Introduction

A chord is a line connecting two points on the circumference of a circle. An example of a chord can be seen in figure 1, where the line segment d is a chord. A well known puzzle regarding chords is Bertrand's paradox. Bertrand asks the following question, suppose there is an equilateral triangle inscribed in a circle. What is the probability that a random chord in that circle is longer than the side lengths of the triangle? The paradox occurs when seemingly sensible ways of constructing a random chord will give different probabilities. These different ways of constructing a random chord will change the distribution of the length of the chord. Here I will present my (not so rigorous) attack on this problem, which will lead to the (correct) distribution of a related variable of chord length, and the answer to Bertrand's question.

Length distribution of a chord

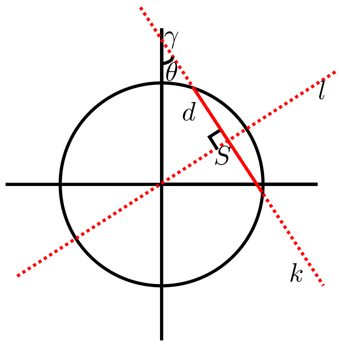

Figure 1: circle

We start with a random line, k, it is given by $y = \tan\left(\theta - \frac{\pi}{2}\right)x + \gamma$ and the line l by $y = \tan(\theta)x$. One gets that they intersect at $S = \left(\frac{\gamma}{2}\sin(2\theta), \gamma\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\right)$ and subsequently that $s = ||S|| = |\gamma\sin(\theta)|$ and the length of the chord is $d = 2\sqrt{1-s^2}$. Now all that remains is determining the distribution of s. Firstly observe that by symmetry we only need to consider positive γ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Secondly, since $\mathbb{P}\left(|\gamma| \leq 1\right) = 0$, so we can focus on $\gamma \in [1, \infty)$.

Let $\gamma \sim \text{Unif}(1, T)$ and $\theta \sim \text{Unif}\left(0, \frac{\pi}{2}\right)$. Now the CDF of s can be computed in the following way.

$$\mathbb{P}(s \leq x) = \lim_{T\to\infty} \mathbb{P}(\gamma\sin(\theta) \leq x | \gamma\sin(\theta) \leq 1)$$

$$= \lim_{T\to\infty} \frac{\int_1^T \int_0^{\frac{\pi}{2}} \mathbb{1}\{\gamma\sin(\theta) \leq x\}\, \mathrm{d}\theta\, \mathrm{d}\gamma}{\int_1^T \int_0^{\frac{\pi}{2}} \mathbb{1}\{\gamma\sin(\theta) \leq 1\}\, \mathrm{d}\theta\, \mathrm{d}\gamma} = \lim_{T\to\infty} \frac{\int_1^T \arcsin\left(\frac{x}{\gamma}\right)\, \mathrm{d}\gamma}{\int_1^T \arcsin\left(\frac{1}{\gamma}\right)\, \mathrm{d}\gamma}$$

$$= \lim_{T\to\infty} \frac{T\arcsin\left(\frac{x}{T}\right) + \frac{1}{2}x\ln\left|\frac{\sqrt{T^2-x^2}}{T} + 1\right| - \frac{1}{2}x\ln\left|\frac{\sqrt{T^2-x^2}}{T} - 1\right| - \arcsin(x) - \frac{1}{2}x\ln\left|\frac{\sqrt{1^2-x^2}}{1} + 1\right| + \frac{1}{2}x\ln\left|\frac{\sqrt{1^2-x^2}}{1} - 1\right|}{T\arcsin\left(\frac{1}{T}\right) + \frac{1}{2}\ln\left|\frac{\sqrt{T^2-1^2}}{T} + 1\right| - \frac{1}{2}\ln\left|\frac{\sqrt{T^2-1^2}}{T} - 1\right| - \arcsin(1) - \frac{1}{2}\ln\left|\frac{\sqrt{1^2-1^2}}{1} + 1\right| + \frac{1}{2}\ln\left|\frac{\sqrt{1^2-1^2}}{1} - 1\right|}$$

$$= \lim_{T\to\infty} \frac{-\frac{1}{2}x\ln\left|\frac{\sqrt{T^2-x^2}}{T} - 1\right|}{-\frac{1}{2}\ln\left|\frac{\sqrt{T^2-1^2}}{T} - 1\right|}, \text{ as they are the dominant terms, all the others are constants w.r.t. } T$$

$$= x\lim_{T\to\infty} \frac{\ln\left|\frac{\sqrt{T^2-x^2}}{T} - 1\right|}{\ln\left|\frac{\sqrt{T^2-1^2}}{T} - 1\right|} = x.$$

So, $\mathbb{P}(s \leq x) = x \Rightarrow s \sim \text{Unif}(0, 1)$. With this we can answer Bertrand's problem $\mathbb{P}(d \geq \sqrt{3}) = \mathbb{P}(s \leq \frac{1}{2}) = \frac{1}{2}$.