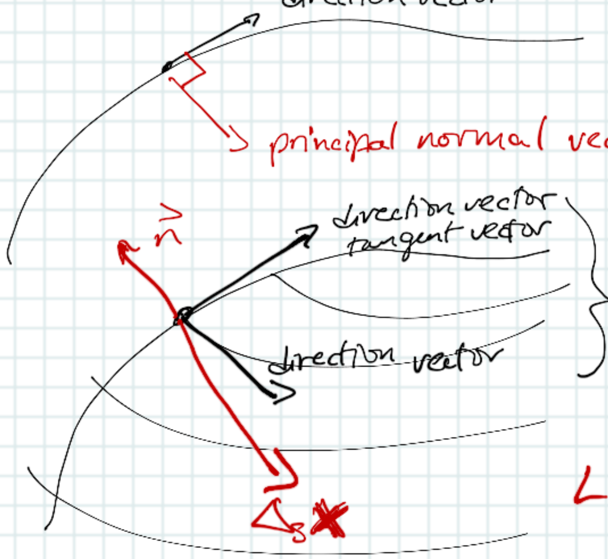
Curve:

direction vector – 1st derivative
- ideally ALP so length is 1 & it's intrinsic

principal normal vector – 2nd derivative
- sideways acceleration
- based on ALP – intrinsic

Surface



1st partial derivatives
- not intrinsic

$$\text{LBO} - \Delta_s \mathbf{X} = \underbrace{\operatorname{div} \nabla \mathbf{X}}_{\text{essentially } 2^{\text{d}} \text{ derivative}}$$

position function

$$\underbrace{\Delta_s \mathbf{X}}_{\text{intrinsic}} = -2H\vec{n}$$

H mean curvature

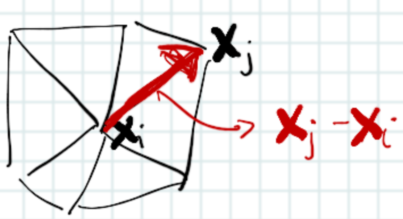
$\vec{n}$ outwards normal

notice that on the curve, the principal normal vector (aka acceleration) pulls the object sideways so that the path curves

→ $\Delta_s \mathbf{X}$ is basically doing the same thing – pulling the surface inwards

– we now have a formula for approximating DLBO for any function. Let's apply it to $\mathbf{X}$, the position function:

$$\Delta \mathbf{X}_i = \frac{1}{N(v_i)} \sum_{v_j \in N_1(v_i)} (\mathbf{X}_j - \mathbf{X}_i)$$



DLBO of $\mathbf{X}$ is the average of the vectors from v_i to it's neighbours v_j

$$\Delta \mathbf{X}_i = \frac{1}{|N(v_i)|} \sum_{v_j \in N_1(v_i)} (\mathbf{X}_j - \mathbf{X}_i)$$

$$= \frac{1}{|N(v_i)|} \sum_{v_j \in N_1(v_i)} \mathbf{X}_j - \frac{1}{|N(v_i)|} \sum_{v_j \in N_1(v_i)} \overset{\text{constant}}{\mathbf{X}_i}$$

$$= \frac{1}{|N(v_i)|} \sum_{v_j \in N_1(v_i)} \mathbf{X}_j - \frac{1}{|N(v_i)|} \cdot \mathbf{X}_i \sum_{v_j \in N_1(v_i)} 1$$

$$= \underbrace{\underbrace{\frac{1}{|N(v_i)|} \sum_{v_j \in N_1(v_i)} \mathbf{X}_j}_{\substack{\text{average of the positions}\\ \text{of the neighbours of } v_i\\ \text{- i.e. centroid / centre of gravity / barycentre}\\ \text{of the 1-ring}}} - \frac{1}{\cancel{|N(v_i)|}} \mathbf{X}_i \cancel{|N_1(v_i)|}}_{\substack{\text{subtract the vertex}\\ \text{to get a vector}}}$$

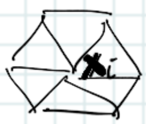
$\Delta \mathbf{X}_i$ is the vector from the vertex to the centroid of it's 1-ring

$$\Delta \mathbf{X}_i = \left(\frac{1}{|N_1(v_i)|} \sum_{v_j \in N_1(v_i)} \mathbf{X}_j \right) - \mathbf{X}_i$$

vector from vertex to centroid of 1-ring

let's consider some examples

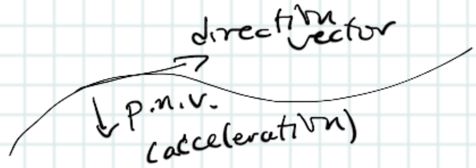
flat mesh (perfect hexagon) - $\mathbf{X}_i$ in the middle

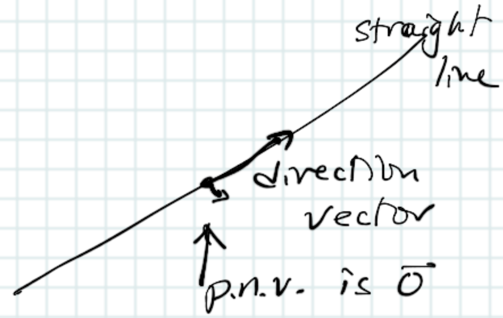


notice x_i is the centroid of its neighbours

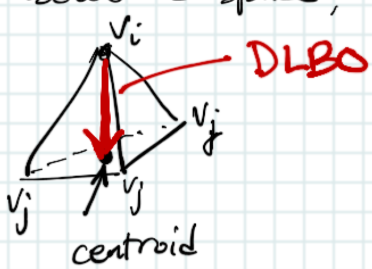
so $\Delta \mathbf{X}_i = \vec{0}$

in a curve





what about a spike?



- vector pointing in.
- the more the surface curves, the longer it is

Curvatures: two principal curvatures κ_1, κ_2

s.t. $\frac{\kappa_1 + \kappa_2}{2} = H$ mean curvature

$\kappa_1 \kappa_2 = K$ Gaussian curvature - intrinsic

- we have the DLBO which is related to H

∴ if we can estimate K as well, we can solve for κ_1, κ_2

Estimating H:
1. compute DLBO estimate of $\Delta_s \mathbf{X}$
2. we know that $\Delta_s \mathbf{X} = -2H\vec{n}$
3. assume that $\|\vec{n}\| = 1$ (perfect unit normal)
4. solve for H as follows

$$\|\Delta_s \mathbf{X}\| = \|-2H\vec{n}\|$$

$$\underbrace{\|\Delta_s \mathbf{X}\|}_{\text{computed } \Delta_s \mathbf{X} \text{ \& length is easy}} = 2H\underbrace{\|\vec{n}\|}_{1}$$

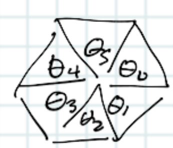
vector to 1-ring centroid

$$\frac{\|\Delta_s \mathbf{X}\|}{2} = H$$

Estimating K:

three angles here each $< \frac{\pi}{2}$

$$\sum_{T_j \in N_1(v_i)} \theta_j < 2\pi$$



← flat mesh

$$\sum_{T_j \in N_1(v_i)} \theta_j = 2\pi$$

Gaussian curvature $K(v_i) \simeq \frac{1}{A_i}\left(2\pi - \sum_j \theta_j\right)$

area of 1 ring

subtract the known angles from 2π

Given estimates of H, K,

$$K_{1,2}(v_i) = H(v_i) \pm \sqrt{H(v_i)^2 - K(v_i)}$$

quadratic formula

If you assume that $\|\vec{n}\| = 1$

and now we have a fairly simple way of estimating curvature on a mesh

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Curves: direction vector
principal normal vector

Surfaces: DLBO to estimate H (mean curvature)
Angle Sum to estimate K (Gaussian curvature)
solve for principal curvatures

– now we can discuss how to use this for mesh quality / repair
mesh smoothing, mesh simplification, &c.