01: Functions & Calculus

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Contents

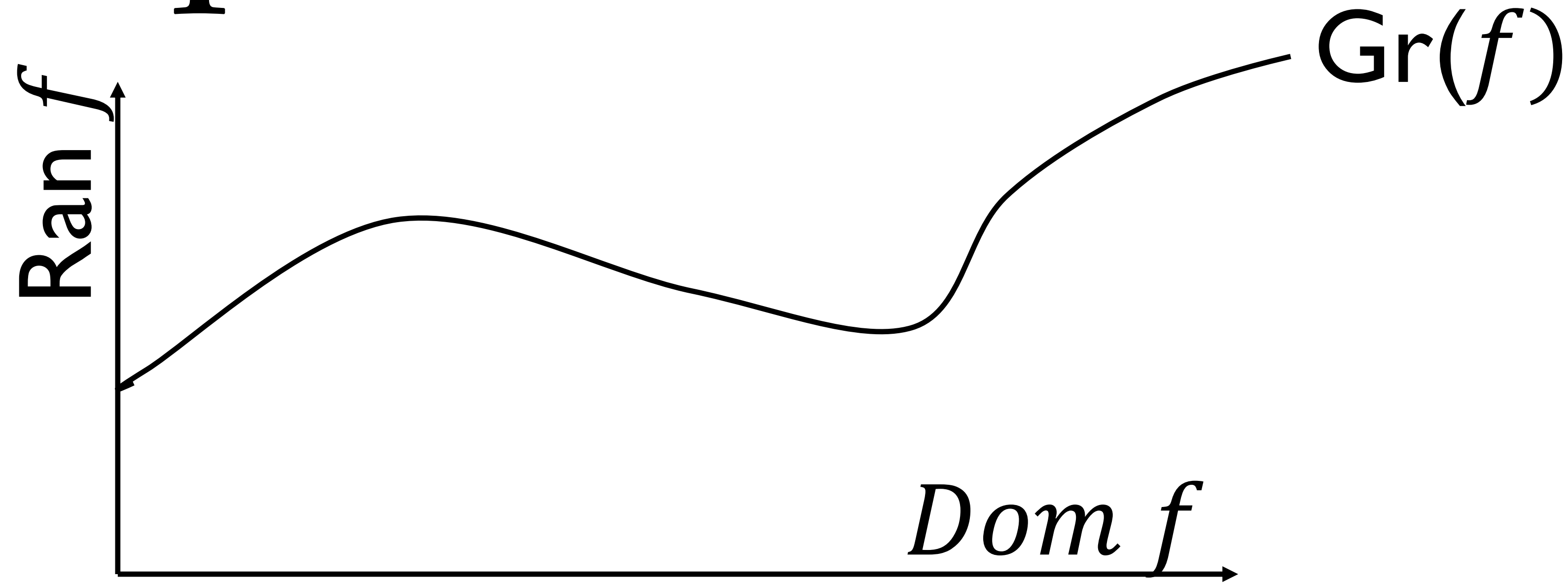
Functions

- Also known as *maps*
- Define a relationship between two sets
  - Domain
  - Range (aka Codomain)
- Given a value in the domain
- Assign a value in the range

Notation

- Most general: $f: Dom\ f \to Ran\ f$
- More specific: $f: \mathbb{R} \to \mathbb{R}$
- Explicit: $f(x) = x^{20} + \sin x$

Graph of a Function



$$Gr(f) = \{(x, y): x \in Dom\ f, y = f(x) \in Ran\ f\}$$

- Set of points defined by the function
- Notice we are now in 2-D
- The *embedding space* of the graph

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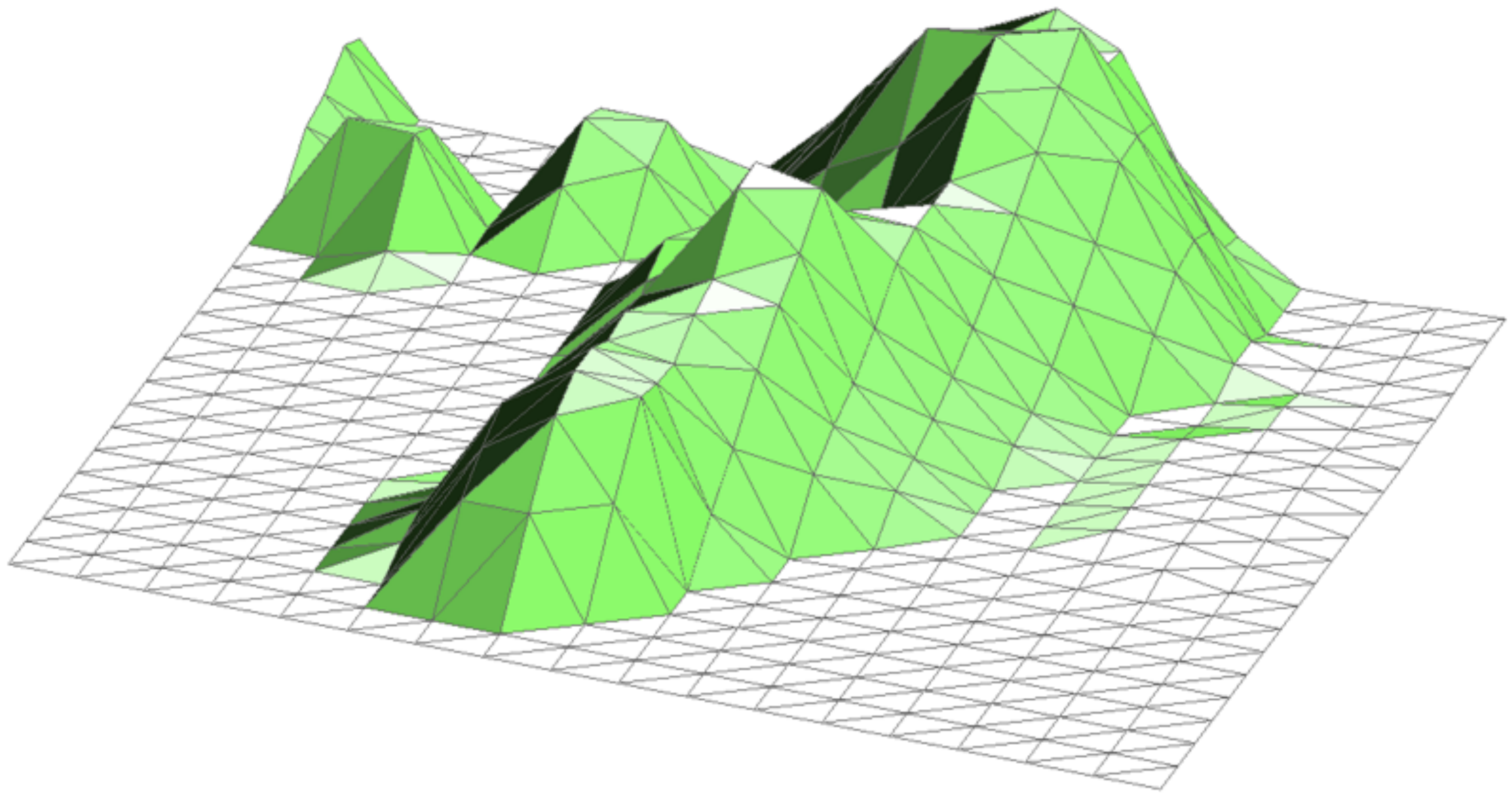
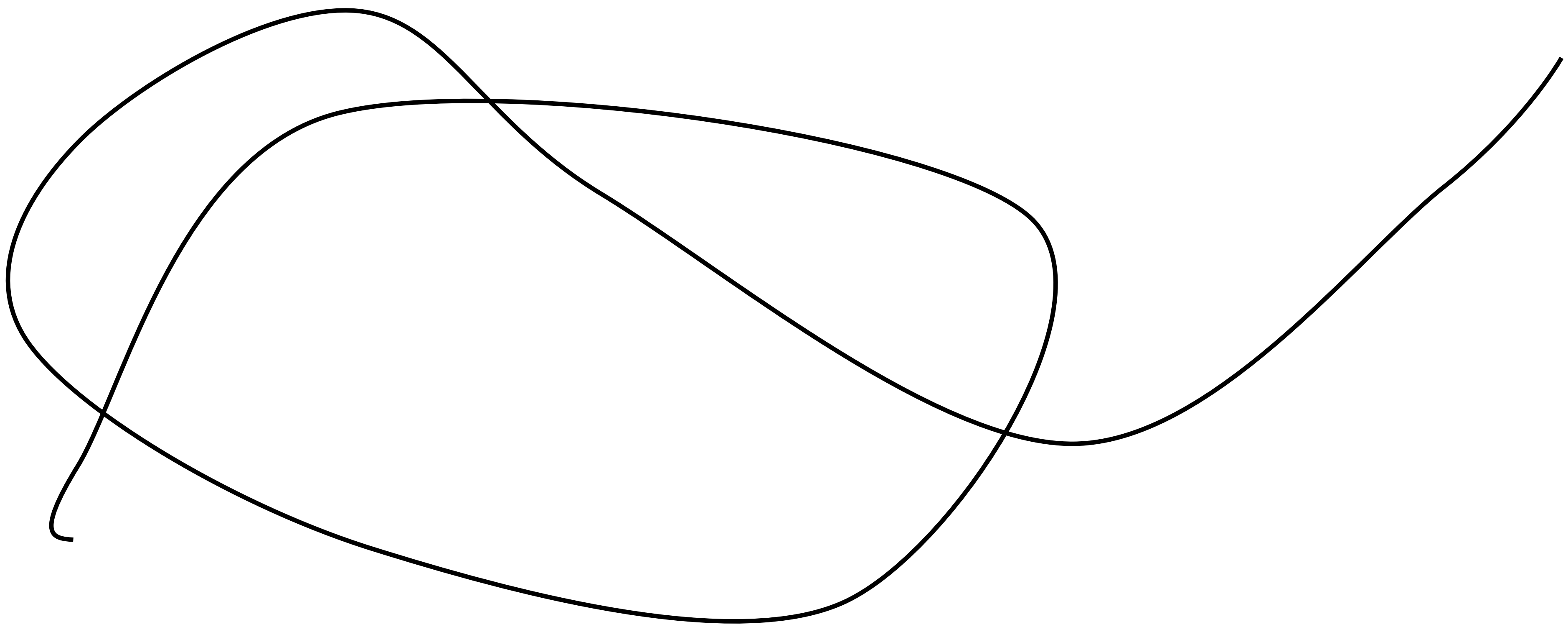
$f: \mathbb{R}^2 \to \mathbb{R}$: Height Field



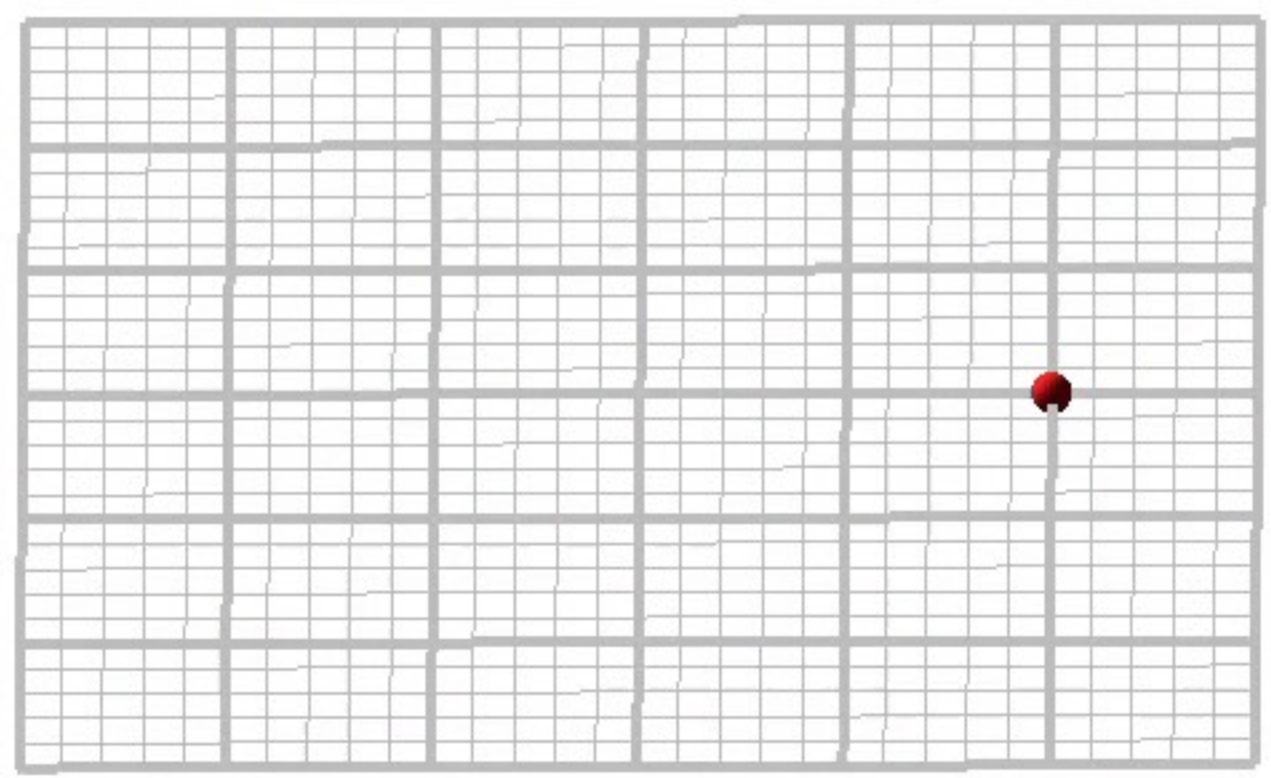
- For every point in the plane, define a *height*
- The graph is then a *terrain*
  - i.e. a surface in a 3D embedding space

$f: \mathbb{R} \to \mathbb{R}^2$: Planar Curve



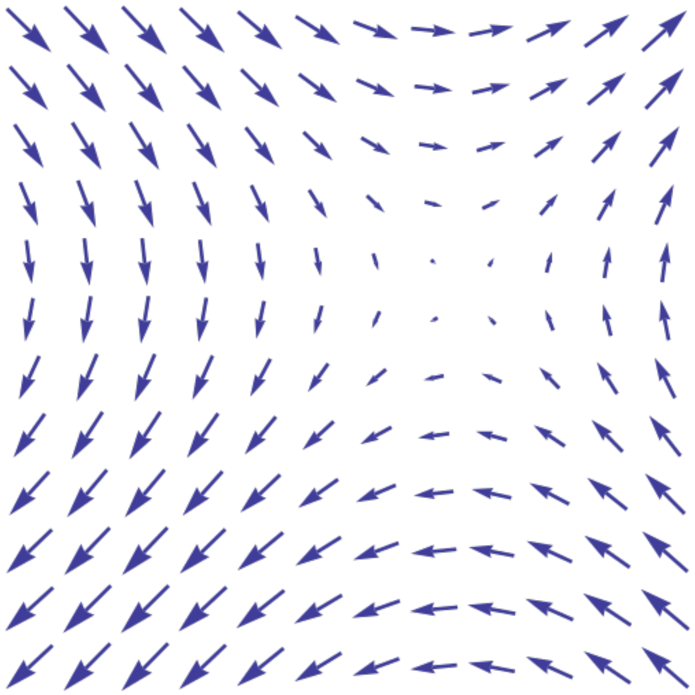
- Also known as a *parametric* curve
- Assigns (x,y) coordinates to each *t* value
- Can self-intersect, &c.

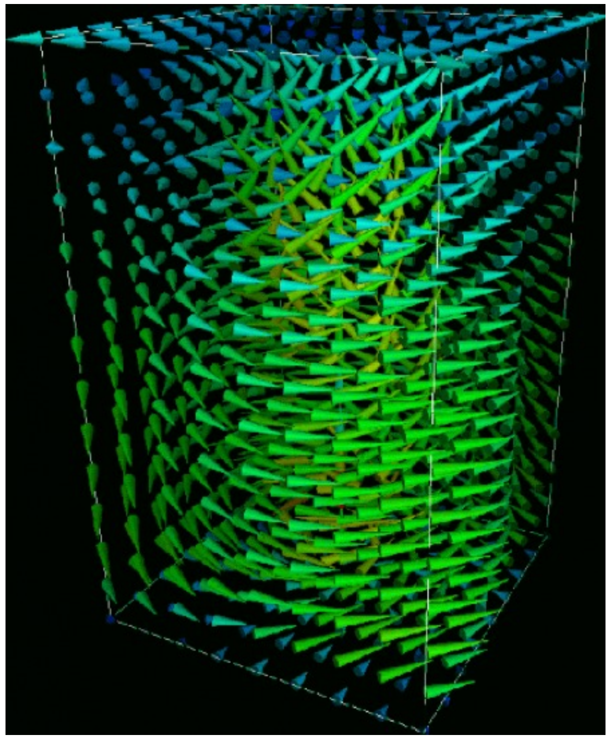
$f: \mathbb{R} \to \mathbb{R}^3$: Space Curve



$$Helix = \left\{ (\cos t, \sin t, t) : 0 \le t \le \frac{5}{2}\pi \right\}$$

Vector Fields





- Fields whose output is a vector
- Range has same dimension as domain
- And there is added *semantic* meaning
- Frequently represents flow

Manifolds

- Generalisation of the idea of surfaces
- Sets that are *locally* equivalent to surfaces
- Defined for *any* dimension
  - 1-manifold – a curve
  - 2-manifold – a surface
  - 3-manifold – a volume
- Always exist in an *embedding space*

Differential Calculus

- The *derivative* of a function
- Represents the *slope* of a function
  - Rate of change of f with respect to x
- Notation: $f'(x)$ or $\frac{df}{dx}$
- Definition: $\mathrm{f}'(\mathrm{x}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- Can be used to construct graph of function

Rules of Differentiation

- Constant: $\frac{d}{dx}c = 0$
- Addition: $\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$
- Multiplication: $\frac{d}{dx}cf = c\frac{df}{dx}$
- Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
- Polynomials are easy:
  - $\frac{d}{dx}7x^3 + 2x^2 - 5 = 21\mathrm{x}^2 + 4\mathrm{x}$

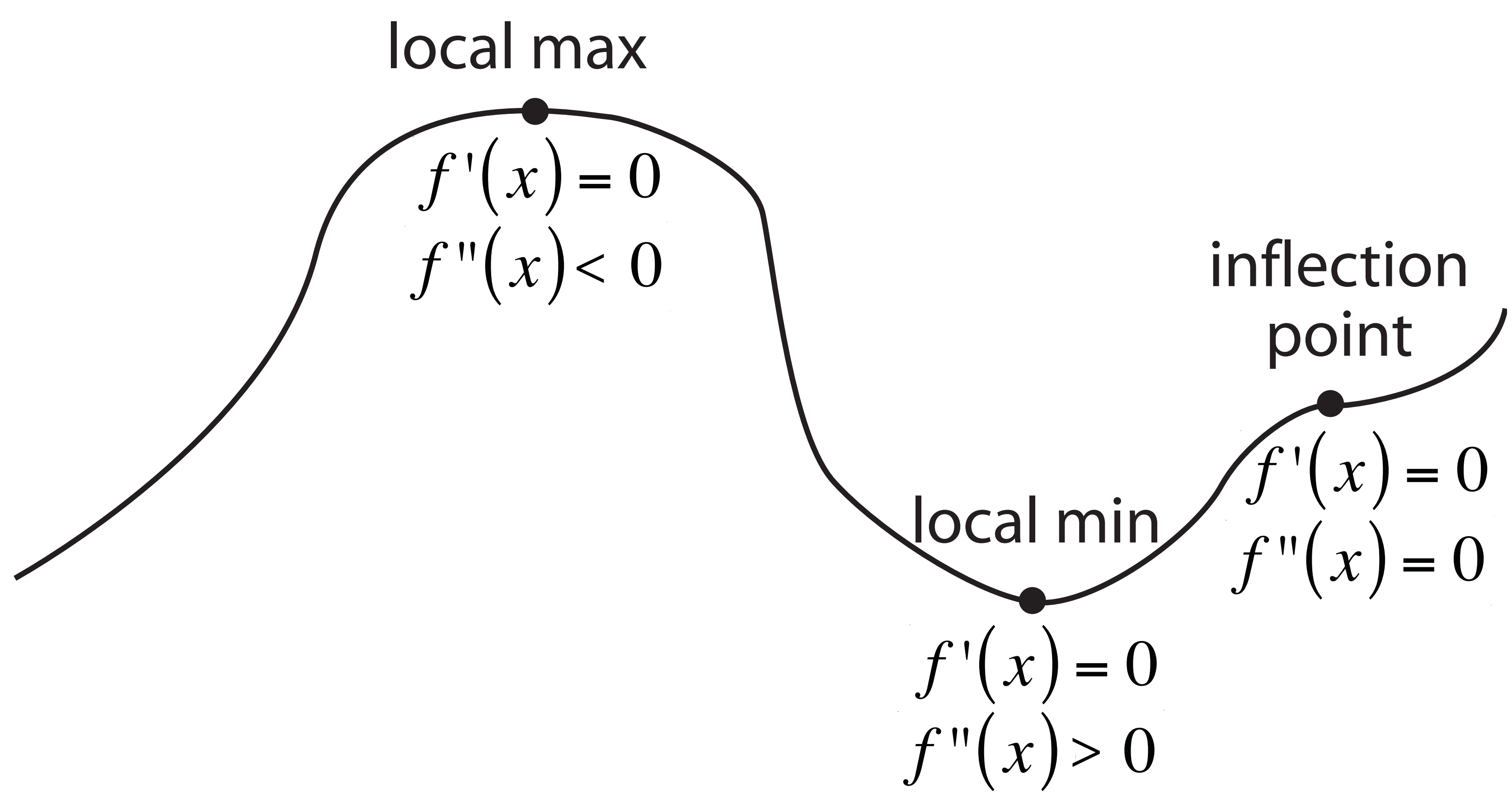
More Rules

- Trigonometry: $\frac{d}{dx}\sin x = \cos x$, &c.
- Exponential: $\frac{d}{dx}e^x = e\hat{}x$
- Logarithmic: $\frac{d}{dx}\ln x = \frac{1}{x}$
- Product Rule: $(\mathrm{fg})' = \mathrm{f}'\mathrm{g} + \mathrm{g}'\mathrm{f}$
- Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
- Chain Rule: $\left(f(g)\right)' = f'(g)g'$

Higher Derivatives

- Notation: $f''(x), f'''(x), f^{(4)}(x), f^{(n)}(x)$
- Or: $\frac{d^2 f}{dx^2}, \frac{d^3 f}{dx^3}, \frac{d^4 f}{dx^4}, \frac{d^n f}{dx^n}$
- Obtained by repeated differentiation

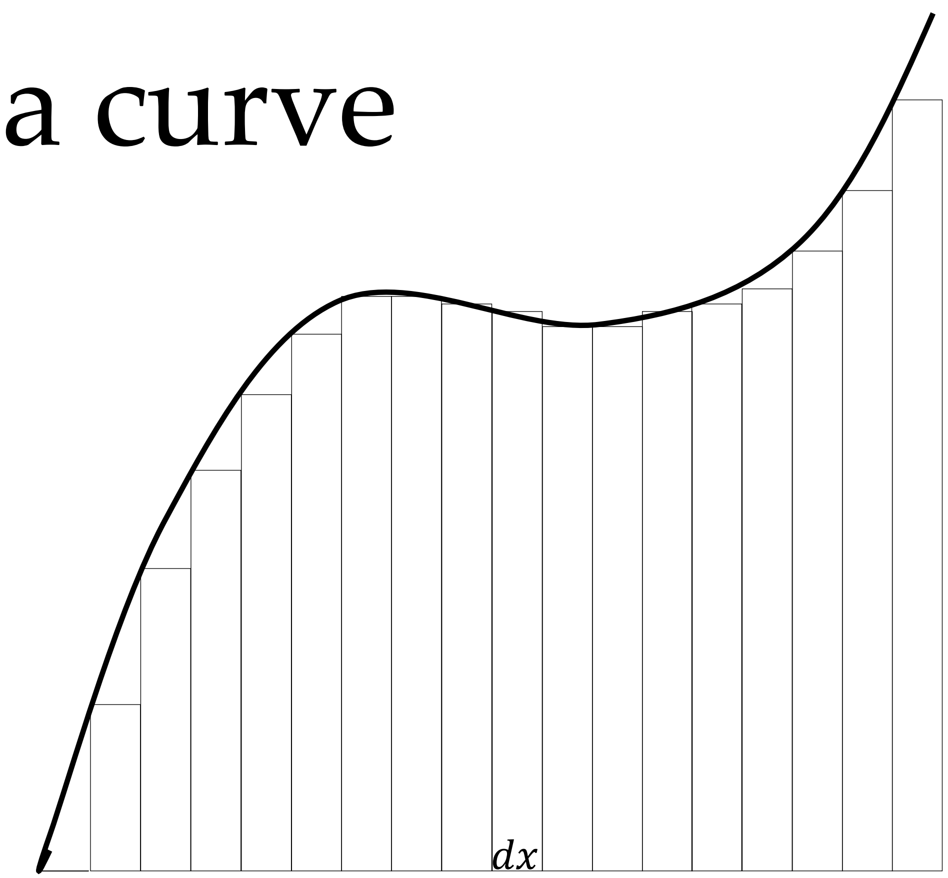
Sketching A Function



- Partition graph into segments
- Defined by roots of derivatives

Integral Calculus

- The *integral* of a function
- Represents the *area* under a curve
- Notation:



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\delta x$$

$$x_i = a + i\delta x$$

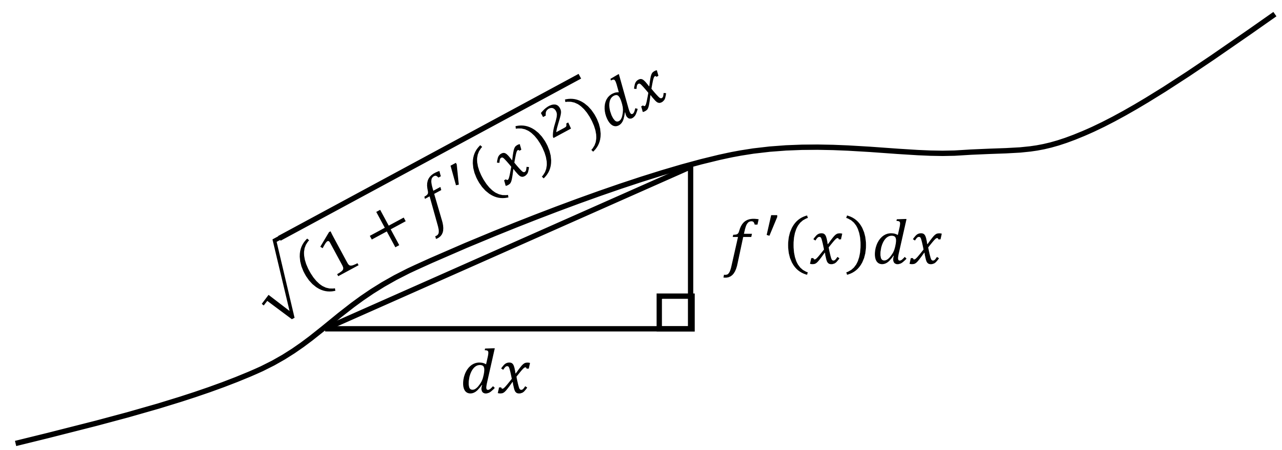
$$\delta x = (b-a)/n$$

- Fundamental Theorem: $\frac{d}{dx}\int f(x)dx = f(x)$

Rules of Integration

- Addition: $\int f + g\, dx = \int f dx + \int g\, dx$
- Multiplication: $\int a\, f(x) dx = a \int f(x) dx$
- Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- Polynomials:
- $\int 7x^3 + x^2 - 7dx = \frac{7}{4}x^4 + \frac{1}{3}x^3 - 7x + c$
- Not all functions are easily integrated
  - Including most of the ones we want

Arc-length Integral



- Computes the *length* of a curve
  - $\int_a^b \sqrt{(1+f'(x)^2)}\,dx$
  - Derivative measures *distortion*
    - *i.e.* difference between curve & domain

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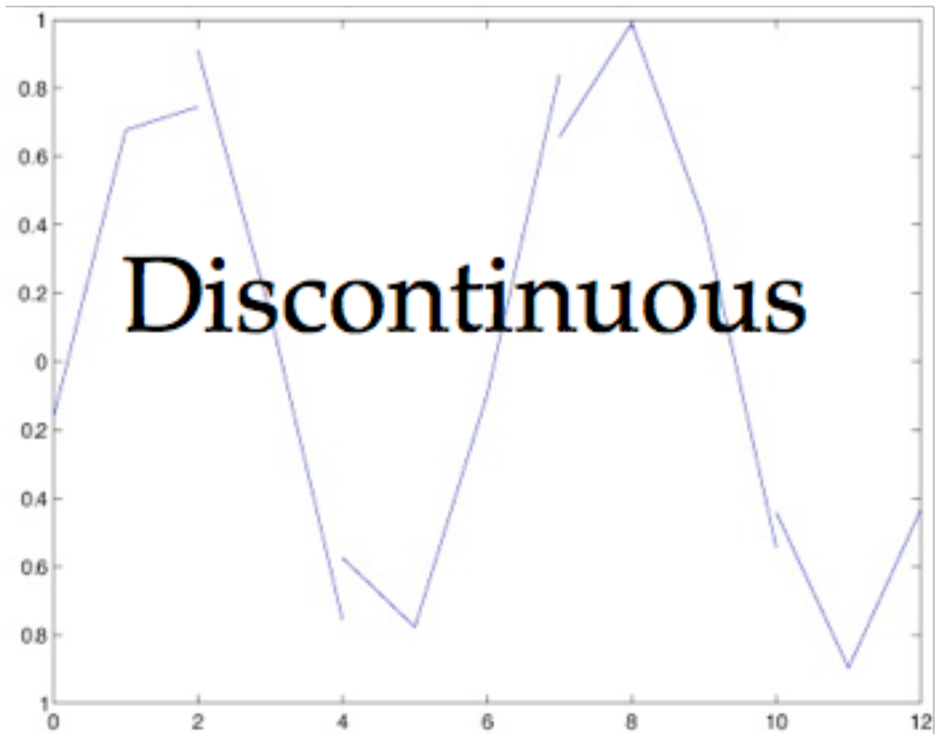
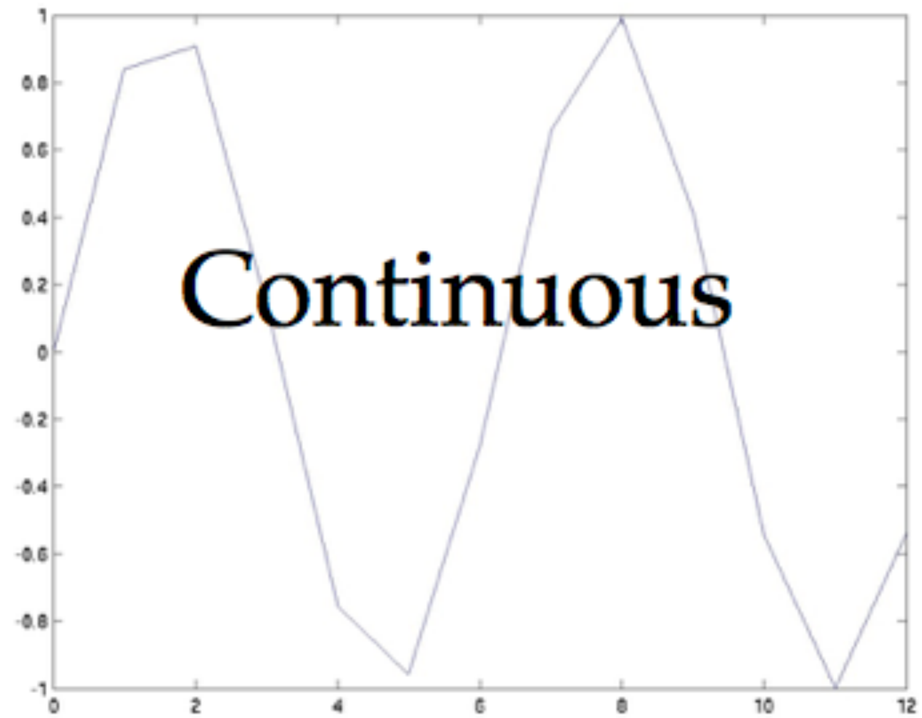
Continuity





- A function *f(x)* is continuous at *a* if:

$$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

- Also known as C^0 continuous

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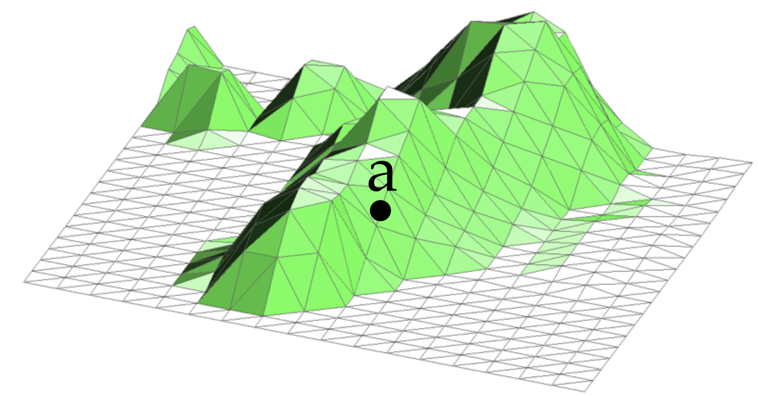
Higher-D Continuity

- We extend the idea of the limit
- The function is the same from any direction
- This leads to *epsilon-delta* proofs
- For our purposes, it generalises

Generalising Limits

- For $f\colon\ \mathbb{R} \to \mathbb{R}$, limits were left or right:
  - $\lim_{x \to a^-} f(x),\ \lim_{x \to a^+} f(x)$



- What about $f\colon \mathbb{R}^2 \to \mathbb{R}$?
  - What does "left" or "right" mean?
- Hmm, let's take a step back
  - And have another look at $f\colon\ \mathbb{R} \to \mathbb{R}$

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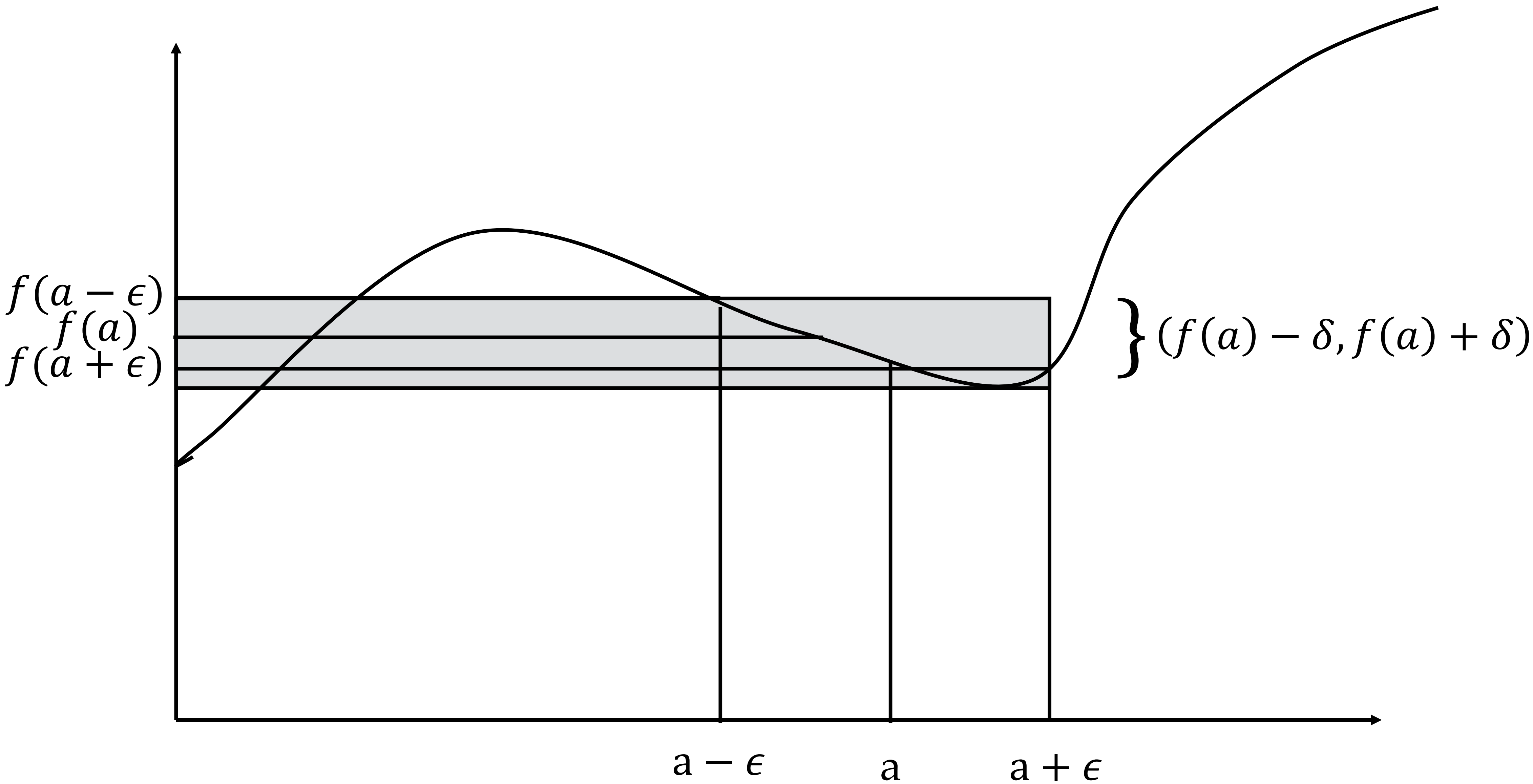
Epsilon Notation

- Limits are *really* about getting close to a point
- So why don't we pick *how close*
- And rewrite it in terms of the distance in x?
  - $\lim_{x \to a^+} f(x) = \lim_{\epsilon \to 0^+} f(a + \epsilon)$
- Now we can express both sides at once:
  - $\lim_{x \to a} f(x) = \lim_{\epsilon \to 0} f(a + \epsilon)$

An Example



An interval in x $(a-\epsilon, a+\epsilon)$

maps to an interval in y

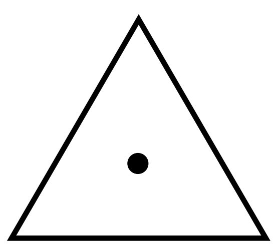
but not necessarily $(f(a-\epsilon), f(a+\epsilon))$

Epsilon-Delta Proofs

- Mathematicians can now generalise
  - You pick the desired $\delta > 0$
  - I choose a suitable $\epsilon > 0$
  - So that if $x \in (a - \epsilon, a + \epsilon)$
  - I guarantee $f(x) \in (f(a) - \delta, f(a) + \delta)$
- And this formalises "f(x) gets close to f(a)"
  - But in an entirely different way

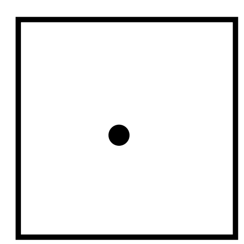
Generalising the Interval

- But how do we do this in 2-D?
- We generalise the idea of an interval first
- But which of the following is best?



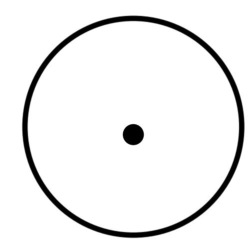
Triangle:
Simplest
Polygon

$$\alpha A + \beta B + \gamma C, \quad \alpha, \beta, \gamma \in (0,1)$$



Square:
Product of
Intervals

$$[a_x - \epsilon, a_x + \epsilon] \times [a_y - \epsilon, a_y + \epsilon]$$



Circle:
Defined
by Distance

$$\{x: dist(x, a) \leq \epsilon\}$$

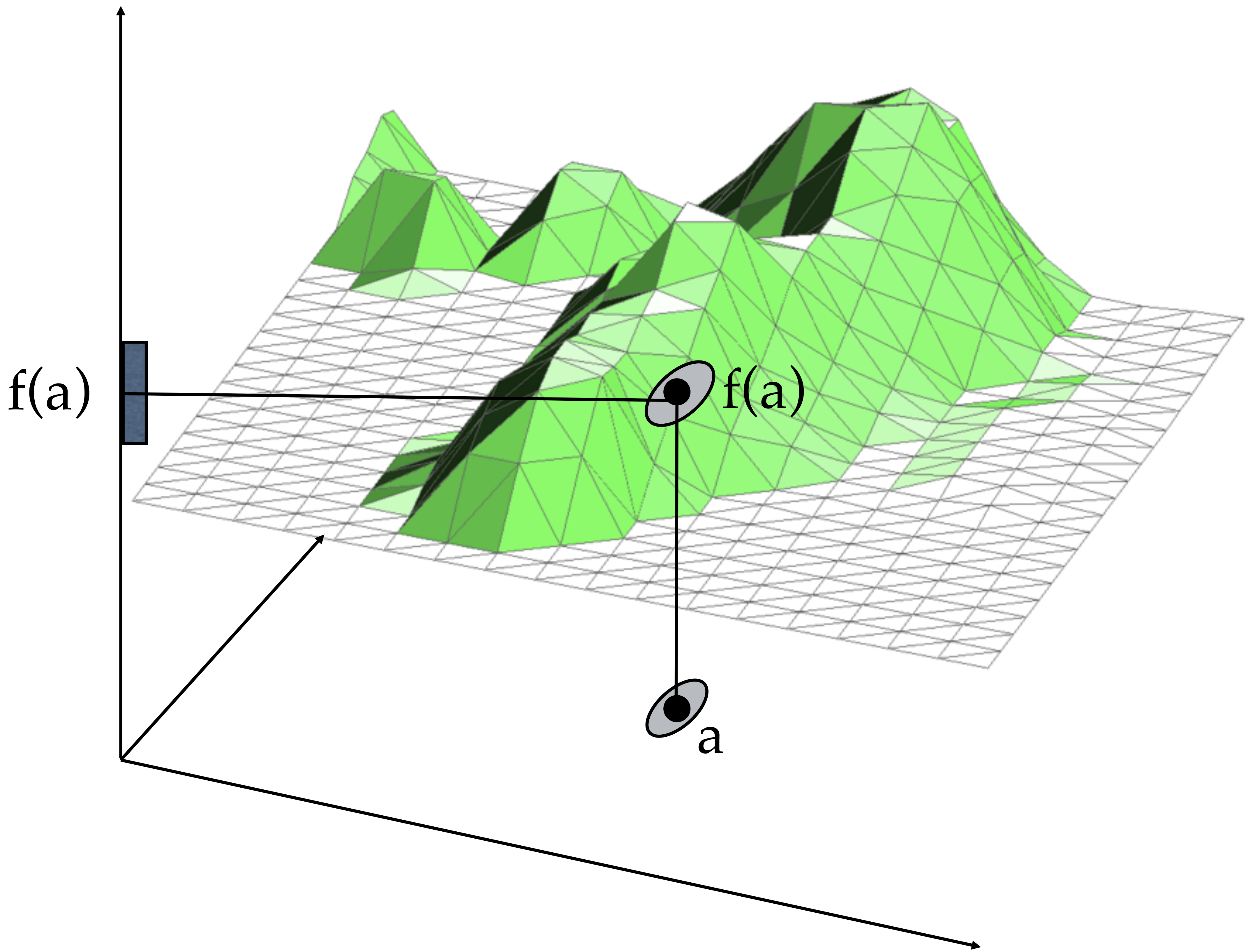
The Neighbourhood

- Epsilon-delta proofs use a circle
  - You pick the desired $\delta > 0$
  - I construct a desired $\epsilon > 0$
  - So that if $dist(x, a) < \epsilon$
  - I guarantee $f(x) \in \ (f(a) - \delta, f(a) + \delta)$
- More generally, we use a *neighbourhood*
- An arbitrary small open set near a point

What does this Mean?

- Well, we needed limits to build calculus
- Limits express how "close" we are
- Which we are *always* interested in
- We can extend calculus to more dimensions
  - Using epsilon-delta proofs

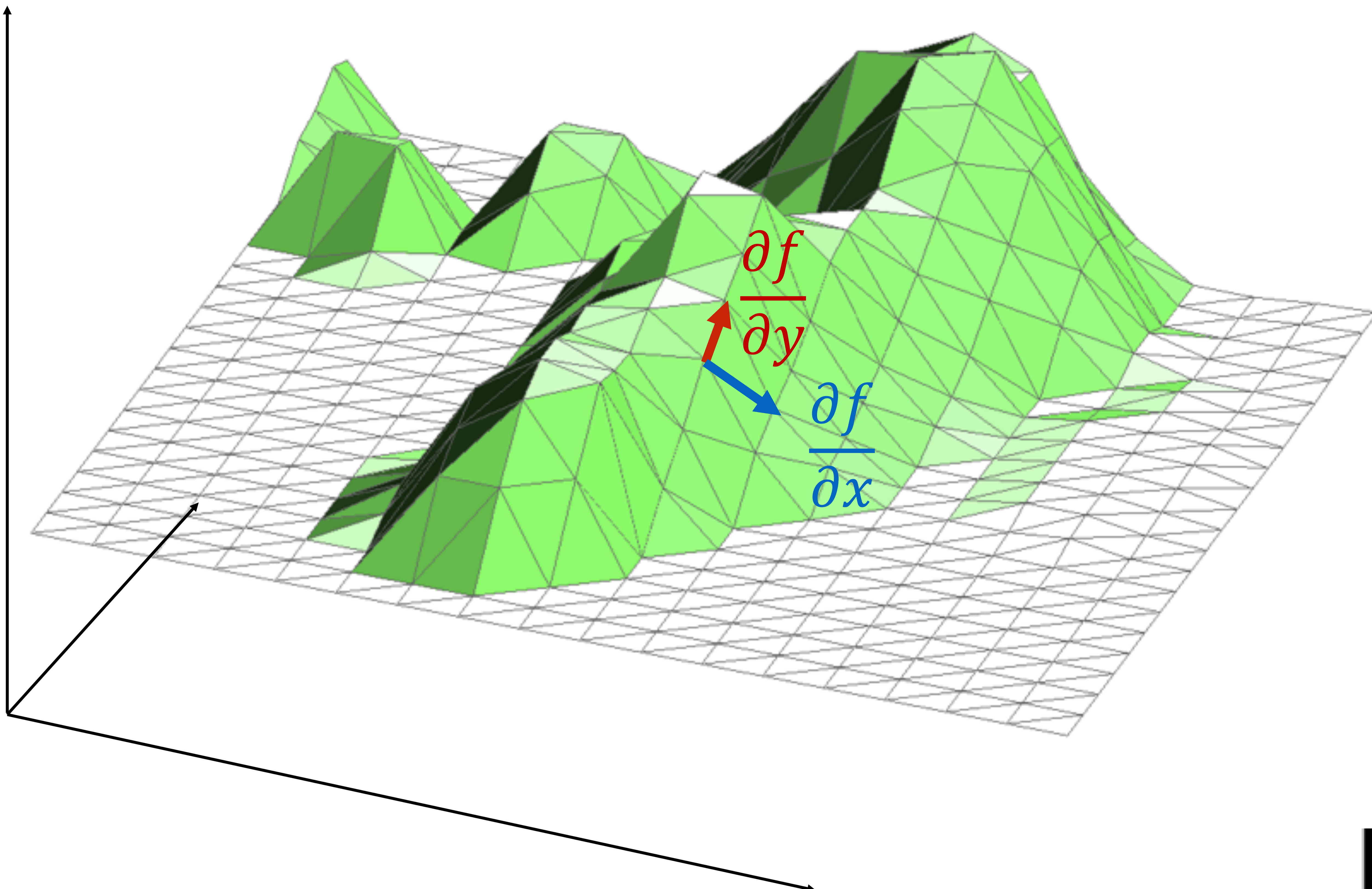
An Example



Multi-D Derivatives

- Slope & rate of change become ambiguous
- We define them *with respect to* a variable
- I.e. rate of change of f as x changes:
  - Notation: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$
- Derived by *fixing* other variables:
  - $\frac{\partial}{\partial x}(x^2y + y^3z - xyz) = 2xy - yz$
  - $\frac{\partial}{\partial y}(x^2y + y^3z - xyz) = x^2 + 3y^2z - xz$
  - $\frac{\partial}{\partial z}(x^2y + y^3z - xyz) = y^3 - xy$

An Example



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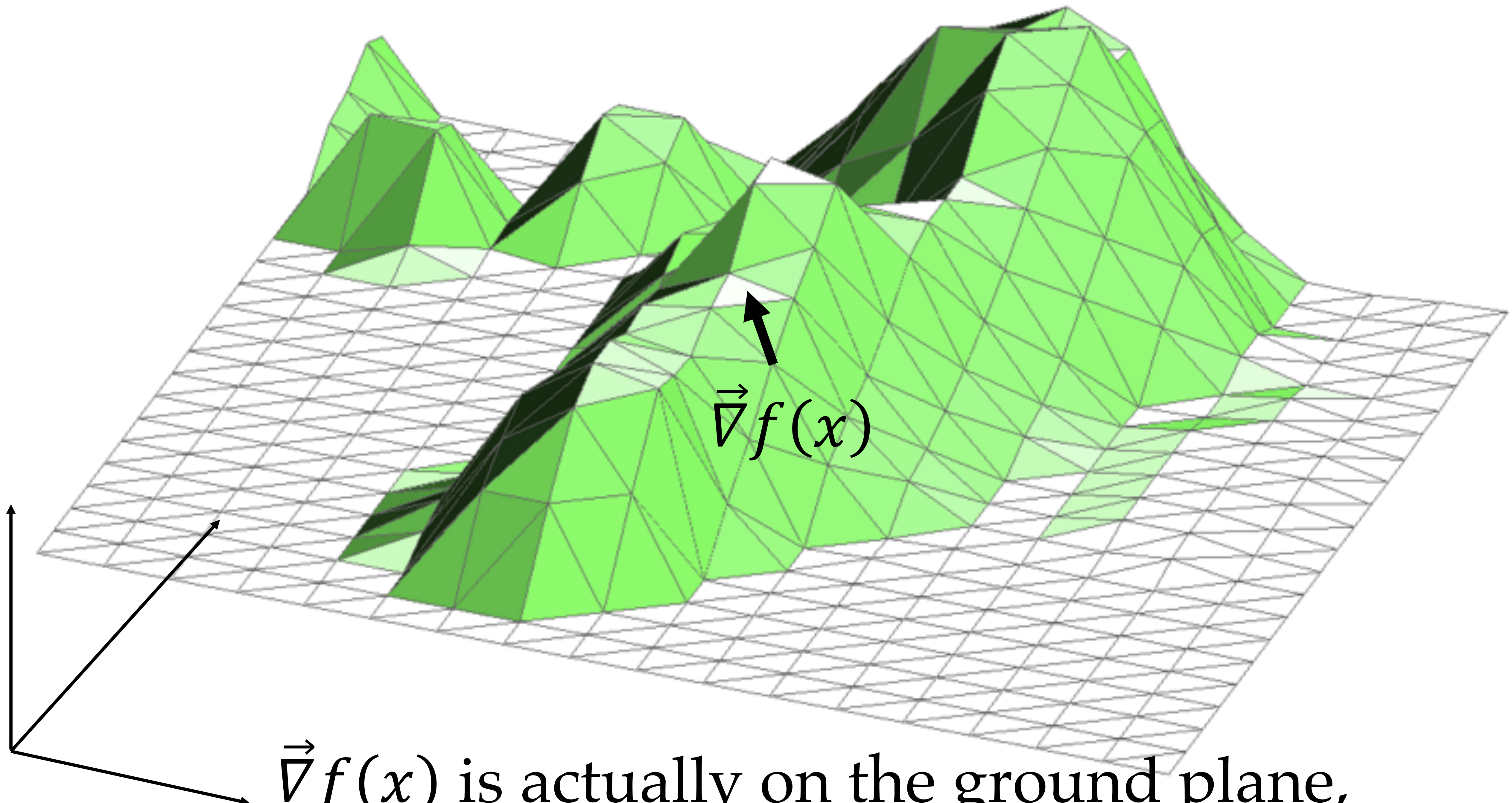
Gradient Vector

- A vector made of the partial derivatives:
  - $\vec{\nabla} f(x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
- Defines a vector field
- Direction & rate of *steepest* ascent
  - So the idea of slope is still with us
  - As is the idea of distortion

An Example



$\vec{\nabla} f(x)$ is actually on the ground plane,
but I'm showing it on the surface
so you can see that it points uphill

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Higher-Order Partials

- Notation specifies *order* of differentiation:
  - $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$
  - $f(x, y) = x^2 y + y^{17} - \sin xy$
  - $\frac{\partial f}{\partial x} = 2xy - y \cos xy$
  - $\frac{\partial f}{\partial y} = x^2 + 17y^{16} - x \cos xy$
  - $\frac{\partial^2 f}{\partial x \partial y} = 2x - \cos xy + xy \sin xy$

Multiple Integrals

- Integration can also be done repeatedly
  - $\int_a^b \left(\int_c^d f(x,y)dx \right) dy$
- Summation is now over a small rectangle
  - $A = [a,b] \times [c,d]$
- And can be rewritten:

$$\int_A f(x,y)dx = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_{i,j}) \delta x \delta y$$

$$x_{i,j} = (a + i\delta x, c + j\delta y)$$

$$\delta x = (b-a)/n, \delta y = (d-c)/n$$

Numerical Calculus

- Most of our functions will *not* be integrable
- We will approximate them numerically
- There are a number of methods:
  - Taylor Series
  - Numerical Integration
  - Table Lookup

Taylor Series

- For any function *f(x)*

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \ldots$$

- We choose an easy value for a.

- For example, look at cosine, with a=0

$$\cos(x) = \cos(0) + \frac{-\sin(0)(x-0)}{1!} + \frac{-\cos(0)(x-0)^2}{2!} + \frac{\sin(0)(x-0)^3}{3!} + \ldots$$

$$= 1 + \frac{-0x}{1} + \frac{-1x^2}{2} + \frac{0x^3}{6} + \ldots$$

- Many functions converge *slowly*

Numerical Integration

- We just go back to our summation:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i)\delta x$$

- And choose a *small* δx
- The rest is standard numerical computation

Table Lookup

- A lot of computations are reused
  - e.g., sin, cos, logarithm
- And we may not need high accuracy
- So store a *lookup table*
  - an array with (e.g.) 1024 values
- Then just take the closest value
- This runs a *lot* faster