# Seminar 10 and 11 - 2025

## Theoretical aspects

A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables converges in probability to a random variable $X$, denoted by $X_n \xrightarrow{\mathbb{P}} X$, if

$$\lim_{n\to\infty} \mathbb{P}\Big(|X_n - X| \le \varepsilon\Big) = 1 \quad \text{for every} \quad \varepsilon > 0\,.$$

A sequence $(X_n)_{n\ge 1}$ of random variables converges in mean square to a random variable $X$ if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0.$$

This convergence is denoted by $X_n \xrightarrow{L^2} X$.

A sequence $(X_n)_{n\ge 1}$ of random variables converges in distribution to a random variable $X$ if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

in each continuity point $x$ of $F_X$. This convergence is denoted by $X_n \xrightarrow{d} X$.

**1.** A program returns a value according to a random variable $X$ with $E(X) = m \in \mathbb{R}$ and $V(X) = \sigma^2$, $\sigma > 0$. Prove that $X$ takes values in the interval $(m - 3\sigma, m + 3\sigma)$ with more than 88% probability.

**2.** The number of items produced in a factory during a day is a random variable with mean 50. If we consider a day, which event is more likely: $E_1$:"the production is more than 100 items in this day" or $E_2$:"the production is at most 100 items in this day"?

**3.** Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables with $Unif[a, b]$ distribution, where $a < b$. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \ldots, X_n\} \;\text{ and }\; Z_n = \min\{X_1, \ldots, X_n\}.$$

Prove that $Y_n \xrightarrow{P} b$ and $Z_n \xrightarrow{P} a$.

**4.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Bernoulli random variables. Prove that $X_n \xrightarrow{P} 0$ if and only if $X_n \xrightarrow{L^2} 0$.

**5.** Let $\lambda > 0$. A calling center has the following property, for every $n \in \mathbb{N}$, $n \ge 100$, during an hour interval $(0, 1]$: the calls arrive independently with at most one call in each time subinterval $\left(\frac{i}{n}, \frac{i+1}{n}\right]$, one call has probability $\frac{\lambda}{n}$ to occur, $i = \overline{0, n-1}$. Let's denote by $X_n$ the corresponding total number of calls. Prove that $X_n \xrightarrow{d} X$, where $X \sim Poiss(\lambda)$.

**6.** Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables with $Unif[0, 1]$ distribution. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \ldots, X_n\} \;\text{ and }\; Z_n = \min\{X_1, \ldots, X_n\}.$$

Prove that $Y_n \xrightarrow{L^2} 1$ and $Z_n \xrightarrow{L^2} 0$.

**7***. Consider a sequence of distinct coins such that the probability of getting a head with the $n$th coin is $\frac{1}{n}$, $n \in \mathbb{N}^*$. Let $X_n$ be 1, if the toss of the $n$th coin shows a head, and 0, otherwise. Do we have $X_n \xrightarrow{a.s.} 0$?

**8.** Let $(X_n)_n$, be a sequence of random variables such that for each $n \in \mathbb{N}^*$: $X_n \sim Exp(n)$, i.e, $X_n$ has the following density function

$$f_{X_n}(t) = \begin{cases} 0, & \text{if } \ t \leq 0 \\ ne^{-nt}, & \text{if } \ t > 0 \, . \end{cases}$$

(a) Prove that $X_n \xrightarrow{P} 0$.
(b) Consider $Y_n = nX_n$, for each $n \in \mathbb{N}^*$. Prove that $(Y_n)_n$ does not converge in probability to 0.
(c) Write the cumulative distribution function (cdf) of $Z_n = \frac{1}{\sqrt{n}} Y_n$, $n \in \mathbb{N}^*$. Does $(Z_n)_n$ converge in probability to 0?