2

The Binomial Model for Stock Options

2.1 The Basic Model

We now discuss a simple **one-step binomial model** in which we can determine the rational price today for a call option. In this model we have two times, which we will call $t = 0$ and $t = 1$ for convenience. The time $t = 0$ denotes the present time and $t = 1$ denotes some future time. Viewed from $t = 0$, there are two states of the world at $t = 1$. For convenience they will be called the **upstate** (written $\uparrow$) and the **downstate** (written $\downarrow$). There is **no special meaning** to be attached to these states. It does not **necessarily** mean that a stock price has a low price in the downstate and a higher value in the upstate, although this will sometimes be the case. The term **binomial** is used because there are **two** states at $t = 1$.

In our model there are two **tradeable assets**; eventually there will be other derived assets:

1. **a risky asset** (e.g. a stock);
2. **a riskless asset**.

By a **tradeable asset** we shall mean an asset that can be bought or sold on demand at any time in any quantity. They are the typical assets used in the construction of portfolios. In Chapter 14 on real options we shall note some problems with this concept.

We assume for each asset that its buying and selling prices are equal.

The risky asset.

At $t = 0$, the risky asset S will have the known value $S(0)$ (often non-negative).

At $t = 1$, the risky asset has two distinct possible values (hence its value is uncertain or risky), which we will call $S(1, \uparrow)$ and $S(1, \downarrow)$. We simply require that $S(1,\uparrow) \neq S(1,\downarrow)$, but without loss of generality (wlog), we may assume that $S(1,\uparrow) > S(1,\downarrow)$.

The riskless asset

At $t = 0$, the riskless asset B will have value $B(0) = 1$.

At $t = 1$, the riskless asset has the same value (hence riskless) in both states at $t = 1$, so we write $B(1,\uparrow) = B(1,\downarrow) \equiv R = 1 + r$. Usually $R \geq 1$ and so $r \geq 0$, which we can call **interest**, is non-negative. It represents the amount earned on $1.

It is easy to show that if $S(1,\uparrow) = S(1,\downarrow)$ there is an arbitrage, unless $S(1,\uparrow) = S(1,\downarrow) = (1 + r)S(0)$.

We also assume that

$$S(1,\downarrow) < RS(0) < S(1,\uparrow). \tag{2.1}$$

We shall see the importance of inequality (2.1) below.

Example 2.1. Here $S(0) = 5$, $S(1,\uparrow) = \frac{20}{3}$ and $S(1,\downarrow) = \frac{40}{9}$. $B(0) = 1$ and $B(1,\uparrow) = B(1,\downarrow) = R = \frac{10}{9}$. So $r = \frac{1}{9}$ and (2.1) clearly holds.

Suppose $X(1)$ is any claim that will be paid at time $t = 1$. In our model $X(1)$ can take one of two values: $X(1,\uparrow)$ or $X(1,\downarrow)$. We shall determine $X(0)$, the premium or price of X at time $t = 0$.

Often the values of $X(1)$ are uncertain because $X(1) = f(S(1))$ (a function of S) and $S(1)$ is uncertain. As X is an asset whose value depends on S, it is a **derived asset** written on S, or a derivative on S. X is also called a **derivative** or a **contingent claim**.

Example 2.2. When we write $X(1) = [S(1) - K]^+$ we mean

$$\begin{aligned} X(1,\uparrow) &= [S(1,\uparrow) - K]^+ \\ X(1,\downarrow) &= [S(1,\downarrow) - K]^+. \end{aligned}$$

Assuming we have a model for S, we can find $X(0)$ in terms of this information. This could be called **relative pricing**. It presents a different methodology than, (though often equivalent to) what the economists call **equilibrium pricing**, for example.

There are two steps to relative pricing.

Step 1

Find H_0 and H_1 so that

$$X(1) = H_0 B(1) + H_1 S(1). \tag{2.2}$$

Both sides here are random quantities and (2.2) means

$$X(1,\uparrow) = H_0 R + H_1 S(1,\uparrow) \tag{2.3a}$$
$$X(1,\downarrow) = H_0 R + H_1 S(1,\downarrow). \tag{2.3b}$$

The interpretation is as follows: H_0 represents the number of dollars held at $t = 0$, and H_1 the number of stocks held at $t = 0$. At $t = 1$, the level of holdings does not change, but the underlying assets do change in value to give $H_0 B(1) + H_1 S(1)$.

Solving (2.3a) and (2.3b) gives

$$H_1 = \frac{X(1,\uparrow) - X(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} \tag{2.4}$$

and

$$\begin{aligned} H_0 &= \frac{X(1,\uparrow) - H_1 S(1,\uparrow)}{R} \\ &= \frac{X(1,\uparrow) - \frac{X(1,\uparrow)-X(1,\downarrow)}{S(1,\uparrow)-S(1,\downarrow)} S(1,\uparrow)}{R} \\ &= \frac{S(1,\uparrow)X(1,\downarrow) - S(1,\downarrow)X(1,\uparrow)}{R\,[S(1,\uparrow) - S(1,\downarrow)]}. \end{aligned} \tag{2.5}$$

Note: It is rather crucial that $S(1,\uparrow) \neq S(1,\downarrow)$.

Example 2.3 (continuation of Example (2.1)*).* If $X(1,\uparrow) = 7$ and $X(1,\downarrow) = 2$, then equations (2.3a) and (2.3b) become

$$7 = H_0 \frac{10}{9} + H_1 \frac{20}{3}$$
$$2 = H_0 \frac{10}{9} + H_1 \frac{40}{9},$$

giving $H_0 = -7.2$ and $H_1 = 2.25$.

Remark 2.4. We should now take a little time to interpret the situation where H_0 or H_1 is negative.

In the previous example, $H_0 = -7.2$ means we borrowed 7.2 and $t = 0$ and we have a liability (a negative amount) of $H_0 R = -8$ at $t = 1$.

Suppose instead that $X(1,\uparrow) = 2$ and $X(1,\downarrow) = 7$, then $H_0 = 15.3$ and $H_1 = -2.25 < 0$. Now $H_1 = -2.25$ means we shorted (borrowed) 2.25 stocks at $t = 0$ and we have a liability at $t = 1$ as we must return the value of the stock at $t = 1$. This value will depend on whether we are in $\uparrow$ or $\downarrow$. By the way, we must also assume that we have a **divisible market**, which is one in which any (real) number of stocks can be bought and sold. If we think of stocks in lots of 1000 shares, then 2.25 is really 2250 shares. This is how we could interpret these "fractional shares".

Short sell means "borrow and sell what you do not own".

There are basically two ways of raising cash: Borrow money at interest (from a bank, say) or short sell an asset. In the former case, you must repay the loan with interest at a future date and in the second case, you must buy back the asset later and return it to its owner.

In an analogous way there are two ways of devolving yourself of cash. You can put money in a bank to earn interest, or you can buy an asset. In the former case you can remove the money later with any interest it has earned, and in the latter case you can sell the asset (at a profit or loss) at a future date.

Step 2

Using the **one price theorem**, which is a consequence of the **no arbitrage axiom**, we must have

$$X(0) = H_0 + H_1 S(0). \tag{2.6}$$

Remark 2.5. This equation is true because the claim X and the portfolio $H_0 B + H_1 S$ have the same value in both possible states of the world at $t = 1$. In this situation, $X(0)$ represents **outflow** of cash at $t = 0$. If $X(0) > 0$, then $X(0)$ represents the amount to be paid at $t = 0$ for the asset with payoff $X(1)$ at $t = 1$. If $X(0) < 0$, then $-X(0)$ represents an amount received at $t = 0$ for the asset with payoff $X(1)$ at $t = 1$.

We shall review for this call option why $X(0)$ must equal $H_0 + H_1 S_0$. First assume (if possible) that

$$X(0) < H_0 + H_1 S(0). \tag{2.7}$$

In fact let us use the numbers from the previous example. Thus (2.7) is

$$2.25 S(0) - 7.2 - X(0) > 0 \tag{2.8}$$

We now perform the following trades at t=0.

Short sell 2.25 shares of stock, put 7.2 in the bank, buy one asset.

Equation (2.8) gives the strategy to adopt. If a quantity is a positive value of assets such as $2.25S(0)$, this suggests one should short sell the assets; if a quantity is a negative value of assets (that is, $-X(0)$), this suggests one should purchase the assets. A positive number alone indicates a borrowing and a negative number, -7.2, an investment of cash in a bank.

In fact

$$2.25S(0) - (7.2 + X(0)) > 0$$

where $2.25S(0)$ is income, $7.2+X(0)$ payouts. Note that because this difference is positive you have a profit from this trading at $t = 0$. Put this profit in your pocket—and do not touch it (at least for the time being).

Note the following: You did not need any of your own money to carry out this trade. The short sale of the borrowed stock was enough to finance the investment of 7.2 and the purchase of X for $X(0)$, and there was money left over.

The consequence at t=1.

There are two cases:

In $\uparrow$

Sell X for $X(1,\uparrow) = 7$, remove the money from the bank with interest $7.2R = 8$. This results in 15 (dollars), which can be used to fund the repurchase (and return) of the $2.25S(1,\uparrow) = 15$. There are no further liabilities. Thus, there are **no unfunded liabilities at t=1**.

In $\downarrow$

Sell X for $X(1,\downarrow) = 2$, remove the money from the bank with interest $7.2R = 8$. This results in 10 (dollars), which can be used to fund the repurchase (and return) of the $2.25S(1,\downarrow) = 10$. There are again no further liabilities. Thus again there are **no unfunded liabilities at** $t = 1$.

In summary, we have made a profit at $t = 0$ and have no unfunded liabilities at $t = 1$. This is making money by taking no risks—by not using your own money. This is an example of an arbitrage opportunity which our fundamental axiom rules out. In efficient markets one assumes that arbitrage opportunities do not exist, and so we have a contradiction to (2.8). In practice, arbitrage opportunities may exist for brief moments, but, due to the presence of arbitrageurs, the markets quickly adjust prices to eliminate these arbitrage opportunities. At least that is the theory.

After this discussion we see that (2.7) cannot hold (at least not in the example, but also more generally). Therefore,

$$X(0) \geq H_0 + H_1 S(0).$$

Assume now, if possible, that

$$X(0) > H_0 + H_1 S(0). \tag{2.9}$$

In the example, this would mean

$$X(0) + 7.2 - 2.25S(0) > 0. \tag{2.10}$$

We now perform the following trades at t=0.

Short sell the asset, borrow 7.2 and buy 2.25 stock.

This yields a positive profit at $t = 0$ which is placed deep in your pocket until after $t = 1$. In other words raising funds from the short sale and borrowings is more than enough to cover the cost of 2.25 shares.

The consequence at t=1.

There are two cases:

In $\uparrow$

Sell the shares for $2.25S(1,\uparrow) = 15.00$, repay the loan with interest $7.2R = 8$, purchase the asset for 7 and return to the (rightful) owner. Everything balances out. Thus, there are **no unfunded liabilities at t=1**.

In $\downarrow$

Sell the shares for $2.25S(1,\downarrow) = 10.00$, repay the loan with interest $7.2R = 8$, purchase the asset for 2 and return to the (rightful) owner. Everything balances out. Thus, there are **no unfunded liabilities at t=1**.

In summary, we have again made a profit at $t = 0$ and have no unfunded liabilities at $t = 1$. This is again an arbitrage opportunity. Therefore, (2.9) is false as well. We then conclude the result claimed in (2.6) must hold.

Let us now substitute (2.4) and (2.5) into (2.6). Then

$$\begin{aligned}
X(0) &= H_0 + H_1 S(0) \\
&= \left[\frac{S(1,\uparrow)X(1,\downarrow) - S(1,\downarrow)X(1,\uparrow)}{R\,(S(1,\uparrow) - S(1,\downarrow))}\right] + \left[\frac{X(1,\uparrow) - X(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)}\right] S(0) \\
&= \frac{X(1,\uparrow)\,[RS(0) - S(1,\downarrow)] + X(1,\downarrow)\,[S(1,\uparrow) - RS(0)]}{R\,[S(1,\uparrow) - S(1,\downarrow)]} \\
&= \frac{1}{R}\,[\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow)]
\end{aligned}$$

where

$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} > 0 \tag{2.11}$$

$$1-\pi = \frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)} > 0.$$

Here $0 < \pi < 1$ follows from the assumption of the model (2.1).

Therefore,

$$X(0) = \frac{\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow)}{R}. \tag{2.12}$$

This is the **general pricing formula** for a contingent claim option in a one-step binomial model.

It was derived by using two ideas:

1. replicating portfolios (step 1);
2. there are no arbitrage opportunities (vital for the step 2 argument).

This method is called **relative pricing** because relative to the given inputs $S(0)$, $S(1,\uparrow)$, $S(1,\downarrow)$, $B(0)$, $B(1,\uparrow)$ and $B(1,\downarrow)$ we can price other assets. We simply calculate π as in (2.11) and then use (2.12). Let us note that even though S was thought of as being a stock, it could have stood for **any** risky asset at all.

The numbers π and $1-\pi$ are called the **risk neutral probabilities** of states $\uparrow$ and $\downarrow$, respectively. We shall see why this name is used.

We can write (2.12) as

$$X(0) = \mathbf{E}^{\pi}\left[\frac{X(1)}{B(1)}\right], \tag{2.13}$$

which is the **risk neutral expectation** of $\frac{X(1)}{B(1)}$. It stands for

$$\pi\frac{X(1,\uparrow)}{B(1,\uparrow)} + (1-\pi)\frac{X(1,\downarrow)}{B(1,\downarrow)}.$$

This is the same as the right hand side of (2.12).

Remark 2.6. It can be shown that there is no arbitrage possible in our binomial model if and only if (iff) a formula of the type (2.13) holds with $0 < \pi < 1$.

Remark 2.7. The author that is credited with the first use of binomial option pricing is Sharpe in 1978 [70, pages 366–373]. He argues as follows: First select h so that

$$hS(1,\uparrow) - X(1,\uparrow) = hS(1,\downarrow) - X(1,\downarrow)\ .$$

Set this common value equal to

$$R(hS(0) - X(0)).$$

This again leads to equation (2.12).

In 1979 Rendleman and Bartter [63] gave a similar argument. First select α so that

$$S(1,\uparrow) + \alpha X(1,\uparrow) = S(1,\downarrow) + \alpha X(1,\downarrow)$$

and set this common value to

$$R(S(0) + \alpha X(0)).$$

This (normally) again leads to equation (2.12). We say this because a choice of α may not always exist. For the Sharpe approach, a choice of h can always be made.

Exercise 2.8. Verify the claims made in this remark.

Not all models that one could write down are arbitrage free.

Example 2.9 (Continuation of Example 2.1*).*

Simply make the change $S(1,\downarrow) = \frac{17}{3}$. Starting with nothing, choose $H_0 = -5$ (borrow 5 stocks), $H_1 = 1$ (buy one stock). Then $H_0 + H_1 S(0) = 0$. At $t = 1$, our position will be $X(1) \equiv -5R + S(1)$ (meaning sell the stock and repay the loan). This is $\frac{10}{9}$ in the upstate and $\frac{1}{9}$ in the down state. So with no start-up capital we have generated a profit (in both states) by simply trading. This is an arbitrage opportunity. Note that condition (2.1) is violated here.

Example 2.10 (On why $0 < \pi < 1$ *should hold).* As in equation (2.11)

$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)}.$$

We assumed in inequality (2.1) that $0 < S(1,\downarrow) < RS(0) < S(1,\uparrow)$. So, for example,

$$0 < RS(0) - S(1,\downarrow) < S(1,\uparrow) - S(1,\downarrow)$$

and the result that (2.1) implies that $0 < \pi < 1$ follows. If we choose X with $X(1,\uparrow) = 1$ and $X(1,\downarrow) = 0$, then $X(0) > 0$ to exclude arbitrage. Then (2.12) implies that $\pi > 0$. A similar argument using X with $X(1,\uparrow) = 0$ and $X(1,\downarrow) = 1$ leads to $1 - \pi > 0$. So the absence of arbitrage opportunities leads to $0 < \pi < 1$.

Notation

It is often useful to use the following notation when $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n$:

1. $\mathbf{x} \geq 0$ if $x_i \geq 0$ for each $i = 1, 2, \ldots, n$.
2. $\mathbf{x} > 0$ if $\mathbf{x} \geq 0$ and $x_i > 0$ for at least one i.
3. $\mathbf{x} \gg 0$ if $x_i > 0$ for each $i = 1, 2, \ldots, n$.

2.2 Why Is π Called a Risk Neutral Probability?

This discussion will take place within the **one-step binomial asset pricing model**.

Some of the steps here will be left to the reader as exercises.

For any $0 \leq p \leq 1$, let $\mathbf{E}^p[X(1)]$ be defined by

$$\mathbf{E}^p[X(1)] = pX(1,\uparrow) + (1-p)X(1,\downarrow). \tag{2.14}$$

Here p could represent a (subjective) probability (viewed from $t = 0$) that the upstate ($\uparrow$) will occur at $t = 1$. Let X be a (tradeable) asset whose value at $t = 0$ is $X(0)$ and whose values at $t = 1$ are $X(1,\uparrow)$ and $X(1,\downarrow)$, depending on whether the upstate or downstate occurs at $t = 1$. From (2.12),

$$X(0) = \frac{1}{R}\left[\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow)\right] \equiv \frac{1}{R}\mathbf{E}^\pi[X(1)]\,. \tag{2.15}$$

For the asset X we can define the **return** r_X by

$$r_X = \frac{X(1) - X(0)}{X(0)}, \tag{2.16}$$

which is shorthand for

$$\begin{aligned} r_X(\uparrow) &= \frac{X(1,\uparrow) - X(0)}{X(0)} \\ r_X(\downarrow) &= \frac{X(1,\downarrow) - X(0)}{X(0)}. \end{aligned}$$

Lemma 2.11. *For any* $0 \leq p, q \leq 1$ *suppose there are associated probabilities. Then*

$$\boldsymbol{E}^p[r_X] - \boldsymbol{E}^q[r_X] = (p-q)\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right]. \tag{2.17}$$

Proof. Exercise. □

Remark 2.12.

$$\mathbf{E}^\pi[r_X] = r \equiv R - 1 \tag{2.18}$$

Proof. Exercise. □

Corollary 2.13.

$$\boldsymbol{E}^p[r_X] - r = (p-\pi)\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right] \tag{2.19}$$

$$\boldsymbol{E}^p[r_X] - r_X(\uparrow) = (p-1)\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right] \tag{2.20}$$

$$\boldsymbol{E}^p[r_X] - r_X(\downarrow) = p\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right] \tag{2.21}$$

Proof. For (2.19), use (2.18) and $q = \pi$ in (2.17). For (2.20), use $q = 1$ in (2.17). For (2.21), use $q = 0$ in (2.17). □

Definition 2.14. *Given probability p, let X and Y be two (tradeable) assets. Their values at $t = 0$ are $X(0)$, $Y(0)$. At $t = 1$ in the $\uparrow$ state (resp., $\downarrow$ state) their values are $X(1,\uparrow), Y(1,\uparrow)$ (resp., $X(1,\downarrow), Y(1,\downarrow)$). Then define $V^p_{X,Y}$ by*

$$\begin{aligned} V^p_{X,Y} &= \boldsymbol{Cov}^p(r_X, r_Y) \\ &\equiv \boldsymbol{E}^p\left[(\boldsymbol{E}^p[r_X] - r_X)(\boldsymbol{E}^p[r_Y] - r_Y)\right] \qquad (2.22) \\ &= \boldsymbol{E}^p[r_X r_Y] - \boldsymbol{E}^p[r_X]\,\boldsymbol{E}^p[r_Y] \qquad (2.23) \end{aligned}$$

Lemma 2.15.

$$V^p_{X,Y} = p(1-p)\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right]\left[\frac{Y(1,\uparrow) - Y(1,\downarrow)}{Y(0)}\right] \tag{2.24}$$

Proof. Use (2.22), (2.14) together with (2.20) and (2.21). We leave the details as an exercise. □

Corollary 2.16. *The variance of X is then*

$$\sigma_X^2 \equiv V^p_{X,X} = p(1-p)\left[\frac{X(1,\uparrow) - X(1,\downarrow)}{X(0)}\right]^2. \tag{2.25}$$

Let us now assume (wlog) that $\mathbf{E}^p[r_X] \geq r$. With this assumption we have the following lemma.

Lemma 2.17. *Suppose that $0 < p < 1$. Then*

$$\boldsymbol{E}^p[r_X] - r = \frac{|p-\pi|}{\sqrt{p(1-p)}}\,\sigma_X \tag{2.26}$$

Proof. This follows from (2.19) and (2.25) and the assumption.

Remark 2.18. Equation (2.26) says something about the expected return from asset X in terms of its volatility (variance). We say that an asset is **riskier** when it has a higher volatility (and hence a higher value of σ_X). By (2.26), if the volatility is zero, then the expected return is just r (the risk free interest), but when the volatility is non-zero we have a higher expected return. This result fits well with reality—if you want a higher expected return you must take on more risk. However, there is one situation where this does not hold. This is when $p = \pi$. In this case your expected return is always r no matter what risk. If your (subjective) probabilities about events at $t = 1$ coincide with π, then you are insensitive to risk, or what is the same thing, you are **risk-neutral**. So π is the upstate probability of a risk neutral person.

Remark 2.19. Equation (2.15) is the usual pricing equation for an asset X, expressing $X(0)$ in terms of the future values of X via the risk neutral probability π. We can also express $X(0)$ via the subjective probability p. In fact suppose the assumption before Lemma 2.17 holds, and

$$\Lambda(p) = \frac{|p - \pi|}{\sqrt{p(1-p)}}.$$

Then by a simple rearrangement

$$X(0) = \frac{\mathbf{E}^p\left[X(1)\right]}{R + \Lambda(p)\ \sigma_X}, \tag{2.27}$$

so the discounting must be **risk adjusted** if you use subjective probabilities. Note that $\Lambda(\pi) = 0$.

Another rearrangement starts with

$$\mathbf{E}^p\left[r_X\right] - r = \beta_{X,Y}\left[\mathbf{E}^p\left[r_Y\right] - r\right]. \tag{2.28}$$

Here

$$\beta_{X,Y} = \frac{V^p_{X,Y}}{V^p_{Y,Y}},$$

which is a regression coefficient for the returns of X onto those of Y. This quantity is called a **beta** in financial circles, and betas are often published information. It is often the case that betas do not change too quickly from time to time. The identity (2.28) follows from (2.19) applied to both X and Y together with (2.24) and (2.25). It is necessary to consider $p \neq \pi$ and $p = \pi$ separately to avoid dividing 0 by 0, which is even invalid in finance! Equation (2.28) looks very much like the **CAPM** formula (CAPM = Capital

Asset Pricing Model), widely used in finance despite its restricted validity. It is valid in our simple model! Equation (2.28) can be arranged to give

$$X(0) = \frac{\mathbf{E}^p\left[X(1)\right]}{R + \beta_{X,Y}\left[\mathbf{E}^p\left[r_Y\right] - r\right]}, \tag{2.29}$$

which is a **relative** pricing formula using subjective probabilities. Given information about Y you can price X provided you also know the correlations between the returns on X and Y (which one sometimes assumes are relatively constant). It is because of the arrangement (2.29) that (2.28) is termed CAPM (read as CAP M). In practice Y is often related to some **index**.

2.3 More on Arbitrage

There are **two forms** of arbitrage opportunities. We suppose neither type exists in efficient markets. If they did exist they would exist only temporarily. An arbitrageur is someone who looks out for such opportunities and exploits them when they do exist.

The **type one** arbitrage opportunity arose in the proof of equation (2.6) in the last section. Indeed, if equation (2.6) did not hold we were able to make a profit at $t = 0$ without any unfunded liabilities at $t = 1$. Here one ends up with a profit at $t = 1$ in all states of the world.

The **type two** arbitrage opportunity arose in Examples 2.9 and 2.10. This is the situation where you start with nothing at $t = 0$, you have no liabilities at $t = 1$, but in one or more states of the world you can make a positive profit.

We now give some more examples:

Example 2.20 (Refer to Example 2.1*).* Here we exhibit a type two arbitrage.

We choose S as in Example 2.1, but suppose

$$B(0) = 1 \text{ and } B(1,\uparrow) = B(1,\downarrow) = R = \frac{4}{3}.$$

Note that condition (2.1) is violated.

Choose $H_0 = 5$ and $H_1 = -1$, then $H_0 + H_1 S(0) = 0$.

At $t = 0$ we short sell one stock and invest the proceeds in a bank.

At $t = 1$ in $\uparrow$ our position is $H_0 R + H_1 S(1,\uparrow) = 0$; in other words our investment gives rise to $5 \times \frac{4}{3} = \frac{20}{3}$, which is enough to cover the repurchase of stock at $\frac{20}{3}$, which is then returned to its owner.

At $t = 1$ in $\downarrow$ our position is $H_0 R + H_1 S(1,\uparrow) = \frac{20}{9}$; in other words our investment gives rise to $5 \times \frac{4}{3} = \frac{20}{3}$, which is enough to cover the repurchase of stock at $\frac{40}{9}$, which is then returned to its owner, and with $\frac{20}{9}$ to spare.

Remark 2.21. Type two non-arbitrage was also used in Example 2.10.

Exercise 2.22 (A Variant of the One Price Theorem). Let X and Y be two assets (or portfolios of assets). Prove

1. if $X(1) > Y(1)$, then $X(0) > Y(0)$;
2. if $X(1) = Y(1)$, then $X(0) = Y(0)$.

Remark 2.23. $X(1) > Y(1)$ has the meaning of the notation on page 20. That is: $X(1) \geq Y(1)$ in all states of the world at $t = 1$, and strict inequality holds in at least one state of the world. [It is possible we are thinking beyond binomial models here.]

In fact, if 1. does not hold, we can obtain a type two arbitrage by short selling X and buying Y; if 2. does not hold, we obtain a type one arbitrage. The principle is this: **(short) sell high, buy low**.

2.4 The Model of Cox-Ross-Rubinstein

We shall now describe the Cox-Ross-Rubinstein model and we shall write **CRR** for **Cox-Ross-Rubinstein**. See [18].

The following notation will be used:

$$\begin{aligned} S(0) &= S > 0 \\ S(1, \uparrow) &= uS \\ S(1, \downarrow) &= dS, \end{aligned}$$

where, as in equation (2.1),

$$0 < d < R < u.$$

Then

$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} = \frac{R-d}{u-d} \tag{2.30}$$

$$1 - \pi = \frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)} = \frac{u-R}{u-d} \tag{2.31}$$

and

$$X(0) = \frac{1}{R}\left[\frac{R-d}{u-d}X(1,\uparrow) + \frac{u-R}{u-d}X(1,\downarrow)\right]. \tag{2.32}$$

Example 2.24 (European call option). Here $X(1) = (S(1) - K)^+$.
Assume $S(1,\downarrow) < K < S(1,\uparrow)$, then

$$\begin{aligned} X(1,\uparrow) &= (S(1,\uparrow) - K)^+ = uS - K \\ X(1,\downarrow) &= (S(1,\downarrow) - K)^+ = 0 \end{aligned}$$

and so

$$\begin{aligned} X(0) &= \frac{\pi\,(uS - K)}{R} \\ &= S\left[\frac{\pi u}{R}\right] - \left[\frac{K}{R}\right]\pi. \end{aligned} \tag{2.33}$$

Remark 2.25. For those familiar with the Black and Scholes formula for pricing call options,

$$C(0) = S(0)\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2), \tag{2.34}$$

we note an obvious similarity. Here $\mathcal{N}$ has the definition

$$\mathcal{N}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy. \tag{2.35}$$

We shall meet these ideas again later. The expressions for d_1 and d_2 are given in Chapter 4.

Continuing, we note that

$$0 < \frac{\pi u}{R} < 1. \tag{2.36}$$

In fact

$$0 < \frac{\pi u}{R} = \left(\frac{R-d}{u-d}\right)\frac{u}{R} = \frac{1-\frac{d}{R}}{1-\frac{d}{u}} < 1$$

as $R < u$ implies $1 - \frac{d}{R} < 1 - \frac{d}{u}$.
If $K \leq S(1,\downarrow)$, then $X(1) = S(1) - K$ and so

$$
\begin{aligned}
X(0) &= \frac{\pi(S(1,\uparrow) - K) + (1-\pi)(S(1,\downarrow) - K)}{R} \\
&= \frac{\pi S(1,\uparrow) - +(1-\pi)S(1,\downarrow)}{R} - \frac{K}{R} \\
&= S(0) - \frac{K}{R}.
\end{aligned}
$$

If $K \geq S(1,\uparrow)$, then $X(1) = 0$ and so $X(0) = 0$.

Example 2.26. Consider the claim $X(1) = (K - S(1))^+$. This is a European put option in the binomial model. Assume $S(1,\downarrow) < K < S(1,\uparrow)$; then

$$
\begin{aligned}
X(1,\uparrow) &= (K - S(1,\uparrow))^+ = 0 \\
X(1,\downarrow) &= (K - S(1,\downarrow))^+ = K - dS
\end{aligned}
$$

and so

$$
\begin{aligned}
X(0) &= \frac{(1-\pi)\,(K - dS)}{R} \\
&= \left[\frac{K}{R}\right](1-\pi) - S\left[\frac{(1-\pi)d}{R}\right]. \qquad (2.37)
\end{aligned}
$$

Remark 2.27. As mentioned before, π is called a **risk-neutral probability** (of being in state $\uparrow$). It is characterized by

$$
S(0) = \frac{\pi S(1,\uparrow) + (1-\pi)S(1,\downarrow)}{R}.
$$

This says that under π, the expected discounted value of $S(1)$ is $S(0)$.

2.5 Call-Put Parity Formula

This is also called **put-call parity**. It applies to **European** style call and put options.

There are several **model-independent** formulae in finance. Clearly, such formulae are very important. We shall meet a number of them. The most well known one is the call-put parity formula, which states:

$$
C(0) - P(0) = S(0) - \frac{K}{R}, \qquad (2.38)
$$

at least in the present framework. We shall discuss generalizations later.

The calls and puts in this formula are assumed to have the same strike price K and the same time to expiry (maturity).

CRR Model-Dependent Proof

Suppose that $S(1,\downarrow) < K < S(1,\uparrow)$. Then with $S = S(0)$ and (2.33) and (2.37),

$$\begin{aligned} C(0) &= \frac{\pi}{R}\,[uS - K] \\ P(0) &= \frac{1-\pi}{R}\,[K - dS] \\ C(0) - P(0) &= \frac{\pi}{R}\,[uS - K] - \frac{1-\pi}{R}\,[K - dS] \\ &= \frac{\pi(uS) + (1-\pi)(dS)}{R} - \frac{\pi K + (1-\pi)K}{R} \\ &= S(0) - \frac{K}{R} \end{aligned}$$

By the way, $\frac{K}{R} = PV(K) \equiv PV_0(K)$, the present value at $t = 0$ of K at $t = 1$ (PV = Present Value).

Model-Independent Proof. Model-independent relations are very important.

We again have two times: now ($t = 0$), and expiry date ($t = T$). Assume (if possible) that

$$C(0) - P(0) - S(0) + PV(K) > 0. \tag{2.39}$$

We shall show there is type one arbitrage. At $t = 0$, we short sell a call option, buy a put option, buy one stock, borrow $PV(K)$. The short sale and the borrowing is enough to cover the put options and stock price, and there is cash left over (by (2.39)), which we pocket.

At expiry ($t = T$), we **cash settle the call**, realize value of the put, sell the stock, repay the loan. The net of all these transactions is

$$-(S(T) - K)^+ + (K - S(T))^+ + S(T) - K = 0. \tag{2.40}$$

The person who let you borrow the call only needs the cash value ($(S(T) - K)^+$) of the call at expiry (called **cash settling**). The assets (put and stock), are just enough to cover the liabilities of the call and loan repayment.

One can demonstrate (2.40) by looking at the two cases: $S(T) > K$ and $S(T) \leq K$. In the first case:

$$-(S(T)-K)^+ + (K-S(T))^+ + S(T) - K = -(S(T)-K) + 0 + S(T) - K = 0$$

and in the latter

$$-(S(T)-K)^+ + (K-S(T))^+ + S(T) - K = 0 + (K-S(T)) + S(T) - K = 0.$$

The other case of (2.39),

$$C(0) - P(0) - S(0) + PV(K) < 0,$$

is treated in a similar way. First write this as

$$-C(0) + P(0) + S(0) - PV(K) > 0. \tag{2.41}$$

At $t = 0$, buy a call option, short sell a put option, short sell a stock and invest $PV(K)$. The two short sales are enough to cover the call options and the investment amount. Further, there is cash left over (by (2.41)), which we pocket.

At expiry ($t = T$), we realize value of the call, cash settle the put, buy a stock and return, realize the investment K. The net cost of all these transactions is

$$(S(T) - K)^+ - (K - S(T))^+ - S(T) + K = 0 \tag{2.42}$$

as before.

In both cases, we can pocket a profit at $t = 0$ and have no unfunded liabilities at expiry. These are type one arbitrages. These financial contradictions show the call-put parity equality must hold. [A reason to prefer the term call-put parity is because it could also be read "call minus put" which is the left hand side of the call-put parity formula. It reminds us which way they are around!].

2.6 Non Arbitrage Inequalities

In the section above we saw the first of these: the call-put parity formula. This was proved in the CRR one-step model and then we gave a model independent proof. It is the fact that it has a model independent proof which makes it a fundamental result. However, note that the call-put parity formula holds for European options. It does not hold for the American style counterparts.

We now investigate other results for which there are model-independent proofs. Consequently, we are no more in the simple two-state, one-period model.

Example 2.28 (Lower bounds for European calls).

Let $0 \leq t < T$. Let $C(t)$ be the value at t of a European call option that expires at time T, whose strike price is K. We also write $PV_t(K)$ for the value at time t of K at time T. This amount could be found by some discounting formula whose precise details do not matter here—as long as interest rates are not random. Then

$$
\begin{aligned}
C(t) &\geq [S(t) - PV_t(K)]^+ && (2.43)\\
&= \max[0, S(t) - PV_t(K)]. && (2.44)
\end{aligned}
$$

Proof. Clearly $C(t) \geq 0$ as $C(T) = (S(T) - K)^+ \geq 0$. [See Exercise (2.22)] So we only need to show

$$C(t) \geq S(t) - PV_t(K). \tag{2.45}$$

Suppose to the contrary that

$$C(t) < S(t) - PV_t(K),$$

which is the same as

$$S(t) - PV_t(K) - C(t) > 0. \tag{2.46}$$

If (2.46) were the case, we show how to create an arbitrage.

At time t we short sell one stock, invest $PV_t(K)$ in a bank, buy a call option, (expiring at T with strike price K). The short sale is enough to cover the purchases and (2.46) says there is a positive amount left over for the pocket.

At time T, the expiry date of the call option, we buy a stock and return it, realize the value of the call, realize the value of the investment in the bank, (take the K out of bank). The net proceeds are given by the left hand side of

$$-S(T) + (S(T) - K)^+ + K \geq 0. \tag{2.47}$$

This implies that there are no unfunded liabilities at time T. To show (2.47) we consider two cases: $S(T) > K$ and $S(T) \leq K$. For the former

$$-S(T) + (S(T) - K)^+ + K = -S(T) + (S(T) - K) + K = 0$$

and in the latter case

$$-S(T) + (S(T) - K)^+ + K = -S(T) + 0 + K \geq 0.$$

Therefore arbitrage has been established. This is a financial contradiction and so (2.45) and hence (2.43) and (2.44) hold. $\square$

Example 2.29 (American call options). It is not optimal to exercise an American call option before expiry if the underlying stock does not pay dividends during the life of the option. In fact, at any time t prior to expiry,

$$C_A(t) > (S(t) - K)^+. \tag{2.48}$$

In other words, before expiry, an American call option is always worth (strictly) more than its exercise value. For this we require positive interest rates.

Proof. First note that

$$C_A(t) \geq C_E(t). \tag{2.49}$$

That is, an American call option is always worth at least the same as the European counterpart. After all the American call option offers all the privileges of the European call option and other benefits besides—the right to exercise the call before expiry, for example. One can also argue more rigorously: Assume that (2.49) is not true for some time t prior to expiry and construct an arbitrage opportunity. Suppose at time t it is true that $C_E(t) - C_A(t) > 0$. At time t short sell the European call and purchase an American call (with the same specifications of strike price and exercise price). Pocket the profit. As you own the American call option you decide when to exercise it. Decide **not** to exercise it early. At expiry the realized value of the American Call Option is $(S(T) - K)^+$, which is just the same as the value of the European call option. So this realized value can be used to cash settle the European call option at time T.

This argument is included to show how non arbitrage arguments can be used to derive financial conclusions.

Suppose now that $S(t) > K$. Then, as interest rates are positive,

$$S(t) - PV_t(K) > S(t) - K > 0. \tag{2.50}$$

So (using Example 2.28)

$$\begin{aligned} C_A(t) &\geq C_E(t) \\ &\geq S(t) - PV_t(K) \\ &> S(t) - K = (S(t) - K)^+. \end{aligned}$$

Thus, (2.48) holds if $S(t) > K$.

Suppose now that $S(t) \leq K$; then $(S(t) - K)^+ = 0$. But $C_A(t) > 0$ for $t < T$, so again (2.48) holds if $S(t) \leq K$. □

Remark 2.30. 1. One consequence of this example is the following: If a stock does not pay a dividend during the life of an option, then the American call options and the European call option have the same value. As dividends are usually paid twice a year, there will be many short term call options (90-day options) for which this condition applies. The financial press usually advertises when dividends are paid, and sometimes predicts when the next dividends will be paid based on what happened the year before.

2. The corresponding result does not hold for American and European put options. Always before expiry at $t < T$

$$P_A(t) > P_E(t), \tag{2.51}$$

where we have assumed that the two puts are the same in every other respect. The difference

$$e(t) \equiv P_A(t) - P_E(t) > 0 \tag{2.52}$$

is called the **early-exercise premium**. This is the extra amount one pays for an American put to have the right to exercise it early.

Example 2.31 (Estimate interest from option prices). For European calls and puts, the call-put parity formula can be rearranged to yield

$$R = \frac{K}{S(0) + P(0) - C(0)}. \tag{2.53}$$

For American puts and calls we have

$$C_A(0) - P_A(0) \leq S(0) - PV(K) \tag{2.54}$$

when dividends are not paid during the life of the options. This can be deduced from the call-put parity formula for European options and using $C_A(0) = C_E(0)$ and $P_A(0) \geq P_E(0)$.

Proof. Exercise. □

Under these circumstances (regarding dividends) we have

$$R \geq \frac{K}{S(0) + P_A(0) - C_A(0)}. \tag{2.55}$$

We should be able to check this holds from data in the financial press (otherwise there are arbitrage opportunities)

Example 2.32 (An AOL example). Time $t = 0$ is 22 July 2003 (the previous trading day).

$S(0) = \$16.85$, $K = \$16.00$, $C(0) = \$1.20$, $P(0) = \$0.45$. Then the right hand side of (2.55) is

$$\frac{16.00}{16.85 + 0.45 - 1.20} = 0.9937. \tag{2.56}$$

The interest rate at the time was around 1.00%, so there is no violation of (2.55).

Remark 2.33. We have seen that an American call and a European call have the same value when there are no dividends paid on the underlying stock. Under these circumstances we saw that it is not optimal to exercise an American call option early, as it is more profitable to sell the option than to exercise it. So the early exercise feature under these circumstances provides no extra value. We shall discuss later what happens when there are dividend payments.

Example 2.34 (Call options are decreasing functions of their time to expiration). Suppose there are two calls which are identical except they have different expiration times T_1 and T_2, with $T_1 > T_2$. "Now" is time t. Their times to expiration are τ_1 and τ_2 with $\tau_i = T_i - t, i = 1, 2$, so $\tau_1 > \tau_2$. The values of these calls at time t are $C^{\tau_i}(t)$, for $i = 1, 2$. We claim

$$C^{\tau_1}(t) \geq C^{\tau_2}(t) \tag{2.57}$$

for $0 \leq t \leq T_2$. (After T_2 the call with shorter time to expiry ceases to exist so (2.57) is either obvious or does not mean much, depending how you view things.) To prove (2.57) assume (if possible) that

$$C^{\tau_2}(t) - C^{\tau_1}(t) > 0$$

for some $0 \leq t \leq T_2$. This leads to an arbitrage opportunity as follows. At t, short the τ_2 call and purchase the τ_1 call. Pocket the profit. At time T_2, the value $V(T_2)$ of the position is

$$\begin{aligned} V(T_2) &= -(S(T_2) - K)^+ + C^{\tau_1}(T_2) \\ &\geq -(S(T_2) - K)^+ + (S(T_2) - K)^+ \\ &= 0, \end{aligned}$$

where we have used Example 2.28. So at time T_2 we have no unfunded liabilities if we sell the longer-dated call and cash settle the shorter-dated one at this time. So we now have a type one arbitrage opportunity. This is a (financial) contradiction, and so our claim holds.

Many more relations can be deduced in this model-independent way.

2.7 Exercises

Exercise 2.35. This exercise refers to Example 2.24. What are the reasons that market players will buy and sell put options. Are buyers and sellers matched?

Exercise 2.36. In Example 2.2 show that $0 \leq H_1 \leq 1$.

Exercise 2.37. Prove the identities (2.17), (2.18), (2.26), (2.27) and (2.28) in Section 2.2.

Exercise 2.38. This exercise refers to Example 3.1. How can you decide whether a futures trader is a speculator or a hedger? Explain the market for futures/forwards.

Exercise 2.39. Let $S = \{S(t) \mid t \geq 0\}$ be the price process of some stock (e.g., AOL shares). Let $C(t)$ denote the value at time t of a (European) call option written on S with maturity date T and exercise price K. Then $C(T) = \max[0, S(T) - K]$. Draw a graph, plotting $C(T)$ versus $S(T)$. This is called the payoff graph for the this call option. The profit graph is the plot of $C(T) - C(0)$ versus $S(T)$. Draw the profit graph. For what values of $S(T)$ will the profit be positive ? (This profit ignores the time value of money.)

Exercise 2.40. Repeat Exercise 2.39 but for the (European) put option. The difference between a put and a call is that with the put you have the right to sell rather than the right to buy. If $P(t)$ is the value of this put with strike price K and expiry date T, explain why $P(T) = \max[0, K - S(T)]$. Plot $P(T)$ versus $S(T)$, and $P(T) - P(0)$ versus $S(T)$. If you want to take a numerical example, choose the AOL/AUG03/16.00/PUT with $P(0) = 0.45$USD. For what values of $S(T)$ will the profit be positive? Explain why the holding of a put option on S is like holding an insurance policy over S.

Exercise 2.41. The current price of a certain stock is \$94 and 3-month call options with a strike price of \$95 currently sell for \$4.70. An investor who feels that the stock price will increase is trying to decide between buying 100 shares and buying 2000 call options (20 contracts). Both strategies would involve an investment of \$9,400. What advice would you give the investor? How high does the stock price have to rise for the option strategy to be more profitable?

Exercise 2.42. Suppose two banks XYZ and ABC are equally rated (as regards risk). Suppose that XYZ offers and charges customers 4% interest on deposits or loans, while ABC offers and charges 6% interest. Seeing this situation, how could you make a riskless profit without using any of your own money? You should provide an explicit strategy for achieving this, and explain any problems you might have carrying it out in practice. If the two banks were not equally rated, what possible reason could you give for the difference in interest rates?

Exercise 2.43. Suppose IBM pays a dividend D on their shares S at time τ. Show that $S(\tau+) = S(\tau-) - D$. Actually, to be precise, τ should be what is called the ex-dividend date. You should again argue your solution from the assumption of no arbitrage. $S(\tau+)$ means the value of S just after τ, and $S(\tau-)$ the value just before.

Exercise 2.44. Let C_i for $i = 1, 2, 3$ be European call options all expiring at T with strike prices K_i, for $i = 1, 2, 3$ all written on the same stock S. The **butterfly spread** is the combination $C_1 - 2C_2 + C_3$ with $K_2 = \frac{1}{2}(K_1 + K_3)$. Graph $C(T)$ against $S(T)$. Show that $C_2(0) < \frac{1}{2}(C_1(0) + C_3(0))$. Discuss which assumptions you make.

Exercise 2.45. We established call-put parity formula holds for European call and put options:

$$C(0) - P(0) = S(0) - PV(K),$$

where $PV(K) = K/R$. With the choices $S(0) = \$10.50$, $K = \$10.00$, $C(0) = \$3.00$, $P(0) = \$1.00$ and $R = 1.0043$, show that the call-put parity formula is violated. Show how to create an arbitrage opportunity of at least \$1000. You must not use any of your own money to fund this arbitrage opportunity.

Exercise 2.46. Read Linear Regression in the Appendix to Exercise 2.47, which can be studied together with this exercise.

Consider the data in Table 2.1 for XYZ/AUG03/CALLs. Suppose the spot price of the XYZ shares is $S = \$16.96$. (Not all strike prices are used here.)

Table 2.1. XYZ/AUG03/CALL for Exercise 2.46

	1	2	3	4	5	Sum
n_i	171	316	475	802	594	$n =$
$\omega_i = \frac{n_i}{n}$						1
x_i	17.50	18.00	18.50	19.00	19.50	
y_i	0.36	0.19	0.12	0.06	0.06	

We had $N = 5$, the x values represent strike prices, and the y values represent (ask) call prices. The n values are for open interest, which gives the number of contracts presently held with a particular strike price. We plot call prices against strike prices and seek the least squares fit line. Find its slope m and intercept c. Using the equations

$$c = \frac{\pi u S}{R} \qquad m = -\frac{\pi}{R} \qquad d = \frac{R - \pi u}{1 - \pi},$$

estimate π, u and d.

Use these values to compute the value of the XYZ/AUG02/17.00/PUT and compare your answer with the market value 0.55 CAD (ask).

These XYZ options are really American options, but as they are rarely exercised (why?) they price like European options. The value of R is to be taken as 1.0042.

Exercise 2.47. Consider the one-step binomial model with stock prices having $d = 1/u$. We can price ATM calls by

$$c = \frac{\pi}{R}(uS - K) = \frac{\pi}{R}(u - 1)S.$$

This leads to

$$u = \frac{1 + Rx}{R(1 - x)},$$

where $x = c/S$. So u can be calculated for a range of ATM calls by this formula. The CRR paper uses $u = \exp(\sigma\sqrt{\Delta t})$ where σ is the volatility and Δt is the time interval between $t = 0$ and $t = 1$, which we will take as $31/365 = 0.08493$ and so $\sqrt{\Delta t} \approx 0.291431$. It may therefore be of interest to plot $\ln u$ versus σ, get the line of best fit, and see if the estimated slope is about 0.29143. This is what you are now asked to do. Here are some "data" for various AUG03 calls. Use $R = 1.0042$ as before.

Table 2.2. ATM Call Prices for Exercise 2.47

Company	S	ATM Call Price	σ
ABC	16.96	0.6800	0.2685
DEF	9.18	0.3567	0.3614
HIJ	29.10	0.9780	0.2345
KLM	31.00	1.0400	0.2841
NOP	8.46	0.6800	0.6765

In making your line of least squares fit, use $w_i = 1/5 = 0.2$ for $i = 1, 2, 3, 4, 5$. Also observe that you should use the zero intercept form of linear regression here.

Appendix to Exercises 2.46 and 2.47

Linear Regression

We are given data points $\{(x_i, y_i) \mid i = 1, 2, \ldots, N\}$ and we want to place a line of best fit through them. The model will be

$$y_i = mx_i + c + \epsilon_i \tag{2.58}$$

for each $i = 1, 2, \ldots, N$. Here ϵ_i denotes an error for each i.

The **least squares fit line** is that line (or choice of m, c) so that

$$\sum_{i=1}^{N} \epsilon_i^2 \tag{2.59}$$

is minimal.

In finance we may have n_i measurements all at (x_i, y_i) and n_i may not be the same for each i. For example we could plot call prices versus strike prices from NYSE data. For the n_i we could use the open interest or the volume of trade.

In either case let us put

$$M = n_1 + n_2 + \ldots + n_N$$

and set

$$w_i \equiv \frac{n_i}{M},$$

which gives the proportion of measurements at (x_i, y_i). We could then minimize

$$\sum_{i=1}^{N} w_i \epsilon_i^2. \tag{2.60}$$

Setting derivatives of this expression with respect to m and c, both to zero yields the estimates for m and c which are

$$m = \frac{\overline{xy} - \overline{x}\ \overline{y}}{\overline{x^2} - \overline{x}^2} \tag{2.61}$$

and

$$c = \overline{y} - m\overline{x}. \tag{2.62}$$

Here we are using

$$\overline{y} = \sum_{i=1}^{N} w_i y_i$$

$$\overline{x} = \sum_{i=1}^{N} w_i x_i$$
$$\overline{xy} = \sum_{i=1}^{N} w_i x_i y_i$$
$$\overline{x^2} = \sum_{i=1}^{N} w_i x_i^2.$$

If $N > 1$, then $\overline{x^2} \neq \overline{x}^2$, so we never divide by zero! In fact

$$\overline{x^2} - \overline{x}^2 = \sum_{i=1}^{N} w_i (x_i - \overline{x})^2$$
$$\overline{xy} - \overline{x}\ \overline{y} = \sum_{i=1}^{N} w_i (x_i - \overline{x})(y_i - \overline{y}),$$

so we can solve for m and hence for c.

Example 2.48 (linear regression). Consider some XYZ call option prices. Suppose N = 5; $x_1 = 10.50$, $x_2 = 11.00$, $x_3 = 11.50$, $x_4 = 12.00$, $x_5 = 12.50$ and $y_1 = 1.36$, $y_2 = 0.95$, $y_3 = 0.62$, $y_4 = 0.38$, $y_5 = 0.20$ (we are using the selling prices). We could weight by **open interest** (open interest is the number of contracts in a particular class of options), and suppose $n_1 = 56$, $n_2 = 662$, $n_3 = 941$, $n_4 = 969$, $n_5 = 268$. Then $M = 2896$ and so $w_1 = \frac{56}{2896}$, $w_2 = \frac{662}{2896}$, $w_3 = \frac{941}{2896}$, $w_4 = \frac{969}{2896}$, $w_5 = \frac{268}{2896}$. Then

$$\begin{aligned}\overline{x} &= \frac{56 \times 10.50 + 662 \times 11.00 + 941 \times 11.50 + 969 \times 12.00 + 268 \times 12.50}{2896} \\ &= 11.62620856\end{aligned}$$

$$\begin{aligned}\overline{y} &= \frac{56 \times 1.36 + 662 \times 0.95 + 941 \times 0.62 + 969 \times 0.38 + 268 \times 0.20}{2896} \\ &= 0.590573204\end{aligned}$$

$$\begin{aligned}\overline{x^2} &= \frac{56 \times 10.50^2 + 662 \times 11.00^2 + \ldots + 969 \times 12.00^2 + 268 \times 12.50^2}{2896} \\ &= 135.4054731\end{aligned}$$

$$\begin{aligned}\overline{xy} &= \frac{56 \times 1.36 \times 10.50 + 662 \times 0.95 \times 11.00 + \ldots + 268 \times 0.20 \times 12.50}{2896} \\ &= 6.73879489\end{aligned}$$

$$
\begin{aligned}
m &= \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2} \\
&= \frac{6.73879489 - 11.62620856 \times 0.590573204}{135.4054731 - 11.62620856^2} \\
&= \frac{-0.127772349}{0.23674762} = -0.539698557
\end{aligned}
$$

$$
\begin{aligned}
c &= \overline{y} - m\overline{x} = 0.590573204 + 0.539698557 \times 11.62620856 \\
&= 6.865221193.
\end{aligned}
$$

We shall see later some interpretation of these estimates of m and c. If you know some statistics about the errors, then you can discuss the confidence intervals of the estimators of m and c given in formulae (2.61) and (2.62).

Many of these calculations can be easily carried out in MS-EXCEL.

Zero-Intercept Linear Regression

The model will now be

$$y_i = mx_i + \epsilon_i \tag{2.63}$$

for each $i = 1, 2, \ldots, N$. Here ϵ_i denotes an error for each i.

The (weighted) **least squares fit line** is that line (or choice of m) making

$$\sum_{i=1}^{N} w_i \epsilon_i^2 \tag{2.64}$$

minimal. Setting the derivative of this expression with respect to m to zero gives:

$$m = \frac{\overline{xy}}{\overline{x^2}} \tag{2.65}$$

with the same notation as above.

Exercise 2.49. Show that the value of a call can never be less than the value of an otherwise identical call with a higher strike price; that is,

$$C(K_1) \geq C(K_2) \quad \text{if} \quad K_2 > K_1,$$

and furthermore

$$K_2 - K_1 \geq C(K_1) - C(K_2) \quad \text{if} \quad K_2 > K_1.$$

http://www.springer.com/978-0-387-25898-0

Binomial Models in Finance
van der Hoek, J.; Elliott, R.J.
2006, XIV, 306 p., Hardcover
ISBN: 978-0-387-25898-0