1 Procedure to derive the asymptotic expansions

This document explains the procedure we used to derive the asymptotic expansions of nodes and weights of classical and modified Gauss-Jacobi and Gauss-Laguerre quadrature rules for an arbitrary order in n. This procedure is implemented in the Sage worksheets that are available with this document. These files accompany the article "Arbitrary-order asymptotic expansions of Gaussian quadrature rules with classical and generalised weight functions" by Daan Huybrechs and Peter Opsomer [5].

1.1 Introduction

The asymptotic formulae in the article have a very similar structure for all regions and also between Laguerre-, Hermite- and Jacobi type orthogonal polynomials. We retain this analogy by using the same symbols for analogous expansion coefficients, while three symbols in the superscript specify them. The first character is q in general and denotes the type of quadrature rule: L for Laguerre-type and J for Jacobi-type. There is no separate symbol for the Hermite case as we treat it in the paper. If present, the second symbol is r and indicates the region: IV for the left boundary region, I for the lens or bulk and III for the right boundary region. If present, the third character is p in general and indicates which polynomial we are computing: n for $p_n(x)$, d for $p'_n(x)$ and s for $p_{n-1}(x)$. The Kronecker delta function is used to distinguish between the cases, and avoids tedious repetitions of similar formulae.

There are four types of asymptotic expansions for the Jacobi polynomials, the Laguerre polynomials, as well as for the Hermite polynomials. As a result, one would expect that 24 long asymptotic formulae should be provided for a complete and accurate characterisation of the nodes and weights. However, the expansions in the outer region are not of interest as this region does not contain zeros. Moreover, Gauss-Hermite quadrature rules can straightforwardly be derived from Gauss-Laguerre rules as elaborated on in the article. Next, only the expansions of the polynomials in the lens involve the evaluation of functions such as J_α, Ai and cos at large arguments, such that we could evaluate the weights near the endpoints by inserting the asymptotic expansion of the polynomial into the formula

$$w_k = \frac{-\gamma_{n+1}}{\gamma_n p'_n(x_k) p_{n+1}(x_k)} = \frac{\gamma_n}{\gamma_{n-1} p'_n(x_k) p_{n-1}(x_k)} > 0. \tag{1.1}$$

without loss of accuracy due to this effect. However, we do also derive explicit formulae for the weights near the endpoints as these are expected to require significantly less computational time. Finally, there is a symmetry between the expansions in the left and right region in the Jacobi case, which we exploit in the paper. Taking all this into account, two times four explicit formulae are sufficient and are presented in the article.

For Laguerre-type polynomials, the only difference between monomial and general function $Q(x)$ is the formula for f_j, the expansion coefficient of the phase function detailed in [4, §6]. The procedure for general polynomial $Q(x)$ yields fractional powers of n, which was considered too technical for the scope of this research. Arbitrary-order expansions near the soft edge were not pursued, as the corresponding weights underflow even for moderate n. As a result, the formulae in this document are not valid for the combination $L\mathrm{III}p$. However, the paper does present the leading order term for nodes and weights there.

For the expansions in the lens, again for Laguerre-type polynomials, we only consider the case where $Q(x)$ is a monomial. Otherwise, we would have to compute Newton iterations on and series expansion coefficients of (an integral of) a contour integral of $Q'(x)$ for z near $z_{1,1}$ for each node x_k, and this may be quite time-consuming. However, for some weights, it might be possible to obtain explicit symbolic expressions for such series expansion coefficients via residue calculus as in [2, §5.2] and [4, §3.4]. Near the endpoints, such coefficients may be reused as they are independent of k and represent a series expansion which is already used to construct higher order terms of the asymptotic expansions of the orthogonal polynomials.

1.2 Powers of z

The unknowns $z_{1,q}^{qrn}$ provide the expansion of the nodes. The z in the asymptotic expansions only changes for the shifted Laguerre polynomial evaluated at the node $p_{n-1}(x_k)$:

$$z^{qrp} = \frac{x_k}{\beta_{n-\delta_{p,s}}} \sim a\delta_{r,\mathrm{IV}} + \frac{\beta_n}{\beta_{n-\delta_{p,s}}} \sum_{q=1}^{\infty} z_{1,q}^{qrp} n^{1-2\delta_{r,\mathrm{IV}}-q}, \tag{1.2}$$

$$\left(\frac{n^{2\delta_{r,\mathrm{IV}}} z^{qrp}}{z_{1,1}^{qrn}} - 1\right)^u \sim \sum_{j=u}^{\infty} \chi_{u,j}^{qrp} n^{-j},$$

$$\chi_{1,j}^{qrp} = \frac{z_{1,j+1}^{qrp}}{z_{1,1}^{qrn}}, \qquad \chi_{u,j}^{qrp} = \sum_{y=1}^{j-1} \chi_{1,j-y}^{qrp} \chi_{u-1,y}^{qrp}.$$

Although expansions of temporary variables were computed as inverse powers of n here, the notation for the expansions of nodes and weights in the paper is considerably shorter and could facilitate computing temporary variables here. Using the notation $|_{z=z^{qrp}}$ for evaluation of the zeroth or first derivative at $z = z^{qrp}$, integer powers and the square root expand as

$$(z^{qrp} - a)^j \sim \sum_{l=1+\delta_{r,\mathrm{IV}}(2j-2)}^{\infty} z_{j,l}^{qrp} n^{1-2\delta_{r,\mathrm{IV}}-l},$$

$$z_{j,l}^{qrp} = \sum_{m=\delta_{r,\mathrm{IV}}(2j-3)}^{\delta_{r,\mathrm{IV}}(l-2)} z_{1,l-m-1}^{qrp} z_{j-1,m}^{qrp},$$

$$\left.\frac{\partial^{(\delta_{p,d})}\sqrt{z-a}}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{qrp}} \sim \sum_{m=1}^{\infty} \omega_m^{qrp} n^{1-m-\delta_{r,\mathrm{IV}}(1+2\delta_{p,d})},$$

$$\omega_m^{qrp} = \left\{z_{1,1}^{qrn} - a\right\}^{1/2-\delta_{p,d}} 2^{-\delta_{p,d}\delta_{q,L}} \left(\delta_{m,1} + \sum_{u=1}^{m-1} \binom{1/2-\delta_{d,d}}{u} \chi_{u,m-1}^{qrp}\right),$$

$$X_{1,j}^{L\mathrm{I}p} = \frac{z_{1,j+1}^{L\mathrm{I}p}}{z_{1,1}^{L\mathrm{I}p} - 1}$$

$$X_{u,j}^{L\mathrm{I}p} = \sum_{y=1}^{j} X_{1,1-y+j}^{L\mathrm{I}p} X_{u-1,y}^{L\mathrm{I}p},$$

$$\left.\frac{\partial^{(\delta_{p,d})}\sqrt{1-z}}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{L\mathrm{I}p}} \sim \sum_{m=1}^{\infty} \Omega_m^{L\mathrm{I}p} n^{1-m},$$

$$\Omega_m^{L\mathrm{I}p} = \{1 - z_{1,1}\}^{1/2-\delta_{p,d}} (-2)^{-\delta_{p,d}} \left(\delta_{m,1} + \sum_{u=1}^{m-1} \binom{1/2-\delta_{p,d}}{u} X_{u,m-u}^{L\mathrm{I}p}\right).$$

1.3 Phase function

The expansion of the trigonometric functions appearing in the left boundary region is

$$\Xi^{q\mathrm{IV}p} = \begin{pmatrix} \sin\left[\mu(z) + \frac{1}{2}\arccos\left(\frac{2z-a-1}{1-a}\right)\right] & \cos\left[\mu(z) + \frac{1}{2}\arccos\left(\frac{2z-a-1}{1-a}\right)\right] \\ \sin\left[\mu(z) - \frac{1}{2}\arccos\left(\frac{2z-a-1}{1-a}\right)\right] & \cos\left[\mu(z) - \frac{1}{2}\arccos\left(\frac{2z-a-1}{1-a}\right)\right] \end{pmatrix}$$

$$\left.\frac{\partial^{(\delta_{p,d})}\Xi^{q\mathrm{IV}p}}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{q\mathrm{IV}p}} \sim \sum_{j=1}^{\infty} \Xi_j^{q\mathrm{IV}p} n^{1+\delta_{p,d}-j}$$

$$
\begin{aligned}
\Xi_j^{q\mathrm{IV}p} &= \sin\begin{pmatrix} \frac{\pi}{2} & \pi \\ \frac{-\pi}{2} & 0 \end{pmatrix}\delta_{j,1} + \sum_{q=1}^{j-1}\frac{1}{q!}\begin{pmatrix} \sin\left[\frac{\pi}{2}-\frac{q\pi}{2}\right] & \sin\left[\pi-\frac{q\pi}{2}\right] \\ \sin\left[\frac{-\pi}{2}-\frac{q\pi}{2}\right] & \sin\left[\frac{-q\pi}{2}\right] \end{pmatrix}\begin{pmatrix} \zeta_{1,q,j-1}^{q\mathrm{IV}p} \\ \zeta_{2,q,j-1}^{q\mathrm{IV}p} \end{pmatrix}, \quad p \neq d, \\
\Xi_i^{q\mathrm{IV}d} &= \sum_{j=1}^{i}\begin{pmatrix} \Xi_{j,1,2}^{q\mathrm{IV}d} & \Xi_{j,1,1}^{q\mathrm{IV}d} \\ \Xi_{j,2,2}^{q\mathrm{IV}d} & \Xi_{j,2,1}^{q\mathrm{IV}d} \end{pmatrix}\zeta_{1,i+1-j}^{q\mathrm{IV}d},
\end{aligned}
$$

The functions that determine the behaviour of the polynomial are according to [2, §2.3] and [5, §2.2] a cosine in the lens and Bessel functions near finite endpoints: their arguments $A_b(z)$ are expanded in § 1.6 and 1.8 as

$$
\begin{aligned}
\left[A_b(z^{qrp}) - \epsilon_{b,1,1}^{qrp}\left\{1+\delta_{r,\mathrm{I}}\left(n-1+\frac{\epsilon_{b,1,2}^{q\mathrm{I}p}}{\epsilon_{b,1,1}^{q\mathrm{I}p}}\right)\right\}\right]^i &\sim \sum_{k=2}^{\infty}\epsilon_{b,i,k}^{qrp} n^{1-k+\delta_{r,\mathrm{I}}}, \\
\epsilon_{b,i,k}^{qrp} = \sum_{j=2+\delta_{r,\mathrm{I}}}^{k-1}\epsilon_{b,1,k+1-j+\delta_{r,\mathrm{I}}}^{qrp}\epsilon_{b,i-1,j}^{qrp},
\end{aligned}
$$

where $\epsilon_{1,1,1}^{qrp}$ is the kth zero of the function determining the behaviour.

In the case of finite endpoints, where the behaviour of the polynomial is given by Bessel functions, we define one vector J where the arguments of the Bessel function are shifted with both negative and positive integers. The maximal order we need is T, we evaluate at the zero of the unshifted Bessel function $j_{\alpha,k}$ for Laguerre or $j_{\beta,k}$ for Jacobi, and we have [6, (10.6.1)]

$$
\begin{aligned}
J_{\nu-1}(z) + J_{\nu+1}(z) &= \frac{2\nu}{z}J_\nu(z), \\
J_T = 0, \quad J_{T-1} &= J_{\alpha-1}(j_{\alpha,k}), \quad J_{T+1} = -J_{T-1}, \\
J_{T\pm j} &= \frac{2(\alpha \pm j \mp 1)}{j_{\alpha,k}}J_{T\pm j\mp 1} - J_{T\pm j\mp 2}.
\end{aligned}
$$

The derivatives of the Bessel function are [6, (10.6.7)]

$$
\begin{aligned}
J_\alpha^{(k)}(z) &= \frac{1}{2^k}\sum_{n=0}^{k}(-1)^n\binom{k}{n}J_{\alpha-k+2n}(z), \\
J_\alpha^{(k)}(y) &\sim \sum_{l=0}^{\infty}\frac{S_{l+k}}{l!}(y-j_{\alpha,k})^l, \\
S_i &= \frac{1}{2^i}\sum_{n=0}^{i}(-1)^n\binom{i}{n}J_{T-i+2n}.
\end{aligned}
$$

This also appeared in [1, (2.16-19)]: all of the J_t are $J_{\alpha-1}(j_{\alpha,k})$ times a rational function in α and $j_{\alpha,k}$, so all of the S_k are as well. We only need zeroth, first and second derivatives, so $k \in \{0,1,2\}$. Using this notation, we expand the Bessel function with the argument $A[z]$ as

$$
\begin{aligned}
\left.\frac{\partial^{(\delta_{p,d})}J_\alpha^{(k)}(A[z])}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{qrp}} &\sim \sum_{j=1}^{\infty}\iota_{k+1,j}^{qrp}n^{1-j+2\delta_{p,d}}, \\
\iota_{k+1,j}^{qrp} &= S_k\delta_{1,j} + \sum_{i=1}^{j-1}\frac{S_{k+i-1}}{i!}\epsilon_{i,j}^{qrp}, \quad p \neq d, \\
\iota_{k,q}^{L\mathrm{IV}d} &= \sum_{j=1}^{q}\iota_{k+1,j}^{L\mathrm{IV}n}\epsilon_{1,1,q+1-j}^{L\mathrm{IV}d}.
\end{aligned}
$$

One might be interested in an infinite endpoint, for which we did not consider arbitrary order expansions here. Then, the behaviour of the polynomial is given by an Airy function, which can be written as a (modified) Bessel function as well [6, §9.6]

$$Ai(-z) = \frac{\sqrt{z}}{3}\left(J_{1/3}\left[\frac{2z^{3/2}}{3}\right] + J_{-1/3}\left[\frac{2z^{3/2}}{3}\right]\right) \tag{1.3}$$

$$Ai(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}}K_{\pm 1/3}\left(\frac{2z^{3/2}}{3}\right) \tag{1.4}$$

$$Ai'(z) = \frac{-z}{\pi\sqrt{3}}K_{\pm 2/3}\left(\frac{2z^{3/2}}{3}\right). \tag{1.5}$$

This seems to prevent writing derivatives of the Airy function evaluated at zeros a_k of $Ai(z)$ as the first derivative $Ai'(a_k)$ times a rational function in a_k. On one hand, (1.3) suggests that a_k is not a zero of a Bessel function of the first kind. On the other hand, taking derivatives of (1.4) gives modified Bessel functions $K_{-2/3}$ and $K_{4/3}$ or $K_{-4/3}$ and $K_{2/3}$, while neither $K_{4/3}$ nor $K_{-4/3}$ appear in (1.5). However, using the ordinary differential equation $Ai''(z) = zAi(z)$, one can deduce by induction that

$$Ai^{(m)}(a_k) = (m-2)Ai^{(m-3)}(a_k) + a_k Ai^{(m-2)}(a_k), \quad m \geq 3.$$

This recursion can be started with $Ai(a_k) = 0$, $Ai^{(1)}(a_k) = Ai'(a_k)$ and $Ai^{(2)}(a_k) = 0$. As a result, $Ai^{(m)}(a_k)/Ai'(a_k)$ is a polynomial in a_k of degree $\leq \lfloor m/2 \rfloor$ with coefficients as in [10, A004747]:

$$\frac{Ai^{(m)}(a_k)}{Ai'(a_k)} = \sum_{i=1}^{\lfloor(\lceil m/2\rceil+m+1-3\lfloor m/3\rceil)/3\rfloor} b_{\lfloor m/3\rceil+i,3i-m-1+3\lfloor m/3\rceil} a_k^{3i-m-2+3\lfloor m/3\rceil},$$

$$b_{n,j} = \frac{(n-1)!}{(j-1)!3^{n-j}}\sum_{k=0}^{n-j}\binom{k}{n-j-k}3^k(-1)^{n-j-k}\binom{n+k-1}{n-1},$$

$$b_{n+1,j} = (3n-j)b_{n,j} + b_{n,j-1}, \qquad b_{1,j} = \delta_{j,1}.$$

1.4 Oscillatory part

For expansions in the lens, the U matrices are used in the expansion of the $R(z)$

$$\left(z^{q\mathrm{I}p} - \frac{1+a\pm 1}{2+a}\right)^{-p} \sim \sum_{l=1}^{\infty} H^{q\mathrm{I}p}_{3/2\mp 1/2,p,l} n^{1-l},$$

$$H^{q\mathrm{I}p}_{3/2\mp 1/2,p,l} = \left(z^{q\mathrm{I}p}_{1,1} - \frac{1+a\pm 1}{2+a}\right)^{-p}\delta_{l,1} + \sum_{j=1}^{l-1}\left(z^{q\mathrm{I}p}_{1,1}\right)^j \chi^{q\mathrm{I}p}_{j,l-1}\left(z^{q\mathrm{I}p}_{1,1} - \frac{1+a\pm 1}{2+a}\right)^{-j-p}\binom{-p}{j},$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\frac{\partial^{\delta_{p,d}} R^{\mathrm{outer}}\left(z^{q\mathrm{I}p}\right)}{\partial z^{\delta_{p,d}}} \sim \sum_{q=1}^{\infty}\begin{pmatrix}\eta^{q\mathrm{I}p}_{1,q} & \eta^{q\mathrm{I}p}_{2,q}\end{pmatrix} n^{1-q+\delta_{p,d}},$$

$$\eta^{q\mathrm{I}p}_{j,m} = \delta_{1,j}\delta_{1,m} + \sum_{k=1}^{m-1}\sum_{p=1}^{\lceil 3/2k\rceil}\sum_{l=1+\delta_{p,n}(m-k-1)}^{m-k}(-1)^{m-k-l}\binom{-k}{m-k-l}\left[U^{\mathrm{right}}_{k,p,1,j}H^{q\mathrm{I}p}_{1,p,l} + U^{\mathrm{left}}_{k,p,1,j}H^{q\mathrm{I}p}_{2,p,l}\right] \qquad p \neq d,$$

$$\eta^{q\mathrm{I}d}_{j,m} = -\sum_{k=1}^{m}\sum_{p=1}^{\lceil 3/2k\rceil} pU^{\mathrm{right}}_{k,p,1,j}H^{q\mathrm{I}n}_{1,p+1,1+m-k} + pU^{\mathrm{left}}_{k,p,1,j}H^{q\mathrm{I}n}_{2,p+1,1+m-k}.$$

Here, we do not provide all formulae by which higher order terms of the orthogonal polynomials can be computed, as equations for these U- and Q-matrices are available in [2, 4]. These are all implemented in

Sage worksheets available at [8, 9]. Note that for $m > \lceil k/2 \rceil$, we have $U_{k,m}^{\text{right/left}} \equiv 0$ for the Jacobi case and $U_{k,m}^{\text{left}} \equiv 0$ for the Laguerre case. For the expansions near the left endpoint (r$=$ IV), the Q matrices are used

$$\begin{aligned}
&\begin{pmatrix}1 & 0\end{pmatrix} R^{\text{right/left}}(z^{qrp}) \sim \begin{pmatrix}1 & 0\end{pmatrix} \left(I + \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} \frac{[z^{qrp}-a]^p}{(n-\delta_{p,s})^k} Q_{k,p}^{\text{right/left}} \right), \\
&\left.\frac{\partial^{(\delta_{p,d})} \begin{pmatrix}1 & 0\end{pmatrix} R^{\text{right/left}}(z)}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{qrp}} \sim \sum_{q=1}^{\infty} n^{1-q-\delta_{p,d}} \begin{pmatrix}\eta_{1,q}^{qrp} & \eta_{2,q}^{qrp}\end{pmatrix}, \\
&\eta_{n,q}^{qrp} = \delta_{1,n}\delta_{1,q}(1-\delta_{p,d}) + Q_{q,1,1,n}^{\text{right/left}}\delta_{p,d} + \sum_{k=1}^{q-1-\delta_{p,d}} \left\{ \left[\delta_{p,n}\delta_{k,q-1} + \delta_{p,s}\binom{-k}{q-1-k}(-1)^{q-1-k}\right] Q_{k,0,1,n}^{\text{right/left}} + \right. \\
&\left. \sum_{i=0}^{\delta_{p,s}(q-k-1)} \binom{-k}{i}(-1)^i \sum_{p=1+\delta_{p,d}}^{\lfloor (q-k-i-1+3\delta_{p,d})/2\rfloor} [1+(p-1)\delta_{p,d}]\, z_{p-\delta_{p,d},q-k-i-2+(5-2p)\delta_{p,d}}^{qrp} Q_{k,p,1,n}^{\text{right/left}} \right\}.
\end{aligned}$$

The oscillatory part $\varepsilon(z)$ in the expansions in [2, §2.3], [4, §4] and [5, §2.2] is $\begin{pmatrix}1 & 0\end{pmatrix} R(z)$ times a column vector and the latter can be expanded as

$$\begin{aligned}
&(-1)^{k\delta_{r,\mathrm{I}}} \sum_{r=1}^{\infty} n^{-r+\delta_{r,\mathrm{I}}+3\delta_{p,d}} \begin{pmatrix} D_\infty \kappa_{1,r}^{qrp} \\ \frac{-i}{D_\infty}\kappa_{2,r}^{qrp}\end{pmatrix}, \\
&\kappa_{n,r}^{qrp} = \sum_{j=1}^{r+1} \Xi_{r+2-j,n,1}^{qrp}\iota_{1,j}^{qrp} + \Xi_{r+2-j,n,2}^{qrp}\iota_{2,j}^{qrp} \qquad r \neq \mathrm{I} \ \&\ p \neq d, \\
&\kappa_{n,r}^{q\mathrm{IV}d} = \left[\sum_{i=1}^{r-1} \Xi_{i,n,1}^{q\mathrm{IV}d}\iota_{1,r-i}^{q\mathrm{IV}n} + \Xi_{i,n,2}^{q\mathrm{IV}d}\iota_{2,r-i}^{q\mathrm{IV}n}\right] + \sum_{k=1}^{r} \Xi_{k,n,1}^{q\mathrm{IV}n}\iota_{1,1-k+r}^{q\mathrm{IV}d} + \Xi_{k,n,2}^{q\mathrm{IV}n}\iota_{2,1-k+r}^{q\mathrm{IV}d}.
\end{aligned}$$

Note that our shortened notation avoids specifying formulae separately for each q, r and p as well as for some indices. For the bulk, this is so technical that we do not try to unify the following into fewer functions, except when the first index is one: there we join the Jacobi and Laguerre cases

$$\begin{aligned}
\kappa_{1,r}^{q\mathrm{I}n} &= 0 + \sum_{j=1}^{r-1} \frac{\cos\left(\frac{\pi}{2}+\frac{\pi}{2}j\right)}{j!}\epsilon_{1,j,1+r}, \\
\kappa_{2,r}^{L\mathrm{I}n} &= 2\sqrt{z_{1,1}^{L\mathrm{I}n}}\sqrt{1-z_{1,1}^{L\mathrm{I}n}}\delta_{r,1} + 2\sqrt{z_{1,1}^{L\mathrm{I}p}}\sqrt{1-z_{1,1}^{L\mathrm{I}n}} \left(\sum_{n=1}^{\lfloor (r-1)/2\rfloor} \frac{(-1)^n}{(2n)!}\epsilon_{2,2n,1+r}^{L\mathrm{I}n}\right) \\
&+ \left(2z_{1,1}^{L\mathrm{I}n}-1\right)\sum_{n=1}^{\lfloor r/2\rfloor} \frac{(-1)^n}{(2n-1)!}\epsilon_{2,2n-1,1+r}^{L\mathrm{I}n}, \\
\kappa_{2,r}^{J\mathrm{I}n} &= \sqrt{1-\left(z_{1,1}^{J\mathrm{I}n}\right)^2}\delta_{r,1} + \sqrt{1-\left(z_{1,1}^{J\mathrm{I}n}\right)^2}\left(\sum_{n=1}^{\lfloor (r-1)/2\rfloor} \frac{(-1)^n}{(2n)!}\epsilon_{2,2n,1+r}^{J\mathrm{I}n}\right) \\
&+ z_{1,1}^{J\mathrm{I}n}\sum_{n=1}^{\lfloor r/2\rfloor} \frac{(-1)^n}{(2n-1)!}\epsilon_{2,2n-1,1+r}^{J\mathrm{I}n}, \\
\kappa_{1,r}^{q\mathrm{I}d} &= -\tilde{\epsilon}_{1,1,r}^{q\mathrm{I}d} - \sum_{q=2}^{r}\tilde{\epsilon}_{1,1,r+1-q}^{q\mathrm{I}d}\sum_{j=1}^{q-1}\frac{\sin\left[\frac{\pi}{2}+\frac{\pi}{2}j\right]}{j!}\epsilon_{1,j,1+q}^{q\mathrm{I}n}, \\
\kappa_{2,r}^{L\mathrm{I}d} &= \tilde{\epsilon}_{2,1,r}^{L\mathrm{I}d}\left(1-2z_{1,1}^{L\mathrm{I}n}\right) + \sum_{q=2}^{r}\tilde{\epsilon}_{2,1,r+1-q}^{L\mathrm{I}d}\left(-\left(2z_{1,1}^{L\mathrm{I}n}-1\right)\left[\sum_{n=1}^{\lfloor (q-1)/2\rfloor}\frac{(-1)^n}{(2n)!}\epsilon_{2,2n,1+q}^{L\mathrm{I}n}\right]\right.
\end{aligned}$$

$$-2\sqrt{z_{1,1}^{L\mathrm{I}n}}\sqrt{1-z_{1,1}^{L\mathrm{I}n}}\sum_{n=1}^{\lfloor q/2\rfloor}\frac{(-1)^{n+1}}{(2n-1)!}\epsilon_{2,2n-1,1+q}^{L\mathrm{I}n}\Bigg),$$

$$\kappa_{2,r}^{J\mathrm{I}d}=\sum_{q=1}^{r}\epsilon_{2,r+1-q}^{J\mathrm{I}d}\Bigg[-t\delta_{1,q}-t\left(\sum_{n=1}^{\lfloor (q-1)/2\rfloor}\frac{(-1)^{n}}{(2n)!}\epsilon_{2,2n,q+1}^{J\mathrm{I}n}\right)$$

$$+\sqrt{1-t^2}\sum_{n=1}^{\lfloor q/2\rfloor}\frac{(-1)^(n)}{(2n-1)!}\epsilon_{2,2n-1,q+1}\Bigg]$$

$$\kappa_{1,r}^{q\mathrm{I}s}=\kappa_{2,r}^{q\mathrm{I}n}$$

$$\kappa_{2,r}^{L\mathrm{I}s}=4\sqrt{z_{1,1}^{L\mathrm{I}s}}\sqrt{1-z_{1,1}^{L\mathrm{I}s}}\left(2z_{1,1}^{L\mathrm{I}s}-1\right)\delta_{r,1}+4\sqrt{z_{1,1}^{L\mathrm{I}s}}\sqrt{1-z_{1,1}^{L\mathrm{I}s}}\left(2z_{1,1}^{L\mathrm{I}s}-1\right)\left(\sum_{n=1}^{\lfloor (r-1)/2\rfloor}\frac{(-1)^{n}}{(2n)!}\epsilon_{2,2n,1+r}^{L\mathrm{I}s}\right)$$

$$+\left(\left[8z_{1,1}^{L\mathrm{I}s}\right]^2-8z_{1,1}^{L\mathrm{I}s}+1\right)\sum_{n=1}^{\lfloor r/2\rfloor}\frac{(-1)^{n}}{(2n-1)!}\epsilon_{2,2n-1,1+r}^{L\mathrm{I}s},$$

$$\kappa_{2,r}^{J\mathrm{I}s}=2z_{1,1}^{J\mathrm{I}s}\sqrt{1-\left(z_{1,1}^{q\mathrm{I}s}\right)^2}\left(\delta_{r,1}+\sum_{n=1}^{\lfloor (r-1)/2\rfloor}\frac{(-1)^{n}}{(2n)!}\epsilon_{2,2n,1+r}^{J\mathrm{I}s}\right)$$

$$+\left(2\left[z_{1,1}^{J\mathrm{I}s}\right]^2-1\right)\sum_{n=1}^{\lfloor r/2\rfloor}\frac{(-1)^{n}}{(2n-1)!}\epsilon_{2,2n-1,1+r}^{J\mathrm{I}s}.$$

Combined together,

$$\left.\frac{\partial^{(\delta_{p,d})}\varepsilon^{qrp}(z)}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{L\mathrm{IV}p}}\sim(-1)^{k\delta_{r,\mathrm{I}}}\sum_{l=0}^{\infty}0_{l}^{qrp}n^{-l+\delta_{r,\mathrm{I}}\left(1-2\delta_{p,d}\right)+3\delta_{p,d}},$$

$$0_{l}^{qrp}=\sum_{q=1}^{l}D_{\infty}\eta_{1,q}^{qrp}\kappa_{1,l+1-q}^{qrp}-\frac{i}{D_{\infty}}\eta_{2,q}^{qrp}\kappa_{2,l+1-q}^{qrp}\qquad p\neq d,$$

$$0_{l}^{qrd}=\left(\sum_{q=1}^{l-4+2\delta_{r,\mathrm{I}}}D_{\infty}\eta_{1,q}^{qrd}\kappa_{1,l-3-q+2\delta_{r,\mathrm{I}}}^{qrn}-\frac{i}{D_{\infty}}\eta_{2,q}^{qrd}\kappa_{2,l-3-q+2\delta_{r,\mathrm{I}}}^{qrn}\right)$$

$$+\sum_{q=1}^{l}D_{\infty}\eta_{1,q}^{qrn}\kappa_{1,l+1-q}^{qrd}-\frac{i}{D_{\infty}}\eta_{2,q}^{qrn}\kappa_{2,l+1-q}^{qrd}.$$

For the left Laguerre nodes, we had $j_{\alpha,k}=4\sqrt{z_{1,1}^{L\mathrm{IV}n}}$ in § 1.6, from which we obtain the leading order behaviour of the node x_k in [5, (5.1)]. Similarly, $n\sqrt{2+2z_{1,1}^{J\mathrm{IV}n}}=j_{\beta,k}$ in § 1.8 for the Jacobi case, giving the leading order behaviour of the node in [5, (6.1)]. Higher order coefficients can be obtained recursively by setting $0_{l}^{L\mathrm{IV}n}=0$. We extract the coefficient $z_{1,j}$ with the highest j from this by only considering those coefficients where it can appear in the coefficients defined before

$$z_{1,l+1}^{q\mathrm{IV}n}=\frac{(-1)^{\delta_{q,J}}\sqrt{(1-a)z_{1,1}^{q\mathrm{IV}n}}}{D_{\infty}f_{\delta_{q,J}}^{q\mathrm{IV}}S_1}\Bigg[D_{\infty}\Bigg\{S_1\Bigg((-1)^{\delta_{q,L}}2^{1/(1-a)}f_{\delta_{q,J}}^{q\mathrm{IV}}\sqrt{z_{1,1}^{q\mathrm{IV}n}}\left\{\delta_{l+1,1}+\sum_{u=2}^{l}\binom{1/2}{u}\chi_{u,l+1-u}^{q\mathrm{IV}n}\right\}$$

$$+(-1)^{\delta_{q,L}}2^{1/(1-a)}\sum_{j=1}^{\lfloor l/2\rfloor}f_{j+\delta_{q,J}}^{q\mathrm{IV}}\sum_{i=2j-1}^{l-1}\omega_{l-i}^{q\mathrm{IV}n}z_{j,i}^{q\mathrm{IV}n}\Bigg)+\left(\sum_{i=2}^{l}\frac{S_{i-\delta_{q,L}}}{i!}\epsilon_{i,l+1}^{q\mathrm{IV}n}\right)$$

$$+\sum_{j=1}^{l}\Xi_{l+2-j,1,1}^{q\mathrm{IV}n}\iota_{1,j}^{q\mathrm{IV}n}+\Xi_{l+2-j,1,2}^{q\mathrm{IV}n}\iota_{2,j}^{q\mathrm{IV}n}\Bigg\}+\sum_{q=2}^{l}D_\infty\eta_{1,q}^{q\mathrm{IV}n}\kappa_{1,l+1-q}^{q\mathrm{IV}n}-iD_\infty^{-1}\eta_{2,q}^{q\mathrm{IV}n}\kappa_{2,l+1-q}^{q\mathrm{IV}n}\Bigg].$$

1.5 Expansion of $p_n'(x_k)$ and $p_{n-1}(x_k)$

In the case $p\neq n$, also the factor that multiplies $\varepsilon(z)$ should be expanded. It does not need to be differentiated for $p=d$ as it gets multiplied with $\varepsilon(z)$ in the derivative of their product and the latter is zero up to the order T up to which the expansion of the node x_k has been calculated. The factor always contains the quartic roots

$$[z^{qrp}-a]^{-1/4}\sim\sum_{k=1}^{\infty}o_k^{qrp}n^{3/2-\delta_{r,\mathrm{I}}/2-k},$$

$$o_k^{qrp}=\left(z_{1,1}^{qrn}-a\right)^{-1/4}\left[\delta_{k,1}+\sum_{u=1}^{k-1}\binom{-1/4}{u}\chi_{u,k-1}^{qrp}\right],$$

$$(1-z^{qrp})^{-1/4}\sim\sum_{l=1}^{\infty}O_l^{qrp}n^{1-l},$$

$$O_m^{qrp}=\left\{1-z_{1,1}^{qrp}\right\}^{-1/4}\left(\delta_{m,1}+\sum_{u=1}^{m-1}\binom{-1/4}{u}X_{u,m-1}^{qrp}\right),$$

$$(z^{qrp}-a)^{-1/4}(1-z^{qrp})^{-1/4}\sim\sum_{m=1}^{\infty}Q_m^{qrp}n^{3/2-\delta_{r,\mathrm{I}}/2-m},$$

$$Q_m^{qrp}=\sum_{k=1}^{m}o_k^{qrp}O_{m-k+1}^{qrp}.$$

There may be an additional factor depending on z for which the expansion is in terms of B_l. For the expansion in the bulk, the additional factor is just 1, so $B_k^{q\mathrm{I}p}=\delta_{k,1}$. In both expansions in the left boundary region, there is an additional square root

$$\sqrt{i\pi[n-\delta_{p,s}]\bar{\phi}_{n-\delta_{p,s}}\left[z^{L\mathrm{IV}p}\right]}\sim\sum_{k=1}^{\infty}B_k^{L\mathrm{IV}p}n^{1-k},$$

$$\sqrt{\arccos(-z^{J\mathrm{IV}p})}\sim\sum_{k=1}^{\infty}B_k^{J\mathrm{IV}p}n^{1/2-k},$$

$$B_k^{q\mathrm{IV}p}=\sqrt{\left(\frac{\pi}{2}\right)^{\delta_{q,L}}\epsilon_{1,1,1}^{q\mathrm{IV}p}}\left(\delta_{1,k}+\sum_{u=1}^{k-1}\binom{1/2}{u}\frac{\epsilon_{u,k}^{q\mathrm{IV}p}}{\left[\epsilon_{1,1,1}^{q\mathrm{IV}p}\right]^u}\right).$$

Multiplying it with $\varepsilon(z)$ yields

$$\sum_{j=1}^{\infty}P_j^{qrp}n^{2\delta_{p,d}-j}\quad\text{with}\quad P_j^{qrp}=\sum_{l=1}^{j}0_l^{qrp}B_{j-l+1}^{qrp}.$$

The expansion of $p_{n-1}(x_k)$ or $p_n'(x_k)$ then is

$$p_{n-\delta_{p,s}}^{(\delta_{p,d})}\left[x_k^{qr}\right]\sim Y^{qrp}w\left[x_k^{qr}\right]^{-1/2}\sum_{k=1}^{\infty}\Delta_k^{qrp}n^{1/2-k+\delta_{r,\mathrm{I}}\left(1/2-2\delta_{p,d}\right)+3\delta_{p,d}},$$

$$\Delta_k^{qrp}=\sum_{m=1}^{k}Q_m^{qrp}P_{1+k-m}^{qrp}.$$

The expansion of the weights is given in § 1.7 and 1.9 because of the difference between the Laguerre and Jacobi cases.

1.6 Arguments in Gauss-Laguerre

The superscripts Lrs, where only the region is unspecified, are the only cases in (1.2) where $\frac{\beta_n}{\beta_{n-\delta_{p,s}}}$ is not identically one. For general functions $Q(x)$, this ratio is calculated as a numerical solution of

$$2\pi n = \int_0^{\beta_n} Q'(x)\sqrt{\frac{x}{\beta_n - x}}\,\mathrm{d}x.$$

For a general polynomial, fractional powers $n^{-1/m}$ can be avoided by numerically evaluating the expansion described in [4, §3.2], inserting $z = x_k/\beta_n$ into $R^{\text{outer}}(z)$ and using the procedure for general functions $Q(x)$ as well. For monomial $Q(x)$,

$$\frac{\beta_n}{\beta_{n-1}} = (1-1/n)^{-1/m} \sim \sum_{j=1}^{\infty} \binom{-1/m}{j-1}(-1/n)^{j-1},$$

$$z_{1,i}^{Lrs} = \sum_{q=1}^{i+\delta_{r,\mathrm{I}}} (-1)^{i-q+\delta_{r,\mathrm{I}}} \binom{-1/m}{i-q+\delta_{r,\mathrm{I}}} z_{1,q}^{Lrn}.$$

The arccosine in the shifted polynomial expands to

$$\left.\frac{\partial^{(\delta_{p,d})}\arccos\left(\frac{2z-a-1}{1-a}\right)}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{qrp}} \sim \arccos\left(\frac{2\,z^{qrp}|_{n=\infty}-a-1}{1-a}\right)(1-\delta_{p,d}) + \sum_{r=1}^{\infty} \zeta_{1,r}^{qrp} n^{1-\delta_{r,\mathrm{IV}}(1-2\delta_{p,d})-r}.$$

The coefficients in this series expansion, respectively for the left boundary region and the bulk, equal

$$\frac{2\zeta^{q\mathrm{IV}p}_{2/3\mp1/2,1,r}}{-\alpha\mp1} = \left[-2^{1+a/2-\delta_{p,d}}\frac{(1/2)_{(r-1)/2}}{[(r-1)/2!]\{r+\delta_{p,d}(1-r)\}}\left(z_{1,1}^{q\mathrm{IV}p}\right)^{r/2-\delta_{p,d}}\right]_{r\text{ odd}}$$
$$-\,2^{1+a/2-\delta_{p,d}}\sum_{u=1}^{r-1}\sum_{l=0}^{(r-1-u)/2}\frac{(1/2)_l\left(z_{1,1}^{q\mathrm{IV}p}\right)^{l+1/2-\delta_{p,d}}}{(l!)(1+2l-\delta_{p,d}2l)}\binom{l+1/2-\delta_{p,d}}{u}\chi^{q\mathrm{IV}p}_{u,r-1-2l},$$

$$\frac{2\zeta^{q\mathrm{I}p}_{3/2\pm1/2,1,r}}{-\alpha\mp1} = \arccos\left(\frac{2z_{1,1}^{q\mathrm{I}p}-a-1}{1-a}\right)\delta_{1,k}(1-\delta_{p,d}) - \delta_{1,k}\delta_{p,d}\left(z_{1,1}^{q\mathrm{I}p}\right)^{5/2}\left(1-z_{1,1}^{q\mathrm{I}p}\right)^{-1/2}$$
$$-\sum_{l=2}^{k}\left(1+\delta_{p,d}[l-1]\right)\chi^{q\mathrm{I}p}_{l-1,k}\sum_{n=0}^{l-2+\delta_{p,d}}\frac{(-1)^n}{l-1+\delta_{p,d}}\left(z_{1,1}^{q\mathrm{I}p}\right)^{n+5/2}$$
$$\binom{-1/2}{l-2-n+\delta_{p,d}}\left(1-z_{1,1}^{q\mathrm{I}p}\right)^{-1/2-n}\binom{-1/2}{n}.$$

Powers of this arccosine appearing in the left boundary region are

$$\left[\arccos\left(\frac{2z^{qrp}-a-1}{1-a}\right)-\pi\delta_{r,\mathrm{IV}}\right]^q \sim \sum_{r=1}^{\infty}\zeta_{q,r}^{qrp}n^{-r}.$$

The phase function is expanded in the following cases

$$\lambda_n(z^{L\mathrm{I}p}) \pm \frac{1}{2}\arccos\left(\frac{2z-a-1}{1-a}\right) = \left[n+\frac{\mu+\nu+1\pm1}{2}\right]\arccos\left(\frac{2z-a-1}{1-a}\right) - \frac{\pi}{4} - \frac{\nu\pi}{2}$$
$$-\,\frac{n}{2}\sqrt{1-z}\sqrt{z-a}\frac{2}{mA_m}\sum_{k=0}^{m-1}A_{m-1-k}z^k,$$

in the bulk for monomial $Q(x)$,

$$\bar{\phi}_{n-1}(z^{L\mathrm{IV}p}) \sim i\theta(z^{L\mathrm{IV}p}) \sum_{j=0}^{\infty} f_j^{L\mathrm{IV}p} \left(z^{L\mathrm{IV}p}\right)^{j+1/2},$$

$$f_j^{L\mathrm{IV}p} = -\frac{(1/2)_j}{j!(1+2j)} - \frac{1}{2mA_m} \sum_{k=0}^{\min(m-1,j)} (-1)^{j-k} \binom{1/2}{j-k} A_{m-k-1}$$

in the left disk for monomial $Q(x)$, while for general functions $Q(x)$ and with (functional) dependences on $n-1$

$$f_j^{L\mathrm{IV}p}(n-1) = \frac{+1}{2(2j+1)} \sum_{l=0}^{j} (-1)^l \binom{1/2}{l} d_{j-l}^{L\mathrm{IV}p}(n-1).$$

The argument of the Bessel functions $A(z^{L\mathrm{IV}p})$ is expanded as follows

$$\left.\frac{\partial^{(\delta_{p,d})} 2i\,(n-\delta_{p,s})\,\bar{\phi}_{n-\delta_{p,s}}(z)}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{L\mathrm{IV}p}} \sim \epsilon_{1,1,1}^{L\mathrm{IV}p} + \sum_{k=2}^{\infty} \epsilon_{1,1,k}^{L\mathrm{IV}p} n^{1-k},$$

$$\begin{aligned}
\epsilon_{1,1,k}^{L\mathrm{IV}p} &= 2f_0^{L\mathrm{IV}p}\left[\omega_{k-1}^{L\mathrm{IV}p}\delta_{p,s}\,(1-\delta_{k,1}) - \omega_k^{L\mathrm{IV}p}\right] - 2\left[\sum_{j=1}^{\lfloor (k-1)/2\rfloor} f_j^{L\mathrm{IV}p} \sum_{i=2j-1}^{k-2} \omega_{k-1-i}^{L\mathrm{IV}p} z_{j,i}^{L\mathrm{IV}p}\right] \\
&\quad + \delta_{p,s}\left\{2\sum_{j=1}^{\lfloor k/2\rfloor -1} f_j^{L\mathrm{IV}p} \sum_{i=2j-1}^{k-3} \omega_{k-i-2}^{L\mathrm{IV}p} z_{j,i}^{L\mathrm{IV}p}\right\} \qquad p \neq d, \\
\epsilon_{1,1,k}^{L\mathrm{IV}d} &= \left\{-kf_{(k-1)/2}[z_{1,1}^{L\mathrm{IV}n}]^{k/2-1}\right\}_{k\text{ odd}} \\
&\quad - 2\sum_{j=0}^{k/2-1} f_j(j+1/2)[z_{1,1}^{L\mathrm{IV}n}]^{j-1/2} \sum_{u=1}^{k-2j-1} \binom{j-1/2}{u} \chi_{u,k-2j-u}^{L\mathrm{IV}n}.
\end{aligned}$$

In the last expression for the derivative of the polynomial at its zero, the term with the subscript "k odd" is only added when k is odd such that $f_{(k-1)/2}$ has an integer index. Here, $\epsilon_{1,1,1}^{L\mathrm{IV}p} = j_{\alpha,k} = \sqrt{4nz_{1,1}^{L\mathrm{IV}n}}$, which means that the leading order term of the asymptotic expansion in the left boundary region is zero when $A(z^{L\mathrm{IV}p})$ equals a zero of the Bessel up to first order.

In the bulk, a similar reasoning gives $\epsilon_{1,1,1}^{L\mathrm{I}p} = t = z_{1,1}^{L\mathrm{I}p}$ as leading order. The expansion of the argument of the cosine can be derived for monomial $Q(x)$ as

$$\left.\frac{\partial^{(\delta_{p,d})} H_n(z)}{\partial z^{(\delta_{p,d})}}\right|_{z=z^{L\mathrm{I}p}} \sim \sum_{l=1}^{\infty} F_l^{L\mathrm{I}p} n^{1-l},$$

$$\begin{aligned}
F_l^{L\mathrm{I}p} &= \frac{2}{mA_m}\sum_{k=0}^{m-1} A_{m-1-k}\Big(\left(z_{1,1}^{L\mathrm{I}p}\right)^{k-\delta_{p,d}} \delta_{l,1} + \sum_{j=1+\delta_{p,d}}^{\min(l-1+\delta_{p,d},k)} (1+(j-1)\delta_{p,d})\binom{k}{j} z_{j-\delta_{p,d},l+1}^{L\mathrm{I}p} \left(z_{1,1}^{L\mathrm{I}p}\right)^{k-j} \Big), \\
\epsilon_{3/2\mp 1/2,1,k}^{L\mathrm{I}p} &= \left(\frac{\alpha \pm 1}{2} - \delta_{p,s}\right)\zeta_{k-1}^{L\mathrm{I}p} - \frac{\pi}{4}\delta_{k,2}\,(1-\delta_{p,d}) + \zeta_k^{L\mathrm{I}p} \\
&\quad - \sum_{l=1}^{k} \frac{F_l^{L\mathrm{I}p}}{2} \sum_{m=1}^{1+k-l} \omega_{2+k-l-m}^{L\mathrm{I}p}\Omega_m^{L\mathrm{I}p}\,[1-\delta_{p,d}] + \omega_{2+k-l-m}^{L\mathrm{I}n}\Omega_m^{L\mathrm{I}n}\delta_{p,d} \\
&\quad + \left(\sum_{l=1}^{k-1} \frac{F_l^{L\mathrm{I}p}}{2} \sum_{m=1}^{k-l} \omega_{1+k-l-m}^{L\mathrm{I}p}\Omega_m^{L\mathrm{I}p}\right)\delta_{p,s} \\
&\quad - \delta_{p,d}\sum_{l=1}^{k} \frac{F_l^{L\mathrm{I}n}}{2} \sum_{m=1}^{1+k-l} \left[\omega_{2+k-l-m}^{L\mathrm{I}d}\Omega_m^{L\mathrm{I}n} + \omega_{2+k-l-m}^{L\mathrm{I}n}\Omega_m^{L\mathrm{I}d}\right].
\end{aligned}$$

1.7 Nodes and weights in Gauss-Laguerre

For the nodes in the bulk, we can choose $n\epsilon_{1,1,1} + \epsilon_{1,1,2}$ to be $\pi/2 + \mathcal{O}(n^{-1})$ plus any multiple of π to obtain the leading order behaviour, which will affect the value of $z_{1,2}$. The following choice corresponds to [11, (57)] with $t = z_{1,1}^{LIn}$, see [5, §3.2 & 4.3], such that the expressions for standard associated Laguerre nodes become shorter:

$$
\begin{aligned}
\frac{4n-4k+3}{4n+2\alpha+2}\pi &= 2\arccos(\sqrt{t}) - \sqrt{t-t^2}\sum_{k=0}^{m-1}\frac{A_{m-1-k}}{mA_m}t^k,\\
z_{1,2}^{LIn} &= \frac{[\alpha+1]\left\{t-t^2\right\}\left\{\sum_{k=0}^{m-1}\frac{A_{m-1-k}}{mA_m}t^k\right\}}{2+\sum_{k=0}^{m-1}\left(1+k-(2+k)t\right)\frac{A_{m-1-k}}{mA_m}t^k},\\
z_{l+2}^{LIn} &= \Bigg(\left[\sum_{j=2}^{l}\frac{\cos\left(\frac{\pi}{2}+\frac{\pi}{2}j\right)}{j!}\epsilon_{1,j,2+l}^{LIn}\right] + D_\infty^{-1}\left\{\sum_{q=2}^{l+1}D_\infty\eta_{1,q}^{LIn}\kappa_{1,2+l-q}^{LIn} - iD_\infty^{-1}\eta_{2,q}^{LIn}\kappa_{2,2+l-q}^{LIn}\right\}\\
&\quad - \left\{\sum_{q=3}^{l+2}\chi_{q-1,l+2}^{LIp}\sum_{n=0}^{q-2}\frac{(-1)^n}{q-1}t^{n+5/2}\binom{-1/2}{q-2-n}(1-t)^{-1/2-n}\binom{-1/2}{n}\right\}\\
&\quad + \frac{\omega_1^{LIn}\Omega_1^{LIn}}{mA_m}\left[\sum_{k=0}^{m-1}A_{m-1-k}\left(t^k\delta_{l,-1} + \sum_{j=2}^{\min(l+1,k)}\binom{k}{j}t^{k-j}z_{j,l+2}^{LIp}\right)\right] - [\alpha+1]\xi_{l+1}^{LIn}/2\\
&\quad + \frac{F_1^{LIn}}{2}\sqrt{t}\sqrt{1-t}\left\{2\delta_{l,-1} + \sum_{u=2}^{l+1}\binom{1/2}{u}\left(\chi_{u,l+1}^{LIn} + X_{u,l+1}^{LIn}\right)\right\}\\
&\quad + \frac{F_1^{LIn}}{2}\left\{\sum_{m=2}^{l+1}\omega_{l+3-m}^{LIn}\Omega_m^{LIn}\right\} + \frac{1}{2}\sum_{q=2}^{l+1}F_q^{LIn}\sum_{m=1}^{l+3-q}\omega_{l+4-q-m}^{LIn}\Omega_m^{LIn}\Bigg)\\
&\quad \left(\frac{-1}{\sqrt{t}\sqrt{1-t}} - \frac{\omega_1^{LIn}\Omega_1^{LIn}}{mA_m}\left[\sum_{k=0}^{m-1}A_{m-1-k}kt^k\right] + \frac{-F_1^{LIn}(1-2t)}{4\sqrt{t}\sqrt{1-t}}\right)^{-1}.
\end{aligned}
$$

For the expansion in the bulk, the additional factor is just 1, so $B_k^{LIp} = \delta_{k,1}$. In the expansion in the left boundary region, there is an additional square root

$$
\begin{aligned}
\sqrt{i\pi[n-\delta_{p,s}]\bar{\phi}_{n-\delta_{p,s}}\left[z^{LIVp}\right]} &\sim \sum_{k=1}^{\infty}B_k^{LIVp}n^{1-k},\\
B_k^{LIVp} &= \sqrt{\frac{\pi\epsilon_{1,1,1}^{LIVp}}{2}}\left(\delta_{1,k} + \sum_{u=1}^{k-1}\binom{1/2}{u}\frac{\epsilon_{u,k}^{LIVp}}{\left[\epsilon_{1,1,1}^{LIVp}\right]^u}\right).
\end{aligned}
$$

For the expansion of the weights, note that $\beta_n^{-\alpha/2}z_k^{-\alpha/2} = x_k^{-\alpha/2}$ and for the shifted polynomial $\beta_{n-1}^{-\alpha/2}\left(\bar{z} = z_k\beta_n/\beta_{n-1}\right)^{-\alpha/2} = x_k^{-\alpha/2}$. Filling the results into (1.1) yields

$$
w_k^{Lr} \sim \frac{1 - i4^{\alpha+1}\sum_{k=1}^{\infty}(n-1)^{-k}(U_{k,1}^{\text{right}} + U_{k,1}^{\text{left}})|_{1,2}}{X^{Lr}\frac{2}{\pi}4^{\alpha}e^{Q(x_k)}x_k^{-\alpha}\left[-\sum_{k=1}^{\infty}\Delta_k^{Lrs}n^{1/2-k}\right]\left\{\sum_{k=1}^{\infty}\Delta_k^{Lrs}n^{7/2-k}\right\}}. \tag{1.6}
$$

The factor X^{Lr} is independent of the region r in the Laguerre case and equals

$$
X^{Lr} = \beta_n^{n-1+\alpha/2}\beta_{n-1}^{n-1+\alpha/2+2-2n-\alpha-1}\exp\left\{(n-1)l_{n-1}/2 + nl_n/2 - (n-1)l_{n-1}\right\}.
$$

One should evaluate this last equation directly in the case of general functions $Q(x)$ as a numerical value is then computed for β_n. We proceed by expanding β_n and other functions in the case of monomial $Q(x)$:

$$X^{Lr} = \frac{[mq_m A_m/2]^{1/m}}{4n\mathrm{e}^{1/m}}[1-1/n]^{-\alpha/2/m} \exp\left(-n/m \log[1-1/n]\right),$$

$$\exp\left(-n/m \log[1-1/n]\right) \sim \mathrm{e}^{1/m} \sum_{j=1}^{\infty} v_{j+1,j} n^{1-j},$$

$$v_{2,j} = \frac{(2m)^{-j}}{j!}, \tag{1.7}$$

$$v_{K,j} = \sum_{i=0}^{\lfloor j/(K-1) \rfloor} \frac{v_{K-1,j-(K-1)i}}{(Km)^i i!}, \tag{1.8}$$

$$X^{Lr} \sim \sum_{l=1}^{\infty} t_l n^{-l},$$

$$t_l = [mq_m A_m/2]^{1/m} 2^{-2} \sum_{i=0}^{l-1} \binom{\frac{-\alpha}{2m}}{i} (-1)^i v_{l-i}.$$

As the $v_{K,j}$ do not change in (1.8) for $j < K-1$, it can be defined as the vector (1.7) and applying (1.8) $T-2$ times, the number of terms in the final expression. In [9, GaussLaguerreLensStd6.sws], we thus define a vector `p[l]` as $v_{2,j}$ and then do steps over K to obtain the $v_{K,j}$ in that vector. We combine the $\Delta^{L\mathrm{IV}s}$ with the $\Delta^{L\mathrm{IV}d}$:

$$E_j^{Lr} = -\sum_{k=1}^{j} \Delta_k^{Lrs} \Delta_{1+j-k}^{Lrd} \Rightarrow \qquad \text{ь}_q^{Lr} = \sum_{l=1}^{q} t_l E_{1-l+q}^{Lr}$$

then gives the expansion in the denominator of (1.6), which we divide the numerator by to obtain the final asymptotic expansion of the weights

$$\text{б}_l^{Lr} = \frac{1}{\text{ь}_1^{Lr}} \left(\delta_{l,1} - \sum_{j=1}^{l-1} \text{б}_j^{Lr} \text{ь}_{l+1-j}^{Lr} \right),$$

$$w_k^{Lr} \sim n^{-1+\delta_{r,\mathrm{I}}} \mathrm{e}^{-Q(x_k)} x_k^{\alpha} \sum_{j=1}^{\infty} W_j^{Lr} n^{1-j},$$

$$W_j^{Lr} = \pi 2^{-1-2\alpha} \left[\text{б}_j^{Lr} - i4^{\alpha+1} \sum_{k=1}^{j-1} (U_{k,1}^{\text{right}} + U_{k,1}^{\text{left}})|_{1,2} \sum_{i=0}^{j-1-k} \binom{-k}{i} (-1)^i \text{б}_{j-i-k}^{Lr} \right].$$

1.8 Arguments in Gauss-Jacobi

As in the Laguerre case, we can reuse coefficients from the computation of the U and Q matrices [2, (40)]:

$$\arccos(-z) \sim (-2v)^{1/2} \sum_{n=0}^{\infty} f_n v^n, \qquad f_n = \frac{(\frac{1}{2})_n}{(+2)^n n!(1+2n)}.$$

The argument of the Bessel functions is then expanded as

$$\left. \frac{\partial^{\delta_{p,d}} n \arccos(-z)}{\partial z^{\delta_{p,d}}} \right|_{z=z^{J\mathrm{IV}p}} \sim \epsilon_{1,1,1}^{J\mathrm{IV}p} + \sum_{k=2}^{\infty} \epsilon_{1,1,k}^{J\mathrm{IV}p} n^{1-k},$$

$$\epsilon_{1,1,k}^{J\mathrm{IV}n} = \frac{f_1 \omega_k}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sum_{j=1} \lfloor (k-1)/2 \rfloor f_{j+1} \sum_{i=1}^{k-2j} \omega_{k+1-2j-i} z_{j,i-2+2j},$$

$$\epsilon_{1,1,l}^{J\mathrm{IV}d} = \sum_{j=1}^{l} \omega_j^{J\mathrm{IV}d} \Omega_{l-j+1}^{J\mathrm{IV}d},$$

$$\epsilon_{1,1,k}^{J\mathrm{IV}s} = \epsilon_{1,1,k}^{J\mathrm{IV}n} - \epsilon_{1,1,k-1}^{J\mathrm{IV}n}, k > 1, \quad \epsilon_{1,1,1}^{J\mathrm{IV}s} = \epsilon_{1,1,1}^{J\mathrm{IV}n}.$$

Here, $\epsilon_{1,1,1}^{J\mathrm{IV}p} = f_1\omega_1/\sqrt{2} = n\sqrt{2+2z_{1,1}^{J\mathrm{IV}n}} = j_{\beta,k}$, which means that the leading order term of the asymptotic expansion in the left boundary region is zero when $n\arccos(z^{J\mathrm{IV}p})$ equals a zero of the Bessel up to first order.

However, the argument of the trigonometric functions is more involved for the left disk with respect to the Laguerre case:

$$\zeta_{3/2\mp1/2,1,j}^{J\mathrm{IV}p} = \sum_{m=1}^{j} \frac{-\alpha-\beta\mp 1}{\sqrt{2}} \omega_m^{J\mathrm{IV}p} F_{j+1-m}^{J\mathrm{IV}p} + \frac{q_m^{J\mathrm{IV}p} D_{j+1-m}^{J\mathrm{IV}p}}{2}, \quad p \neq d$$

$$\zeta_{3/2\mp1/2,1,j}^{J\mathrm{IV}d} = -\frac{-\alpha-\beta\mp 1}{2}\epsilon_{1,1,j}^{J\mathrm{IV}d} + \left(\sum_{m=1}^{j} \frac{q_m^{J\mathrm{IV}d} D_{j+1-m}^{J\mathrm{IV}n}}{2}\right) + \sum_{m=1}^{j-2} \frac{q_m^{J\mathrm{IV}n} D_{j+1-m}^{J\mathrm{IV}d}}{2},$$

$$\frac{\arccos(z_k)-\pi}{-\sqrt{2+2z_k}} \sim \sum_{j=1}^{\infty} F_j^{J\mathrm{IV}p} n^{1-j},$$

$$F_j^{J\mathrm{IV}p} = f_1^{J\mathrm{IV}p}\delta_{m,1} + \sum_{n=1}^{\lfloor (m-1)/2 \rfloor} f_{n+1}^{J\mathrm{IV}p} zn, m-2^{J\mathrm{IV}p},$$

$$\left(-z_k^{Jrp}\right)^{\delta_{p,d}} \left[1-\left(z_k^{Jrp}\right)^2\right]^{1/2-\delta_{p,d}} \sim \sum_{m=1}^{\infty} q_m^{Jrp} n^{\delta_{r,\mathrm{I}}-m},$$

$$q_m^{Jrp} = \sum_{n=1}^{m} \omega_n^{Jrp} \Omega_{m-n+1}^{Jrp}, \quad p \neq d$$

$$q_l^{J\mathrm{IV}d} = \epsilon_{1,1,l}^{J\mathrm{IV}d} - \sum_{j=1}^{l-2} \epsilon_{1,1,j}^{J\mathrm{IV}d} z_{1,l-j-1}^{J\mathrm{IV}d},$$

$$q_m^{J\mathrm{I}d} = -\sum_{n=1}^{m} z_{1,m-n+1}^{J\mathrm{I}n} \sum_{j=1}^{n} \omega_j^{J\mathrm{I}d} \Omega_{n-j+1}^{J\mathrm{I}d},$$

$$\left.\frac{\partial^{\delta_{p,d}} \sum_{m=0}^{\infty} d_{m+1}^{J\mathrm{IV}p}(z+1)^m}{\partial z^{\delta_{p,d}}}\right|_{z=z_k^{J\mathrm{IV}n}} \sim \sum_{m=1}^{\infty} D_m^{J\mathrm{IV}p} n^{1-m},$$

$$D_m^{J\mathrm{IV}p} = d_{1+\delta_{p,d}}^{J\mathrm{IV}p}\delta_{m,1} + \sum_{n=1}^{m-2} d_{n+1+\delta_{p,d}}^{J\mathrm{IV}p} z_{n,m-2}^{J\mathrm{IV}p}.$$

For the bulk, $\epsilon_{3/2\mp1/2,1,1}^{J\mathrm{I}p} = \xi_1^{J\mathrm{I}p}$ and

$$\epsilon_{3/2\mp1/2,1,r}^{J\mathrm{I}p} = \frac{\alpha+\beta\pm 1-\delta_{p,s}}{2}\xi_{r-1}^{J\mathrm{I}p} - \frac{2\alpha+1}{4}\pi\delta_{r,2}(1-\delta_{p,d}) + \xi_r^{J\mathrm{I}p}$$
$$+ \frac{1}{2}\sum_{l=1}^{r-1} q_l^{J\mathrm{I}p} C_{r-l}^{J\mathrm{I}n} + \delta_{p,d} q_l^{J\mathrm{I}n} C_{r-l}^{J\mathrm{I}d}$$

$$\left.\frac{\partial^{\delta_{p,d}} \arccos(x)}{\partial x^{\delta_{p,d}}}\right|_{x=x_k^{J}\mathrm{I}} \sim \sum_{k=1}^{\infty} \xi_k^{J\mathrm{I}p} n^{1-k},$$

$$\xi_k^{J\mathrm{I}p} = \arccos(t)\delta_{k,1} + \sum_{l=2}^{k} \zeta_l^{J\mathrm{I}p} z_{l-1,k}^{J\mathrm{I}p}, \quad p \neq d$$

$$\xi_k^{J\mathrm{I}d} = \zeta_2^{J\mathrm{I}n}\delta_{k,1} + \sum_{l=2}^{k} l\zeta_{l+1}^{J\mathrm{I}p} z_{l-1,k}^{J\mathrm{I}p},$$

$$\zeta_l^{J\mathrm{I}p} = \arccos(t)\delta_{l,1} + (t+1)^{3/2-l}(1-t)^{3/2-l}\sum_{n=0}^{l-2}\frac{-1}{l-1}(1+t)^n$$

$$\binom{-1/2}{l-2-n}(-1)^n(1-t)^{l-2-n}\binom{-1/2}{n},$$

$$\left.\frac{\partial^{\delta_{p,d}}\sum_{n=0}^{\infty} h_{n+1}(x-t)^n}{\partial x^{\delta_{p,d}}}\right|_{x=x_k^{J\mathrm{I}}} \sim \sum_{m=1}^{\infty} C_m^{J\mathrm{I}p} n^{1-m},$$

$$C_m^{J\mathrm{I}p} = h_{1+\delta_{p,d}}\delta_{m,1} + \sum_{i=2}^{m}(1+\delta_{p,d}[i-1])h_{i+\delta_{p,d}} z_{i-1,m}.$$

1.9 Nodes and weights in Gauss-Jacobi

For the nodes in the bulk, we can choose $n\epsilon_{1,1,1} + \epsilon_{1,1,2}$ to be $\pi/2 + \mathcal{O}(n^{-1})$ plus any multiple of π to obtain the leading order behaviour, which will affect the value of $z_{1,2}$. The following choice corresponds to [3, Th. 2.1] with $t = z_{1,1}^{L\mathrm{I}n}$, see [5, §3.2 & 6.5]:

$$z_{1,l+1}^{J\mathrm{I}n} = -\sqrt{1-t^2}\Bigg[\left(\sum_{q=2}^{l+1}\eta_{1,q}^{J\mathrm{I}n}\kappa_{1,l+1-q}^{J\mathrm{I}n} - \frac{i}{D_\infty^2}\eta_{2,q}^{J\mathrm{I}n}\kappa_{2,l+1-q}^{J\mathrm{I}n}\right) + \left(\sum_{j=2}^{l-1}\frac{\cos(\pi/2+j\pi/2)}{j!}\epsilon_{1,j,l+1}^{J\mathrm{I}n}\right)$$

$$-\left(\sum_{i=3}^{l+1}\zeta_i z_{i-1,l+1}^{J\mathrm{I}n}\right) - \frac{\xi_l^{J\mathrm{I}n}}{2}(\alpha+\beta+1) + (2\alpha+1)\frac{\pi}{4}\delta_{l,1} - \frac{1}{2}\sum_{i=1}^{l} q_i^{J\mathrm{I}n} C_{l+1-i}^{J\mathrm{I}n}\Bigg].$$

Filling the results into (1.1) yields

$$\frac{w(x_k^{J\mathrm{IV}})}{w_k^{J\mathrm{IV}}} \sim -\left(2^{-1}D_\infty^{-2} + i\sum_{k=1}^{\infty}\frac{U_{k,1,2,1}^{\mathrm{right}} + U_{k,1,2,1}^{\mathrm{left}}}{(n)^k}\right)\sqrt{n(n-1)}\left[\sum_{k=1}^{\infty}\Delta_k^{J\mathrm{IV}s}n^{-k}\right]\left\{\sum_{l=1}^{\infty}\Delta_l^{J\mathrm{IV}d}n^{3-l}\right\},$$

$$\frac{w(x_k^{J\mathrm{I}})}{w_k^{J\mathrm{I}}} \sim \left(D_\infty^{-2} + 2i\sum_{k=1}^{\infty}\left.\frac{U_{k,1}^{\mathrm{right}} + U_{k,1}^{\mathrm{left}}}{(n-1+1)^k}\right|_{2,1}\right)\pi^{-1}\left[\sum_{k=1}^{\infty}\Delta_k^{J\mathrm{I}s}n^{1-k}\right]\left\{\sum_{l=1}^{\infty}\Delta_l^{J\mathrm{I}d}n^{2-l}\right\},$$

$$E_j^{Jr} = \sum_{k=1}^{j}\Delta_k^{Jrs}\Delta_{1+j-k}^{Jrd},$$

$$t_l^{J\mathrm{IV}} = \frac{\binom{1/2}{l-1}(-1)^{l-1}}{2D_\infty^2} + i\sum_{k=1}^{l-1}\binom{1/2}{l-1-k}(-1)^{l-1-k}\left(U_{k,1,2,1}^{\mathrm{right}} + U_{k,1,2,1}^{\mathrm{left}}\right),$$

$$\mathfrak{6}_q^{J\mathrm{IV}} = \sum_{l=1}^{q} t_l^{J\mathrm{IV}} E_{1-l+q}^{J\mathrm{IV}},$$

$$\mathfrak{6}_q^{J\mathrm{I}} = \frac{E_q^{J\mathrm{I}}}{D_\infty^2} + 2i\sum_{k=1}^{q-1}E_{q-k}^{J\mathrm{I}}\left(U_{k,1,2,1}^{\mathrm{right}} + U_{k,1,2,1}^{\mathrm{left}}\right).$$

We can then obtain the asymptotic expansion of the weight

$$\frac{w_k^{Jr}}{w(x_k^{Jr})} \sim \sum_{l=1}^{\infty} W_l^{Jr} n^{-l-\delta_{\mathrm{I},\mathrm{IV}}},$$

$$W_l^{Jr} = \frac{1}{\mathfrak{6}_1^{Jr}}\left([\pi - (\pi+1)\delta_{r,\mathrm{IV}}]\delta_{l,1} - \sum_{j=1}^{l-1} W_j^{Jr}\mathfrak{6}_{1+l-j}^{Jr}\right).$$

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