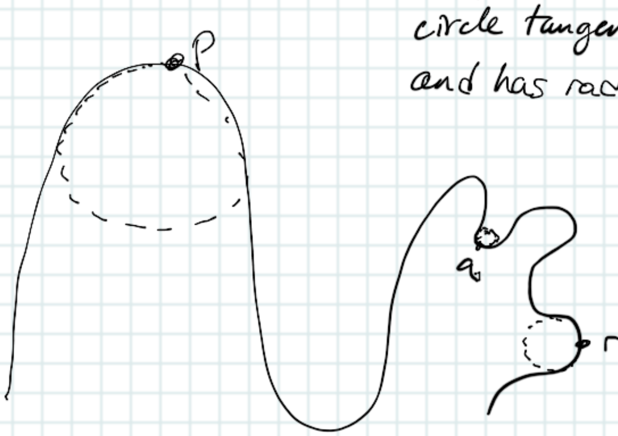
Given a curve $\mathbf{X}(s)$ in the arc length parameterization (ALP) and a point p on the curve, fit a circle tangent to the curve at p. In general, the radius of the circle tangent at p changes as we travel along the curve and has radius



$$\frac{1}{K} = \frac{1}{\|\mathbf{X}''(s)\|}$$

"Kappa" is the "curvature"

principal normal vector

In 3D, this is still true, since we showed how to construct a Frénet frame

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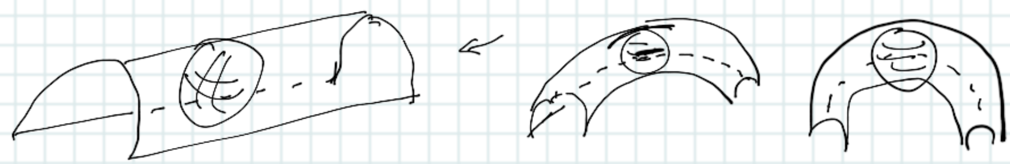
What about a surface S?

curved in one direction but not the other



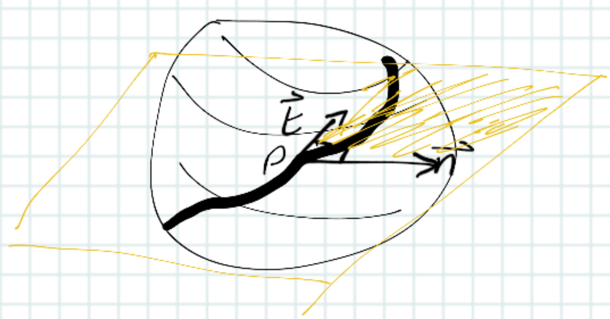
we can fit a sphere into this one way, but not the other

→ this is why there is no ALP for surfaces

Let $\vec{t} = u_t \mathbf{X}_u + v_t \mathbf{X}_v$ be a tangent vector at $p \in S$ corresponding to a vector $\vec{t} \in \Omega$

We know that S has a normal vector $\vec{n}$ at p. Define P to be the plane through p with vectors $\vec{n}, \vec{t}$. The curve

$$P = p + \alpha\vec{n} + \beta\vec{t}$$



we can consider the curve of intersection between P & S

$$C_{PS} = P \cap S$$

which depends on $\vec{t}$

$K_n(t)$ is the "normal curvature" of S at p along a curve C

$$K_n(t) = \frac{\bar{t}^T \mathbf{II}\, \bar{t}}{\bar{t}^T \mathbf{I}\, \bar{t}}$$

$\mathbf{II}$ is the second fundamental form

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \underbrace{\mathbf{X}_{uu}^T \vec{n}} & \mathbf{X}_{vu}^T \vec{n} \\ \mathbf{X}_{uv}^T \vec{n} & \mathbf{X}_{vv}^T \vec{n} \end{bmatrix}$$

dot product between $\vec{n}$ and a second derivative

$$K_n(t) = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2}$$

u_t, v_t are the components of $\bar{t}$ in Ω

rational quadratic which can be solved.

you get two values – a maximum value K_1 } the "principal curvatures"
a minimum value K_2 }

with corresponding directions $\bar{t}_1, \bar{t}_2 \in \Omega$

$\vec{t}_1, \vec{t}_2 \in \mathbb{R}^3$ on S

The ellipsoid that fits the surface at p will be oriented along $\vec{t}_1, \vec{t}_2$ with radii $\frac{1}{K_1}, \frac{1}{K_2}$

Euler Theorem: The normal curvature $K_n(t)$ and the principal curvatures $K_1(t), K_2(t)$ are related by:

$K_n(t) = K_1 \cos^2 \psi + K_2 \sin^2 \psi$ where ψ is the angle between $\bar{t}$ and $\bar{t}_1$ (direction of max. curvature)

i.e. how a surface curves is fully described by

K_1 K_2 $\vec{t}_1, \vec{t}_2$ and $\bar{t}_1 \perp \bar{t}_2$ (mathematically guaranteed)

We can define a "curvature tensor" – a matrix whose eigenvectors are $\vec{t}_1, \vec{t}_2, \vec{n}$ with eigenvalues $\kappa_1, \kappa_2, 0$

$$M\vec{v} = \lambda\vec{v}$$

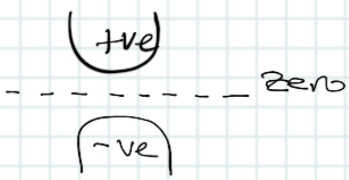
$$C = PDP^{-1}$$

$$P = \begin{bmatrix} \vec{t}_1 & \vec{t}_2 & \vec{n} \end{bmatrix} \qquad D = \begin{bmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ basis 3x3 matrix

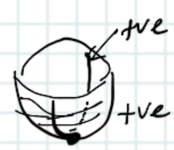
scale matrix

Consider κ_1, κ_2, which can be +ve, 0, or -ve



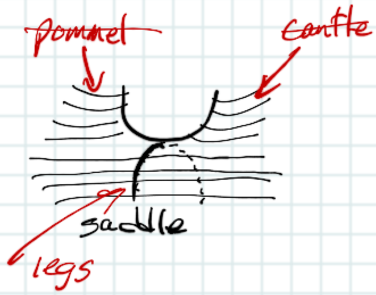
+ 0 −

+



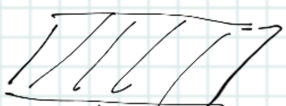
hollow/concave surface

half pipe



saddle

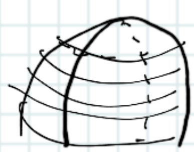
0 half pipe



plane

half pipe

− saddle half pipe



upside down hollow

$K > 0$ elliptical

$K < 0$ hyperbolic

$K = 0$ parabolic

define H, the mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

K, the Gaussian curvature

$$K = \kappa_1 \kappa_2$$

Notice that K, κ_1, κ_2, H are independent of the function chosen to describe the surface, i.e. that they are similar in nature to the ALP
- they depend on the surface itself, not on the description (function) we use

Properties of the surface that depend only on the surface are called "intrinsic"

Gaussian Theorem Egregium:

Gaussian curvature is intrinsic.

Problem: we want to find Gaussian curvature

Gaussian curvature exists only for smooth surfaces

We have a mesh that is not smooth (& usually triangles)

so if we want to take advantage of intrinsic properties such as curvature for editing, improving or simplifying meshes, we will need to work out how to approximate curvature for an existing triangle mesh