Seminar 13_14. Dynamics of the systems of material points

Moments of inertia.

Let be (*S*) a system of material points (discrete – finite number of points M_i (m_i), $i = 1,\dots, N$ or continuous – rigid body) in an orthogonal frame of reference$Ox_1x_2x_3$ and V a simple manifold in $\mathbf{R}^3$ (point, line or plane)

Definition: The scalar value I(V) given by:

$$I(V)=\begin{cases}\sum_{i=1}^{N} m_i d_i^{\,2} & \text{, (S) discrete system (N points)}\\ \int_V d^2 dm = \int_D d^2 \rho\, d\tau & \text{, (S) rigid body (domain D, } d\tau \text{ is the element of volume, area or length)}\end{cases}$$

is the moment of inertia of the system (S) with respect to the manifold V. Depending on the manifold the moments can be polar axial or planar.

Steiner's theorem:

Let be G the center of mass of the material system (S). Consider Δ and Δ_G two parallel axis, such that $G\in \Delta_G$. We note d = distance (Δ, Δ_G). Then for the moment of inertia of the system with respect to Δ and Δ_G we have:

$$I(\Delta)=I(\Delta_G)+md^2$$

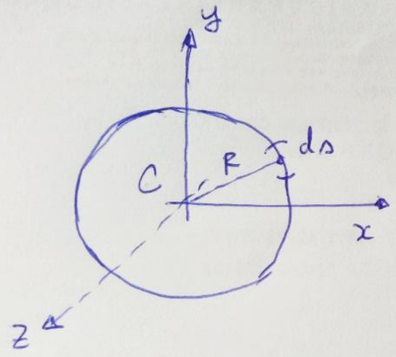
where m is the mass of the system.

The kinetic energy (rotation about the axis) and the moment of momentum

$$T=\frac{1}{2}I(\Delta)\omega^2$$

$$\vec{K}_0 = [\mathbf{I}]\cdot[\omega].$$

1) Calculating the moment of inertia of a thin ring about the central axis



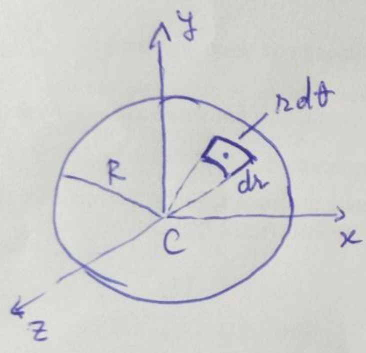
$$I_C = \int_{(\mathcal{C})} R^2\, dm =$$

$$dm = \rho\, ds = \rho R\, d\theta$$

$$= \rho R^3 \int_0^{2\pi} d\theta = 2\pi \rho R^3 = \underbrace{2\pi R \rho}_{m} \cdot R^2 =$$

$$= mR^2$$

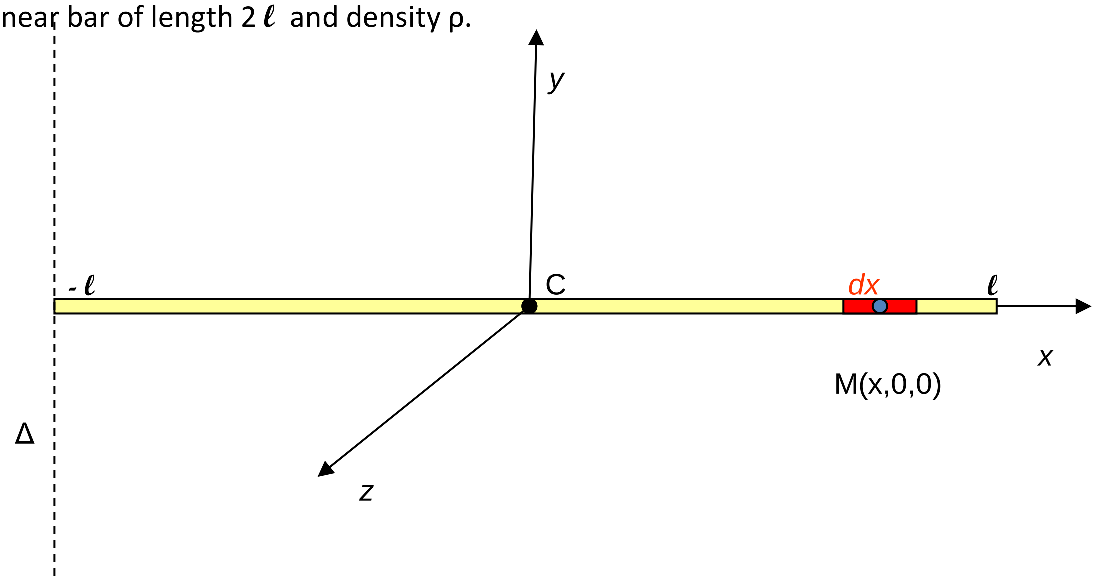
2) Calculating the moment of inertia of a disk about the central axis



$$I_C = \int_D r^2\, dm = \rho \int_0^R \int_0^{2\pi} r^3\, dr\, d\theta = \rho \pi R^2 \cdot \frac{R^2}{2} = \frac{mR^2}{2}$$

$$dm = \rho \cdot d\sigma = \rho\, r\, dr\, d\theta$$

3. Calculate the moment of inertia relative to the centre of inertia C and relative to the axis Δ for a linear bar of length 2 ℓ and density ρ.



We use:

$$dm = \rho dl = \rho dx$$

$$I_{Cy} = \int_D (x^2 + \underbrace{z^2}_{=0}) dm = \int_D x^2 dm = \rho \int_{-l}^{l} x^2 dx = \frac{ml^2}{3}$$

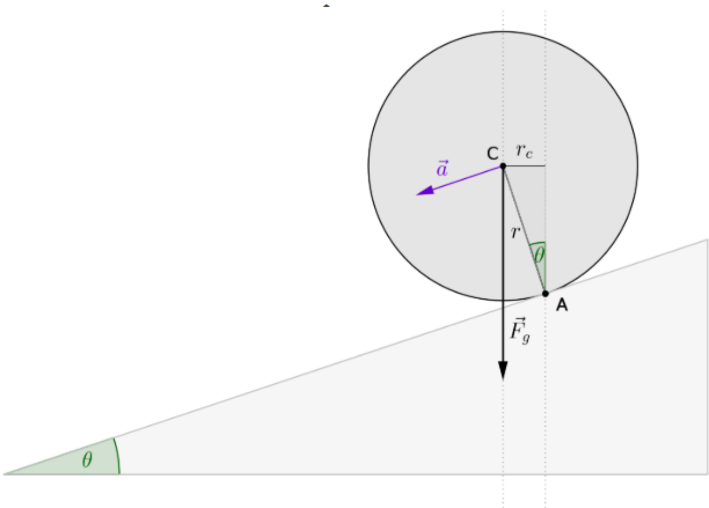
$$I_{Cz} = \int_D (x^2 + \underbrace{y^2}_{=0}) dm = \int_D x^2 dm = \rho \int_{-l}^{l} x^2 dx = \frac{ml^2}{3}$$

$$I_{zy} = \int_D yz dm = 0, I_{xz} = \int_D xz dm = 0, I_{xy} = \int_D xy dm = 0$$

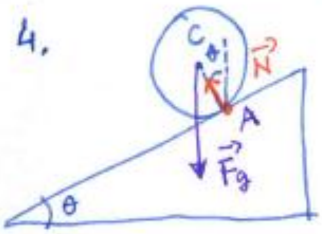
In order to calculate the moment of inertia relative to the axis Δ we use the Steiner's theorem:

$$I_\Delta = I_G + Md^2 = I_{Cy} + ml^2 = \frac{4ml^2}{3}$$

4. A sphere of mass m and radius r is rolling down a slope of inclination θ without slipping under the action of its own weight $\vec{F}_g = m\vec{g}$. At what rate does the sphere accelerate down the slope?



4.



$$\vec{M}_A = \vec{AC} \times \vec{F_g} + \underbrace{\vec{AA}}_{0} \times \vec{N}$$

$$M_A = r mg \sin\theta$$

$$\frac{d\vec{K}_A}{dt} = \vec{M}_A \quad (*)$$

$$\vec{K}_A = I_A \cdot \vec{\omega} \underset{\text{T. Steiner}}{=} (I_c + mr^2)\vec{\omega} = \left(\frac{2}{5}mr^2 + mr^2\right)\vec{\omega} = \frac{7m}{5} r^2 \vec{\omega}$$

$$(*) \Rightarrow \frac{7}{5}\cancel{m} r^2 \dot{\omega} = r\cancel{m}g \sin\theta \Rightarrow \dot{\omega} = \frac{5}{7}\frac{g}{r}\sin\theta$$

$$\Rightarrow \vec{a}_c = \dot{\vec{\omega}} \times \vec{r} \Rightarrow |\vec{a}_c| = \dot{\omega} r = \frac{5}{7} g \sin\theta$$