow E_y; \quad \epsilon_{yz} \leftrightarrow E_x$$

In this way, we can replace the electric field by the shear strain, the Hamiltonian can be written as

$$H = \tfrac{p^2}{2m} + B\,\mathrm{Tr}\,\epsilon + \tfrac{1}{2}\tfrac{C_3}{\hbar}[(\epsilon_{xy}p_y - \epsilon_{xz}p_z)\sigma_x + \quad +(\epsilon_{zy}p_z - \epsilon_{xy}p_x)\sigma_y + (\epsilon_{zx}p_x - \epsilon_{yz}p_y)\sigma_z]$$

Where $\frac{C_3}{\hbar} = 8 \times 10^5 m/s$ for GaAs.

From this point of view, we successfully construct the *effective* Hamiltonian for QSHE. They presume that the $\mathrm{Tr}\,\epsilon$ is zero which would effect the physic of the Hamiltonian. Then, they assume that particles are confined in the xy plane which means that p_z is zero, and they also assume that $\epsilon_{xy} = 0$, $\epsilon_{xz}(\leftrightarrow E_y) = gy$ and $\epsilon_{yz}(\leftrightarrow E_x) = gx$. With this strain configuration they add the quatum well which is parabolic, the Hamiltonian which is described above can be expressed as

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}\frac{C_3}{\hbar}g(yp_x - xp_y)\sigma_z + D(x^2 + y^2)$$

They make the change of the variables that $x \to (2mD)^{-1/4}x$, $y \to (2mD)^{-1/4}y$ and $R = \frac{1}{2}\frac{C_3}{\hbar}\sqrt{\frac{2m}{D}}g$. After the change of the variabels and let $R = 2$ or $D = D_0 \equiv \frac{2mgC_3^{@}}{16\hbar^2}$ the Hamiltonian would become

$$H = \frac{1}{2m}(\vec{p} - e\vec{A}\sigma_z)^2$$

with $\vec{A} = \frac{mC_3g}{2\hbar e}(y, -x, 0)$.

The Hamiltonian above which is equivalent to the electon in the magnetic field but it is charaterize by the spin in the verctor potential term. Therefore, we can image that the particle with different spin infleuce by the different direction magnetic field. The Hamiltonian in matrix form that can be expressed as

$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix}$$

$$H_{\downarrow,\uparrow} = \sqrt{\frac{D}{2m}}[p_x^2 + p_y^2 + x^2 + y^2 \pm R(xp_y - yp_x)]$$

Then the next step, they construct the raising operator and lowering operator which is simple harmonic oscillator in xy plane. Then, choosing $z = x + iy$ we have operators for two different spins

$$\begin{aligned} a = \partial_{z^\star} + \tfrac{z}{2}, \quad a^\dagger &= -\partial_z + \tfrac{z^\star}{2} \\ b = \partial_z + \tfrac{z^\star}{2}, \quad b^\dagger &= -\partial_{z^\star} + \tfrac{z}{2} \end{aligned}$$

in terms of which the Hamiltonian decouples

$$H_{\downarrow,\uparrow} = 2\sqrt{\frac{D}{2m}}\left[(1\frac{R}{2})aa^\dagger + (1 \pm \frac{R}{2})bb^\dagger + 1\right]$$

The eigenvalues of the system can be expressed as

$$E_{m,n}^{\downarrow,\uparrow} = \frac{1}{2}\sqrt{\frac{D}{2m}}\left[(1 \mp \frac{R}{2})m + (1 \pm \frac{R}{2})n + 1\right]$$

where $aa^\dagger = m$ and $bb^\dagger = n$. In this model, they choose $R \approx 2$ for which can reduce the Hamiltonian to the oparticle in the quantum Hall state with one spin. For spin up, Hamiltionian can be dexcribed by $H_\uparrow = \frac{1}{2}\frac{C_3}{\hbar}g(2aa^\dagger + 1)$ with charge Hall conductance which is e^2/h. For spin down, the Hamiltonian is same as spin up which just replace operator $aa^\dagger$ by $bb^\dagger$, but it carries charge Hall conductance with minus sign $-e^2/h$. This model can be regarded as two copies Haldane's model with opposite charge Hall conductance, the total charge Hall conductance is zero, but two spins experience opposite direction of *effective* magnetic field, so we get quantized spin conductance with $2(e^2/h)(\hbar/2e) = 2(e/4\pi)$ which is same as QSHE what Kane and Mele derived.

Bernevig and Zhang construct the QSHE which replaces the electric field by the stress strain. In the figrue **??** shows that the system they proposed is equivalent to a bilayer system; one layer have a spin up electrons in the presence of a down-magnetic

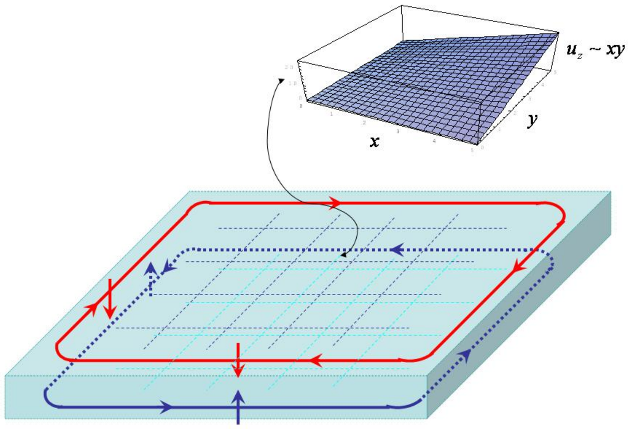

Figure 3: Spin up and down electrons have opposite chirality as they feel the opposite spin-orbit coupling force. Total charge conductance vanishes but spin conductance is quantized. The inset shows the lattice displacement leading to the strain configuration.

field, the other layer have spin downs electrons in presence of a up-magnetic field. Because of the time reversal invariant, it causes the magnetic is alternative in different layer.

Then, they discuss how to realized the stress strain tensor form in the Hamiltonian proposed in their paper. The strain tensor is related to the diplacement of the lattice atoms from their equilibrium position u_i which can be expressed as $\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$. They choose the equlibrium positions of the form $\vec{u} = (0, 0, 2gxy)$, then substitutes it into the expression above that we can get the strain configuration which is same as the form what they proposed for QSHE($\epsilon_{zx} = gy$ and $\epsilon_{yx} = gx$). The above discussion can be applied on the GaAs that we pulverize it on a substate in MBE at a rate which is funciton of the position. The pulverization rate should vary as xy which is described in the inset of the fiugure **??**. In this way, they can construct any kinds of gauge in the magnetic-field language in terms of the different strain configurations. For Landau gauge, they give a method to realize which is growing the quantum well in the [110] direction. The strain what they created for Landau gauge is $\epsilon_{xy} = \frac{1}{4}S_{44}T$, and $\epsilon_{xz} = \epsilon_{yz} = 0$ where T is the impurity concentration, S_{44} is a material constant and x, y, z are the cubic axes. The spin-orbit part of the Hamiltonian is now $\frac{C_3}{\hbar}\epsilon_{xy}(p_x\sigma_y - p_y\sigma_x)$. Since they growth in the [110] direction, they transform to the right coordinate and let $\epsilon_{xy} = gy'$, the Hamiltonian after the transformation would become

$$H = \frac{p^2}{2m} + \frac{C_3}{\hbar} gy' p_{x'}\sigma_{z'} + Dy^2$$

This is Landau gauge Hamiltonian that they add the confining term in this Hamiltonian. Therefore, we can construct any kind of physical gauge what we want to research by this method.

Concolusion

In this term paper, we introduce Haldane's model which give rise to the quantum Hall effect with periodic magnetic field and total zero flux generating in this model. Haldane's find that it can use this model to describe the quantum Hall state. Then, Kane and Mele proposed the model which introduce the spin that which doesn't destroy the time reversal symmetry in the graphene. They found that this is another kind of topological phase which is different from the quantum Hall state with time reversal symmetry protecting. After Kane and Mele proposed the QSHE model, Bernevig and Zhang give a idea that they replace the electric field in spin orbital coupling term by stress strain tensor, they argue that how to use this method to derived the same results that what Kane and Mele got in QSHE. They also give the experimental method to realize this method in lab, they proposed that this method can create any kind of gauge that we want if we can find the corresponding stress strain tensor.