Optimal Strategies for Olympic Weightlifting

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Abstract

In competitive weightlifting competitions, athletes make strategic decisions regarding which weights to lift. Currently, there is limited literature on weightlifting strategies that aim to maximize the winning probability. This paper addresses this gap by developing a computationally efficient model for Olympic weightlifting. We model Olympic weightlifting with a Markov decision process (MDP) and solve it using game theory. Specifically, we identify the subgames in the MDP, prove they have a unique perfect equilibrium, and provide an efficient algorithm for finding this equilibrium. Our model is robust and scalable, providing a starting point for determining optimal weightlifting strategies.

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1 Introduction

Weightlifting is commonly associated with impressive demonstrations of strength. While this captures the essence of the sport, it does overlook an essential aspect: the strategy behind weight selection. In competitions weightlifters have three attempts to lift the heaviest weight possible. Therefore, they need to decide whether to attempt heavier weights, risking failure, or attempt lighter weights and rely on others failing.

The strategies in weightlifting are unique due to the sport's characteristics. Unlike track and field events with fixed difficulties, weightlifters choose their own difficulty. Moreover, weightlifters' performances are censored, since there is no partial credit for incomplete lifts, this is unlike gymnastics and figure skating where partial credit is awarded. While this censoring is shared with high jumping and pole vaulting, these sports do not have the same scarcity of attempts, since there athletes can continue trying until they fail three times (IAAF, 2024).

The literature on optimal weightlifting strategies is sparse. Lilien (1974) explores the problem from an individual perspective but does not consider interactions with other athletes. To the author's knowledge, there is no literature that examines these interactions specifically in the context of weightlifting. Games of timing (see Radzik (1996) for a comprehensive history and introduction) have similar strategic considerations to those in weightlifting. However, analysis of these games is typically focused on the two player case. Extensions to cases with more than two players, such as those explored by Presman and Sonin (2006), tend to be highly technical.

This paper aims to fill this gap by developing a model for Olympic weightlifting. This is done by modelling the problem as a Markov decision process (MDP) and solving it with game theory. Our main contribution is providing a scalable model for Olympic weightlifting. We achieve this by decomposing our MDP into many smaller games. We then provide an efficient algorithm to solve these games. Moreover, we prove that this algorithm produces the sole well-behaved solution. Lastly, we compare the model with actual weightlifting strategies and suggest possible future extensions of our work.

This paper is organized as follows. Section 2 covers the rules of Olympic weightlifting. Section 3 formulates the mathematical model. An algorithm for solving this model and proof of its well-behaved solution are provided in Section 4. Section 5 considers the problem of estimating the ability of the athletes. The model is compared to competition data in Section 6. Lastly, Section 7 discusses the limitations of the model, outlines avenues for future research, and concludes the paper.

2 Problem Definition

The International Weightlifting Federation (IWF) sets the standards and guidelines for Olympic weightlifting (Catalyst Athletics, 2022; IWF, 2023). All IWF events follow the same format, consisting of a weigh-in session, a warm-up period, and the main competition, which includes two disciplines: the Snatch, and the Clean and Jerk.

At the beginning of the competition day, athletes are issued an athlete's card (AC), containing information such as weight class, entry total (what an athlete expects to lift in the first round, this is used for preliminary sorting), a random lot number, and personal details. The event starts with the weigh-in session, where athletes are weighed and sorted into their respective weight classes. At the weigh-in, the athlete's bodyweight and the weight of their starting attempts are recorded on the AC. The weight an athlete intends to lift is called the attempt weight. The combined attempt weights of the first Snatch and Clean and Jerk attempts must not be less than the entry total minus 20kg. This is commonly known as the 20 kg rule and incentivizes athletes to give a representative entry total. The starting attempt weights can be changed up to two times. After the weigh-in, the athletes proceed to the warm-up area.

The main competition starts with the Snatch discipline. During the competition, the barbell starts light and is progressively loaded with heavier weights. Athletes take turns attempting lifts according to the calling order, which is determined based on the attempt weight (lowest first), the number of remaining attempts (most attempts first), the sequence of previous attempts (least recent first), and the athlete's lot number (lowest first). The success of each lift is determined by three judges who evaluate technique and adherence to the rules. After each attempt, the athlete declares the attempt weight for their next attempt. This weight cannot be less than the weight of the barbell, but can be changed up to two times before the next attempt. The Snatch discipline is finished when each athlete has had three attempts or opted to stop earlier. After a short break, the procedure is repeated for the Clean and Jerk discipline. The overall winner is determined by the combined weight lifted in both disciplines. In the event of a tie, the athlete with the lowest bodyweight is declared the winner.

3 Mathematical Modelling

In this section, we develop a mathematical model for Olympic weightlifting. We begin by discussing the challenges that make modelling the IWF rules difficult and impractical. Then, to address these challenges we introduce a simplification of the IWF competition, the (n, k)-Olympic competition. We formalize this competition as a multi-agent Markov decision process (MDP), and describe this MDP in Sections 3.3 through 3.8. We cover the solution of the MDP in Section 4.

3.1 (n, k)-Olympic Competition

An Olympic weightlifting model under the IWF rules would have an enormous action and state spaces. The action space would be large due to athletes being able to adjust their attempt weights at any time, while the state space is large because the amount lifted in each discipline needs to be tracked to determine the winner.

Therefore, we introduce the (n, k)-Olympic competition as a more manageable alternative. The (n, k)-Olympic competition has n athletes competing in a single discipline. During the competition, the barbell is loaded with weights from the set $W = \{w_1, \ldots, w_m\}$ where $w_1 < \cdots < w_m$. These weights are such that all athletes can lift w_1 and that no one can lift w_m, an example would be $w_1 = 1$ kg and $w_m = 500$ kg. The competition starts with w_1 loaded onto the barbell. The athletes with attempts left can indicate whether they want to attempt a lift. If there are multiple athletes wanting to attempt, then one is selected according to the calling order. The calling order depends first on the number of attempts left (highest first) and then on the lot number (lowest first). After each attempt the weight is kept the same and the athletes can indicate again if they want to lift. It is important to remark that all previous indications are reset. If no one wants to lift, the weight on the barbell is incremented. This process repeats until all athletes have had k attempts or the loaded weight is w_m. The athlete who has lifted the most is the winner. In the event of a tie, the athlete with the lowest bodyweight is declared the winner.

The (n, k)-Olympic competition differs from the IWF competition in three ways. The first, and most important, difference is how the attempt weights are selected. In the (n, k)-Olympic competition, athletes indicate whether they want to attempt to lift the current weight on the barbell, as opposed to the IWF competition, where athletes select their attempt weight from W. The weight selection mechanism in the (n, k)-Olympic competition is preferred from a modelling perspective because it results in a smaller action space. Moreover, the weight selection mechanism of the (n, k)-Olympic competition is still similar to that of the IWF competition. To see this, consider an IWF competition where athletes can adjust their attempt weights without restrictions (not just two times). In such a competition, athletes can effectively indicate, for each weight loaded on the barbell, whether they want to attempt or not. This means that the weight selection mechanism of the (n, k)-Olympic competition is an unrestricted case of the weight selection mechanism of the IWF competition.

Secondly, in the (n, k)-Olympic competition, entry totals and the associated 20 kg rule are dropped, since their sole purpose is to sort athletes into groups of comparable skill. Not considering the 20 kg rule has minimal impact on the choices of the athletes and makes the action space more regular. Lastly, the (n, k)-Olympic competition only features one type of lift. As a consequence, we do not need to keep track of the exact amounts that the athletes have lifted, only their ranking, to determine the winner. This greatly reduces the state space.

3.2 Example

Consider athletes A, B, and C, with lot numbers 2, 3 and 1 respectively, competing in a $(3, 2)$-Olympic competition. Let $w_1 = 50$ kg, the barbell then weights 50 kg initially. B and C intend to lift. C, with the lowest lot number, attempts and succeeds. Subsequently, no other athlete expresses intent to lift 50 kg, so the weight is incremented to 51 kg, however again no athlete expresses intent to lift, leading to an increase to 52 kg. A and B intend to lift 52 kg, and A, with the lowest lot number, attempts but fails. A wishes to attempt again but fails once more. B and C do not want to attempt 52 kg, so the barbell is incremented to 53 kg. Both B and C express intent to lift. B, having more attempts left, tries and succeeds. With no further expressions of intent, the weight increases to 54 kg. C intends to lift and succeeds. Afterwards, B attempts and succeeds as well. The competition is finished because no athletes have attempts left. The final results are that A, B, and C lifted 0, 54, and 54 kg, respectively. A is therefore last, C is second by tiebreak, and B wins.

3.3 State Space

The state space, S, comprises tuples of the form $s = (w, L, d, k_1, \ldots, k_n)$ where: $w \in W$ represents the current weight of the barbell. L denotes the current player in the lead, that is, the player who has lifted the most so far. If no player has lifted anything, then $L = \emptyset$. $d \in \{\text{False}, \text{True}\}$ indicates whether the player in the lead, L, has lifted w. This is important for players whose tiebreaks are worse than the leader's, since if the leader lifted w, that is $d = \text{True}$, then the players with inferior tiebreaks cannot gain the lead by lifting the current weight loaded on the barbell. Lastly, $k_i \in \{0, \ldots, k\}$ indicates the number of lifting attempts player i has left.

The starting state of the MDP is $(w_1, \emptyset, \text{False}, k, \dots, k)$. The terminal states of the MDP are the states where a lead change is impossible. This is at least the case when the barbell is so heavy it cannot be lifted, i.e, $w = w_m$, or when all non-leading players have no attempts left, i.e., $k_i = 0$ for all $i \neq L$. With this notation, the example of Section 3.2 can be represented succinctly as the following sequence of states:

$$
\begin{aligned}
&(50, \emptyset, \text{False}, 2, 2, 2) \to (50, C, \text{True}, 2, 2, 1) \to \\
&(51, C, \text{False}, 2, 2, 1) \to \\
&(52, C, \text{False}, 2, 2, 1) \to (52, C, \text{False}, 1, 2, 1) \to (52, C, \text{False}, 0, 2, 1) \to \\
&(53, C, \text{False}, 0, 2, 1) \to (53, B, \text{True}, 0, 1, 1) \to \\
&(54, B, \text{False}, 0, 1, 1) \to (54, C, \text{True}, 0, 1, 0) \to (54, B, \text{True}, 0, 0, 0).
\end{aligned}
$$

This example makes clear that either the weight increases, or the amount of attempts left for one player decreases. This means that no state can be revisited and that there can be at most $m + k \times n$ transitions. The MDP therefore has a finite horizon and a solution. This solution can, in principle, be found with dynamic programming.

3.4 Action Space

In any state s, all players with remaining attempts can signal intent to lift. Player i's possible actions in state s are therefore,

$$
A_{s,i} = \begin{cases} \{\text{Pass}, \text{Lift}\}, & \text{if } k_i > 0, \\ \{\text{Pass}\}, & \text{if } k_i = 0. \end{cases}
$$

The action space, A_s, is subsequently given by $A_s = A_{s,1} \times \cdots \times A_{s,n}$. We will denote a generic tuple of actions by $a \in A_s$.

3.5 Player Selection

The action a expresses the intentions of the players, not who lifts next. To determine that we also need the calling order.

We formalize the calling in state s as a permutation π_s. It depends on the number of remaining attempts and the lot numbers. The lot numbers are also a permutation over the players and will be denoted by π_{lot}. By the rules laid out in Section 3.1, π_s is such that $\pi_s(i) < \pi_s(j)$ if and only if one of the following is true: $k_i > k_j$, or $k_i = k_j$ and $\pi_{\text{lot}}(i) < \pi_{\text{lot}}(j)$.

The function Next, which depends on the calling order, maps the intentions of the players to which player will lift next and is defined as,

$$
\text{Next}(s, a) = \begin{cases} \emptyset, & \text{if } a_i = \text{Pass for all } i, \\ \underset{i \in \{1,\dots,n\}}{\operatorname{argmin}} \{\pi_s(i) : a_i = \text{Lift}\}, & \text{otherwise.} \end{cases}
$$

3.6 Lifting Probabilities

When an athlete attempts to lift a weight, they are not always successful. To model this let $G_{i,l}(w)$ denote the probability of player i lifting weight w in their l-th attempt. To ensure that $G_{i,l}$ is sensible we make regularity assumptions. Firstly, we assume that

$G_{i,l}(w)$ is non-increasing and continuous in w. Secondly, we assume that $G_{i,l}(0) = 1$ and that $G_{i,l}(w_m) = 0$. Under these assumptions $G_{i,l}$ is the survival function of a non-negative continuous random variable. This also leads to a nice interpretation, if the ability of a player is distributed according to $1 - G_{i,l}$, then they are successful at lifting the weight w if their realised ability is greater than w.

While the ability distribution can depend on the attempt number, this will not be done in this paper. Here we assume that player i their ability is independent and identically distributed according to $F_i = 1 - G_i$ for all attempts. With these lifting success probabilities the transition probabilities of the MDP can be determined.

3.7 Transition Probabilities

Consider a non-terminal state $s = (w_l, L, d, k_1, \ldots, k_n)$, where the intentions are a. If $\text{Next}(s, a) = \emptyset$, then the weight of the barbell will be incremented and the resulting state will therefore be $s' = (w_{l+1}, L, \text{False}, k_1, \ldots, k_n)$ with probability one. If $\text{Next}(s, a) = i$, then there are two possible subsequent states, depending on the success of player i.

If player i is unsuccessful, then only k_i is decremented, so the resulting state is $s' = (w_l, L, d, k_1, \ldots, k_i - 1, \ldots, k_n)$. This happens with probability $F_i(w)$.

If player i is successful, then the state is updated as follows: w does not change, i becomes the player in the lead if they have preferential tiebreaks or if the current player in the lead did not lift w, the player in the lead lifted w successfully so $d = \text{True}$, lastly, k_i is decremented. Summarized: $s' = (w_l, L', \text{True}, k_1, \ldots, k_i - 1, \ldots, k_n)$, where

$$L' = \begin{cases} i, & \text{if } d = \text{False or if } i < L, \\ L, & \text{otherwise.} \end{cases}$$

Player i being successful happens with probability $G_i(w)$. The probability of transitioning from state s to s' given action a will be denoted as $P(s'|s, a)$.

3.8 One Step Rewards

In this model we assume that the athletes' only goal is to win. Therefore, athletes only receive a reward when they win. Winning occurs when the lead cannot change, that is, only when the current state is terminal. The reward of player i in state s is therefore given by:

$$r_i(s) = \begin{cases} \mathbb{1}\{L = i\}, & \text{if } s \text{ is terminal,} \\ 0, & \text{otherwise.} \end{cases}$$

We have now defined all the components of the MDP.

4 Game Theoretic MDP Solution

Solving a multi-agent MDP differs fundamentally from solving a single-agent (or typical) MDP due to the need to account for interactions between agents. In a typical MDP, the optimal policy maximizes the expected cumulative reward for the single agent. In contrast, a solution to a multi-agent MDP consists of a set of policies, such that each agent's policy maximizes its expected cumulative reward, given the policies of the other agents.

For typical MDPs, it is evident that an optimal policy exists, since it is a maximizer over a finite set of policies. However, for multi-agent MDPs, the existence of an optimal policy is not as evident, despite the finiteness of the policy set. This is because it is not trivial whether there exists a policy which satisfies the optimality condition. In our case, the MDP has a special structure: it is finite, and states cannot be revisited. This allows us to interpret the MDP as a finite game where the policies are the strategies. Nash (1951) informs us that such games, and thus our MDP, have a solution, a Nash equilibrium (NE). However, determining NEs is a hard problem in general.

To address this, we use the special structure of our MDP to decompose it into several subgames, which can be solved more easily. Specifically, let $S_0 \subset S$ denote all terminal states. These states have a known value, and can be used to define and solve the subgames associated with the states S_1, the states which solely transition to states within S_0. Repeating this process for states in $S_2, S_3, \ldots, S_{m+k\times n}$ enables us to solve the entire MDP.

In this section, we formally describe these subgames using the strategic form, present an algorithm to solve them efficiently, and prove that this solution is the sole well-behaved NE.

4.1 Strategic Form

The strategic form of a game consists of a set of players, pure-strategy spaces, and payoff functions (Fudenberg and Tirole, 1991).

The set of players is $I = \{1, \ldots, n\}$. The pure-strategy space for player i in state s is $A_{s,i}$.

We define $u_i(j; s)$ to be player i's winning probability in state s if player j lifts. The payoff function of player i in state s for strategy a is then $u_i(\text{Next}(s, a); s)$, we denote this concisely as $u_i(a; s)$.

The payoff values are determined as follows: Suppose that $s \in S_h$, then s transitions to a state in $N_h = \bigcup_{l=0}^{h-1} S_l$. The values of states in N_h, denoted by $\eta_i(s')$ for $s' \in N_h$, are known when state s is considered. Thus,

$$u_i(a; s) = \sum_{s' \in N_h} P(s'|s, a)\eta_i(s').$$

For a mixed strategy $A \sim \sigma = (p_1, \ldots, p_n)$, where $\mathbb{P}(A_i = \text{Lift}) = p_i$, the expected payoff is,

$$\begin{aligned}\mathbb{E}[u_i(\text{Next}(s, A); s)] &= \sum_{j=1}^{n} u_i(j; s) \cdot \mathbb{P}(\text{Next}(s, A) = j) + u_i(\emptyset; s) \cdot \mathbb{P}(\text{Next}(s, A) = \emptyset) \\ &= \sum_{j=1}^{n} u_i(j; s) \cdot p_j \prod_{l \in \{x: \pi_s(x) < \pi_s(j)\}} (1 - p_l) + u_i(\emptyset; s) \cdot \prod_{l=1}^{n} (1 - p_l).\end{aligned}$$

For brevity, we denote $\mathbb{E}[u_i(\text{Next}(s, A); s)]$ by $u_i(\sigma; s)$. Lastly, if $x = (x_1, \ldots, x_m)$ then we write (y, x_{-i}) to denote $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m)$.

4.2 Solution Algorithm

It remains to determine an NE for each of these subgames. To do this, first note that if the player first in the calling order intends to lift, then the intentions of the other players do not matter. Moreover, the player last in the calling order can only lift if no other players decide to lift. This fact endows the payoff function with a sequential structure, which is visualised in Figure 1. In Figure 1, $\pi_s^{-1}(i)$ is the index of the i-th player in the calling order, and $u(j; s) = (u_1(j; s), \ldots, u_n(j; s))$.

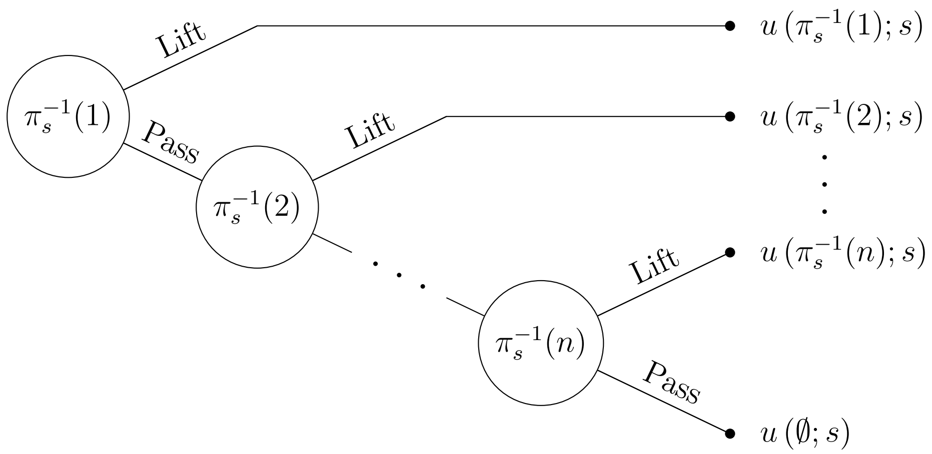

Figure 1: Graphical representation of the subgame.

This sequential structure suggests that an NE can be found systematically with backward induction. This method is implemented in Algorithm 1.

Algorithm 1 Efficient solution algorithm

$s \leftarrow (w, L, d, k_1, \ldots, k_n)$ ▷ (Section 3.3)
$a^* \leftarrow (\text{Pass}, \ldots, \text{Pass})$
for $i \in \{\pi_s^{-1}(n), \pi_s^{-1}(n-1), \ldots, \pi_s^{-1}(1)\}$ **do**
  if $k_i = 0$ **then**
    continue
  end if
  $a^*_{\text{alt}} \leftarrow (\text{Lift}, a^*_{-i})$
  if $u_i(a^*_{\text{alt}}; s) \geq u_i(a^*; s)$ **then**
    $a^* \leftarrow a^*_{\text{alt}}$
  end if
end for

In this method players select the action that maximizes their payoff given that their choice affects the outcome. For example, the last player in the calling order, denoted by i, will choose to lift if $u_i(i; s) \geq u_i(\emptyset; s)$.

Theorem 4.1. *Algorithm 1 computes a Nash Equilibrium.*

Proof. Recall that a strategy profile a^* is an NE, if for all players $i \in I$, $u_i((a_i^*, a_{-i}^*); s) \geq u_i((a_i, a_{-i}^*); s)$, for all $a_i \in A_{s,i}$. That is, the strategies are best responses to each other. Consider player $i = \pi_s^{-1}(n)$ and suppose that no one intends to lift. Player i's best response then is,

$$a_i^* = \begin{cases} \text{Pass}, & \text{if } u_i(\emptyset; s) > u_i(i; s), \\ \text{Lift}, & \text{if } u_i(\emptyset; s) \leq u_i(i; s). \end{cases}$$

Here it assumed that players choose to lift when the payoffs are equal. The rational is that players prefer to have easier (lighter) lifts. However, strictly there is not a unique best response in this case.

If another player does intend to lift then, by the calling order, player i's strategy does not influence the payoff. In that case, every strategy, including a_i^*, is a best response.

Now consider player $j = \pi_s^{-1}(n-1)$. If someone before j in the calling order indicates to lift then every strategy is a best response. If no one before them in the calling order lifts then player j will choose to lift, if $u_j(j; s)$ is greater than or equal to the payoff they get under the choice of player i. This is precisely what Algorithm 1 does. This inductive argument can be extended to show that a myopic method results in an NE. $\square$

4.3 Multiple NEs Problem

The subgame can have more NEs than the one Algorithm 1 computes. These NEs may also have different payoffs. Consider the game in Figure 2, here $a_1^* = (\text{Pass}, \text{Pass}, \text{Pass})$ is the NE that Algorithm 1 finds. However, according to the definition of the NE,

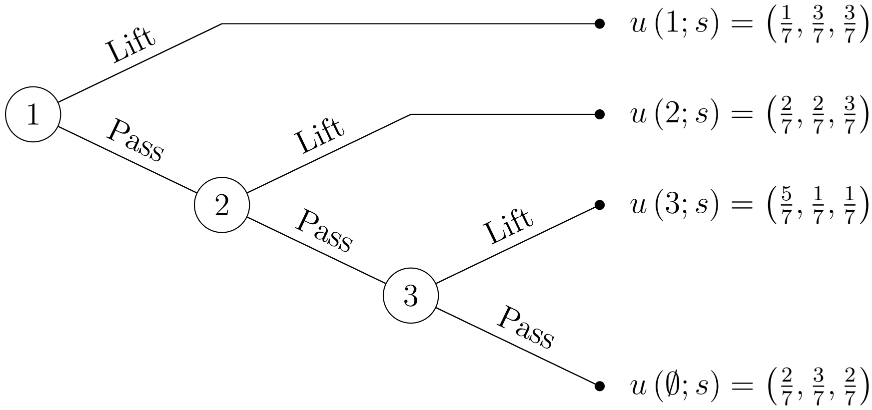

Figure 2: Example subgame with multiple Nash Equilibria.

$a_2^* = (\text{Pass}, \text{Lift}, \text{Lift})$ is also an NE.

Having multiple NEs is problematic because all players need to follow the same NE for the game to reach a stable outcome. This requires coordination, but this is not possible because the players are competitive.

4.4 Trembling Hand Perfect Refinement

Because a single equilibrium is desired, we consider a refinement of the NE, the Trembling-Hand Perfect (THP) equilibrium, as introduced by Selten (1975). This refinement strengthens the notion of rationality by being robust to small errors.

Definition 4.2 (THP equilibrium)**.** *A (mixed) strategy profile σ is a THP equilibrium if there exists a sequence of totally mixed strategy profiles $\{\sigma^m\}_{m=1}^{\infty}$ which converge to σ such that, for all players $i \in I$ and m, σ_i is a best response to σ_{-i}^m (Fudenberg and Tirole, 1991).*

Applying this refinement to the game in Figure 2, we find that a_2^* does not constitute a THP equilibrium. This is because player 3 obtains a strictly larger utility from choosing Pass over Lift, since players 1 and 2 both select Pass with a non zero probability. We in fact prove the following.

Theorem 4.3. *Algorithm 1 computes the only Trembling-Hand Perfect equilibrium.*

Proof. Let $\sigma^m = (p_1^m, \ldots, p_n^m)$ and for ease of notation assume without loss of generality that $\pi_s(i) = i$ (the calling order is in order). By the definition of the THP equilibrium p_n is such that the following payoff is maximized,

$$u_n((p_n, \sigma_{-n}^m); s) = \sum_{j=1}^{n-1} u_n(j; s) p_j^m \prod_{l<j} (1 - p_l^m) + (u_n(n; s) p_n + u_n(\emptyset; s)(1 - p_n)) \prod_{l=1}^{n-1} (1 - p_l).$$

For the purposes of maximization the first term can be ignored since it is constant with respect to p_n. The resulting relevant payoff is,

$$(u_n(n; s) p_n + u_n(\emptyset; s)(1 - p_n)) \prod_{l=1}^{n-1} (1 - p_l).$$

The relevant payoff is linear in p_n and non-zero, because all $p_l^m \in (0, 1)$, therefore,

$$p_n = \begin{cases} 0, & \text{if } u_n(\emptyset; s) > u_n(n; s), \\ 1, & \text{if } u_n(\emptyset; s) \le u_n(n; s), \end{cases}$$

is the unique maximizer (note that there is no unique maximizer at equality, but here the early lifting assumption is invoked). Consequently,

$$p_n^m = \begin{cases} \varepsilon_n^m, & \text{if } u_n(\emptyset; s) > u_n(n; s), \\ 1 - \varepsilon_n^m, & \text{if } u_n(\emptyset; s) \le u_n(n; s), \end{cases}$$

where $\varepsilon_n^m \to 0$ as $m \to \infty$. Now, p_{n-1} is such that the following relevant payoff is maximized,

$$(u_{n-1}(n-1; s) p_{n-1} + (u_{n-1}(n; s) p_n^m + u_{n-1}(\emptyset; s)(1 - p_n^m))(1 - p_{n-1})) \prod_{l=1}^{n-2} (1 - p_l^m).$$

The relevant payoff is again linear and non-zero, so

$$p_{n-1} = \begin{cases} 0, & \text{if } u_{n-1}(n; s) p_n^m + u_{n-1}(\emptyset; s)(1 - p_n^m) > u_{n-1}(n-1; s), \\ 1, & \text{if } u_{n-1}(n; s) p_n^m + u_{n-1}(\emptyset; s)(1 - p_n^m) \le u_{n-1}(n-1; s), \end{cases}$$

is the unique maximizer (we again make use of the early lifting assumption). p_{n-1} depends on the trembles, but these can be made so small that they do not affect the inequality. Therefore,

$$p_{n-1} = \begin{cases} 0, & \text{if } u_{n-1}(n; s) p_n + u_{n-1}(\emptyset; s)(1 - p_n) > u_{n-1}(n-1; s), \\ 1, & \text{if } u_{n-1}(n; s) p_n + u_{n-1}(\emptyset; s)(1 - p_n) \le u_{n-1}(n-1; s). \end{cases}$$

Recursively applying the above method will yield a unique THP equilibrium. This equilibrium is the same equilibrium as the NE in Theorem 4.1, so Algorithm 1 computes the only Trembling-Hand Perfect equilibrium of the subgame.

□

5 Estimating the Lifting Probabilities

To compute the policies described in Section 4, we need the ability distributions of the players. In this section we present methods for estimating these distributions from competition data. We start by considering the specifics of competition data, and then discuss stationarity and independence problems. We finish this section by comparing the estimation methods.

5.1 Competition Data

The competition data we use was obtained through web scraping from the International Weightlifting Results Project (IWRP) website. This data consists of pairs (w_{ti}, z_{ti}), where w_{ti} is the attempt weight and z_{ti} is the success indicator for the t-th competition and the i-th lift. Under the assumptions of Section 3.6, the athlete's ability can be represented as a latent random variable X_t such that the probability that the athlete lifts weight w_{ti} is $\mathbb{P}(X_t > w_{ti})$. The data therefore contains interval censored information about the athlete's ability. Unfortunately, this means that each X_t only has three censored data points associated with it.

To overcome this limitation we examine the data more closely. In Figure 3 we have visualised the competition data of an athlete's entire career. This athlete was selected for

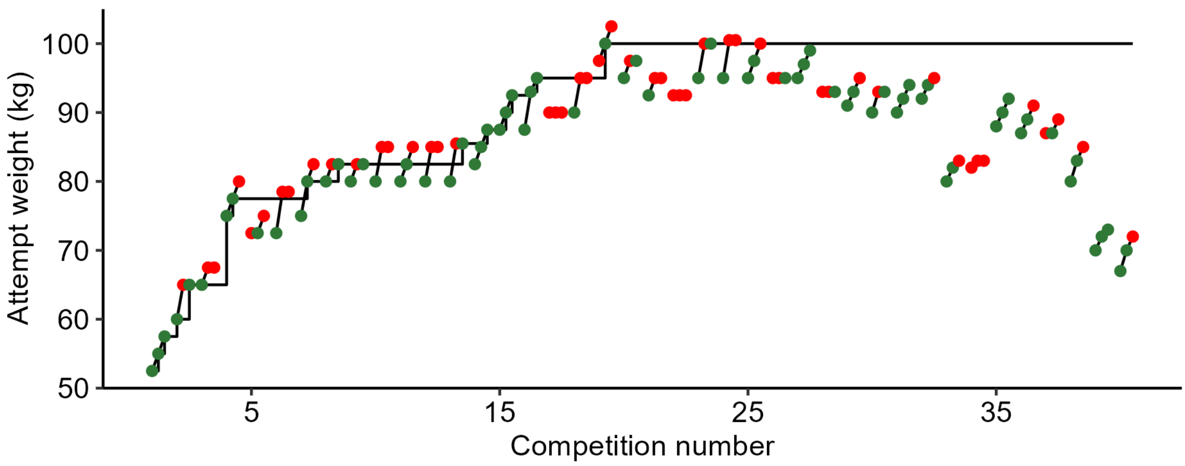

Figure 3: Athlete's career performance: Each set of three represents a competition. Successful lifts (green), unsuccessful lifts (red), and personal best (black).

their typical career trajectory: improvement in the beginning, a mid-career plateau, and a decline as they age. The career in Figure 3 spans from age 15 to 43, which is longer than usual. Most athletes retire earlier and exhibit a less pronounced decline.

Focusing on the individual competitions, we see that the first attempt is usually successful, the second attempt is mostly successful, and the last attempt often fails. This pattern indicates that the athletes have an excellent awareness of their ability and that their ability has minimal variation during a competition, as the gap between certain lifts and failures is only a few weight increments.

This minimal variance informs us that the most important information regarding an athlete's ability is the first attempt weight. This suggests that a Bayesian approach could be fruitful for estimating an athlete's ability. This is, however, beyond the scope of this paper. We will instead remove the non-stationary trend to pool the data.

5.2 Stationarity

The ability distribution is non-stationary for three main reasons: athletes' bodyweight changes, athletes become more (or less) capable by training (or lack thereof), and athletes age.

An athlete's bodyweight is very important for their lifting capacity, as can be seen in Figure 4, where 1000 successful Snatch attempts from the IWRP database for each 5 kg weight range are plotted. The IWRP dataset lacks a gender indicator, so the performances in Figure 4 are mixed.

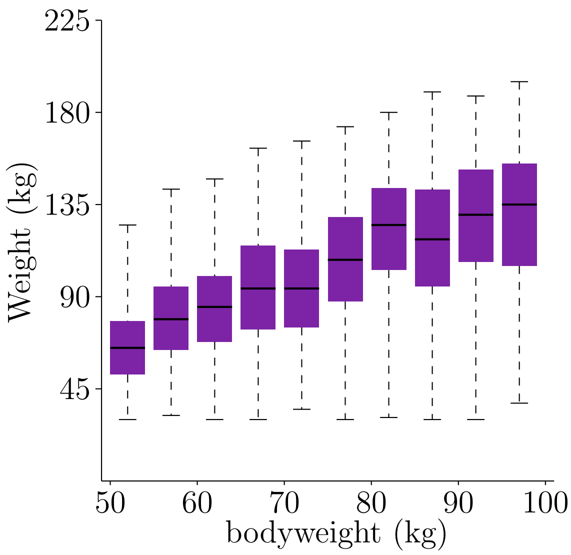

Figure 4: Bodyweight against snatch attempt weight.

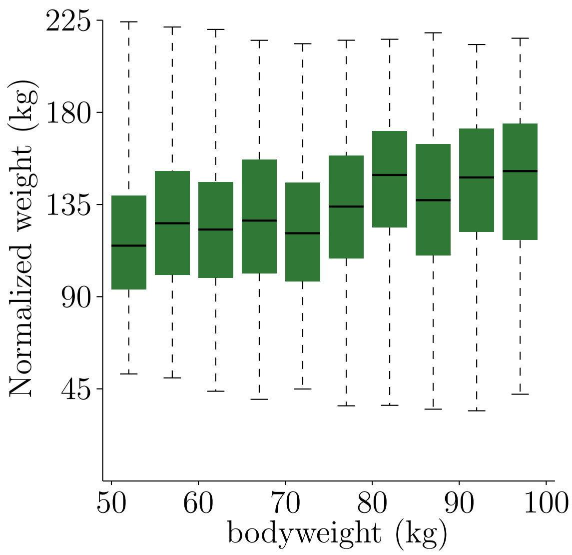

Figure 5: Bodyweight against normalized snatch attempt weight.

Kauhanen et al. (2002) extensively compare multiple methods for standardizing weightlifting results, including the Sinclair method used by the IWF, and conclude that their non-parametric method, which they call the Golden Standard (GS), performs best. They encapsulate their GS in a simple regression equation: to counteract the bodyweight effect, scale the results by a factor of $\mathrm{GS}(\text{bodyweight}) = 1 + 160800 \cdot \text{bodyweight}^{-3.087}$. For example, if someone weighing 75 kg lifts 150 kg then their normalized lift is 189.3 kg.

In Figure 5 this normalization has been applied to results of Figure 4. We see that top athletes in each category obtain a similar normalized performance and that there is an upward trend in the median. The upward trend in the median is likely a byproduct of the gender mixing. This is because men are stronger on average than women and constitute a larger part of the heavier population. The trend should therefore disappear with properly labeled data. Still, the fact that the top athletes' performances are similar indicates that the normalization removes the bodyweight effect.

It is obvious that athletes can become more or less skilled over time. To deal with this non-stationarity, we assume that X_t depends linearly on the athlete's skill and bodyweight effect. Specifically, we assume that,

$$X_t \sim \frac{\text{skill}_t}{\text{GS}(\text{bodyweight}_t)} X,$$

where X is stationary.

We propose three metrics for skill_t, a constant, which will serve as the null, the personal best (PB), and an exponentially weighted moving average of the attempt weights of the first attempts (FA). The latter two can formally be written as, $\text{PB}_T = \max\{w_{ti} : t < T \text{ and } z_{ti} = 1\}$ and $\text{FA}_T = \frac{1-\delta}{\delta-\delta^T} \sum_{t=1}^{T-1} \delta^{T-t} \cdot w_{t1}$ respectively, where δ is a smoothing parameter.

The PB as a skill metric is self-explanatory. The FA skill metric is motivated by the following. Figure 3 informs us that athletes know their own ability well, and that they want to succeed in their first attempt. This means that the first attempt can be seen as a quantile, for example 20% quantile, of the ability distribution. If the specific quantile does not change significantly over time then the first attempts encode relevant information about the athlete's skill. Moreover, it is also more flexible than the PB metric, since it can decrease.

If there are observations for which no data is available then, the median skill metric value of the bodyweight category is used. Lastly, we denote the normalized data pairs by $(\tilde{w}_{ti}, z_{ti})$.

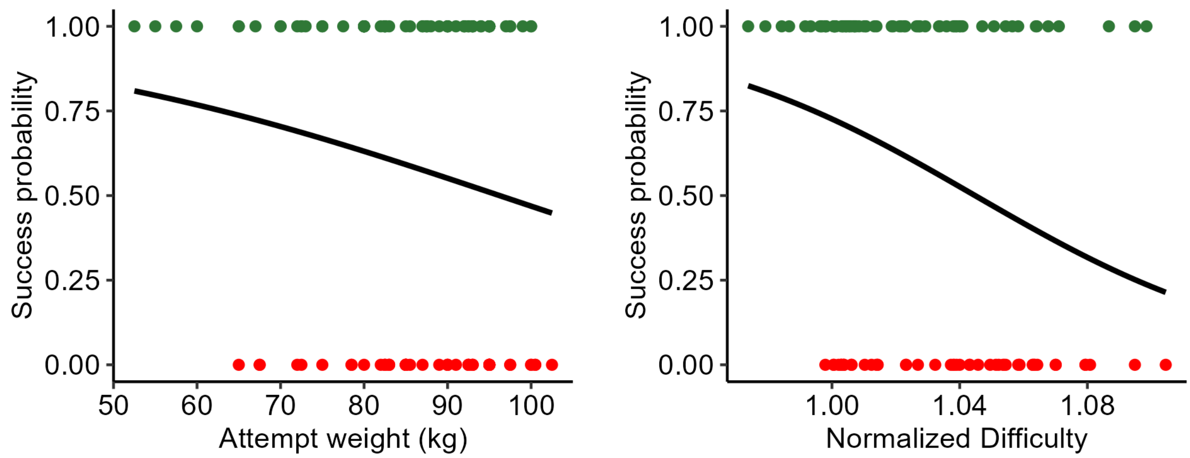

Figure 6: Logit model with (right) and without (left) FA normalization.

In Figure 6, we fitted a logit model to both the non-normalized and FA normalized data. As expected, there are few failures for attempts below a normalized difficulty of one, which correspond to the weight of the first attempt. Furthermore, we see that the logit curve is steeper after normalization. This indicates that normalization reduces variance. This is desirable, since Figure 3 suggests that there is a low variance during competitions.

5.3 Dependence Structure

Fitting a logit model to the data, as depicted in Figure 6, is somewhat questionable, because it implicitly assumes that the ability is independent of the (normalized) attempt weight. This assumption is problematic because athletes can choose their attempt weights. We find it likely that when athletes feel strong (weak), they select a heavier (lighter) weight. We therefore posit that there is a small positive correlation between the attempt weight and the athlete's ability.

To see the impact of erroneously using the independence assumption, consider the case where $X \sim \text{Gamma}(40, 0.5)$ and where the attempt weight is uniformly distributed between 85% and 105% of the realized ability (x_i). In this case, athletes can anticipate their ability to some extent and choose weights they expect to succeed with. This results in a positive correlation between the ability and the attempt weight.

In Figure 7 we see the Weibull and isotonic non-parametric estimates of the ability under the incorrect assumption of independence. The non-parametric estimator is included to show that the parametric assumption is not at fault. The estimators are clearly inconsistent, since their bias does not decrease for a larger sample size.

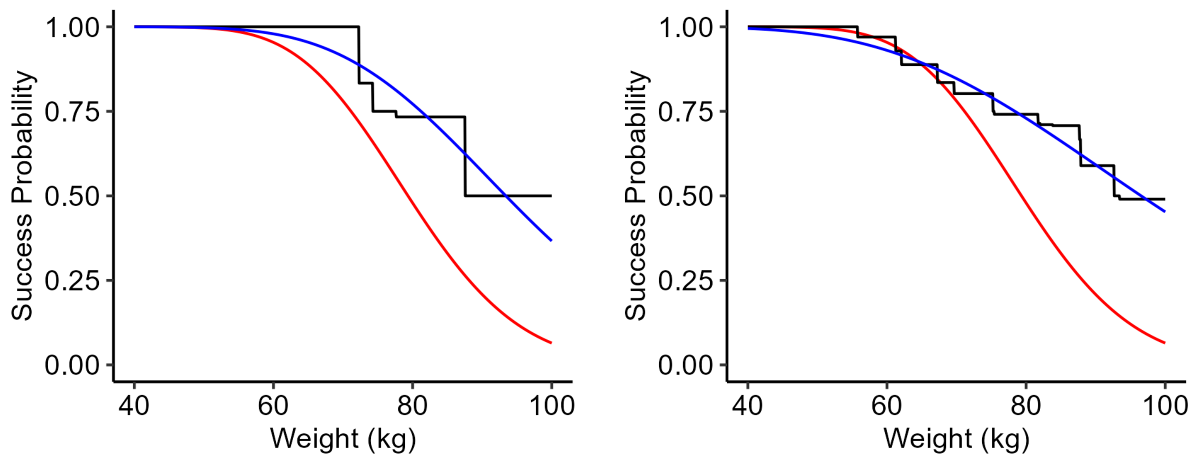

Figure 7: Comparison underlying ability (red) against inconsistent parametric (blue) and non-parametric (black) estimates for 50 (left) and 1000 (right) observations.

Since biased estimators are undesirable and because we suspect dependence between attempt weight and ability, we would like to test this dependence. Unfortunately, and surprisingly, this cannot be done, since the dependence structure of this type of data cannot, in general, be identified (Sun, 2006, Chapter 10). Therefore, the dependence structure between the attempt weight and ability becomes a model choice.

Given our reasons to suspect a positive correlation, we choose to include a dependence structure using copulas. Specifically, we select a Frank copula with a moderately positive rank correlation coefficient of 0.25. This choice of copula is motivated by the fact that the Frank copula has no tail dependencies, which would be obfuscated due to the censoring (McNeil et al., 2015; Titman, 2014).

5.4 Shape Estimation

To estimate the marginal distribution of the normalized ability under this dependence structure, we use Titman (2014)'s maximum likelihood procedure. To start, denote the joint and marginal distributions of attempt weight and the ability by $F_{X,W}$, F_X and F_W respectively. By design, $F_{X,W}(x, w) = C(F_X(x), F_W(w))$ where C is the Frank copula.

Using this, the probability of the event $(\tilde{w}_{ti}, z_{ti})$ can be written as,

$$\begin{aligned}
\mathbb{P}(z_{ti} = 0, W = \tilde{w}_{ti}) &= \mathbb{P}(X \leq \tilde{w}_{ti}, W = \tilde{w}_{ti}) \\
&= \lim_{h \to 0} \frac{C(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti} + h)) - C(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti}))}{h} \\
&= C_v(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti})) f_w(\tilde{w}_{ti}) \\
\mathbb{P}(z_{ti} = 1, W = \tilde{w}_{ti}) &= (1 - C_v(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti}))) f_w(\tilde{w}_{ti}),
\end{aligned}$$

where C_v is the partial derivative of C with respect to its second argument. With these probabilities the log-likelihood of the competition data is,

$$\sum_{i,t} \left(\log(f_w(\tilde{w}_{ti})) + z_{ti} \log(1 - C_v(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti}))) + (1 - z_{ti}) \log(C_v(F_X(\tilde{w}_{ti}), F_W(\tilde{w}_{ti})))\right).$$

Titman proposes to replace F_W by the empirical CDF, since it is a consistent estimator. After this substitution the log-likelihood becomes,

$$\sum_{i,t} \left(z_{ti} \log(1 - C_v(F_X(\tilde{w}_{ti}), \hat{F}_W(\tilde{w}_{ti}))) + (1 - z_{ti}) \log(C_v(F_X(\tilde{w}_{ti}), \hat{F}_W(\tilde{w}_{ti}))) \right).$$

This can be maximized non-parametrically with Titman's method, or parametrically with an appropriate choice of parametric family. The parametric distribution we consider are the gamma and Weibull distributions. These are chosen, because they are both non-negative and have different tail behaviour. Moreover, we also consider a kernel-smoothed version of the non-parametric estimator.

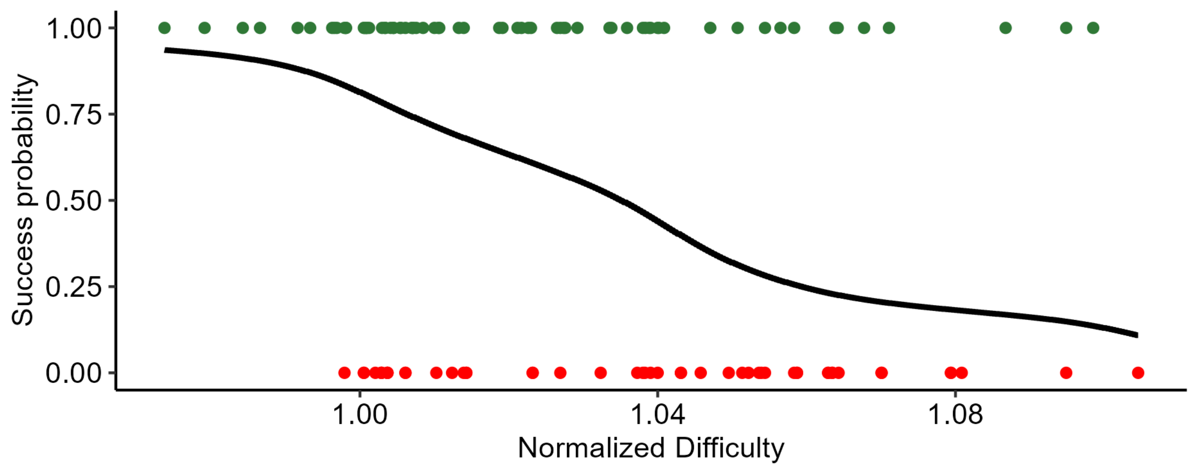

Figure 8: Kernel smoothed non-parametric estimator with copula dependence structure.

In Figure 8 we see the kernel smoothed non-parametric estimate of the success probability. The biggest difference with the estimate in Figure 6, is that the range of the ability is now almost covered by the competition data. This brings it further in line with the empirical observations from Figure 3. It should be noted that the non-parametric estimator is less efficient than the parametric estimators. This is not a problem when a reasonable amount of data is available, but for athlete's with shorter careers this can cause overfitting.

5.5 Results

To determine the most effective method for estimating the ability distributions, we compare all combinations of shape and skill estimation techniques. The comparison was conducted on a representative subset of 150 athletes, where the data was split into training and testing subsets for each individual athlete. The performance of the methods was evaluated based on their median log-likelihood statistic, with the non-parametric model serving as the null model for comparison.

Shape	Skill Metric	Training	Test
Weibull	Constant	0.268 (0.111)	0.415 (0.234)
Gamma	Constant	0.326 (0.175)	0.458 (0.231)
Smoothed NP	Constant	0.269 (0.090)	0.364 (0.207)
Weibull	PB	0.267 (0.165)	0.426 (0.297)
Gamma	PB	0.322 (0.197)	0.439 (0.273)
Smoothed NP	PB	0.266 (0.130)	0.362 (0.203)
Weibull	FA	0.284 (0.153)	0.420 (0.273)
Gamma	FA	0.507 (0.281)	0.621 (0.368)
Smoothed NP	FA	0.382 (0.172)	0.456 (0.249)

Table 1: Median log-likelihood statistic with the IQR in parenthesis.

The results in Table 1 show that the PB based skill metric is marginally better than the null and significantly better than the FA metric. This is surprising, as it indicates that skill is relatively constant throughout an athlete's career.

The fact that the gamma distribution, a thick tailed distribution, performs poorly further strengthens our earlier observations from Figure 3 that the ability distribution has a low variance. The Weibull and non-parametric estimator give similar results on the training data, but the non-parametric estimator generalizes better. Therefore, in the subsequent analysis, we will use the PB metric and the non-parametric estimator.

6 Model Results

In this section, we compare the weight selection strategies of Section 4 with those observed in actual weightlifting competitions, which we will refer to as the MDP and empirical policies respectively. Furthermore, we investigate the impact of the number of attempts on the outcome of the competition.

6.1 Strategy Comparison

Athletes have to balance between choosing heavy weights and successfully lifting them. Therefore, the most natural way to summarize an athlete's selection policy is by considering the probabilities of them lifting their selected weights. An athlete's policy is therefore a triplet of probabilities (q_1, q_2, q_3), where $q_1 \geq q_2 \geq q_3$. We map these values to the unit cube using the transformation $\left(1 - q_1, 1 - \frac{q_2}{q_1}, 1 - \frac{q_3}{q_2}\right)$ to better visualize them. Here, the first entry represents the initial difficulty, while the second and third entries indicate the first and second increases in difficulty, respectively.

We compare the MDP and empirical policies for all tournaments in the IWRP dataset which have up to seven participants, as this is our computational limit for calculating the MDP policies. For each tournament, we first estimate the athletes' ability distribution using the method of Section 5. This directly yields the empirical policies. Using these ability distributions, we construct and solve the MDP of Section 3. We then generate a sample tournament from the MDP solution and extract its policies.

The resulting MDP and empirical policies are visualized in Figure 9. The figure

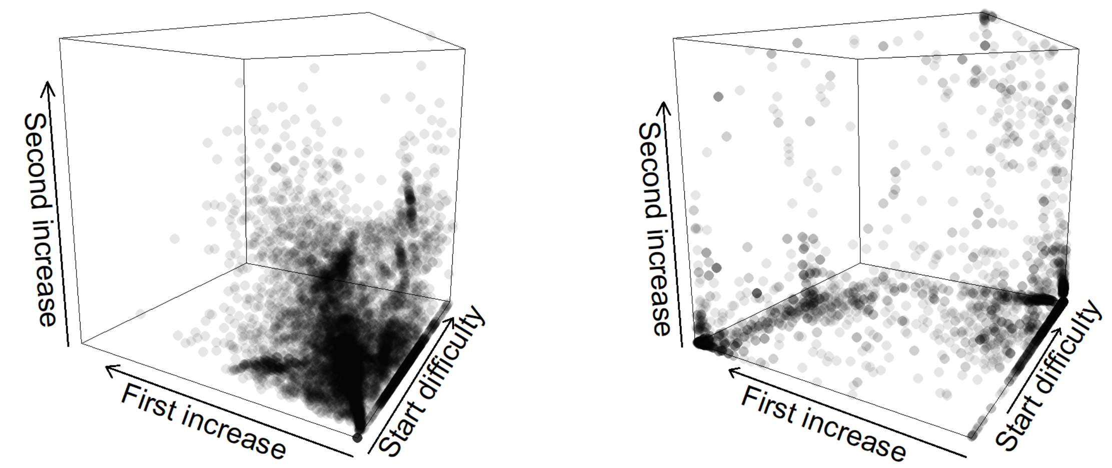

Figure 9: Scatter plot of empirical policies (left) and MDP policies (right)

shows us that the policies differ significantly. The empirical policies generally start with a relatively low difficulty, 0.23 on average, and then increase the difficulty with small steps. This is inline with our expectations from Figure 3. In contrast, the MDP policies begin with a much higher average difficulty of 0.77. Moreover, the MDP policies are show concentration at the extremes: 62.9% of the time MDP policies involve lifting the same weight three times, compared to only 8.2% for the empirical policies. Additionally, the MDP policies increase the weight on all attempts only 9.2% of the time, whereas the empirical policies do so 59.0% of the time.

The fact that the predominant MDP policy is to lift the same weight three times makes sense when you consider the fact that the model these policies are based on only rewards first place. This incentivizes the weaker athletes to go all out, since bombing out is equivalent to second place. Moreover, this extra competition pushes the stronger athletes to also consider riskier strategies.

This also suggests an extension of the model, namely adding rewards for other placements. This was not pursued here because it would significantly increase the state space, because more than just the current athlete in the lead would have to be tracked.

6.2 Conversion Rate

Another way to compare the MDP and empirical policies is by considering how often the best athlete wins.

To analyse this, we first clarify the definition of best athlete. Let p_i denote the probability that athlete i realises the greatest ability, that is $p_i = \mathbb{P}(X_i > X_j \text{ for all } j \neq i)$. We then define the best player to be the player with the highest p_i. Using order statistic notation we denote the p_i's of the best and second athlete by $p_{(1)}$ and $p_{(2)}$ respectively. Moreover, to measure the extent of the best player's dominance, we consider how $p_{(1)}$ relates to $p_{(2)}$.

For a tournament with n athletes we have the following constraints on $p_{(1)}$ and $p_{(2)}$: $p_{(1)} \geq p_{(2)}$, $p_{(1)} + p_{(2)} \leq 1$ and $p_{(2)} \geq \frac{1-p_{(1)}}{n-1}$. Therefore, $(p_{(1)}, p_{(2)})$ is constrained to the triangle $T_n = \{(x_1, x_2) : x_1 + x_2 \leq 1, x_1 \geq x_2, x_2(n-1) \geq 1 - x_1\}$.

For all tournaments with up to seven participants, we first estimate the athletes' ability distribution, and use them to compute the p_i's. This results in a $(p_{(1)}, p_{(2)})$ pair and an indicator whether the best player won. We fit a function η to this data, which gives the winning (conversion) rate of the best player. This is done non-parametrically under the constraint that a higher $p_{(1)}$ and $p_{(2)}$ correspond to a higher and lower conversion rate respectively. Figure 10 presents the results of these non-parametric regressions. We

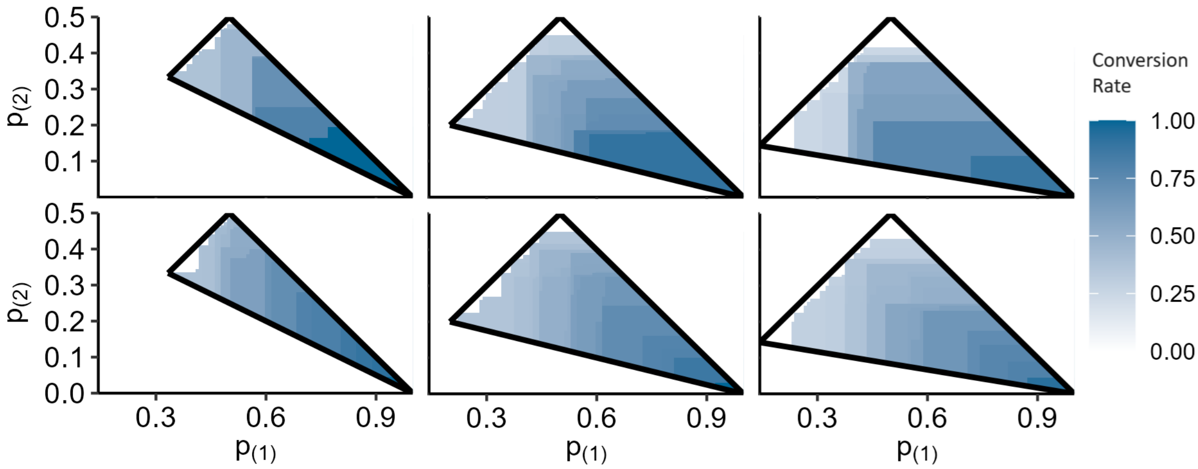

Figure 10: Conversion rate for tournaments with three (left), five (middle) and seven (right) athletes under the empirical policies (top) and the MDP policies (bottom).

observe that the conversion rate is lower under the MDP policies. This is likely because, unlike in actual competitions where athletes might be satisfied with second or third place, in the MDP model all athletes are trying to win. This extra competition naturally results in a lower winning probability for the best player. This further provides evidence for the hypothesis that the rewards are not entirely correct.

6.3 Asymptotic Conversion rate

Although the MDP model does not entirely capture empirical weightlifting policies, it is still a useful tool for understanding overall competition dynamics. Here we investigate how the mean conversion rate changes with the number of attempts.

The conversion rate depends on the athletes' ability distributions in the tournament. To evaluate the mean conversion rate, we first define a random tournament with realistic ability distributions.

For our random tournament we draw the ability distributions from the Weibull family, since Table 1 informs us that the Weibull distribution is a good parametric model for this. Moreover, we also want the skill gaps between the athletes to be sensible. Therefore we adjust our Weibull parameter sampling such that the $p_{(1)}$ and $p_{(2)}$ of the tournament are uniform over T_n. This is achieved by first generating a large set of tournaments and then selecting a subsample of them which uniformly cover T_n by means of farthest point sampling.

The mean conversion rate for this type of random tournament can be seen in Figure 11. We see that the conversion rate is about 60% for competitions with three attempts.

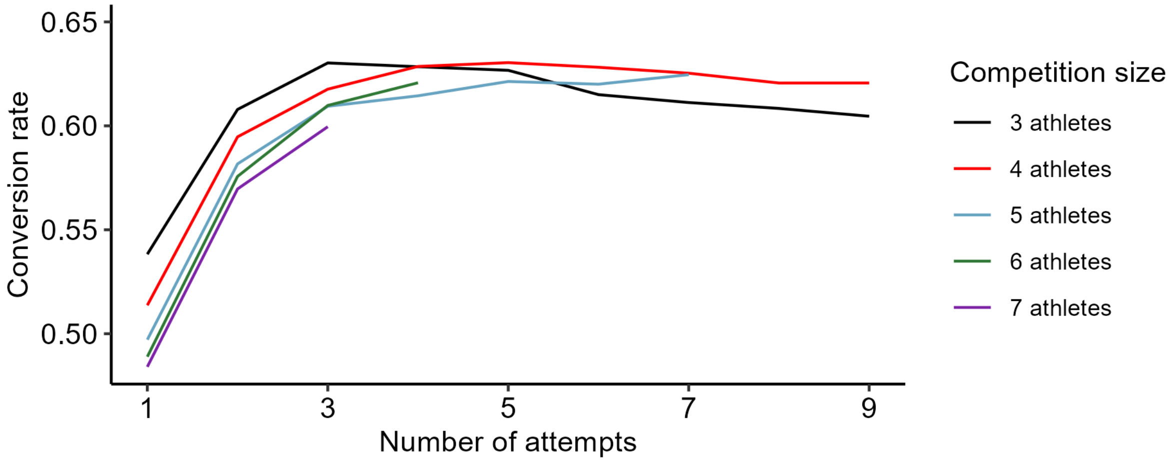

Figure 11: Conversion rate for various competition sizes

Moreover, the conversion rate also attains a maximum around three to five attempts. This means that competitions with many rounds do not effectively determine the best athlete, as we have defined it. This is because, as the amount of attempts increases, the athletes that win are those who have the thickest right tail, not necessarily those that can give a consistent performance.

7 Discussion and Conclusion

In this paper we addressed the gap in the literature on weightlifting strategies aimed at maximizing winning probabilities by presenting a scalable model for Olympic weightlifting. Additionally, we provided methods for estimating athletes' abilities from competition data.

Our primary contributions are the (n, k)-Olympic competition and Theorem 4.3, which together form the foundation of our model. The key feature of the (n, k)-Olympic competition is that the decisions regarding weight selection are binary. As a consequence, the action space of the subgames of the MDP is relatively small, allowing for a complete analysis. Specifically, we showed that these subgames have a unique Trembling-Hand Perfect equilibrium and provided an efficient algorithm for finding this equilibrium.

A limitation of our model is that it hinges on the assumption that all athletes only want to win. This is a strong assumption, and in Section 6 we presented evidence that this assumption is violated. Figure 9 shows that athletes' strategies do not align with this assumption, since they seem more focused on personal goals. Similarly, Figure 10 illustrates that actual competitions do not match the expected competitiveness under the assumption that all athletes only want to win.

These insights suggest a clear direction for future research: include rewards for placements besides first in the model. Such an extension can build on our work but will have to reconsider the state space of the MDP. Currently, the model only tracks which athlete is in the lead, not the rankings of the athletes. While this is a conceptually straightforward addition, it does necessitate more research to obtain a computationally tractable model.

We have also conducted preliminary research (not included in this paper) into powerlifting, an alternative form of weightlifting. In powerlifting competitions, weight selection is simultaneous (IPF, 2023). This simultaneity makes powerlifting resemble the timing games discussed by Radzik (1996). Like those games, the strategies in powerlifting are mixed strategies. This introduces an extra layer of complexity, which makes powerlifting unattractive for investigating large weightlifting competitions. However, it is still an interesting direction for future research.

Regarding the estimation of athletes' abilities, we discussed several characteristics of the competition data that make this difficult, namely non-stationarity and the dependence between weight selection and ability. We addressed this by removing the non-stationary trend and explicitly introducing a dependence structure through copulas.

However, we do suspect that this method approach does not fully make use of the data, since it does not make extensive use of the attempt weights. We therefore suggest that a Bayesian model, possibly including covariates such as age and gender, could be used instead to model the athletes' abilities.

Such a Bayesian approach would be incompatible with our current MDP, as we currently assume that ability is the same for all lifts. A partially observable MDP could remedy this, but would introduce further problems regarding the large state space. If a Bayesian approach is taken, then we will have to likely model the competition in an entirely new way.

In conclusion, this paper has laid the groundwork for understanding weightlifting strategies from a game theoretic perspective, through the novel (n, k)-Olympic competition model. With further research, we wish to see this work contribute to the sport of Olympic weightlifting.

References

Catalyst Athletics (2022). Understanding the Athlete Card for Olympic Weightlifting Competition. https://www.youtube.com/watch?v=T0mH_a8LEsI. Accessed: 08-04-2024.

Fudenberg, D. and J. Tirole (1991). *Game theory.* MIT Press.

IAAF (2024). IAAF Competition rules 2024: Book C2.1 Technical Rules (as of 17 January 2024).

IPF (2023). Technical Rules book of the International Powerlifting Federation (as of January 2023).

IWF (2023). International Weightlifting Federation (IWF) Technical and Competition Rules & Regulations (as of 26 July 2023).

Kauhanen, H., P. Komi, and K. Häkkinen (2002). Standardization and validation of the body weight adjustment regression equations in Olympic weightlifting. *The Journal of Strength & Conditioning Research 16*, 58–74.

Lilien, G. (1974). Optimal weightlifting. In S. P. Ladany and R. E. Machol (Eds.), *Optimal Strategies in Sports*, pp. 101–112. New York: North-Holland.

McNeil, A., R. Frey, and P. Embrechts (2015). *Quantitative risk management: concepts, techniques and tools-revised edition.* Princeton University Press.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics 54*, 286–295.

Presman, E. and I. Sonin (2006). The existence and uniqueness of Nash equilibrium point in an m-player game "shoot later, shoot first!". *International Journal of Game Theory 34*, 185–205.

Radzik, T. (1996). Results and problems in games of timing. In *Statistics, Probability and Game Theory*, Volume 30, pp. 269–292. Institute of Mathematical Statistics.

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory 4*, 25–55.

Sun, J. (2006). *The statistical analysis of interval-censored failure time data*, Volume 3. Springer.

Titman, A. (2014). A pool-adjacent-violators type algorithm for non-parametric estimation of current status data with dependent censoring. *Lifetime data analysis 20*, 444–458.