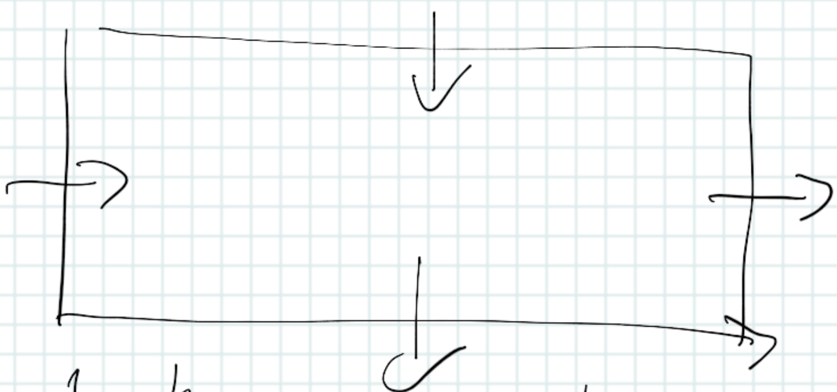
Geometric Processing Oct. 15, 2020 p. 1

Curve : $f: \mathbb{R} \to \mathbb{R}^3$

Surface : $f: \mathbb{R}^2 \to \mathbb{R}^3$

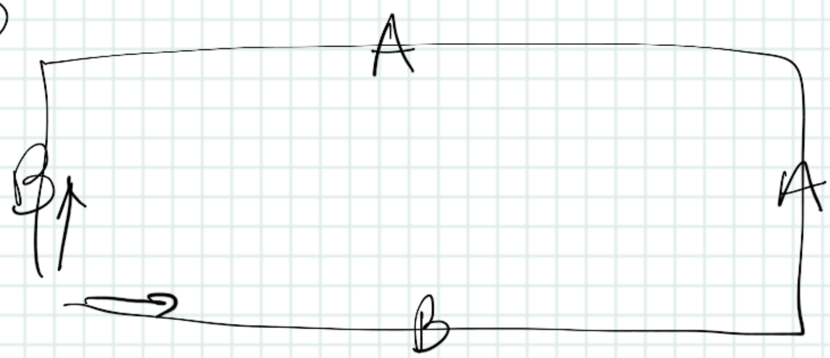


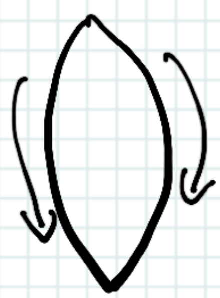
Torus

genus 1

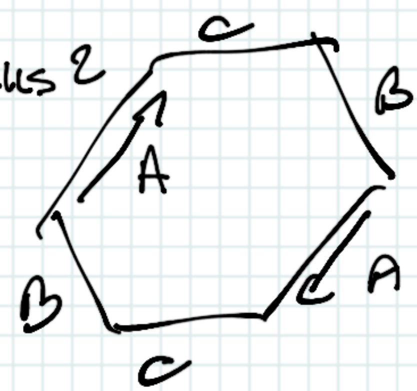
Genus 1 has wrap around in u & v

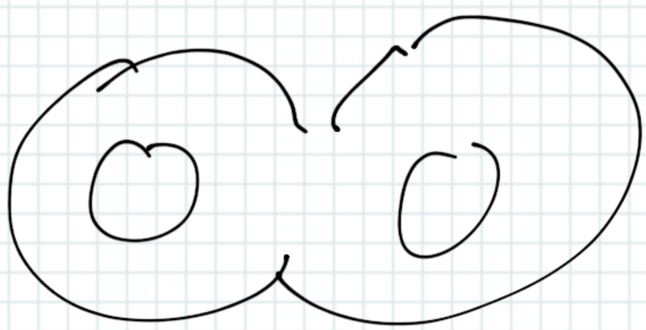
Genus 0





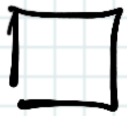
Genus 2





In practice, assume we have a bunch of patches each of which is a topological disk

often square



sometimes arbitrary

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pretend that we have a manifold Ω that is locally equivalent to a disk

a surface is a set S defined by a function

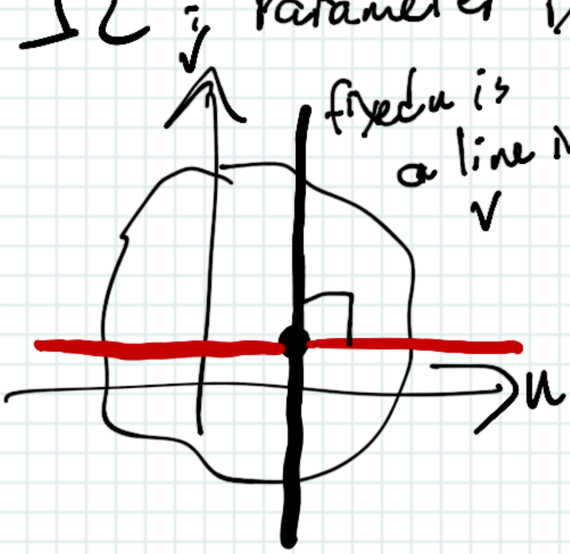
$$f : \Omega \to \mathbb{R}^3 \quad \text{where } \Omega \text{ is a 2-manifold}$$

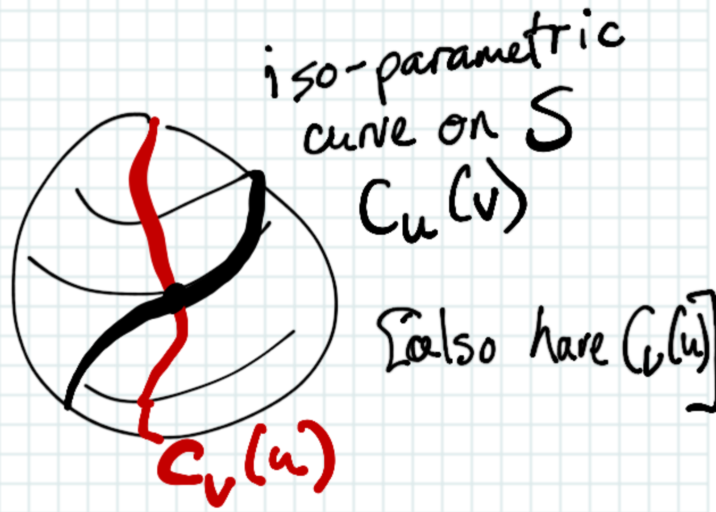
$$f : (u,v) \to x,y,z$$

$$f : \mathbf{u} \to \mathbf{x}$$

What happens if we set u to a constant and vary v?

Ω : Parameter Domain





Also written as:

$$\mathbf{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

define $\mathbf{X}_u = \frac{\delta \mathbf{X}}{\delta u} = \begin{bmatrix} \delta x/\delta u \\ \delta y/\delta u \\ \delta z/\delta u \end{bmatrix}$

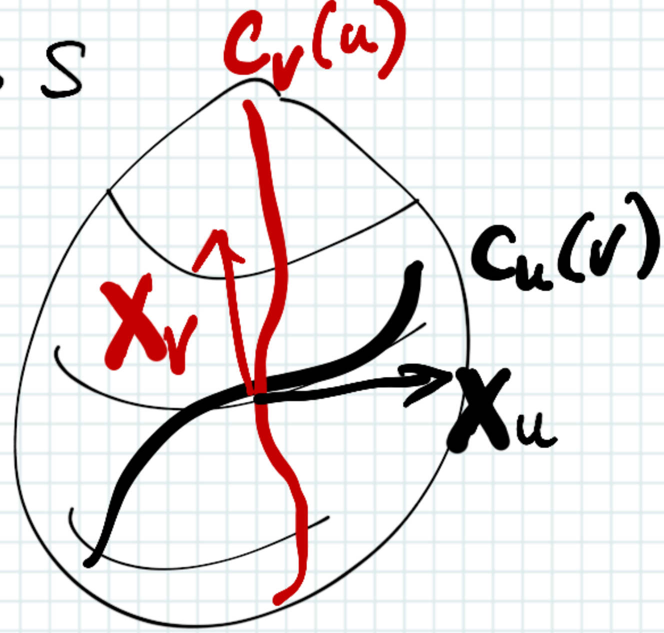
↑ vector of partial derivatives

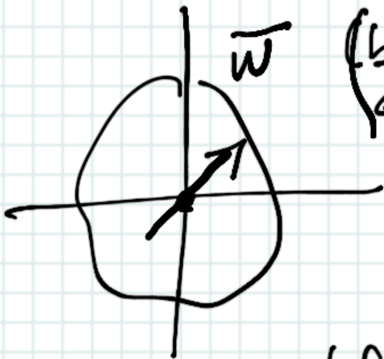
so $\mathbf{X}_u$ is the direction vector of $C_v(u)$

$\mathbf{X}_v$ " " " " " $C_u(v)$

and they are both tangent to the surface

and $\mathbf{X}_u \times \mathbf{X}_v$ is normal to S





(bar is a vector in Ω
arrow is a vector in $\mathbb{R}^3$)

map the line $(u_0, v_0) + \bar{w}t$
to the surface

define $C_w(t) = \mathbf{X}(u_0 + t u_w, v_0 + t u_v)$

$$\vec{w} = \frac{\delta C_w(t)}{\delta t} = \delta \mathbf{X}(u_0 + t u_w, v_0 + t u_w)/\delta t$$

$$= \begin{bmatrix} \delta x(u_0 + t u_w, v_0 + t v_w)/\delta t \\ \delta y(u_0 + t u_w, v_0 + t v_w)/\delta t \\ \delta z(u_0 + t u_w, v_0 + t v_w)/\delta t \end{bmatrix}$$

Then apply the chain rule

derivative of $f(g)$ is $f'(g) \cdot g'$

$$\delta x(u + t u_w, v_0 + t v_w)/\delta t \cdot (u_w, v_w)$$

$$= \underbrace{\begin{bmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} \end{bmatrix}}_{J} \bar{w}$$

J – the Jacobian matrix

The Jacobian matrix measures distortion between Ω & S

For example, let

$$\mathbf{X}(u,v) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} \text{ - just a flat sheet}$$

$$J_{\mathbf{X}} = \begin{bmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

But if $\mathbf{X}(u,v) = \begin{pmatrix} 2u \\ v \\ 0 \end{pmatrix}$, $J = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

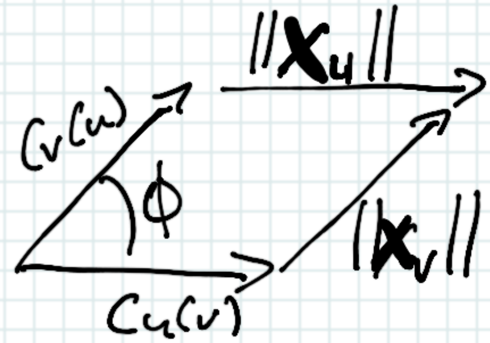
Let $\bar{w}_1, \bar{w}_2$ be two unit vectors in Ω

$$\cos\theta = \bar{w}_1 \cdot \bar{w}_2 = \bar{w}_1^T \bar{w}_2 \quad , \theta \text{ is the angle}$$

What is the angle ϕ between $\vec{w}_1$ and $\vec{w}_2$?

$$\cos\phi = \vec{w}_1 \cdot \vec{w}_2 \, / \, \|\vec{w}_1\| \, \|\vec{w}_2\|$$

$$\begin{aligned} \|\vec{w}_1\| \, \|\vec{w}_2\| \cos\phi &= \vec{w}_1 \cdot \vec{w}_2 \\ &= J\bar{w}_1 \cdot J\bar{w}_2 \\ &= (J\bar{w}_1)^T (J\bar{w}_2) \\ &= \bar{w}_1^T \underbrace{J^T J} \bar{w}_2 \end{aligned}$$



$$\mathbf{I} = J^T J$$

first fundamental form
- a square matrix
- so it has a determinant

$$\mathbf{I} = J^T J = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{X}_u^T \mathbf{X}_u & \mathbf{X}_u^T \mathbf{X}_v \\ \mathbf{X}_v^T \mathbf{X}_u & \mathbf{X}_v^T \mathbf{X}_v \end{bmatrix}$$

E is the distortion squared along $C_u(v)$
G " " " " " $C_v(u)$
F is 0 iff $\mathbf{X}_u \perp \mathbf{X}_v$

To measure the length of a curve on the surface

$$\ell(a,b) = \int_a^b \sqrt{(u_t\ v_t)\,\mathrm{I}\,(u_t\ v_t)^T}$$

$$= \int_a^b \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2}\ dt$$

Area $\quad A = \iint_U \sqrt{\det(\mathrm{I})}\ du\,dv$

U is a region $U \subset \Omega$

$$= \iint_U \sqrt{EG - F^2}\ du\,dv$$