COMP 5821M Geometric Processing

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I. We need to understand smooth surfaces mathematically

II. It's easiest to start with smooth curves & build up from there

III. In a parameterized curve, the direction vector is the first derivative dt

IV. Unfortunately, there are multiple equations describing any given curve

V. The solution to this is to pick one parameterization based on the distance along the curve - i.e. the Arc Length Parameterization (ALP)

VI. Working with the ALP means you have no distortion in your description

VIa. We used the derivatives to find the curvature of the curve

VII. We want to do the same for surfaces

VIII. We can study surfaces by looking at them as a set of curves - eg cross-sections

IX. When we do this, we look isoparametric curves - curves following u or v across the surface.

X. We discovered the Jacobian matrix J, and the first & second fundamental forms **I** and **II** which capture forms of distortion

XI. We saw that these describe the relationship between a small circle in Ω and a small ellipse on the surface called the ellipse of anisotropy.

XII. We observed that ideally we would have a surface parameterization where the ellipse was always a circle.

! XIII. But it can't be done, so we need to look at "intrinsic" properties of the surface instead.

XIV. We defined the two principal curvatures of the surface using a variation on the isoparametric curves. K_1 K_2 combine to give K, H, the Gaussian & mean curvatures, of which K is intrinsic.

Now we start simplifying & turn it into something we can use

Define an "operator" over the group of functions to be a mapping that takes a function f as a parameter and returns a new function g

e.g. $| \ |$ is the absolute value operator
Pass in f and it returns $|f|$

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $|f|: \mathbb{R} \to \mathbb{R}$

$f(x)$ $\qquad$ $|f|(x) \neq |f(x)|$
semantically

The operator ∇ takes a function f and returns

vector where each element has as many components as f

$$\nabla f = \left(\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z} \right)$$ a vector-valued function

The operator div takes a function f and returns

→ adds the components $$\operatorname{div} f = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} + \frac{\delta f}{\delta z}$$ div - divergence

Define Δf the Laplace operator ("del f") to be

$$\Delta f = \operatorname{div} \nabla f$$

$$= \operatorname{div} \left(\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z} \right)$$

$$= \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2}$$ → really just a variation on Pythagoras $x^2 + y^2 + z^2$

On a surface S, defined by $\mathbf{X}(u,v)$ [the position function $\mathbf{X}: \Omega \to \mathbb{R}^3$]

$$\Delta \mathbf{X} = \frac{\delta^2 \mathbf{X}}{\delta u^2} + \frac{\delta^2 \mathbf{X}}{\delta v^2}$$

$$= \mathbf{X}_{uu} + \mathbf{X}_{vv}$$

Related to this is the "Laplace-Beltrami" operator

$\Delta_S f = \operatorname{div}_S \nabla_S f$, but everyone is sloppy and writes

$\Delta f = \operatorname{div} \nabla f$ instead

Note that the Laplace-Beltrami operator applies to *any* function f on the surface - e.g. colour
position

so if we apply Laplace-Beltrami to the position function $\mathbf{X}$, it has been proved that:

$$\underbrace{\Delta_s \mathbf{X}}_{\text{intrinsic}} = -2 \underset{\text{mean curvature}}{H} \underset{\text{normal vector}}{\vec{n}}$$

[stated without proof]

we will estimate this

Now, in an ideal world, there would be a perfect parameterization, and a side effect of that would be that the direction vectors of the isoparametric curves would be $\perp$ to each other, would be units, and the first fundamental form $\mathbf{I}$ would be the identity matrix. But then the normal vector would be length 1.

We cheat. We assume that $\|\vec{n}\| = 1$, since we usually have a unit normal available. Then if we have approximated $\Delta_s \mathbf{X}$ and assumed $\|\vec{n}\| = 1$, we can solve for H.

$$\Delta_s \mathbf{X} = \operatorname{div} \nabla \mathbf{X}_s = \operatorname{div}\left(\frac{\delta \mathbf{X}}{\delta u}, \frac{\delta \mathbf{X}}{\delta v}\right)$$

vector quantities

$$= \frac{\delta^2 \mathbf{X}}{\delta u} + \frac{\delta^2 \mathbf{X}}{\delta v}$$

$=$ another vector

All of this is for smooth surfaces.

We have triangulated meshes, so we have to do it again [more or less] to work out what this means in practice