

Seminar 1

Combinatorics

1. *The basic principle of counting:* If m experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment, and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment, and if..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_m$ possible outcomes of the m experiments.

Example: In how many ways can a person choose to wear 3 shirts, 2 jeans and 4 pairs of shoes?

A: $3 \cdot 2 \cdot 4 = 24$.

2. *Arrangements of n objects taken k ($n \geq k$):* selections of k **distinct** and **ordered** items from n distinct given objects.

$$A_n^k = \text{number of arrangements of } n \text{ objects taken } k \\ = n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

Example: How many codes with 3 distinct digits can be generated with the digits: 0, 1, 2, 3, 4?

A: $A_5^3 = \frac{5!}{2!} = 60$.

3. *Permutations of n objects:* arrangements of n objects taken n .

$$P_n = \text{number of permutations of } n \text{ objects} = A_n^n = n!$$

Remark: By convention, $0! = 1$.

Example: In how many ways can 4 persons sit on a bench? A: $P_4 = 4!$

4. *Combinations of n objects taken k ($n \geq k$):* selections of k **distinct** and **not ordered** items from n distinct given objects, i.e., subsets with k elements of a set with n elements.

$$C_n^k = \text{number of combinations of } n \text{ objects taken } k \\ = \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!}$$

Remark: $C_n^k = C_n^{n-k}$.

Example: How many football teams can be formed out of a group of 15 persons? A: $C_{15}^{11} = C_{15}^4 = 1365$.

5. *Number of functions* from a set A with k elements to a set B with n elements: n^k ($k, n \in \mathbb{N}^*$).

Remark: A function can be identified with selections of k items that are **not** necessarily distinct (i.e., an item can be chosen multiple times), but **ordered**, from n distinct objects. Hence, we can say that the *functions* are *arrangements with repetitions*.

Example: How many codes with 3 digits can be generated with the digits: 0, 1, 2, 3, 4? A: $5^3 = 125$.

6. *Permutations with repetitions:* Consider n objects such that n_1 are alike, n_2 are alike, ..., n_m are alike ($n, m \in \mathbb{N}^*, m \leq n; n_1, \dots, n_m \in \mathbb{N}, n_1 + \dots + n_m = n$). The number of distinct permutations of these n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_m!}$$

Example: In how many ways can we order 3 white marbles, 4 black marbles and 2 gray marbles on a line?

Example: $\bullet \circ \bullet \circ \bullet \circ \bullet \bullet \bullet$ A: $\frac{9!}{3!4!2!} = 1260$.

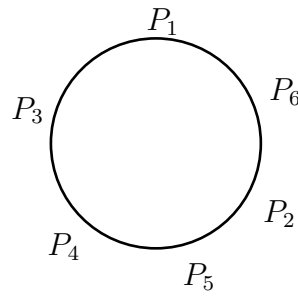
Example: In how many ways can we order 3 white marbles and 4 black marbles on a line?

Example: $\circ \bullet \circ \circ \bullet \bullet \bullet$ A: $\frac{7!}{3!4!} = 35$.

Remark: In the above example we get the same value as $C_7^3 = C_7^4$, because the number of permutations with repetitions for $m = 2$ types of objects is the same as the number of ways to choose the distinct and not ordered positions (i.e., combinations), in each permutation, for the objects of the first type (the remaining positions are automatically left for the second type objects).

Problems

1. How many different 7-place license plates are possible in Romania, given that the digit groups 00 and 000 are not allowed, Q cannot be used at all and all 3-letter groups starting with O or I are forbidden? Examples of 7-place license plates in Romania: CJ-01-YYR, IF-12-XYZ, B-101-ABC.
2. In how many ways can we put 5 maths books, 3 programming books and 4 novels on a shelf, given that each book has a different author and
 - a) the books of the same type are grouped together?
 - b) only the novels are necessarily grouped together?
3. X goes to the theater with 10 girls and 9 boys. In how many ways can they occupy an entire row such that X seats between two girls?
4. a) How many 5-place codes contain exactly 3 decimal digits and 2 capital latin letters? Examples: 01DD0, A21Z0, 999XX.
 b) How many 5-place codes contain exactly 3 distinct decimal digits and 2 distinct capital latin letters? Example: A21Z0.
5. How many anagrams does the word MISSISSIPPI have?
6. In how many ways can 6 persons sit at a round table? Example:



7. a) In how many distinct ways can the characters: A, A, A, B, B, 0, 0, 0, 1 be ordered on a line? Example: 1 B A 0 A 0 0 B A.
 b) In how many distinct ways can the characters: A, A, A, B, B, 0, 0, 0, 1 be ordered on a circle? Example:

