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Moduli Spaces of Flat Connections

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Preface

The goal of this thesis is to construct and study *moduli spaces of flat connections*. These moduli spaces are the spaces of all possible flat principal bundles over a given manifold with a given structure group, up to isomorphism. This defines the space as a mere set, and our main goal is to endow the moduli space with extra structure (a topology, a smooth structure, a symplectic structure).

This thesis was written in the 2012-2013 academic year at two different universities. From September 2012 until January 2013 I was at Utrecht University as an exchange student, where I started learning about moduli spaces of flat connections under the supervision of professor Marius Crainic and the guidance of doctor Florian Schätz. From February 2012 until June 2013, my thesis research continued at the KU Leuven. Supervision was provided by professor Franki Dillen, whose sad passing away in April 2013 meant a great loss to the department of mathematics. Later supervision was provided by professor Joeri Van der Veken. I am grateful to Marius, Florian, Franki and Joeri for their patience, help and guidance. I would also like to thank professor Johan Quaegebeur for his help in practical matters.

The text assumes only that the reader has a working knowledge of differential geometry, is familiar with the basic ideas behind fundamental groups and covering spaces and knows what morphisms in a category are. In particular, we will make use of manifolds, smooth maps and their derivatives, Lie brackets of vector fields, differential forms, fundamental groups of surfaces and lifts of curves to covering spaces without explanation. In contrast, the reader need not know anything about Lie groups, Lie algebras, bundles and vector-valued differential forms as their basics are explained in the first few chapters.

Summary (samenvatting)

(Dutch version on next page – Nederlandstalige versie op volgende pagina)

The aim of this work is to understand moduli spaces of flat connections. To a given manifold Σ and Lie group G we can associate the moduli space $\mathcal{M}(\Sigma, G)$ of flat principal G-bundles over M. This moduli space can be equipped with a topology. We also equip it with a smooth structure in a certain sense, although the moduli space typically has singularities. Inspired by the procedure of *symplectic reduction*, we describe how the moduli space may in some cases be equipped with a symplectic structure.

Chapters 1, 2, 3 and 4 contain material that is necessary to understand what the moduli spaces $\mathcal{M}(\Sigma, G)$ are. The first chapter discusses Lie groups and Lie algebras on a basic level. The second chapter introduces principal bundles and their connections starting from the more general concept of a fibre bundle. Vector bundles are also treated separately. The third chapter contains some of the theory of vector-valued differential forms. These are necessary in later chapters to understand the symplectic structure on the moduli space. The fourth chapter talks about symplectic and Poisson geometry. The most important concept here is *symplectic reduction*, a procedure we will later use to equip the moduli space with a symplectic structure.

Chapters 5, 6 and 7 study the moduli spaces themselves. The fifth chapter is the heart of the text, as it explains how the moduli space can be identified with $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, the space of all morphism from the fundamental group of Σ to G up to conjugation. This leads to a topology and a smooth structure on $\mathcal{M}(\Sigma, G)$. In this chapter we also explain how the moduli space may be regarded as a symplectic quotient of an infinite-dimensional symplectic manifold under a Hamiltonian action if Σ is a compact orientable surface. The sixth chapter contains a topological characterization of the reducible part of the moduli space for compact orientable surfaces. The case $G = \mathrm{SU}(2)$ is treated in greater detail, where it is shown that a large part of the moduli space is smooth. The last chapter derives an explicit formula for the symplectic structure on the moduli space, which expresses the symplectic form as a integral over a 1-chain instead of a 2-chain. Finally, the abelian case is discussed, in which the new formula simplifies considerably and yields an integral-free expression for the symplectic form.

Het doel van dit werk is het begrijpen van moduliruimten van vlakke connecties. Aan een gegeven differentiaalvariëteit Σ *en Lie-groep* G *kunnen we de moduliruimte* $\mathcal{M}(\Sigma, G)$ *van vlakke* G*-hoofdvezelbundels over* M *associëren. Deze moduliruimte kan voorzien worden van een topologie. In zekere zin voorzien we ze daarenboven van een gladde structuur, hoewel de moduliruimte singulariteiten heeft. Naar analogie met* symplectische reductie *beschrijven we hoe de moduliruimte soms uitgerust kan worden met een symplectische structuur.*

Hoofdstukken 1, 2, 3 en 4 bevatten theorie die nodig is om te begrijpen wat de moduliruimten $\mathcal{M}(\Sigma, G)$ *zijn. Het eerste hoofdstuk behandelt de basis van Lie-groepen en Lie-algebra's. Het tweede hoofdstuk introduceert hoofdvezelbundels en connecties daarop vanuit het algemenere concept van vezelbundels. Vectorbundels worden apart behandeld. Het derde hoofdstuk bevat een deel van de theorie van vectorwaardige differentiaalvormen. Deze zijn nodig om in latere hoofdstukken de symplectische structuur op de moduliruimte te begrijpen. Het vierde hoofdstuk gaat over symplectische meetkunde en Poisson-meetkunde. Het belangrijkste concept hier is* symplectische reductie*, een procedure die we later zullen gebruiken om de moduliruimte uit te rusten met een symplectische strucuur.*

Hoofstukken 5, 6 en 7 behandelen de eigenlijke moduliruimten. Het vijfde hoofdstuk is de spil van deze tekst en legt uit hoe de moduliruimte geïdentificeerd kan worden met $\mathrm{Hom}(\pi_1(\Sigma), G)/G$*, de ruimte van alle morfismen van de fundamentaalgroep van* Σ *naar* G *modulo conjugatie. Dit resulteert in een topologie en een gladde structuur op* $\mathcal{M}(\Sigma, G)$*. In dit hoofdstuk leggen we ook uit hoe de moduliruimte geïnterpreteerd kan worden als een symplectisch quotiënt van een oneindigdimensionale symplectische ruimte onder een Hamiltoniaanse actie indien* Σ *een compact orienteerbaar oppervlak is. Het zesde hoofdstuk bevat een topologische karakterisatie van het reducibele gedeelte van de moduliruimte voor compacte oriënteerbare oppervlakken. Het geval* $G = \mathrm{SU}(2)$ *wordt in meer detail behandeld, en er wordt getoond dat een groot deel van de moduliruimte glad is. Het laatste hoofdstuk leidt een expliciete formule af voor de symplectische structuur op de moduliruimte, die de symplectische vorm uitdrukt als een integraal over een 1-keten in plaats van een 2-keten. Tenslotte wordt het abelse geval besproken, waarin de nieuwe formule vereenvoudigt en een integraal-vrije uitdrukking geeft voor de symplectische vorm.*

List of symbols and conventions

$\mathcal{A}(P)$	Set of connections on P
Ad, ad, Ad_g, ad_g	Adjoint action (of g)
Ad^*, ad^*, Ad^*_g, ad^*_g	Coadjoint action (of g)
$[A_1, \ldots, A_n]$	Equivalence class of the tuple $(A_1, \ldots, A_n)$
$[\alpha \wedge \beta]$	Wedge product followed by Lie bracket
$\langle \alpha \wedge \beta \rangle$	Wedge product followed by bilinear pairing
A_p	Connection A at p regarded as 1-form
d_A	Covariant exterior derivative
F_A	Curvature of connection A
f^*E	Pullback of E along f
$\mathcal{F}(M)$	Set of smooth real-valued functions on M
$\Gamma(E)$	Set of sections of the bundle E
$\mathcal{G}(P)$	Gauge group of P
$\mathfrak{g}_P$	Adjoint bundle of P
Hol_q	Extended holonomy for base point q
$\mathrm{Hol}_p(A, \gamma)$	Holonomy of A along γ for base point p
H_p, V_p	Horizontal space at p, vertical space at p
Id_A	Identity map $A \to A$
λ_g	Left translation by g
$\Lambda^k(E)$	k-th exterior power of vector bundle E
$\mathrm{Lie}(G)$, $\mathfrak{g}$	Lie algebra of G
$\mathcal{L}_X$	Lie derivative along X
$\mathcal{M}(\Sigma, G)$, $\mathcal{M}$	Moduli space of flat connections on principal G-bundles over Σ
$\mathcal{M}_P(\Sigma, G)$	P-block of $\mathcal{M}(\Sigma, G)$
μ_ξ	ξ-component of moment map μ
ω_{AB}	Atiyah-Bott 2-form
$\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$	Space of horizontal equivariant $\mathfrak{g}$-valued k-forms on P

$\Omega^k(M)$, $\Omega^k(M, E)$	Space of differential k-forms on M (with values in E)
Pt_γ, $\mathrm{Pt}_{\gamma,t}$	Parallel transport along γ (for time t)
Tf, T_xf	Tangent map of the map f (at x)
θ	Maurer-Cartan form
$W(T)$	Weyl group of torus T
X_p	Vector field X at point p
$[\cdot, \cdot]$	Lie bracket of vector fields, Lie bracket on Lie algebra
$\{\cdot, \cdot\}$	Poisson bracket
$\nabla_v(s)$	Covariant derivative of s along v

By *smooth* we mean *of class* C^∞. Group units are denoted by e. However, sometimes a more specific notation is adopted, such as writing 1 for the group $(\mathbb{C}^\times, \cdot)$ and 0 for $(\mathbb{R}, +)$, or Id for matrix groups. Manifolds are required to be Hausdorff and second countable. They need not be connected. By a *submanifold* we mean an embedded submanifold (not merely an immersed one).

We will make extensive use of group actions. To let the group element g act on a from the left, we will write $g \cdot a$. To let it act from the right, we write $a \cdot g$. Suppose G is a group acting on sets A and B. A map $f : A \to B$ will be called *equivariant (with respect to the G-actions)* iff

- both actions are left actions and $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$ and $g \in G$.
- both actions are right actions and $f(a \cdot g) = f(a) \cdot g$ for all $a \in A$ and $g \in G$.
- the action on A is a left action and the action on B is a right action and $f(g \cdot a) = f(a) \cdot g^{-1}$ for all $a \in A$ and $g \in G$.
- the action on A is a right action and the action on A is a left action and $f(a \cdot g) = g^{-1} \cdot f(a)$ for all $a \in A$ and $g \in G$.

Note that to any right action $A \times G \to A : (a, g) \mapsto a \cdot g$ we can associate a left action $G \times A \to A : (g, a) \mapsto a \cdot g^{-1}$ and vice versa. Under this association, all four of the above conditions boil down to the same requirement.

Contents

Chapter 1

Lie groups and Lie algebras

In this first chapter, we discuss Lie groups and Lie algebras. Lie groups are in some sense "smooth" groups, and they are often appropriate to capture symmetries in differential geometry. Lie algebras are the infinitesimal, algebraic counterparts to Lie groups, as they describe what a Lie group looks like locally, but fail to describe global topological features. Our main reference for the first two sections of this chapter is the book by John Lee [Lee03]. In the last section, we base the definitions of (co)adjoint actions and representations on the book by William Fulton and Joe Harris [FH91].

1.1 Lie groups

Definition 1. *A* Lie group *is a group G that is also a smooth manifold, in such a way that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps.*

Definition 2. *A* morphism of Lie groups *is a smooth map between Lie groups that is also a group morphism.*

These two definitions define a category, the category of Lie groups. This means we also have a notion of *isomorphisms* between Lie groups (a Lie group morphism that has an inverse that is also a Lie group morphism), and of *automorphisms* of a Lie group (an isomorphism of a Lie group with itself). Let us consider a few examples of Lie groups.

- The group of real numbers $\mathbb{R}$ under addition.
- The group of positive real numbers $\mathbb{R}_+$ under multiplication. This Lie group is isomorphic to the previous one, and an isomorphism is given by $\exp : \mathbb{R} \to \mathbb{R}_+$.
- The group of complex numbers of modulus 1 under multiplication, the so-called *circle group*. As a manifold, this is the circle S^1. This is a subgroup of the Lie group of non-zero complex numbers.

- $\mathrm{GL}(V)$, the group of automorphisms of a real vector space V of finite dimension n. Fixing a basis of V, we can identify this group with the group of invertible $(n \times n)$-matrices. It is common practice to denote this group by $\mathrm{GL}(n)$.

  Note that the group operations are indeed smooth: the entries of a matrix product are polynomial functions of the entries of the factors, and the entries of the inverse of a matrix are rational functions of the entries of this matrix with non-zero denominator (the denominator is the determinant of the matrix).

- Many subgroups of $\mathrm{GL}(n)$ are important Lie groups. Examples are $\mathrm{SL}(n)$, the group of $(n \times n)$-matrices of determinant 1, $\mathrm{O}(n)$, the group of orthogonal $(n \times n)$-matrices, and $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n)$.

- The group of unitary $(n \times n)$-matrices (with complex entries) is the Lie group $\mathrm{U}(n)$. The subgroup of $\mathrm{U}(n)$ consisting of those matrices that have determinant 1 is called $\mathrm{SU}(n)$, the special unitary group. Note that $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ are isomorphic, and are isomorphic to the circle group.

Definition 3. *Let G be a Lie group. A morphism $\mathbb{R} \to G$ is called a* one-parameter subgroup *of G.*

Often, the image of the morphism rather than the morphism itself is called one-parameter subgroup. As an example, consider $G = \mathbb{C}^\times$, the Lie group of non-zero complex numbers. The positive real numbers form a one-parameter subgroup of G by the morphism

$$\mathbb{R} \to \mathbb{C} : t \mapsto \exp(t).$$

Also the circle group $\mathrm{U}(1)$ is a one-parameter subgroup of $\mathbb{C}^\times$, by the morphism

$$\mathbb{R} \to \mathbb{C} : t \mapsto \exp(2\pi i t).$$

In the next section, we shall see that all one-parameter subgroups of $\mathbb{C}^\times$ are exactly the morphisms of the form

$$\mathbb{R} \to \mathbb{C} : t \mapsto \exp(t\xi)$$

for some $\xi \in \mathbb{C}$ (the images of these one-parameter subgroups are logarithmic spirals passing through 1, with the cases $\xi \in \mathbb{R}$ and $\xi \in i\mathbb{R}$ leading to a degenerate spiral). An example is shown in figure 1.

1.2 Lie algebras

We now discuss Lie algebras, the "infinitesimal versions" of Lie groups.

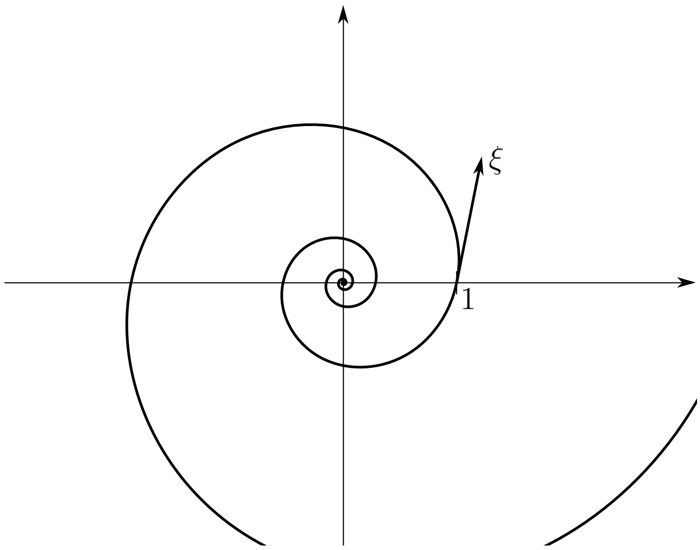

Figure 1: A one-parameter subgroup of $\mathbb{C}$, given by $t \mapsto \exp(t\xi)$. The image is a logarithmic spiral passing through 1. In this case $\xi = 0.2 + i$, which is the velocity of the curve when passing through 1.

1.2.1 The functor Lie

Given a Lie group G and an element $h \in G$, we can consider *left translation by h*

$$\lambda_h : G \to G : g \mapsto hg.$$

All of the maps λ_h are diffeomorphisms of the manifold G.

A vector field on a Lie group G is called *left-invariant* if it equals its own pushforward along any left multiplication map. In symbols, a vector field X on G is left-invariant iff

$$(T_e\lambda_h)(X_e) = X_h \quad \text{for all } h \in G.$$

A left-invariant vector field is completely determined by its value at the identity (which is an element of T_eG). Indeed, given a tangent vector $v \in T_eG$, we define the vector field X by

$$X_g = (T_e\lambda_g)(v).$$

This does indeed define a smooth vector field on G, because in local coordinates the Jacobian of the map λ_g varies smoothly with g. It is also clear that this equality is necessary for X to be left-invariant.

The set of left-invariant vector fields on a Lie group G is a vector space (of the same dimension as G, because it is essentially T_eG as shown above). We denote it by $\mathrm{Lie}(G)$, but very often it is also denoted by the same letter as the group, in lowercase fraktur (for example $\mathrm{Lie}(G) = \mathfrak{g}$). Note that the Lie bracket of two left-invariant vector fields is again left-invariant (indeed, the Lie bracket commutes with diffeomorphisms). This turns $\mathrm{Lie}(G)$ into a so-called *Lie algebra*.

Definition 4. *A* Lie bracket *on a vector space L is a bilinear operation*

$$[\cdot,\cdot] : L \times L \to L$$

that is antisymmetric and satisfies the Jacobi identity

$$[\xi,[\eta,\nu]] + [\eta,[\nu,\xi]] + [\nu,[\xi,\eta]] = 0$$

for all $\xi, \eta, \nu \in L$.

To clarify: by antisymmetry we mean that $[\xi,\xi] = 0$ for all $\xi \in L$, which is equivalent to $[\xi,\eta] = -[\eta,\xi]$ for all $\xi,\eta \in L$ if the characteristic of the field is not 2 (as always in this text).

Definition 5. *A* Lie algebra *is a finite-dimensional real vector space L equipped with a Lie bracket.*

Definition 6. *A* morphism of Lie algebras *is a linear map $\phi : L \to L'$ such that*

$$\phi([\xi,\eta]) = [\phi(\xi),\phi(\eta)] \quad \textit{for all } \xi,\eta \in L.$$

Note that in this equality, the bracket on the left hand side is the one of L, while the bracket on the right hand side is the bracket of L'.

We have thus defined the category of Lie algebras. As an example, let us discuss the Lie algebra of $\mathrm{GL}(n)$. Because $\mathrm{GL}(n)$ is an open subset of the vector space of $(n \times n)$-matrices, the tangent space to each point can be identified with this vector space. In particular, the tangent space at e can be identified with the space of $(n \times n)$-matrices, and hence the Lie algebra $\mathfrak{gl}(n)$ of $\mathrm{GL}(n)$ is just the n^2-dimensional vector space of $(n \times n)$ square matrices. This characterizes $\mathfrak{gl}(n)$ as a vector space. To characterize it as a Lie algebra, we also need to know what the bracket operation is.

Theorem 7. *The bracket operation $[\cdot,\cdot]$ on the Lie algebra $\mathfrak{gl}(n)$ of $\mathrm{GL}(n)$ is the commutator of matrices, meaning that*

$$[\xi,\eta] = \xi\eta - \eta\xi \quad \textit{for all } A, B \in \mathfrak{gl}(n)$$

where the multiplication on the right is ordinary matrix multiplication after the identification of $\mathfrak{gl}(n)$ with the space of $(n \times n)$-matrices.

Proof. Let $\nu \in T_e \mathrm{GL}(n)$ be an element of the Lie algebra. If $p \in \mathrm{GL}(n)$ is an element of the Lie group, the tangent space $T_p \mathrm{GL}(n)$ can be identified with the space of $(n \times n)$-matrices in the usual way, and under this identification, the vector at p of the left-invariant vector field associated to ν is $p\nu$ (an ordinary matrix product).

Let us call an $\mathbb{R}$-valued function on $\mathrm{GL}(n)$ *linear* if it is the restriction to $\mathrm{GL}(n)$ of a linear function on the space of $(n \times n)$-matrices. Note that any covector at e is the differential of some linear function on $\mathrm{GL}(n)$. To check equality of two vector fields on

$\mathrm{GL}(n)$ at e, it therefore suffices to show that they act the same way on linear functions on $\mathrm{GL}(n)$.

Now pick a linear function A on $\mathrm{GL}(n)$. If $\nu \in \mathfrak{gl}(n)$ is a left-invariant vector field on $\mathrm{GL}(n)$ (considered as a matrix by identifying with $T_e \mathrm{GL}(n)$), we can let ν act on A and we get

$$(\nu(A))(p) = \frac{d}{dt}_{|t=0} (A(p + tp\nu)) = A(p\nu).$$

Notice that $\nu(A)$ is again a linear function on $\mathrm{GL}(n)$. Applying this to $\nu \in \{\xi, \eta\}$, we find

$$\begin{aligned} [\xi, \eta](A)(p) &= \xi(\eta(A))(p) - \eta(\xi(A))(p) \\ &= \eta(A)(p\xi) - \nu(A)(p\eta) \\ &= A(p\xi\eta) - A(p\eta\xi) \\ &= A(p(\xi\eta - \eta\xi)) \\ &= (\xi\eta - \eta\xi)(A)(p). \end{aligned}$$

This proves that $[\xi, \eta]$ and $(\xi\eta - \eta\xi)$ act the same way on linear functions, establishing the result. $\square$

Let us give a few more examples of Lie algebras.

- The Lie algebra of $\mathbb{R}$ is $\mathbb{R}$ with the zero bracket.
- The circle group has as its Lie algebra the tangent space to the circle at 1. As a Lie algebra, this is just $\mathbb{R}$ (indeed, there is only one 1-dimensional Lie algebra up to isomorphism; its bracket is identically zero).
- The Lie algebra of $\mathrm{SL}(n)$ is written $\mathfrak{sl}(n)$ and consists of all $(n \times n)$-matrices with zero trace. To see this, note that the differential of the determinant at the identity is the trace.
- The Lie algebra of $\mathrm{O}(n)$ consists of all $(n \times n)$ skew-symmetric matrices. Because $\mathrm{SO}(n)$ is the component of the identity of $\mathrm{O}(n)$ and only this component plays a role in determining the Lie algebra of a Lie group, $\mathrm{SO}(n)$ has the same Lie algebra as $\mathrm{O}(n)$.
- The Lie algebra of $\mathrm{SU}(n)$ consists of the traceless anti-hermitian $(n \times n)$ complex matrices.

Note that in all these cases, the bracket is just the commutator.

Lie algebras are algebraic objects, and we have seen that to any Lie group G we can associate a Lie algebra $\mathrm{Lie}(G)$. In fact, to any morphism $f : G \to H$ of Lie groups, we can associate a morphism of Lie algebras $\mathrm{Lie}(f)$, turning Lie into a functor from the category of Lie groups to the category of Lie algebras. The map $\mathrm{Lie}(f)$ is just defined by

$$\mathrm{Lie}(f) = T_e(f),$$

the differential of the map f at the identity (which is a map from $T_eG \cong \mathrm{Lie}(G)$ to $T_eH \cong \mathrm{Lie}(H)$).

Theorem 8. *If $f : G \to H$ is a morphism of Lie groups, the map $\mathrm{Lie}(f) : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a morphism of Lie algebras.*

The proof of this theorem rests on a simple observation.

Lemma 9. *Let $\nu \in \mathrm{Lie}(G)$ be a left-invariant vector field on G. Take ν' to be the left-invariant vector field on H associated to $(T_ef)(\nu) \in T_eH$. In other words, ν' is the image of ν under $\mathrm{Lie}(f)$. Then ν and ν' are f-related.*

Proof. We have to check that for any $g \in G$ it holds that

$$(T_gf)(\nu(g)) = \nu'(f(g)).$$

By left-invariance, the left hand side is

$$(T_gf)(\nu(g)) = (T_gf)((T_e\lambda_g)(\nu(e))) = (T_e(f \circ \lambda_g))(\nu(e)).$$

Because f is a morphism of Lie groups, we have $f \circ \lambda_g = \lambda_{f(g)} \circ f$. This implies

$$\begin{aligned}(T_gf)(\nu(g)) &= (T_e(\lambda_{f(g)} \circ f))(\nu(e)) \\ &= (T_e\lambda_{f(g)})(T_e(f)(\nu(e)) \\ &= (T_e\lambda_{f(g)})(\nu'(e)) \\ &= \nu'(f(g))\end{aligned}$$

where in the last step we used left-invariance of ν'. This proves the lemma. □

Proof of theorem 8. Let $\xi, \eta \in \mathrm{Lie}(G)$. We have to show that

$$\mathrm{Lie}(f)[\xi, \eta] = [\mathrm{Lie}(f)(\xi), \mathrm{Lie}(f)(\eta)].$$

By the lemma, $\mathrm{Lie}(f)(\xi)$ is f-related to ξ and $\mathrm{Lie}(f)(\eta)$ is f-related to η. This implies that $[\mathrm{Lie}(f)(\xi), \mathrm{Lie}(f)(\eta)]$ is f-related to $[\xi, \eta]$. In other words,

$$T_gf([\xi, \eta](g)) = [\mathrm{Lie}(f)(\xi), \mathrm{Lie}(f)(\eta)](f(g)).$$

In particular, for $g = e$ we get

$$\mathrm{Lie}(f)([\xi, \eta]) = [\mathrm{Lie}(f)(\xi), \mathrm{Lie}(f)(\eta)].$$

This proves the result. □

An important facet of Lie theory is describing the extent to which Lie is an equivalence of categories. We will not discuss this issue in detail. Let us just mention one result. A proof can be found in [Lee03] at theorem 15.32 (page 396) and theorem 15.35 (page 398), keeping in mind that we have already established theorem 8.

Theorem 10. *The functor* Lie *from the category of simply connected Lie groups (a full subcategory of the category of Lie groups) to the category of Lie algebras is an equivalence of categories.*

1.2.2 The exponential map

We now introduce a crucial ingredient to understanding Lie groups and Lie algebras: the exponential map.

Suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of the Lie group G. By differentiation we can associate to this one-parameter subgroup the element $\gamma'(0) \in T_eG$. This map

$$\{\text{one-parameter subgroups of } G\} \to \mathfrak{g} = T_eG$$

is in fact a bijection.

Theorem 11. *The map*

$$\varphi : \{\textit{one-parameter subgroups } \gamma \textit{ of } G\} \to T_eG : \gamma \mapsto \gamma'(0)$$

is a bijection.

Proof. Let $\xi \in T_eG$ be any element of the Lie algebra of G. Consider this element as a left-invariant vector field on G. This is the vector field X with $X_g = (T_e\lambda_g)(\xi)$.

If γ is to be a one-parameter subgroup that is mapped to ξ, then it has to satisfy the conditions

$$\gamma'(0) = \xi$$
$$\gamma(t+s) = \gamma(t)\gamma(s) = \lambda_{\gamma(t)}(\gamma(s)) \quad \text{for all } t, s \in \mathbb{R}.$$

By differentiating the second condition with respect to s at $s = 0$, we find that γ has to satisfy

$$\gamma'(t) = (\lambda_{\gamma(t)})_*(\xi). \tag{$*$}$$

This is exactly the condition that γ is an integral curve of the left-invariant vector field associated to ξ. By uniqueness of integral curves, the map φ is injective.

To show surjectivity, let $\gamma : (-\varepsilon, \varepsilon) \to G$ be the local solution to $(*)$ with $\gamma(0) = e$. For a fixed $t \in (-\varepsilon, \varepsilon)$, the curve $s \mapsto \gamma(t+s)$ is the integral curve of the vector field X through $\gamma(t)$ for $s = 0$. The curve $s \mapsto \gamma(t)\gamma(s)$ is an integral curve of the vector field $(\lambda_{\gamma(t)})_*X = X$ through $\gamma(t)$ for $s = 0$. By uniqueness of integral curves, we conclude that

$$\gamma(t+s) = \gamma(t)\gamma(s) \quad \text{for all } t, s \text{ sufficiently small.}$$

So far γ is only defined in a neighbourhood of 0, but by demanding that $\gamma(t+s) = \gamma(t)\gamma(s)$ for all $s, t \in \mathbb{R}$, we can extend γ in a unique way to a one-parameter subgroup of G. This gives us a one-parameter subgroup that is mapped to ξ under the given map, showing surjectivity of φ. ☐

Note that we have implicitly shown that any left-invariant vector field is complete (meaning its flow can be extended to all of the real line).

We now define the *exponential map* for the Lie group G as

$$\exp : \mathfrak{g} \to G : \xi \mapsto (\varphi^{-1}(\xi))(1).$$

In other words, given a left-invariant vector field ξ on G, the point $\exp(\xi)$ is where you end up after flowing along ξ for one unit of time (starting at e).

Using standard results in ordinary differential equations, one proves

Theorem 12. *The exponential map* $\exp : \mathfrak{g} \to G$ *is smooth.*

Proof. Consider the manifold $G \times \mathfrak{g}$ and the vector field X on it, such that at the point (g, ξ) the vector is $((\lambda_g)_*(\xi), 0)$. Then $\exp(\xi)$ is just a component of the value at 1 of the integral curve of X under the initial conditions (e, ξ). By smooth dependence on initial conditions of the solutions to differential equations, exp is smooth. $\square$

If we denote by $\gamma_\xi : \mathbb{R} \to G$ the integral curve of the left-invariant vector field ξ starting at e, then it is clear that

$$\exp(t\xi) = \gamma_\xi(t).$$

Differentiating this with respect to t at $t = 0$ gives us

Theorem 13. *The differential of the exponential map at zero is the identity.*

As a basic example, let us consider the exponential map of the Lie group $\mathbb{C}^\times$, the group of non-zero complex numbers. Note that the tangent space to $\mathbb{C}^\times$ at any point can be identified with $\mathbb{C}$. The Lie algebra of $\mathbb{C}^\times$ is thus $T_1\mathbb{C}^\times \cong \mathbb{C}$. This means that the exponential map is a map

$$\exp : \mathbb{C} \to \mathbb{C}^\times.$$

Note that by exp we mean the exponential map in the Lie group sense, not the usual exponential map on the complex numbers (although they will turn out to be identical). Let $z \in \mathbb{C}$ be an element of the Lie algebra. The left-invariant vector field on $\mathbb{C}^\times$ associated to z is

$$\mathbb{C}^\times \to \mathbb{C} : w \mapsto wz.$$

The differential equation for the integral curve $\gamma : \mathbb{R} \to \mathbb{C}$ of this left-invariant vector field is then

$$\gamma'(t) = \gamma(t)z.$$

Solving this differential equation with initial condition $\gamma(0) = 1$ gives

$$\gamma(t) = e^{tz}$$

where $e^{\cdots}$ is the usual exponential map on the complex numbers. Evaluating this at $t = 1$ gives

$$\exp(z) = e^z,$$

proving that the exponential map (in the Lie sense) on $\mathbb{C}$ is the same as the usual exponential map. By theorem (11), the one-parameter subgroups of $\mathbb{C}^\times$ are exactly of the form

$$t \mapsto \exp(tz)$$

for $z \in \mathbb{C}$.

For matrix groups (Lie groups of invertible matrices with multiplication as the group operation, for example $\mathrm{SU}(n)$), the exponential map turns out to be the classical exponential map of matrices as given by the usual power series. A proof can be found in [Lee03] on page 383 (proposition 15.18).

Theorem 14. *Let G be a matrix group. Then*

$$\exp(A) = \sum_{k\geq 0} \frac{A^k}{k!}$$

where the right-hand side is an absolutely convergent power series.

1.3 The (co)adjoint representation

Let us now study some important *representations* of Lie groups and Lie algebras.

Definition 15. *A* representation *of a Lie group G on a finite-dimensional real vector space V is a morphism of Lie groups*

$$G \to \mathrm{GL}(V).$$

Definition 16. *A* representation *of a Lie algebra $\mathfrak{g}$ on a finite-dimensional real vector space V is a morphism of Lie algebras*

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

Notice that a representation of a Lie group induces a representation of the corresponding Lie algebra by application of the functor Lie.

1.3.1 The adjoint representation

Let G be a Lie group. Given an element $h \in G$, we can consider the map

$$G \to G : g \mapsto hgh^{-1}.$$

It is an automorphism of G, and therefore the differential of this map is an automorphism of $\mathrm{Lie}(G) = \mathfrak{g}$. Associating to h this differential gives a representation of G on its own Lie algebra called the *adjoint representation*. We write Ad for the adjoint representation, so

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : h \mapsto \mathrm{Ad}_h$$

where

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g} : \xi \mapsto \frac{d}{dt}_{|t=0} \left(h \exp(t\xi) h^{-1} \right).$$

This adjoint representation is our first example of a representation of a Lie group.

Let us now apply the functor Lie (essentially, we will differentiate) to this representation to get a representation of the Lie algebra. The resulting representation of $\mathfrak{g}$ will also be called *adjoint representation* but will be denoted by ad. We calculate:

$$\begin{aligned}
\mathrm{ad}_\xi(\eta) &= \frac{d}{dt}_{|t=0} \left(\mathrm{Ad}_{\exp(t\xi)}(\eta)\right) \\
&= \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (\exp(t\xi)\exp(s\eta)\exp(-t\xi)) \\
&= [\xi, \eta]
\end{aligned}$$

where in the last step we recognized the geometric definition of the Lie bracket (using flows of vector fields). This gives us a Lie algebra representation

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : \xi \mapsto \mathrm{ad}_\xi .$$

By theorem (8), we know that this is a Lie algebra morphism. In this case, however, explicit verification is easy. Indeed, the equality

$$\mathrm{ad}([\xi, \eta]) = [\mathrm{ad}(\xi), \mathrm{ad}(\eta)]$$

is equivalent to

$$[[\xi, \eta], \alpha] = [\xi, [\eta, \alpha]] - [\eta, [\xi, \alpha]] \quad \text{for any } \alpha \in \mathfrak{g},$$

which is just the Jacobi identity for the Lie bracket.

Let us state one easy result that will be useful later on.

Lemma 17. *Suppose G is a matrix group. Then the Lie algebra $\mathfrak{g}$ is a vector space of matrices. The adjoint action of G on $\mathfrak{g}$ is given by conjugation.*

Proof. Let $M \in G$ be a matrix and $\xi \in \mathfrak{g}$ a matrix in the Lie algebra. Then

$$\begin{aligned}
\mathrm{Ad}_M(\xi) &= \frac{d}{dt}_{|t=0} \left(M \exp(t\xi) M^{-1}\right) \\
&= M \left(\frac{d}{dt}_{|t=0} \exp(t\xi)\right) M^{-1} \\
&= M\xi M^{-1}.
\end{aligned}$$

□

1.3.2 The coadjoint representation

Let G be a Lie group and $h \in G$. By dualizing the map

$$\mathrm{Ad}_{h^{-1}} : \mathfrak{g} \to \mathfrak{g}$$

we get a map

$$\mathrm{Ad}_h^* : \mathfrak{g}^* \to \mathfrak{g}^*.$$

(Note that by this notation we mean $(Ad^*)_h$ rather than $(Ad_h)^*$. In cases where we mean the latter, we will always write the brackets explicitly.) This gives us a representation of G on the dual of its Lie algebra,

$$\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*) : h \mapsto \mathrm{Ad}^*_h = (\mathrm{Ad}_{h^{-1}})^*$$

where the last star means dualization of linear maps. This definition turns Ad^* into a representation of the Lie group, because

$$\mathrm{Ad}^*_{gh} = (\mathrm{Ad}_{(gh)^{-1}})^* = (\mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_{g^{-1}})^* = (\mathrm{Ad}_{g^{-1}})^* \circ (\mathrm{Ad}_{h^{-1}})^* = \mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h \,.$$

This representation Ad^* of G on $\mathfrak{g}^*$ is called the *coadjoint representation* of G.

We now want to give an appropriate definition of the *coadjoint representation* of $\mathfrak{g}$. There are two reasonable approaches:

- we can dualize the adjoint representation of $\mathfrak{g}$;
- or we can differentiate the coadjoint representation of G.

Fortunately, both approaches give the same result. Let us define the coadjoint representation of $\mathfrak{g}$ using the first approach. In other words, we set

$$\mathrm{ad}^*_\xi = (\mathrm{ad}_{-\xi})^*$$

and call

$$\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*) : \xi \mapsto \mathrm{ad}^*_\xi$$

the coadjoint representation of $\mathfrak{g}$. This yields a representation of $\mathfrak{g}$ as is easily verified (using the fact that ad is a representation, or in other words using the Jacobi identity). Let us now differentiate the coadjoint representation of G to get a representation of $\mathfrak{g}$ and check that this gives ad^* (this renders the verification that ad^* is a representation unnecessary because representations of G automatically differentiate to representations of $\mathfrak{g}$).

We want to establish that

$$\frac{d}{dt}_{|t=0} \left(\mathrm{Ad}^*_{\exp(t\xi)} \right) = \mathrm{ad}^*_\xi \,.$$

To check their equality, we evaluate both of them on $\alpha \in \mathfrak{g}^*$. We are done if the results (which are elements of $\mathfrak{g}^*$) are the same. To check equality of the results, we evaluate them both on an element $\eta \in \mathfrak{g}$. In other words, what we will check is

$$\langle \mathrm{ad}^*_\xi(\alpha), \eta \rangle = \langle \frac{d}{dt}_{|t=0} \left(\mathrm{Ad}^*_{\exp(t\xi)} \right) (\alpha), \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The left hand side is (by definition of the coadjoint representation) equal to $\langle \alpha, \mathrm{ad}_{-\xi}(\eta) \rangle = -\langle \alpha, [\xi, \eta] \rangle$. The right hand side is equal to

$$\begin{aligned}\langle \frac{d}{dt}_{|t=0} \left(\mathrm{Ad}^*_{\exp(t\xi)}\right)(\alpha), \eta\rangle &= \langle \frac{d}{dt}_{|t=0} \left(\mathrm{Ad}^*_{\exp(t\xi)}(\alpha)\right), \eta\rangle \\ &= \frac{d}{dt}_{|t=0} \langle \mathrm{Ad}^*_{\exp(t\xi)}(\alpha), \eta\rangle \\ &= \frac{d}{dt}_{|t=0} \langle \alpha, \mathrm{Ad}_{\exp(-t\xi)}(\eta)\rangle \\ &= \langle \alpha, \frac{d}{dt}_{|t=0} \mathrm{Ad}_{\exp(-t\xi)}(\eta)\rangle \\ &= \langle \alpha, \mathrm{ad}_{-\xi}(\eta)\rangle \\ &= \langle \alpha, [-\xi, \eta]\rangle.\end{aligned}$$

This shows that the coadjoint representation of $\mathfrak{g}$ is indeed obtained by differentiating the coadjoint representation of G.

1.4 Compact Lie groups

In this section we mention several results about compact Lie groups without proofs. They introduce some concepts that we will make use of later, notably *maximal tori* and the *Weyl group*. These results can be found in [DK99] (corollary 1.12.4 on page 61 and theorem 3.7.1 on page 152).

Theorem 18. *Every connected abelian Lie group is isomorphic to* $\mathbb{R}^m \times \mathrm{U}(1)^n$ *for some natural numbers* m, n*. In particular, every compact connected abelian Lie group is isomorphic to* $\mathrm{U}(1)^n$ *for some* n*.*

Definition 19. *If* G *is a Lie group, then a* torus in G *is a compact connected abelian Lie subgroup of* G*. A* maximal torus in G *is a torus in* G *that is not strictly contained in any other torus in* G*.*

Theorem 20. *Every maximal torus in a compact connected Lie group is a maximal abelian subgroup (an abelian subgroup not strictly contained in any other abelian subgroup). Every element of a compact connected Lie group* G *is contained in some maximal torus, and all maximal tori are conjugate (so if* T *and* T' *are maximal tori, there is some* $g \in G$ *such that* $T' = gTg^{-1}$*).*

Definition 21. *If* G *is a compact connected Lie group and* T *is a maximal torus in it, then we define its* Weyl group $W(T)$ *to be*

$$W(T) = \frac{N(T)}{T},$$

where $N(T)$ *is the normalizer of* T*, meaning that* $N(T) = \{g \in G \mid gTg^{-1} = T\}$*.*

Theorem 22. *If G is a compact connected Lie group and T a maximal torus in it, then the Weyl group $W(T)$ is finite.*

Note that the Weyl group $W(T)$ acts on T via group automorphisms (by conjugation) and that this action is faithful because T is a maximal abelian subgroup of G. The following lemma is lemma 4.33 in [Ada69] (page 96).

Lemma 23. *Suppose G is a compact connected Lie group and T a maximal torus in it. If $t, t' \in T$ are conjugate (so that $t' = gtg^{-1}$ for some $g \in G$), then there is an element of the Weyl group $W(T)$ sending t to t'.*

For the case $G = \mathrm{SU}(2)$, a maximal torus is given by

$$T = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \mid \omega \in S^1 \right\}.$$

Its Weyl group is the group of two elements, with the non-trivial element acting by

$$\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \mapsto \begin{pmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{pmatrix}.$$

Chapter 2

Bundles and connections

In this chapter we study fibre bundles and connections on them. Starting from the general concept of a fibre bundle with a connection, we later focus on two important special cases: vector bundles, where the fibre is a vector space, and principal bundles, where the fibre is a principal homogeneous space. Our references for this chapter are [Hec12] (for definitions of the bundles) and [KN63] and [Mic88] (for connections).

2.1 Fibre bundles

We start by studying fibre bundles in general. We also treat connections and parallel transport, both of which will later be specialized to the cases of vector bundles and principal bundles.

2.1.1 Definition

A fibre bundle is a space that locally looks like a product of two manifolds, but may be different from an ordinary product globally. Let us give the formal definition.

Definition 24. *A (smooth)* fibre bundle *is a tuple* (E, Σ, π, F) *where* E*,* Σ *and* F *are smooth manifolds and* $\pi : E \to \Sigma$ *is a smooth surjection. Furthermore, we require that every* $x \in \Sigma$ *has an open neighbourhood* $U \subset \Sigma$ *such that* $\pi^{-1}(U)$ *is diffeomorphic to* $U \times F$ *via a map* $\varphi : \pi^{-1}(U) \to U \times F$ *in such a way that the following diagram commutes (here,* $proj_1$ *is projection onto the first factor):*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

We call E the total space*, Σ the* base space*, π the* projection *and F the* fibre*. The set of the $\{(U_i, \varphi_i)\}$ is called a* local trivialization *of the bundle. If $x \in \Sigma$, the preimage $\pi^{-1}(\{x\})$ is called the* fibre over x*, denoted by E_x.*

Very often, we will refer to "the fibre bundle E" instead of mentioning the whole tuple. We also speak of a bundle *over* Σ to indicate that Σ is the base space.

Let us look at a few examples to gain some intuition.

- An obvious example is the *trivial fibre bundle*. Let Σ and F be two smooth manifolds, and consider $E = \Sigma \times F$ with π the projection onto the first factor. Then (E, Σ, π, F) is clearly a fibre bundle (indeed, in the definition we can take $U = \Sigma$ for all x and $\varphi = \mathrm{Id}$).

- A familiar example of a fibre bundle is the tangent bundle of a manifold. If M is a smooth m-dimensional manifold, then the tangent bundle TM is a fibre bundle over M (with the usual projection $TM \to M$).

- An interesting example is $(S^1, S^1, z \mapsto z^2, \{0, 1\})$. This is a fibre bundle that is not trivial, because the trivial fibre bundle would be $S^1 \times \{0, 1\}$, which is not connected, whereas S^1 is.

- Another example of a fibre bundle is the Möbius strip, pictured in figure 2. It is a fibre bundle over the circle with fibre $(0, 1)$ (or you could include the boundary, resulting in a fibre bundle with fibre $[0, 1]$, which would require a slight generalization of our definition of fibre bundle to allow manifolds with boundary).

The last two examples are in fact intimately related. To see this, note that we could consider the boundary of the Möbius strip, which is diffeomorphic to S^1, and look at the projection of this boundary circle onto the red one. This gives exactly the fibre bundle $(S^1, S^1, z \mapsto z^2, 0, 1)$.

Definition 25. *Let (E, Σ, π, F) be a fibre bundle. A* section *of this fibre bundle is a smooth map $s : \Sigma \to E$ such that $\pi \circ s = \mathrm{Id}_\Sigma$. The set of all sections of the fibre bundle is written $\Gamma(E)$.*

A fibre bundle is a way of associating a copy of the space F to every point of Σ and a section is just a choice of a point in every such copy (in a smooth way). Let us also define morphisms of fibre bundles.

Definition 26. *Let (E, Σ, π, F) and (E', Σ', π', F') be two fibre bundles. A* morphism of fibre bundles*, also called a* bundle map *is smooth map $\phi : E \to E'$ such that there exists a smooth map $\varphi : \Sigma \to \Sigma'$ such that following diagram commutes.*

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ \Sigma \arrow[r, "\varphi"'] & \Sigma' \end{tikzcd}$$

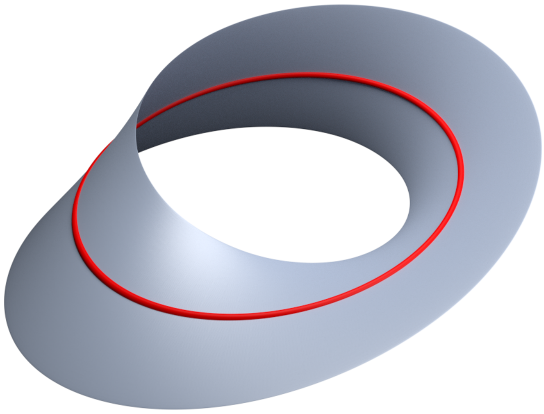

Figure 2: The Möbius strip, obtained by gluing two ends of a stroke of paper together with a half-turn twist. It is a fibre bundle over the circle S^1 with fibre $(0,1)$. The base space can be identified with the circle marked in red. Projection onto the base space then corresponds to mapping any point of the Möbius strip to the nearest red point.

In other words, a morphism of fibre bundles is a smooth map that preserves fibres. Note that ϕ determines φ uniquely by surjectivity of π. We say that ϕ is a bundle map *covering* φ.

2.1.2 Structure groups

Very often, a fibre bundle will carry some extra structure called a *structure group*. Let (E, Σ, π, F) be a fibre bundle and G a Lie group, and suppose that G acts on the fibre F from the left. A local trivialization of the fibre bundle is called a *G-atlas* if for any two overlapping charts (U_i, φ_i) and (U_j, φ_j) the transition function

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$(x, \xi) \mapsto (x, t_{ij}(x)\xi)$$

for some smooth function $t_{ij} : (U_i \cap U_j) \to G$. Two G-atlases are called *equivalent* if their union is also a G-atlas.

Definition 27. *A* fibre bundle with structure group G *(a G-*bundle *for short) is a fibre bundle with an equivalence class of G-atlases.*

Non-rigorously stated, equipping a fibre bundle with a structure group G is a way of demanding that the different locally trivial pieces are glued together only in ways given by the action of G on F, not by arbitrary diffeomorphisms of F. It is easy to check that the Möbius strip admits a $(\mathbb{Z}/2\mathbb{Z})$-atlas. Note that a fibre bundle is trivial exactly if it admits the trivial group as structure group.

2.1.3 Pullback of a bundle

Suppose M and N are two smooth manifolds, and $f : M \to N$ is a smooth map. Let (E, N, π, F) be a fibre bundle over N. We can then pull back this bundle via f to get a fibre bundle f^*E over M. This *pullback bundle* is defined as

$$f^*E = \{(x, e) \in M \times E \mid f(x) = \pi(e)\} \subset M \times E.$$

It is a submanifold of $M \times E$ by the implicit function theorem (by noting that the differential of π has full rank). The projection τ for the pullback bundle is given by projection onto the first factor. In symbols:

$$\tau : f^*E \to M : (x, e) \mapsto x.$$

The fibre of f^*E over $x \in M$ is essentially the fibre of E over $f(x)$. Using the full notation, the pullback bundle is (f^*E, M, τ, F). It is not hard to check that this is indeed a fibre bundle. Moreover, pullback of bundles respects structure groups: if E is a fibre bundle with structure group G, then so is f^*E. Projection onto E induces a map $\tilde{f} : f^*E \to E : (x, e) \mapsto e$. This is a bundle map covering f. Sections can be pulled back too. Given a section $s \in \Gamma(E)$, we have $f^*s = s \circ f \in \Gamma(f^*E)$.

Let us look at an example. Consider the map $f : S^1 \to S^1 : z \mapsto z^2$. We have seen that the Möbius strip E is a fibre bundle over S^1, so that we can pull it back via f to obtain a fibre bundle f^*E over S^1. The situation is illustrated in figure 3. Note that *the pullback f^*E is trivial.* If one were to make a paper model of the bundle as shown in the figure, the model would be a stroke of paper with both ends glued together will a *full twist* (the two half-turns don't cancel out). Nevertheless, the bundle is trivial. And indeed, the bundle is diffeomorphic (as manifolds) to the untwisted version, even though they are not *isotopic* as embedded submanifolds of Euclidean space.

2.1.4 Connections

If (E, Σ, π, F) is a fibre bundle, then all the fibres are disjoint copies of F that have nothing to do with each other except for being glued together in a smooth fashion. We now want to identify nearby fibres in a stronger way. To this end, we will consider *connections* on E.

First observe that if $p \in E$, then there is a distinguished subspace V_p of T_pE, called the *vertical space at p*, that is the tangent space to the fibre p is contained in. In symbols, this can be expressed as

$$V_p = \ker(T_p\pi) \subseteq T_pE.$$

The dimension of this vertical space is $\dim(F)$. So given a fibre bundle, we can tell whether a vector is "vertical" by checking whether it is mapped to zero by the differential of π. On a mere fibre bundle, however, we cannot tell whether a tangent vector is "horizontal". This is precisely what a connection provides.

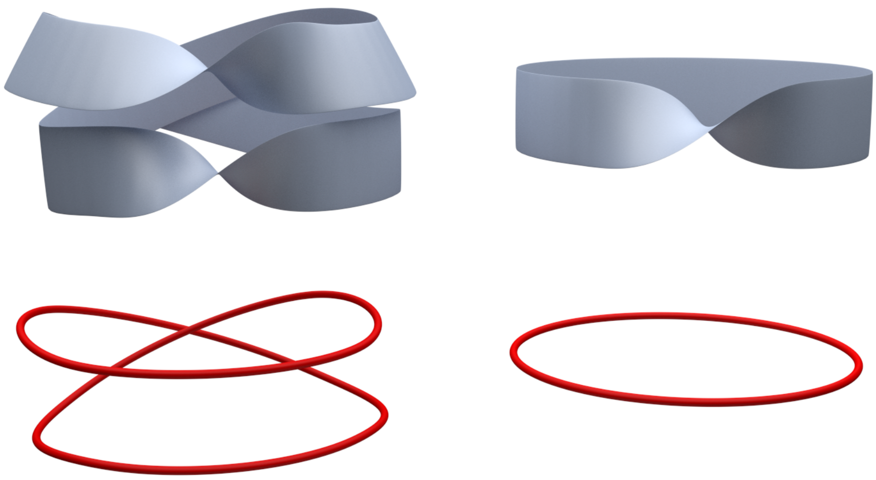

Figure 3: Pulling back the Möbius strip via the map $S^1 \to S^1 : z \mapsto z^2$. On the right the Möbius strip in blue, with the base space in red. On the left the pullback bundle. We visualize the map $z \mapsto z^2$ as "squashing down vertically" the base space on the left to get the circle on the right.

Definition 28. *A* connection *on a fibre bundle* (E, Σ, π, F) *is a choice of linear complement to the vertical space at every point of* E*, in a smooth way. In other words, a connection is a smooth choice of subspaces* H_p *of* T_pE *such that*

$$T_pE = V_p \oplus H_p \quad \textit{for every } p \in E.$$

We call H_p *the* horizontal space at p.

If a connection on E is given, the map $T_p\pi$ is an isomorphism between H_p and $T_{\pi(p)}\Sigma$. If a smooth vector field X on Σ is then given, we can lift this to a smooth *horizontal* vector field on E in a unique way. This lift is the unique vector field $\tilde{X}$ on E such that $\tilde{X}_p \in H_p$ and $T_p\pi(\tilde{X}_p) = X_{\pi(p)}$ for every $p \in E$.

Note that a connection can also be encoded by specifying at every point the projection $A_p : T_pE \to V_p$ with kernel H_p. Conversely, if at every point we are given a linear map $A_p : T_pE \to V_p$ that restricts to the identity on V_p, then we can recover a connection from this by defining

$$H_p = \ker(A_p).$$

A connection also allows us to locally lift curves horizontally. Suppose that $\gamma : I \to \Sigma$ is a smooth curve in Σ (or a piecewise smooth one by a straightforward extension of what we explain here). We want to lift γ horizontally, which means we are looking for a curve $\tilde{\gamma}$ in E such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}'(t)$ is horizontal for all t. Let $t_0 \in (0, 1)$ and pick a point $p \in \pi^{-1}(\gamma(t_0))$. In a neighbourhood U of $\gamma(t_0)$, the bundle is the trivial bundle $U \times F$. Take local coordinates on E around p by picking local coordinates x_i on U and y_i on F. The horizontal lift of the tangent vector $(v_1, \ldots, v_{\dim(\Sigma)})$ at the point $q \in E$ is given by

$$(v_1, \ldots, v_{\dim(\Sigma)}, f_1(q, v), \ldots, f_{\dim(F)}(q, v)),$$

where the f_i are smooth functions of q and $v = (v_1, \ldots, v_{\dim(\Sigma)})$ (they are in fact linear in v). The differential equations

$$\begin{cases} x'(t) = \gamma'(t) \\ y_i'(t) = f_i(\gamma'(t)) \end{cases}$$

together with the initial condition $(x(t_0), y(t_0)) = p$ then specify a local horizontal lift of γ (and any two such lift coincide on their common domain of definition).

We are mainly interested in connections where any curve can in fact be lifted globally, meaning that the lift is defined on the whole domain of definition of γ.

Definition 29. *A connection on a fibre bundle (E, Σ, π, F) is called* complete *if any curve $\gamma : I \to \Sigma$ defined on some interval I has a horizontal lift $\tilde{\gamma} : I \to E$. In this case, the connection is also called an* Ehresmann connection*.*

The point in this definition is that $\tilde{\gamma}$ is defined on all of I.

Let us give an example of a connection that is not complete. Consider the trivial bundle over $\mathbb{R}$ with fibre $\mathbb{R}$, which is $\mathbb{R}^2$ with projection onto the first coordinate. Write

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ or } xy \leq 1\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ and } xy \geq 2\}.$$

These are two disjoint closed subsets of $\mathbb{R}^2$. Pick a smooth function $\mathbb{R}^2 \to \mathbb{R}$ that is 0 on A and 1 on B (it is a standard result that any two closed disjoint sets in a manifold can be separated in this way; it follows immediately from proposition A.8 on page 224 of [MrT97]). Now consider the vector field

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto \begin{cases} (1, 0) & \text{if } x \geq 0, \\ \left(1, f(x, y) \cdot \frac{-3}{x^2}\right) & \text{otherwise.} \end{cases}$$

It is smooth and defines a connection on the bundle by demanding that it is horizontal. However, the curve $\mathbb{R}_{<0} \to \mathbb{R}^2 : t \mapsto (t, 3/t)$ is an integral curve of this vector field. This shows that the curve $\mathbb{R} \to \mathbb{R} : t \mapsto t$ cannot be lifted for this connection if we start at $(-3, -1)$, because lifting it on $\mathbb{R}_{<0}$ first results in the map $\mathbb{R}_{<0} \to \mathbb{R}^2 : t \mapsto (t, 3/t)$, which cannot even be continuously extended to all of $\mathbb{R}$.

Just as vector fields on a compact manifold are always complete, we have the following nice result. A proof can be found in [Mic88] on page 41, under paragraph 9.10. We will not need this result.

Theorem 30. *If a fibre bundle has compact fibre, any connection on it is complete.*

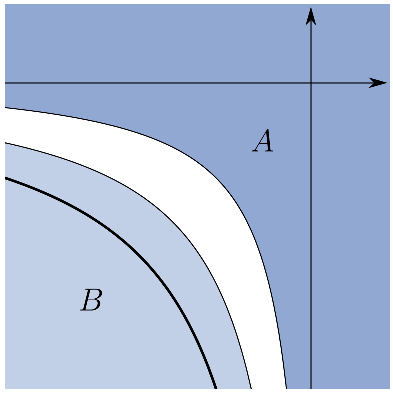

Figure 4: The third quadrant in the plane. In dark blue the region A, in light blue the region B. To construct an incomplete connection, we start with a curve in E which we want to be the lift of some curve in the base space and that runs off to infinity in finite time. Here, we pick a branch of a hyperbola, shown on the picture as a thick line and completely contained in B. Locally around the curve, we take a nowhere-vertical vector field tangent to it. We then extend this vector field to all of the plane in a nowhere-vertical way, thus defining an incomplete connection.

2.1.5 Parallel transport and holonomy

Suppose the fibre bundle (E, Σ, π, F) is equipped with a complete connection. Pick a curve $\gamma : I \to \Sigma$ and assume $0 \in I$. Using completeness of the connection, we get a map

$$\mathrm{Pt}_\gamma : E_{\gamma(0)} \times I \to E : (p, t) \mapsto (\text{lift of } \gamma \text{ at } p)(t).$$

Moving along a horizontal curve in a fibre bundle with a connection is called *parallel transport*. We say that $\mathrm{Pt}_\gamma(p, t)$ is the result of parallel transport of p along γ for a time t. Often, we will leave out t from the notation and just assume that $t = 1$ (then we must of course have $1 \in I$). Note that Pt_γ is a smooth map since the solutions of smooth differential equations depend smoothly on the initial conditions. Also note that reparametrization of γ does not change parallel transport along it.

If we fix t we get a map

$$\mathrm{Pt}_{\gamma,t} : E_{\gamma(0)} \to E_{\gamma(t)}.$$

This map is a diffeomorphism, since an inverse is given by parallel transport along γ in the reverse direction. Suppose now that $\gamma(0) = \gamma(t)$, so that we are doing parallel transport along a *loop* in the base space. Then the resulting $\mathrm{Pt}_{\gamma,t}$ is an automorphism of $E_{\gamma(0)}$ called the *holonomy of the connection around the loop* γ.

2.2 Vector bundles

We now treat a very important type of fibre bundle: vector bundles. These are roughly speaking fibre bundles where the fibre is a vector space.

2.2.1 Definition

Definition 31. *A* vector bundle *is a fibre bundle* (E, Σ, π, F) *where the fibre* F *is a finite-dimensional real vector space with structure group* $\mathrm{GL}(F)$ *acting on* F *in the usual way. The dimension of* F *is called the* rank *of the vector bundle.*

Note that every fibre of a vector bundle is a vector space. Indeed, the operations of addition and scalar multiplication can be carried out in any chart of the $\mathrm{GL}(F)$-atlas, and the result will not depend on the chosen chart because the transition maps are fibrewise linear. Also note that any vector bundle has a section, the zero section mapping the point $x \in \Sigma$ to the origin of the vector space E_x. Locally, the vector bundle is just $U \times F$ for some open $U \subseteq \Sigma$. This means that we can pick local coordinates on E by picking coordinates on U and identifying the vector space F with $\mathbb{R}^n$. The resulting local coordinates on E will be called *linear coordinates*.

We want morphisms between vector bundles to preserve the extra structure.

Definition 32. *A* vector bundle morphism *is a bundle map that is linear on each fibre.*

A well-known example of a vector bundle is the tangent bundle to a smooth manifold, with the projection given by mapping a vector to its base point. If $f : M \to N$ is a smooth map between manifolds, then $Tf : TM \to TN$ is a morphism of vector bundles covering f. The space of sections of a vector bundle is an $\mathcal{F}(\Sigma)$-module (and a real vector space).

If E and F are two vector bundles over Σ and E_x is a linear subspace of F_x for every $x \in \Sigma$, then we say that E is a *subbundle* of F. As an example, on any fibre bundle with a connection the horizontal spaces at each point together form a subbundle of the tangent bundle to the given fibre bundle.

Many constructions with vector spaces can be generalized to vector bundles. For example, if E and F are vector bundles defined over some manifold Σ, then we can form their direct sum $E \oplus F$, which as a set is just

$$E \oplus F = \bigcup_{x \in \Sigma} E_x \oplus F_x.$$

It carries the structure of a vector bundle where the projection maps the elements of $E_x \oplus F_x$ to x. To show this, observe that over some neighbourhood U of x both E and F are trivial. Take local coordinates $(x_1, \ldots, x_{\dim(\Sigma)})$ on E and extend these to local linear coordinates $(x_1, \ldots, x_{\dim(\Sigma)}, e_1, \ldots, e_{\mathrm{rank}(F)})$ on E and $(x_1, \ldots, x_{\dim(\Sigma)}, f_1, \ldots, f_{\mathrm{rank}(F)})$ on F. Then $(x_1, \ldots, x_{\dim(\Sigma)}, e_1, \ldots, e_{\mathrm{rank}(E)}, f_1, \ldots, f_{\mathrm{rank}(F)})$ form local linear coordinates on $E \oplus F$.

In an analogous fashion one defines the tensor product of vector bundles and the dual and the exterior powers of a vector bundle. We leave the details to the reader. The reader can also check the following: if E and F are vector bundles over a common base space, then a bundle map $E \to F$ covering the identity is essentially a section of $E^* \otimes F$. The proof is a slight adaptation of the one used to show that $\mathrm{Hom}(V, W) \cong V^* \otimes W$ for finite-dimensional real vector spaces V and W.

2.2.2 Linear connections

On vector bundles, we are interested in complete connections that preserve the linear structure under parallel transport. In other words, we want $\mathrm{Pt}_{\gamma,t} : E_{\gamma(0)} \to E_{\gamma(t)}$ to be linear for all γ and t. A moment's thought shows that this is equivalent to the following definition.

Definition 33. *Suppose (E, Σ, π, F) is a vector bundle equipped with a complete connection. If γ is a curve in Σ and p is a point of the fibre $E_{\gamma(0)}$, then write $\tilde{\gamma}_p$ for the horizontal lift of γ starting at p. The connection on E is called* linear *if*

$$\tilde{\gamma}_{t \cdot p} = t \cdot \tilde{\gamma}_p \quad \textit{for all curves } \gamma\textit{, points } p \in E_{\gamma(0)} \textit{ and } t \in \mathbb{R}$$

and

$$\tilde{\gamma}_{p+q} = \tilde{\gamma}_p + \tilde{\gamma}_q \quad \textit{for all curves } \gamma\textit{, points } p, q \in E_{\gamma(0)}.$$

Recall that a connection can also be specified by giving projections $A_p : T_pE \to V_p$, the link between these projections and the horizontal spaces being

$$H_p = \ker(A_p).$$

Note that we can identify the vertical space at p with the fibre through p because the tangent space to a vector space can be identified with this vector space. We can thus regard A_p as a map from T_pE to $E_{\pi(p)}$.

2.2.3 The covariant derivative

We now introduce the *covariant derivative*, which measures how much a section deviates from being "constant" (where constancy is measured by horizontalness in a sense we will make precise). To motivate the definition of the covariant derivative, we first try to quantify to what extent curves fail to be horizontal.

Take a curve $\gamma : I \to E$ with $0 \in I$. We will try to quantify how much γ deviates from horizontalness at time $t = 0$. If γ were horizontal, we would have

$$\mathrm{Pt}^{-1}_{\pi\circ\gamma,t}(\gamma(t)) = \gamma(0) \quad \text{for all } t \in I.$$

This suggests a way to quantify the deviation of γ from horizontalness at time $t = 0$. If we write this deviation as $D_0\gamma$, then a reasonable definition would be

$$D_0\gamma = \frac{d}{dt}_{|t=0} \left(\mathrm{Pt}^{-1}_{\pi\circ\gamma,t}(\gamma(t)) - \gamma(0)\right). \tag{$*$}$$

This definition is not very easy to work with, so let us reformulate it. Write $p = \gamma(0)$ and $x = \pi(p)$. Note that $(\pi \circ \gamma)^* E$ is a submanifold of $\mathbb{R} \times E$. We can consider the map

$$\varphi : \mathbb{R} \times E_x \to (\pi \circ \gamma)^* E : (t, q) \mapsto (t, \mathrm{Pt}_{\pi \circ \gamma, t}(q)).$$

Its differential at $(0, p)$ is an isomorphism, implying that φ is a diffeomorphism between some neighbourhood $U \subseteq \mathbb{R} \times E_x$ of $(0, p)$ and a neighbourhood $V \subseteq (\pi \circ \gamma)^* E$ of $(0, p)$. Composing its inverse with projection onto E_x results in a map

$$\varpi : V \to E_x.$$

Now for $|t|$ small enough we can consider $t \mapsto \varpi(t, \gamma(t))$, which by construction equals $\mathrm{Pt}^{-1}_{\pi \circ \gamma, t}(\gamma(t))$. Differentiating with respect to t at $t = 0$ gives

$$\begin{aligned} D_0 \gamma &= \frac{d}{dt}_{|t=0} \varpi(t, \gamma(t)) \\ &= (T_{(0,p)} \varpi)(1, \gamma'(0)) \end{aligned}$$

Now the map $T_p E \to E_x : v \mapsto (T_{(0,p)} \varpi)(1, v)$ is a linear map that vanishes on horizontal vectors and is the identity on V_p by $(*)$. This uniquely determines the linear map, which therefore equals the projection A_p. We find that

$$D_0 \gamma = A_p(\gamma'(0)).$$

This is the reformulation of $(*)$ we were looking for, and it is this definition we use for differentiating sections along vector fields. If we are given a tangent vector $v \in T_x \Sigma$ and a section s, then we want to define $\nabla_v(s)$ to quantify how much the section s deviates from horizontalness in the direction of v. To this end, we can pick a curve γ in Σ such that $\gamma'(0) = v$ and then calculate

$$\begin{aligned} D_0(s \circ \gamma) &= A_{s(x)}((s \circ \gamma)'(0)) \\ &= A_{s(x)}((T_x s)(v)). \end{aligned}$$

Definition 34. *Suppose (E, Σ, π, F) is a vector bundle equipped with a linear connection. Let s be a section of the bundle and let v be a tangent vector to Σ at x. Then we define the* covariant derivative of s along v *to be*

$$\nabla_v(s) = A_{s(x)}((T_x s)(v)) \in E_x.$$

If X is a vector field on Σ, we define the covariant derivative of s along X *to be the section $\nabla_X(s)$ given by*

$$\nabla_X(s)(x) = \nabla_{X_x}(s).$$

The covariant derivative satisfies the following properties for X and Y vector fields on Σ, f and g smooth functions on Σ and s and s' sections of the bundle:

- $\nabla_{fX+gY}(s) = f\nabla_X(s) + g\nabla_Y(s)$,
- $\nabla_X(s + s') = \nabla_X(s) + \nabla_X(s')$,
- $\nabla_X(fs) = f\nabla_X(s) + df(X)s$.

The first of these is immediate from the definition. The second one follows from linearity of D_0 as a map from the vector space of curves lifting $\pi \circ \gamma$ to E_x (and this linearity is clear from $(*)$ on page 22). The third property follows from the definition by a calculation using local coordinates. Every covariant derivative with these properties defines a unique linear connection on E.

2.3 Principal bundles

We now come to the most important type of fibre bundle for our purposes: principal bundles. To get some intuition for these, let us first discuss the notion of a *principal homogeneous space*.

2.3.1 Principal homogeneous spaces

Definition 35. *Let G be a Lie group. A principal homogeneous space for G is a manifold on which G acts smoothly (say from the right), in such a way that the action is both free and transitive.*

This is a technical definition to formalize what is intuitively *a group that has forgotten where its identity is.* Indeed, given a point x of the principal homogeneous space X, we can identify X with G by identifying $x \cdot g$ with g. However, this identification depends on a choice of this point x and there is no canonical way of turning X into a group.

An example is enlightening. Consider $G = \mathrm{U}(1)$, the circle group. Pick any circle $X = \Gamma$ in the plane, which is a smooth manifold. There is no canonical way of turning X into a group. However, there is nice action of $\mathrm{U}(1)$ on Γ: the element $e^{i\theta}$ rotates the circle over an angle θ. This action is free and transitive. What is the difference between $\mathrm{U}(1)$ and Γ? Not much: the essential difference is that G has a "special point", 1. If we pick any point $\gamma \in \Gamma$ and declare this to be special, Γ is converted into a group by the rule

$$(e^{i\theta}\gamma) * (e^{i\theta'}\gamma) = e^{i\theta}e^{i\theta'}\gamma.$$

In a principal homogeneous space, there is no special point (hence homogeneous) and only "differences" (or "quotients") between elements are group elements. A principal homogeneous space is to a group as an affine space is to a vector space.

Note that G is a principal homogeneous space for itself, with the action given by right multiplication.

Definition 36. *Let G be a Lie group and X and Y be two principal homogeneous spaces for G. A* morphism of principal homogeneous spaces *is a smooth map $\phi : X \to Y$ such that $\phi(x \cdot g) = \phi(x) \cdot g$ for all $x \in X$ and $g \in G$.*

Any morphism of principal homogeneous spaces is an isomorphism. This means that the category of principal homogeneous spaces of G is a groupoid. Considering G as a principal homogeneous space for itself with G acting on G by right multiplication, the automorphisms of G as a principal homogeneous space are exactly given by *left* multiplication by some group element. In the next subsection, we will define a principal bundle roughly to be a bundle with principal homogeneous spaces as fibres. It is the observation that automorphisms of G as a principal homogeneous space are given by left multiplication that motivates our demand for G to be the structure group by left multiplication.

2.3.2 Definition

Definition 37. *Let G be a Lie group. A* principal bundle *is a fibre bundle (P, Σ, π, G) with structure group G which acts on the fibre G via left multiplication.*

To emphasize the structure group, one often speaks of a principal G-bundle. Given a principal bundle (P, Σ, π, G), the group G acts on this bundle from the *right* in a natural way, by right multiplication in any trivialization (which is well-defined). This right G-action turns the fibres into principal homogeneous spaces for G.

Definition 38. *A* morphism of principal bundles *is a bundle map that is a morphism of principal homogeneous spaces on each fibre. In other words, if both (P, Σ, π, G) and (P', Σ', π', G) are principal bundles (with equal structure groups), then a bundle map $\phi : P \to P'$ is a morphism of principal bundles iff*

$$\phi(p \cdot g) = \phi(p) \cdot g \quad \textit{for all } p \in P \textit{ and } g \in G.$$

Let us consider some examples of principal bundles.

- The trivial principal bundle $\Sigma \times G$ (with the equivalence class of G-atlases given by the obvious trivialization).
- Let (E, Σ, π, F) be a vector bundle. Let n denote the dimension of the vector space F. Then we associate to this vector bundle E a principal $\mathrm{GL}(n)$-bundle called the *frame bundle*. The fibre over $x \in \Sigma$ of this principal bundle is the set of all ordered bases of E_x, on which $\mathrm{GL}(n)$ acts from the right by change of basis. We will not discuss this in detail. If we take E to be the tangent bundle to Σ, then the manifold Σ is called *parallelizable* iff this associated frame bundle is trivial. In general, this is a strong condition on manifolds, but all Lie groups are parallelizable.
- Consider $\Sigma = S^2$ and take P to be the set of all unit tangent vectors to Σ, with π the projection $P \to \Sigma$. Letting S^1 act on P by rotation (so that $e^{i\theta}$ acts on $v \in P$ by rotating it over an angle θ around the outward normal), we have a principal bundle (P, Σ, π, S^1). Note that this principal bundle has no sections: a section would correspond to a smooth choice of unit tangent vector at every point of the sphere, which does not exist by the famous Hairy Ball Theorem. By the following theorem,

this shows that we have a non-trivial principal bundle. This bundle is called the *Hopf fibration.*

This example can be generalized by replacing S^2 by a closed orientable surface of genus g. The bundle will be trivial precisely if $g = 1$ ([Hir76], theorem 2.2 on page 133 and theorem 2.10 on page 137).

In strong contrast with the situation for vector bundles (which always have sections), we have the following result.

Theorem 39. *A principal bundle is trivial iff it admits a section.*

Proof. It is clear that the trivial bundle admits a section. Let (P, Σ, π, F) be a principal bundle admitting a section $s : \Sigma \to P$. Consider the map

$$\Sigma \times G \to P : (x, g) \mapsto (s(x) \cdot g).$$

This is clearly a bundle map, and it is surjective by transitivity of the group action on the fibres of P. It is injective by freeness of the group action on the fibres of P. It is therefore an isomorphism of bundles. This proves the theorem. $\square$

Let us introduce some more notation that will be useful later on. If $\xi \in \mathfrak{g}$, then we write

$$p \cdot \xi = \frac{d}{dt}_{|t=0} (p \cdot \exp(t\xi)) .$$

Notice that

$$\{p \cdot \xi \mid \xi \in \mathfrak{g}\} = \ker(T_p\pi) = V_p$$

If $v \in T_pP$ and $g \in G$, then we write $v \cdot g$ for the pushforward of v under the action of g.

An interesting result is the following theorem. Its proof requires the use of homotopy theory and classifying spaces, for which we refer to [Hor13] (and to [Hus66] section 4.13 for background on universal bundles).

Theorem 40. *If G is a compact simply connected Lie group and Σ is a compact orientable surface, then every principal G-bundle over Σ is trivial.*

2.3.3 The gauge group

In the study of moduli spaces of flat connections, *gauge transformations* play an important role.

Definition 41. *Let (P, Σ, π, G) be a principal bundle. A* gauge transformation *of this bundle is an automorphism of P covering the identity* Id_Σ.

Very often gauge transformations are encoded as maps $P \to G$ instead of $P \to P$. Indeed, if $\tilde{u} : P \to P$ is a gauge transformation, then we can associate to it a unique smooth map $u : P \to G$ such that

$$\tilde{u}(p) = p \cdot u(p) \quad \text{for all } p \in P$$

because $\tilde{u}$ is fibre-preserving.

On the other hand, if we are given a map $u : P \to G$, we can associate to it the map $\tilde{u} : P \to P : p \mapsto p \cdot u(p)$. This associated map is a bundle map covering the identity, but it need not be a morphism of principal bundles. The condition for $\tilde{u}$ to be a morphism of principal bundles is

$$\tilde{u}(p \cdot g) = \tilde{u}(p) \cdot g \quad \text{for all } p \in P,\ g \in G,$$

which is equivalent to demanding that

$$u(p \cdot g) = g^{-1} u(p) g \quad \text{for all } p \in P,\ g \in G.$$

This shows that the group of automorphisms of P covering the identity is essentially

$$\mathcal{G}(P) = \left\{ u : P \to G \mid u(p \cdot g) = g^{-1} u(p) g \text{ for all } p \in P \text{ and } g \in G \right\}.$$

This set is called the *gauge group* of the principal bundle. Note that we have not specified the operation turning $\mathcal{G}(P)$ into a group. We want this operation to correspond to composition in terms of gauge transformations, so that $\widetilde{uv} = \tilde{u} \circ \tilde{v}$. This requires

$$p \cdot (uv)(p) = \tilde{u}(p \cdot v(p)) = p \cdot v(p) \cdot u(p \cdot v(p)) = p \cdot u(p) \cdot v(p)$$

so that the group operation on $\mathcal{G}(P)$ is just pointwise multiplication.

2.3.4 Principal connections

Just like vector bundles, principal bundles carry a class of connections that are compatible with their extra structure. For vector bundles, we demanded that parallel transport induce morphisms of vector spaces between fibres. For principal bundles, we demand that parallel transport induces morphisms of principal homogeneous spaces. In other words, we want $\mathrm{Pt}_{\gamma,t} : P_{\gamma(0)} \to P_{\gamma(t)}$ to be a morphism of principal homogeneous spaces for all γ and t. A moment's thought shows that this is equivalent to the following definition.

Definition 42. *Suppose (P, Σ, π, F) is a principal bundle equipped with a complete connection. If for any curve γ in Σ with horizontal lift $\tilde{\gamma}$ starting at p, the curve*

$$\tilde{\gamma} \cdot g : t \mapsto \tilde{\gamma}(t) \cdot g$$

is the horizontal lift of γ starting at $p \cdot g$, the connection on P is called a principal connection*.*

Lemma 43. *A complete connection on a principal bundle is principal iff its horizontal subspaces are invariant under the right G-action on the bundle, by which we mean that*

$$H_p \cdot g = H_{p \cdot g} \quad \textit{for all } p \in P \textit{ and } g \in G.$$

Here, $H_p \cdot g$ is the set $\{v \cdot g \mid v \in H_p\}$, recalling that $v \cdot g$ is the pushforward of v under the action of g.

Proof. If the horizontal subspaces are invariant under the G-action, then clearly the connection is principal. To prove the converse, suppose that the connection is principal and pick $p \in P$ and $g \in G$. Taking any $v \in H_p$, it suffices to show that $v \cdot g \in H_{p \cdot g}$, because this shows that $H_p \cdot g \subseteq H_{p \cdot g}$ and repeating the argument for $p \cdot g$ and g^{-1} instead of p and g will then yield the other inclusion.

It thus suffices to show that $v \cdot g \in H_{p \cdot g}$. Pick any horizontal curve through p with initial velocity v (for example, lift a curve in Σ with initial velocity $(T_p\pi)(v)$). Applying g to this curve yields a curve through $p \cdot g$ with initial velocity $v \cdot g$, and it is horizontal because the connection is principal. □

Note that in the definition of principal connection, we demand that the connection be complete. In fact, any connection on a principal bundle whose horizontal subspaces are invariant under the right G-action is automatically complete and thus a principal connection ([KN63], proposition 3.1 on page 69).

If $p \in P$ is a point in a principal bundle, the vertical space V_p can be identified with $\mathfrak{g}$ by identifying $\xi \in \mathfrak{g}$ with $p \cdot \xi$. This allows us to regards the projections $A_p : T_pP \to V_p$ associated to any connection as linear maps

$$A_p : T_pP \to \mathfrak{g}.$$

We shall do this from now on. Specifying a (not necessarily principal) connection on a principal bundle can then be done by specifying a linear map $A_p : T_pP \to \mathfrak{g}$ for every p in a smooth way, such that $A_p(p \cdot \xi) = \xi$ for all $\xi \in \mathfrak{g}$ (the latter condition amounts to demanding that $A_p : T_pP \to V_p$ restricts to the identity on V_p).

When is a connection specified by such A_p principal? We have the following result.

Lemma 44. *Let (P, Σ, π, G) be a principal bundle and suppose that the kernels of the linear maps $A_p : T_pP \to \mathfrak{g}$ specify a connection on it, meaning that A_p depends smoothly on p and $A_p(p \cdot \xi) = \xi$ for all $p \in P$ and $\xi \in \mathfrak{g}$. Then the connection specified by these kernels is principal iff*

$$A_{p \cdot g}(v \cdot g) = \mathrm{Ad}_{g^{-1}}(A_p(v)) \tag{$*$}$$

for all $p \in P$, $g \in G$, $v \in T_pP$.

Proof. The connection is principal precisely if for any $p \in P$ and $g \in G$, the map $A_{p \cdot g} : T_{p \cdot g}P \to \mathfrak{g}$ is the unique linear map that maps $(p \cdot g) \cdot \xi$ to ξ and has kernel $H_p \cdot g$. This unique linear map is in fact the map

$$T_{p \cdot g}P \to \mathfrak{g} : v \mapsto \mathrm{Ad}_{g^{-1}}(A_p(v \cdot g^{-1}))$$

as one verifies by evaluating on horizontal and vertical vectors. We conclude that the connection is principal iff for all $p \in P$, $g \in G$, $v \in T_{p \cdot g}P$ we have

$$A_{p \cdot g}(v) = \mathrm{Ad}_{g^{-1}}(A_p(v \cdot g^{-1})),$$

which is equivalent to $(*)$ by replacing v by $v \cdot g$. $\square$

For principal connections, we will use this specification of a connection by its associated projections A_p frequently. In fact, we will use the name A for the principal connection with projections A_p, and the set of all principal connections on a given principal bundle P will be denoted by $\mathcal{A}(P)$.

If P and Q are two principal bundles over Σ, equipped with connections A and B respectively, then these bundles are called *gauge equivalent* if there is an isomorphism of principal bundles $f : P \to Q$ such that $(T_p f)(H_p^A) = H_{f(p)}^B$ for all $p \in P$ (here, H_p^A denotes the horizontal subspace of T_pP as given by the connection A and $H_{f(p)}^B$ denotes the horizontal subspace of $T_{f(p)}Q$ as given by B). Gauge equivalences are the isomorphisms of principal bundles that are equipped with a connection.

2.3.5 Holonomy in principal bundles

Suppose (P, Σ, π, G) is a principal bundle equipped with a principal connection A. If γ is a loop in Σ, then the holonomy of A around γ is a morphism of principal homogeneous spaces

$$\mathrm{Pt}_\gamma : P_{\gamma(0)} \to P_{\gamma(0)}.$$

For every point $p \in P_{\gamma(0)}$ there is then a unique $g \in G$ such that $\mathrm{Pt}_\gamma(p) = p \cdot g$. This element is written $\mathrm{Hol}_p(A, \gamma)$. In symbols,

$$\mathrm{Pt}_\gamma(p) = p \cdot \mathrm{Hol}_p(A, \gamma) \quad \text{for all } p \in P_{\gamma(0)}.$$

It follows from the fact that Pt_γ is a morphism of principal homogeneous spaces that

$$\mathrm{Hol}_{p \cdot g}(A, \gamma) = g^{-1} \mathrm{Hol}_p(A, \gamma) g. \tag{†}$$

Now suppose that $x, y \in \Sigma$ are in the same component of Σ. Taking a curve δ from x to y and lifting it horizontally at a point $p \in P_x$, we get a curve in P that ends in $q \in P_y$. For any loop γ in Σ at y we then have

$$\mathrm{Hol}_p(\delta \star \gamma \star \bar{\delta}) = \mathrm{Hol}_q(\gamma).$$

In the above equation, the star means concatenation of the curves from left to right. Note that reparametrizing a curve does not influence the holonomy of the connection around it. We conclude that

$$\mathrm{Hol}(A) = \{\mathrm{Hol}_p(A, \gamma) \mid \gamma \text{ is a piecewise smooth curve in } \Sigma\}$$

only depends on the base point p up to conjugation. We can therefore speak of *the holonomy of the connection* A as the conjugacy class of the subgroup $\mathrm{Hol}(A) \subseteq G$.

Chapter 3

Vector-valued differential forms

An ordinary differential k-form on a manifold M is the specification of a linear map $\Lambda^k(T_pM) \to \mathbb{R}$ at every point $p \in M$ depending smoothly on p. In this chapter we extend this notion to differential forms taking values in some vector bundle over M. We then reformulate some of the material in this new terminology, and we discuss the covariant exterior derivative on principal bundles that are equipped with a connection. Our references for this chapter are [Jan05] and [KN63] (section II.5).

3.1 Definitions and properties

Let M be a smooth manifold. The space of differential forms of degree k on M is denoted $\Omega^k(M)$. A differential k-form on M consists of a smooth choice of linear maps $\Lambda^k(T_pM) \to \mathbb{R}$ at every point. We can state this as follows: a differential k-form on M is a morphism of vector bundles from $\Lambda^k(TM)$ to $M \times \mathbb{R}$ covering the identity. This motivates the following definition.

Definition 45. *Let M be a smooth manifold and E a vector bundle over M. An E-*valued differential k-form on M *is a morphism of vector bundles from $\Lambda^k(TM)$ to E covering the identity* Id_M.

The space of E-valued k-forms over M will be denoted by $\Omega^k(M, E)$. If V is a fixed finite-dimensional vector space, we can speak of V-valued differential forms on M, in which case we mean differential forms with values in the trivial bundle $M \times V$. For example, if A is a principal connection on a principal bundle, the projections $A_p : T_pP \to \mathfrak{g}$ specify a $\mathfrak{g}$-valued 1-form on P, the so-called *connection 1-form*.

3.1.1 Pullback

Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Suppose E is a vector bundle over N and α is an E-valued k-form on N. Then we can pull back α to M via f

to get $f^*\alpha$. It is an (f^*E)-valued form on M. The pullback form is defined by

$$(f^*\alpha)_x(v_1, \dots, v_k) = (x, \alpha(T_x f(v_1), \dots, T_x f(v_k))) \in (f^*E)_x.$$

The gauge group of a principal bundle acts on a connection on this bundle by pullback of the connection 1-form.

3.1.2 Wedge product

Let M be a smooth manifold and D, E vector bundles over M. Assume that α is a D-valued k-form over M and β is an E-valued l-form over M. We then define their *wedge product* in a similar way as the wedge product of ordinary differential forms, but we use the tensor product as the bilinear operation. The wedge product $\alpha \wedge \beta$ is a $D \otimes E$-valued $(k+l)$-form on M given by

$$\begin{aligned}(\alpha \wedge \beta)_x&(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \otimes \beta_x(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).\end{aligned}$$

for all $x \in M$ and $v_i \in T_x M$. Calculation shows that this product is associative (just like the ordinary wedge product). The wedge product is compatible with pullbacks in the sense that

$$(f^*\alpha) \wedge (f^*\beta) = f^*(\alpha \wedge \beta).$$

We will often be concerned with differential forms on a principal G-bundle that take values in the Lie algebra $\mathfrak{g}$. Suppose α, β are $\mathfrak{g}$-valued forms on a smooth manifold, of degree k and l respectively. We can then form a new $\mathfrak{g}$-valued form $[\alpha \wedge \beta]$ given by

$$\begin{aligned}[\alpha \wedge \beta]_x&(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) [\alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \beta_x(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})].\end{aligned}$$

Note that $[\alpha \wedge \beta]$ is essentially the composition of $\alpha \wedge \beta$ with the bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. From the definition one checks that

$$[\alpha \wedge \beta] = (-1)^{kl+1} [\beta \wedge \alpha].$$

In particular, if k is even, we have $[\alpha \wedge \alpha] = 0$. There is also an analogue of the Jacobi identity for this product. If γ is a $\mathfrak{g}$-valued j-form, then

$$(-1)^{kj} [\alpha \wedge [\beta \wedge \gamma]] + (-1)^{kl} [\beta \wedge [\gamma \wedge \alpha]] + (-1)^{lj} [\gamma \wedge [\alpha \wedge \beta]] = 0.$$

An analogous construction works for a symmetric bilinear form on any vector space. Suppose $\langle -, - \rangle$ is some fixed symmetric bilinear form on some finite-dimensional real vector space V, and let α and β be V-valued forms on a smooth manifold of degree k and

l respectively. We can then form an ordinary $(k+l)$-form $\langle \alpha \wedge \beta \rangle$ on this manifold by defining

$$\begin{aligned}&\langle \alpha \wedge \beta \rangle_x(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l})\\ &\qquad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \langle \alpha_x(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \beta_x(v_{\sigma(k+1)}, v_{\sigma(k+l)}) \rangle.\end{aligned}$$

We have the identity

$$\langle \alpha \wedge \beta \rangle = (-1)^{kl} \langle \beta \wedge \alpha \rangle.$$

3.1.3 Exterior derivative

We want to define an exterior derivative for vector-valued forms. Here, we shall only do so for forms taking values in a trivial vector bundle. Suppose V is a finite-dimensional real vector space and M is a smooth manifold. Let α be a V-valued k-form on M. Then we define the exterior derivative $d\alpha$ by the usual formula

$$\begin{aligned}(d\alpha)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\alpha(X_0, \ldots, X_{i-1}, \hat{X}_i, X_{i+1}, \ldots, X_k))\\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned}$$

The usual properties

$$d(d\alpha) = 0 \quad d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$$

still hold for any k-form α and any l-form β. This exterior derivative commutes with pullbacks. In section 3.3 we will discuss exterior derivatives in greater detail.

If G is a Lie group, there is an important $\mathfrak{g}$-valued 1-form on G that we now briefly discuss.

Definition 46. *If G is a Lie group, the $\mathfrak{g}$-valued 1-form on G given by*

$$\theta : TG \to \mathfrak{g} : v \in T_gG \mapsto (T_g(\lambda_{g^{-1}}))(v) \in T_eG = \mathfrak{g}$$

is called the Maurer-Cartan form *on G.*

In words, the Maurer-Cartan form identifies every tangent space T_gG with $T_eG = \mathfrak{g}$ through left translation. To illustrate the exterior derivative of a differential form with values in a fixed vector space (and because we will need it later), we prove the *Maurer-Cartan structure equation.*

Lemma 47 (Maurer-Cartan structure equation)**.** *If G is a Lie group with Maurer-Cartan form θ, then for any two vectors $v, w \in T_gG$ we have*

$$(d\theta)_g(v, w) = -[\theta_g(v), \theta_g(w)].$$

This equation is often written as $d\theta + \frac{1}{2}[\theta, \theta] = 0$.

Proof. Extend v and w to left-invariant vector fields V and W on G (these are the left-invariant vector fields associated to $\theta(v)$ and $\theta(w)$). Then $\theta(V)$ and $\theta(W)$ are constant on G. We find

$$\begin{aligned}(d\theta)_g(v,w) &= (d\theta)_g(V,W)\\ &= V(\theta(W)) - W(\theta(V)) - \theta_g([V,W])\\ &= -\theta_g([V,W]).\end{aligned}$$

By the definition of the Lie bracket on $\mathfrak{g}$, the latter equals $-[\theta_g(V),\theta_g(W)]$ and the lemma is proved. $\square$

The following result is useful for differentiation on Lie groups. We leave the proof to the reader.

Lemma 48. *Let G be a Lie group and $\alpha : (-\varepsilon,\varepsilon) \to G$ and $\beta : (-\varepsilon,\varepsilon) \to G$ two curves in G. Then*

- $\theta\left(\frac{d}{dt}_{|t=0}(\alpha(t)\beta(t))\right) = \mathrm{Ad}_{\beta(0)^{-1}}\left(\theta\left(\frac{d}{dt}_{|t=0}\alpha(t)\right)\right) + \theta\left(\frac{d}{dt}_{|t=0}\beta(t)\right)$,
- $\theta\left(\frac{d}{dt}_{|t=0}\alpha(t)^{-1}\right) = -\mathrm{Ad}_{\alpha(0)}\left(\theta\left(\frac{d}{dt}_{|t=0}\alpha(t)\right)\right)$.

3.2 Associated bundles

Suppose (P,Σ,π,G) is a principal bundle and V is a finite-dimensional vector space. Assume $f : G \to \mathrm{GL}(V)$ is a representation of G. We can then construct the *associated bundle* $P \times_f V$, which is a vector bundle over Σ with fibre V. This associated bundle is

$$P \times_f V = (P \times V)/\sim$$

where $\sim$ is the equivalence relation $(p\cdot g, v) \sim (p, f(g)(v))$. The corresponding projection is

$$\tau : P \times_f V \to \Sigma : [p,v] \mapsto \pi(p).$$

We now consider an important special case.

Definition 49. *Given a principal bundle (P,Σ,π,G), we define the* adjoint bundle $\mathfrak{g}_P$ *to be $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$.*

Note that as a set, this bundle is the quotient of $P \times \mathfrak{g}$ under the equivalence $(p\cdot g,\xi) \sim (p, \mathrm{Ad}_g(\xi))$.

Suppose now that we have given a connection on the principal bundle. We will be interested in $\mathfrak{g}$-valued differential forms on this bundle P that are both horizontal and equivariant.

Definition 50. *Let (P, Σ, π, G) be a principal bundle. A $\mathfrak{g}$-valued k-form α on P is called* horizontal *if at every point $p \in P$ the map α_p returns zero whenever one of its arguments is vertical. It is called* equivariant *if for all $g \in G$, $p \in P$ and vectors $v_1, \dots, v_k \in T_pP$ we have*

$$\alpha_{p\cdot g}(v_1 \cdot g, \dots, v_k \cdot g) = \mathrm{Ad}_{g^{-1}}(\alpha_p(v_1, \dots, v_k)).$$

As an example, connection 1-forms are equivariant. The set of $\mathfrak{g}$-valued horizontal, equivariant k-forms on P will be denoted by $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$. The space $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$ can be identified with the space $\Omega^k(\Sigma, \mathfrak{g}_P)$ of $\mathfrak{g}_P$-valued forms on Σ as we now explain. Suppose $\alpha \in \Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$. Then we can create $\beta \in \Omega^k(\Sigma, \mathfrak{g}_P)$ as follows: given $x \in \Sigma$ and vectors $v_1, \dots, v_k \in T_x\Sigma$, we define

$$\beta_x(v_1, \dots, v_k) = [p, \alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)] \in (\mathfrak{g}_P)_x \tag{$*$}$$

where p is any point in the fibre over x and $\tilde{v}_i$ is the horizontal lift of v_i. This is well-defined by equivariance of α. Conversely, if $\beta \in \Omega^k(\Sigma, \mathfrak{g}_P)$, then we can construct $\alpha \in \Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$ by demanding $(*)$. From now on, we will identify $\Omega^k(\Sigma, \mathfrak{g}_P)$ and $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$.

3.3 The covariant exterior derivative

In this section we discuss the *covariant exterior derivative*, also known as the exterior covariant derivative. Our goal is to find an appropriate way of differentiating an element of $\Omega^k(\Sigma, \mathfrak{g}_P)$ to obtain an element of $\Omega^{k+1}(\Sigma, \mathfrak{g}_P)$.

Let us pick $\alpha \in \Omega^k(\Sigma, \mathfrak{g}_P)$, regarded as a horizontal and equivariant $\mathfrak{g}$-valued k-form on P. The obvious candidate for a derivative of α is $d\alpha$, which is indeed an equivariant $\mathfrak{g}$-valued $(k+1)$-form on P. It need not be horizontal, however. We will try to define the covariant exterior derivative $d_A\alpha$ as the "horizontal version" of $d\alpha$.

Given a principal connection A on a principal bundle P, there is a bundle map

$$h : TP \to TP$$

covering the identity $P \to P$ that sends a vector to its horizontal part (in fact, this is a bundle map from TP to its subbundle H of horizontal subspaces). It induces a bundle map

$$h_k : \Lambda^k(TP) \to \Lambda^k(TP)$$

for every k by setting $h(\alpha \wedge \beta) = h(\alpha) \wedge h(\beta)$. These are bundle maps covering the identity, and by abuse of notation we will just write $h = h_k$.

If E is a vector bundle over P, then an E-valued k-form over P is a bundle map $\Lambda^k(TP) \to E$ covering the identity. We can compose such a form with h (applying h first) to obtain a new differential form over P. This new form is clearly horizontal. We are thus led to define d_A as follows.

Definition 51. *Given a principal bundle* (P, Σ, π, G) *equipped with a connection* A*, we define the* covariant exterior derivative d_A *as*

$$d_A : \Omega^k(\Sigma, \mathfrak{g}_P) \to \Omega^{k+1}(\Sigma, \mathfrak{g}_P) : \alpha \mapsto d\alpha \circ h.$$

In words, to apply $d_A\alpha$ to a set of vectors $v_1, \dots, v_{k+1} \in T_pP$, we first take the horizontal parts $h(v_i)$ of these vectors and then apply $d\alpha$ to the resulting horizontal vectors. The covariant exterior derivative can also be defined by a more hands-on formula.

Theorem 52. *For* $\alpha \in \Omega^k(\Sigma, \mathfrak{g}_P)$ *we have*

$$d_A\alpha = d\alpha + [A \wedge \alpha]. \qquad (*)$$

Here, we consider the connection A *as a* $\mathfrak{g}$*-valued 1-form on* P *and* α *as an equivariant horizontal* $\mathfrak{g}$*-valued* k*-form on* P*.*

Proof. We will check the equality $(*)$ by applying both sides to vectors. By linearity, it suffices to check this on a set of vectors $v_0, \dots, v_k$ all of which are either horizontal or vertical. We may assume that the first m of these are vertical and the others are horizontal.

The case $m = 0$. In this case all the vectors are horizontal and

$$(d_A\alpha)(v_0, \dots, v_k) = (d\alpha)(v_0, \dots, v_k).$$

The right hand side of $(*)$ yields the same result because

$$[A \wedge \alpha](v_0, \dots, v_k) = \sum_j (-1)^j [A(v_j), \alpha(v_0, \dots, \hat{v}_j, \dots, v_k)]$$

and all the factors $A(v_j)$ are zero. This means the term $[A \wedge \alpha]$ will not contribute in $(*)$.

The case $m \geq 1$. We want to apply the right hand side of $(*)$ to

$$p \cdot \xi_0 = v_0, \dots, p \cdot \xi_{m-1} = v_{m-1}, v_m, \dots, v_k$$

where the vectors $v_m, \dots, v_k$ are horizontal (and all of the above are elements op T_pP). Since $m \geq 1$ we expect to find zero. We may assume that the $v_m, \dots, v_k$ are linearly independent (and by horizontalness then so are their projections onto Σ).

Because we want to use the coordinate-free definition of the exterior derivative, we will extend the vectors v_i to local vector fields around p. We extend the vectors $p \cdot \xi_i$ by taking $q \cdot \xi_i$ at every point $q \in P$ (these are in fact global extensions). We extend the vectors v_i ($i \geq m$) by extending their projection onto Σ locally in a mutually commuting way and lifting these vector fields to horizontal vector fields around p.

We can now calculate:

$$\begin{aligned}
&(d\alpha)_p(p \cdot \xi_0, \dots, p \cdot \xi_{m-1}, v_m, \dots, v_k) + [A \wedge \alpha](p \cdot \xi_0, \dots, p \cdot \xi_{m-1}, v_m, \dots, v_k) \\
&\qquad = \sum_j (-1)^j v_j(\alpha(v_0, \dots, \hat{v}_j, \dots, v_k)) \\
&\qquad\quad + \sum_{i<j} (-1)^{i+j} \alpha([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k) \\
&\qquad\quad + \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma)[A(v_{\sigma(0)}), \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})].
\end{aligned}$$

If $m \geq 2$, then every term above contains a factor of α with at least one vertical vector among its arguments. Because α is horizontal, the result is zero in this case. All that remains is the case $m = 1$.

We inspect the three terms above for the case $m = 1$. The second term will not contribute: either v_0 (the only vertical vector) appears in the Lie brackets, in which case the bracket is zero because v_0 commutes with all the other fields (as is easily verified, keeping in mind that the horizontal spaces are G-invariant), or v_0 appears as an argument to α which results in zero because α is horizontal.

The first term has only one contribution: the case where the vertical vector is not supplied to α (and is hence upfront). The third term only has a contribution in the case where the vertical vector is not supplied to α. After noticing that the $k!$ compensates for the repeated terms in the third term, what remains is

$$\begin{aligned}(d_A\alpha)_p(p \cdot \xi_0, v_1, \ldots, v_k) &= (p \cdot \xi_0)(\alpha(v_1, \ldots, v_k)) + [A(p \cdot \xi_0), \alpha(v_1, \ldots, v_k)] \\ &= (p \cdot \xi_0)(\alpha(v_1, \ldots, v_k)) + [\xi_0, \alpha(v_1, \ldots, v_k)].\end{aligned}$$

Now notice that $v_i(p \cdot g) = v_i(p) \cdot g$ by the way we extended the v_i. We get

$$\begin{aligned}(d_A\alpha)_p(p \cdot \xi_0, v_1, \ldots, v_k) &= \frac{d}{dt}_{|t=0} (\alpha(v_1(p) \cdot \exp(t\xi_0), \ldots, v_k(p) \cdot \exp(t\xi_0))) \\ &\quad + [\xi_0, \alpha(v_1, \ldots, v_k)] \\ &= \frac{d}{dt}_{|t=0} \mathrm{Ad}_{\exp(-t\xi_0)}(\alpha(v_1, \ldots, v_k)) + [\xi_0, \alpha(v_1, \ldots, v_k)] \\ &= \mathrm{ad}_{-\xi_0}(\alpha(v_1, \ldots, v_k)) + [\xi_0, \alpha(v_1, \ldots, v_k)] \\ &= 0.\end{aligned}$$

This shows that the formula $(*)$ is correct. $\square$

3.4 Curvature

We now have a suitable notion of exterior derivative for $\Omega^k(\Sigma, \mathfrak{g}_P)$. One may wonder whether the important relation $d^2 = 0$ still holds for this new exterior derivative d_A, so whether $d_A^2 = 0$. Suppose that $\alpha \in \Omega^k(\Sigma, \mathfrak{g}_P)$. Then

$$\begin{aligned}d_A d_A \alpha &= d_A(d\alpha + [A \wedge \alpha]) \\ &= d^2\alpha + d[A \wedge \alpha] + [A \wedge d\alpha] + [A \wedge [A \wedge \alpha]] \\ &= [dA \wedge \alpha] + \frac{1}{2}[[A \wedge A] \wedge \alpha] \\ &= [(dA + \frac{1}{2}[A \wedge A]) \wedge \alpha].\end{aligned}$$

This motivates the following definition.

Definition 53. *Let A be a connection on a principal bundle. The two-form*

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

is called the curvature *of the connection A. A connection is called* flat *if its curvature vanishes everywhere on the principal bundle. Given a principal bundle, the set of all flat connections on it is denoted by $\mathcal{A}_0$. A* flat principal bundle *is a tuple (P, A) where P is a principal bundle and A a flat connection on it.*

We will often refer to the flat principal bundle (P, A) as P, keeping in mind that it is equipped with a certain flat connection. It is easy to verify that

$$F_A = dA \circ h$$

where h is the projection onto the horizontal subbundle, which is analogous to the definition of the covariant exterior derivative for $\Omega^k(\Sigma, \mathfrak{g}_P)$.

There is another very important motivation for considering the curvature of a connection. Let us describe it briefly. We first state the import Frobenius theorem. For a proof, we refer to [Lee03], page 359 (theorem 14.5).

Theorem 54 (Frobenius)**.** *Let H be a rank k subbundle of the vector bundle TP. Then the following conditions are equivalent:*

- *for every two local sections X and Y of H (local vector fields on P with $X_p, Y_p \in H_p$ for every p), the Lie bracket is also a local section of H;*
- *around every point p in P there exists a chart such that H is spanned by the first k coordinate vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$.*

If either (and hence both) of these conditions is satisfied, the bundle H is called integrable*.*

Now let X and Y be local horizontal vector fields around a point $p \in P$. Consider the following calculation:

$$\begin{aligned} F_A(X_p, Y_p) &= (dA)(X_p, Y_p) + \frac{1}{2}[A \wedge A](X_p, Y_p) \\ &= (dA)(X_p, Y_p) + [A(X_p), A(Y_p)] \\ &= (dA)(X_p, Y_p) \\ &= X_p(A(Y)) - Y_p(A(X)) - A([X, Y]_p) \\ &= -A([X, Y]_p). \end{aligned}$$

This calculation shows that flatness of the connection is equivalent to the first condition mentioned in the Frobenius theorem, and hence to the second. In other words, flatness of the connection is precisely integrability of the horizontal subbundle. In other words, a connection is flat if its horizontal subspaces line up to be tangent to the leaves of a foliation.

Chapter 4

Symplectic and Poisson structures

Symplectic geometry is the branch of geometry concerned with manifolds that are equipped with a closed and non-degenerate two-form. Poisson geometry is concerned with the study of manifolds that are equipped with a Poisson structure, and is in a sense a generalization of symplectic geometry. We discuss some of the basic ideas. Our main references are [Hec12] (for symplectic manifolds) and [Wat07] (for Poisson manifolds).

4.1 Symplectic manifolds

Definition 55. *A* symplectic structure *on a smooth manifold M is a closed and non-degenerate two-form on M.*

Definition 56. *A* symplectic manifold *is a smooth manifold equipped with a symplectic structure.*

Definition 57. *A map $\phi : M \to N$ between symplectic manifolds (M, ω) and (N, ν) is a* morphism of symplectic manifolds *(also called a* symplectic map *or a* symplectomorphism*) if $\phi^*\nu = \omega$.*

A non-degenerate two-form on a manifold M gives an isomorphism between T_p^*M and T_pM at each point $p \in M$. We will denote this isomorphism by

$$\sharp : T_p^*M \to T_pM : v \mapsto v^\sharp$$

and its inverse by

$$\flat : T_pM \to T_p^*M : v \mapsto v^\flat.$$

These isomorphisms are completely determined by the condition

$$v^\flat(w) = \omega(v, w) \quad \text{for all } v, w \in T_pM.$$

A different way to write this is

$$v^\flat = i_v\omega$$

where i denotes the interior product.

To any smooth function $f \in \mathcal{F}(M)$ we can associate a vector field X_f by

$$X_f = (df)^\sharp.$$

Definition 58. *A vector field X on M is called* Hamiltonian *if it equals X_f for some smooth function f. The flow of this vector field is then called the* (Hamiltonian) flow *of f.*

Definition 59. *A vector field on M is called* symplectic *if its flow preserves the symplectic structure. In other words, a vector field X on M is symplectic iff*

$$\mathcal{L}_X \omega = 0.$$

Lemma 60. *Every Hamiltonian vector field is symplectic.*

Proof. Let $f \in \mathcal{F}(M)$. Using Cartan's formula, we have

$$\begin{aligned}
\mathcal{L}_{X_f}(\omega) &= (di_{X_f} + i_{X_f}d)(\omega) \\
&= d(i_{X_f}\omega) \\
&= d(X_f^\flat) \\
&= d(df) \\
&= 0.
\end{aligned}$$

□

We can also find an expression for the extent to which the flow of f preserves $g \in \mathcal{F}(M)$:

$$\begin{aligned}
X_f(g) &= (dg)(X_f) \\
&= (dg)((df)^\sharp) \\
&= \omega((dg)^\sharp, (df)^\sharp) \\
&= -\omega(X_f, X_g).
\end{aligned}$$

In particular, the function f is constant along its own flow. In classical mechanics, this is a very important observation: the evolution of a system will be given by the Hamiltonian flow of a certain function H (called the Hamiltonian), which is in many contexts the total energy of the system. The observation that H is constant along its own flow is just the law of *conservation of energy.*

4.2 Symplectic reduction

In classical mechanics, many physical systems can be described as a tuple (M, ω, H), called a *Hamiltonian system*, where (M, ω) is a symplectic manifold and $H \in \mathcal{F}(M)$ is a function called the Hamiltonian. The manifold M is the phase space of the system, and the dynamics of the system (how the system evolves through time) are given by the flow of the Hamiltonian vector field X_H. In many cases, the physical system will exhibit some kind of symmetry, in the sense that a Lie group acts on the symplectic manifold by symplectomorphisms which leave H invariant. Under certain technical assumptions, the symmetry of the physical system will lead to conserved quantities and to symplectic reduction, in which the dynamics of the original system are reduced to those of a Hamiltonian system of lower dimension. Let us discuss symplectic reduction.

4.2.1 Moment maps

Suppose (M, ω) is a symplectic manifold and G is a Lie group that acts smoothly on M from the left (in the sense that the map $G \times M \to M$ is smooth). We can then associate to any $\xi \in \mathfrak{g}$ a vector field X_ξ on M by setting

$$X_\xi(p) = \frac{d}{dt}_{|t=0} (\exp(t\xi) \cdot p) .$$

Definition 61. *A smooth action $G \times M \to M$ of a Lie group on the symplectic manifold M is called* Hamiltonian *if the vector fields X_ξ are Hamiltonian for all $\xi \in \mathfrak{g}$.*

Suppose now that G acts on (M, ω) in a Hamiltonian fashion. To any $\xi \in \mathfrak{g}$ we can then associate the smooth function f such that $X_\xi = X_f$. This choice of f is not unique, but it is unique up to a locally constant function. We get a map

$$\nu : \mathfrak{g} \to \mathcal{F}(M) : \xi \mapsto (f \text{ such that } X_f = X_\xi)$$

that can be made linear by appropriate choices for the functions f. Indeed, pick a basis $\{\xi_1, \ldots, \xi_n\}$ for $\mathfrak{g}$ and pick arbitrary smooth functions $f_1, \ldots, f_n$ such that $X_{\xi_i} = X_{f_i}$ for $i \in \{1, \ldots, n\}$. Define ν by $\nu(\xi_i) = f_i$ and extending linearly.

Note that we can view M as a subset of the dual of $\mathcal{F}(M)$ by identifying a point $p \in M$ with the map

$$\mathrm{eval}_p : \mathcal{F}(M) \to \mathbb{R} : f \mapsto f(p).$$

Let us now take the dual of the linear map ν and restrict it to $M \subset \mathcal{F}(M)^*$. The result is a map $\mu : M \to \mathfrak{g}^*$, and it is an example of a so-called *moment map*.

Definition 62. *Let G act on (M, ω) in a Hamiltonian fashion. A linear map*

$$\mu : M \to \mathfrak{g}^*$$

is called a moment map *for this action if for any $\xi \in \mathfrak{g}$ the Hamiltonian vector field of the function*

$$\mu_\xi : M \to \mathbb{R} : p \mapsto \mu(p)(\xi)$$

is equal to X_ξ.

The map μ_ξ is called the *ξ-component of μ*.

Definition 63. *Let G act on (M, ω) in a Hamiltonian fashion, with moment map $\mu : M \to \mathfrak{g}^*$. The moment map is called* equivariant *if*

$$\mu(g \cdot p) = \mathrm{Ad}^*_g(\mu(p)) \quad \textit{for all } g \in G,\, p \in M.$$

In this case, (M, ω, μ) is called a Hamiltonian G-space.

4.2.2 Symplectic quotients

If (M, ω, μ) is a Hamiltonian G-space, then the inverse image $N = \mu^{-1}(0)$ is invariant under G. Assume that 0 is a regular value of μ. Then N is a submanifold of M. Write $i : N \hookrightarrow M$ for the inclusion of N into M. We can restrict ω to a two-form $i^*\omega$ on N. This two-form will be closed (recall that pullback and exterior derivative commute). However, in general it need not be non-degenerate (and indeed, N could even be odd-dimensional, forcing $i^*\omega$ to be degenerate everywhere).

Let us try to understand what prevents $i^*\omega$ from being a symplectic form. To make this more precise, at every point $p \in N$ the two-form $i^*\omega$ gives a map $T_pN \to T_p^*N$. Non-degeneracy of the two-form is just injectivity of this map, so we shall try to find its kernel.

Let us first describe T_pN. Note that $T_pN = \ker(T_p\mu)$. Take $\xi \in \mathfrak{g}$. By the definition of a moment map, we have that

$$X_\xi = (d\mu_\xi)^\sharp.$$

This implies

$$\omega_p(v, (X_\xi)_p) = \omega_p(v, ((d\mu_\xi)^\sharp)_p) = -(d\mu_\xi)_p(v) = -\langle (T_p\mu)(v), \xi \rangle$$

for any vector $v \in T_pM$. This shows that the kernel of $T_p\mu$ is exactly $\{(X_\xi)_p \mid \xi \in \mathfrak{g}\}^\omega$ where the superscript ω indicates the orthogonal complement with respect to ω, so

$$T_pN = \{(X_\xi)_p \mid \xi \in \mathfrak{g}\}^\omega.$$

We are now in a position to find the kernel of the map $T_pN \to T_p^*N$: it is just

$$T_pN \cap (T_pN)^\omega = \{(X_\xi)_p \mid \xi \in \mathfrak{g}\}^\omega \cap \{(X_\xi)_p \mid \xi \in \mathfrak{g}\}.$$

Take $\xi, \eta \in \mathfrak{g}$ and note that

$$\omega_p(X_\xi, X_\eta) = -\langle (T_p\mu)(X_\eta), \xi \rangle.$$

Now $(T_p\mu)(X_\eta) = 0$ because it is just $\frac{d}{dt}_{|t=0}(\mu(\exp(t\eta \cdot p))$ and μ is identically zero on N. We conclude that

$$\ker(T_pN \to T_p^*N) = \{(X_\xi)_p \mid \xi \in \mathfrak{g}\}.$$

Suppose now that G is compact and acts freely. We can then apply the following result about group actions. A proof can be found in [Lee03] on page 153 (theorem 7.10).

Lemma 64. *If a compact Lie group G acts freely on a manifold N, then the quotient space N/G has a natural manifold structure such that $\pi : N \to N/G$ is a smooth submersion.*

In this case, we see that the kernel of $T_pN \to T_p^*N$ is exactly the kernel of $d_p\pi$ with π the projection $N \to N/G$ as in the lemma. This means that N/G carries a symplectic structure ω_0 defined by the equation

$$\pi^*\omega_0 = i^*\omega.$$

The space $(N/G, \omega_0)$ is called the *symplectic quotient* of M by the action of G. It is a symplectic manifold of dimension $\dim(M) - 2\dim(G)$.

4.3 Poisson manifolds

We now introduce Poisson manifolds. We explain why every symplectic manifold can be seen as a Poisson manifold, and we show that a general Poisson manifold is a foliation-like collection of symplectic leaves (though different leaves can have different dimensions).

4.3.1 Poisson structures

Definition 65. *Let A be an algebra. A* Poisson structure *on A is a bilinear map*

$$\{\cdot,\cdot\} : A \times A \to A : (f, g) \mapsto \{f, g\}$$

that is a Lie bracket on the vector space A and that satisfies the Leibniz rule

$$\{f, gh\} = g\{f, h\} + \{f, g\}h$$

for all $f, g, h \in A$. If the algebra A is equipped with a Poisson structure, it is called a Poisson algebra.

Definition 66. *A* Poisson structure *(or* Poisson bracket*) on a smooth manifold M is a Poisson structure on the algebra $\mathcal{F}(M)$ of smooth functions on M.*

Definition 67. *A* Poisson manifold *is a smooth manifold equipped with a Poisson structure.*

Definition 68. *Let M and N be two Poisson manifolds. A smooth map $\phi : M \to N$ is a* morphism of Poisson manifolds *(also called a* Poisson map*) if for all $f, g \in \mathcal{F}(N)$ we have*

$$\{f \circ \phi, g \circ \phi\} = \{f, g\} \circ \phi.$$

Suppose M is a Poisson manifold. By the Leibniz rule, for any fixed $f \in \mathcal{F}(M)$ the map $g \mapsto -\{f, g\}$ is a derivation of $\mathcal{F}(M)$. It is known from differential geometry that every derivation of $\mathcal{F}(M)$ is in fact a vector field. This means that we can associate to f a vector field X_f corresponding to the derivation, i.e. the vector field satisfying

$$\{f, g\} = -X_f(g) \quad \text{for all } g \in \mathcal{F}(M).$$

This vector field X_f is called the *Hamiltonian vector field* associated to f.

Note that the calculation

$$\{f, g\}(p) = -(X_f)_p(g) = -(d_p g)(X_f) = (X_g)_p(f) = (d_p f)(X_g)$$

shows that the Poisson bracket of f and g at p only depends on their differentials $d_p f$ and $d_p g$. This means that a Poisson structure gives rise to a linear mapping

$$T_p^* M \to T_p M : d_p f \mapsto X_f$$

at every point of the manifold. The rank of this map will be called the *rank of the Poisson manifold at the point p.*

4.3.2 Symplectic manifolds are Poisson

Let (M, ω) be a symplectic manifold. We will show that it is also a Poisson manifold. Consider

$$\{\cdot, \cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M) : (f, g) \mapsto \{f, g\} = \omega(X_f, X_g) = -X_f(g).$$

We claim that this map is a Poisson structure on M. Bilinearity and antisymmetry of the bracket follow from bilinearity and antisymmetry of ω.

The Jacobi identity for this bracket follows immediately from the following lemma. Its proof is based on page 67 of [Hec12].

Lemma 69. *Let M be a smooth manifold and ω a non-degenerate two-form on it. Use this two-form to associate to any smooth function f a vector field $X_f = (df)^\sharp$. Define a bracket on M by setting $\{f, g\} = \omega(X_f, X_g)$. Then this bracket satisfies the Jacobi identity iff ω is closed.*

Proof. First note that the Jacobi identity for the bracket is equivalent to

$$[X_f, X_g] = -X_{\{f,g\}} \quad \text{for all } f, g \in \mathcal{F}(M).$$

Indeed, taking f and g in $\mathcal{F}(M)$ we have $[X_f, X_g] = -X_{\{f,g\}}$ exactly if $[X_f, X_g](h) = -X_{\{f,g\}}(h)$ for all $h \in \mathcal{F}(M)$. The latter is equivalent to $X_f(X_g(h)) - X_g(X_f(h)) = \{\{f, g\}, h\}$, or in other words $\{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\}$, which is just the Jacobi identity.

Note that $\{f, g\} = \omega(X_f, X_g) = -X_f(g) = -\mathcal{L}_{X_f}(g)$. Consider now the calculation

$$d\{f, g\} = -d(\mathcal{L}_{X_f}(g)) = -\mathcal{L}_{X_f}(dg) = -\mathcal{L}_{X_f}(i_{X_g}\omega) = -i_{[X_f,X_g]}\omega - i_{X_g}(\mathcal{L}_{X_f}\omega)$$

where we have used that $\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}$. This shows that $[X_f, X_g] = -X_{\{f,g\}}$ for all f, g if and only if $\mathcal{L}_{X_f}\omega = 0$ for all f. This is equivalent to $d\omega = 0$ since $\mathcal{L}_{X_f}\omega = i_{X_f}(d\omega)$. $\square$

We now know that every symplectic manifold is also Poisson. If f is a smooth function on a symplectic manifold M, we have two notions of *Hamiltonian vector field associated to* f: one for M as a symplectic manifold, one for M as a Poisson manifold. By their definitions, these coincide. Note that the linear map $T_p^*M \to T_pM$ induced by the Poisson structure on a symplectic manifold is just the identification between T_p^*M and T_pM using the symplectic structure (and in the case of a symplectic manifold, this is an isomorphism because the symplectic structure is non-degenerate).

The converse is also true: if the map $T_p^*M \to T_pM$ is an isomorphism everywhere, then the manifold is symplectic.

Theorem 70. *Suppose M is a Poisson manifold that has full rank everywhere. Then M is a symplectic manifold.*

Proof. The symplectic structure on M can be reconstructed by inverting the map $T_p^*M \to T_pM$ to get a map $T_pM \to T_p^*M$ (which we then view as a bilinear map from $T_pM \times T_pM$ to $\mathbb{R}$). This bilinear map is then a symplectic structure: antisymmetry follows easily from antisymmetry of the Poisson bracket, non-degeneracy follows immediately from the assumption of full rank and closedness follows from the Jacobi identity for the Poisson bracket according to the previous lemma. □

4.3.3 The splitting theorem and symplectic leaves

Some Poisson manifolds are not symplectic, and we have seen that the obstruction to being symplectic is degeneracy of the map $T_p^*M \to T_pM$. We now prove a result originally by Weinstein about the local structure of Poisson manifolds: near a point p, a Poisson manifold is the product of a symplectic manifold and a Poisson manifold of rank zero at p. The original reference for this result is [Wei83].

Definition 71. *Given two Poisson manifolds M and N, we define their product to be the smooth manifold $M \times N$ equipped with the bracket such that the two projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Poisson maps, and such that*

$$\{f \circ \pi_M, g \circ \pi_N\} = 0 \quad \textit{for all } f \in \mathcal{F}(M) \textit{ and } g \in \mathcal{F}(N).$$

This product Poisson manifold admits a more conceptual description if we look at the maps

$$\pi_M^* : \mathcal{F}(M) \to \mathcal{F}(M \times N) : f \mapsto f \circ \pi_M$$

and

$$\pi_N^* : \mathcal{F}(N) \to \mathcal{F}(M \times N) : g \mapsto g \circ \pi_N.$$

It is just the smooth manifold $M \times N$ equipped with the bracket such that π_M^* and π_N^* are bijective Lie algebra homomorphisms onto commuting subalgebras of $\mathcal{F}(M \times N)$. One verifies that this determines the bracket on $M \times N$ completely.

If we take local coordinates x_i on M and y_i on N, then the product manifold is defined by

$$\{x_i \circ \pi_M, y_j \circ \pi_N\} = 0$$

and

$$\{x_i \circ \pi_M, x_j \circ \pi_M\} = \{x_i, x_j\} \circ \pi_M \quad \{y_i \circ \pi_N, y_j \circ \pi_N\} = \{y_i, y_j\} \circ \pi_N.$$

We now come to the splitting theorem, which clarifies what Poisson manifolds look like locally. The proof given here is based on [Wat07] and [Vai94] (theorem 2.16 on page 27).

Theorem 72 (Splitting theorem)**.** *Let M be a Poisson manifold and take $p \in M$. Then there exists an open neighbourhood U of p with coordinates $q_1, \ldots, q_r$, $p_1, \ldots, p_r$, and $n_1, \ldots, n_{\dim(M)-2r}$ such that*

$$\{q_i, q_j\} = \{p_i, p_j\} = \{q_i, n_j\} = \{p_i, n_j\} = 0$$

for all i, j *and*

$$\{q_i, p_j\} = \delta_{ij}$$

and

$$\{n_i, n_j\}(p) = 0.$$

These relations are called the canonical commutation relations *and the coordinates are called* canonical coordinates around the point p. *Taking* S *to be the* $2r$*-dimensional local submanifold of* M *determined by the* q_i *and* p_i *and* N *to be the* $2r$*-codimensional local submanifold of* M *determined by the* n_i*, we have that*

$$M = S \times N \quad \text{locally around } p$$

as Poisson manifolds. Note that S *is a symplectic manifold and that* N *is a Poisson manifold of rank zero at* p*.*

If M is symplectic, then the canonical coordinates are called *Darboux coordinates*, and the splitting theorem is known as the *Darboux theorem.* Note that this theorem highlights an important fundamental difference between Riemannian and symplectic geometry: any two points in symplectic manifolds have neighbourhoods that are isomorphic as symplectic manifolds. This is far from true in the Riemannian case, where *curvature* is a local invariant. This makes symplectic geometry inherently global, since there is nothing interesting to say locally (except for the linear algebra of symplectic forms).

Proof of the splitting theorem. We prove this by induction on $\dim(M)$. The theorem is clear for $\dim(M) \in \{0, 1\}$. Suppose therefore that $\dim(M) \geq 2$ and that the result is true for all Poisson manifolds of lower dimension.

If M has rank zero at p, we can take local coordinates n_i and we are done. If M has positive rank at p, there is a function $p_1 \in \mathcal{F}(M)$ such that $X_{p_1}(p) \neq 0$. By a standard result in differential geometry, there exist local coordinates q_i around p such that $X_{p_1} = \frac{\partial}{\partial q_1}$ in a neighbourhood of p.

Considering q_1 as a smooth function around p, we have $\{q_1, p_1\} = X_{p_1}(q_1) = 1$ locally. Note that X_{p_1} and X_{q_1} are independent at p, for otherwise we would have

$$1 = \{q_1, p_1\} = -X_{q_1}(p_1) = -\lambda X_{p_1}(p_1) = \lambda\{p_1, p_1\} = 0.$$

Note also that the vector fields X_{p_1} and X_{q_1} commute, because

$$[X_{p_1}, X_{q_1}] = -X_{\{p_1,q_1\}} = -X_{\text{constant function } 1} = 0.$$

We can therefore find new coordinates $\alpha_1, \ldots, \alpha_{\dim(M)}$ in a neighbourhood of p such that

$$X_{p_1} = \frac{\partial}{\partial \alpha_1} \quad \text{and} \quad X_{q_1} = \frac{\partial}{\partial \alpha_2}.$$

Because $\frac{\partial p_1}{\partial \alpha_1} = X_{p_1}(p_1) = 0$, $\frac{\partial p_1}{\partial \alpha_2} = X_{q_1}(p_1) = -1$, $\frac{\partial q_1}{\partial \alpha_1} = X_{p_1}(q_1) = 1$ and $\frac{\partial q_1}{\partial \alpha_2} = X_{q_1}(q_1) = 0$, the Jacobian of $(p_1, q_1, \alpha_3, \dots, \alpha_{\dim(M)})$ in this new coordinate system is given by

$$\left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & * \\ \hline \multicolumn{2}{c|}{0} & \mathrm{Id}_{\dim(M)-2} \end{array}\right)$$

which is invertible (here, the star denotes an appropriately sized block with unknown entries, the zero in the lower left is a block of zeroes and Id_n denotes the $n \times n$ identity matrix). By the inverse function theorem, this means that $p_1, q_1, \alpha_3, \dots, \alpha_{\dim(M)}$ form local coordinates in some neighbourhood of p. We have the bracket relations

$$\{p_1, \alpha_i\} = \{q_1, \alpha_i\} = 0 \quad (i \geq 3).$$

Using the Jacobi identity, we get

$$\frac{\partial \{\alpha_i, \alpha_j\}}{\partial \alpha_1} = \{p_1, \{\alpha_i, \alpha_j\}\} = 0 \quad \text{for } i, j \geq 3,$$

with an analogous result for α_2 and p_2, implying the bracket relations

$$\{\alpha_i, \alpha_j\} = \varphi^{ij}(\alpha_3, \dots, \alpha_{\dim(M)}) \quad \text{for } i, j \geq 3$$

for some smooth functions φ^{ij}.

Write S_1 for the two-dimensional local submanifold through p determined by p_1 and q_1, and N_1 for the local submanifold through p of codimension two determined by α_i ($i \geq 3$). By the bracket relations established above, M is locally the product of the Poisson manifolds S_1 and N_1. Note that S_1 is in fact symplectic.

By induction, N_1 has local coordinates $q_2, \dots, q_r, p_2, \dots, p_r, n_1, \dots, n_{\dim(M)-2r}$ satisfying the canonical commutation relations. Taking the coordinates $q_1, \dots, q_r, p_1, \dots, p_r$ and $n_1, \dots, n_{\dim(M)-2r}$ around p then proves the result. $\square$

Let M be a Poisson manifold. Define an equivalence relation on M as follows: two points p and q are equivalent iff there is a piecewise smooth curve from p to q such that every segment of this curve is a integral curve of a Hamiltonian vector field. The equivalence classes of this relation are called the *symplectic leaves* of M. These symplectic leaves are in fact injectively immersed submanifolds of M, as we now show.

Let L be a symplectic leaf of M. Take a point $p \in M$ where the rank of M is r. We can find an open neighbourhood that is of the form $S \times N$ with S a symplectic manifold and N a Poisson manifold of rank zero at p. Without loss of generality, we shall assume that p has all its canonical coordinates zero. Using canonical coordinates, one sees that $S \times \{0\} \subset L$. If we use the coordinates q_i, p_i on S, this makes L into a smooth manifold. The inclusion $L \hookrightarrow M$ is an immersion by the very definition of the coordinates on L. This shows that M is the disjoint union of injectively immersed symplectic manifolds. Note that the symplectic leaves are by definition path connected. Also note that the symplectic leaves can have varying dimension (indeed, the dimension of the symplectic leaf through point p is just the rank of the Poisson manifold at p). The symplectic leaves of a symplectic manifold are its components. Because the local submanifold S in the splitting theorem carries a symplectic structure, so do the symplectic leaves.

4.3.4 Example: coadjoint orbits

Let G be a Lie group. The dual of its Lie algebra is naturally a Poisson manifold for the bracket

$$\{f, g\}(\alpha) = \langle \alpha, [(df)_\alpha, (dg)_\alpha] \rangle \quad \text{for all } f, g \in \mathcal{F}(\mathfrak{g}^*) \text{ and } \alpha \in \mathfrak{g}^*.$$

Here, $T_\alpha f$ and $T_\alpha g$ are seen as elements of the Lie algebra $\mathfrak{g}$: they are both linear maps $T_\alpha \mathfrak{g}^* \cong \mathfrak{g}^* \to \mathbb{R}$. The bracket $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Antisymmetry and the Jacobi identity follow from the corresponding properties of the Lie bracket. The Leibniz rule follows immediately from the usual Leibniz rule $d(gh) = g\,dh + h\,dg$.

Lemma 73. *Suppose G is a connected Lie group. The symplectic leaves of $\mathfrak{g}^*$ are the orbits of the coadjoint action Ad^*.*

Proof. Pick a point $\alpha \in \mathfrak{g}^*$ and write L for the symplectic leaf through α. In canonical coordinates, one sees that the tangent space $T_\alpha L$ consists precisely of the vectors of the form $X_f(\alpha)$ for some function $f \in \mathcal{F}(\mathfrak{g}^*)$ (that is, the tangent space to the symplectic leaf is precisely the image of the map $T^*_\alpha \mathfrak{g}^* \to T_\alpha \mathfrak{g}^*$ induced by the Poisson structure). Because any covector at α is in fact the differential at α of some linear function $f \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$, the tangent space consists of the vectors at α of Hamiltonian vector fields of linear functions on $\mathfrak{g}^*$. Take two linear functions $f, g \in \mathfrak{g}$. Then

$$\{f, g\}(\alpha) = \langle \alpha, [(df)_\alpha, (dg)_\alpha] \rangle = \langle \alpha, \mathrm{ad}_f(g) \rangle = -\langle \mathrm{ad}^*_f(\alpha), g \rangle.$$

This shows that $X_f(\alpha) = \mathrm{ad}^*_f(\alpha)$ and so the tangent space to the symplectic leaf at α is

$$T_\alpha L = \{\mathrm{ad}^*_f(\alpha) \mid f \in \mathfrak{g}\}.$$

The coadjoint orbit of α is the image of the map $\varphi : G \to \mathfrak{g}^* : g \mapsto \mathrm{Ad}^*_g(\alpha)$. For a point $g \in G$ we have $T_g\varphi : f \mapsto \mathrm{Ad}^*_g(\mathrm{ad}^*_f(\alpha))$. This demonstrates that

$$\mathrm{Im}(T_g \varphi) = T_{\varphi(g)} L.$$

From this, we conclude two things:

- The inverse image $\varphi^{-1}(L)$ is open. Then of course also the inverse images of all other leaves are open. However, because $\varphi^{-1}(L)$ is the complement of the union of the inverse images of the other leaves, it is also closed. By connectedness of G. we have $\varphi^{-1}(L) = G$. In words: orbits are subsets of symplectic leaves.

- The image of φ (this is the orbit of α) is open in L. Then of course also any other orbits that lie in L are open subsets of it. But again, the orbit of α is the complement of the union of the other orbits in L, and hence closed. By connectedness of symplectic leaves, the orbit of α is exactly L. $\square$

As an example, let us consider the case $G = \mathrm{SU}(2)$. It follows from the definition of SU(2) that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Note that the map $(\alpha, \beta) \mapsto \left(\begin{smallmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{smallmatrix}\right)$ is an embedding of the four-dimensional real vector space into the eight-dimensional real vector space. Its restriction to the 3-sphere is an embedding of the 3-sphere whose image is SU(2). We conclude that SU(2) is a Lie group diffeomorphic to the 3-sphere.

The Lie algebra of SU(2) is three-dimensional, with basis

$$u = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, w = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and Lie brackets

$$[u, v] = 2w \quad [v, w] = 2u \quad [w, u] = 2v.$$

Lemma 74 ((Co)adjoint action of SU(2))**.** *Relative to the basis (u, v, w) of $\mathfrak{su}(2)$, the adjoint action of $M = \left(\begin{smallmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{smallmatrix}\right)$ is given by*

$$N = \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & -2ab + 2cd & 2ac + 2bd \\ 2ab + 2cd & a^2 - b^2 + c^2 - d^2 & -2ad + 2bc \\ -2ac + 2bd & 2ad + 2bc & a^2 + b^2 - c^2 - d^2 \end{pmatrix}$$

where $\alpha = a + bi$ and $\beta = c + di$ (implying that $a^2 + b^2 + c^2 + d^2 = 1$). Relative to the dual basis of (u, v, w), the coadjoint action of M is given by the same matrix. This is an orthogonal matrix, so that we have a map $\chi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. This map is a double covering map.

Proof. Let us write $M = \left(\begin{smallmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{smallmatrix}\right)$. We investigate the adjoint action of M on the basis (u, v, w) of $\mathfrak{su}(2)$. Recalling lemma 17, we see that

$$\mathrm{Ad}_M(u) = MuM^{-1} = \begin{pmatrix} -i\alpha\beta - i\bar{\alpha}\bar{\beta} & i\alpha^2 - i\bar{\beta}^2 \\ -i\beta^2 + i\bar{\alpha}^2 & i\alpha\beta + i\bar{\alpha}\bar{\beta} \end{pmatrix},$$

$$\mathrm{Ad}_M(v) = MvM^{-1} = \begin{pmatrix} \alpha\beta - \bar{\alpha}\bar{\beta} & -\alpha^2 - \bar{\beta}^2 \\ \beta^2 + \bar{\alpha}^2 & -\alpha\beta + \bar{\alpha}\bar{\beta} \end{pmatrix},$$

and

$$\mathrm{Ad}_M(w) = MwM^{-1} = \begin{pmatrix} i\alpha\bar{\alpha} - i\beta\bar{\beta} & 2i\alpha\bar{\beta} \\ 2i\beta\bar{\alpha} & -i\alpha\bar{\alpha} + i\beta\bar{\beta} \end{pmatrix}.$$

Note that it is easy to write these right hand sides as linear combinations of the u, v and w: the coefficient for u is just the imaginary part of the lower left entry, the coefficient for v is the real part of the lower left entry, and the coefficient for w is the upper left entry divided by i. We can thus write down the matrix for the adjoint action of M relative to the basis (u, v, w), resulting in N. To verify that this matrix is orthogonal, one checks that

$$N^T N = \begin{pmatrix} (a^2+b^2+c^2+d^2)^2 & 0 & 0 \\ 0 & (a^2+b^2+c^2+d^2)^2 & 0 \\ 0 & 0 & (a^2+b^2+c^2+d^2)^2 \end{pmatrix}$$

and the right hand side is the identity matrix by the condition $a^2+b^2+c^2+d^2 = 1$. By the definition of coadjoint action, $\mathrm{Ad}^*_M = (\mathrm{Ad}_{M^{-1}})^* = (\mathrm{Ad}^{-1}_M)^*$, which shows that relative to the dual basis, the matrix of Ad^*_M is given by the transpose of the inverse of N, which is just N.

It only remains to prove that the map $\chi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a double covering map. We first prove that the map is surjective. Observe that we can pick $b = c = 0$, $a = \cos(\theta)$ and $d = \sin(\theta)$ to get

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

so that all rotations about the first coordinate axis are in the image of χ. An analogous argument shows that all rotations about the other two coordinate axes are also contained in the image of χ. Because rotations about the coordinate axes generate $\mathrm{SO}(3)$, the map χ is surjective.

Straightforward calculation shows that

$$T_{\mathrm{Id}}\chi(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \quad T_{\mathrm{Id}}\chi(v) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \quad T_{\mathrm{Id}}\chi(w) = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

proving that χ has full rank at Id. Now let $M \in \mathrm{SU}(2)$. Write $\lambda_M : \mathrm{SU}(2) \to \mathrm{SU}(2)$ for left multiplication by M and write $\lambda_{\chi(M)} : \mathrm{SO}(3) \to \mathrm{SO}(3)$ for left multiplication by $\chi(M)$. Then consider the following commutative diagram:

$$\begin{tikzcd} T_{\mathrm{Id}}\,\mathrm{SU}(2) \arrow[r, "T_{\mathrm{Id}}\chi"] \arrow[d, "T_{\mathrm{Id}}\lambda_M"'] & T_{\mathrm{Id}}\,\mathrm{SO}(3) \arrow[d, "T_{\mathrm{Id}}\lambda_{\chi(M)}"] \\ T_M\,\mathrm{SU}(2) \arrow[r, "T_M\chi"'] & T_{\chi(M)}\,\mathrm{SO}(3) \end{tikzcd}$$

Because $T_{\mathrm{Id}}\chi$, $T_{\mathrm{Id}}\lambda_M$ and $T_{\mathrm{Id}}\lambda_{\chi(M)}$ are isomorphisms, so is $T_M\chi$. This proves that χ has full rank everywhere and is therefore a local diffeomorphism.

Let us now find the kernel of χ. For $\chi(M)$ to equal the identity, we need

$$a^2 - b^2 - c^2 + d^2 = a^2 - b^2 + c^2 - d^2 = a^2 + b^2 - c^2 - d^2 = a^2 + b^2 + c^2 + d^2 = 1,$$

which is equivalent to

$$b = c = d = 0 \quad \text{and} \quad a^2 = 1.$$

This shows that the kernel of χ is exactly $\{\pm \mathrm{Id}\}$.

We can finally prove that χ is a double covering. Take any point $N \in \mathrm{SO}(3)$. Its inverse image under χ consists of two points $\pm M$. Take disjoint neighbourhoods of M and $-M$ that are mapped diffeomorphically to neighbourhoods of N. The intersection of these neighbourhoods of N is clearly evenly covered by χ, proving that χ is a covering map. Because the kernel of χ has cardinality 2, it is a double cover. $\square$

Chapter 5

Constructing the moduli space

In this chapter we consider the *moduli space of flat connections* for a fixed structure group G and base space Σ. This moduli space $\mathcal{M}(\Sigma, G)$ is the set of all flat principal G-bundles over Σ up to gauge equivalence. Slightly more formally, we shall regard $\mathcal{M}(\Sigma, G)$ as a quotient of the set of all flat principal G-bundles (P, A) on Σ, the equivalence relation being gauge equivalence. We shall denote the gauge equivalence class of (P, A) by $[P, A]$. Throughout the chapter, we assume that Σ is connected.

5.1 The moduli space as a space of representations

Our first approach to studying the moduli space will be built on the idea that *a flat principal G-bundle over Σ up to gauge equivalence is essentially a group morphism $\pi_1(\Sigma) \to G$ up to conjugation*. Let us sketch the idea.

The holonomy of a flat connection around a loop only depends on the homotopy class of this loop. Picking a base point p in a flat principal bundle, we can construct a group morphism $\pi_1(\Sigma) \to G$ by mapping any loop to the holonomy around it (up to inversion). Changing base points conjugates the resulting morphism, so that we can associate to any flat principal bundle a group morphism $\pi_1(\Sigma) \to G$ up to conjugation. It turns out that this group morphism actually determines the flat bundle up to gauge equivalence.

At the end of this section we will be able to identify

$$\mathcal{M}(\Sigma, G) \cong \frac{\mathrm{Hom}(\pi_1(\Sigma), G)}{G},$$

where G acts on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugation. This allows us to consider $\mathcal{M}(\Sigma, G)$ not just as a set, but as a topological space (and in many cases, the moduli space will inherit even more structure). The space $\mathrm{Hom}(\pi_1(\Sigma), G)$ is often called a *representation variety*, as for example in [Lab08].

5.1.1 The correspondence...

We will formalize the correspondence outlined above. Our goal is a bijective map

$$\Psi : \mathcal{M}(\Sigma, G) \to \frac{\mathrm{Hom}(\pi_1(\Sigma), G)}{G}.$$

We would like to define $\Psi([P, A])$ as the equivalence class of the group morphism

$$\pi_1(\Sigma) \to G : [\gamma] \mapsto \mathrm{Hol}_p(A, \gamma)^{-1}$$

where p is some fixed point of P. Many things need to be checked to construct Ψ:

- the above map is well-defined, meaning that $\mathrm{Hol}_p(A, \gamma)$ only depends on the homotopy class of γ,
- the above map is a group morphism,
- the above map depends on p only up to conjugation,
- the above map only depends on the gauge equivalence class of (P, A).

The first of these uses flatness of the connection and follows from the lemma below. The second is a relatively easy verification using (†) on page 29 and we leave it to the reader. The third is a direct consequence of that same equation. The fourth should be clear from the definition of gauge equivalence.

Lemma 75. *The holonomy of a flat principal connection around a contractible loop is trivial.*

Proof. Let (P, Σ, π, G) be a principal bundle with a flat connection A, and fix a point p of P. Write Q for the subset of P of all points that are the endpoint of piecewise smooth horizontal curves in P that start at p. Around any point $q \in Q$, there is a chart (of P) such that the horizontal space at every point in this chart is spanned by the first k coordinate vector fields. This shows that Q is an immersed submanifold of P, and the tangent space to Q consists of vectors that are horizontal in P. Because horizontal vectors are never vertical, the projection $\tau : Q \to \Sigma$ (the restriction of π to Q) has bijective differential at every point. By the inverse function theorem, this shows that τ is locally a diffeomorphism at every point. We see that τ is a surjective local diffeomorphism. In fact, it is a covering map: take a point $q \in Q$. Then there is a neighbourhood U of q and a neighbourhood V of $\tau(q)$ such that $\tau : U \to V$ is a diffeomorphism. This neighbourhood V of $\tau(q)$ is evenly covered by τ because for any other preimage $q \cdot g$ of $\tau(q)$ the map τ gives a diffeomorphism between $U \cdot g$ and V.

We can now use the theory of covering maps. Taking a null-homotopic loop γ in Σ starting at x, it lifts to a *loop* in Q, meaning that the horizontal lift of γ starting at p also ends at p. This shows that

$$\mathrm{Hol}_p(A, \gamma) = \{e\},$$

proving the lemma. □

5.1.2 ... is a bijection

Let us now prove that the correspondence Ψ is indeed a bijection. We first prove injectivity.

Theorem 76. *The correspondence Ψ is injective.*

Proof. Suppose (P, A) and (Q, B) are two flat principal G-bundles on Σ and $\Psi([P, A]) = \Psi([Q, B])$. We shall find a gauge equivalence between the two flat bundles. Pick points $p \in P$ and $q \in Q$ lying above the same point $x \in \Sigma$. By assumption, there is some $g \in G$ such that

$$\mathrm{Hol}_p(A, \gamma) = g \, \mathrm{Hol}_q(B, \gamma) g^{-1}$$

for every loop γ in Σ starting at x. By replacing q by $q \cdot g$, we may assume that in fact

$$\mathrm{Hol}_p(A, \gamma) = \mathrm{Hol}_q(B, \gamma).$$

We will construct a gauge equivalence $f : P \to Q$. Pick a point $y \in \Sigma$. Take a curve δ in Σ from x to y. Lift δ to an A-horizontal curve δ_A in P and to a B-horizontal curve δ_B in Q. The idea is to demand that

$$f(\delta_A(1)) = \delta_B(1).$$

(Indeed, this is the only possibility for $f(\delta_A(1))$ because $f \circ \delta_A$ should be horizontal according to B, so that we want $f \circ \delta_A = \delta_B$.) This completely determines f on the fibre above y because f is to be a bundle map. In fact, this determines f on the fibre above y in a way that is *independent* of the chosen curve δ as we now verify.

Suppose γ is another curve from x to y. Lift γ to curves γ_A and γ_B. We have to show that f as constructed above (using δ) satisfies $f(\gamma_A(1)) = \gamma_B(1)$. This follows from the following calculation:

$$\begin{aligned} f(\gamma_A(1)) &= f(\delta_A(1) \cdot \mathrm{Hol}_{\delta_A(1)}(A, \bar{\delta} \star \gamma)) \\ &= f(\delta_A(1)) \cdot \mathrm{Hol}_{\delta_A(1)}(A, \bar{\delta} \star \gamma) \\ &= \delta_B(1) \cdot \mathrm{Hol}_p(A, \gamma \star \bar{\delta}) \\ &= \delta_B(1) \cdot \mathrm{Hol}_q(B, \gamma \star \bar{\delta}) \\ &= \delta_B(1) \cdot \mathrm{Hol}_{\delta_B(1)}(B, \bar{\delta} \star \gamma) \\ &= \gamma_B(1). \end{aligned}$$

This shows that we can indeed define f using the procedure above without worrying about which curve to pick for every $y \in \Sigma$. From considerations in local coordinates, it is easy to see that f is smooth. It is thus a bundle map covering the identity (and therefore an isomorphism of principal bundles). By construction, Tf maps horizontal vectors to horizontal vectors. We conclude that f is a gauge equivalence. □

We have thus shown that a flat principal bundle is essentially determined by its holonomy. We now prove that every group morphism $\pi_1(\Sigma) \to G$ actually occurs as the holonomy of some flat bundle.

Theorem 77. *The correspondence Ψ is surjective.*

Proof. Suppose $\psi : \pi_1(\Sigma) \to G$ is a group morphism. We will construct a flat principal G-bundle P over Σ and a point $p \in P$ such that

$$\mathrm{Hol}_p(A, \gamma)^{-1} = \psi([\gamma])$$

for every loop γ in Σ starting at $\pi(p)$. Here, A is the flat connection on P and π is the projection $P \to \Sigma$, as always.

Let $\tilde{\Sigma}$ be the universal cover of Σ with projection $\tau : \tilde{\Sigma} \to \Sigma$. Pick a point $x \in \tilde{\Sigma}$. Note that $\pi_1(\Sigma, \tau(x))$ acts on $\tilde{\Sigma}$ via monodromy (from the left). Note also that we can let $\pi_1(\Sigma)$ act on G from the left via

$$\pi_1(\Sigma) \times G \to G : ([\gamma], g) \mapsto \psi([\gamma])g.$$

We can then consider the quotient

$$P = \frac{\tilde{\Sigma} \times G}{\pi_1(\Sigma)}$$

where $[\gamma] \cdot (x, g) = ([\gamma] \cdot x, [\gamma] \cdot g)$ for $(x, g) \in \tilde{\Sigma} \times G$. Because $\pi_1(\Sigma)$ acts properly and freely on $\tilde{\Sigma} \times G$, this quotient is a smooth manifold such that

$$\tilde{\Sigma} \times G \to P$$

is a smooth submersion (even a local diffeomorphism). In fact, P is a principal G-bundle for the projection $[x, g] \mapsto \tau(x)$ (which is well-defined), and the right G-action on P is given by $[x, g] \cdot h = [x, gh]$ (which is also well-defined).

We will now construct a suitable connection on P. Call a tangent vector to $\tilde{\Sigma} \times G$ *horizontal* if it only has a component in the $\tilde{\Sigma}$-direction, not in the G-direction (in other words, if it is of the form $(v, 0)$ where $v \in T\tilde{\Sigma}$). The action of $\pi_1(\Sigma)$ on $\tilde{\Sigma} \times G$ will map horizontal vectors to horizontal vectors, so that the notion of horizontalness descends to the quotient P. We thus define horizontal spaces at each point of P, determining a connection A on P. This connection is easily seen to be flat by integrability of the horizontal subbundle.

Write $p = [x, e]$. We claim that $\psi(A, p) = \psi$, proving the result. Let $\gamma : I \to \Sigma$ be any loop starting at $\tau(x)$. Lift γ to a curve $\tilde{\gamma}$ on $\tilde{\Sigma}$ starting at x. The curve

$$t \mapsto [\tilde{\gamma}(t), e]$$

is then the horizontal lift of γ to P starting at p. It ends at

$$[[\gamma] \cdot x, e] = [x, \psi([\gamma])^{-1}] = p \cdot \psi([\gamma])^{-1},$$

showing that $\mathrm{Hol}_p(A, \bar{\gamma}) = \mathrm{Hol}_p(A, \gamma)^{-1} = \psi([\gamma])$. The result is proved. $\square$

5.1.3 A topology on the moduli space

By the results of the previous subsection, we can identify the moduli space with a set of equivalence classes of group morphisms:

$$\mathcal{M}(\Sigma, G) \cong \frac{\mathrm{Hom}(\pi_1(\Sigma), G)}{G}.$$

This identification allows us to endow $\mathcal{M}(\Sigma, G)$ with more structure than that of a mere set. The first extra structure we will study is a topology. To topologize the moduli space, we will ask that $\pi_1(\Sigma, G)$ be finitely generated, a mild assumption.

If the group $\pi_1(\Sigma)$ is finitely generated, we can pick generators $a_1, \ldots, a_n$ of $\pi_1(\Sigma)$. A group morphism from $\pi_1(\Sigma)$ to G is then completely determined by the images of the a_i, so that we can consider $\mathrm{Hom}(\pi_1(\Sigma), G)$ to be a subset of G^n. This turns $\mathrm{Hom}(\pi_1(\Sigma), G)$ into a topological space, and $\mathcal{M}(\Sigma, G)$ then inherits the quotient topology. The following result is straightforward and ensures that our way of topologizing the moduli space makes sense.

Lemma 78. *The topology on* $\mathrm{Hom}(\pi_1(\Sigma), G)$ *as given above is independent of the choice of generators.*

Proof. Suppose $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are two sets of generators for $\pi_1(\Sigma)$. Write H^a for $\mathrm{Hom}(\pi_1(\Sigma), G)$ equipped with the topology inherited from G^n using the a_i, and write H^b for $\mathrm{Hom}(\pi_1(\Sigma), G)$ equipped with the topology inherited from G^m using the b_i. Note that $H^a = H^b$ as sets. We will prove that the identity map from H^a to H^b is continuous, implying that the topology on H^a is finer (stronger, has more opens) than the topology on H^b. Reversing the roles of the a_i and b_i then shows that the topologies are in fact equal, establishing the lemma.

The situation can be illustrated in a diagram.

$$\begin{tikzcd} G^n \arrow[r, "f"] & G^m \\ H^a \arrow[u, hook] & H^b \arrow[u, hook] \end{tikzcd}$$

Since the a_i generate $\pi_1(\Sigma)$, there is a way to express b_j as

$$b_j = f_j(a_1, \ldots, a_n),$$

where $f_j(g_1, \ldots, g_n)$ is a product of (inverses of) the g_i. There will in general be many ways to do so, but pick one for each b_j. We then get a map

$$f : G^n \to G^m$$

whose components are the f_j. This map is smooth because G is a Lie group, so it is most certainly continuous. Its restriction to H^a is exactly the identity $H^a \to H^b$, which is therefore continuous. $\square$

Write ϖ for the quotient map $\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathcal{M}(\Sigma, G)$. Suppose $S \subseteq \mathcal{M}(\Sigma, G)$ is a subset of the moduli space. Then S inherits a topology from $\mathcal{M}(\Sigma, G)$. However, one could propose a second way to topologize S, namely by topologizing $\varpi^{-1}(S)$ as a subset of $\mathrm{Hom}(\pi_1(\Sigma), G)$ and then putting the corresponding quotient topology on S. So there are two ways to construct a topology on S:

- it inherits a subspace topology from $\mathcal{M}(\Sigma, G)$ (the "subspace-of-quotient topology"),
- it inherits a quotient topology from $\varpi^{-1}(S) \subseteq \mathcal{M}(\Sigma, G)$ (the "quotient-of-subspace topology").

It is known from general topology that for arbitrary topological spaces the subspace-of-quotient topology and the quotient-of-subspace topology need not be equal (the latter is always at least as fine as the former). Fortunately, we need not worry about this in our situation, as shown below. This rather technical result will be useful later.

Lemma 79. *Suppose S is a subset of $\mathcal{M}(\Sigma, G)$. It inherits a subspace topology from $\mathcal{M}(\Sigma, G)$. This topology is equal to the topology of S as a quotient of $\varpi^{-1}(S)$.*

Proof. We will prove that every open of the quotient-of-subspace topology is also open in the subspace-of-quotient topology. The converse statement is well-known for general topological spaces and is verified easily.

Suppose therefore that $U \subseteq S$ is open for the quotient-of-subspace topology. This means that $\varpi^{-1}(U)$ is an open subset of $\varpi^{-1}(S)$, so that there is an open $V \subseteq \mathrm{Hom}(\pi_1(\Sigma), G)$ such that

$$\varpi^{-1}(U) = \varpi^{-1}(S) \cap V.$$

We have to find an open $W \subseteq \mathcal{M}(\Sigma, G)$ such that $U = S \cap W$. We claim that choosing $W = \varpi(V)$ does the trick. Note that $\varpi(V)$ is open in $\mathcal{M}(\Sigma, G)$ because quotient maps of group actions are always open. It only remains to verify that

$$U = S \cap \varpi(V).$$

To show that $U \subseteq S \cap \varpi(V)$ we pick an element $x \in U$. This is obviously in S, and picking $y \in \varpi^{-1}(\{x\})$ we observe that $y \in V$ so that $x \in \varpi(V)$. The first inclusion is proved.

To show that $U \supseteq S \cap \varpi(V)$ we pick an element $x \in S \cap \varpi(V)$. There is an $y \in V$ such that $\varpi(y) = x$, and clearly $y \in \varpi^{-1}(S)$. This shows that $y \in \varpi^{-1}(U)$ so that $x \in U$. The lemma is proved. □

5.2 A smooth structure on the moduli space

In this section, we will explain how the moduli space can be equipped with more structure than a mere topology. In particular, we will identify some points as *smooth* and others as *singular*.

5.2.1 Singular points

We start this chapter with a fairly general and straightforward discussion of the notion of a singularity. This subsection is not specific to the study of moduli spaces of flat connections.

Suppose $f : M \to \mathbb{R}$ is a smooth function. Then its zero set $S = f^{-1}(0)$ is a closed subset of M. If f is submersive at every point of S, then S is a submanifold of M. In general, however, f need not be submersive at every point of S. The points of S where f is not submersive are called *singular points of f*. As an example, consider the map $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^3 - y^2$. Its zero set is the so-called *cusp*. There is exactly one point on the cusp that is singular for f: the origin. The cusp in shown in figure 5, and it shows that the cusp is "not smooth" in the origin.

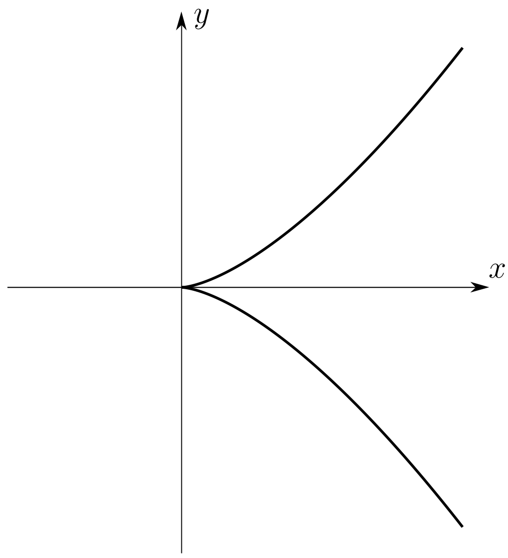

Figure 5: The cusp. It is the zero set of the polynomial $x^3 - y^2$ and has a singularity at the origin.

Now consider the map $g : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^2$. Its zero set is the y-axis, which is a smooth submanifold of the plane. However, every point on the y-axis is singular for g. If we look at the map $h : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$, we also have the y-axis as zero set, but this time the map is a submersion.

This leads us to the following question: how can we distinguish "bad" points of a zero set from "good" ones in a way that is geometrically sensible? So given a closed subset S of a manifold M, which criterion can tell us whether a point in S is a "singularity"? A geometrically satisfying definition may be the following.

Definition 80. *If S is a closed subset of a manifold M, then a point $p \in S$ is called* smooth of dimension n *if there is a neighbourhood U of p in M such that $S \cap U$ is a submanifold of U of dimension n. If p is not smooth of any dimension, p is called a* singularity *of S.*

In other words, a point is smooth if around the point the inclusion $S \hookrightarrow M$ looks like an inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$. The reader may verify that according to this definition the origin is the only singularity of the cusp.

Definition 81. *If S is a closed subset of a manifold M, N is a manifold and U is an open subset of S, then a map $f : U \to N$ is called* smooth *if there is an open neighbourhood V of U in M and a map $\bar{f} : V \to N$ such that $\bar{f}_{|U} = f$.*

Lemma 82. *Let S be a closed subset of a manifold M and $p \in S$. Then p is smooth of dimension n iff there is a neighbourhood U of p in S, an open subset V of $\mathbb{R}^n$ and a homeomorphism $f : V \to U$ such that a map $g : U \to \mathbb{R}$ is smooth precisely if $g \circ f$ is smooth.*

Proof. Suppose p is smooth of dimension n. Then locally around p the inclusion $S \hookrightarrow M$ looks like $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$, and a function $\mathbb{R}^n \to \mathbb{R}$ is smooth precisely if it can be extended to a smooth function $\mathbb{R}^m \to \mathbb{R}$. This proves one of the two implications.

Now suppose there exist U, V and f as in the lemma. Then $f : V \to U$ is smooth and a homeomorphism. We claim that f is an embedding, and the only thing that is left to prove is that f is an immersion. Now $f^{-1} : U \to V$ is smooth: indeed, its i'th component $f_i^{-1} : U \to \mathbb{R}$ is smooth because $g \circ f_i^{-1}$ is just projection $\mathbb{R}^n \to \mathbb{R}$ onto the i'th component. This means that there is a map $\bar{f}^{-1} : \bar{U} \to V$ (with $\bar{U}$ some open neighbourhood of U in M) that restricts to f^{-1} on U. By construction $\bar{f}^{-1} \circ f$ is the identity on V. Its differential is therefore injective, which means that f must have injective differential. $\square$

The lemma says that smoothness of points of S can be detected just by looking at the smooth functions on (open subsets of) S. It is this lemma that will motivate our definition of singularities of the moduli space. We can also reformulate the lemma using language more commonly used in algebraic geometry: if we turn S into a ringed space using its sheaf of smooth functions, then a point of S is smooth of dimension n iff it has a neighbourhood that is isomorphic to an open part of $\mathbb{R}^n$ as ringed spaces. This point of view using sheaves is standard in algebraic geometry, but is not widely used in differential geometry. For a treatment of differential geometry in the language of sheaves, we recommend [GdS03].

5.2.2 Application to the moduli space

Inspired by the previous subsection, we make the following definition.

Definition 83. *Pick a set of n generators of $\pi_1(\Sigma)$ to consider $\mathrm{Hom}(\pi_1(\Sigma), G) \subseteq G^n$. Let U be an open subset of $\mathrm{Hom}(\pi_1(\Sigma), G)$. A map $U \to \mathbb{R}$ is called* smooth *if it is the restriction to U of a smooth function on some open subset V of G^n.*

The reader is encouraged to check that this is independent of the choice of generators by imitating the proof of lemma 78. We can now say when a real-valued function on $\mathcal{M}(\Sigma, G)$ is smooth.

Definition 84. *Let U be an open subset of $\mathcal{M}(\Sigma, G)$. Write $\varpi : \mathrm{Hom}(\pi_1(\Sigma), G) \to \mathcal{M}(\Sigma, G)$ for the quotient map. A map $f : U \to \mathbb{R}$ is called* smooth *if $f \circ \varpi : \varpi^{-1}(U)$ is smooth.*

Definition 85. *A point $p \in \mathcal{M}(\Sigma, G)$ is called* smooth of dimension n *if there is a neighbourhood U of p, an open subset of $\mathbb{R}^n$ and a homeomorphism $\phi : U \to V$ such that a function $f : V \to \mathbb{R}$ is smooth exactly if $f \circ \phi$ is smooth.*

A point that is not smooth is called singular*.*

5.3 The moduli space as a symplectic reduction

If Σ is a compact orientable surface and $\mathfrak{g}$ admits an Ad-invariant non-degenerate symmetric bilinear form, then the moduli space of flat connections can in a sense be obtained as a symplectic reduction of some infinite-dimensional symplectic space (or in fact a disjoint union of several such reductions). The goal of this section is to explain this point of view. It will allow us to find a symplectic structure on (parts of) the moduli space later on, when we are studying concrete examples of moduli spaces. The material in this section is based on chapter 25 of [dS01] and on [Jan05].

It should be noted that this section is far from mathematically rigorous. We will be working with an infinite-dimensional symplectic manifold without bothering to define what that actually is. The role of Lie group acting on the space will be played by the gauge group, which is also infinite-dimensional. This means that many assumptions in the symplectic reduction procedure are violated: even if one builds up a well-founded theory of infinite-dimensional manifolds, for example, we demanded in the original recipe for symplectic reduction that the group acting on the manifold be compact. There is little hope for this in our infinite-dimensional context. We will ignore these issues entirely and regard this section as a *source of inspiration* for finding a symplectic structure on the moduli space in concrete cases, without claiming mathematical correctness. This symplectic structure can be placed on firm footing using algebraic techniques from group cohomology, as done in [Kar92]. We do not discuss this. Let us get started.

5.3.1 Picking Σ and G

To construct $\mathcal{M}(\Sigma, G)$ as a symplectic reduction, we need several restrictions on Σ and G. For Σ, the restriction is easy: since we will later want to integrate 2-forms over Σ, we will require that Σ be a compact orientable surface. It is well-known that compact orientable surfaces are classified by a single natural number, the genus g of the surface. For $g = 0$ the surface is the sphere S^2 and for $g \geq 1$ the surface is the connected sum of g tori. The first few of these surfaces are shown in figure 6.

For us, the case $g = 0$ is not interesting, since the moduli space $\mathcal{M}(S^2, G)$ is a singleton. We will henceforth assume that Σ is a compact orientable surface of genus $g \geq 1$. The

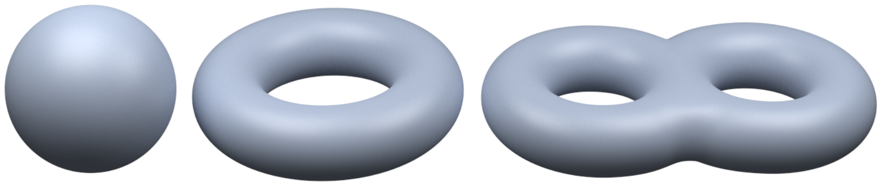

Figure 6: From left to right the compact orientable surfaces of genus 0, 1 and 2. The compact orientable surface of genus g is the connected sum of g tori, or the sphere if $g = 0$. To construct $\mathcal{M}(\Sigma, G)$ as a symplectic reduction, we will assume Σ is a compact orientable surface.

standard presentation of the fundamental group of this surface is

$$\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e \rangle.$$

In other words, the fundamental group of Σ is generated by $2g$ generators a_i, b_i, subject to the single relation that $\prod_i a_i b_i a_i^{-1} b_i^{-1}$ be the group unit.

We also restrict our choice of structure group G somewhat, but not as severely as our choice of base space. The Lie algebra $\mathfrak{g}$ should admit an Ad-invariant non-degenerate symmetric bilinear form. This means there should be some symmetric bilinear map

$$\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

that is non-degenerate and satisfies

$$\langle \mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\eta) \rangle = \langle \xi, \eta \rangle.$$

This will allow us to produce an ordinary ($\mathbb{R}$-valued) differential form on Σ given two $\mathfrak{g}_P$-valued forms: if $\alpha \in \Omega^k(\Sigma, \mathfrak{g}_P)$ and $\beta \in \Omega^l(\Sigma, \mathfrak{g}_P)$, then we can form their wedge product

$$\alpha \wedge \beta \in \Omega^{k+l}(\Sigma, \mathfrak{g}_P \otimes \mathfrak{g}_P).$$

In a local trivialization, $\alpha \wedge \beta$ assumes values in $\mathfrak{g} \otimes \mathfrak{g}$, so that we can apply the bilinear form to obtain an ordinary ($\mathbb{R}$-valued) differential form $\langle \alpha \wedge \beta \rangle \in \Omega^{k+l}(\Sigma)$. The Ad-invariance of the pairing ensures that the result is independent of the chosen trivialization, so that we have a non-degenerate bilinear pairing on every fibre of the bundle $\mathfrak{g}_P$ given by

$$\langle [p, \xi], [p, \eta] \rangle = \langle \xi, \eta \rangle.$$

If we restrict ourselves to Lie groups whose Lie algebra admits an Ad-invariant non-degenerate symmetric bilinear form, we should ask ourselves how severe this restriction is. Are there many such Lie groups? Fortunately, Lie groups admitting an Ad-invariant non-degenerate symmetric bilinear form on their Lie algebra are not rare. All compact Lie groups admit an appropriate form, because we can average an arbitrary inner product on $\mathfrak{g}$ over the group (resulting in a form that is even positive definite). Moreover, there is another large class of Lie groups admitting the kind of form we want: the so-called *semisimple* Lie groups. These semisimple Lie groups have a non-degenerate *Killing form*, which satisfies all our requirements. We will not study the theory of semisimple Lie groups any further.

The following lemma will be useful to us in the future. The proof is a long calculation involving lots of manipulation of permutations, and is analogous to the usual proof that the exterior derivative is a derivation of degree 1.

Lemma 86. *If $\alpha \in \Omega^k(\Sigma, \mathfrak{g}_P)$ and $\beta \in \Omega^l(\Sigma, \mathfrak{g}_P)$, then*

$$d\langle \alpha \wedge \beta \rangle = \langle d_A \alpha \wedge \beta \rangle + (-1)^k \langle \alpha \wedge d_A \beta \rangle,$$

analogous to the situation for the ordinary exterior derivative.

5.3.2 The plan

Let us explain in more detail how we plan to equip $\mathcal{M}(\Sigma, G)$ with a symplectic structure. Pick Σ and G as explained in the previous subsection. The moduli space $\mathcal{M}(\Sigma, G)$ contains equivalence classes $[P, A]$ of flat bundles. If two flat bundles (P, A) and (Q, B) are gauge equivalent, then in particular the bundles P and Q are isomorphic (as mere bundles without connections). This shows that we can split up $\mathcal{M}(\Sigma, G)$ into disjoint parts, one for each isomorphism class of principal G-bundles over Σ. Some of these parts may be empty, because not all principal G-bundles over Σ need to admit a flat connection.

We will use the notation

$$\mathcal{M}_P(\Sigma, G) = \{[Q, B] \in \mathcal{M}(\Sigma, G) \mid Q \text{ is isomorphic to } P \text{ as a principal bundle}\}$$

for these disjoint parts. Every such part will be called a *block* of the moduli space. In particular, we shall call $\mathcal{M}_P(\Sigma, G)$ the *P-block* of the moduli space. We have the following result. For a proof we refer to [Bel10].

Lemma 87. *Every block $\mathcal{M}_P(\Sigma, G)$ is a union of path components of $\mathcal{M}(\Sigma, G)$.*

The plan is to construct $\mathcal{M}_P(\Sigma, G)$ as a symplectic reduction of $\mathcal{A}$, the space of all connections on P. In this way $\mathcal{M}_P(\Sigma, G)$ will inherit a symplectic structure, and thus $\mathcal{M}(\Sigma, G)$ will also have a symplectic structure. We start by equipping $\mathcal{A}$ with a symplectic structure. Later we show that $\mathcal{M}_P(\Sigma, G)$ is a symplectic reduction of $\mathcal{A}$ for the moment map $A \mapsto F_A$.

5.3.3 The Atiyah-Bott 2-form

Let (P, Σ, π, G) be a principal bundle, as always. Suppose A and B are two principal connections on P. Recall that these principal connections both specify an equivariant $\mathfrak{g}$-valued 1-form on P through their projections $A_p : T_pP \to \mathfrak{g}$. The difference α between these is again equivariant, and because the projections A_p and B_p have equal restrictions to V_p, the 1-form α is also horizontal. This means that $\alpha \in \Omega^1_{\mathrm{Ad},H}(P, \mathfrak{g}) = \Omega^1(\Sigma, \mathfrak{g}_P)$.

Conversely, given a connection A on P and an $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$, the sum of α and the connection 1-form of A will be the connection 1-form of some principal connection B. We conclude that the set $\mathcal{A}$ of all principal connections on P is an affine space in the sense that

$$\mathcal{A} = A + \Omega^1(\Sigma, \mathfrak{g}_P).$$

We will equip $\mathcal{A}$ with a symplectic structure and show that $\mathcal{M}_P(\Sigma, G)$ can be obtained as a symplectic reduction of $\mathcal{A}$. Of course, it is not entirely clear what is meant by a symplectic structure on $\mathcal{A}$, but since the space is an affine space as explained above, we can say that

$$T_A\,\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g}_P)$$

for every $A \in \mathcal{A}$. The symplectic structure we will construct on $\mathcal{A}$ is called the *Atiyah-Bott 2-form* after Sir Michael Atiyah and Raoul Bott, who constructed this form in their paper [AB83]. It is given by

$$\omega_{AB}(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle.$$

This is an antisymmetric pairing $T_A\,\mathcal{A} \times T_A\,\mathcal{A} \to \mathbb{R}$. It is non-degenerate as shown in the following lemma. It is closed because it has constant coefficients (although this is just an analogy from the finite-dimensional case). This is why we consider ω_{AB} to be a symplectic structure on $\mathcal{A}$.

Lemma 88. *The Atiyah-Bott 2-form*

$$\omega_{AB} : \Omega^1(\Sigma, \mathfrak{g}_P) \times \Omega^1(\Sigma, \mathfrak{g}_P) \to \mathbb{R}$$

is non-degenerate.

Proof. Suppose $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$ is not zero. We will show that there is some $\beta \in \Omega^1(\Sigma, \mathfrak{g}_P)$ such that $\omega_{AB}(\alpha, \beta) \neq 0$.

Because $\alpha \neq 0$, there is some $p \in \Sigma$ and some $v \in T_p\Sigma$ such that $\alpha(v) \neq 0$. In some neighbourhood of p the bundle $\mathfrak{g}_P$ is trivial. Working in this neighbourhood we can find a chart around p such that coordinates on $\mathfrak{g}_P$ are given by $(x, y, \xi_1, \ldots, \xi_{\dim(G)})$ where (x, y) are coordinates on Σ and the ξ_i are linear coordinates on $\mathfrak{g}$. Changing coordinates on Σ we may assume that $v = \frac{\partial}{\partial x}$ at p.

Using these coordinates we can consider $\alpha(v)$ as an element of $\mathfrak{g}$, and because the pairing $\langle -, - \rangle$ is non-degenerate there is some $\xi \in \mathfrak{g}$ such that $\langle \alpha(v), \xi \rangle \neq 0$. In fact, working in an even smaller neighbourhood U of p and replacing ξ by $-\xi$ if necessary, we may assume that $\langle \alpha(\frac{\partial}{\partial x}), \xi \rangle > 0$ in U. Now take β to be the 1-form that vanishes outside U and is given in U by

$$\beta = f(x, y)\xi \otimes dy$$

where f is a smooth non-negative function with $f(p) = 1$ and support contained in U. Then by construction we have

$$\begin{aligned}\omega_{AB}(\alpha, \beta) &= \int_\Sigma \langle \alpha \wedge \beta \rangle \\ &= \int_U f(x,y) \langle \alpha \left(\frac{\partial}{\partial x} \right), \xi \rangle \, dxdy > 0.\end{aligned}$$

This proves the result. $\square$

5.3.4 The curvature as a moment map

We have seen in the previous subsection that $\mathcal{A}$ is a symplectic space. Our goal is to construct $\mathcal{M}_P(\Sigma, G)$ as a symplectic reduction of this space under the action of the gauge group.

Recall that the gauge group is

$$\mathcal{G}(P) = \{u : P \to G \mid u(p \cdot g) = g^{-1} u(p) g \text{ for all } p \in P \text{ and } g \in G\}$$

and that it acts on $\mathcal{A}$ by pullback. We can identify the Lie algebra of the gauge group with

$$\mathrm{Lie}(\mathcal{G}(P)) = \{h : P \to \mathfrak{g} \mid h(p \cdot g) = \mathrm{Ad}_{g^{-1}}(h(p)) \text{ for all } p \in P \text{ and } g \in G\}.$$

In other words,

$$\mathrm{Lie}(\mathcal{G}(P)) = \Omega^0(\Sigma, \mathfrak{g}_P).$$

Integration provides a non-degenerate bilinear pairing

$$\Omega^2(\Sigma, \mathfrak{g}_P) \times \Omega^0(\Sigma, \mathfrak{g}_P) \to \mathbb{R} : (F, h) \mapsto \int_\Sigma \langle F \wedge h \rangle,$$

and we will use it to identify $\mathrm{Lie}(\mathcal{G}(P))^* = \Omega^2(\Sigma, \mathfrak{g}_P)$ (the proof of non-degeneracy is similar to and slightly easier than the proof of lemma 88). We claim that the action of $\mathcal{G}(P)$ on $\mathcal{A}$ is Hamiltonian with equivariant moment map

$$\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G}(P))^* = \Omega^2(\Sigma, \mathfrak{g}_P) : A \mapsto -F_A.$$

For this claim to hold, we have to verify that

- the map μ is equivariant for the action of $\mathcal{G}(P)$,
- the Hamiltonian vector field of the function μ_h is the vector field X_h on $\mathcal{A}$ (where X_h is the vector field associated to $h \in \Omega^0(\Sigma, \mathfrak{g}_P)$ as in subsection 4.2.1).

We formulate the verifications as lemmas.

Lemma 89. *The curvature moment map μ is equivariant.*

Proof. To check that μ is equivariant, we have to check that

$$F_{u^*A} = \mathrm{Ad}_u^*(F_A) \quad \text{for all } A \in \mathcal{A} \text{ and } u \in \mathcal{G}(P).$$

We check this by pairing with an arbitrary $h \in \Omega^0(\Sigma, \mathfrak{g}_P)$. We then have to check that

$$\langle F_{u^*A} \wedge h \rangle = \langle F_A \wedge \mathrm{Ad}_u(h) \rangle.$$

This is an equality of ordinary 1-forms on Σ, which we check by evaluating on the vectors $v, w \in T_x\Sigma$. To evaluate the left hand side, we pick some fixed point $p \in P_x$. Then we lift v and w to u^*A-horizontal elements $\bar{v}$ and $\bar{w}$ of $T_{p\cdot u(p)^{-1}}P$ and to A-horizontal elements $\tilde{v}, \tilde{w}$ of T_pP. Then we have

$$\begin{aligned}
\langle F_{u^*A} \wedge h \rangle(v, w) &= \langle F_{u^*A}(v, w), h(x) \rangle \\
&= \langle [p \cdot u(p)^{-1}, F_{u^*A}(\bar{v}, \bar{w})], [p, h(p)] \rangle \\
&= \langle [p \cdot u(p)^{-1}, F_A(T_{p\cdot u(p)^{-1}}\tilde{u}(\bar{v}), T_{p\cdot u(p)^{-1}}\tilde{u}(w))], [p, h(p)] \rangle \\
&= \langle [p \cdot u(p)^{-1}, F_A(\tilde{v}, \tilde{w})], [p, h(p)] \rangle \\
&= \langle [p \cdot u(p)^{-1}, F_A(\tilde{v}, \tilde{w})], [p \cdot u(p)^{-1}, \mathrm{Ad}_{u(p)}(h(p))] \rangle \\
&= \langle F_A(\tilde{v}, \tilde{w}), \mathrm{Ad}_{u(p)}(h(p)) \rangle
\end{aligned}$$

where in the fourth step we observed that F_A is horizontal and the differences $T\tilde{u}(\bar{v}) - \tilde{v}$ and $T\tilde{u}(\bar{w}) - \tilde{w}$ are vertical. Evaluating the right hand side on (v, w) is easier:

$$\begin{aligned}
\langle F_A \wedge \mathrm{Ad}_u(h) \rangle(v, w) &= \langle [p, F_A(\tilde{v}, \tilde{w}], [p, \mathrm{Ad}_u(h)(p)] \rangle \\
&= \langle [p, F_A(\tilde{v}, \tilde{w})], [p, \mathrm{Ad}_{u(p)}(h(p))] \rangle \\
&= \langle F_A(\tilde{v}, \tilde{w}), \mathrm{Ad}_{u(p)}(h(p)) \rangle.
\end{aligned}$$

This proves equivariance of μ. □

Lemma 90. *The curvature moment map μ is a moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$.*

Proof. This will boil down to Stokes' theorem. Suppose $h \in \mathrm{Lie}(\mathcal{G}(P)) = \Omega^0(\Sigma, \mathfrak{g}_P)$. Then $\mu_h = \langle \mu, h \rangle$ is a component of μ (the brackets denote the pairing of $\mathrm{Lie}(\mathcal{G}(P))$ with its dual). We have to check that $d\mu_h$ equals $i_{X_h}\omega_{AB}$, where X_h is the Hamiltonian vector field on $\mathcal{A}$ associated to h. We will verify this by applying both to an arbitrary tangent vector $\alpha \in T_A\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g}_P)$. We will therefore check that

$$(d\mu_h)_A(\alpha) = \omega_{AB}(X_h, \alpha). \tag{‡}$$

First of all, note that $\mu(A) = -dA \circ h$ so that $\mu(A + t\alpha) = -d(A + t\alpha) \circ h = -F_A - td_A\alpha$. We have thus found the left hand side of (‡):

$$(d\mu_h)_A(\alpha) = -\int_\Sigma \langle d_A\alpha \wedge h \rangle.$$

We now investigate the right hand side of (‡). We claim that

$$X_h = d_A h.$$

It suffices to check this on horizontal tangent vectors to P, so pick $v \in H_p$. Then

$$\begin{aligned}(X_h)(v) &= \frac{d}{dt}_{|t=0}((\exp(th)^*A)_p(v) - A_p(v))\\ &= \frac{d}{dt}_{|t=0}(\exp(th)^*A)_p(v)\\ &= \frac{d}{dt}_{|t=0}A_{p\cdot\exp(th(p))}((T_p\tilde{u}_t)(v)).\end{aligned}$$

Here, $\tilde{u}_t$ is the gauge transformation associated to the element $u_t = \exp(th)$ of the gauge group. We now calculate $(T_p\tilde{u}_t)(v)$. To this end, we take γ to be a curve in P with initial velocity v. Then

$$\begin{aligned}(T_p\tilde{u}_t)(v) &= \frac{d}{ds}_{|s=0}(\gamma(s)\cdot u_t(\gamma(s)))\\ &= v\cdot u_t(\gamma(0)) + \frac{d}{ds}_{|s=0}(\gamma(0)\cdot u_t(\gamma(s))).\end{aligned}$$

The first of these two terms is horizontal by invariance of the horizontal subbundle. We thus find

$$\begin{aligned}(X_h)(v) &= \frac{d}{dt}_{|t=0}A_{p\cdot\exp(th(p))}\left(\frac{d}{ds}_{|s=0}[\gamma(0)\cdot u_t(\gamma(s))]\right)\\ &= \frac{d}{dt}_{|t=0}A_{p\cdot\exp(th(p))}\left(\frac{d}{ds}_{|s=0}[(p\cdot\exp(th(p)))\cdot(u_t(\gamma(0))^{-1}u_t(\gamma(s)))]\right)\\ &= \frac{d}{dt}_{|t=0}\theta\left(\frac{d}{ds}_{|s=0}u_t(\gamma(0))^{-1}u_t(\gamma(s))\right).\end{aligned}$$

Write f for the map $(-\varepsilon,\varepsilon)^2 \to G : (s,t) \mapsto u_t(\gamma(0))^{-1}u_t(\gamma(s))$. Then the above expression is exactly

$$\frac{d}{dt}_{|t=0}(f^*\theta)\left(\frac{d}{ds}_{|s=0}\right),$$

which by the Maurer-Cartan structure equation and the fact that $\frac{d}{dt}$ and $\frac{d}{ds}$ commute, equals

$$\frac{d}{ds}_{|s=0}(f^*\theta)\left(\frac{d}{dt}_{|t=0}\right) + \left[\theta\left(T_{(0,0)}f\left(\frac{d}{ds}\right)\right), \theta\left(T_{(0,0)}f\left(\frac{d}{dt}\right)\right)\right].$$

The first vector in the Lie brackets is zero. We continue our calculation noting that

$u_0(p) = e$ for all $p \in P$.

$$\begin{aligned}(X_h)(v) &= \frac{d}{ds}_{|s=0} \theta \left(\frac{d}{dt}_{|t=0} u_t(\gamma(0))^{-1} u_t(\gamma(s)) \right) \\ &= \frac{d}{ds}_{|s=0} \left(\mathrm{Ad}_{u_0(\gamma(s)) u_0(\gamma(0))} (h(\gamma(0))) + h(\gamma(s)) \right) \\ &= (dh)(v).\end{aligned}$$

This proves that $X_h = d_A h$. To prove the lemma, it thus suffices to show that

$$-\int_\Sigma \langle d_A \alpha \wedge h \rangle = \int_\Sigma \langle d_A h \wedge \alpha \rangle.$$

By Stokes' theorem and the fact that Σ has no boundary, it suffices to show that

$$d\langle \alpha \wedge h \rangle = \langle d_A \alpha \wedge h \rangle - \langle \alpha \wedge d_A h \rangle.$$

This follows from lemma 86. $\square$

If Σ is a compact orientable surface with boundary, the moduli space $\mathcal{M}(\Sigma, G)$ has a Poisson structure whose symplectic leaves are obtained by fixing the conjugacy class of the holonomy around every boundary component (see [Wei95], page 9).

Chapter 6

A topological observation

In this chapter we will split the moduli space into two parts. We then completely describe the topology of one of these, the so-called *reducible part* (which is the smaller and worse-behaved part, in a sense). Throughout the chapter, G is a compact connected Lie group and Σ is a compact orientable surface of genus $g \geq 1$. We write $\mathcal{M} = \mathcal{M}(\Sigma, G)$.

6.1 The reducible part

The fundamental group of Σ has the well-known presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e \rangle,$$

so it is generated by $2g$ generators satisfying a single relation. Consider the map

$$\mu : G^{2g} \to G : (A_1, B_1, \dots, A_g, B_g) \mapsto \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}.$$

Then $\mathrm{Hom}(\pi_1(\Sigma), G)$ is the subset $\mu^{-1}(e)$ of G^{2g} (and this is how we equipped the set $\mathrm{Hom}(\pi_1(\Sigma), G)$ with a topology). The moduli space $\mathcal{M}$ is the quotient of $\mathrm{Hom}(\pi_1(\Sigma), G)$ under conjugation and therefore contains equivalence classes of $2g$-tuples of elements of G. We will write $[A_1, B_1, \dots, A_g, B_g]$ for the equivalence class represented by the tuple $(A_1, B_1, \dots, A_g, B_g) \in \mathrm{Hom}(\pi_1(\Sigma), G) \subseteq G^{2g}$.

Definition 91. *A tuple $(A_1, \dots, B_g) \in G^{2g}$ is called* reducible *if all of its $2g$ entries commute pairwise. Otherwise the tuple is called* irreducible.

Note that conjugation does not change irreducibility of a tuple, so that we can make analogous definitions for the moduli space.

Definition 92. *A point $[A_1, \dots, B_g] \in \mathcal{M}$ is called* reducible *if the tuple $(A_1, \dots, B_g)$ is. Otherwise the point is called* irreducible.

Write $\mathcal{R}$ for the subset of $\mathcal{M}$ containing all reducible points. Write $\mathcal{S}$ for the subset of $\mathcal{M}$ containing all irreducible points. Then

$$\mathcal{M} = \mathcal{R} \sqcup \mathcal{S}.$$

Recall the notions of maximal torus and Weyl group from section 1.4 on page 12. The main result of this section is the following.

Theorem 93. *Suppose G is a compact Lie group and Σ is a compact orientable surface of genus g. Then the reducible part $\mathcal{R}$ of the moduli space is homeomorphic to*

$$\frac{T^{2g}}{W(T)}$$

where T is a maximal torus in G and $W(T)$ its Weyl group, acting on all copies of T simultaneously.

Proof. Write $W = W(T)$. We will construct a homeomorphism $\Pi : \mathcal{R} \to T^{2g}/W$. Pick any point $[A_1, \dots, B_g] \in \mathcal{R}$. The fact that $(A_1, \dots, B_g)$ is irreducible is exactly the fact that all the A_i and B_i lie in a common torus in G, and hence in some maximal torus. Because all maximal tori are conjugate, we can conjugate to obtain a tuple in T^{2g}. We take this tuple to be a representative of $\Pi([A_1, \dots, B_g])$.

Of course, we still have to check that this map Π is well-defined, because there may be multiple ways to conjugate $(A_1, \dots, B_g)$ to get a tuple in T^{2g}, and we have to verify that all possible resulting tuples in fact represent the same equivalence class in T^{2g}/W. This is lemma 23 on page 13.

Recall that a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism. We will prove that $\mathcal{R}$ is compact and that Π is continuous, surjective and injective. Because T^{2g}/W is Hausdorff (recall that W is finite), the result will follow.

Let us first show that $\mathcal{R}$ is compact. Let $\tilde{\mathcal{R}}$ denote $\varpi^{-1}(\mathcal{R})$ where $\varpi : \mu^{-1}(e) \to \mathcal{M}$ is the projection. Because $\mathcal{R}$ is a quotient of $\tilde{\mathcal{R}}$ (by lemma 79), it suffices to prove that $\tilde{\mathcal{R}}$ is compact. Now $\tilde{\mathcal{R}}$ is a closed subset of G^{2g}, which itself is compact. This proves that $\tilde{\mathcal{R}}$ is compact. We have showed that $\mathcal{R}$ is compact.

It is easy to see that Π is surjective: the equivalence class of $(A_1, \dots, B_g)$ is the image of $[A_1, \dots, B_g]$ under Π. It is also easy to see that Π is injective: if $\Pi([A_1, \dots, B_g]) = \Pi([A'_1, \dots, B'_g])$ then by definition of W the tuples $(A_1, \dots, B_g)$ and $(A'_1, \dots, B'_g)$ are conjugate, proving that $[A_1, \dots, B_g] = [A'_1, \dots, B'_g]$.

It remains to prove that $\Pi : \mathcal{R} \to T^{2g}/W$ is continuous. Because $\mathcal{R}$ is a quotient of $\tilde{\mathcal{R}}$, it suffices to show that the composition

$$\tilde{\mathcal{R}} \xrightarrow{\text{projection}} \mathcal{R} \xrightarrow{\Pi} T^{2g}/W$$

is continuous. We will call this composition $\tilde{\Pi}$, and we will prove its continuity using sequences. Let $(A_1, B_1, \dots, A_g, B_g)$ be a point of $\tilde{\mathcal{R}}$. Take a sequence $((A_1^n, \dots, B_g^n))_n$ in

$\tilde{\mathcal{R}}$ converging to this point, indexed by an upper index for notational reasons. For every n, take an element $C^n \in G$ that conjugates the tuple $(A_1^n, \ldots, B_g^n)$ to lie in T^{2g}, i.e. such that

$$(C^n A_1^n (C^n)^{-1}, \ldots, C^n B_g^n (C^n)^{-1}) \in T^{2g}.$$

This means that

$$\tilde{\Pi}(A_1^n, \ldots, B_g^n) = [C^n A_1^n (C^n)^{-1}, \ldots, C^n B_g^n (C^n)^{-1}].$$

By compactness of G, we can take a subsequence $(C^{m(n)})_n$ that converges to $C \in G$. Then because $A_i^{m(n)} \to A_i$, $B_i^{m(n)} \to B_i$ and $C^{m(n)} \to C$ we find that

$$\lim_{n \to +\infty} C^{m(n)} A_i^{m(n)} (C^{m(n)})^{-1} = C A_i C^{-1}$$

and

$$\lim_{n \to +\infty} C^{m(n)} B_i^{m(n)} (C^{m(n)})^{-1} = C B_i C^{-1}.$$

As limits of sequences in T, all CA_iC^{-1} and CB_iC^{-1} lie in T, which means that

$$\tilde{\Pi}(A_1, \ldots, B_g) = [CA_1C^{-1}, \ldots, CB_gC^{-1}].$$

We conclude that

$$\lim_{n \to +\infty} \tilde{\Pi}(A_1^{m(n)}, \ldots, B_g^{m(n)}) = \tilde{\Pi}(A_1, \ldots, B_g). \tag{$*$}$$

We have now proved the following statement: any sequence $((A_1^n, \ldots, B_g^n))_n$ in $\tilde{\mathcal{R}}$ has a subsequence such that $(*)$ holds. This suffices to establish continuity of $\tilde{\Pi}$ at $(A_1, \ldots, B_g)$: suppose $\tilde{\Pi}(A_1^n, \ldots, B_g^n)$ does not converge to $\tilde{\Pi}(A_1, \ldots, B_g)$. Then there is a subsequence whose image points under $\tilde{\Pi}$ stay out of a certain neighbourhood of $\tilde{\Pi}(A_1, \ldots, B_g)$. This subsequence then contradicts the statement. We have shown that $\tilde{\Pi}$ (and hence Π) is continuous, finishing the proof. $\square$

The above theorem completely determines the topology of the moduli space in the case $g = 1$. Indeed, in the case $\Sigma = S^1 \times S^1$ we immediately have

$$\mathcal{R} = \mathcal{M} \quad \text{and} \quad \mathcal{S} = \emptyset$$

from the definition of reducibility (because $\pi_1(\Sigma)$ is abelian).

Corollary 94. *Suppose G is a compact connected Lie group. Then $\mathcal{M}(S^1 \times S^1, G)$ is homeomorphic to*

$$\frac{T^2}{W(T)}$$

where T is a maximal torus in G and $W(T)$ its Weyl group, acting on both copies of T at the same time.

As an example, consider $G = \mathrm{SU}(2)$. The maximal torus is then diffeomorphic to S^1, with $W \cong \mathbb{Z}_2$ acting by complex conjugation. The moduli space of flat connections on the principal $\mathrm{SU}(2)$-bundle over $S^1 \times S^1$ is then homeomorphic to

$$\frac{S^1 \times S^1}{\mathbb{Z}_2}$$

where $\mathbb{Z}_2$ acts by simultaneous complex conjugation of both circles. After cutting and pasting appropriately, one sees that this moduli space is homeomorphic to the sphere S^2. In the next section, we discuss the example $G = \mathrm{SU}(2)$ in greater detail, because in this case we can also say something about the irreducible part.

6.2 The case $G = \mathrm{SU}(2)$

Let us study the moduli space of flat connections on the trivial (the only) $\mathrm{SU}(2)$-bundle over a compact orientable surface of genus g. The topology of the case $g = 1$ was already discussed in the previous section: the moduli space is homeomorphic to S^2.

In the last section, we split the moduli space into a reducible part $\mathcal{R}$ and an irreducible part $\mathcal{S}$. We already know the reducible part topologically by theorem 93. Let us state the result for the sake of concreteness.

Theorem 95. *If Σ is a compact orientable surface of genus g, then the reducible part of $\mathcal{M}(\Sigma, \mathrm{SU}(2))$ is homeomorphic to*

$$(\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{2g \text{ copies}})/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts on the circles by complex conjugation.

Our goal for this section is the following result.

Theorem 96. *If Σ is a compact orientable surface of genus g, then the irreducible part of $\mathcal{M}(\Sigma, \mathrm{SU}(2))$ is a smooth manifold of dimension $6g - 6$.*

Note that $6g - 6$ is the dimension one expects the moduli space to have. Indeed, we have

$$\mathcal{M} = \mu^{-1}(e)/\,\mathrm{SU}(2),$$

(where $\mu : G^{2g} \to G$ is the map defined at the start of this chapter) and since μ is a map from $\mathrm{SU}(2)^{2g}$ to $\mathrm{SU}(2)$, we expect $\mu^{-1}(e)$ to have dimension

$$2g \cdot \dim(\mathrm{SU}(2)) - \dim(\mathrm{SU}(2)) = 6g - 3,$$

leading to a quotient of dimension

$$6g - 3 - \dim(\mathrm{SU}(2)) = 6g - 6.$$

This dimension count also serves as a rough sketch of our proof strategy: we first show that μ is submersive at all irreducible tuples (so that $\mu^{-1}(e)$ is a smooth manifold), and then we prove that $\mathcal{M}$ is a quotient of this manifold under a free action of a compact group of the same dimension as $\mathrm{SU}(2)$. Our first task is to investigate the rank of μ.

The following is a slight generalization of lemma 18 in [Jan05].

Lemma 97. *Suppose $G = \mathrm{SU}(2)$ and let $\mu : G^{2g} \to G$ be as before. Then the rank of μ is 3 at all irreducible points, 0 at the points (A_i, B_i) with all A_i and B_i equal to $\pm\,\mathrm{Id}$ and 2 at all other points. In particular, μ has full rank exactly at the irreducible points.*

In fact, [Gol84] mentions the following generalization (section 3.7). A proof of this generalization can also be found there.

Theorem 98. *Let G be a Lie group and*

$$\mu : G^{2g} \to G : (A_1, B_1, \ldots, A_g, B_g) \mapsto \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}.$$

Then the rank of μ at $(A_1, \ldots, B_g)$ equals the codimension of the centralizer of $\{A_1, \ldots, B_g\}$ in G.

To prove lemma 97, we will use the following result.

Lemma 99. *Two elements A and B of $\mathrm{SU}(2)$ commute iff their (co)adjoint actions are rotations about the same axis or either of them is $\pm\,\mathrm{Id}$.*

Proof. First suppose that A and B are commuting elements of $\mathrm{SU}(2)$. After conjugation, we may assume that they are both diagonal. From lemma 74 we conclude that their coadjoint action are both rotations about the third coordinate axis.

Now suppose that A and B act on $\mathfrak{su}(2)$ (or $\mathfrak{su}(2)^*$) by rotation about a common axis. After conjugation, we may assume that this axis is the third coordinate axis. From lemma 74 we deduce that A and B are both diagonal and therefore commute. $\square$

We now prove lemma 97. The proof is a rewritten and expanded version of the one given by Janner in [Jan05].

Proof of lemma 97. We shall differentiate μ at the point $(A_1, \ldots, B_g)$ in the direction $(a_1, \ldots, b_g) \in \mathfrak{g}^{2g}$ for a general Lie group G (so for now, we don't use the fact that $G = \mathrm{SU}(2)$). More precisely, consider a curve $(A_1(t), \ldots, B_g(t))$ in G^{2g} with $A_i(0) = A_i$ and $B_i(0) = B_i$ such that

$$\theta\left(\frac{d}{dt}_{|t=0} A_i(t)\right) = a_i \quad \text{and} \quad \theta\left(\frac{d}{dt}_{|t=0} B_i(t)\right) = b_i$$

where θ is the Maurer-Cartan form. Applying μ gives the curve

$$t \mapsto \gamma(t) = \prod_{i=1}^{g} A_i(t) B_i(t) A_i(t)^{-1} B_i(t)^{-1}.$$

Using lemma 48 we can differentiate this curve. This results in

$$\begin{aligned}\theta\left(\frac{d}{dt}_{|t=0}\gamma(t)\right) &= \sum_{i=1}^{g}\Bigg(\mathrm{Ad}_{\left(\prod_{j>i}A_jB_jA_j^{-1}B_j^{-1}\right)^{-1}}\Big(\mathrm{Ad}_{B_iA_iB_i^{-1}}(a_i)\\ &\quad + \mathrm{Ad}_{B_iA_i}(b_i)\\ &\quad - \mathrm{Ad}_{B_iA_i}(a_i)\\ &\quad - \mathrm{Ad}_{B_i}(b_i)\Big)\Bigg)\\ &= \sum_{i=1}^{g}\left(\mathrm{Ad}_{\left(\prod_{j>i}A_jB_jA_j^{-1}B_j^{-1}\right)^{-1}B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}}-1)a_i + (1-\mathrm{Ad}_{A_i^{-1}})b_i\right)\right).\end{aligned}$$

The rank of μ at $(A_1,\ldots,B_g)$ is therefore the rank of the linear map $\mathfrak{g}^{2g}\to\mathfrak{g}$ given by

$$(a_1,\ldots,b_g)\mapsto\sum_{i=1}^{g}\left(\mathrm{Ad}_{\left(\prod_{j>i}A_jB_jA_j^{-1}B_j^{-1}\right)^{-1}B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}}-1)a_i + (1-\mathrm{Ad}_{A_i^{-1}})b_i\right)\right).$$

Suppose now that μ does not have full rank at $(A_1,\ldots,B_g)$. Then in particular for any fixed i the map

$$\mathfrak{g}^2\to\mathfrak{g} : (a,b)\mapsto(\mathrm{Ad}_{B_i^{-1}}-1)a + (1-\mathrm{Ad}_{A_i^{-1}})b$$

is not surjective, for otherwise take $a_j=b_j=0$ for $j\neq i$ and we obtain all of $\mathfrak{g}$ as the image of $T_{(A_1,\ldots,B_g)}\mu$ already by just varying a_i and b_i.

Let us now specialize to the case $G=\mathrm{SU}(2)$. We claim that the non-surjectivity of the above map $\mathfrak{g}^2\to\mathfrak{g}$ implies that A_i and B_i commute. Indeed, if either is $\pm\,\mathrm{Id}$, we are done. Otherwise, $(\mathrm{Ad}_{B_i^{-1}}-1)$ has as image the plane perpendicular to the rotation axis of Ad_{B_i}, and $(1-\mathrm{Ad}_{A_i^{-1}})$ has as image the plane perpendicular to the rotation axis of Ad_{A_i}. The sum of these planes (as linear subspace of three-dimensional space $\mathfrak{su}(2)$) must have dimension less than three, implying that they coincide. We conclude that Ad_{A_i} and Ad_{B_i} have the same axes of rotation, so that A_i and B_i commute. Because this argument can be made for any i, we conclude that if μ does not have full rank at $(A_1,\ldots,B_g)$, then

$$\theta\left(\frac{d}{dt}_{|t=0}\gamma(t)\right) = \sum_{i=1}^{g}\mathrm{Ad}_{B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}}-1)a_i + (1-\mathrm{Ad}_{A_i^{-1}})b_i\right).$$

The j'th term in this sum takes as values all elements of the plane perpendicular to the rotation axis of A_j (and of B_j, because these are the same axes), unless A_j and B_j are both $\pm\,\mathrm{Id}$ (then it is always zero). For μ not to have full rank, these planes must coincide for all possible j's, so that all the A_i and B_j that are not $\pm\,\mathrm{Id}$ have the same axis of rotation. In particular, all the A_i and B_j commute, proving that μ has full rank at all irreducible points. We also conclude that in the reducible case, μ does not have full rank, and that this rank is 0 if all A_i and B_i are $\pm\,\mathrm{Id}$ and exactly 2 otherwise. This proves the lemma. □

Proof of theorem 96. Notice that the action of $\mathrm{SU}(2)$ on $\mu^{-1}(e)$ is not free, because $-\mathrm{Id} \in \mathrm{SU}(2)$ acts trivially. We can, however, let the quotient $\mathrm{SU}(2)/\{\pm\mathrm{Id}\} \cong \mathrm{SO}(3)$ act on $\mu^{-1}(\mathrm{Id})$. We claim that this action is free, implying that $\mathcal{S}$ is the quotient of a manifold under a free action of a compact group, hence a manifold of dimension

$$\dim(\mu^{-1}(e)) - \dim(\mathrm{SO}(3)) = \dim(\mathrm{SU}(2)^{2g}) - \dim(\mathrm{SU}(2)) - \dim(\mathrm{SO}(3)) = 6g - 6$$

(recall lemma 64). Suppose that $[C] \in \mathrm{SU}(2)/\{\pm\mathrm{Id}\}$ acts trivially on $(A_i, B_i) \in \mu^{-1}(e)$. This means that C commutes with all A_i and B_i. Because (A_i, B_i) is irreducible, there are two matrices in $A_1, \ldots, B_g$ that do not commute. Call these matrices X and Y.

The situation is as follows: C commutes with both X and Y, but X and Y do not commute. The rotation axes of Ad_X and Ad_Y are different. The rotation axis of Ad_C cannot equal both of these axes, so Ad_C must be the identity, proving that $C = \pm\mathrm{Id}$. This proves that the action of $\mathrm{SU}(2)/\{\pm\mathrm{Id}\}$ on $\mu^{-1}(e)$ is free, establishing the result. □

Chapter 7

The symplectic structure

In this chapter we study in greater detail the symplectic structure on the block of the moduli space corresponding to the trivial bundle. We first build up some theory about the moduli space for simply connected base spaces. In this case, a flat connection has no holonomy, which simplifies the theory a lot. Recall theorem 40, which shows that restricting ourselves to the trivial bundle is not necessarily as drastic as it may seem.

7.1 Flat connections on trivial bundles over simply connected bases

We discuss flat connections on trivial principal bundles over simply connected base spaces. Suppose Σ is a simply connected manifold and G a Lie group. If we are given a flat connection A on the trivial bundle $\Sigma \times G$, then we can encode this connection as a map $E : P \to G$ as follows.

Definition 100. *Suppose A is a flat connection on the trivial principal bundle $\Sigma \times G$, with the base space Σ simply connected and G any Lie group. Then we define the* elevation with respect to p *of this connection as the map $E : P \to G$ such that*

$$q = (\text{end point of horizontal lift of curve from } \pi(p) \text{ to } \pi(q) \text{ starting at } p) \cdot E(q)$$

for all $q \in \Sigma \times G$.

Note that the elevation E is well-defined and smooth by flatness of the connection and simply connectedness of Σ. The elevation has a nice interpretation: it tells you how much a point differs from p if you identify all the fibres by parallel transport. In particular, a curve is horizontal precisely if it is contained in a level set of E.

We have the following obvious result.

Lemma 101. *If E is the elevation of a flat connection on a trivial bundle over a simply connected base space, then*

$$E(p \cdot g) = E(p) \cdot g$$

for all $p \in \Sigma \times G$ and $g \in G$.

The elevation of a connection determines the connection completely, as shown by the following lemma.

Lemma 102. *If A is a connection with elevation E, then we can recover A as*

$$A_p(v) = \theta(T_pE(v)) \quad \textit{for all } v \in T_p(\Sigma \times G).$$

Here, θ is the Maurer-Cartan form as before (see page 32).

Proof. Both sides of the equality are linear in v, so it suffices to check it on horizontal and vertical vectors. On horizontal vectors, both sides are zero (the left one by definition, the right one because horizontal curves have constant elevation). On vertical vectors, the result follows from the previous lemma. $\square$

We will be interested in tangent vectors to the space of all flat connections. What we thus want to do is to consider a family of flat connections A^t parametrized by a parameter t. We want to consider this as a curve in $\mathcal{A}_0$, the space of all flat connections on our trivial bundle.

We can define the tangent vector to this curve in $\mathcal{A}_0$ to be $A' : TP \to \mathfrak{g}$ given by

$$A'(v) = \frac{d}{dt}_{|t=0} A^t(v) \quad \text{for all } v \in TP.$$

Earlier we saw that $A' \in \Omega^1(\Sigma, \mathfrak{g}_P)$. Our goal for now is to recover A' from the elevations E^t of the connections A^t. To this end, define $E' : P \to \mathfrak{g}$ as

$$E'(p) = \theta\left(\frac{d}{dt}_{|t=0} E^t(p)\right).$$

We would like to find a link between A' and E'. Notice that E' is equivariant, in the sense that

$$E'(p \cdot g) = \mathrm{Ad}_{g^{-1}}(E'(p)) \quad \text{for all } p \in \Sigma \times G \text{ and } g \in G.$$

This means we can consider $E' \in \Omega^0(\Sigma, \mathfrak{g}_P)$, and we already have $A' \in \Omega^1(\Sigma, \mathfrak{g}_P)$. The link between A' and E' is now as nice as one could hope for.

Lemma 103. *We have*

$$A' = d_A E'$$

where $A = A^0$, the flat connection at time $t = 0$.

Proof. We will show that

$$A'(v) - dE'(v) = [A_p(v), E'(p)] \quad \text{for all } v \in T_p(\Sigma \times G),$$

where we consider A' and E' as $\mathfrak{g}$-valued forms on $\Sigma \times G$.

Let v be a tangent vector to $\Sigma \times G$ at p. Take a curve γ in $\Sigma \times G$ with $\gamma'(0) = v$. We evaluate the left hand side of the above equality. We find

$$\begin{aligned}(A')_p(v) - (dE')_p(v) &= \frac{d}{dt}_{|t=0} \left(A^t_p(v) \right) - \frac{d}{ds}_{|s=0} \left(E'(\gamma(s)) \right) \\ &= \frac{d}{dt}_{|t=0} \left(\theta \left(\frac{d}{ds}_{|s=0} E^t(\gamma(s)) \right) \right) - \frac{d}{ds}_{|s=0} \left(\theta \left(\frac{d}{dt}_{|t=0} E^t(\gamma(s)) \right) \right).\end{aligned}$$

Consider the map $f : (-\varepsilon, \varepsilon)^2 \to G$ given by $f(s,t) = E^t(\gamma(s))$. Now note that

$$\theta \left(\frac{d}{ds}_{|s=0} E^t(\gamma(s)) \right) = (f^*\theta)_{(0,t)}(\frac{d}{ds}),$$

where we have pulled back the $\mathfrak{g}$-valued Maurer-Cartan form via f. An analogous result holds for the t-variable. We have found

$$(A')_p(v) - (dE')_p(v) = \frac{d}{dt}_{|t=0} \left((f^*\theta)_{(0,t)} \left(\frac{d}{ds} \right) \right) - \frac{d}{ds}_{|s=0} \left((f^*\theta)_{(s,0)} \left(\frac{d}{dt} \right) \right).$$

Since $\frac{d}{ds}$ and $\frac{d}{dt}$ commute, this is the same thing as

$$(A')_p(v) - (dE')_p(v) = (d(f^*\theta))_{(0,0)} \left(\frac{d}{dt}, \frac{d}{ds} \right).$$

This leads to

$$\begin{aligned}(A')_p(v) - (dE')_p(v) &= f^*(d\theta)_{(0,0)} \left(\frac{d}{dt}, \frac{d}{ds} \right) \\ &= (d\theta) \left(T_{(0,0)} f \left(\frac{d}{dt} \right), T_{(0,0)} f \left(\frac{d}{ds} \right) \right).\end{aligned}$$

Using the Maurer-Cartan structure equation (lemma 47 on page 32) this reduces to

$$\begin{aligned}(A')_p(v) - (dE')_p(v) &= - \left[\theta \left(\frac{d}{dt}_{|t=0} E^t(\gamma(0)) \right), \theta \left(\frac{d}{ds}_{|s=0} E^0(\gamma(s)) \right) \right] \\ &= - \left[E'(p), A_p(v) \right].\end{aligned}$$

This is precisely what we wanted, and the lemma is proved. □

7.2 Application to compact orientable surfaces

Suppose Σ is a compact orientable surface of genus $g \geq 1$ and G is a Lie group admitting an Ad-invariant non-degenerate symmetric bilinear form on its Lie algebra (and a fixed such form is chosen). Take two families A^t_1 and A^t_2 of flat connections on the trivial

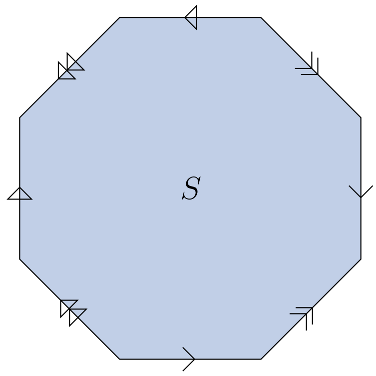

Figure 7: The fundamental domain S we will integrate over, here for genus 2. It is a polygon with $4g$ sides. To recover the surface Σ topologically, we can appropriately identify the sides pairwise. Integrating a differential 2-form over the compact orientable surface Σ can be accomplished by integrating the pullback of this form to $\tilde{\Sigma}$ over this fundamental domain. As opposed to Σ, the fundamental domain S has a boundary (but trivial fundamental group).

bundle $P = \Sigma \times G$ parametrized by a real parameter t in a neighbourhood of 0, in such a way that $A_1^0 = A_2^0 = A$. In other words, take two curves of flat connections on $\Sigma \times G$ which pass through A at time $t = 0$. Associate to these curves their initial velocities $A_1', A_2' \in \Omega^1(\Sigma, \mathfrak{g}_P)$. Explicitly, define

$$A_1'(v) = \frac{d}{dt}_{|t=0} \left(A_1^t(v) \right) \quad \text{for all } v \in T(\Sigma \times G),$$

with an analogous definition for A_2'. We shall try to find an expression for

$$\omega_{AB}(A_1', A_2').$$

The idea is to pull back the bundle to the universal cover of Σ to get rid of all holonomy.

7.2.1 Pulling back to the universal cover

The universal cover of a compact orientable surface of genus $g \geq 1$ is diffeomorphic to an open disk. This universal cover can be tiled with $4g$-gons in such a way that τ maps the interior of each polygon injectively to Σ and the edges of the polygon get mapped to curves on Σ (the standard generators of the fundamental group). All the vertices of such a polygon get mapped to the same point. Restricting τ to one of these polygons results in a topological quotient map from the polygon to Σ. This kind of polygon in $\tilde{\Sigma}$ is called a *fundamental domain*, and integrating a 2-form over Σ can be accomplished by integrating its pullback to $\tilde{\Sigma}$ over a fundamental domain. This is the strategy we will use to study the Atiyah-Bott 2-form. The case $g = 2$ is illustrated in figure 7.

Let $\tau : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Then we have a map

$$\tau \times \mathrm{Id}_G : \tilde{\Sigma} \times G \to \Sigma \times G.$$

This map is a local diffeomorphism and allows us to pull back the connections A_1^t and A_2^t to flat connections $\tilde{A}_1^t$ and $\tilde{A}_2^t$ on $\tilde{P} = \tilde{\Sigma} \times G$. Moreover, we can pull back A_1' and A_2' to get $\tilde{A}_1', \tilde{A}_2' \in \Omega^1(\tilde{\Sigma}, \mathfrak{g}_{\tilde{P}})$. Note that $\tilde{A}_1'$ and $\tilde{A}_2'$ are the initial velocities of the curves $\tilde{A}_1^t$ and $\tilde{A}_2^t$. The 2-form $\langle \tilde{A}_1' \wedge \tilde{A}_2' \rangle$ is the pullback of $\langle A_1' \wedge A_2' \rangle$ via the map τ.

We have

$$\begin{aligned} \omega_{AB}(A_1', A_2') &= \int_\Sigma \langle A_1' \wedge A_2' \rangle \\ &= \int_S \langle \tilde{A}_1' \wedge \tilde{A}_2' \rangle. \end{aligned}$$

Now associate to $\tilde{A}_1^t$ and $\tilde{A}_2^t$ the elevations E_1^t and E_2^t for a fixed base point $p \in \tilde{\Sigma} \times G$. Write $E_1^0 = E_2^0 = E$, so that E is the elevation of $\tilde{A} = \tilde{A}_1^0 = \tilde{A}_2^0$. Then we have

$$\omega_{AB}(A_1', A_2') = \int_S \langle d_A E_1' \wedge d_A E_2' \rangle.$$

By lemma 86 and Stokes' theorem, this means

$$\omega_{AB}(A_1', A_2') = \int_{\partial S} \langle E_1' \wedge d_A E_2' \rangle.$$

7.2.2 Integrating over a horizontal domain

We will try to simplify the above expression for $\omega_{AB}(A_1', A_2')$ further. Our next step is to get rid of the covariant exterior derivative and replace it with an ordinary exterior derivative. The idea is to consider $\langle E_1' \wedge d_A E_2' \rangle$ as a $\mathfrak{g}$-valued 1-form on $\tilde{P}$ and integrate this over a horizontally embedded version of S in $\tilde{P}$.

Recall that we fixed a base point p when we constructed the elevations E_1^t and E_2^t. The set of all points in $\tilde{P} = \tilde{\Sigma} \times G$ that can be reached from p using an A-horizontal curve forms an embedded submanifold of $\tilde{P}$. Indeed, it is the graph of the map

$$\tilde{\Sigma} \to G : x \mapsto E(x, e)^{-1}$$

since the points in this set are exactly the points for which the elevation is zero (there is exactly one such point in $\tilde{P}$ above every point of $x \in \tilde{\Sigma}$, and it is the point $E(x, e)^{-1}$).

Write $f : \tilde{\Sigma} \to \tilde{P} : x \mapsto (x, E(x, e)^{-1})$. We can consider E_1' and $d_A E_2'$ as $\mathfrak{g}$-valued forms on $\tilde{P}$ of degrees 0 and 1 respectively. Their pullbacks $f^*(E_1')$ and $f^*(d_A E_2')$ are $\mathfrak{g}$-valued forms on $\tilde{\Sigma}$. Note that $\langle f^*(E_1') \wedge f^*(d_A E_2') \rangle$ is exactly the 1-form $\langle E_1' \wedge d_A E_2' \rangle$ on $\tilde{\Sigma}$. However, because Tf maps every vector in $T\tilde{\Sigma}$ to a *horizontal* vector in $\tilde{P}$, we have $f^*(d_A E_2') = d(f^* E_2')$. This shows that in fact

$$\omega_{AB}(A_1', A_2') = \int_{\partial S} \langle f^* E_1' \wedge d(f^* E_2') \rangle.$$

Writing $F_i = f^*(E_i')$ for $i \in \{1,2\}$ results in

$$\omega_{AB}(A_1', A_2') = \int_{\partial S} \langle F_1 \wedge dF_2 \rangle.$$

Note that F_i is a smooth $\mathfrak{g}$-valued function on $\tilde{\Sigma}$ given by

$$F_i(x) = E_i'(x, E(x,e)^{-1}) = \mathrm{Ad}_{E(x,e)}(E'(x,e)).$$

7.2.3 Bringing in the holonomy

Pick $q = (x,g) \in \tilde{P}$. For any $y \in \tilde{\Sigma}$, we can pick a curve γ_y from x to y in $\tilde{\Sigma}$ and lift it to an $\tilde{A}$-horizontal curve $\gamma_{y,q}$ in $\tilde{\Sigma}$ starting at q. Then $(\tau \times \mathrm{Id}_g) \circ \gamma_{y,q}$ is the A-horizontal lift of $\tau \circ \gamma$ (which is a curve in Σ from $\tau(x)$ to $\tau(y)$). It starts at the point $(\tau(x), g)$ and ends at some point $(\tau(y), g_y)$.

Definition 104. *Suppose $\tau : \tilde{\Sigma} \to \Sigma$ is the universal cover of the manifold Σ and suppose that G is a Lie group. Given a flat connection on the principal bundle $\Sigma \times G$, we define its* extended holonomy with respect to $q \in \tilde{\Sigma} \times G$ *to be the map*

$$\mathrm{Hol}_q : \tilde{\Sigma} \to G : y \mapsto g^{-1} g_y. \tag{$*$}$$

If y happens to lie above $\tau(x)$, then $\mathrm{Hol}_q(y)$ is exactly the holonomy of the connection A around the curve $\tau \circ \gamma_y$ for the base point $(\tau(x), g)$. Note that $\mathrm{Hol}_q(y)$ does not depend on the chosen curve γ_y: all possible choices of such a curve are homotopic by simply connectedness of $\tilde{\Sigma}$, and horizontal lifts of homotopic curves have equal end points by flatness of the connection.

Lemma 105. *The extended holonomy as defined above is given by*

$$\mathrm{Hol}_q(y) = E(q)^{-1} E(x,e) E(y,e)^{-1} E(q).$$

Note that this is in fact equal to $E(q)^{-1}E(x,g)E(y,g)^{-1}E(q)$ for any $g \in G$.

Proof. If we lift γ_y to a horizontal curve $\gamma_{y,(x,E(x,e)^{-1})}$ in $\tilde{P}$ starting at $(x, E(x,e)^{-1})$ (where the elevation is e), then it will end in $(y, E(y,e)^{-1})$ (where the elevation is also e). By invariance of the horizontal subbundle, the curve $\gamma_{y,(x,E(x,e)^{-1})} \cdot E(x,e)g$ is a horizontal lift of γ_y starting at q, so it is $\gamma_{y,q}$. In particular, $\gamma_{y,q}$ ends in $(y, E(y,e)^{-1}E(x,e)g)$, so that $g_y = E(y,e)^{-1}E(x,e)g$. The result now follows from the definition of Hol_q and the observation that $E(q) = E(x,e)g$. □

Recall the situation we were in: we have connections A_i^t on P and $\tilde{A}_i^t$ on $\tilde{P}$ (for $i \in \{1,2\}$). The connection $\tilde{A}_i^t$ has elevation E_i^t with respect to some fixed base point $p \in \tilde{P}$. We fix some point $q \in \tilde{P}$, which we will later take to equal p, but for now we will do our calculations for general q. Associate extended holonomies $\mathrm{Hol}_{i,q}^t$ to the connections $\tilde{A}_i^t$.

Write Hol_q for the extended holonomy map of $\tilde{A}$ (so that $\mathrm{Hol}_q = \mathrm{Hol}^0_{1,q} = \mathrm{Hol}^0_{2,q}$). We want to find a link between the quantities E'_i and the holonomies. We do this by differentiating

$$\mathrm{Hol}^t_{i,q}(y) = E^t_i(q)^{-1} E^t_i(x,e) E^t_i(y,e)^{-1} E^t_i(q)$$

with respect to time at $t = 0$. Using lemma 48 gives

$$\begin{aligned}\theta\left(\frac{d}{dt}_{|t=0} \mathrm{Hol}^t_{i,q}(y)\right) &= -\,\mathrm{Ad}_{E(q)^{-1}E(y,e)E(x,e)^{-1}E(q)}(E'_i(q)) \\ &\quad + \mathrm{Ad}_{E(q)^{-1}E(y,e)}(E'_i(x,e)) \\ &\quad - \mathrm{Ad}_{E(q)^{-1}E(y,e)}(E'_i(y,e)) \\ &\quad + E'_i(q).\end{aligned}$$

Let us now take $q = p = (x, g)$ to simplify matters. Because all our elevations have p as a base point, $E^t_i(p) = e$ and therefore $F_i(x) = E'_i(p) = 0$ and also $E'_i(x,e) = 0$. We get

$$\begin{aligned}\theta\left(\frac{d}{dt}_{|t=0} \mathrm{Hol}^t_{i,p}(y)\right) &= -\,\mathrm{Ad}_{E(y,e)}(E'_i(y,e)) \\ &= -F_i(y).\end{aligned}$$

We conclude the following.

Theorem 106. *Given*

- *a Lie group G with a fixed* Ad*-invariant symmetric non-degenerate bilinear form on its Lie algebra,*
- *a compact orientable surface Σ of genus $g \geq 1$,*
- *two families A^t_1 and A^t_2 of flat connections on $\Sigma \times G$ smoothly parametrized by t such that $A^0_1 = A^0_2 = A$,*

write $A'_i \in \Omega^1(\Sigma, \mathfrak{g}_P)$ for the initial velocity of the curve A^t_i, and pick a fundamental domain S in the universal cover $\tau : \tilde{\Sigma} \to \Sigma$ as before. Then

$$\omega_{AB}(A'_1, A'_2) = \int_{\partial S} \langle F_1 \wedge dF_2 \rangle$$

where

$$F_i : \tilde{\Sigma} \to G : y \mapsto -\theta\left(\frac{d}{dt}_{|t=0} \mathrm{Hol}^t_{i,p}(y)\right)$$

and $\mathrm{Hol}^t_{i,p}$ is the extended holonomy of A^t_i.

7.2.4 The abelian case

In this subsection we will apply theorem 106 to the case where G is abelian. We write the group operation additively.

We start with a lemma. It is a direct consequence of the definitions and the fact that changing base points within a fibre does not change the holonomy.

Lemma 107. *Suppose G is an abelian Lie group and Σ a manifold. Let $\tau : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ, and let A be a flat connection on the trivial bundle $\Sigma \times G$. Let $\mathrm{Hol}_p : \tilde{\Sigma} \to G$ be the extended holonomy of A for some base point $p \in \tilde{\Sigma} \times G$. If $\gamma : [0,1] \to \tilde{\Sigma}$ is a curve in $\tilde{\Sigma}$ such that $\tau \circ \gamma$ is a loop, then*

$$\mathrm{Hol}_p(\gamma(1)) - \mathrm{Hol}_p(\gamma(0))$$

is the holonomy of A around $\tau \circ \gamma$ (for any base point lying above $\tau(\gamma(0)) = \tau(\gamma(1))$; note that changing base points within a fibre of $\Sigma \times G$ does not change the holonomy in the abelian case).

Proof. Write $p = (x, g)$. Take a curve δ in $\tilde{\Sigma}$ from x to $\gamma(0)$. Lift δ to an $\tilde{A}$-horizontal curve δ_p in $\tilde{\Sigma} \times G$. Then δ_p ends at $(\gamma(0), g + \mathrm{Hol}_p(\gamma(0)))$ by definition of the extended holonomy. Lift γ to an $\tilde{A}$-horizontal curve $\gamma_{\delta_p(1)}$ starting at $\delta_p(1)$. Then the concatenation $\delta_p \star \gamma_{\delta_p(1)}$ is a horizontal lift of a curve in Σ from x to $\gamma(1)$, so it ends at $(\gamma(1), g + \mathrm{Hol}_p(\gamma(1)))$.

This shows that $\gamma_{\delta_p(1)}$ is a horizontal lift of γ starting at $(\gamma(0), g + \mathrm{Hol}_p(\gamma(0)))$ and ending at $(\gamma(1), g + \mathrm{Hol}_p(\gamma(1)))$. Then $(\tau \times \mathrm{Id}) \circ \gamma_{\delta_p(1)}$ is a horizontal lift of $\tau \circ \gamma$ starting at $(\tau(\gamma(0)), g + \mathrm{Hol}_p(\gamma(0)))$ and ending at $(\tau(\gamma(1)), g + \mathrm{Hol}_p(\gamma(1)))$. The holonomy of this curve is therefore

$$\mathrm{Hol}_p(\gamma(1)) - \mathrm{Hol}_p(\gamma(0))$$

as was to be proved. □

We are now ready to apply theorem 106 to the case of an abelian structure group G. As in the theorem, we have

$$\omega_{AB}(A_1', A_2') = \int_{\partial S} \langle F_1 \wedge dF_2 \rangle.$$

Let us integrate over the edges of ∂S separately. The polygon S has $4g$ edges, which we will call $a_1, b_1, \bar{a}_1, \bar{b}_1, \ldots, a_g, b_g, \bar{a}_g, \bar{b}_g$ in counterclockwise order. Under the map $\tau : \tilde{\Sigma} \to \Sigma$, the edge a_i gets identified with $\bar{a}_i$ (in the opposite direction), and the edge b_i gets identified with $\bar{b}_i$ (also in the opposite direction). The case $g = 2$ was shown before in figure 7 on page 77.

Parametrize the edges a_i and $\bar{a}_i$ as curves $\gamma_a^i : [0,1] \to \tilde{\Sigma}$ and $\gamma_{\bar{a}}^i : [0,1] \to \tilde{\Sigma}$ in such a way that $\tau(\gamma_a^i(\lambda)) = \tau(\gamma_{\bar{a}}^i(1-\lambda))$ for all $\lambda \in [0,1]$. Take analogous parametrizations for the b_i and $\bar{b}_i$. The situation is illustrated in figure 8.

For any $\lambda \in [0,1]$, we can consider the curve γ as indicated in the right hand side picture in figure 8. This is a curve in ∂S running from $\gamma_a^i(\lambda)$ to $\gamma_{\bar{a}}^i(1-\lambda)$. Split this curve into three parts:

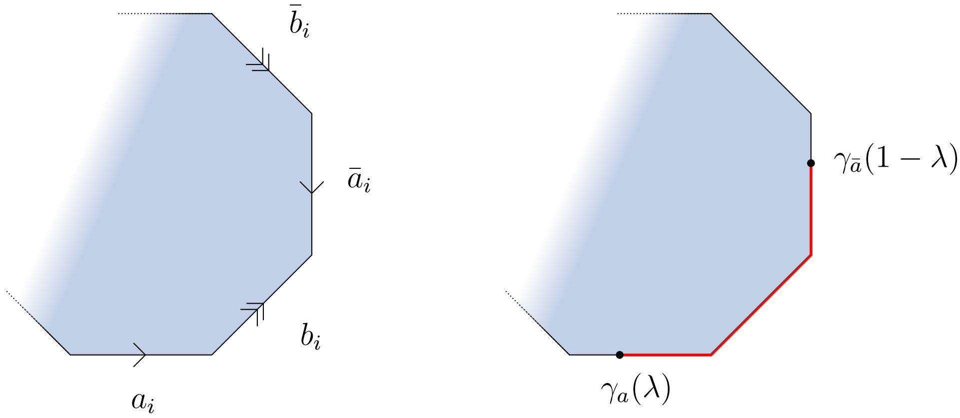

Figure 8: A group of four edges a_i, b_i, $\bar{a}_i$, $\bar{b}_i$ on the boundary of S (the others left out). On the left: the naming and identification conventions for the edges. On the right: the parametrizations γ_a^i and $\gamma_{\bar{a}}^i$ illustrated. The two marked points coincide under $\tau : \tilde{\Sigma} \to \Sigma$. If γ_a^i and $\gamma_{\bar{a}}^i$ are parametrized by arc-length on the drawing, then $\lambda \approx \frac{1}{3}$. Also indicated is the curve γ in red, running from $\gamma_a^i(\lambda)$ to $\gamma_{\bar{a}}^i(1-\lambda)$ along ∂S.

- γ_1 running from $\gamma_a^i(\lambda)$ to $\gamma_a^i(1) = \gamma_b^i(0)$,
- γ_2, which is just γ_b^i,
- γ_3 running from $\gamma_b^i(1) = \gamma_{\bar{a}}^i(0)$ to $\gamma_{\bar{a}}^i(1-\lambda)$.

Then by lemma 107 we have (for $j \in \{1, 2\}$)

$$\mathrm{Hol}_{p,j}^t(\gamma_{\bar{a}}^i(1-\lambda)) - \mathrm{Hol}_{p,j}^t(\gamma_a^i(\lambda)) = \mathrm{Hol}(A_j^t, \tau \circ \gamma)$$

where we have conveniently forgotten a base point on the right hand side because the choice of base point does not matter. Using $\gamma = \gamma_1 \star \gamma_2 \star \gamma_3$ we get

$$\mathrm{Hol}_{p,j}^t(\gamma_{\bar{a}}^i(1-\lambda)) - \mathrm{Hol}_{p,j}^t(\gamma_a^i(\lambda)) = \mathrm{Hol}(A_j^t, \tau \circ \gamma_1) + \mathrm{Hol}(A_j^t, \tau \circ \gamma_2) + \mathrm{Hol}(A_j^t, \tau \circ \gamma_3).$$

However, $\tau \circ \gamma_1$ and $\tau \circ \gamma_3$ are *the same curve traversed in the opposite direction*, so that their contributions cancel. Moreover, $\tau \circ \gamma_2$ is just the generator b_i of $\pi_1(\Sigma)$. Writing $b_{i,j}^t$ for the holonomy of A_j^t around this generator, we find

$$\mathrm{Hol}_{p,j}^t(\gamma_{\bar{a}}^i(1-\lambda)) - \mathrm{Hol}_{p,j}^t(\gamma_a^i(\lambda)) = b_{i,j}^t.$$

An analogous relation holds after switching the roles of a and b (with an additional minus sign because $\bar{a}_i$ is enclosed between b_i and $\bar{b}_i$, not a_i):

$$\mathrm{Hol}_{p,j}^t(\gamma_{\bar{b}}^i(1-\lambda)) - \mathrm{Hol}_{p,j}^t(\gamma_b^i(\lambda)) = -a_{i,j}^t.$$

We now differentiate these relations with respect to t at $t=0$. Writing $\theta\left(\frac{d}{dt}_{|t=0} b^t_{i,j}\right) = b'_{i,j}$ and $\theta\left(\frac{d}{dt}_{|t=0} a^t_{i,j}\right) = a'_{i,j}$ this gives

$$F_j(\gamma^i_{\bar{a}}(1-\lambda)) - F_j(\gamma^i_a(\lambda)) = b'_{i,j} \tag{†}$$

and

$$F_j(\gamma^i_{\bar{b}}(1-\lambda)) - F_j(\gamma^i_b(\lambda)) = -a'_{i,j}. \tag{‡}$$

Let us now further evaluate the integral in theorem 106. We use the parametrizations to integrate over all edges separately. This leads to

$$\begin{aligned}\omega_{AB}(A'_1, A'_2) = &\sum_{i=1}^{g}\left(\int_0^1 \langle F_1(\gamma^i_a(\lambda)), (dF_2)(\gamma^{i\prime}_a(\lambda))\rangle \, d\lambda\right)\\ &+\sum_{i=1}^{g}\left(\int_0^1 \langle F_1(\gamma^i_{\bar{a}}(\lambda)), (dF_2)(\gamma^{i\prime}_{\bar{a}}(\lambda))\rangle \, d\lambda\right)\\ &+\sum_{i=1}^{g}\left(\int_0^1 \langle F_1(\gamma^i_b(\lambda)), (dF_2)(\gamma^{i\prime}_b(\lambda))\rangle \, d\lambda\right)\\ &+\sum_{i=1}^{g}\left(\int_0^1 \langle F_1(\gamma^i_{\bar{b}}(\lambda)), (dF_2)(\gamma^{i\prime}_{\bar{b}}(\lambda))\rangle \, d\lambda\right).\end{aligned}$$

In the second and fourth of these terms, we substitute $\lambda \to (1-\lambda)$. By differentiating (†) and (‡) with respect to λ, we see that

$$dF_j(\gamma^{i\prime}_{\bar{a}}(1-\lambda)) = -dF_j(\gamma^{i\prime}_a(\lambda))$$

and analogously for b. Also by (†) and (‡), the first two terms can then be combined, as can the last two terms. We get

$$\begin{aligned}\omega_{AB}(A'_1, A'_2) = &\sum_{i=1}^{g}\left(\int_0^1 \langle -b'_{i,1}, (dF_2)(\gamma^{i\prime}_a(\lambda))\rangle \, d\lambda\right)\\ &+\sum_{i=1}^{g}\left(\int_0^1 \langle a'_{i,1}, (dF_2)(\gamma^{i\prime}_b(\lambda))\rangle \, d\lambda\right).\end{aligned}$$

This can be transformed to

$$\begin{aligned}\omega_{AB}(A'_1, A'_2) = &\sum_{i=1}^{g}\langle -b'_{i,1}, F_2(\gamma^i_a(1)) - F_2(\gamma^i_a(0))\rangle\\ &+\sum_{i=1}^{g}\langle a'_{i,1}, F_2(\gamma^i_b(1)) - F_2(\gamma^i_b(0))\rangle.\end{aligned}$$

Now by lemma 107 this is equivalent to

$$\begin{aligned}\omega_{AB}(A'_1, A'_2) &= \sum_{i=1}^{g}\left(\langle -b'_{i,1}, -a'_{i,2}\rangle + \langle a'_{i,1}, -b'_{i,2}\rangle\right)\\ &= \sum_{i=1}^{g}\langle b'_{i,1}, a'_{i,2}\rangle - \langle b'_{i,2}, a'_{i,1}\rangle.\end{aligned}$$

This explicit formula for the Atiyah-Bott 2-form is as nice as we could have hoped for. Notice that the antisymmetry of ω_{AB} is clear from this formula, whereas it wasn't apparent in theorem 106. All that is left for us to do in this chapter is to reformulate the result a little bit.

Theorem 108 (Symplectic structure in the abelian case, trivial bundle)**.** *Suppose Σ is a compact orientable surface of genus $g \geq 1$, and suppose G is an abelian Lie group with a fixed symmetric non-degenerate bilinear form on its Lie algebra. Then the moduli space $\mathcal{M}(\Sigma, G)$ is diffeomorphic to G^{2g} as shown before.*

Write $a_1, b_1, \ldots, a_g, b_g$ for the $2g$ projections to the copies of G composed with the inversion map $G \to G$ so that by abuse of notation $a_i \in G$ is the holonomy around the generator a_i of the fundamental group (and analogously for b). Then $\theta \circ (Ta_i) : T\,\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathfrak{g}$ and $\theta \circ (Tb_i) : T\,\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathfrak{g}$ are $\mathfrak{g}$-valued 1-forms on $\mathrm{Hom}(\pi_1(\Sigma), G)$. Call these 1-form da_i and db_i. Then the symplectic structure on the block $\mathcal{M}_{\Sigma \times G}(\Sigma, G)$ of the moduli space is given by

$$\omega_{AB} = \sum_{i=1}^{g} \langle db_i \wedge da_i \rangle.$$

Note that this formula reminds us of Darboux coordinates, and in the case of a 1-dimensional abelian group (there are two of those: the reals and the circle group), the b_i and a_i are in fact Darboux coordinates.

Conclusions and further research

This thesis has given the reader an introduction to moduli spaces of flat connections. After introducing the basic tools necessary for their study, we have constructed the moduli spaces $\mathcal{M}(\Sigma, G)$ as sets. We were able to equip the spaces with a topology and a smooth structure (in the sense that we have a notion of smooth function on the moduli spaces) by characterizing flat connections using their holonomy. The topology of the reducible part of the moduli space was found explicitly as a quotient of the product of maximal tori. In the special case $G = \mathrm{SU}(2)$ the irreducible part is smooth.

Based on the procedure of symplectic reduction, we have introduced a symplectic form on the moduli space. We have derived a formula for this symplectic structure as integral over a 1-chain instead of a 2-chain. In the case of an abelian structure group, this yields an explicit integral-free formula for the symplectic form.

The work done in this thesis can be extended in various interesting ways for future research. First of all, there are various directions in which the moduli spaces can be investigated without regard for their symplectic structure: it may be possible to extend the result obtained for SU(2) (theorem 96) to other structure groups such as SO(3), which is very similar so SU(2). Another interesting question is whether theorem 93 holds in the smooth sense, so whether the reducible part is actually diffeomorphic to the given quotient instead of merely homeomorphic. Besides these generalizations that seem to be close to the work already done, it may be worthwhile investigating the topology of the moduli space using the tools of cohomology theory and homotopy theory (as is done in part in [Jan05]).

Also the symplectic structure allows for many routes into further research. The most obvious one is to rigorously define the symplectic structure on the moduli space, for example using group cohomology (as in [Gol84]). As a second option, one could try to make the formula for the symplectic structure as given in theorem 106 even more explicit as was done in the abelian case. Indeed, all of chapter 7 was originally written by the author only for the abelian case, and most of it was later generalized to the non-abelian situation, so the same may be possible for the explicit formula. Finally, it may be interesting to allow for Σ a compact orientable surface with boundary, in which case the symplectic structure degenerates into a Poisson structure ([Wei95]).

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Vulgariserende samenvatting

In dit werk behandelen we zogenaamde *moduliruimten van vlakke connecties*. Deze hogerdimensionale ruimten duiken op bij de studie van hoofdvezelbundels, objecten die belangrijke toepassingen hebben in de topologie en de differentiaalmeetkunde.

De tekst veronderstelt een zekere achtergrond in de differentiaalmeetkunde, maar bouwt ook veel van de benodigde voorkennis zelf op. Ongeveer de helft van de tekst bestaat uit vereiste voorkennis uit de differentiaalmeetkunde en aanverwante gebieden. Lie-groepen, bundels, vectorwaardige differentiaalvormen, symplectische meetkunde en Poissonmeetkunde komen aan bod.

De tweede helft van de tekst is gewijd aan de studie van de moduliruimten van vlakke connecties zelf. Als eerste wordt uitgelegd wat deze ruimten juist zijn, en hoe we ze precies kunnen beschouwen als (topologische) ruimte. Ook wordt er uiteengezet in welke zin de moduliruimten singulariteiten bevatten. Verder wordt de topologie van deze moduliruimten bestudeerd. Ten slotte wordt uitgelegd hoe de moduliruimten van vlakke connecties in sommige gevallen uitgerust kunnen worden met een symplectische structuur, en wordt er een expliciete formule opgesteld voor deze symplectische structuur in speciale gevallen.

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