15.05.2025 - Homework 6

A

$A\,X = b$

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \\ 15 \\ 3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 + 2x_4 = 3 \\ 2x_2 + x_3 = 4 \\ x_1 + 3x_2 + 4x_3 + 4x_4 = 15 \\ 2x_2 + x_3 - x_4 = 3 \end{cases}$$

(1)

$$\tilde{A} = \left(\begin{array}{cccc|c} \boxed{1} & -1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 4 \\ 1 & 3 & 4 & 4 & 15 \\ 0 & 2 & 1 & -1 & 3 \end{array}\right) \xrightarrow{L_3 - L_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 4 \\ 0 & \boxed{4} & 3 & 2 & 12 \\ 0 & 2 & 1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{L_2 \leftrightarrow L_3} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 3 \\ 0 & \boxed{4} & 3 & 2 & 12 \\ 0 & 2 & 1 & 0 & 4 \\ 0 & 2 & 1 & -1 & 3 \end{array}\right) \xrightarrow[L_4 - \frac{L_2}{2}]{L_3 - \frac{L_2}{2}} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 3 \\ 0 & 4 & 3 & 2 & 12 \\ 0 & 0 & \boxed{-\frac{1}{2}} & -1 & -2 \\ 0 & 0 & -\frac{1}{2} & -2 & -3 \end{array}\right)$$

$$\xrightarrow{L_4 - L_3} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 3 \\ 0 & 4 & 3 & 2 & 12 \\ 0 & 0 & -\frac{1}{2} & -1 & -2 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right) \Rightarrow \begin{cases} x_1 - x_2 + x_3 + 2x_4 = 3 \\ 4x_2 + 3x_3 + 2x_4 = 12 \\ -\frac{1}{2}x_3 - x_4 = -2 \\ -x_4 = -1 \end{cases}$$

$\boxed{x_4 = 1} \Rightarrow -\frac{1}{2}x_3 = -1 \Rightarrow \boxed{x_3 = 2}$

$4x_2 = 12 - 2 - 6 \Rightarrow x_2 = \frac{4}{4} \Rightarrow \boxed{x_2 = 1}$

$x_1 = 3 + 1 - 2 - 2 \Rightarrow \boxed{x_1 = 0}$

$$\Rightarrow \boxed{\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 2 \\ x_4 = 1 \end{cases}}$$

② $x^{(0)} = (0\ 0\ 0\ 0\ 0)^T$

$$\begin{cases} x_1 - x_2 + x_3 + 2x_4 = 3 \\ 2x_2 + x_3 = 4 \\ x_1 + 3x_2 + 4x_3 + 4x_4 = 15 \\ 2x_2 + x_3 - x_4 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 3 + x_2 - x_3 - 2x_4 \\ x_2 = \frac{4 - x_3}{2} \\ x_3 = \frac{15 - x_1 - 3x_2 - 4x_4}{4} \\ x_4 = -3 + 2x_2 + x_3 \end{cases}$$

$$\begin{cases} x_1^{(1)} = 3 + x_2^{(0)} - x_3^{(0)} - 2x_4^{(0)} = 3 + 0 - 0 - 0 = 3 \\ x_2^{(1)} = \frac{4 - x_3^{(0)}}{2} = \frac{4-0}{2} = 2 \\ x_3^{(1)} = \frac{15 - x_1^{(0)} - 3x_2^{(0)} - 4x_4^{(0)}}{4} = \frac{15-0-0-0}{4} = \frac{15}{4} \\ x_4^{(1)} = -3 + 2x_2^{(0)} + x_3^{(0)} = -3 + 0 + 0 = -3 \end{cases}$$

$$\begin{cases} x_1^{(2)} = 3 + x_2^{(1)} - x_3^{(1)} - 2x_4^{(1)} = 3 + 2 - \frac{15}{4} - 2 \cdot (-3) = 11 - \frac{15}{4} = \frac{29}{4} \\ x_2^{(2)} = \frac{4 - x_3^{(1)}}{2} = \frac{4 - \frac{15}{4}}{2} = \frac{16-15}{8} = \frac{1}{8} \\ x_3^{(2)} = \frac{15 - x_1^{(1)} - 3x_2^{(1)} - 4x_4^{(1)}}{4} = \frac{15 - 3 - 6 + 12}{4} = \frac{18}{4} = \frac{9}{2} \\ x_4^{(2)} = -3 + 2x_2^{(1)} + x_3^{(1)} = -3 + 4 + \frac{15}{4} = \frac{19}{4} \end{cases}$$

③ $x^{(0)} = (0\ 0\ 0\ 0)^T$

$$\begin{cases} x_1^{(1)} = 3 + x_2^{(0)} - x_3^{(0)} - 2x_4^{(0)} = 3 + 0 - 0 - 0 = 3 \\ x_2^{(1)} = \frac{4 - x_3^{(0)}}{2} = 2 \\ x_3^{(1)} = \frac{15 - x_1^{(1)} - 3x_2^{(1)} - 4x_4^{(0)}}{4} = \frac{15 - 3 - 6 - 0}{4} = \frac{6}{4} = \frac{3}{2} \\ x_4^{(1)} = -3 + 2x_2^{(1)} + x_3^{(1)} = -3 + 4 + \frac{6}{4} = 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2} \end{cases}$$

$$\begin{cases} x_1^{(2)} = 3 + x_2^{(1)} - x_3^{(1)} - 2x_4^{(1)} = 3 + 2 - \frac{6}{4} - 5 = -\frac{6}{4} = -\frac{3}{2} \\ x_2^{(2)} = \frac{4 - x_3^{(1)}}{2} = \frac{4 - \frac{6}{4}}{2} = \frac{10}{8} = \frac{5}{4} \\ x_3^{(2)} = \frac{15 - x_1^{(2)} - 3x_2^{(2)} - 4x_4^{(1)}}{4} = \frac{15 + \frac{6}{4} - \frac{15}{4} - 10}{4} = -1 + \frac{3}{2} = \frac{11}{16} \\ x_4^{(2)} = -3 + 2x_2^{(2)} + x_3^{(2)} = -3 + \frac{5}{2} + \frac{1}{16} = \frac{-48 + 40 + 11}{16} = \frac{3}{16} \end{cases}$$