Assume we have a curve

$\mathbf{X}$ is a point in $\mathbb{R}^2$ or $\mathbb{R}^3$

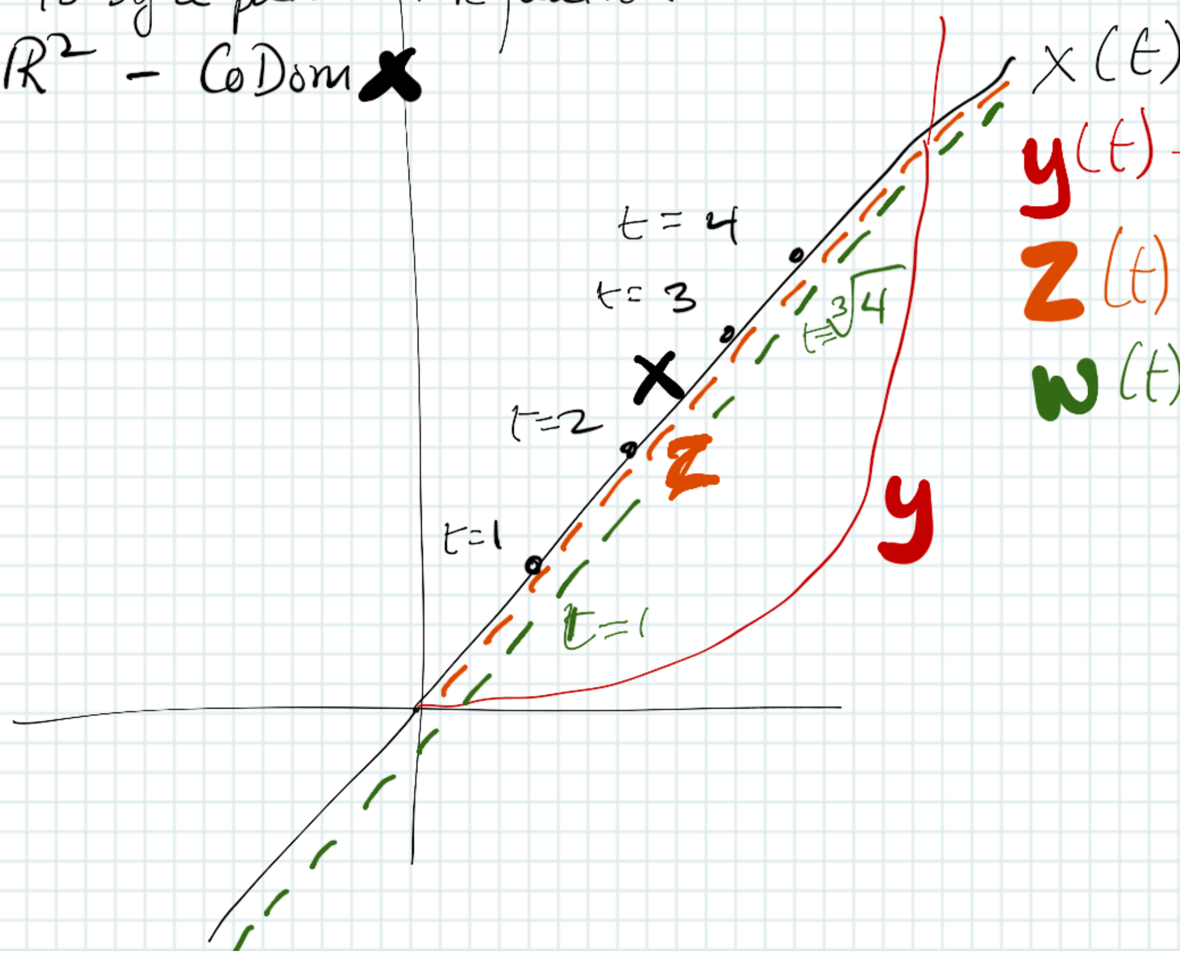
$\mathbf{X}(t)$ is a function (i.e. a parametric description)

$$\mathbf{X}(t) : \mathbb{R} \longrightarrow \mathbb{R}^d$$

t	$\mathbf{X}$
Dom $\mathbf{X}$.	Range $\mathbf{X}$
Domain	Range

A curve is the set of points mapped to by a parametric function from $\mathbb{R}^2$ - CoDom $\mathbf{X}$

$$\mathbf{X}(t) = (t, t)$$

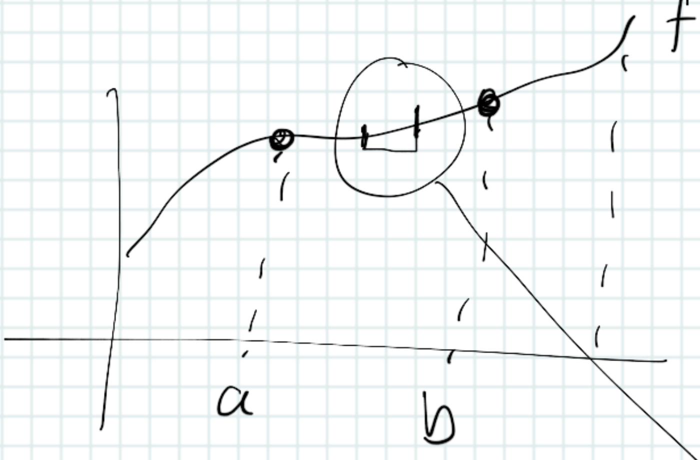


$\mathbf{y}(t) = (t, t^2)$

$\mathbf{Z}(t) = (t^2, t^2)$

$\mathbf{w}(t) = (t^3, t^3)$

Problem: Which description should we use for a curve?



Length $= \int_a^b \sqrt{1+f'(x)^2}\,dx$

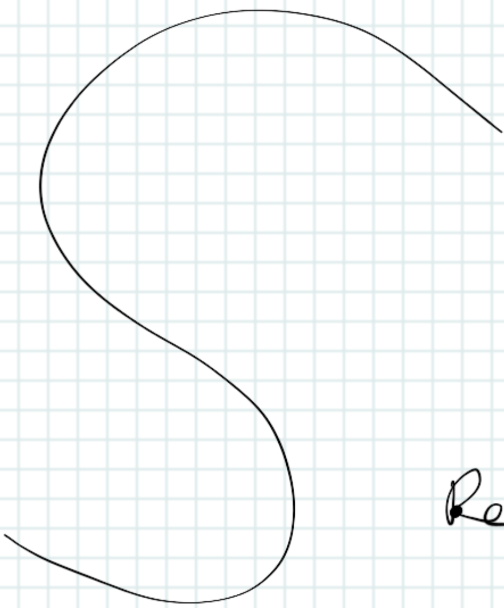
$\frac{h}{\Delta x} = \frac{\text{rise}}{\text{run}}$

$f'(x)$

$h = f'(x)\cdot \Delta x$

$d = \sqrt{\Delta x^2 + (f'(x)\Delta x)^2}$

$= \Delta x \sqrt{1+f'(x)^2}$



Rewrite $f(x)$ as $(t, f(t))$

$\mathbf{f}(t) = (t, f(t))$

& then it's parameterized & I'm ok

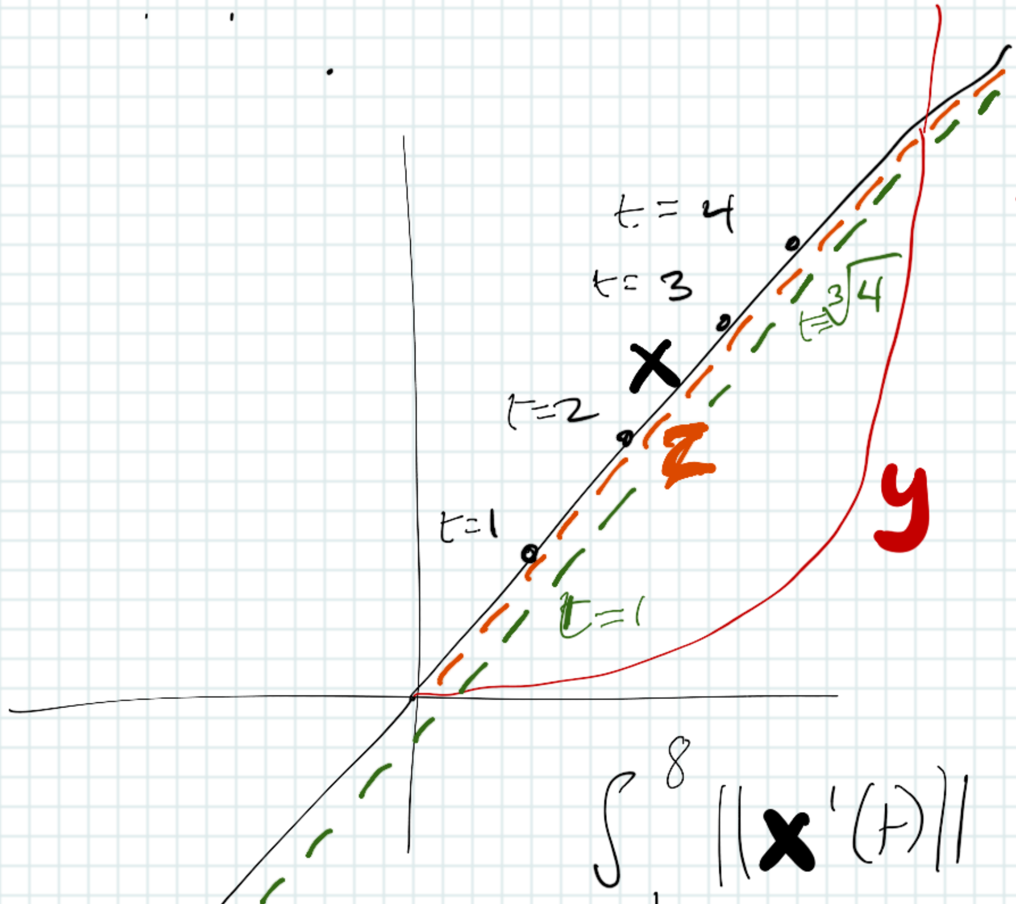
Consider $\frac{d}{dt}\mathbf{f}'(t) = (1, f'(t))$

velocity is $\mathbf{f}'(t)$, speed is $\|\mathbf{f}'(t)\|$

aka direction vector of the curve

distance travelled (length) $= \int_a^b \|\mathbf{f}'(t)\|\,dt$

$$\mathbf{x}(t) = (t, t)$$



$$\mathbf{y}(t) = (t, t^2)$$

$$\mathbf{z}(t) = (t^2, t^2)$$

$$\mathbf{w}(t) = (t^3, t^3)$$

$$\int_1^8 \|\mathbf{x}'(t)\|\, dt = \int_1^8 \|(1,1)\|\, dt = \int_1^8 \sqrt{2}\, dt = \sqrt{2}t\Big|_1^8 = 7\sqrt{2}$$

Length is independent of parameterization

$$\int_1^2 \|\mathbf{w}'(t)\|\, dt = \int_1^2 \|(3t^2, 3t^2)\|\, dt = \int_1^2 3\sqrt{2}t^2\, dt = \sqrt{2}t^3\Big|_1^2 = 7\sqrt{2}$$

Define the arc-length parameterization of a curve (ALP) to be the parameterization where the arc length is the parameter, i.e. given

u is our new AL parameter

$$s(u) = \int_a^u \| X'(t) \| \, dt$$

the t-value where $u = 0$

define $\mathbf{X}(s)$ to be the ALP s.t.

$$\int_a^s \| \mathbf{X}'(t) \| \, dt = s$$

i.e. the arc-length under ALP gives your parameter back.

In general, an ALP exists, but may be hard to construct.

The ALP is the parameterization of the curve where your speed is a constant of $\underline{1}$

i.e. $\| \mathbf{X}'(s) \| = 1$ unit direction vector
unit tangent vector

$$\| \mathbf{X}'(s) \| = \sqrt{\mathbf{X}'(s) \cdot \mathbf{X}'(s)} = 1$$

$$\mathbf{X}'(s) \cdot \mathbf{X}'(s) = 1$$

Product Rule

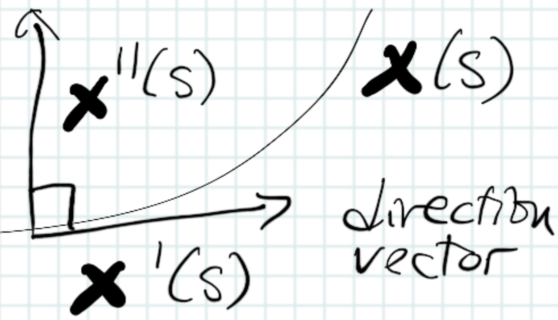
$(fg)' = f'g + g'f$

$$\mathbf{X}''(s)\mathbf{X}'(s) + \mathbf{X}'(s)\mathbf{X}''(s) = 0$$

OR $\cancel{2}\,\mathbf{X}''(s)\cdot\mathbf{X}'(s) = 0$

$$\mathbf{X}''(s) \perp \mathbf{X}'(s)$$

$\mathbf{X}''(s)$ is <u>normal</u> to the curve
& is called principal normal vector



$\mathbf{X}(s)$ arc length parameterization
$\mathbf{X}'(s)$ direction vector
$\mathbf{X}''(s)$ principal normal vector

Can we have $\|\mathbf{X}''(s)\| = 0$?

Consider $\mathbf{V}(t) = \left(\frac{1}{\sqrt{2}}t, \frac{1}{\sqrt{2}}t\right)$ the ALP of $y=x$

$$\mathbf{V}(s) = \left(\frac{1}{\sqrt{2}}s, \frac{1}{\sqrt{2}}s\right)$$

$$\mathbf{V}'(s) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{V}''(s) = (0, 0)$$

as required

When is $\|\mathbf{X}''(s)\| = 1$?

Let $\mathbf{X}(s) = (\sin s, \cos s)$ $\quad \|\mathbf{X}(s)\| = 1$

$\mathbf{X}'(s) = (\cos s, -\sin s)$ $\quad \|\mathbf{X}'(s)\| = 1$

$\mathbf{X}''(s) = (-\sin s, -\cos s)$ $\quad \|\mathbf{X}''(s)\| = 1$

a circle has a constant length p.n.v. of 1 p

How about a circle of radius 2?

The magnitude of the p.n.v. $\mathbf{X}''(s)$ is the radius of the tangent circle at s. and is known as the curvature

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if you have $\mathbf{X}(t)$, $\quad \mathbf{X}'(s) = \dfrac{\mathbf{X}'(t)}{\|\mathbf{X}'(t)\|}$

you can find $\mathbf{X}''(t)$

If we do have ALP, then take $\mathbf{n} = \mathbf{X}'(s) \times \mathbf{X}''(s)$ normalize them & you get a coord. system or <u>frame</u> in particular, the Frénet frame for every point on the curve

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If we have $\mathbf{X}(t)$, take $\mathbf{n} = \mathbf{X}'(t) \times \mathbf{X}''(t)$

$$\frac{\mathbf{X}''(s)}{\|\mathbf{X}''(s)\|} = \frac{\mathbf{n}}{\|\mathbf{n}\|} \times \frac{\mathbf{X}'(t)}{\|\mathbf{X}'(t)\|}$$