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Abstract

We are interested in the near flat regime of the radial basis function periodic interval $[0, 2\pi)$. Therefore we use radial basis functions on the sphere S^1

$$\phi(\theta; \theta_0) = \phi_\epsilon(t) = \psi_\epsilon(\sqrt{2-2t}), \quad t \in [-1, 1] \tag{1}$$

with t = $\cos(\theta - \theta_0) = x^T x_0, x = e^{i\theta}, x_0 = e^{i\theta_0}, \theta, \theta_0 \in \mathbb{R}$ and $\psi_\epsilon(r) : [0, \infty) \to \mathbb{R}$ is the usual radial basis function. Hubbert and Baxter [1] give analytic expressions for the Fourier coefficients $c_{k,\epsilon}$ in

$$\phi_\epsilon(t) = c_{k,\epsilon} + \sum_{k=1}^{\infty} \epsilon^{2k} c_{k,\epsilon} \left(e^{i\theta} e^{-i\theta_0} + e^{-i\theta} e^{i\theta_0} \right) \tag{2}$$

For Gaussians, i.e.,

$$\phi_\epsilon(t) = e^{-\epsilon^2(2-2t)}, \tag{3}$$

these coefficients are

$$c_{0,\epsilon} = 0, \quad c_{k,\epsilon} = \frac{k I_k(2\epsilon^2) e^{-2\epsilon^2}}{\epsilon^{2k}} \tag{4}$$

where I_k is the modified Bessel function of the first kind. These coefficients converge to finite numbers when ϵ grows small. Since $I_k(z) \sim \frac{(z/2)^k}{\Gamma(k+1)}$ for small z (HMF, ...), $\lim_{\epsilon \searrow 0} c_{k,\epsilon} = \frac{1}{\Gamma(k)}$.

Keeping only the first K terms of (2) and filling in $x_m = 2\pi m/M$ and $x_n = 2\pi n/N$, the vector of basis elements can be written as

$$\begin{bmatrix} \phi(\frac{2\pi m}{M}; \frac{2\pi 0}{N}) \\ \phi(\frac{2\pi m}{M}; \frac{2\pi 1}{N}) \\ \vdots \\ \phi(\frac{2\pi m}{M}; \frac{2\pi(M-2)}{N}) \\ \phi(\frac{2\pi m}{M}; \frac{2\pi(M-1)}{N}) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W^0 & W_N^0 & \cdots & W_N^0 & W_N^0 \\ W_N^0 & W_N^{-1} & \cdots & W_N^{-K} & W_N^K & \cdots & W_N^2 & W_N^1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ W_N^0 & W_N^{-1(N-2)} & \cdots & W_N^{-K(N-2)} & W_N^{K(N-2)} & \cdots & W_N^{2(N-2)} & W_N^{1(N-2)} \\ W_N^0 & W_N^{-1(N-1)} & \cdots & W_N^{-K(N-1)} & W_N^{K(N-1)} & \cdots & W_N^{2(N-1)} & W_N^{1(N-1)} \end{bmatrix} E_{K,\epsilon} C_{K,\epsilon} \begin{bmatrix} W_M^0 \\ W_M^1 \\ \vdots \\ W_M^K \\ W_M^{-K} \\ \vdots \\ W_M^{-2} \\ W_M^{-1} \end{bmatrix} \tag{5}$$

where $W_N = e^{\frac{i2\pi}{N}}$, $W_M = e^{\frac{i2\pi}{M}}$ and

$$E_{K,\epsilon} = \operatorname{diag}\{\epsilon^0, \epsilon^2, \ldots, \epsilon^K, \epsilon^K, \ldots, \epsilon^4, \epsilon^2\} \tag{6}$$

$$C_{K,\epsilon} = \operatorname{diag}\{c_{0,\epsilon}, c_{1,\epsilon}, \ldots, c_{K,\epsilon}, c_{K,\epsilon}, \ldots, c_{2,\epsilon}, c_{1,\epsilon}\}. \tag{7}$$

The collocation matrix $A_{m,n} = \phi(x_m; x_n)$ for $m = 1, \ldots, M$, $n = 1, \ldots, N$ becomes

$$A = \begin{bmatrix} W_M^0 & W_M^0 & \cdots & W_M^0 & W_M^0 \\ W_M^0 & W_M^{-1} & \cdots & W_M^{-(M-2)} & W_M^{-(M-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ W_M^0 & W_M^{-K} & \cdots & W_M^{-K(M-2)} & W_M^{-K(M-1)} \\ W_M^0 & W_M^{K} & \cdots & W_M^{K(M-2)} & W_M^{K(M-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ W_M^0 & W_M^{2} & \cdots & W_M^{2(M-2)} & W_M^{2(M-1)} \\ W_M^0 & W_M^{1} & \cdots & W_M^{(M-2)} & W_M^{M-1} \end{bmatrix}^* \times \tag{8}$$

$$E_{K,\epsilon} \times C_{K,\epsilon} \times \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 & W_N^0 \\ W_N^0 & W_N^{-1} & \cdots & W_N^{-1(N-2)} & W_N^{-1(N-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ W_N^0 & W_N^{-K} & \cdots & W_N^{-K(N-2)} & W_N^{-K(N-1)} \\ W_N^0 & W_N^{K} & \cdots & W_N^{K(N-2)} & W_N^{K(N-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ W_N^0 & W_N^{2} & \cdots & W_N^{2(N-2)} & W_N^{2(N-1)} \\ W_N^0 & W_N^{1} & \cdots & W_N^{1(N-2)} & W_N^{1(N-1)} \end{bmatrix}. \tag{9}$$

Note the adjoint applied at the first matrix. The first and last matrix are very similar to the DFT matrix of size $N \times N$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & W_N^{-3} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & W_N^{-6} & \cdots & W_N^{-2(N-1)} \\ 1 & W_N^{-3} & W_N^{-6} & W_N^{-9} & \cdots & W_N^{-3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & W_N^{-3(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix} \tag{10}$$

References

[1] S. Hubbert and B. Baxter, *Radial basis functions for the sphere*, Recent Progress in Multivariate Approximation. 4th International Conference, September 2000, (2001), pp. 33–47.