Seminars 8 and 9 - 2025 - Solutions

1. A player tosses three coins into the air. He wins 1 dollar for the number of heads he will get. However, he will lose 8 dollars if neither coin is a head. (a) Calculate the expected value of this game.
(b) Is the game favorable for the player?
(c) What is the expected profit or loss after playing 10 times this game?
A: N: the number of heads the player gets after tossing the three coins; X: profit or loss after playing one game
(a) $N \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \Longrightarrow X \sim \begin{pmatrix} -8 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \Longrightarrow E(X) = (-8) \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{4}{8} = 0.5\,.$
(b) $E(X) > 0 \Longrightarrow$ the game is favorable for the player.
(c) The expected profit of the player after playing 10 times this game is $10 \cdot 0.5 = 5\$\,.$

2. The function *numpy.random.rand* returns in Python, for each independent call, a random value from the interval $[0,1]$, according to the uniform distribution $Unif[0,1]$. Let $n \in \mathbb{N}^*$ and $p \in (0,1)$. If the function is called n times, what is the expectation and variance of the number of values less than p?
A: Let's call a *success* the event that a value returned by *numpy.random.rand* is less than p. Let X be number of values less than p. Since a value generated by *numpy.random.rand* is less than p with probability $\int_0^p 1dx = p$, we have $X \sim Bino(n,p)$. We will show that $E(X) = np$ and $V(X) = np(1-p)$.
For $i \in \{1,\dots,n\}$ let $X_i \sim Bernoulli(p)$ be such that $X_i = 1$, if the ith trial is a success, and $X_i = 0$, if the ith trial is not a success (in particular, $P(X_i = 1) = p$, $P(X_i = 0) = 1-p$). Then $X = X_1 + \ldots + X_n \sim Bino(n,p)$, so

$$E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = p + ... + p = np.$$

Since the random variables $X_1, \dots, X_n$ are independent,

$$V(X) = V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n) = np(1-p) = np(1-p).$$

3. The length of time for one individual to be served at a cafeteria is a random variable T having an exponential distribution with a mean of 5 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days?

$$T \sim Exp\,(\lambda) \Longleftrightarrow f_T(t) = \begin{cases} 0, & \text{if } t \le 0 \\ \lambda e^{-t\lambda}, & \text{if } t > 0\,. \end{cases}$$

A: $E(T) = \int_{-\infty}^{\infty} t f_T(t)dt = \int_0^{\infty} t\lambda e^{-\lambda t}dt = -te^{-\lambda t}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda t}dt = -\frac{1}{\lambda}e^{-\lambda t}\Big|_0^{\infty} = \frac{1}{\lambda}$. Since $E(T) = 5$, $\lambda = \frac{1}{5}$. $p = P(T < 3) = \int_0^3 \frac{1}{5}e^{-\frac{t}{5}}dt = 1 - e^{-\frac{3}{5}}$. Let X be number of days, out of 6 days, when a person is served less than 3 minutes. Then $X \sim Bino(6,p)$. So, $P(X \ge 4) = \sum_{k=4}^{6} C_6^k p^k (1-p)^{6-k} = \sum_{k=4}^{6} C_6^k (1 - e^{-\frac{3}{5}})^k (e^{-\frac{3}{5}})^{6-k}$.

4. Peter has a bike combination lock with a 3 digit cipher code. He remembers that the first digit is 0 and the other two are even, but he forgot them. Compute the expected number of unsuccessful trials *before* opening the lock for the first time for the following cases:
(a) unsuccessful codes are not eliminated from further selections (Peter is not a well organized person);
(b) unsuccessful codes are eliminated (Peter likes systematical search and notes the already tried codes).

A: Let X denote the number of Peter's unsuccessful trials *before* opening the lock for the first time.
In case (a), the probability to open the lock in one trial is $\frac{1}{5^2}$. Since X is the number of unsuccessful trials before opening the lock for the first time $\Longrightarrow X \sim Geo(p)$ with $p = \frac{1}{25}$.

Using the *Ratio Test for convergent series* (Analysis) we have that $\sum\limits_{k=0}^{\infty} kp(1-p)^k < \infty$. We compute

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{\infty} kp(1-p)^k = (1-p)\sum_{k=1}^{\infty} kp(1-p)^{k-1} \\
&= (1-p)\sum_{j=0}^{\infty}(j+1)p(1-p)^j \\
&= (1-p)\sum_{j=0}^{\infty} jp(1-p)^j + (1-p)\sum_{j=0}^{\infty} p(1-p)^j \\
&= (1-p)E(X) + (1-p) \implies E(X) = \frac{1-p}{p} = \frac{1-\frac{1}{25}}{\frac{1}{25}} = 24.
\end{aligned}
$$

In case (b), where unsuccessful codes are eliminated, it can take at least 0 and at most 24 unsuccessful attempts before opening the lock

$$P(X=0) = \frac{1}{25}, P(X=1) = \frac{24}{25}\cdot\frac{1}{24}, P(X=2) = \frac{24}{25}\cdot\frac{23}{24}\cdot\frac{1}{23}, \cdots,$$

$$P(X=24) = \frac{24}{25}\cdot\frac{23}{24}\cdot\ldots\cdot\frac{1}{2}\cdot 1.$$

We obtain

$$E(X) = \frac{0+1+\cdots+24}{25} = 12.$$

5. It is known that the density function of the random time T (in minutes) in which a device performs a certain task is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \begin{cases} a(b-x), & \text{if } x \in (0,b) \\ \\ 0, & \text{otherwise,} \end{cases}$
where $a, b \in \mathbb{R}$ are parameters. Compute a and b if the expected time to complete the task is 1 minute.
A: By the properties of a density function it follows that $a, b > 0$ and $\int_{\mathbb{R}} f(x)dx = 1$

$$\Longrightarrow \int_0^b a(b-x)dx = 1 \Longrightarrow a = \frac{2}{b^2};$$

$$E(T)=1 \Longrightarrow \int_{\mathbb{R}} xf(x)dx = 1 \Longrightarrow \frac{2}{b^2}\int_0^b x(b-x)dx = 1 \Longrightarrow b=3 \Longrightarrow a=\frac{2}{9}.$$

6. A factory produces graphite-core wood pencils with the standard length of 190 mm and an error (in mm) which follows the normal distribution $N(1, 0.25)$. Compute the expectation and the variance of the length of a pencil.

$X \sim N(\mu, \sigma^2)$, then the density function of X is $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, x \in \mathbb{R}$.

A: For $X \sim N(\mu, \sigma^2)$, we will show that $E(X) = \mu$, $V(X) = \sigma^2$.

When $\mu = 0$ and $\sigma = 1$ it is known that *the density function of the standard normal distribution* is

$$\varphi(t) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{t^2}{2}\right\}, t \in \mathbb{R}, \text{ having the property (from the course): } \int_{-\infty}^{\infty}\varphi(t)dt = 1.$$

In the following calculations one uses the variable change $t = \dfrac{x-\mu}{\sigma}$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf_X(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\exp\left\{-\frac{t^2}{2}\right\}dt + \mu\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{t^2}{2}\right\}dt = 0 + \mu\int_{-\infty}^{\infty}\varphi(t)dt = \mu\,. \end{aligned}$$

Use the same change of variable as before and then partial integration

$$\begin{aligned} \Longrightarrow V(X) &= E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2\exp\left\{-\frac{t^2}{2}\right\}dt = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\left(-\exp\left\{-\frac{t^2}{2}\right\}\right)'dt \\ &= t\left(-\exp\left\{-\frac{t^2}{2}\right\}\right)\Bigg|_{-\infty}^{\infty} - \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1\cdot\left(-\exp\left\{-\frac{t^2}{2}\right\}\right)dt \\ &= 0 - 0 + \sigma^2\int_{-\infty}^{\infty}\varphi(t)dt = \sigma^2. \end{aligned}$$

Back to our problem: let X be the error. Then $E(X) = 1$ and $V(X) = (0.5)^2 = 0.25$. So, the expectation of the *length* L of a pencil is $E(L) = E(X+190) = 190 + E(X) = 191$ and its variance is $V(L) = V(X+190) = V(X) = 0.25$.

7. The life, in years, of a certain type of electrical switch has an exponential distribution with an average life of $\beta = 2$ years. If 100 of these switches function independently in a system, what is the expected value of the number of switches that fail during the first year?

A: Let X be the number of switches that fail during the first year. Then $X \sim Bino(100, p)$, where the probability that any of the switches fails during the first year is $p = \int_0^1 \frac{1}{\beta}e^{-\frac{t}{\beta}}dt = 1 - e^{-\frac{1}{2}}$ (see the solution of Problem 3, if a random variable has the exponential distribution with expectation (average) equal to $\beta = 2$, then its parameter is $\lambda = \frac{1}{\beta}$). Taking into account that $E(X) = 100\cdot p$ (see the solution of Problem 2), we have $E(X) = 100(1 - \frac{1}{\sqrt{e}}) \approx 39$ (it is expected that approximately 39 switches, out of 100, fail during the first year).