Dynamics of the relative motion of a particle

Relative motion:

$$\vec{a} = \vec{a}_t + \vec{a}_r + \vec{a}_c$$

transport acceleration ($\vec{a}_t$), relative acceleration ($\vec{a}_r$), Coriolis acceleration ($\vec{a}_c$)

Equation of motion:

$$m\cdot\vec{a}_r = \overbrace{\vec{F} \underbrace{- m\cdot\vec{a}_t}_{\vec{F}_t} \underbrace{- m\cdot\vec{a}_c}_{\vec{F}_c}}^{\vec{F}_r}$$

$\vec{F}_r$ - relative force

$\vec{F}$ - direct applied forces

$\vec{F}_t$ - transport inertial force

$\vec{F}_c$ - centrifugal inertial force

where:

$$\vec{F}_c = -m\cdot\vec{a}_c = -2m\cdot\vec{\omega}\times\vec{v}_r$$

$$\vec{F}_t = -m\cdot\vec{a}_t = -m\left[\vec{a}_0 + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]$$

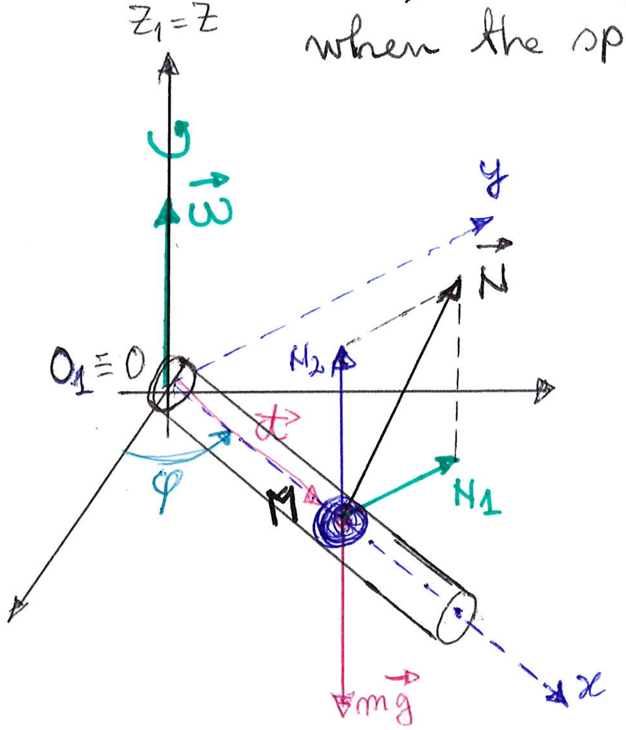
Theorem (of the relative motion) The equation of motion of a particle with respect to an inertial frame of reference maintains its form with respect to a non-inertial one if the given force is replaced by the force relative to the later frame.

$$m\cdot\vec{a}_r = \vec{F}_r$$

① A smooth horizontal cylinder of length $2a$ rotates about a vertical axis passing through the extremity of the cylinder with the angular velocity $\omega = \text{const}$. Inside the cylinder there is a sphere of mass m. At the initial moment ($t = 0$) the distance between the sphere and the rotation axis is a, and its velocity relative to the cylinder is zero.

Find: a) The relative equation of motion of the sphere along the cylinder and the reaction $\vec{N}$ of the cylinder upon the sphere.

b) The absolute trajectory, the absolute velocity $\vec{v}_a$, the relative velocity $\vec{v}_r$ and the time when the sphere leaves the cylinder.



a) **Remark**: The sphere can move inside the tube only in the Ox direction, it means we have two restrictions

$$Y = 0, \quad Z = 0$$

and thus, we have two normal reactions:

$$\vec{N}_1 = N_1 \cdot \vec{j}, \quad N_2 = N_2 \cdot \vec{k}. \quad (1)$$

The total normal reaction will be:

$$\vec{N} = \vec{N}_1 + \vec{N}_2 = N_1 \cdot \vec{j} + N_2 \cdot \vec{k}. \quad (2)$$

The equation of the relative motion is:

$$m\vec{a}_r = m\vec{g} + \vec{N} + \vec{F}_t + \vec{F}_c \quad (3)$$

where

$$\vec{F}_t = -m \cdot \vec{a}_t = -m\left[\underset{0}{\cancel{\vec{a}_O}} + (\vec{r} \times \underset{0}{\cancel{\dot{\vec{\omega}}}}) + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{-\omega^2 \cdot \vec{r}}\right] =$$

$\boxed{\vec{r} = x \cdot \vec{i}}$

$$= -m(-\omega^2 \cdot x \cdot \vec{i}) = m\omega^2 x \cdot \vec{i}. \quad (4)$$

$$\vec{F_c} = -m\vec{a_c} = -m\cdot 2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t} = -2m\,(\vec{\omega}\times\dot{x}\vec{i}) = -2m\omega\cdot\dot{x}\,(\vec{k_0}\times\vec{i}) \Rightarrow$$

$$\vec{r} = x\cdot\vec{i}\quad,\quad \vec{v_r} = \frac{\partial\vec{r}}{\partial t} = \dot{x}$$

$$\vec{F_c} = -2m\omega\dot{x}\vec{j}\qquad(5)$$

Next we consider eq. (3) on the $Oxyz$ axis:

$$\begin{cases} Ox: & m\ddot{x} = m\omega^2 x \\ Oy: & 0 = N_1 - 2m\omega\dot{x} \\ Oz: & 0 = N_2 - mg \end{cases} \Rightarrow \begin{cases} x(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} \\ N_1 = 2m\omega\dot{x} \\ N_2 = mg \end{cases}\qquad(6)$$

$$\ddot{x} - \omega^2 x = 0 \Rightarrow \rho^2 - \omega^2 = 0 \Rightarrow \rho_{1,2} = \pm\omega$$

At $t=0$ (initial conditions):

$$(7)\quad \begin{cases} x(0) = a \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = a \\ \omega C_1 - \omega C_2 = 0 \end{cases} \Rightarrow C_1 = C_2 = \frac{a}{2} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} x(t) &= \frac{a}{2}\left(e^{\omega t} + e^{-\omega t}\right) = a\cdot ch(\omega t) \\ y(t) &= 0 \\ z(t) &= 0 \end{aligned}\right\}\ (8)\ \text{equations of motion (relative).}$$

Next from (6 b) and (6 c) we have:

$$N_1 = 2m\omega\frac{a}{2}\omega\left(e^{\omega t} - e^{-\omega t}\right) \Rightarrow N_1 = 2m\omega^2 a\, sh(\omega t)\quad(9)$$

$$N_2 = mg$$

$$\Rightarrow \boxed{\vec{N} = 2m\omega^2 a\, sh(\omega t)\cdot\vec{j} + mg\vec{k}}\quad(10)$$

b) The absolute motion of the sphere is composed by a rotation and a translation.

Let be

$$\left.\begin{aligned} \rho &= \rho(t) = x(t) = a\, ch(\omega t) \\ \varphi &= \varphi(t) = \omega\cdot t \end{aligned}\right\} \Rightarrow \boxed{\rho = ch(\varphi)}\quad(11)$$

$$\left(\frac{d\varphi}{dt} = \omega \Rightarrow \varphi = \omega\cdot t + \underset{0}{\underset{\shortparallel}{\varphi_0}}\right)$$

the absolute trajectory of the sphere (spiral).

The absolute velocity:

$$\vec{v}_a = \vec{v}_r + \vec{v}_t = \dot{x}\vec{i} + (\vec{\omega} \times \vec{r}) = \dot{x}\vec{i} + (\omega \vec{k} \times x\cdot\vec{i}) = \dot{x}\vec{i} + \omega x\vec{j}$$

$$\Rightarrow \vec{v}_a = a\omega\, sh(\omega t)\cdot\vec{i} + a\omega\, ch(\omega t)\cdot\vec{j} = a\omega\left[sh(\omega t)\cdot\vec{i} + ch(\omega t)\vec{j}\right] \quad (12)$$

$$\dot{x}(t) = a\omega \frac{e^{\omega t} - e^{-\omega t}}{2} = a\omega\, sh(\omega t).$$

The relative velocity:

$$\vec{v}_r = \dot{x}\vec{i} = a\omega\, sh(\omega t)\cdot\vec{i}. \quad (13)$$

$$|\vec{v}_a| = a\omega\sqrt{sh^2(\omega t) + ch^2(\omega t)} \underset{ch^2 - sh^2 = 1}{=} a\omega\sqrt{2ch^2(\omega t) - 1}$$

$$|\vec{v}_r| = a\omega\sqrt{ch^2(\omega t) - 1}$$

When the sphere leaves the tube, we have:

$$x(t_e) = 2a \underset{(8)}{=} a\, ch(\omega t_e) \Rightarrow \boxed{ch(\omega t_e) = 2} \quad (14)$$

Thus $v_a(t_e) = a\cdot\omega\sqrt{8-1} = a\omega\sqrt{7}$

$v_r(t_e) = a\omega\sqrt{4-1} = a\omega\sqrt{3}$

In order to find the exit time, t_e, we have to solve eq. (14).

$$\frac{e^{\omega t_e} + e^{-\omega t_e}}{2} = 2 \Rightarrow e^{\omega t_e} + \frac{1}{e^{\omega t_e}} = 4 \Rightarrow u + \frac{1}{u} = 4$$

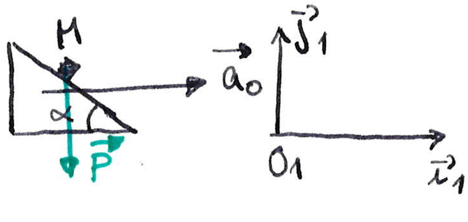
$$\Rightarrow u^2 - 4u + 1 = 0 \Rightarrow u_{1,2} = 2 \pm \sqrt{3} \Rightarrow e^{\omega t_e} = 2 \pm \sqrt{3} \Rightarrow$$

$$\Rightarrow \omega t_e = \ln(2 \pm \sqrt{3}) \Rightarrow \omega t_e = \ln(2+\sqrt{3}) \Rightarrow \boxed{t_e = \frac{1}{\omega}\ln(2+\sqrt{3})} \quad (15)$$

for „−" we obtain a negative time.

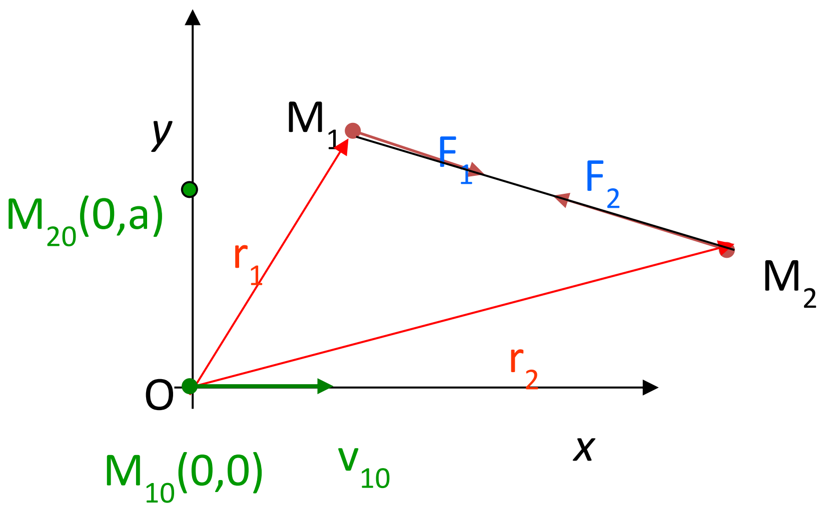
(home-work)

② A particle M of weight $\vec{P} = m\vec{g}$ is moving with friction (the friction coefficient is f) on the face of a triangular prism. The prism has a translation motion with the acceleration $\vec{a}_0$ in a fixed system $O_1\vec{i}_1\vec{j}_1$. Find the relative acceleration of M and the pressure exerted by M on the prism's face. The angle between the prism's face and $O_1\vec{i}_1$ is α.



Seminar 12. Dynamics of the systems of material points

1) Two particles M_1 and M_2 with m = 1 attract each other with a force equal to the distance between them the coefficient of proportionality being 1. At t = 0 the point M_1 was in the origin O and had the velocity $v_1 = a\sqrt{2}$ oriented along the Ox axis, while M_2 was on Oy axis having the velocity $v_2 = 0$ and the ordinate *a*. Find the equation of motion of the system formed by M_1 and M_2.



Observation:
The motion takes place in a plane (in Oxy).
Indeed, we can consider the force acting between M1 and M2 as central (for example the center is M2) and then, according to the theory of the central forces, the motion is in a plane.

$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_1 = \overrightarrow{M_1M_2} \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2 = \overrightarrow{M_2M_1} = -\overrightarrow{M_1M_2} \end{cases} \quad (1)$$

We preoject equations (1) on the Oxy axes ($m_1 = m_2 = 1$) and we add the initial conditions:

$$\begin{cases} \ddot{x}_1 = x_2 - x_1; \ x_1(0) = 0; \ \dot{x}_1(0) = a\sqrt{2} \\ \ddot{y}_1 = y_2 - y_1; \ y_1(0) = 0; \ \dot{y}_1(0) = 0 \\ \ddot{x}_2 = x_1 - x_2; \ x_2(0) = 0; \ \dot{x}_2(0) = 0 \\ \ddot{y}_2 = y_1 - y_2; \ y_2(0) = a; \ \dot{y}_2(0) = 0 \end{cases} \quad (2)$$

From (2) we have:

$$\begin{cases} \frac{d^2}{dt^2}(x_1 + x_2) = 0 \\ \frac{d^2}{dt^2}(x_1 - x_2) = -2(x_1 - x_2) \\ \frac{d^2}{dt^2}(y_1 + y_2) = 0 \\ \frac{d^2}{dt^2}(y_1 - y_2) = -2(y_1 - y_2) \end{cases} \tag{3}$$

Integrating (3)we obtain:

$$\begin{cases} x_1 + x_2 = C_1 t + C_2; \quad x_1 - x_2 = C_5 \cos(\sqrt{2}\,t) + C_6 \sin(\sqrt{2}\,t); \\ y_1 + y_2 = C_3 t + C_4; \quad y_1 - y_2 = C_7 \cos(\sqrt{2}\,t) + C_8 \sin(\sqrt{2}\,t); \end{cases} \tag{4}$$

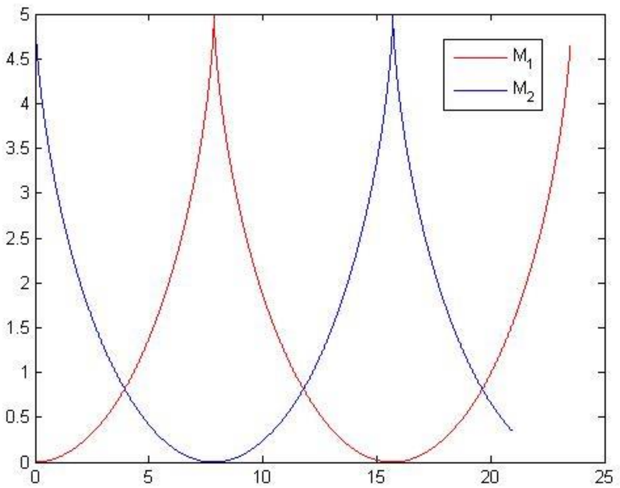
and using the intial conditions (2) one can find the constants of integration C_1,..., C_8:

$$\begin{cases} x_1 + x_2 = a\sqrt{2}\,t; \quad x_1 - x_2 = a \sin(\sqrt{2}\,t); \\ y_1 + y_2 = a; \quad y_1 - y_2 = -a \cos(\sqrt{2}\,t) \end{cases} \tag{5}$$

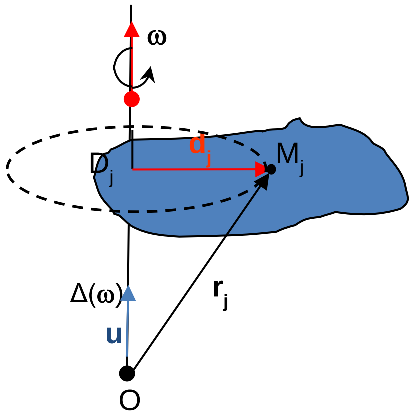
We get:

$$\begin{cases} x_1 = \frac{a}{2}\left[\sqrt{2}\,t + \sin(\sqrt{2}\,t)\right] \\ y_1 = \frac{a}{2}\left[1 - \cos(\sqrt{2}\,t)\right] \end{cases} \qquad \begin{cases} x_2 = \frac{a}{2}\left[\sqrt{2}\,t - \sin(\sqrt{2}\,t)\right] \\ y_2 = \frac{a}{2}\left[1 + \cos(\sqrt{2}\,t)\right] \end{cases} \tag{6}$$

Equations (6) represent two cycloids formed by two diametrically opposed points of a circle that rolls to the line of equation y = a.



2) Find the expression of the kinetic energy of a rotating rigid system (rigid body), (S) about a fixed axis $\Delta(O,\vec{u})$ with an angular velocity $\vec{\omega}$, where $\vec{u}$ is the unit vector of the axis Δ.



Consider the rigid discrete system:

$$(S): M_j(m_j), \vec{r}_j = \overrightarrow{OM}_j, j = 1, \dots, N$$

The velocity $\vec{v}_j$ of the point M_j is given by:

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j \qquad (1)$$

Let be $D_j = pr_\Delta M_j$. Thus, we have

$$\vec{r}_j = \overrightarrow{OD}_j + \vec{d}_j \qquad (2)$$

where $\vec{d}_j = \overrightarrow{D_jM_j}$. Using (1) and (2) we get:

$$\vec{v}_j = \vec{\omega} \times (\overrightarrow{OD}_j + \vec{d}_j) \underbrace{=}_{\vec{\omega} \parallel \overrightarrow{OD}_j} \vec{\omega} \times \vec{d}_j => \vec{v}_j = \vec{\omega} \times \vec{d}_j \qquad (3)$$

Now we can calculate the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 = \frac{1}{2}\sum_{j=1}^{N} m_j (\vec{\omega} \times \vec{d}_j)^2 = \frac{1}{2}\sum_{j=1}^{N} m_j \omega^2 d_j^2 \underbrace{\sin^2(\vec{\omega}, \vec{d}_j)}_{=1(\vec{\omega} \perp \vec{d}_j)} = \frac{1}{2}\omega^2 \sum_{j=1}^{N} m_j d_j^2 = \frac{1}{2} I(\Delta)\omega^2$$

where

$$I(\Delta) = \sum_{j=1}^{N} m_j d_j^2$$

is the moment of inertia of the system (S) with respect to Δ.

Therefore, the kinetic energy of the rigid body rotating about the axis Δ is given by:

$$\boxed{T = \frac{1}{2} I(\Delta)\omega^2}$$