20. $v(\{1\}) = 3, v(\{2\}) = 5, v(\{3\}) = 4, v(\{1,2\}) = 7, v(\{1,3\}) = 9, v(\{2,3\}) = 8, v(\{1,2,3\}) = 21$

Find the C-core

The C-core is the set of all imputations ($x1$, $x2$, $x3$) that satisfy the following conditions:

1. **Efficiency**:

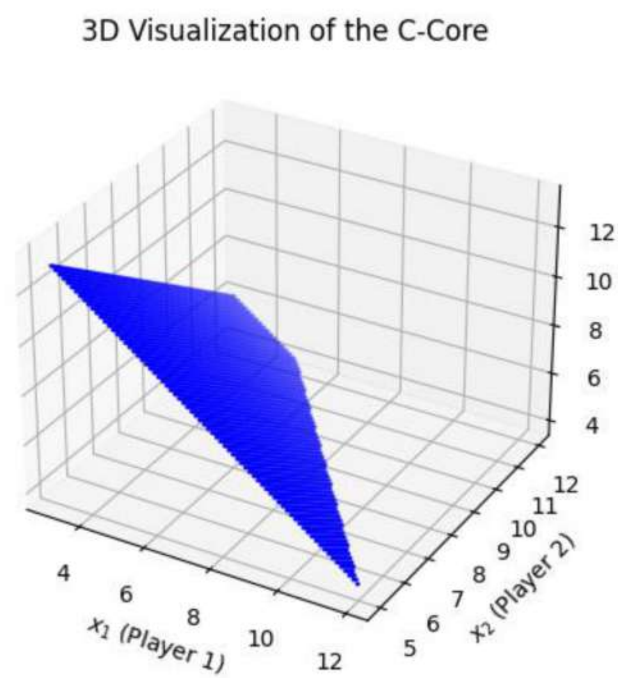
$$x_1 + x_2 + x_3 = 21$$

2. **Individual rationality**:

$$\begin{cases} x_1 \geq 3 \\ x_2 \geq 5 \\ x_3 \geq 4 \end{cases}$$

3. **Group rationality**:

$$\begin{cases} x_1 + x_2 \geq 7 \\ x_1 + x_3 \geq 9 \\ x_2 + x_3 \geq 8 \end{cases}$$



Find the Shapley vector using any two methods (from the presentation)

The Shapley value ϕi for player i is given by:

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} [v(S \cup \{i\}) - v(S)]$$

Player 1:

Contribution when S=∅:

$$\frac{0!(3-0-1)!}{3!}[v(\{1\}) - v(\emptyset)] = \frac{1 \cdot 2}{6}[3-0] = \frac{1}{3} \cdot 3 = 1$$

Contribution when S={2}:

$$\frac{1!(3-1-1)!}{3!}[v(\{1,2\}) - v(\{2\})] = \frac{1 \cdot 1}{6}[7-5] = \frac{1}{6} \cdot 2 = \frac{2}{6} = \frac{1}{3}$$

Contribution when S={3}:

$$\frac{1!(3-1-1)!}{3!}[v(\{1,3\}) - v(\{3\})] = \frac{1 \cdot 1}{6}[9-4] = \frac{1}{6} \cdot 5 = \frac{5}{6}$$

Contribution when S={2,3}:

$$\frac{2!(3-2-1)!}{3!}[v(\{1,2,3\}) - v(\{2,3\})] = \frac{2 \cdot 1}{6}[21-8] = \frac{2}{6} \cdot 13 = \frac{26}{6} = \frac{13}{3}$$

Summing up all contributions:

$$\phi_1 = 1 + \frac{1}{3} + \frac{5}{6} + \frac{13}{3} = 1 + 0.3333 + 0.8333 + 4.3333 = 6.5$$

Player 2:

Contribution when S=∅:

$$\frac{0!(3-0-1)!}{3!}[v(\{2\}) - v(\emptyset)] = \frac{1 \cdot 2}{6}[5-0] = \frac{1}{3} \cdot 5 = \frac{5}{3}$$

Contribution when S={1}:

$$\frac{1!(3-1-1)!}{3!}[v(\{1,2\}) - v(\{1\})] = \frac{1 \cdot 1}{6}[7-3] = \frac{1}{6} \cdot 4 = \frac{4}{6} = \frac{2}{3}$$

Contribution when S={3}:

$$\frac{1!(3-1-1)!}{3!}[v(\{2,3\})-v(\{3\})]=\frac{1\cdot 1}{6}[8-4]=\frac{1}{6}\cdot 4=\frac{4}{6}=\frac{2}{3}$$

Contribution when S={1,3}:

$$\frac{2!(3-2-1)!}{3!}[v(\{1,2,3\})-v(\{1,3\})]=\frac{2\cdot 1}{6}[21-9]=\frac{2}{6}\cdot 12=\frac{24}{6}=4$$

Summing up all contributions:

$$\phi_2=\frac{5}{3}+\frac{2}{3}+\frac{2}{3}+4=\frac{5}{3}+\frac{2}{3}+\frac{2}{3}+4=\frac{9}{3}+4=3+4=7$$

Player 3:

Contribution when S=∅:

$$\frac{0!(3-0-1)!}{3!}[v(\{3\})-v(\emptyset)]=\frac{1\cdot 2}{6}[4-0]=\frac{1}{3}\cdot 4=\frac{4}{3}$$

Contribution when S={1}:

$$\frac{1!(3-1-1)!}{3!}[v(\{1,3\})-v(\{1\})]=\frac{1\cdot 1}{6}[9-3]=\frac{1}{6}\cdot 6=1$$

Contribution when S={2}:

$$\frac{1!(3-1-1)!}{3!}[v(\{2,3\})-v(\{2\})]=\frac{1\cdot 1}{6}[8-5]=\frac{1}{6}\cdot 3=0.5$$

Contribution when S={1,2}:

$$\frac{2!(3-2-1)!}{3!}[v(\{1,2,3\})-v(\{1,2\})]=\frac{2\cdot 1}{6}[21-7]=\frac{2}{6}\cdot 14=\frac{28}{6}=\frac{14}{3}$$

Summing up all contributions:

$$\phi_3=\frac{4}{3}+1+0.5+\frac{14}{3}=1.3333+1+0.5+4.6667=7.5$$

Final Shapley Values:

- Player 1: ϕ1=6.5

- Player 2: $\phi 2=7$
- Player 3: $\phi 3=7.5$

Verification of Shapley Vector in the C-core

1. **Efficiency**:
   $6.5+7+7.5=21$ (satisfied)
2. **Individual rationality**:

   $6.5\geq 3$ (satisfied)

   $7\geq 5$ (satisfied)

   $7.5\geq 4$ (satisfied)

3. **Group rationality**:

   $6.5+7\geq 7$ (satisfied)

   $6.5+7.5\geq 9$ (satisfied)

   $7+7.5\geq 8$ (satisfied)

Since all conditions are satisfied, the Shapley vector (6.5,7,7.5) belongs to the C-core.