Globalizability of local Lie groupoids

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Definition of local Lie groupoids

A *local Lie groupoid* is a manifold G together with

1. a submanifold M (space of object, identities)
2. surjective submersions $s, t : G \to M$
3. an inversion map $\mathcal{V} \subseteq G \to G$
4. a multiplication $\mathcal{U} \subseteq G \ {}_s\!\times_t \ G \to G$

such that

1. $\mathcal{V}$ contains M
2. $\mathcal{U}$ contains $(G \ {}_s\!\times_t \ M) \cup (M \ {}_s\!\times_t \ G)$

and the usual axioms hold. In particular:

$$(fg)h = f(gh) \quad \text{whenever both sides make sense.}$$

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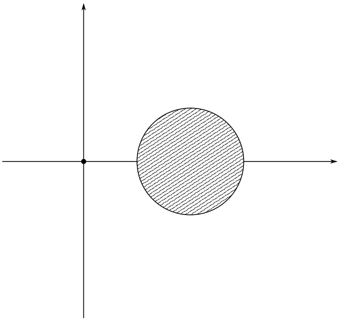
Is this it? Are these all of them? **No.**

Example 2: non-associative local group

$$G = \text{universal cover of } \mathbb{R}^2 \setminus \{(x,y) \mid (x-2)^2 + y^2 \leq 1\}$$



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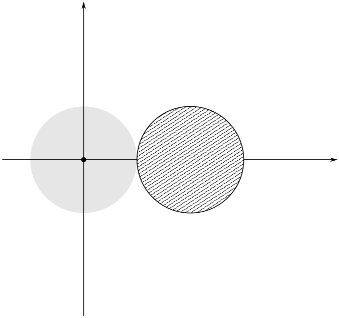
$g \cdot h$ defined if g or h close to identity



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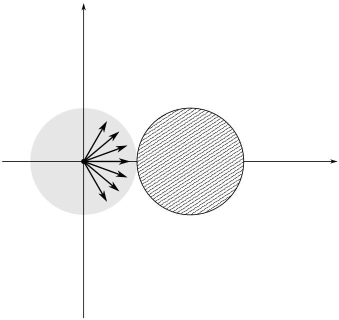

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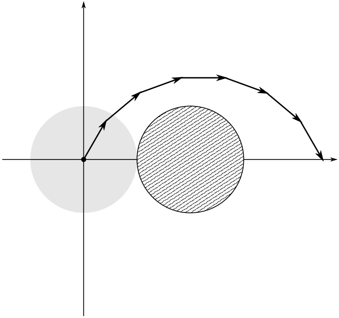

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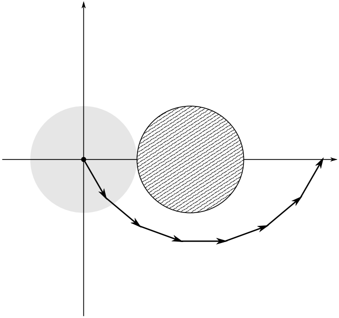
$(((((ab)c)d)e)f)g$
$\neq a(b(c(d(e(fg)))))$



Associative,

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We required 3-associativity:

$$(fg)h = f(gh) \quad \text{whenever both sides make sense.}$$

For local groupoids:

$$\text{3-associative} \not\Rightarrow n\text{-associative}$$

Theorem (Mal'cev)

A local Lie group is globalizable iff it is n-associative for all n.

Proof of Mal'cev's theorem

Take a local Lie group G that is n-associative for all n. Form

$$W(G) = \bigsqcup_{n=1}^{\infty} G^n.$$

Equivalence relation on $W(G)$ generated by $(\ldots, g, h, \ldots) \sim (\ldots, gh, \ldots)$. Set

$$\mathcal{AC}(G) = W(G)/\sim.$$

Prove that $\mathcal{AC}(G)$ is smooth, and that $G \hookrightarrow \mathcal{AC}(G)$.

Mal'cev's theorem for local groupoids

Take local Lie groupoid $G \rightrightarrows M$. Form

$$W(G) = \coprod_{n=1}^{\infty} \underbrace{G \ {}_s\times_t \cdots {}_s\times_t \ G}_{n \text{ times}}.$$

Let $(\ldots, g, h, \ldots) \sim (\ldots, gh, \ldots)$. Set

$$\mathcal{AC}(G) = W(G)/\sim.$$

Map $G \to \mathcal{AC}(G)$.

Lemma

The map $G \to \mathcal{AC}(G)$ is injective iff G is n-associative for all n.

Say that $g \in G_x$ is an *associator* if there is a word

$$(g_1, \ldots, g_n) \in W(G)$$

that can be evaluated to both g and x. Subset $\mathrm{Assoc}(G) \subset G$.

Theorem (M.)

$\mathcal{AC}(G)$ is smooth iff G has uniformly discrete associators (i.e. there is open neighborhood O of M such that $\mathrm{Assoc}(G) \cap O = M$).

Corollary

A local Lie groupoid is globalizable iff it is n-associative for all n.

Classification of local Lie groupoids

Theorem (M.)

Suppose G is a local Lie groupoid with integrable Lie algebroid A. Let $\tilde{G}$ be the s-simply connected cover of G. Let $\mathcal{G}$ be the s-simply connected integration of A. Then there is a diagram

$$\begin{tikzcd}
& \tilde{G} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
G \arrow[dr] & & U \subseteq \mathcal{G} \arrow[dl] \\
& \mathcal{AC}(G) &
\end{tikzcd}$$

Thank you.
Questions?

Another example of a local Lie groupoid

Take $M = S^2$. Set

$$G = \{(y, x, a) \in M \times M \times \mathbb{R} \mid x + y \neq 0\},$$

with $s(y, x, a) = x$ and $t(y, x, a) = y$. Multiplication is

$$(z, y, a) \cdot (y, x, a') = (z, x, a + a' + \text{area of } \Delta xyz),$$

whenever this area is in $(-\pi, \pi)$.