Seminars 12 and 13 - 2025

Theoretical aspects:

Definition:
A sequence $(X_n)_n$ of random variables with $E|X_n| < \infty \, \forall \, n \in \mathbb{N}$, obeys the weak law of large numbers (WLLN), if

$$\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - E(X_k)\Big) \xrightarrow{P} 0.$$

Definition:
A sequence $(X_n)_n$ of random variables with $E|X_n| < \infty, \forall \, n \geq 1$, obeys the strong law of large numbers (SLLN) if

$$\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - E(X_k)\Big) \xrightarrow{a.s.} 0.$$

Theorem 1. *Let $(X_n)_n$ be a sequence of pairwise independent random variables satisfying the condition*

$$V(X_n) \leq L, \text{ for all } n \in \mathbb{N}^*,$$

where $L > 0$ is a constant. Then $(X_n)_n$ obeys the WLLN.

Theorem 2. *If $(X_n)_n$ is a sequence of independent random variables such that $\sum_{n=1}^{\infty}\frac{1}{n^2}V(X_n) < \infty$, then*

$$\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - E(X_k)\Big) \xrightarrow{a.s.} 0,$$

i.e. $(X_n)_n$ obeys the SLLN.

Theorem 3. *Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables such that $E(X_n) = m$ for all $n \in \mathbb{N}$. Then*

$$\frac{1}{n}\sum_{k=1}^{n}X_k \xrightarrow{a.s.} m,$$

i.e. $(X_n)_{n\in\mathbb{N}}$ obeys the SLLN.

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1. Consider the sequence of independent identically distributed random variables $(X_n)_{n\geq 1}$ such that $X_n \sim Unif[1,3]$ for each $n \geq 1$. Compute the a.s. limit of the sequence which is
i) the arithmetic mean of $X_1, ..., X_n$, as $n \to \infty$;
ii) the geometric mean of $X_1, ..., X_n$, as $n \to \infty$;
iii) the harmonic mean of $X_1, ..., X_n$, as $n \to \infty$.

2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $P(X_n = n^2) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$, for all $n \geq 1$. Prove that:
a) $X_n \xrightarrow{P} 0$.
b) $(X_n)_{n\geq 1}$ does not converge in mean square.

3. A bank cashier serves customers in the queue one by one. It is known that the expected service time for each customer is 3 minutes, with a standard deviation of 2 minutes. We assume that the service times for the bank customers are independent. Let T be the total time the bank cashier spends serving 100 customers. Estimate the probability $P(240 < T < 320)$ by using values from the table below.

Hint: Let F denote the cdf of the $N(0,1)$ distribution. In the table below there are computed the values $F(x)$ for $x \in \{-3,-2,-1,0,1,2,3\}$ in Python with `scipy.stats.norm.cdf`$(x, 0, 1)$

x	-3	-2	-1	0	1	2	3
$F(x)$	0.00135	0.02275	0.15866	0.5	0.84134	0.97725	0.99865

4. If $(X_n)_n$ is a sequence of independent normally distributed random variables such that $X_n \sim N(0, \frac{1}{n})$, for each $n \geq 1$. Prove that $(X_n)_n$ obeys the SLLN.

5. The measurement error (in millimeters) of a certain object produced in a factory is a continuous random variable X with the cumulative distribution function $F : \mathbb{R} \to [0,1]$,

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}\left(2 + 3x - x^3\right), & x \in [-1,1] \\ 1, & x > 1. \end{cases}$$

Find: $P(-\frac{1}{2} < X < \frac{1}{2})$, $P(X < \frac{1}{2} | X > -\frac{1}{2})$, $E(X)$.

6. A random value X is generated according to the density function $f_X : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{2}e^{-|x|}$, for all $x \in \mathbb{R}$.
Compute:
a) the cumulative distribution function of X;
b) the cumulative distribution function of the random value X^2;
c) $P(X^2 \geq 1)$;
d) the mean value and the variance of X.

7. For each $n \in \mathbb{N}, n \geq 2$, consider

$$X_n \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix}$$

such that $(X_n)_{n \geq 2}$ is a sequence of pairwise independent random variables.

(a) Does $(X_n)_{n \geq 2}$ obey the weak law of large numbers?

(b) Compute $\lim\limits_{n \to \infty} V\left(\frac{1}{2}(X_{n-1} + X_n)\right)$.

8. Consider a binary communication channel transmitting codes of n bits each. Assume that the probability of successful transmission of a single bit is $p \in (0,1)$ and that the probability of an error is $1-p$. Assume also that the channel is capable of correcting up to m errors, where $0 < m < n$. If we assume that the transmission of successive bits is independent, compute the probability of successful code transmission.