On the Approximability of Weighted Model Integration over DNF Structures

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Weighted Model Counting

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Applications

- Probabilistic Graphical Models
- Probabilistic Logic Programming
- Probabilistic Databases
- Probabilistic Knowledge Bases

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$\mathbf{X}$ set of n real variables

$\mathbf{V}$ set of m Boolean variables

Weighted Model Integration

$c_1 x_1 + \ldots + c_i\, x_i \bowtie c$

LRA Atom, $x_i \in \mathrm{X}$, $\bowtie \in \{<, \leq, >, \geq, =, \neq\}$

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$\mathrm{WMI}(\phi)$ $$\sum_{v} \int_{x_\phi} w(x, v)\, dx$$

where v is a Boolean assignment over V, x_ϕ denotes valuations of X satisfying ϕ.

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For weight functions in WMI, it is common to factorize [4] w as a product of m Boolean literal weights and a density function over real variables, i.e.,:

$$w(x, v) = w_x(x) \prod_{i=1}^{m} w_b(p_i).$$

Special Cases

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➡ Special cases WMI(CNF), WMI(DNF)

Both WMI and WMC are #P-hard for exact solving. Hence, we study WMI within the context of approximate solving.

Approximation Hardness

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	WMC		WMI
CNF	NP-hard [1]	$\rightarrow$	NP-hard
DNF	FPRAS [2]	$\rightarrow$	?

Result:

Show that WMI(DNF) admits an FPRAS for concave weight functions

Result builds on existing FPRAS algorithms for WMC(DNF) and volume computation for the union of convex bodies

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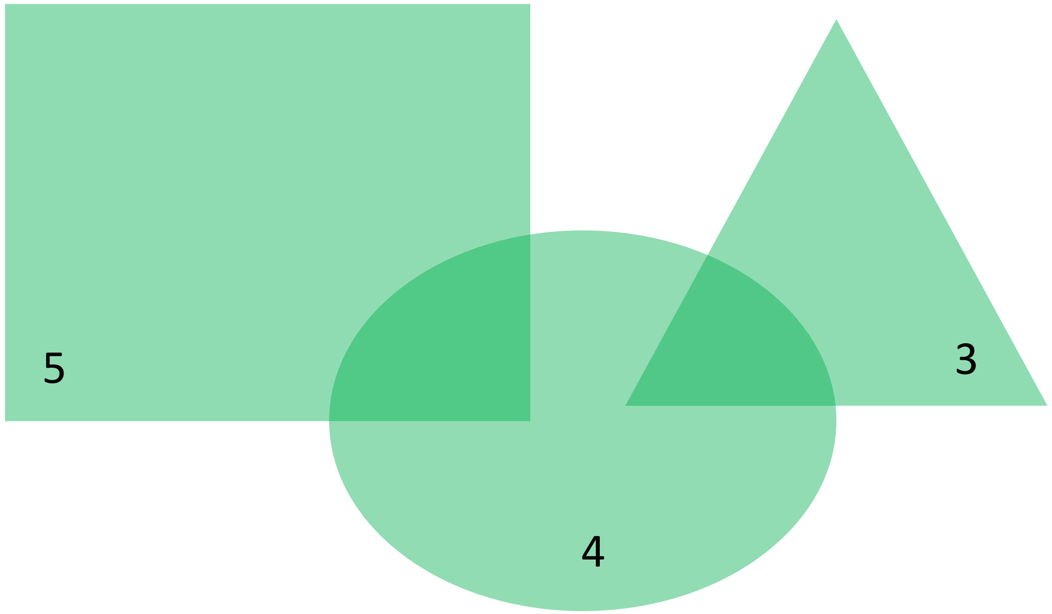

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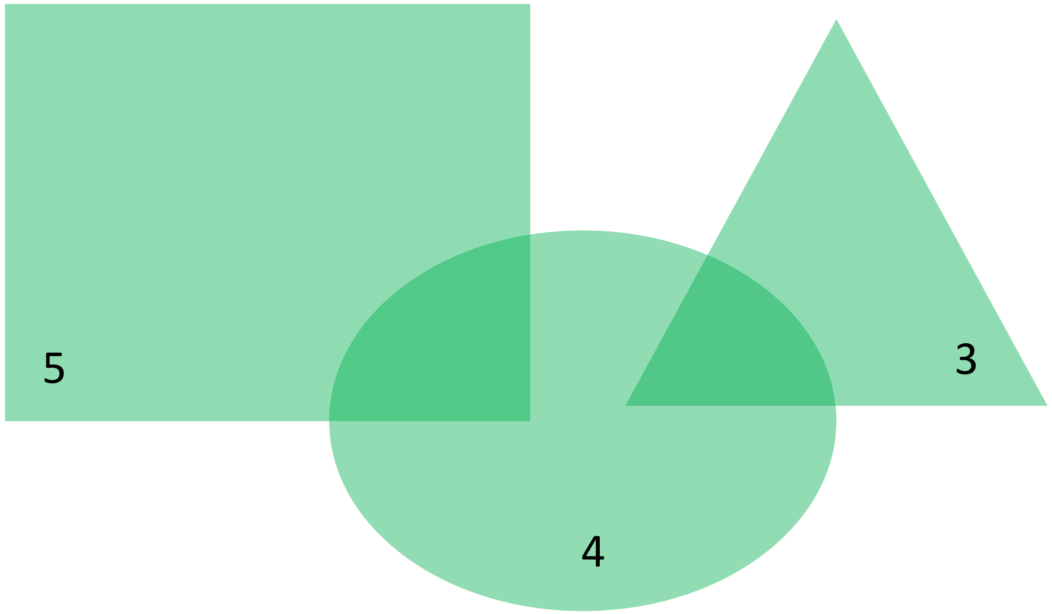

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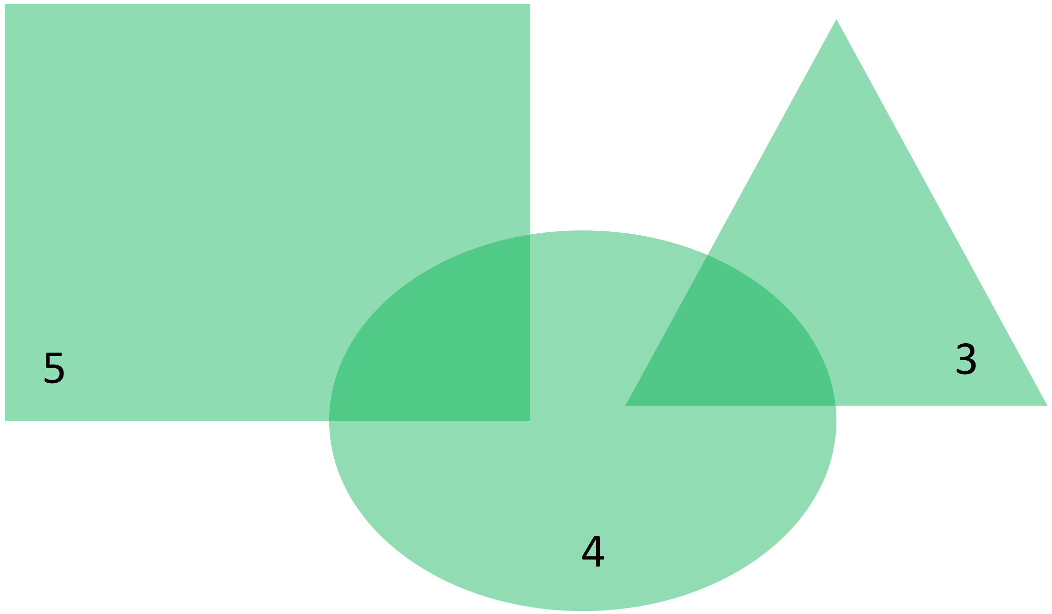

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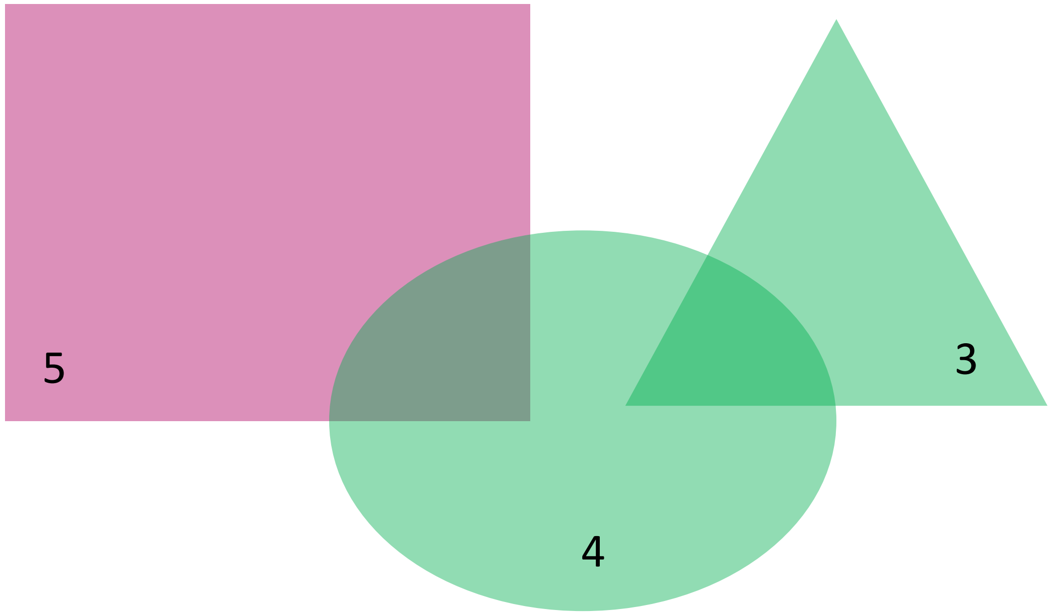

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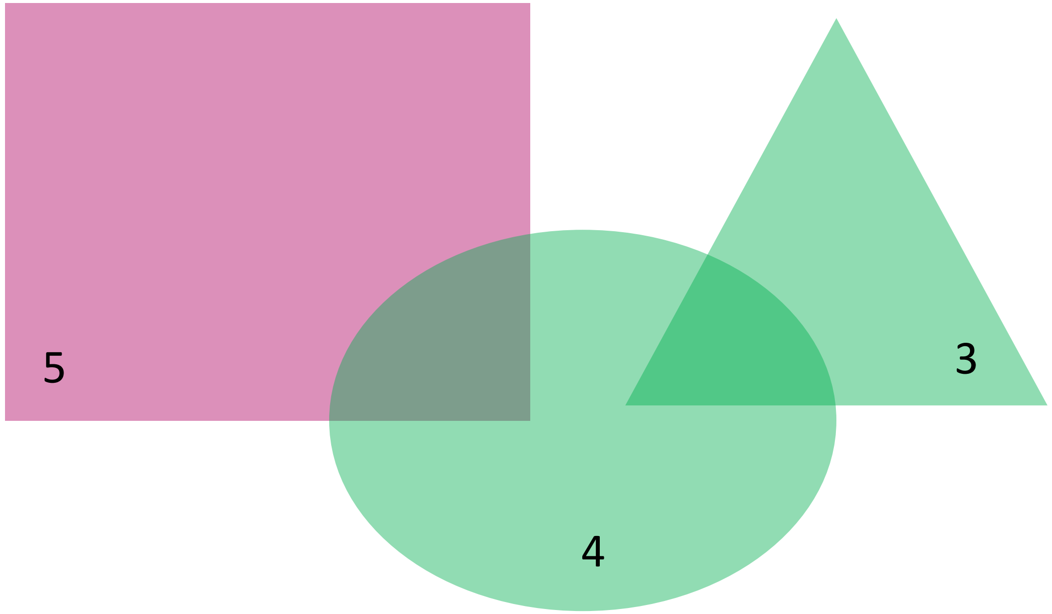

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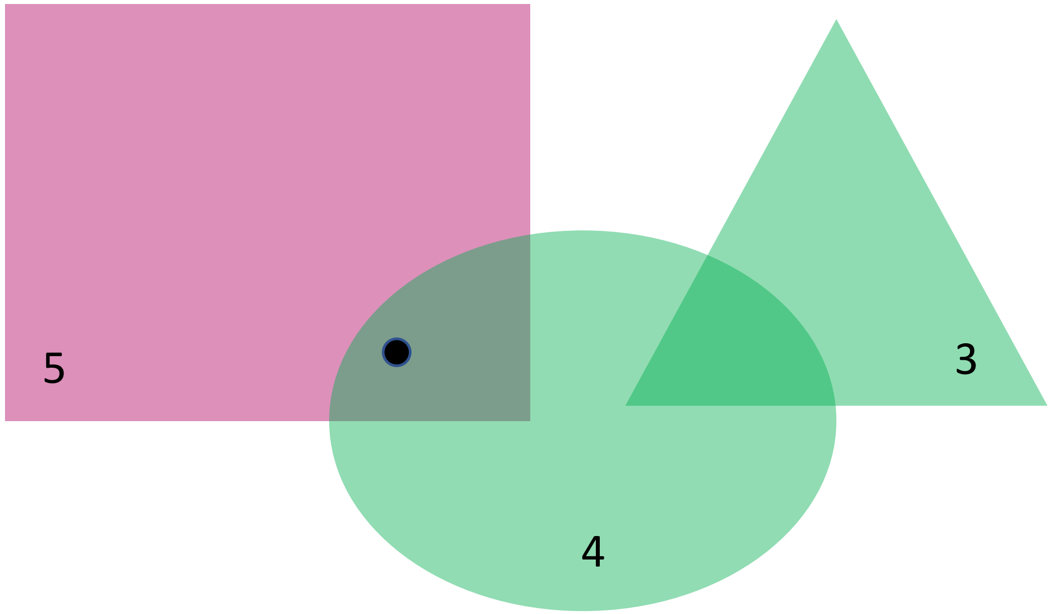

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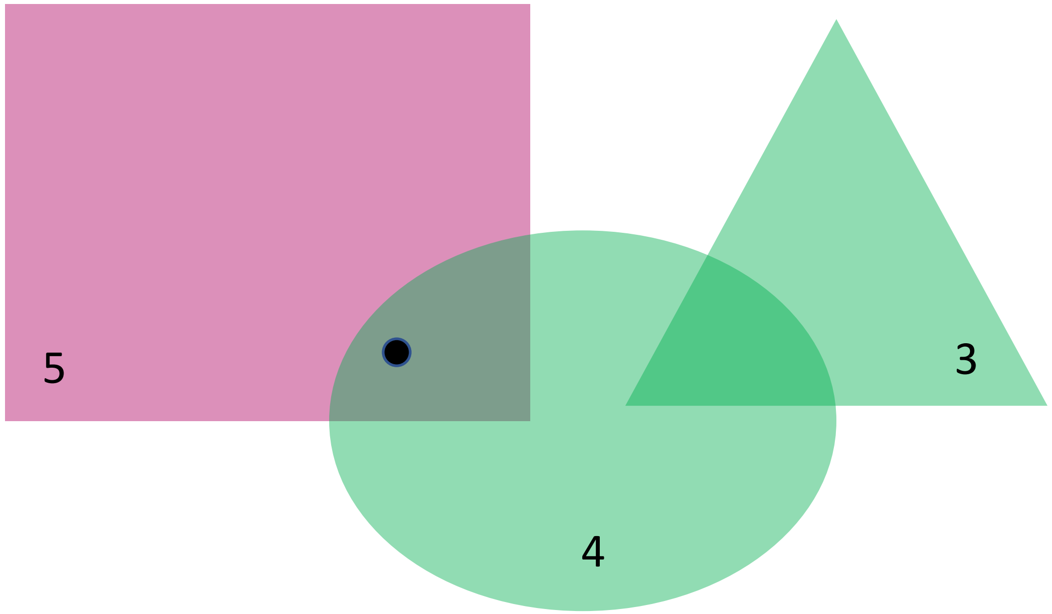

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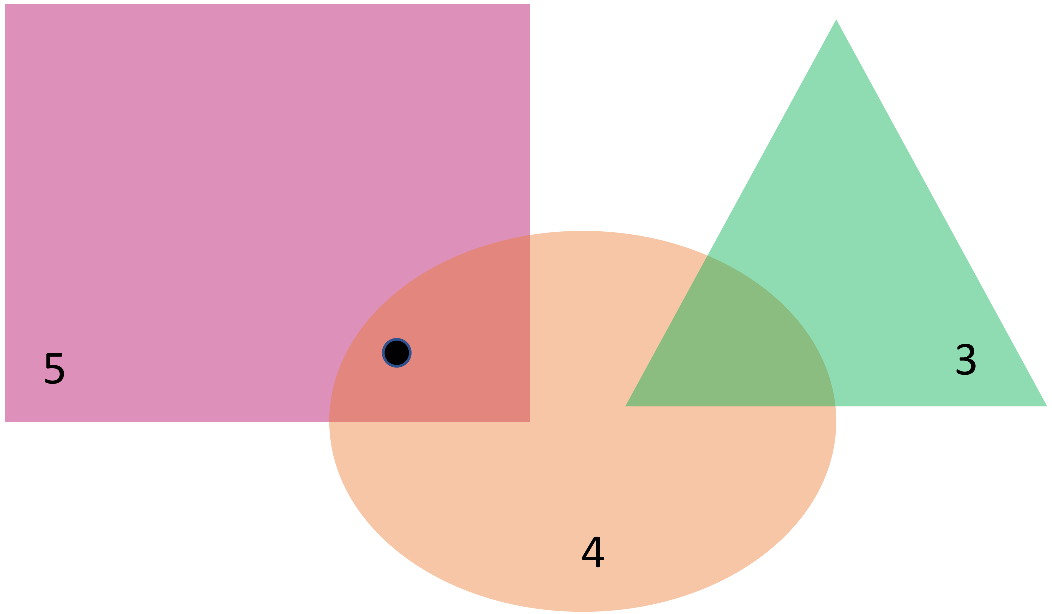

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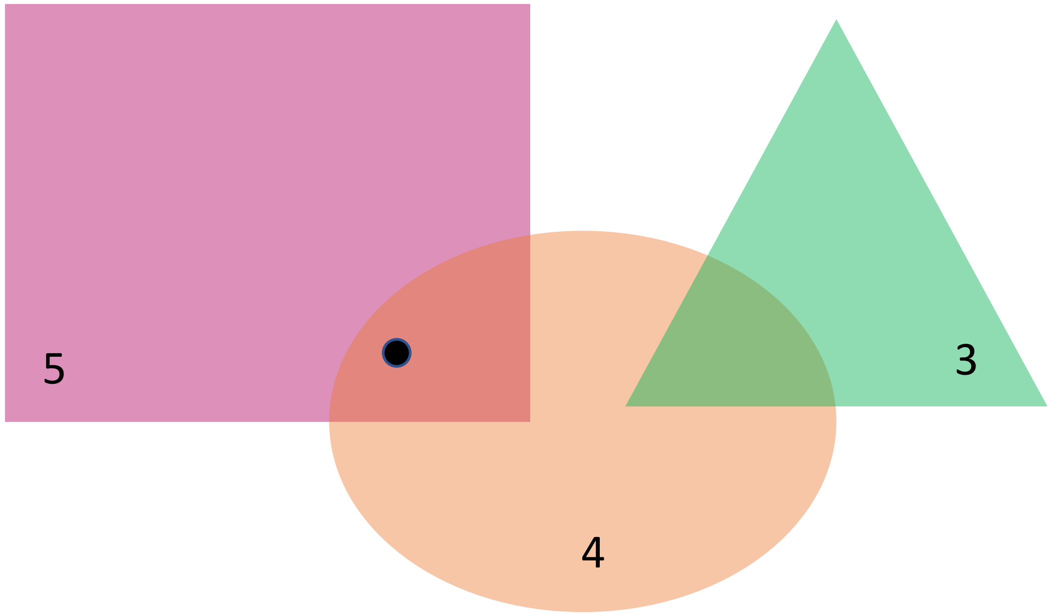

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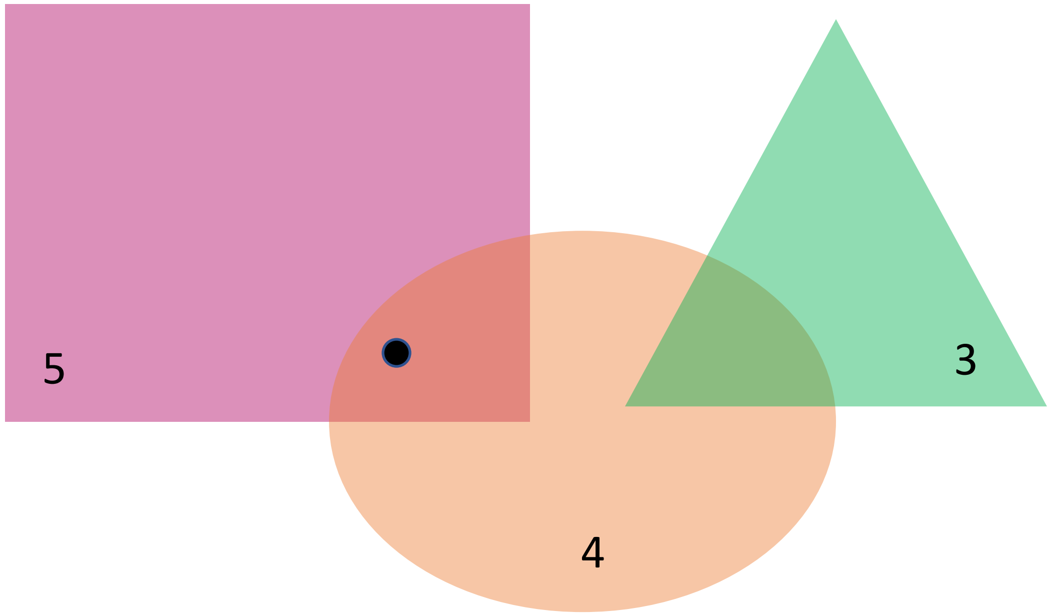

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Number of successes then yields an unbiased estimator for volume

ApproxUnion is an FPRAS

1. Volume Computation: FPRAS with error ϵ_V, confidence δ_V
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for $\varepsilon_V, \varepsilon_S \leq \frac{\varepsilon^2}{47k}, \varepsilon_P \leq \frac{\varepsilon^2}{47k^2}, \delta_V \leq \frac{\delta}{4k}, \delta_S + \delta_P \leq \frac{\delta}{2276 \ln(\frac{8}{\delta})\frac{k}{\varepsilon^2}}$.

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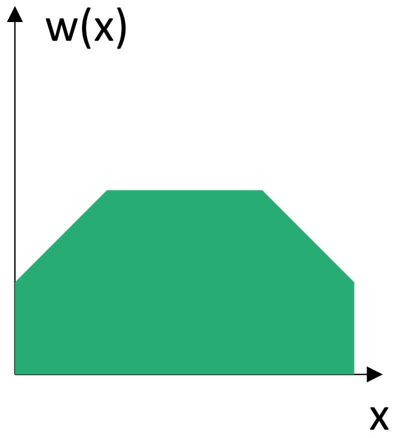

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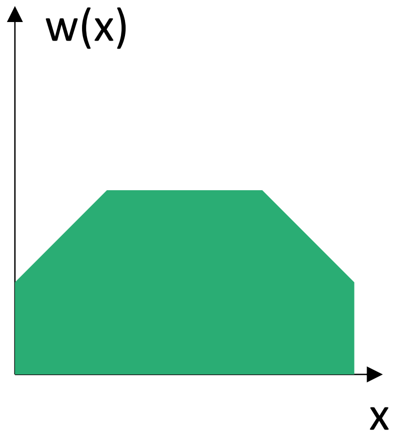

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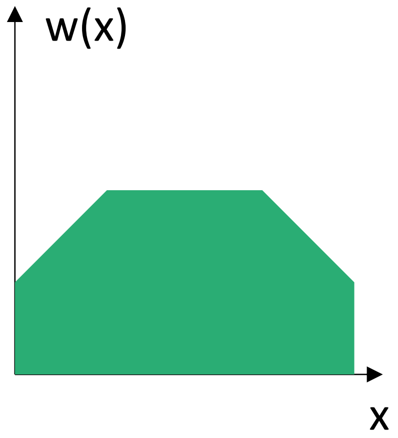

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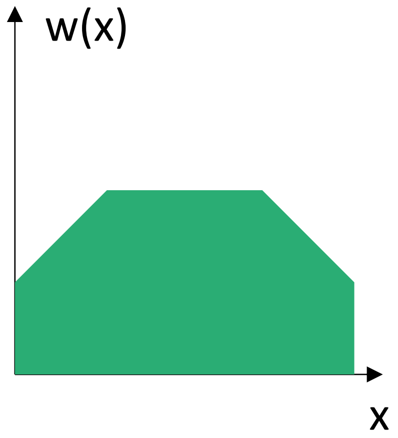

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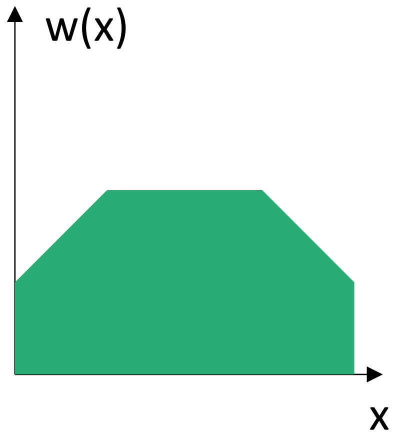

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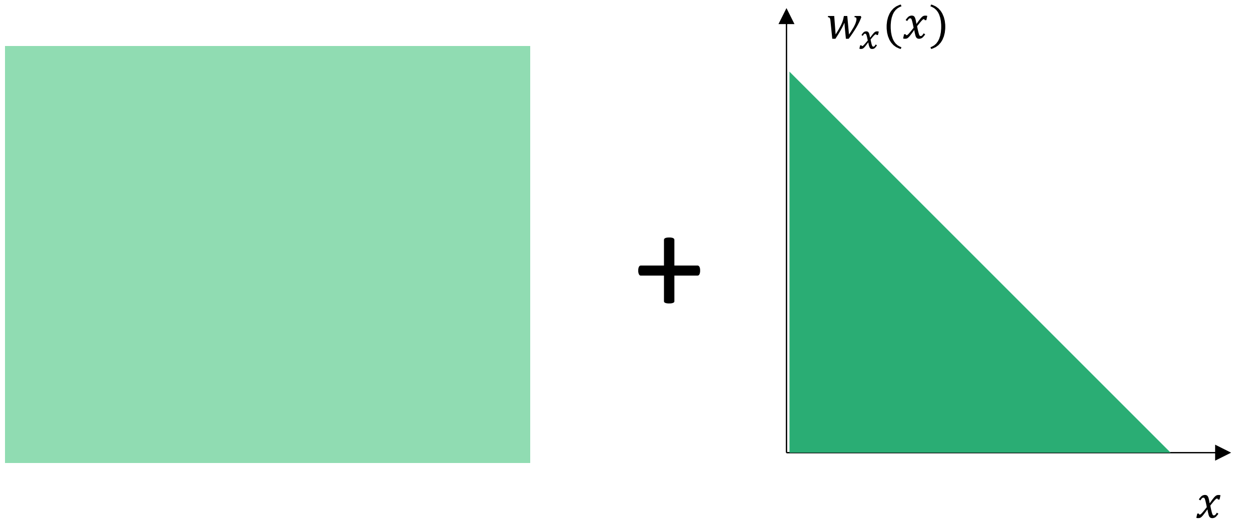

2D Convex Polytope   Concave Weight Function

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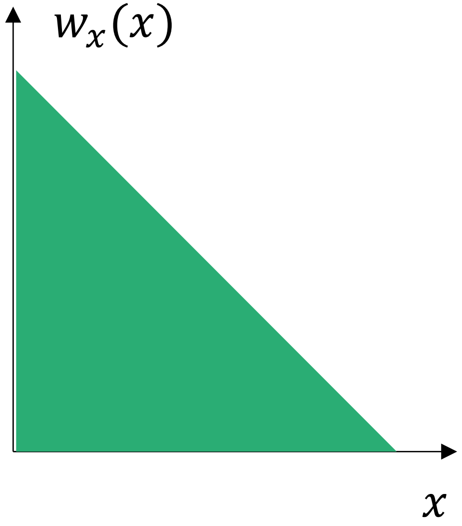

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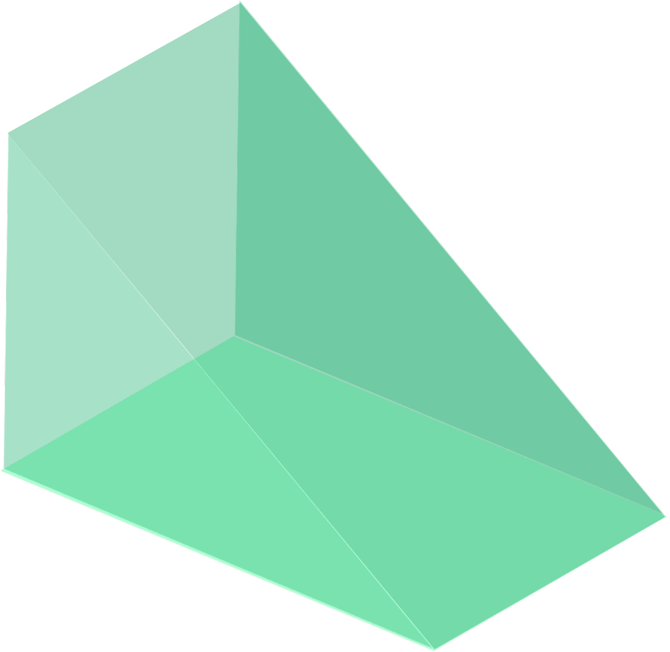

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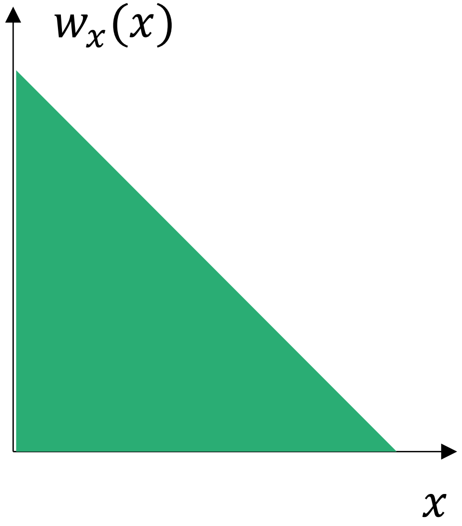

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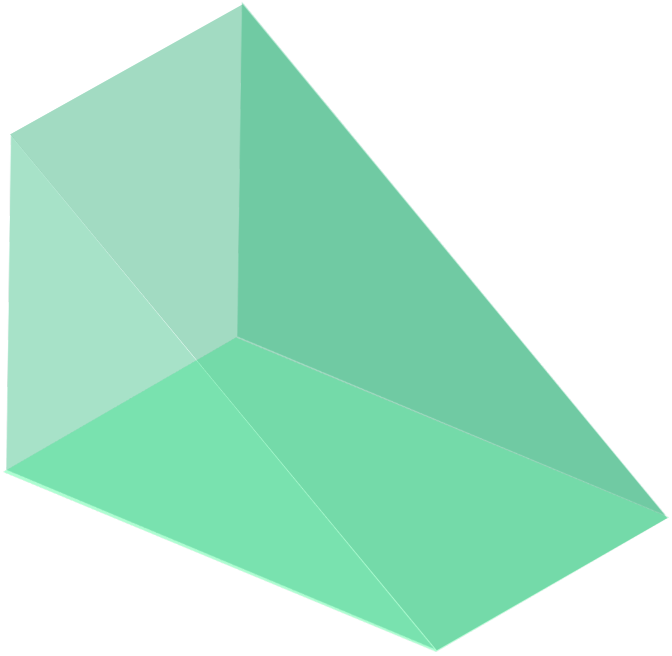

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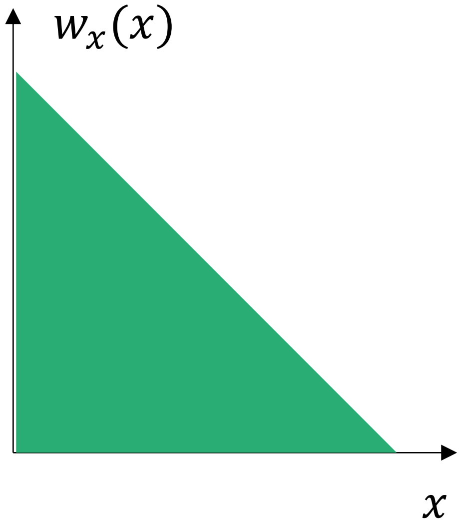

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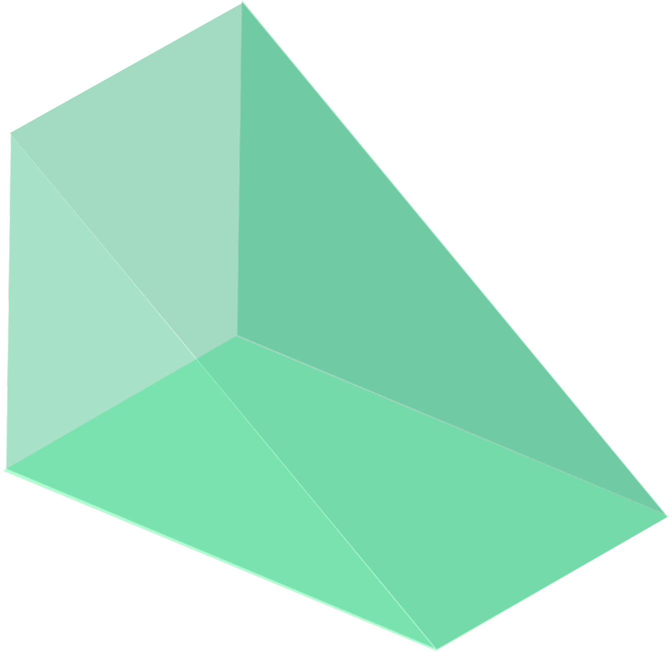

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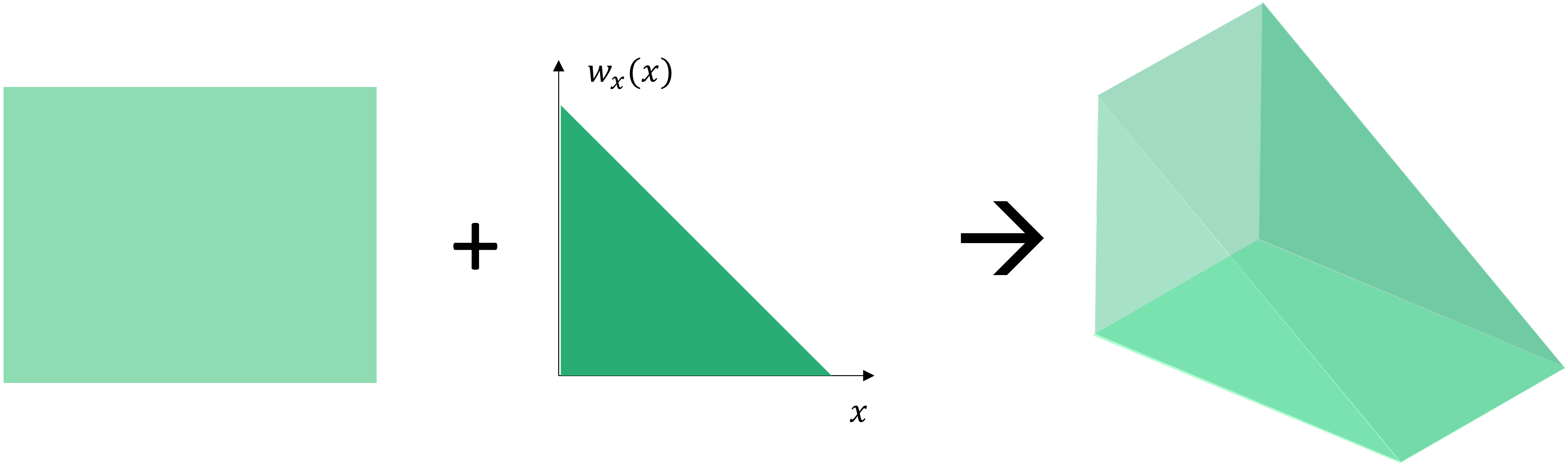

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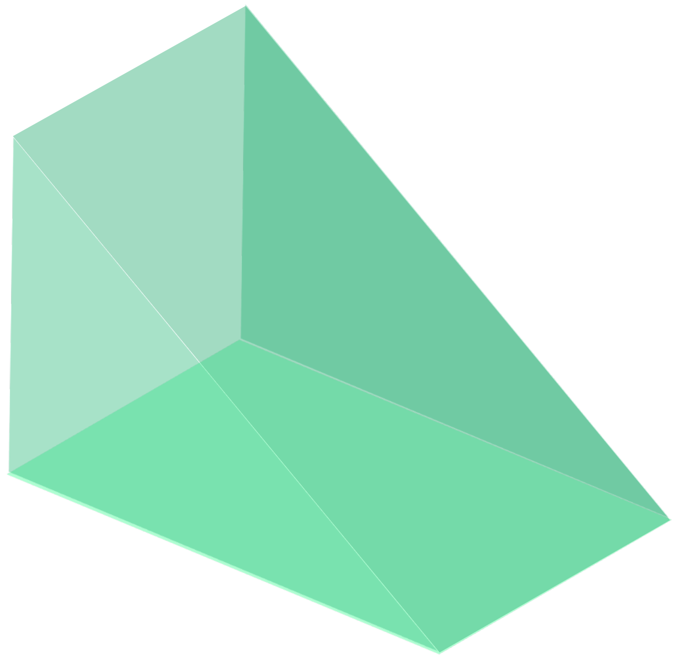

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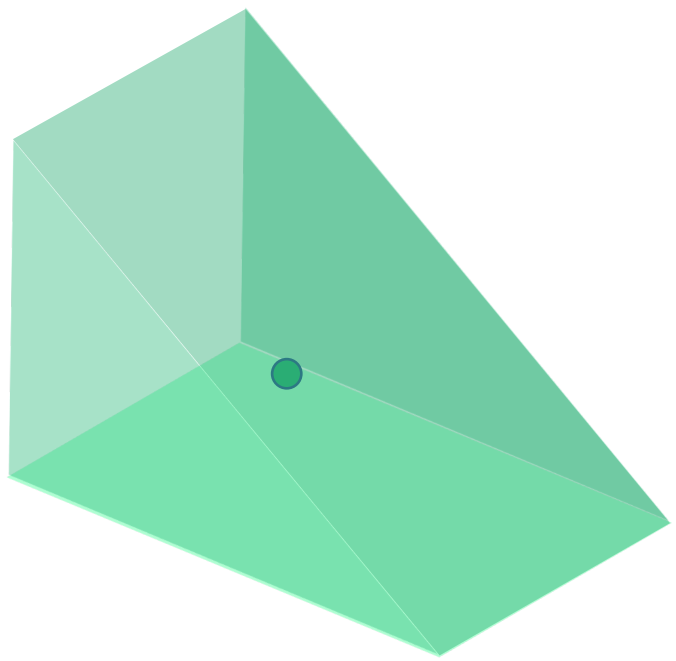

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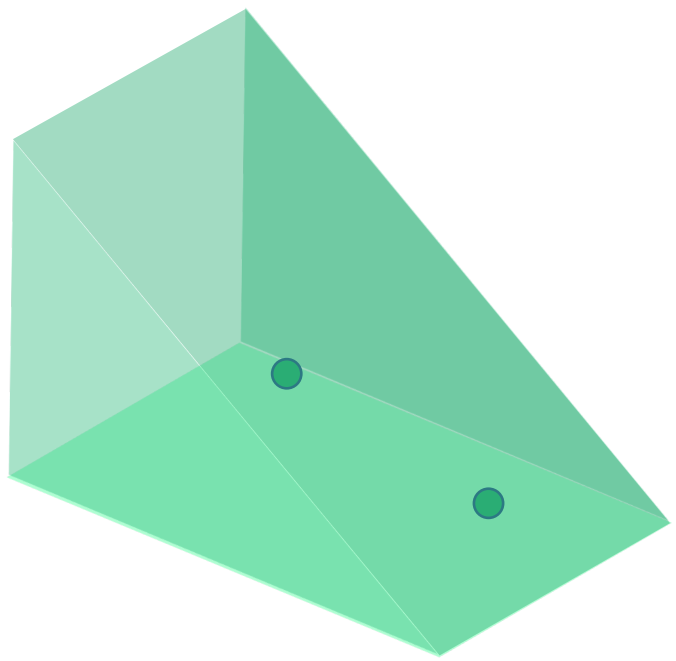

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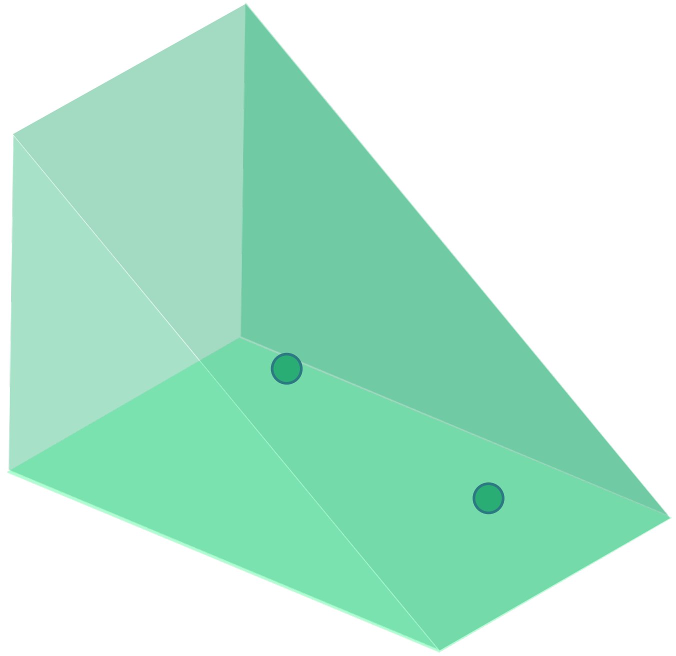

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- Real weight function is a concave polynomial with degree up to 5.

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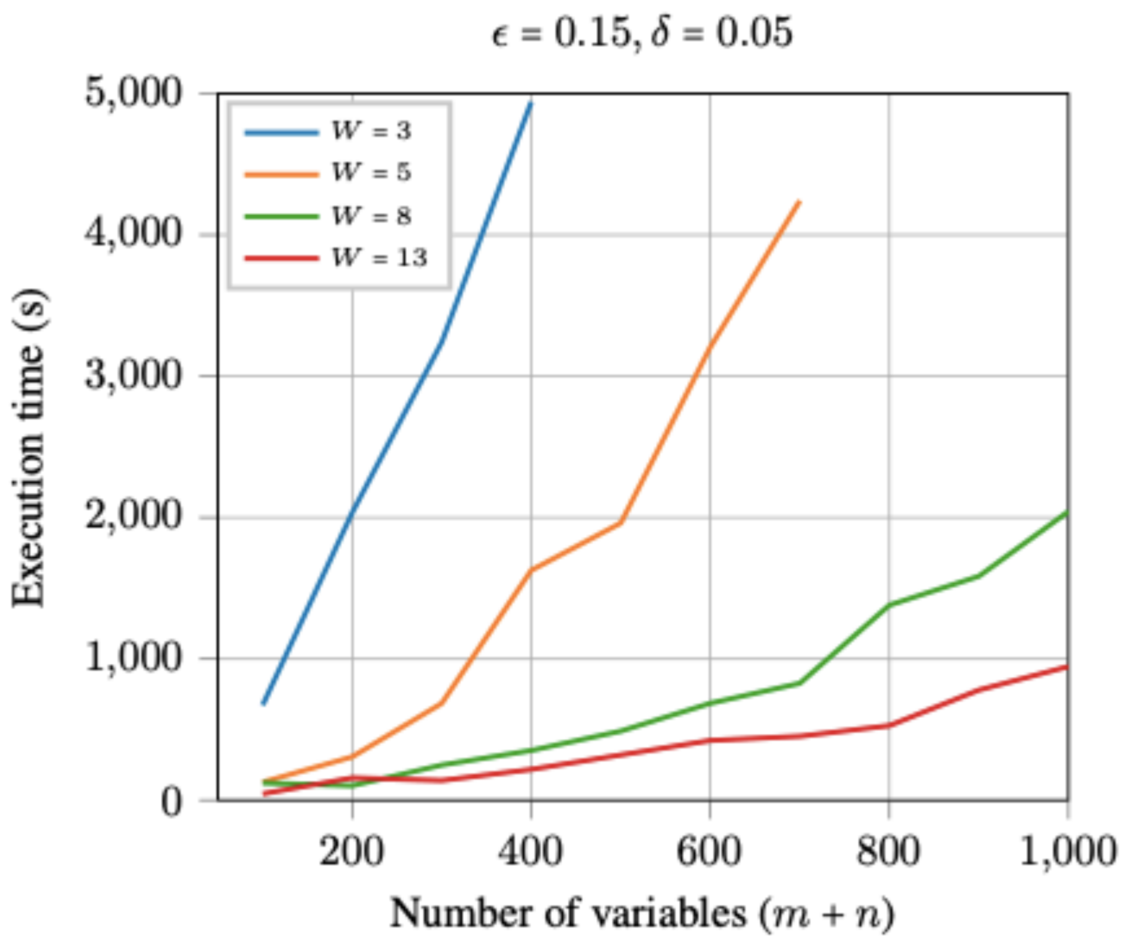
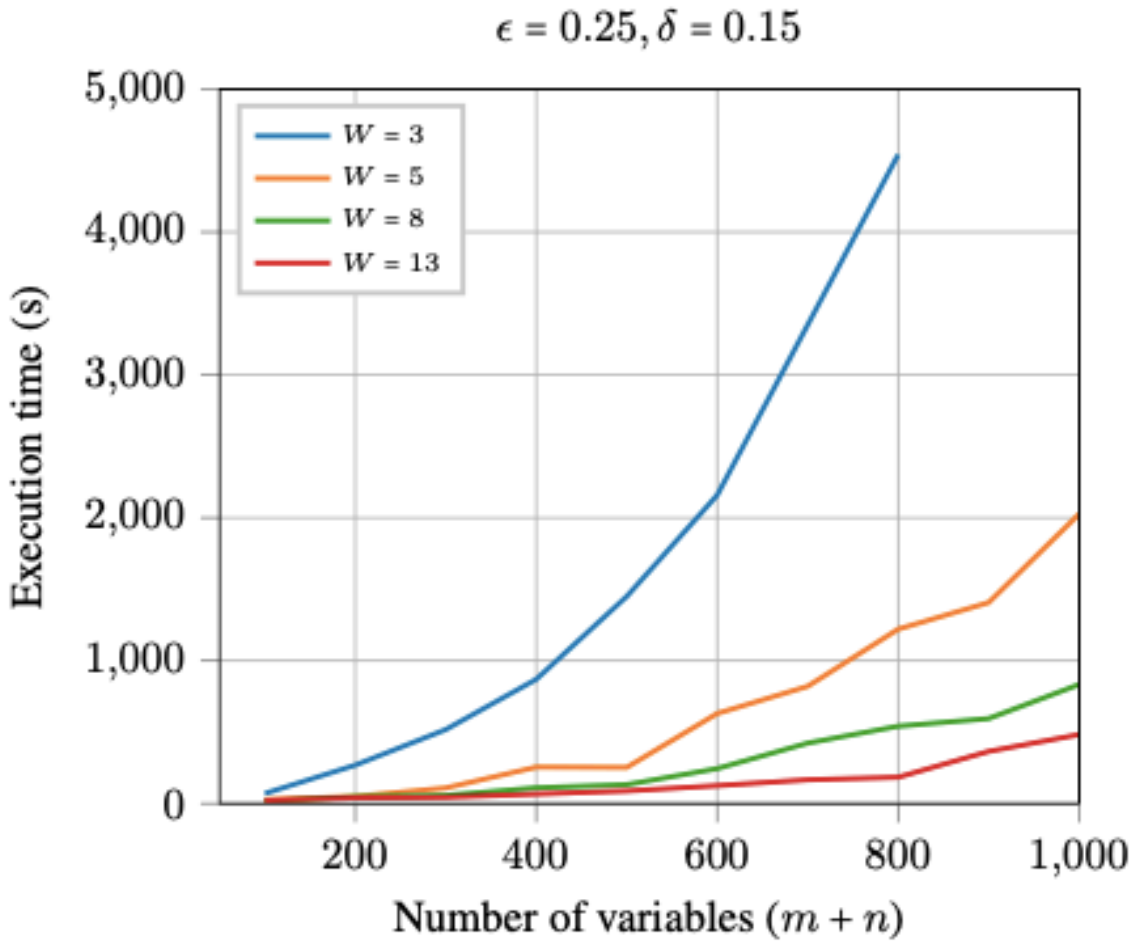
- Target confidence δ: Set to 0.05, 0.15 and 0.25.

Results



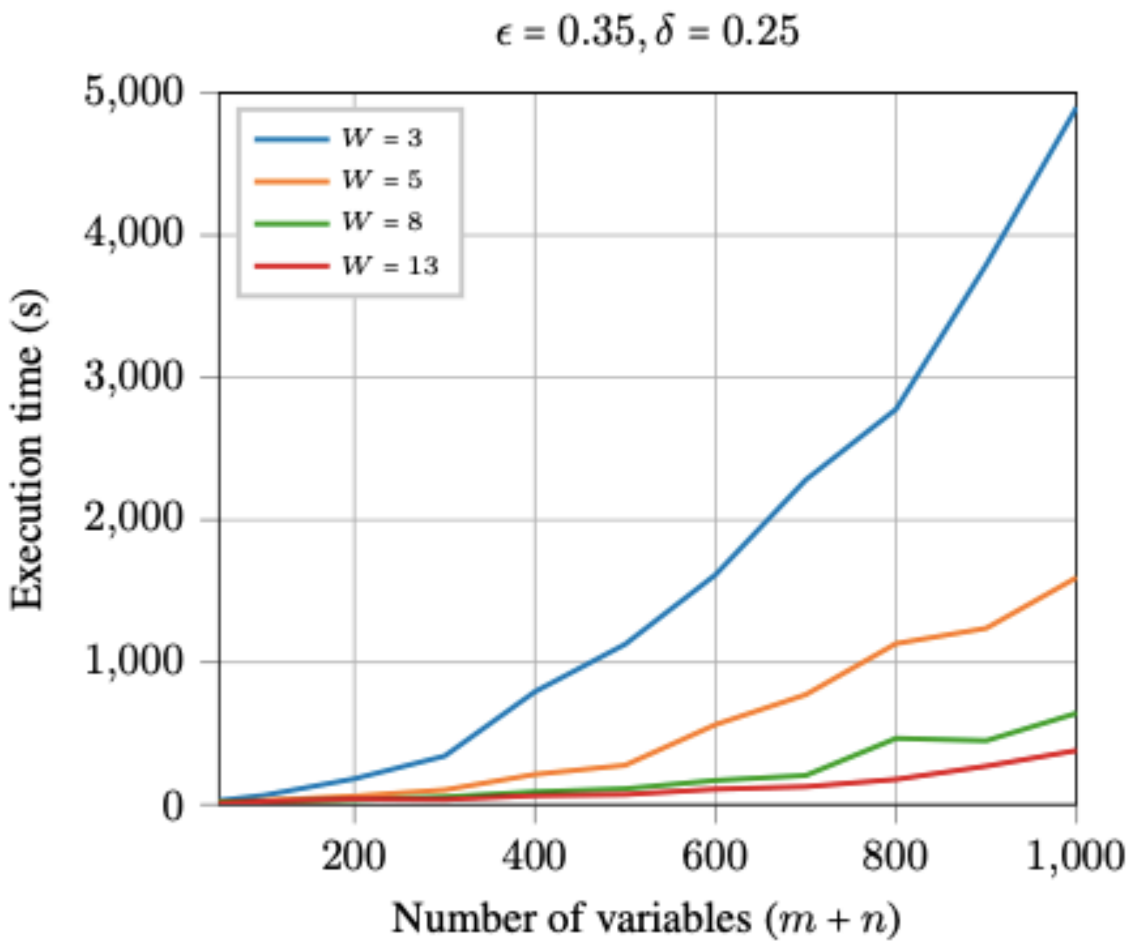
$\epsilon = 0.15, \delta = 0.05$
5,000
4,000
3,000
2,000
1,000
0
Execution time (s)
$W = 3$
$W = 5$
$W = 8$
$W = 13$
200
400
600
800
1,000
Number of variables $(m + n)$

Results



$\epsilon = 0.25, \delta = 0.15$
$W = 3$
$W = 5$
$W = 8$
$W = 13$
Execution time (s)
5,000
4,000
3,000
2,000
1,000
0
200
400
600
800
1,000
Number of variables $(m + n)$

Results



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5,000
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0
Execution time (s)
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- ApproxWMI is scalable, and can efficiently solve instances with up to 1K variables, which is out of reach for existing WMI solvers.
- ApproxWMI can be extended to more general factorizations enabling real-Boolean dependency.
- ApproxWMI is a useful tool for efficient probabilistic inference in hybrid domains.

Thank You!

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