03: Quaternions

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Agenda

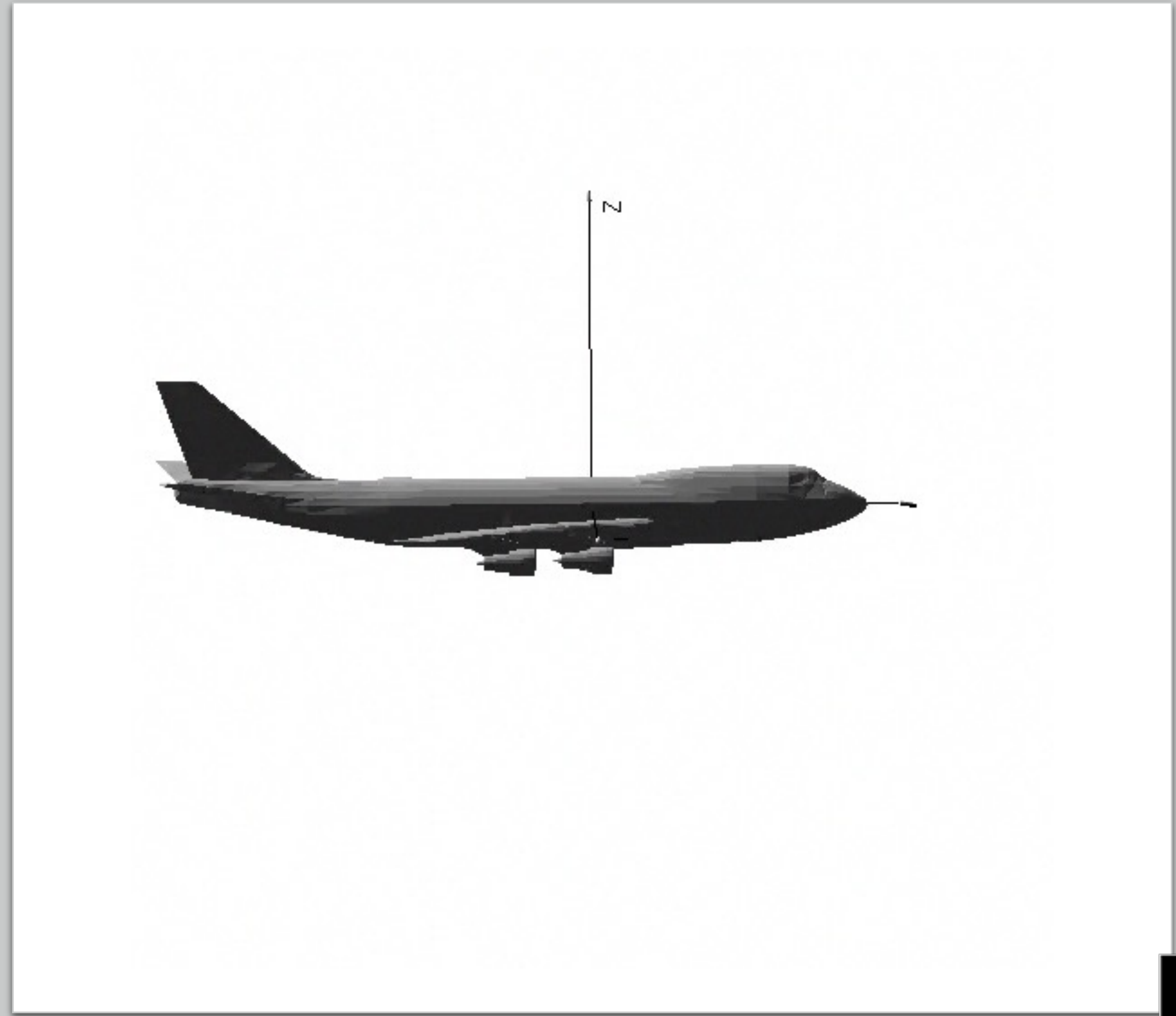
- Shortcomings of other rotation methods
- Quaternion operations
- Rotations with quaternions

Euler Angles

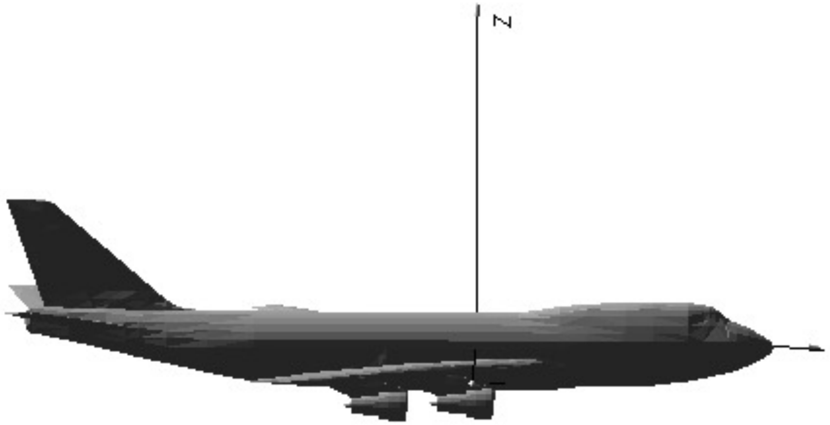
- Euler angles: rotate around global x, y, z
- But we get *gimbal lock:* ambiguous roll & yaw
  - interpolation becomes degenerate

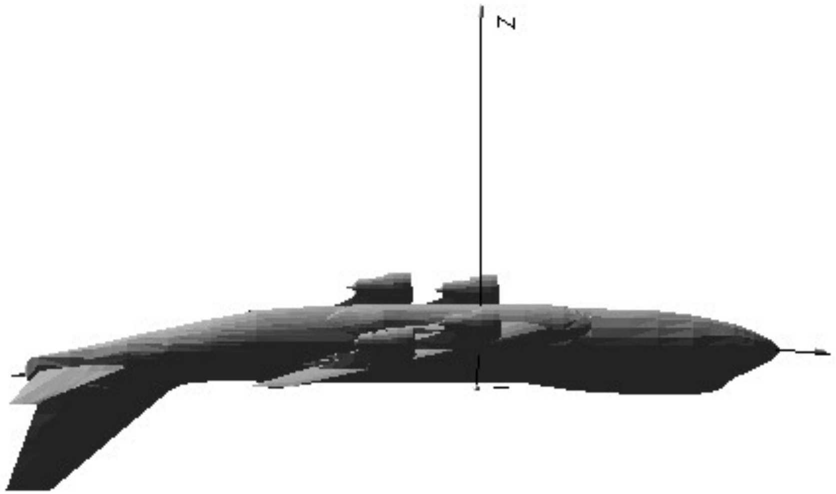
Cardan Angles

- Rotation around x, y, z in fixed *order*
- 180º pitch + 180º yaw = 180º roll



Keyframe Interpolation





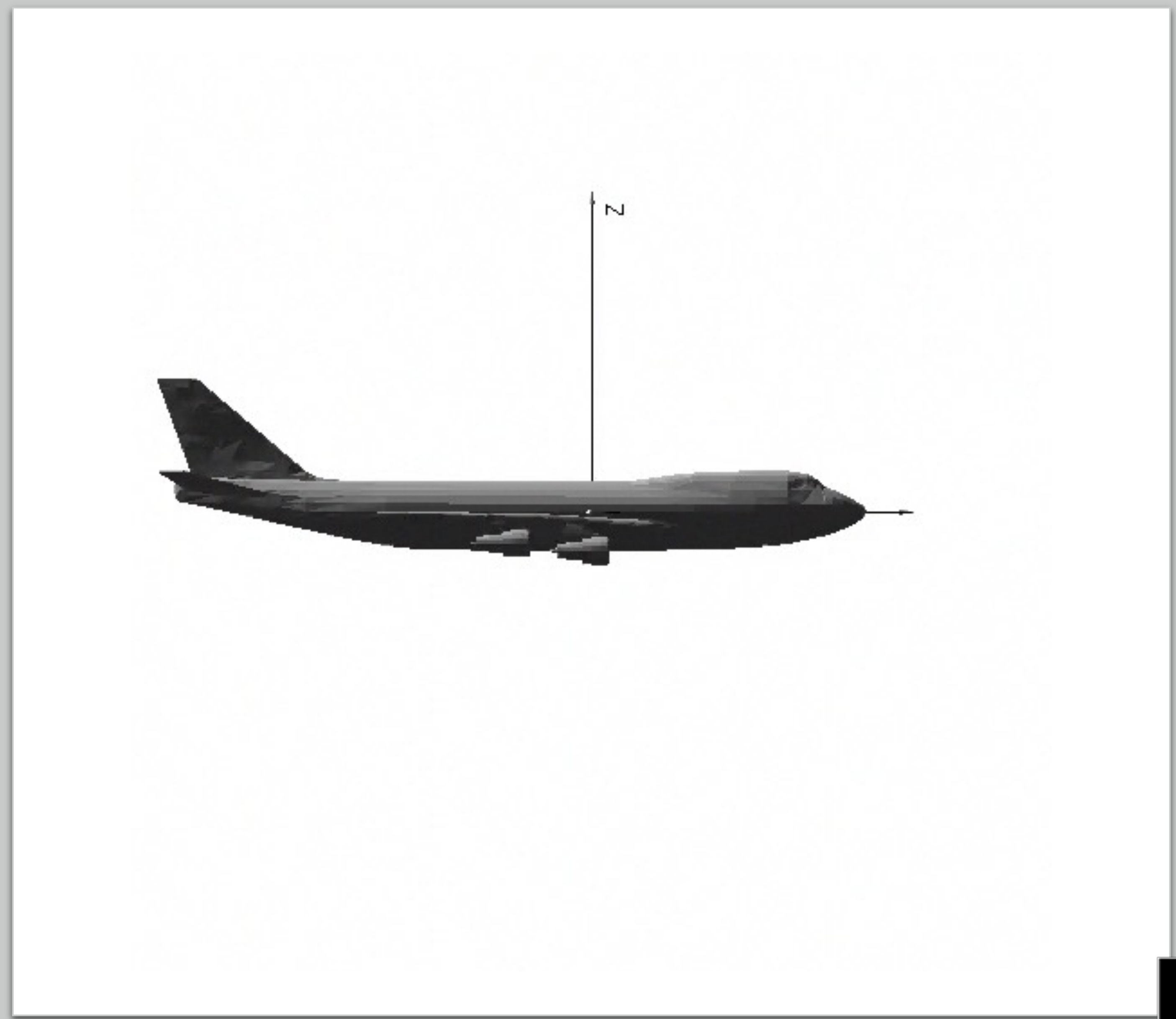
Initial Rotation:
Pitch = 0
Yaw = 0
Roll = 0

Final Rotation:
Pitch = 180
Yaw = 180
Roll = 0

Intermediate Rotation:
Pitch = n
Yaw = n
Roll = 0

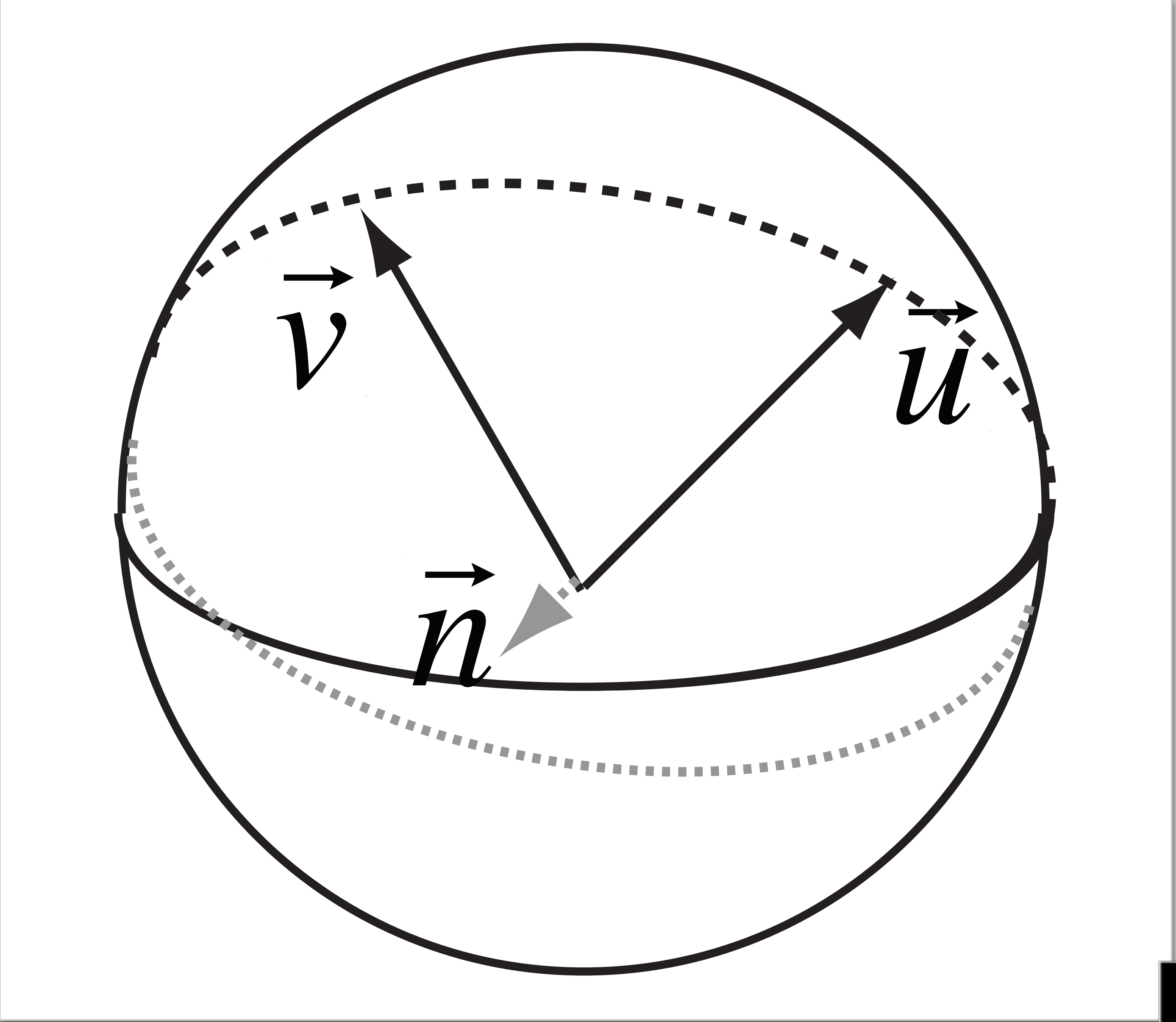
Cardan Interpolation

- Cardan angles are not unique
- One orientation, multiple representations
  - Worst-case behaviour is pretty awful



Great Circle Rotation

- A great circle goes through the origin, cutting the sphere in half.
  - one point rotates *to* a second
  - so actually, two vectors
  - convert to orthonormal basis
    - $\vec{n} = \vec{u} \times \vec{v}, \vec{w} = \vec{v} \times \vec{n}$
  - transform into basis and rotate around n
- But what if we want to roll at the same time?



Quaternions

- Homogeneous rotation coordinates
- Based on complex numbers:
  - a+*bi*
  - a is the *real* part
  - b is the *imaginary* part
- $i = \sqrt{-1}$

Complex Operations

Addition:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$$

Multiplication:

$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 (-1) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i\end{aligned}$$

Conjugation:

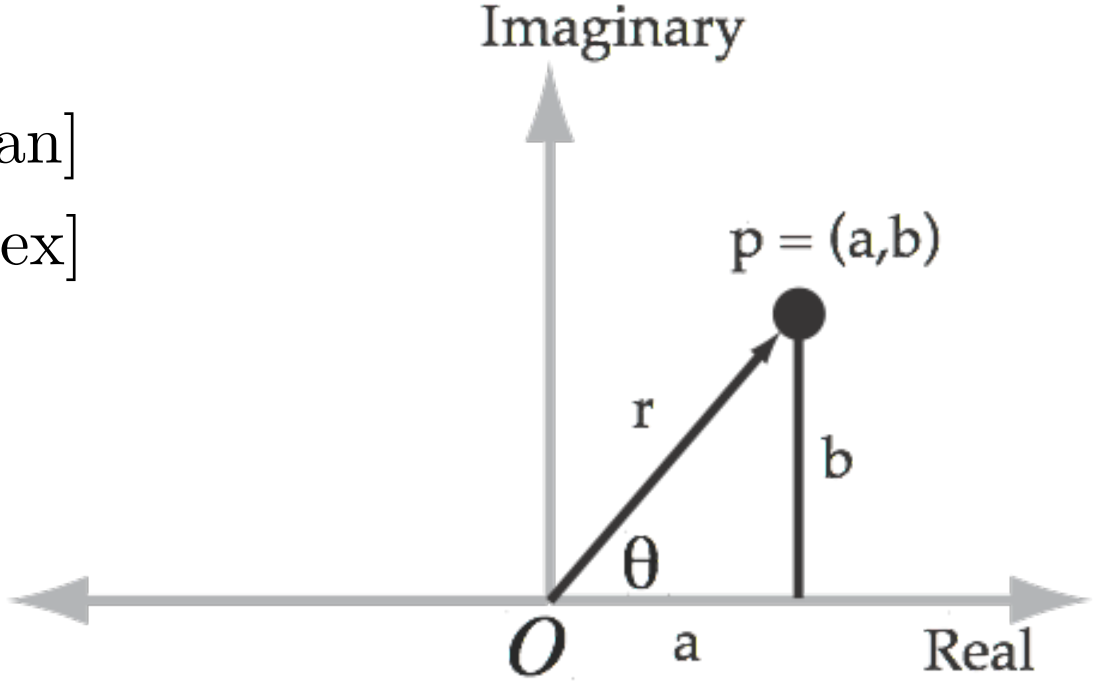
$$(a + bi)^* = a - bi$$

$$\begin{aligned}(a + bi)(a + bi)^* &= (a + bi)(a - bi) \\ &= (aa - b(-b)) + (ab + (-b)a) \\ &= a^2 + b^2\end{aligned}$$

Spatial Interpretation

Treat as *points* in 2D:

$$\begin{aligned} p &= (a,b)[\text{Cartesian}] \\ &= a+bi[\text{Complex}] \\ &= (r,\theta)[\text{Polar}] \\ r &= \sqrt{a^2+b^2} \\ &= \sqrt{pp^*} \\ \theta &= arctan(\frac{b}{a}) \end{aligned}$$



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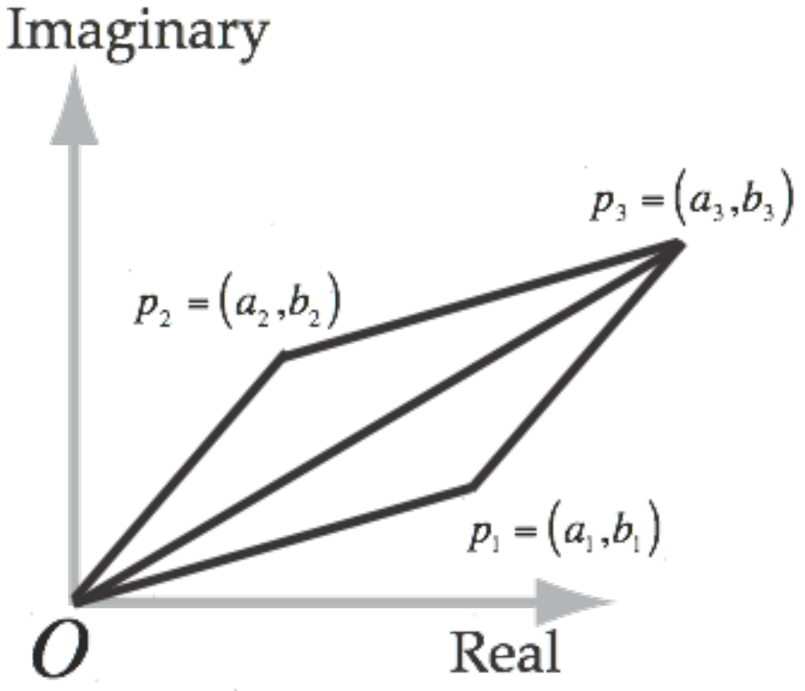
Spatial Addition

Complex Addition:

$$
\begin{aligned}
p_3 &= p_1 + p_2 \\
&= (a_1 + b_1 i) + (a_2 + b_2 i) \\
&= (a_1 + a_2) + (b_1 + b_2) i \\
a_3 &= a_1 + a_2 \\
b_3 &= b_1 + b_2
\end{aligned}
$$



Is translation!
(or vector addition)

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Polar Multiplication

$$
\begin{aligned}
p_3 &= p_1 p_2 \\
&= (a_1 + b_1 i)(a_2 + b_2 i) \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \\
a_3 &= a_1 a_2 - b_1 b_2 \\
b_3 &= a_1 b_2 + b_1 a_2
\end{aligned}
$$

What is the spatial interpretation of p_3?

Consider the polar form (r_3, θ_3), where

$$
\begin{aligned}
r_3 &= \sqrt{a_3^2 + b_3^2} \\
\theta_3 &= \arctan\left(\frac{b_3}{a_3}\right)
\end{aligned}
$$

Polar Multiplication

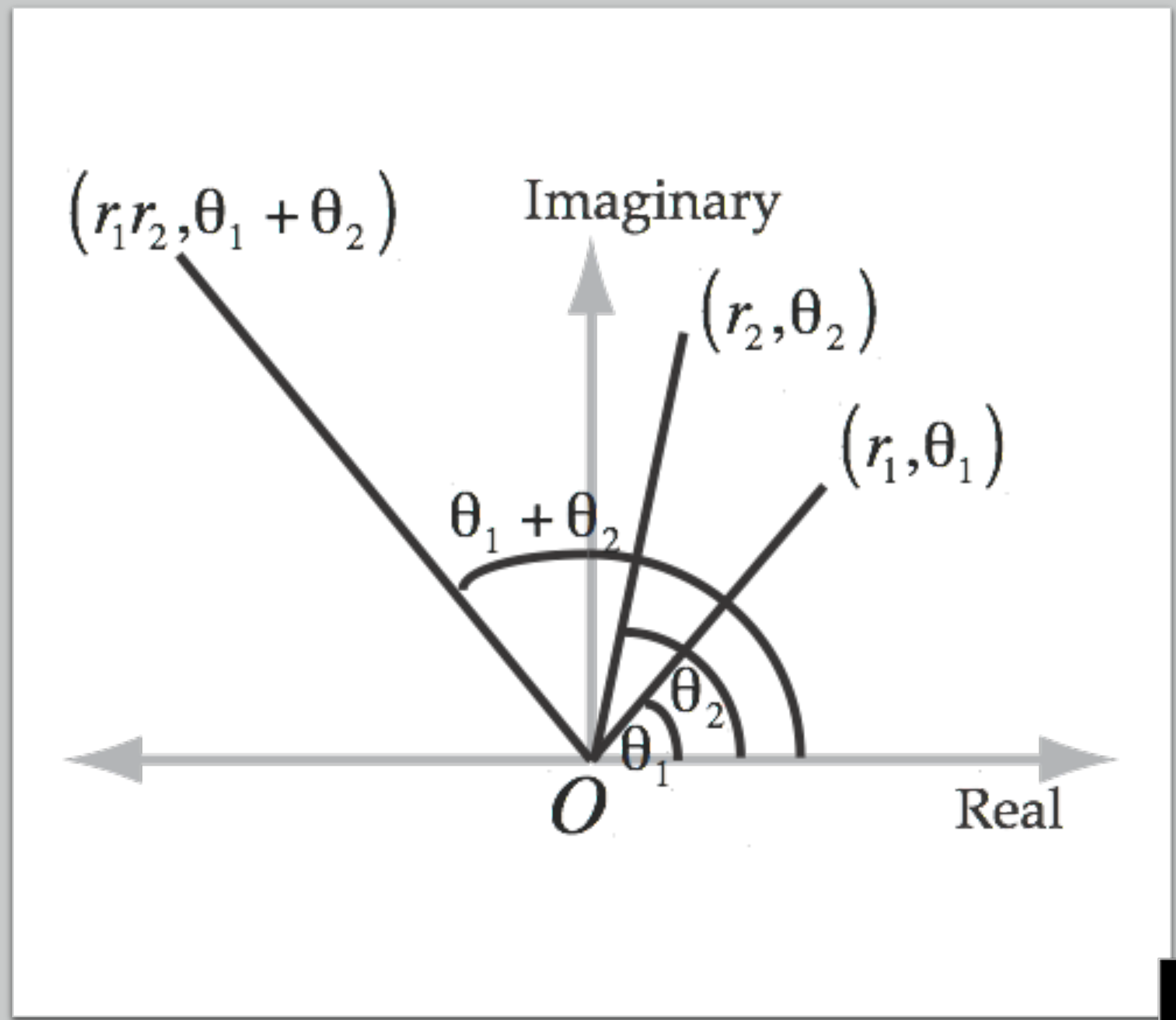
$$
\begin{aligned}
r_3 &= \sqrt{a_3^2 + b_3^2} \\
&= \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2} \\
&= \sqrt{a_1^2a_2^2 - 2a_1a_2b_1b_2 + b_1^2b_2^2 + a_1^2b_2^2 + 2a_1a_2b_1b_2 + a_2^2b_1^2} \\
&= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2} \\
&= \sqrt{a_1^2a_2^2 + a_1^2b_2^2 + b_1^2a_2^2 + b_2^2b_2^2} \\
&= \sqrt{a_1^2(a_2^2 + b_2^2) + b_1^2(a_2^2 + b_2^2)} \\
&= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \\
&= \sqrt{(a_1^2 + b_1^2)}\sqrt{(a_2^2 + b_2^2)} \\
&= r_1r_2
\end{aligned}
$$

What about Theta?

$$\begin{aligned}
\theta_3 &= \arctan\left(\frac{b_3}{a_3}\right) \\
&= \arctan\left(\frac{a_1 b_2 + a_2 b_1}{a_1 a_2 - b_1 b_2}\right) \\
&= \arctan\left(\frac{\frac{a_1 b_2}{a_1 a_2} + \frac{a_2 b_1}{a_1 a_2}}{\frac{a_1 a_2}{a_1 a_2} - \frac{b_1 b_2}{a_1 a_2}}\right) \\
&= \arctan\left(\frac{\frac{b_2}{a_2} + \frac{b_1}{a_1}}{1 - \frac{b_1}{a_1}\frac{b_2}{a_2}}\right) \\
&= \arctan\left(\frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}\right) \\
&= \arctan\left(\tan\left(\theta_1 + \theta_2\right)\right) \\
&= \theta_1 + \theta_2
\end{aligned}$$

Geometric Interpretation

- Multiplication gives
  - Scaling
  - Rotation
  - Based on polar notation



Extending to 3-D

- We'll look at complex conjugates:

$$\begin{aligned}(a+bi)(a+bi)^* &= (a^2+b^2)\\ &= r^2\end{aligned}$$

- Does this work for 3 coordinates?

$$\begin{aligned}(a+bi+cj)(a+bi+cj)^* &= (a+bi+cj)(a-bi-cj)\\ &= a^2-abi-acj+abi-b^2i^2-bcij+acj-bcji-c^2j^2\\ &= a^2-b^2i^2-c^2j^2-bcij-bcji\\ &= a^2+b^2+c^2-bcij-bcji\end{aligned}$$

The Fourth Coordinate

- We need bcij and bcji to cancel out
- So we add a fourth coordinate:

$$
\begin{array}{lllllllllllll}
ij & & & & & & & & & = & k & = & -ji \\
jk & = & j(-ji) & = & -j^2 i & = & -(-1)i & = & & = & i & = & -kj \\
ki & = & (-ji)i & = & -ji^2 & = & -j(-1) & = & & = & j & = & -ik
\end{array}
$$

- And get *quaternions* with four coordinates:
  - a + bi + cj + dk

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With Quaternions

$$
\begin{aligned}
(a+bi+cj+dk)(a+bi+cj+dk)^* &= (a+bi+cj+dk)(a-bi-cj-dk) \\
&= a^2 - abi - acj - adk \\
&\quad +abi - b^2i^2 - bcij - bdik \\
&\quad +acj - bcji - c^2j^2 - cdjk \\
&\quad adk - bdki - cdkj - d^2k^2 \\
&= a^2 + b^2 + c^2 + d^2 \\
&\quad -abi + abi - cdjk - cdkj \\
&\quad -acj - bdik + acj - bdki \\
&\quad -adk - bcij - bcji + adk \\
&= a^2 + b^2 + c^2 + d^2
\end{aligned}
$$

- Now it all works!

Quaternion Operations

Addition:

$$\begin{aligned} q_1 + q_2 &= (a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \end{aligned}$$

Multiplication:

$$\begin{aligned} q_1 q_2 &= (a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)1 \\ &\quad (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)i \\ &\quad (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)j \\ &\quad (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k \end{aligned}$$

Geometric Interpretation

i,j,k are different from 1:

i becomes j becomes k becomes i

looks like rotating between x, y, z

In a quaternion q=(w,x,y,z)

w: the scalar part

(x,y,z): the vector part

Notation

- Scalar s:

$$\begin{aligned} s &= (s, \vec{0}) \\ &= (s, 0, 0, 0) \end{aligned}$$

- Vector v:

$$\begin{aligned} \vec{v} &= (0, \vec{v}) \\ &= (0, x, y, z) \end{aligned}$$

- Quaternion q:

$$\begin{aligned} q &= (w, \vec{v}) \\ & \quad (w, x, y, z) \end{aligned}$$

Properties

- Scalar multiplication:

$$\begin{aligned} sq &= qs \\ &= (s, \vec{0})(w.\vec{v}) \\ &= (sw, s\vec{v}) \end{aligned}$$

- Associativity & Distributivity:

$$\begin{aligned} pq(r) &= p(qr) \\ p(q+r) &= pq + pr \end{aligned}$$

- Anti-commutativity:

$$(pq)^* = q^* p^*$$

Conjugation

$$
\begin{aligned}
(w, x, y, z)^* &= (w, -x, -y, -z) \\
(w, \vec{v})^* &= (w, -\vec{v}) \\
(pq)^* &= q^* p^* \\
(w, \vec{v}) + (w, \vec{v})^* &= (w + w, \vec{v} + (-\vec{v})) \\
&= (2w, \vec{0})
\end{aligned}
$$

Vector Multiplication

$$\begin{aligned}
\vec{v_1}\vec{v_2} &= (0 + x_1 i + y_1 j + z_1 k)(0 + x_2 i + y_2 j + z_2 k) \\
&= (0 - x_1 x_2 - y_1 y_2 - z_1 z_2) \\
&\quad (0 + 0 + y_1 z_2 - z_1 y_2) i \\
&\quad (0 - x_1 z_2 - 0 - z_1 x_2) j \\
&\quad (0 - x_1 y_2 - y_1 x_2 - 0) k \\
&= (-x_1 x_2 - y_2 y_2 - z_1 z_2) \\
&\quad +(y_1 z_2 - z_1 y_2) i \\
&\quad +(-x_1 z_2 + z_1 x_2) j \\
&\quad (x_1 y_2 - y_1 x_2) k \\
&= (-\vec{v1} \cdot \vec{v}_2, \vec{v_1} \times \vec{v}_2)
\end{aligned}$$

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More Multiplication

$$\begin{aligned}
q_1 q_2 &= (w_1, \vec{v}_1)(w_2, \vec{v}_2) \\
&= \Big((w_1, \vec{0}) + (0, \vec{v}_1)\Big)\Big((w_2, \vec{0}) + (0, \vec{v}_2)\Big) \\
&= (w_1, \vec{0})(w_2, \vec{0}) + (w_1, \vec{0})(0, \vec{v}_2) + (0, \vec{v}_1)(w_2, \vec{0}) + (0, \vec{v}_1)(0, \vec{v}_2) \\
&= (w_1 w_2, \vec{0}) + ((0, w_1 \vec{v}_2) + (0, w_2 \vec{v}_1) + (-v\vec{1} \cdot \vec{v}_2, \vec{v_1} \times \vec{v}_2) \\
&= (w_1 w_2 - \vec{v}_1 \cdot \vec{v}_2, , \vec{v_1} \times \vec{v}_2 + w_1 \vec{v}_2 + w_2 \vec{v}_1)
\end{aligned}$$

Norm & Inverse

$$
\begin{aligned}
N(q) &= qq^* \\
&= q^*q \\
&= w^2 + x^2 + y^2 + z^2 \\
&= w^2 + \vec{v} \cdot \vec{v} \\
N(pq) &= N(p)N(q) \\
N(q^*) &= N(q) \\
q^{-1} &= q^*/N(q) \\
qq^{-1} &= qq^*/N(q) \\
&= N(q)/N(q) \\
&= 1
\end{aligned}
$$

Action of a Quaternion

Let p be a point in homogeneous coordinates.

Let q be a quaternion.

Claim:

qpq^{-1}, the action of q on p, rotates p by an angle 2θ around an axis $\vec{v}$

AND:

$$q = \left(k\cos\theta, k\vec{v}\sin\theta\right)$$

Lemma 1

Assume $q = (w, x, y, z)$ is a unit quaternion. Then there exists some angle θ and some unit vector $\vec{v}$ so that $q = (\cos\theta, \vec{v}\sin\theta)$

Proof: $N(q) = w^2 + x^2 + y^2 + z^2 = 1$, so $-1 \le w \le 1$, and $\theta = \arccos(w)$ always exists, and $w = \cos(\theta)$. Also define:

$\vec{v} = \left(\frac{x}{\sin\theta}, \frac{y}{\sin\theta}, \frac{z}{\sin\theta} \right)$. Then:

$$
\begin{aligned}
(\cos\theta, \vec{v}\sin\theta) &= \left(\cos\theta, \frac{x}{\sin\theta}\sin\theta, \frac{y}{\sin\theta}\sin\theta, \frac{z}{\sin\theta}\sin\theta \right) \\
&= (w, x, y, z) \\
&= q
\end{aligned}
$$

Lemma 2

- For any *s≠0, q* and *sq* have the same action

$$
\begin{aligned}
p' &= (sq)\,p\,(sq)^{-1} \\
&= sqpq^{-1}s^{-1} \\
&= ss^{-1}qpq^{-1} \\
&= qpq^{-1}
\end{aligned}
$$

- I.e. *quaternions* are *homogeneous* in nature

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$$qsq^{-1} = sqq^{-1}$$
$$= s$$

Lemma 3

Assume q is a unit quaternion. It’s action on a scalar is:

Lemma 4

Assume q is a unit quaternion. It's action on a vector $\vec{u}$ is another vector $\vec{v}$, i.e. a quaternion $p = (0, \vec{v})$

Proof: Let $p = q\vec{u}q^{-1}$. What is it's scalar part p_w?

$$
\begin{aligned}
2p_w &= p + p^* \\
&= q\vec{u}q^{-1} + \left(q\vec{u}q^{-1}\right)^* \\
&= q\vec{u}q^* + \left(q\vec{u}q^*\right)^* \\
&= q\vec{u}q^* + q^{**}\vec{u}^* q^*
\end{aligned}
$$

$$
\begin{aligned}
2p_w &= q\vec{u}q^* + q\vec{u}^* q^* \\
&= q\left(\vec{u} + \vec{u}^*\right)q^* \\
&= q0q^* \\
&= 0
\end{aligned}
$$

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Lemma 5

Let $p = (w, x, y, z)$ be a point in 3-D space in homogeneous coordinates, and let q be any unit quaternion. Then q's *action* on p, $p' = qpq^{-1}$, takes $p = (w, \vec{u})$ to $p' = (w, \vec{v})$, with $N(\vec{v}) = N(\vec{u})$.

Proof: Applying Lemmas 3 & 4, we get:

$$\begin{aligned} p' &= qpq^{-1} \\ &= q(w + \vec{u})q^{-1} \\ &= qwq^{-1} + q\vec{u}q^{-1} \\ &= w + \vec{v} \end{aligned}$$

$$\begin{aligned} N(\vec{v}) &= N(p') - w^2 \\ &= N(q)N(p)N(q^{-1}) - w^2 \\ &= N(p) - w^2 \\ &= N(\vec{u}) \end{aligned}$$

Theorem

Assume $q = \left(k\cos\theta, k\vec{v}\sin\theta\right)$ is any quaternion. Then the action of q on any homogeneous point $p = \left(w, \vec{u}\right)$ rotates p around the axis $\vec{v}$ by 2θ.

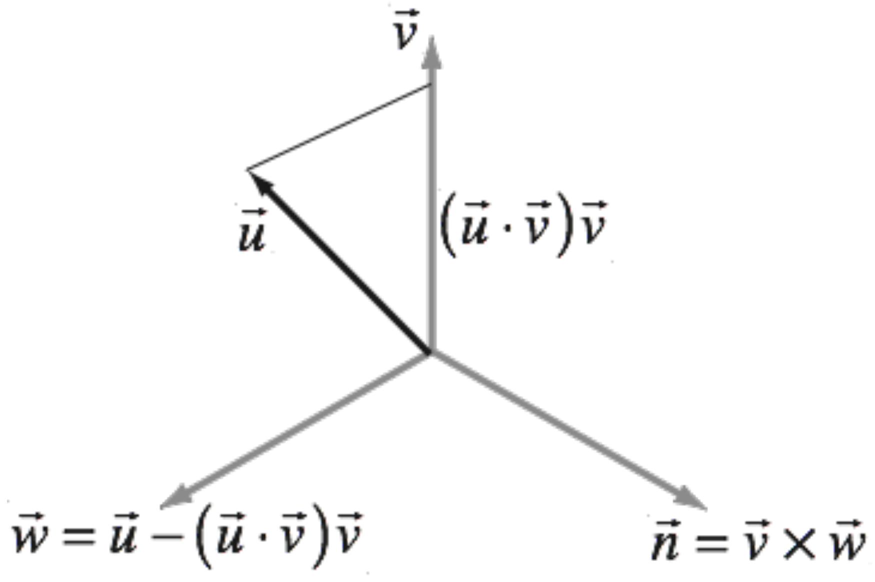
Proof: By Lemma 1, assume that $\vec{v}$ is a unit vector. By Lemma 2, assume that $q = \left(\cos\theta, \vec{v}\sin\theta\right)$ is a unit quaternion. Because p is in homogeneous coordinates, assume $\vec{u}$ is a unit vector. Finally, by Lemma 5, we ignore w and consider $\vec{u}$.

Finding a Basis

Let $\vec{w} = \vec{u} - (\vec{u} \cdot \vec{v})\vec{v}$. This is perpendicular to $\vec{v}$ and coplanar with $\vec{u}$. For simplicity, assume that $\vec{w}$ is a unit vector.

Compute $\vec{n} = \vec{v} \times \vec{w}$ to get an ortho-normal basis $(\vec{v}, \vec{w}, \vec{n})$.

We will look at how q acts on $\vec{v}$ and $\vec{w}$ to prove our result.



Action on $\vec{v}$

$$
\begin{aligned}
q\vec{v}q^{-1} &= (\cos\theta, \sin\theta\vec{v})(0,\vec{v})(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(0\cos\theta - (\sin\theta\vec{v})\cdot\vec{v}, (\sin\theta\vec{v})\times\vec{v} + \cos\theta\vec{v} + 0(\sin\theta\vec{v})\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(0 - \sin\theta(\vec{v}\cdot\vec{v}), \sin\theta(\vec{v}\times\vec{v}) + \cos\theta\vec{v} + \vec{0}\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(-\sin\theta(1), \sin\theta\left(\vec{0}\right) + \cos\theta\vec{v}\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= (-\sin\theta, \cos\theta\vec{v})(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(-\sin\theta\cos\theta - (\cos\theta\vec{v})\cdot(-\sin\theta\vec{v}), (\cos\theta\vec{v})\times(-\sin\theta\vec{v}) + (-\sin\theta)(-\sin\theta\vec{v}) + \cos\theta(\cos\theta\vec{v})\right) \\
&= \left(-\sin\theta\cos\theta + \sin\theta\cos\theta(\vec{v}\cdot\vec{v}), \left(-\sin\theta\cos\theta(\vec{v}\times\vec{v}) + \sin^2\theta\vec{v} + \cos^2\theta\vec{v}\right)\right) \\
&= \left(-\sin\theta\cos\theta + \sin\theta\cos\theta(1), \left(-\sin\theta\cos\theta\left(\vec{0}\right) + \sin^2\theta\vec{v} + \cos^2\theta\vec{v}\right)\right) \\
&= \left(0, \left(\sin^2\theta + \cos^2\theta\right)\vec{v}\right) \\
&= (0, 1\vec{v}) \\
&= (0, \vec{v})
\end{aligned}
$$

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Action on $\vec{w}$

$$
\begin{aligned}
q\vec{w}q^{-1} &= (\cos\theta, \sin\theta\vec{v})(0,\vec{w})(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(0\cos\theta - (\sin\theta\vec{v})\cdot\vec{w}, (\sin\theta\vec{v})\times\vec{w} + \cos\theta\vec{w} + 0(\sin\theta\vec{v})\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(0 - \sin\theta(\vec{v}\cdot\vec{w}), \sin\theta(\vec{v}\times\vec{w}) + \cos\theta\vec{w} + \vec{0}\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(-\sin\theta(0), \sin\theta(\vec{n}) + \cos\theta\vec{w}\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \left(0, \sin\theta\vec{n} + \cos\theta\vec{w}\right)(\cos\theta, -\sin\theta\vec{v}) \\
&= \begin{pmatrix} 0\cos\theta - (\sin\theta\vec{n} + \cos\theta\vec{w})\cdot(-\sin\theta\vec{v}), \\ \quad (\sin\theta\vec{n} + \cos\theta\vec{w})\times(-\sin\theta\vec{v}) + 0(-\sin\theta\vec{v}) + \cos\theta(\sin\theta\vec{n} + \cos\theta\vec{w}) \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\theta(\vec{n}\cdot\vec{v}) - \sin\theta\cos\theta(\vec{w}\cdot\vec{v}), \\ \quad -\sin^2\theta(\vec{n}\times\vec{v}) - \sin\theta\cos\theta(\vec{w}\times\vec{v}) + 0(-\sin\theta\vec{v}) + \sin\theta\cos\theta\vec{n} + \cos^2\theta\vec{w} \end{pmatrix} \\
&= \left(\sin^2\theta(0) - \sin\theta\cos\theta(0), -\sin^2\theta(\vec{w}) - \sin\theta\cos\theta(-\vec{n}) + \sin\theta\cos\theta\vec{n} + \cos^2\theta\vec{w}\right) \\
&= \left(0, (\cos^2\theta - \sin^2\theta)\vec{w} + (2\sin\theta\cos\theta)\vec{n}\right) \\
&= \left(0, \cos 2\theta\vec{w} + \sin 2\theta\vec{n}\right)
\end{aligned}
$$

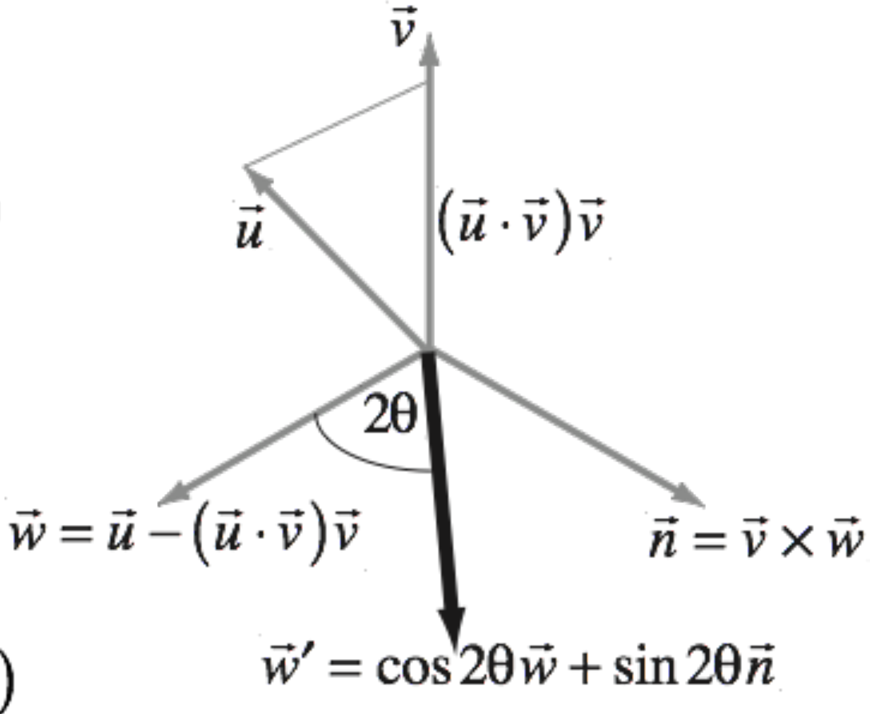
Interpretation

Note that $\vec{w}'$ is perpendicular to $\vec{v}$:

$$\begin{aligned}\vec{w}' \cdot \vec{v} &= \left(\cos 2\theta \vec{w} + \sin 2\theta \vec{n}\right) \cdot \vec{v} \\ &= \cos 2\theta \left(\vec{w} \cdot \vec{v}\right) + \sin 2\theta \left(\vec{n} \cdot \vec{v}\right) \\ &= \cos 2\theta (0) + \sin 2\theta (0) \\ &= 0\end{aligned}$$

And the angle from $\vec{w}$ to $\vec{w}'$ is 2θ:

$$\begin{aligned}\vec{w}' \cdot \vec{w} &= \left(\cos 2\theta \vec{w} + \sin 2\theta \vec{n}\right) \cdot \vec{w} \\ &= \cos 2\theta \left(\vec{w} \cdot \vec{w}\right) + \sin 2\theta \left(\vec{n} \cdot \vec{w}\right) \\ &= \cos 2\theta (1) + \sin 2\theta (0) \\ &= \cos 2\theta\end{aligned}$$



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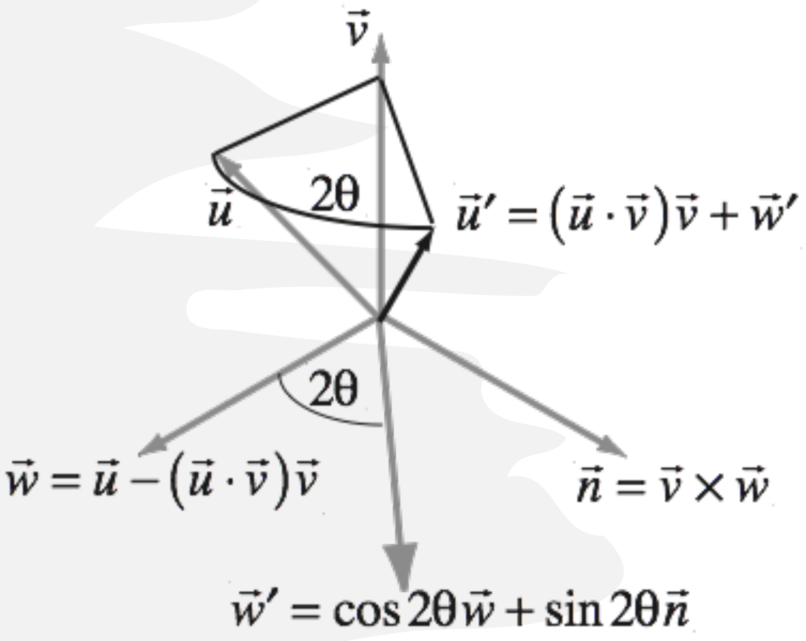
We know that the $\vec{v}$ component of $\vec{u}$ is not changed, and that the $\vec{w}$ component is rotated by an angle of 2θ around $\vec{v}$, so we are done.



Action in General

Uniqueness

A unit quaternion $q = (\cos\theta, \sin\theta\vec{v})$ is the *only* unit quaternion that gives the specified rotation. Moreover, the product of unit quaternions is another unit quaternion, so we can combine these rotations easily. Just like rotation matrices.

But quaternions can be interpolated more easily than matrices, and are more efficient and numerically stable.

Quaternion to Matrix

Let $q = (w, x, y, z)$ be a quaternion, and $p = (p_x, p_y, p_z, p_w)$ be a point in homogeneous coordinates. Note that with quaternions, we have been writing the w coordinate first, but in homogeneous coordinates, it comes last. This is why some authors put the w coordinate last in quaternions, but that leads to writing quaternions as $(\vec{v}, w)$ instead of $(w, \vec{v})$, which obscures the similarity to complex numbers.

Left-Multiplication

$$\begin{aligned}
qp &= (xi + yj + zk + w) * (p_x i + p_y j + p_z k + p_w) \\
\\
&= xip_x i + xip_y j + xip_z k + xip_w + \\
&\quad yjp_x i + yjp_y j + yjp_z k + yjp_w + \\
&\quad zkp_x i + zkp_y j + zkp_z k + zkp_w + \\
&\quad wp_x i + wp_y j + wp_z k + wp_w \\
\\
&= xp_x i^2 + xp_y ij + xp_z ik + xp_w i + \\
&\quad yp_x ji + yp_y j^2 + yp_z jk + yp_w j + \\
&\quad zp_x ki + zp_y kj + zp_z k^2 + zp_w k + \\
&\quad wp_x i + wp_y j + wp_z k + wp_w \\
\\
&= xp_x(-1) + xp_y(k) + xp_z(-j) + xp_w(i) + \\
&\quad yp_x(-k) + yp_y(-1) + yp_z(i) + yp_w(j) + \\
&\quad zp_x(j) + zp_y(-i) + zp_z(-1) + zp_w(k) + \\
&\quad wp_x(i) + wp_y(j) + wp_z(k) + wp_w(1)
\end{aligned}$$

$$\begin{aligned}
&= (wp_x - zp_y + yp_z + xp_w)(i) + \\
&\quad (zp_x + wp_y - xp_z + yp_w)(j) + \\
&\quad (-yp_x + xp_y + wp_z + zp_w)(k) + \\
&\quad (-xp_x - yp_y - zp_z + wp_w)(1)
\end{aligned}$$

$$= \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Converting the quaternion to a matrix multiplication:

Right-Multiplication

$$
\begin{aligned}
pq^{-1} &= (p_x i + p_y j + p_z k + p_w) * (xi - yj - zk + w) \\
&= -p_x ixi - p_x iyj - p_x izk + p_x iw + \\
&\quad -p_y jxi - p_y jyj - p_y jzk + p_y jw + \\
&\quad -p_z kxi - p_z kyj - p_z kzk + p_z kw + \\
&\quad -p_w xi - p_w yj - p_w zk + p_w w) \\
&= -p_x xi^2 - p_x yij - p_x zik + p_x wi + \\
&\quad -p_y xji - p_y yj^2 - p_y zjk + p_y wj + \\
&\quad -p_z xki - p_z ykj - p_z zk^2 + p_z wk + \\
&\quad -p_w xi - p_w yj - p_w zk + p_w w \\
&= -p_x x(-1) - p_x y(k) - p_x z(-j) + p_x w(i) + \\
&\quad -p_y x(-k) - p_y y(-1) - p_y z(i) + p_y w(j) + \\
&\quad -p_z x(j) - p_z y(-i) - p_z z(-1) + p_z w(k) + \\
&\quad -p_w xi - p_w yj - p_w zk + p_w w
\end{aligned}
$$

$$
\begin{aligned}
&= (wp_x - zp_y + yp_z - xp_w)(i) + \\
&\quad (zp_x + wp_y - xp_z - yp_w)(j) + \\
&\quad (-yp_x + xp_y + wp_z + wp_w)(k) + \\
&\quad (xp_x + yp_y + zp_z + wp_w)(1)
\end{aligned}
$$

$$
= \begin{bmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}
$$

Full Action

$$
\begin{aligned}
qpq^{-1} &= \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix} \left(\begin{bmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix} \begin{bmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{bmatrix} \right) \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix} \\
&= \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) & 0 \\ 2(xy + wz) & w^2 + y^2 - x^2 - z^2 & 2(yz - xw) & 0 \\ 2(xz - wy) & 2(xw + yz) & w^2 + z^2 - x^2 - y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) & 0 \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - xw) & 0 \\ 2(xz - wy) & 2(xw + yz) & 1 - 2(x^2 + y^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}
\end{aligned}
$$

Rot. Matrix to Quaternion

$$\text{Let } R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-2\left(y^2+z^2\right) & 2\left(xy-wz\right) & 2\left(xz+wy\right) & 0 \\ 2\left(xy+wz\right) & 1-2\left(x^2+z^2\right) & 2\left(yz-wx\right) & 0 \\ 2\left(xz-wy\right) & 2\left(yz+wx\right) & 1-2\left(x^2+y^2\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It therefore follows that for a pure rotation matrix R,

$$\begin{aligned} a_{11}+a_{22}+a_{33}+1 &= 1-2\left(y^2+z^2\right)+1-2\left(x^2+z^2\right)+1-2\left(x^2+y^2\right)+1 \\ &= 4\left(1-x^2-y^2-z^2\right) \\ &= 4\left(w^2\right) \\ w &= \frac{1}{2}\sqrt{a_{11}+a_{22}+a_{33}+1} \end{aligned}$$

Finding x, y, z:

$$
\begin{aligned}
a_{32} - a_{23} &= (yz + wx) - (yz - wx) \\
&= 2wx
\end{aligned}
$$

$$x = \frac{a_{32} - a_{23}}{2w}$$

Similarly,

$$y = \frac{a_{13} - a_{31}}{2w}$$

$$z = \frac{a_{21} - a_{12}}{2w}$$

Spherical Interpolation

Quaternions rotate on *great circles*

Assume that:

q defines the entire rotation	we want to interpolate in n steps

Exponential Solution

Let $q = \left(\cos\theta, \vec{v}\sin\theta\right)$ be the quaternion for the entire rotation.

We want to find a quaternion r such that applying it n times

gives us a cumulative result of q:

$$\underbrace{rrrr\ldots rrr}_{n\text{ copies}}\, p\, \underbrace{r^{-1}r^{-1}r^{-1}r^{-1}\ldots r^{-1}r^{-1}r^{-1}}_{n\text{ copies}} = qpq^{-1}$$

$$r^n p\left(r^{-1}\right)^n = qpq^{-1}$$

$$r = \sqrt[n]{q}$$

Because quaternions have multiplication and division, we can

actually define and compute square roots, the exponential

function and so on. However, it's messy, and nobody bothers.

Interpolation Hack

Let $q = (\cos\theta, \vec{v}\sin\theta)$ be the quaternion for the entire rotation.

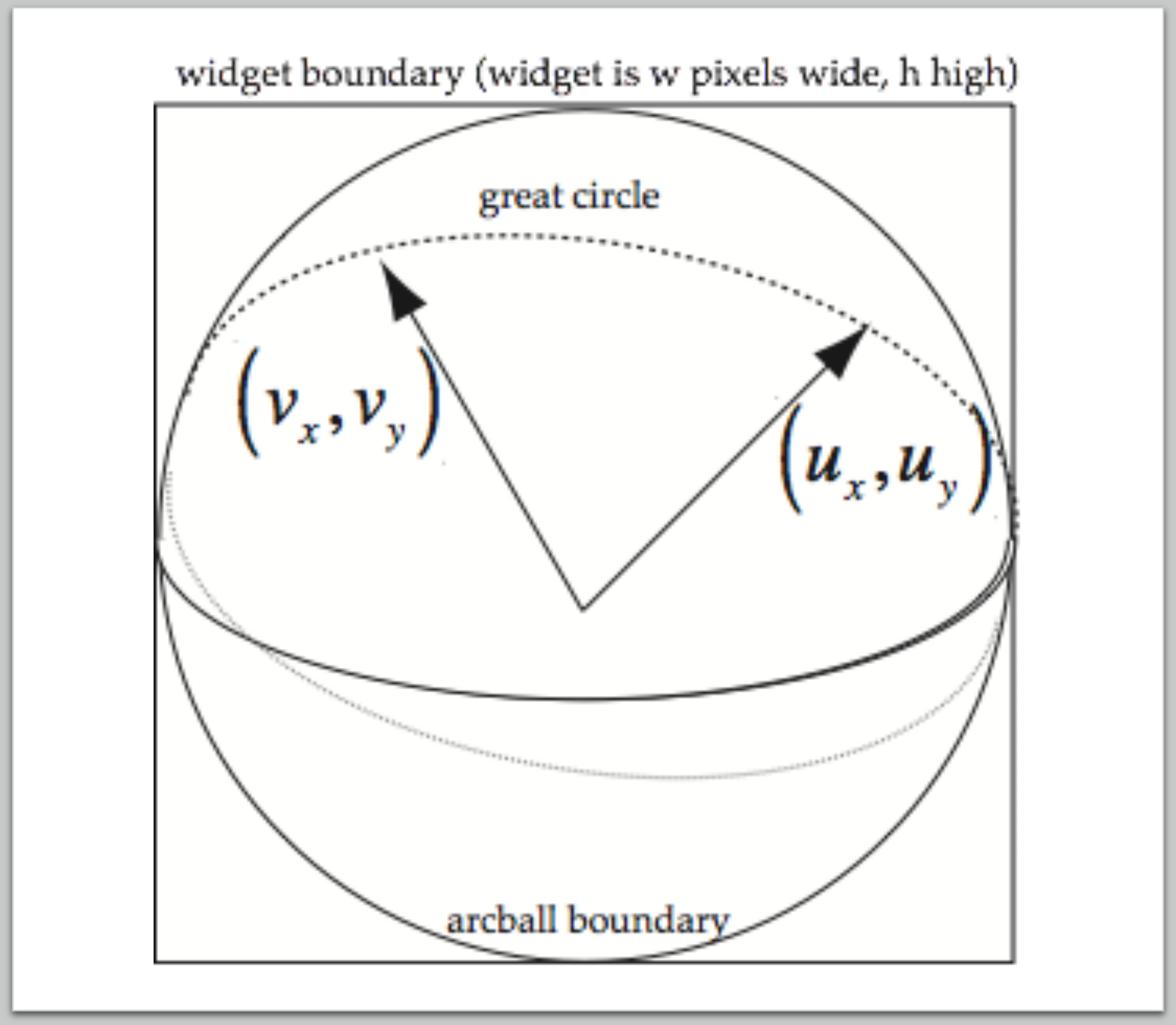
We want to find a quaternion r such that applying it n times gives us a cumulative result of q:

$$\underbrace{rrrr\ldots rrr}_{n \text{ copies}}\, p\, \underbrace{r^{-1}r^{-1}r^{-1}r^{-1}\ldots r^{-1}r^{-1}r^{-1}}_{n \text{ copies}} = qpq^{-1}$$

Let $r = \left(\cos\dfrac{\theta}{n}, \vec{v}\sin\dfrac{\theta}{n}\right)$. Provided we have trig. functions or tables to compute θ, we're done!

Arcball Controller

- An arcball rotation consists of two mouse-clicks: *start (u)* and *end (v)*
- This gives a rotation along the great circle between *u* and *v*



2D to 3D

Both $\vec{u}$ and $\vec{v}$ are given in screen space, i.e. $\vec{u} = \left(u_x, u_y, 0\right)$, $\vec{v} = \left(v_x, v_y, 0\right)$. We scale and translate the coordinates into the range [-1..+1]:

$$u_x{}' = 2.0 - \frac{u_x}{0.5w},$$

$$u_y{}' = 2.0 - \frac{u_y}{0.5h}$$

and push any points outside the circle onto the boundary.

Now set $u_z = \sqrt{1 - \left(u_x^2 + u_z^2\right)}$ to get a unit vector on the virtual trackball.

Do likewise with v_z.

Computing a Quaternion

Now set $u_z = \sqrt{1-\left(u_x^2+u_z^2\right)}$ to get a unit vector on the virtual trackball.

Do likewise with v_z.

Let $q_u = \left(0,\vec{u}\right)$ and $q_v = \left(0,\vec{v}\right)$. Now define our rotation quaternion:

$$q = \sqrt{-q_v q_u} = \sqrt{-q_v}\sqrt{q_u}$$

Look at slide 46!

Since $\vec{u}$ is a unit vector, $q_u^{-1} = q_u{}^* = \left(0,\vec{u}\right)^* = \left(0,-\vec{u}\right) = -q_u$

$$\begin{aligned} q q_u q &= \left(\sqrt{-q_v q_u}\right) q_u \left(\sqrt{-q_v q_u}\right)^{-1} \\ &= \left(\sqrt{q_v}\sqrt{-q_u}\right) q_u \left(\sqrt{\left(-q_v q_u\right)^{-1}}\right) \\ &= \sqrt{q_v}\sqrt{q_u^{-1}}\, q_u \sqrt{q_u^{-1}}\sqrt{-q_v^{-1}} \\ &= \sqrt{q_v}\,(1)\sqrt{q_v} \\ &= q_v \end{aligned}$$

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Arcball Version

You can save yourself some hassle here by using:

$q = q_v q_u^{-1}$

This will rotate the object by 2θ instead of θ, but is much easier to calculate. This is in fact what Ken Shoemake's arcball does for you, so your object spins twice as fast visually as the arcball. Why is this useful?