Moduli Spaces of Flat Connections

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Moduli Spaces of Flat Connections

Define them

Chapters 1 - 5

Today: shortcut

Study them

Chapters 6 - 7

Two levels:

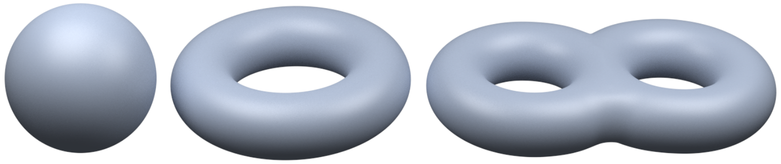
- topology
- symplectic structure

Defining the Moduli Spaces $\mathcal{M}(\Sigma, G)$

Ingredients:

Surface

Base space Σ



Lie group

Structure group G

$$\boxed{\mathcal{M}(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G}$$

Acts by conjugation

Defining the Moduli Spaces $\mathcal{M}(\Sigma, G)$

$$\boxed{\mathcal{M}(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G}$$

Picking generators of $\pi_1(\Sigma)$ we have

$$\mathrm{Hom}(\pi_1(\Sigma), G) \subseteq G^n$$

$\Longrightarrow$ Topology on $\mathcal{M}(\Sigma, G)$

Studying $\mathcal{M}(\Sigma, G)$: topology

We will assume G compact, connected.

Then: Maximal torus T

Maximal product of circles

Weyl group W = Normalizer(T)/T

Finite group acting on the torus

Studying $\mathcal{M}(\Sigma, G)$: topology

Consider $\mathcal{R} \subseteq \mathcal{M}(\Sigma, G)$

Assumed compact

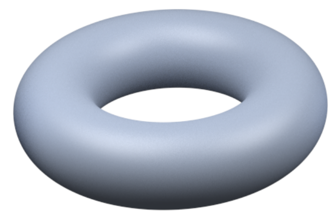
of all morphisms whose image is abelian.

Up to conjugation

$$\boxed{\mathcal{R} \cong \frac{T^{2g}}{W}}$$

Genus of base space

Studying $\mathcal{M}(\Sigma, G)$: topology

Example: $\Sigma =$ 

Then $\mathcal{R} = \mathcal{M}(\Sigma, G)$

$G = \mathrm{SU}(2)$

$$\boxed{\mathcal{M}(\Sigma, G) = \mathcal{R} \cong \frac{S^1 \times S^1}{\mathbb{Z}/2\mathbb{Z}} \cong S^2}$$

Studying $\mathcal{M}(\Sigma, G)$: symplectic structure

A symplectic structure is a two-form ω that is

Closed

$d\omega = 0$

Non-degenerate

At every point the bilinear pairing is non-degenerate

Find one on $\mathcal{M}(\Sigma, G)$?

Studying $\mathcal{M}(\Sigma, G)$: symplectic structure

Necessary: extra structure on G

Ad-invariant inner product on the Lie algebra

For abelian G: explicit formula

For part of the moduli space

If $G = \mathbb{R}$ then $\mathcal{M}(\Sigma, G) = \mathbb{R}^{2g}$ and

$$\boxed{\omega = \sum_{i=1}^{g} dx_i \wedge dx_{i+g}}$$