Contents

Can I jump here?

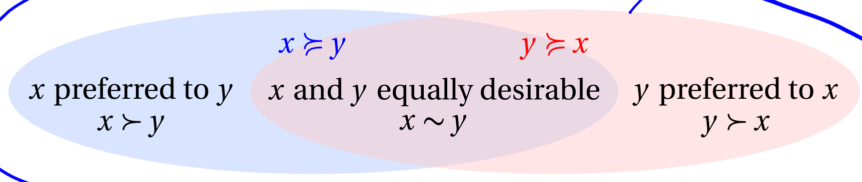


Figure 1.1 An individual's preference between x and y.

individual's answers to the questionnaire: if $x \sim y$ then $x \succsim y$ and $y \succsim x$, so that the individual's answer to the questionnaire is "I regard x and y as equally desirable", and if $x \succ y$ then the individual's answer is "I prefer x to y".

We assume that the individual answers all the questions on the questionnaire. Given our interpretation of the binary relation $\succsim$ as a description of responses to the questionnaire, this assumption means that for all distinct alternatives x and y either $x \succsim y$ or $y \succsim x$. We assume in addition that the same is true if x and y are the *same* alternative. That is, we assume that $x \succsim x$ for every alternative x, a property called reflexivity. The questionnaire does not ask "how do you compare x and x?", so the reflexivity of an individual's preferences cannot be deduced from her answers. We assume it because it fits the interpretation of the binary relation: it says that the individual regards every alternative to be at least as desirable as itself.

The property that for all alternatives x and y, distinct or not, either $x \succsim y$ or $y \succsim x$, is called completeness.

Definition 1.1: Complete binary relation

A binary relation R on the set X is *complete* if for all members x and y of X, either $x\,R\,y$ or $y\,R\,x$ (or both). A complete binary relation is, in particular, *reflexive*: for every $x \in X$ we have $x\,R\,x$.

For a binary relation $\succsim$ to correspond to a preference relation, we require not only that it be complete, but also that it be consistent in the sense that if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. This property is called transitivity.

Definition 1.2: Transitive binary relation

A binary relation R on the set X is *transitive* if for any members x, y, and z of X for which $x\,R\,y$ and $y\,R\,z$, we have $x\,R\,z$.

In requiring that a preference relation be transitive, we are restricting the permitted answers to the questionnaire. If the individual's response to the question regarding x and y is either "I prefer x to y" or "I am indifferent between x and y", and if her response to the question regarding y and z is "I prefer y to z" or "I am

1.2 Preference formation

When we model individuals, we endow them with preference relations, which we take as given; we do not derive these preference relations from any more basic considerations. We now briefly describe a few such considerations, some of which result in preference relations and some of which do not.

Value function The individual has in mind a function v that attaches to each alternative a number, interpreted as her subjective "value" of the alternative; the higher the value, the better the individual likes the alternative. Formally, the individual's preference relation $\succsim$ is defined by $x \succsim y$ if and only if $v(x) \geq v(y)$. The binary relation $\succsim$ derived in this way is indeed a preference relation: it is complete because we can compare any two numbers (for any two numbers a and b either $a \geq b$ or $b \geq a$ (or both)) and it is transitive because the binary relation $\geq$ is transitive (if $x \succsim y$ and $y \succsim z$ then $v(x) \geq v(y)$ and $v(y) \geq v(z)$, and hence $v(x) \geq v(z)$, so that $x \succsim z$).

*P

Distance function One alternative is "ideal" for the individual; how much she likes every other alternative is determined by the distance of that alternative from the ideal, as given by a function d. That is, the individual's preference relation $\succsim$ is defined by $x \succsim y$ if and only if $d(x) \leq d(y)$. This scheme is an example of a value function, with $v(x) = -d(x)$.

*?

Lexicographic preferences An individual has in mind two or more complete and transitive binary relations, $\succsim_1$ and $\succsim_2$, each of which relates to one feature of the alternatives. For example, if X is a set of computers, the features might be the size of the memory and the resolution of the screen. The individual gives priority to the first feature, breaking ties by the second feature. Formally, the individual's preference relation $\succsim$ is defined by $x \succsim y$ if (i) $x \succ_1 y$ or (ii) $x \sim_1 y$ and $x \succsim_2 y$.

The binary relation $\succsim$ defined in this way is a preference relation. Its completeness follows from the completeness of $\succsim_1$ and $\succsim_2$. Now consider its transitivity. Suppose that $x \succsim y$ and $y \succsim z$. There are two cases. (i) The first feature is decisive when comparing x and y: $x \succ_1 y$. Given $y \succsim z$ we have $y \succsim_1 z$, so by the transitivity of $\succsim_1$ we obtain $x \succ_1 z$ (see Problem 1b) and thus $x \succsim z$. (ii) The first feature is not decisive when comparing x and y: $x \sim_1 y$ and $x \succsim_2 y$. If the first feature is decisive for y and z, namely $y \succ_1 z$, then from the transitivity of $\succ_1$ we obtain $x \succ_1 z$ and therefore $x \succsim z$. If the first feature is not decisive for y and z, then $y \sim_1 z$ and $y \succsim_2 z$. By the transitivity of $\sim_1$ we obtain $x \sim_1 z$ and by the transitivity of $\succsim_2$ we obtain $x \succsim_2 z$. Thus $x \succsim z$.

Unanimity rule The individual has in mind n considerations, represented by the complete and transitive binary relations $\succsim_1, \succsim_2, \ldots, \succsim_n$. For example, a parent may take in account the preferences of her n children. Define the binary relation $\succsim$ by $x \succsim y$ if $x \succsim_i y$ for $i = 1, \ldots, n$. This binary relation is transitive but not necessarily complete. Specifically, if two of the relations $\succsim_i$ disagree ($x \succsim_j y$ and $y \succ_k x$), then $\succsim$ is not complete.

Majority rule The individual uses three criteria to evaluate the alternatives, each of which is expressed by a complete, transitive, and antisymmetric binary relation $\succsim_i$. (The antisymmetry of the relations implies that no two alternatives are indifferent according to any relation.) Define the binary relation $\succsim$ by $x \succsim y$ if and only if a majority (at least two) of the binary relations $\succsim_i$ rank x above y. Then $\succsim$ is complete: for all alternatives x and y either $x \succsim_i y$ for at least two criteria or $y \succsim_i x$ for at least two criteria. But the relation is not necessarily transitive, as an example known as the Condorcet paradox shows. Let $X = \{a, b, c\}$ and suppose that $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. Then $a \succ b$ (a majority of the criteria rank a above b) and $b \succ c$ (a majority rank b above c), but $c \succ a$ (a minority rank a above c).

1.3 An experiment

The assumption that preferences are transitive seems natural. When people are alerted to intransitivities in their preferences they tend to be embarrassed and change their evaluations. However, it is not difficult to design an environment in which most of us exhibit some degree of intransitivity. In Section 1.1 we suggested you respond to a long and exhausting questionnaire, with 36 questions, each asking you to compare a pair of alternatives taken from a set of nine alternatives. Each alternative is a description of a vacation package with four parameters: the city, hotel quality, food quality, and price.

As of April 2018, only 15% of the approximately 1,300 responses to the questionnaire do not exhibit any violation of transitivity. We count a set of three alternatives as a violation of transitivity if the answers to the three questions comparing pairs of alternatives from the set are inconsistent with transitivity. Among participants, the median number of triples that violate transitivity is 6 and the average is 9.5. (As a matter of curiosity, the highest number of intransitivities for any participant is 66. There are 84 sets of three alternatives, but the highest possible number of intransitivities is less than 84.)

A quarter of the participants' expressed preferences violate transitivity among the following alternatives.

1. A weekend in Paris, with 4 star hotel, food quality 17, for \$574.
2. A weekend in Paris, for \$574, food quality 17, with 4 star hotel.
3. A weekend in Paris, food quality 20, with 3–4 star hotel, for \$560.

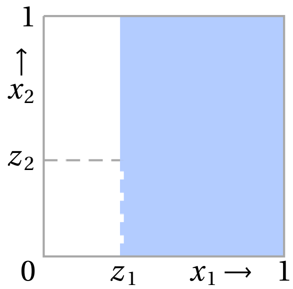


Figure 1.4 The set of alternatives preferred to (z_1, z_2) according to the lexicographic preference relation described in the text is the area shaded blue, excluding the part of the boundary indicated by a dashed line.

interval $(u(b,0), u(b,1))$ from which $f(b)$ is selected. The contradiction now follows from Cantor's diagonal argument, which shows that there is no one-to-one function from the set $[0,1]$ into a countable set (like the set of rational numbers).

If a utility function represents a given preference relation, then many other utility functions do so too. For example, if the function u represents a given preference relation then so does the function $3u-7$ or any other function of the form $au+b$ where a is a positive number. In fact, we have the following result.

Proposition 1.3: Increasing function of utility function is utility function

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. If u represents the preference relation $\succsim$ on X, then so does the function w defined by $w(x) = f(u(x))$ for all $x \in X$.

Proof

We have $w(x) \geq w(y)$ if and only if $f(u(x)) \geq f(u(y))$ if and only if $u(x) \geq u(y)$ (given that f is increasing), which is true if and only if $x \succsim y$.

Problems

1. *Properties of binary relations*. Assume that $\succsim$ is a preference relation.

   a. Show that the binary relation $\succ$ defined by $x \succ y$ if $x \succsim y$ and not $y \succsim x$ is transitive and antisymmetric.

   b. Show that if $x \succsim y$ and $y \succ z$ (or $x \succ y$ and $y \succsim z$) then $x \succ z$.

2. *Minimal element.* Let $\succsim$ be a preference relation over a finite set X.

   a. Show that a is minimal with respect to $\succsim$ in X if and only if there is no $x \in X$ such that $a \succ x$.

   b. Give an example to show that if X is not finite then a preference relation relation may have no minimal and maximal elements in X.

3. *Similarity relations.* Consider the following preference formation scheme. An individual has in mind a function $v : X \to \mathbb{R}$ that attaches to each alternative a number, but is sensitive only to significant differences in the value of the function; she is indifference between alternatives that are "similar". Specifically, the individual prefers x to y if $v(x) - v(y) > 1$ and is indifferent between x and y if $-1 \leq v(x) - v(y) \leq 1$. Is the individual's preference relation necessarily transitive?

4. *Equivalence relations.*

   a. Give two examples of equivalence relations on different sets.

   b. Show that the binary relation R on the set of positive integers defined by $x\,R\,y$ if $x + y$ is even is an equivalence relation.

   c. A partition of the set X is a set of nonempty subsets of X such that every member of X is a member of one and only one subset. For example, the set of sets $\{\{1,3,5\},\{2,4,6\}\}$ is a partition of the set $\{1,2,3,4,5,6\}$. Show that every equivalence relation R on X induces a partition of the set X in which x and y are in the same member of the partition if and only if $x\,R\,y$.

5. *Independence of properties.* Find an example of a binary relation that is complete and transitive but not symmetric. Find also an example of a binary relation that is reflexive, transitive, and symmetric but not complete.

6. *Shepard scale and Escher.* Listen to the Shepard scale and look at a picture of Penrose stairs. (The video at `http://techchannel.att.com/play-video.cfm/2011/10/10/AT&T-Archives-A-Pair-of-Paradoxes` combines them. The lithograph *Ascending and descending* by M. C. Escher is a rendering of Penrose stairs.) Explain the connection between these two examples and the concept of transitivity.

7. *Utility representation.* Let X be the set of all positive integers.

   a. An individual prefers the number 8 to all other numbers. Comparing a pair of numbers different from 8 she prefers the higher number. Construct a utility function that represents these preferences.

2 Choice

2.1 Choice and rational choice

In the previous chapter we discuss an individual's preference relation, a formal concept that describes her mental attitude to all relevant alternatives. We now develop a formal tool to describe an individual's behavior. The two concepts, preferences and choice, are building blocks of the economic models we develop later.

Recall that the notion of a preference relation refers only to the individual's mental attitude, not to the choices she may make. In this chapter, we describe a concept of choice, independently of preferences. This description specifies her decision in any possible choice problem she may confront within the context we are modeling. Suppose, for example, that we want to model a worker who is applying for a job. Then a complete description of her behavior specifies not only which job she chooses if all jobs in the world are open to her, but also her choice from any subset of jobs that she might be offered.

Formally, let X be the set of all the alternatives an individual might face. A choice problem is a nonempty subset A of X, from which the individual chooses an alternative. A choice function describes the individual's choice for every possible choice problem.

Definition 2.1: Choice problem and choice function

Given a set X, a *choice problem for X* is a nonempty subset of X and a *choice function for X* associates with every choice problem $A \subseteq X$ a single member of A (the member chosen).

"Rational

Usually in economics we connect the individual's behavior and her mental attitude by assuming that the individual is *rational* in the sense that

- she has a preference relation over X
- whenever she has to make a choice, she is aware of the set of possible alternatives
- she chooses an alternative that is best according to her preference relation over the set of possible alternatives.

Chapter of *Models in Microeconomic Theory* by Martin J. Osborne and Ariel Rubinstein. Version 2020.3.26 (s).
 https://doi.org/10.11647/OBP.0211.02

Example 2.1

Let $X = \{a, b, c\}$. The choice function that assigns a to $\{a, b, c\}$, a to $\{a, b\}$, a to $\{a, c\}$, and b to $\{b, c\}$ is rationalized by the preference relation $\succsim$ for which $a \succ b \succ c$. That is, we can describe the behavior of an individual with this choice function *as if* she always chooses the best available alternative according to $\succsim$.

On the other hand, any choice function that assigns a to $\{a, b\}$, c to $\{a, c\}$, and b to $\{b, c\}$ is not rationalizable. If this choice function could be rationalized by a preference relation $\succsim$, then $a \succ b$, $b \succ c$, and $c \succ a$, which contradicts transitivity.

Of the 24 possible choice functions for the case in which X contains three alternatives, only six are rationalizable.

We now give some examples of choice procedures and examine whether the resulting choice functions are rationalizable.

Example 2.2: The median

← P

usually middle option is not rationalizable

An individual has in mind an ordering of the alternatives in the set X from left to right. For example, X could be a set of political candidates and the ordering might reflect their position from left to right. From any set A of available alternatives, the individual chooses a median alternative. Precisely, if the number of available alternatives is odd, with $a_1 < a_2 < \cdots < a_{2k+1}$ for some integer k, the individual chooses the single median a_{k+1}, and if the number of alternatives is even, with $a_1 < a_2 < \cdots < a_{2k}$, then the individual chooses a_k, the leftmost of the two medians.

No preference relation rationalizes this choice function. Assume that A contains five alternatives, $a_1 < a_2 < a_3 < a_4 < a_5$. From this set, she chooses a_3. If she has to choose from $\{a_3, a_4, a_5\}$, she chooses a_4. If a preference relation $\succsim$ rationalizes this choice function then $a_3 \succ a_4$ from her first choice and $a_4 \succ a_3$ from her second choice, a contradiction.

Note that the individual's behavior has a rationale of a different type: she always prefers the central option. But this rationale cannot be described in terms of choosing the best alternative according to a preference relation over the set of available alternatives. The behavior can be rationalized if we view the set of alternatives to be the positions $Y = \{$median, one left of median, one right of median, two left of median, two right of median$\}$. Then the first choice problem is Y and the second choice problem is $\{$one left of median, median, one right of median$\}$. The preference relation $\succsim$ given by

$$\text{median} \succ \text{one left of median} \succ \text{one right of median} \succ \dots$$

rationalizes the choice function.

Example 2.3: Steak and salmon

Luce and Raiffa (1957, 288) give an example of a person entering a restaurant in a strange city.

> The waiter informs him that there is no menu, but that this evening he may have either broiled salmon at $2.50 or steak at $4.00. In a first-rate restaurant his choice would have been steak, but considering his unknown surroundings and the different prices he elects the salmon. Soon after the waiter returns from the kitchen, apologizes profusely, blaming the uncommunicative chef for omitting to tell him that fried snails and frog's legs are also on the bill of fare at $4.50 each. It so happens that our hero detests them both and would always select salmon in preference to either, yet his response is "Splendid, I'll change my order to steak".

Consider a set X that consists of the four main courses, salmon, steak, snails, and frog's legs. No preference relation over X rationalizes the person's behavior, because such a preference relation would have to rank salmon above steak by his choice from $\{\text{salmon}, \text{steak}\}$ and steak above salmon by his choice from X.

A reasonable explanation for the person's behavior is that although steak appears in both choice problems, he does not regard it to be the same dish. The availability of snails and frog's legs tells him that the steak is likely to be of high quality. Without this information, he views steak as low quality and chooses salmon.

No preference relation on X rationalizes the person's behavior, but a preference relation on $\{\text{salmon}, \text{low quality steak}, \text{high quality steak}, \text{snails}, \text{frog's legs}\}$ does so:

$$\text{high quality steak} \succ \text{salmon} \succ \text{low quality steak} \succ \text{snails} \succ \text{frog's legs}.$$

An underlying assumption behind the concept of a choice function is that an alternative is the same in every choice set in which it appears. The choice function in the example cannot be rationalized because the example identifies two different options as the same alternative.

Example 2.4: Partygoer

Each of the people in the set $X = \{A, B_1, B_2\}$ organizes a party. A person might be invited to a subset of those parties and can attend only one party. Individuals B_1 and B_2 are both good friends of the partygoer but the relations between B_1 and B_2 are tense. The person's behavior is as follows. If she is invited by A and B_1, she accepts B_1's invitation. If she is invited by all three individuals, she accepts A's invitation. She does so because she is worried that accepting the invitation of B_1 or B_2 will be interpreted negatively by the other individual. Obviously such behavior is not rationalizable by a preference relation over X. As in the previous example, the meaning of choosing one alternative (B_1) is affected by the presence or absence of another alternative (B_2).

2.3 Property α

We say that a choice function satisfies property α if whenever the choice from A is in a subset B then the alternative chosen from A is chosen also from B. We show that (i) any choice function that selects the best alternative according to a preference relation satisfies this property and (ii) any choice function that satisfies the property is rationalizable.

Definition 2.3: Property α

Given a set X, a choice function c for X *satisfies property* α if for any sets A and B with $B \subset A \subseteq X$ and $c(A) \in B$ we have $c(B) = c(A)$.

Notice that property α is not satisfied by the choice functions in Examples 2.2, 2.3, and 2.4.

Proposition 2.1: Rationalizable choice function satisfies property α

Every rationalizable choice function satisfies property α.

Proof

Let c be a rationalizable choice function for X and let $\succsim$ be a preference relation such that for every set $A \subseteq X$, $c(A)$ is the best alternative according to $\succsim$ in A. Assume that $B \subset A$ and $c(A) \in B$. Since $c(A) \succsim y$ for all $y \in A$ we have $c(A) \succsim y$ for all $y \in B$ and thus $c(B) = c(A)$.

Now make another choice.

Which of the following cameras do you choose?

Camera A	Average rating 9.1, 6 megapixels
Camera B	Average rating 8.3, 9 megapixels
Camera C	Average rating 8.1, 7 megapixels

Each question was answered by about 1,300 participants on the website http://gametheory.tau.ac.il. The results are given in the following tables.

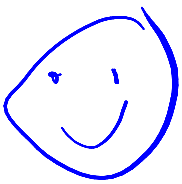
Choice between A and B	
Camera A	48%
Camera B	52%

Choice between A, B, and C	
Camera A	30%
Camera B	68%
Camera C	2%

Thus the appearance of C does not lead people to choose C, but rather causes a significant fraction of participants to choose B, which dominates C, even though in a choice between A and B they choose A. One explanation of this result is that the availability of C directs the participants' focus to B, the alternative that dominates it. An alternative explanation is that the dominance of B over C provides a reason to choose B, a reason that does not apply to A.

2.6.2 Framing effects

Sometimes individuals' choices depend on the way in which the alternatives are described.



You have to spin either roulette A or roulette B. The outcomes of spinning each roulette are given in the following table.

	White	Red	Green	Yellow
roulette A	90%	6%	1%	3%
	\$0	\$45	\$30	−\$15
roulette B	90%	7%	1%	2%
	\$0	\$45	−\$10	−\$15

Which roulette do you choose?

Subjects' choices in this experiment are generally split more or less equally between the two roulettes. About 51% of around 4,000 participants at the website http://gametheory.tau.ac.il have chosen A.

A common explanation for the choice of A is that the problem is complicated and participants simplify it by "canceling" similar parameters. The outcomes of White in the two roulettes are identical and the outcomes of Red and Yellow are very similar; ignoring these colors leaves Green, which yields a much better outcome for roulette A.

Here is another choice problem.

> You have to spin either roulette C or roulette D. The outcomes of spinning each roulette are given in the following table.
>
> | | White | Red | Black | Green | Yellow |
> |---|---|---|---|---|---|
> | roulette C | 90% | 6% | 1% | 1% | 2% |
> | | \$0 | \$45 | \$30 | −\$15 | −\$15 |
> | roulette D | 90% | 6% | 1% | 1% | 2% |
> | | \$0 | \$45 | \$45 | −\$10 | −\$15 |
>
> Which roulette do you choose?

It is clear that D dominates C, and indeed almost all participants (93%) at `http://gametheory.tau.ac.il` have chosen D.

Now notice that A and C differ only in their presentation (the color Yellow in A is split in C into two contingencies). The same is true of B and D (the color Red in B is split in D into two contingencies). The different presentations seem to cause at least half of the participants to apply different choice procedures: they reduce the complicated problem to a simpler one in the choice between A and B and apply a domination criterion in the choice between C and D.

2.6.3 Mental accounting

> Imagine that you have bought a ticket for a show for \$40. When you reach the theatre you discover that you have lost the ticket. You can buy another ticket at the same price. Will you do so?

Now think about another situation.

> Imagine that you intend to go to a show. When you take your wallet out of your pocket to pay for the \$40 ticket, you discover that you have lost \$40, but you still have enough cash to buy a ticket. Will you do so?

3 Preferences under uncertainty

3.1 Lotteries

In Chapter 1 we discuss a model of preferences over an arbitrary set of alternatives. In this chapter we study an instance of the model in which an alternative in the set involves randomness regarding the consequence it yields. We refer to these alternatives as *lotteries*. For example, a raffle ticket that yields a car with probability 0.001 and nothing otherwise is a lottery. A vacation on which you will experience grey weather with probability 0.3 and sunshine with probability 0.7 can be thought of as a lottery as well.

The set X in the model we now discuss is constructed from a set Z of objects called prizes. A lottery specifies the probability with which each prize is realized. For simplicity, we study only lotteries for which the number of prizes that can be realized is finite.

Definition 3.1: Lotteries

Let Z be a set (of *prizes*). A *lottery over* Z is a function $p : Z \to \mathbb{R}$ that assigns a positive number (probability) $p(z)$ to a finite number of members of Z and 0 to all other members, with $\sum_{z \in Z} p(z) = 1$. The *support* of the lottery p, denoted $\text{supp}(p)$, is the set of all prizes to which p assigns positive probability, $\{z \in Z : p(z) > 0\}$.

We denote the set of all lotteries over Z by $L(Z)$, the lottery that yields the prize z with probability 1 by $[z]$, and the lottery that yields the prize z_k with probability α_k for $k = 1, \ldots, K$ by $\alpha_1 \cdot z_1 \oplus \alpha_2 \cdot z_2 \oplus \cdots \oplus \alpha_K \cdot z_K$.

If Z consists of two prizes, z_1 and z_2, then each member p of $L(Z)$ is specified by a pair (p_1, p_2) of nonnegative numbers with sum 1, where $p_1 = p(z_1)$ and $p_2 = p(z_2)$ are the probabilities of the prizes. Thus in this case $L(Z)$ can be identified with the blue line segment in Figure 3.1a. If Z includes three options, $L(Z)$ can similarly be identified with the triangle in Figure 3.1b.

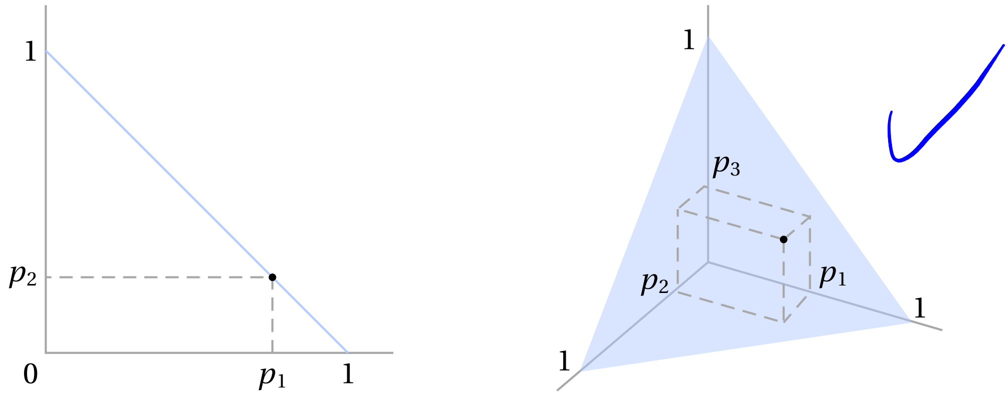
$L(Z)$:

3.2 Preferences over lotteries

We are interested in preference relations over $L(Z)$. In terms of the model in Chapter 1, the set X is equal to $L(Z)$. Here are some examples.

Chapter of *Models in Microeconomic Theory* by Martin J. Osborne and Ariel Rubinstein. Version 2020.3.26 (s).
 https://doi.org/10.11647/OBP.0211.03

$p_1 \cdot z_1 \oplus p_2 \cdot z_2 \cdots$



(a) The set $L(\{z_1, z_2\})$ of lotteries. The point (p_1, p_2) on the line segment represents the lottery p for which $p(z_1) = p_1$ and $p(z_2) = p_2$.

(b) The set $L(\{z_1, z_2, z_3\})$ of lotteries. The point (p_1, p_2, p_3) in the triangle represents the lottery p for which $p(z_1) = p_1$, $p(z_2) = p_2$, and $p(z_3) = p_3$.

Figure 3.1

Example 3.1: A pessimist

An individual has a strict preference relation $\succsim^*$ over the set Z of prizes and (pessimistically) evaluates lotteries by the worst prize, according to the preference relation, that occurs with positive probability. That is, she prefers the lottery $p \in L(Z)$ to the lottery $q \in L(Z)$ if she prefers the worst prize that occurs with positive probability in p to the worst prize that occurs with positive probability in q. Formally, define $w(p)$ to be a prize in $\text{supp}(p)$ such that $y \succsim^* w(p)$ for all $y \in \text{supp}(p)$. Then the individual's preference relation $\succsim$ over $L(Z)$ is defined by $p \succsim q$ if $w(p) \succsim^* w(q)$.

Note that there are many pessimistic preference relations, one for each preference relation over the set of prizes.

For any such preferences, the individual is indifferent between two lotteries whenever she is indifferent between the worst prizes that occur with positive probability in the lotteries. In one variant of the preferences that breaks this tie, if two lotteries share the same worst possible prize then the one for which the probability of the worst prize is lower is preferred.

Example 3.2: Good and bad

An individual divides the set Z of prizes into two subsets, *good* and *bad*. For any lottery $p \in L(Z)$, let $G(p) = \sum_{z \in good} p(z)$ be the total probability that a prize in *good* occurs. The individual prefers the lottery $p \in L(Z)$ to the lottery $q \in L(Z)$ if the probability of a prize in *good* occurring is at least as high for p as it is for q. Formally, $p \succsim q$ if $G(p) \geq G(q)$.

Different partitions of Z into *good* and *bad* generate different preference relations.

Example 3.3: Minimizing options

An individual wants the number of prizes that might be realized (the number of prizes in the support of the lottery) to be as small as possible. Formally, $p \succsim q$ if $|\text{supp}(p)| \leq |\text{supp}(q)|$. This preference relation makes sense for an individual who does not care about the realization of the lottery but wants to be as prepared as possible (physically or mentally) for all possible outcomes.

Preference relations over lotteries can take an unlimited number of other forms. To help us organize this large set, we now describe two plausible properties of preference relations and identify the set of all preference relations that satisfy the properties.

3.2.1 Properties of preferences

Continuity Suppose that for the prizes a, b, and c we have $[a] \succ [b] \succ [c]$, and consider lotteries of the form $\alpha \cdot a \oplus (1-\alpha) \cdot c$ (with $0 \leq \alpha \leq 1$). The continuity property requires that as we move continuously from $\alpha = 1$ (the degenerate lottery $[a]$, which is preferred to $[b]$) to $\alpha = 0$ (the degenerate lottery $[c]$, which is worse than $[b]$) we pass (at least once) some value of α such that the lottery $\alpha \cdot a \oplus (1-\alpha) \cdot c$ is indifferent to $[b]$.

Definition 3.2: Continuity

For any set Z of prizes, a preference relation $\succsim$ over $L(Z)$ is *continuous* if for any three prizes a, b, and c in Z such that $[a] \succ [b] \succ [c]$ there is a number α with $0 < \alpha < 1$ such that $[b] \sim \alpha \cdot a \oplus (1-\alpha) \cdot c$.

When Z includes at least three prizes, pessimistic preferences are not continuous: if $[a] \succ [b] \succ [c]$ then $[b] \succ \alpha \cdot a \oplus (1-\alpha) \cdot c$, for every number $\alpha < 1$. Good and bad preferences and minimizing options preferences satisfy the continuity condition vacuously because in each case there are no prizes a, b and c for which $[a] \succ [b] \succ [c]$.

Independence To define the second property, we need to first define the notion of a compound lottery. Suppose that uncertainty is realized in two stages. First

$[A] \succ [B] \succ [C]$

α $(1-\alpha)$

the lottery p_k is drawn with probability α_k, for $k = 1, \ldots, K$, and then each prize z is realized with probability $p_k(z)$. In this case, the probability that each prize z is ultimately realized is $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$. Note that $\sum_{k=1,\ldots,K} \alpha_k p_k(z) \geq 0$ for each z and the sum of these expressions over all prizes z is equal to 1. We refer to the lottery in which each prize z occurs with probability $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$ as a compound lottery, and denote it by $\alpha_1 \cdot p_1 \oplus \cdots \oplus \alpha_K \cdot p_K$. For example, let $Z = \{W, D, L\}$, and define the lotteries $p = 0.6 \cdot W \oplus 0.4 \cdot L$ and $q = 0.2 \cdot W \oplus 0.3 \cdot D \oplus 0.5 \cdot L$. Then the compound lottery $\alpha \cdot p \oplus (1-\alpha) \cdot q$ is the lottery

$$(\alpha 0.6 + (1-\alpha)0.2) \cdot W \oplus ((1-\alpha)0.3) \cdot D \oplus (\alpha 0.4 + (1-\alpha)0.5) \cdot L.$$

} written in distributed format

Definition 3.3: Compound lottery

Let Z be a set of prizes, let $p_1, \ldots, p_K$ be lotteries in $L(Z)$, and let $\alpha_1, \ldots, \alpha_K$ be nonnegative numbers with sum 1. The *compound lottery* $\alpha_1 \cdot p_1 \oplus \cdots \oplus \alpha_K \cdot p_K$ is the lottery that yields each prize $z \in Z$ with probability $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$.

sum prob of Z across all lotteries $P_1 \ldots P_k$

We can now state the second property of preference relations over lotteries.

Definition 3.4: Independence

Let Z be a set of prizes. A preference relation $\succsim$ over $L(Z)$ satisfies the *independence property* if for any lotteries $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and $\beta \cdot a \oplus (1-\beta) \cdot b$ we have

$$[z_k] \succsim \beta \cdot a \oplus (1-\beta) \cdot b$$

$$\Longleftrightarrow$$

$$\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$$
$$\succsim \alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot z_K.$$

$A \gtrsim B$
$\Longleftrightarrow$
$C \gtrsim D$

$A > B$
$\Longleftrightarrow$
$C > D$

The logic of the property is procedural: the only difference between the lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and the compound lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta a \oplus (1-\beta) b) \oplus \cdots \oplus \alpha_K \cdot z_K$ is in the kth term, which is z_k in the first case and $\beta \cdot a \oplus (1-\beta) \cdot b$ in the second case. Consequently it is natural to compare the two lotteries by comparing $[z_k]$ and $\beta \cdot a \oplus (1-\beta) \cdot b$.

Pessimistic preferences do not satisfy this property. Let $[a] \succ [b]$ and consider, for example, the lotteries

$$p = 0.6 \cdot a \oplus 0.4 \cdot b \quad \text{and} \quad q = 0.6 \cdot b \oplus 0.4 \cdot b = [b].$$

These lotteries differ only in the prize that is realized with probability 0.6. Given that $[a] \succ [b]$, the independence property requires that $p \succ q$. However, for a

$\Longleftrightarrow$

note that already extends $\succsim$ over pessimist the two lotteries are indifferent since the worst prize in the lotteries is the same (b).

Minimizing options preferences also violate the independence property: for any prizes a and b, the lotteries $[a]$ and $[b]$ are indifferent, but $0.5 \cdot a \oplus 0.5 \cdot b \prec 0.5 \cdot b \oplus 0.5 \cdot b$.

Good and bad preferences satisfy the independence property. Let p be the lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and let q be the compound lottery

$$\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot z_K.$$

Note that $G(p) - G(q) = \alpha_k G([z_k]) - \alpha_k G(\beta \cdot a \oplus (1-\beta) \cdot b)$, so that since $\alpha_k > 0$, the sign of $G(p) - G(q)$ is the same as the sign of $G([z_k]) - G(\beta \cdot a \oplus (1-\beta) \cdot b)$. Thus the preferences compare p and q in the same way that they compare $[z_k]$ and $\beta \cdot a \oplus (1-\beta) \cdot b$.

Monotonicity Consider lotteries that assign positive probability to only two prizes a and b, with $[a] \succ [b]$. We say that a preference relation over $L(Z)$ is monotonic if it ranks such lotteries by the probability that a occurs. That is, monotonic preferences rank lotteries of the type $\alpha \cdot a \oplus (1-\alpha) \cdot b$ according to the value of α.

The next result says that any preference relation over $L(Z)$ that satisfies the independence property is monotonic.

Lemma 3.1: Independence implies monotonicity

Let Z be a set of prizes. Assume that $\succsim$, a preference relation over $L(Z)$, satisfies the independence property. Let a and b be two prizes with $[a] \succ [b]$, and let α and β be two probabilities. Then

$$\alpha > \beta \quad \Leftrightarrow \quad \alpha \cdot a \oplus (1-\alpha) \cdot b \succ \beta \cdot a \oplus (1-\beta) \cdot b.$$

"a more likely" "a less"

P_α

Proof

Let $p_\alpha = \boxed{\alpha \cdot a} \oplus (1-\alpha) \cdot b$. Because $\succsim$ satisfies the independence property, $p_\alpha \succ \boxed{\alpha \cdot b} \oplus (1-\alpha) \cdot b = [b]$. Using the independence property again we get

bc $[a] \succ [b]$

$$p_\alpha = (\beta/\alpha) \cdot p_\alpha \oplus (1-\beta/\alpha) \cdot p_\alpha \succ (\beta/\alpha) \cdot p_\alpha \oplus (1-\beta/\alpha) \cdot b = \beta \cdot a \oplus (1-\beta) \cdot b.$$

bc $p_a \succ [b]$ as shown

algebra steps

3.3 Expected utility

We now introduce the type of preferences most commonly assumed in economic theory. These preferences emerge when an individual uses the following scheme to compare lotteries. She attaches to each prize z a number, which we refer to as the value of the prize (or the Bernoulli number) and denote $v(z)$; when evaluating a lottery p, she calculates the expected value of the lottery, $\sum_{z\in Z} p(z)v(z)$. The individual's preferences are then defined by

$$p \succsim q \quad \text{if} \quad \sum_{z\in Z} p(z)v(z) \geq \sum_{z\in Z} q(z)v(z).$$

Definition 3.5: Expected utility

For any set Z of prizes, a preference relation $\succsim$ on the set $L(Z)$ of lotteries is *consistent with expected utility* if there is a function $v: Z \to \mathbb{R}$ such that $\succsim$ is represented by the utility function U defined by $U(p) = \sum_{z\in Z} p(z)v(z)$ for each $p \in L(Z)$. The function v is called the *Bernoulli function* for the representation.

*P

We first show that a preference relation consistent with expected utility is continuous and satisfies the independence property.

Proposition 3.1: Expected utility is continuous and independent

3.1

A preference relation on a set of lotteries that is consistent with expected utility satisfies the continuity and independence properties.

EU → C&I

Proof

Let Z be a set of prizes, let $\succsim$ be a preference relation over $L(Z)$, and let $v: Z \to \mathbb{R}$ be a function such that the function U defined by $U(p) = \sum_{z\in Z} p(z)v(z)$ for each $p \in L(Z)$ represents $\succsim$.

easy PE.

Continuity Let a, b, and $c \in Z$ satisfy $[a] \succ [b] \succ [c]$. For every $z \in Z$, $U([z]) = v(z)$. Thus $v(a) > v(b) > v(c)$. Let α satisfy $\alpha v(a) + (1-\alpha)v(c) = v(b)$ (that is, $0 < \alpha = (v(b) - v(c))/(v(a) - v(c)) < 1$). Then $\alpha \cdot a \oplus (1-\alpha) \cdot c \sim [b]$.

Independence Consider lotteries $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_K \cdot z_K$ and $\beta \cdot a \oplus (1-\beta) \cdot b$. We have

$$\begin{aligned} &\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K \\ &\qquad \succsim \alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot q_K \\ &\Leftrightarrow \text{(by the formula for } U\text{, which represents } \succsim \text{)} \end{aligned}$$

Skip Proofs?

$$\begin{aligned}
&\alpha_1 v(z_1)+\cdots+\alpha_k v(z_k)+\cdots+\alpha_K v(z_K)\\
&\qquad \geq \alpha_1 v(z_1)+\cdots+\alpha_k \beta v(a)+\alpha_k(1-\beta)v(b)+\cdots+\alpha_K v(z_K)\\
&\qquad \Longleftrightarrow \text{(by algebra)}\\
&\qquad \alpha_k v(z_k)\geq \alpha_k \beta v(a)+\alpha_k(1-\beta)v(b)\\
&\qquad \Longleftrightarrow \text{(since } \alpha_k>0)\\
&\qquad v(z_k)\geq \beta v(a)+(1-\beta)v(b)\\
&\qquad \Longleftrightarrow \text{(by the formula for } U\text{, which represents } \succsim)\\
&\qquad [z_k]\succsim \beta\cdot a\oplus(1-\beta)\cdot b.
\end{aligned}$$

The next result, the main one of this chapter, shows that any preference relation that satisfies continuity and independence is consistent with expected utility. That is, we can attach values to the prizes such that the comparison of the expected values of any two lotteries is equivalent to the comparison of the lotteries according to the preference relation.

Proposition 3.2: Continuity and independence implies expected utility

A preference relation on a set of lotteries with a finite set of prizes that satisfies the continuity and independence properties is consistent with expected utility.

3.2 C & I → EU (w/ finite set)

Proof

Let Z be a finite set of prizes and let $\succsim$ be a preference relation on $L(Z)$ satisfying continuity and independence. Label the members of Z so that $[z_1]\succsim\cdots\succsim[z_K]$. Let $z_1=M$ (the best prize) and $z_K=m$ (the worst prize).

First suppose that $[M]\succ[m]$. Then by continuity, for every prize z there is a number $v(z)$ such that $[z]\sim v(z)\cdot M\oplus(1-v(z))\cdot m$. In fact, by monotonicity this number is unique. Consider a lottery $p(z_1)\cdot z_1\oplus\cdots\oplus p(z_K)\cdot z_K$. By applying independence K times, the individual is indifferent between this lottery and the compound lottery

$$p(z_1)\cdot\underbrace{(v(z_1)\cdot M\oplus(1-v(z_1))\cdot m)}_{z_1}\oplus\cdots\oplus p(z_K)\cdot\underbrace{(v(z_K)\cdot M\oplus(1-v(z_K))\cdot m)}_{z_K}.$$

This compound lottery is equal to the lottery

$$\left(\sum_{k=1,\ldots,K}p(z_k)v(z_k)\right)\cdot M\oplus\left(1-\sum_{k=1,\ldots,K}p(z_k)v(z_k)\right)\cdot m.$$

Given $[M]\succ[m]$, Lemma 3.1 implies that the comparison between the

Skip? ? I.e., which, decomposed

this is like an EU formula?

how more of the "less st"

? how

lotteries p and q is equivalent to the comparison between the numbers $\sum_{k=1,\dots,K} p(z_k)v(z_k)$ and $\sum_{k=1,\dots,K} q(z_k)v(z_k)$.

Now suppose that $[M] \sim [m]$. Then by independence, $p \sim [M]$ for any lottery p. That is, the individual is indifferent between all lotteries. In this case, choose $v(z_k) = 0$ for all k. Then the function U defined by $U(p) = \sum_{z \in Z} p(z)v(z) = 0$ for each $p \in L(Z)$ represents the preference relation.

Comment invariant to affine trans'n

Note that if the function $v : Z \to \mathbb{R}$ is the Bernoulli function for an expected utility representation of a certain preference relation over $L(Z)$ then for any numbers $\alpha > 0$ and β so too is the function w given by $w(z) = \alpha v(z) + \beta$ for all $z \in Z$. In fact the converse is true also (we omit a proof): if $v : Z \to \mathbb{R}$ and $w : Z \to \mathbb{R}$ are (both) Bernoulli functions for representations of a certain preference relation then for some numbers $\alpha > 0$ and β we have $w(z) = \alpha v(z) + \beta$ for all $z \in Z$.

3.4 Theory and experiments

We now briefly discuss the connection (and disconnection) between the model of expected utility and human behavior. The following well-known pair of questions demonstrates a tension between the two.

Allais

Imagine that you have to choose between the following two lotteries.

L_1: you receive \$4,000 with probability 0.2 and zero otherwise.
R_1: you receive \$3,000 with probability 0.25 and zero otherwise.

Which lottery do you choose?

Imagine that you have to choose between the following two lotteries.

L_2: you receive \$4,000 with probability 0.8 and zero otherwise.
R_2: you receive \$3,000 with certainty.

Which lottery do you choose?

The responses to these questions by 7,932 students at http://gametheory.tau.ac.il are summarized in the following table.

	L_2	R_2
L_1	20%	44%
R_1	5%	31%

2x2 choices

In our notation, the lotteries are

$$L_1 = 0.2 \cdot [\$4000] \oplus 0.8 \cdot [\$0] \quad \text{and} \quad R_1 = 0.25 \cdot [\$3000] \oplus 0.75 \cdot [\$0]$$
$$L_2 = 0.8 \cdot [\$4000] \oplus 0.2 \cdot [\$0] \quad \text{and} \quad R_2 = [\$3000].$$

Note that $L_1 = 0.25 \cdot L_2 \oplus 0.75 \cdot [0]$ and $R_1 = 0.25 \cdot R_2 \oplus 0.75 \cdot [0]$. Thus if a preference relation on $L(Z)$ satisfies the independence property, it should rank L_1 relative to R_1 in the same way that it ranks L_2 relative to R_2. So among individuals who have a strict preference between the lotteries, only those whose answers are (*i*) L_1 and L_2 or (*ii*) R_1 and R_2 have preferences that can be represented by expected utility. About 51% of the participants are in this category.

Of the rest, very few (5%) choose R_1 and L_2. The most popular pair of answers is L_1 and R_2, chosen by 44% of the participants. Nothing is wrong with those subjects (which include the authors of this book). But such a pair of choices conflicts with expected utility theory; the conflict is known as the Allais paradox.

One explanation for choosing R_2 over L_2 is that the chance of getting an extra $1,000 is not worth the risk of losing the certainty of getting $3,000. The idea involves risk aversion, which we discuss in the next section.

Many of us use a different consideration when we compare L_1 and R_1. There, we face a dilemma: increasing the probability of winning versus a significant loss in the prize. The probabilities 0.25 and 0.2 seem similar whereas the prizes $4,000 and $3,000 are not. Therefore, we ignore the difference in the probabilities and focus on the difference in the prizes, a consideration that pushes us to choose L_1.

Experimentalists usually present the two questions to different groups of people, randomly assigning each participant to one of the questions. They do so to avoid participants guessing the object of the experiment, in which case a participant's answer to the second question might be affected by her answer to the first one. However, even when the two questions are given to the same people, we get similar results.

Findings like the ones we have described have led to many suggestions for alternative forms of preferences over the set of lotteries. In experiments, the behavior of many people is inconsistent with any of these alternatives; each theory seems at best to fit some people's behavior in some contexts.

3.5 Risk aversion

We close the chapter by considering attitudes to risk. We assume that the set Z of prizes is the set of nonnegative real numbers, and think of the prize z as or integral

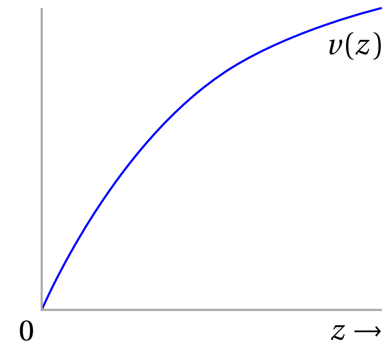


Figure 3.2 A concave Bernoulli function.

the monetary reward of \$$z$. We denote the expected value of any lottery p by $E(p) = \sum_{z \in \text{supp}(Z)} p(z)z$.

An individual is risk-neutral if she cares only about the expectation of a lottery, so that her preferences over lotteries are represented by $E(p)$. Such preferences are consistent with expected utility—take $v(z) = z$. An individual is risk-averse if for every lottery p she finds the prize equal to the expectation of p at least as good as p. That is, an individual with preference relation $\succsim$ is risk-averse if $[E(p)] \succsim p$ for every p. If for every lottery p that involves more than one prize, the individual strictly prefers $[E(p)]$ to p, she is strictly risk-averse.

Definition 3.6: Risk aversion and risk neutrality

If $Z = \mathbb{R}_+$, a preference relation $\succsim$ on the set $L(Z)$ of lotteries over Z is *risk-averse* if $[E(p)] \succsim p$ for every lottery $p \in L(Z)$, is *strictly risk-averse* if $[E(p)] \succ p$ for every lottery $p \in L(Z)$ that involves more than one prize, and is *risk-neutral* if $[E(p)] \sim p$ for every lottery $p \in L(Z)$, where $E(p) = \sum_{z \in Z} p(z)z$.

A strictly risk-averse individual is willing to pay a positive amount of money to replace a lottery with its expected value, so that the fact that an individual buys insurance (which typically reduces but does not eliminate risk) suggests that her preferences are strictly risk-averse. On the other hand, the fact that an individual gambles, paying money to replace a certain amount of money with a lottery with a lower expected value, suggests that her preferences are not risk-averse.

The property of risk aversion applies to any preference relation, whether or not it is consistent with expected utility. We now show that if an individual's preference relation is consistent with expected utility, it is risk-averse if and only if it has a representation for which the Bernoulli function is concave. (Refer to Figure 3.2.)

Proposition 3.3: Risk aversion and concavity of Bernoulli function

Let $Z = \mathbb{R}_+$, assume $\succsim$ is a preference relation over $L(Z)$ that is consistent with expected utility, and let v be the Bernoulli function for the representation. Then $\succsim$ is risk-averse if and only if v is concave.

Proof

Let x and y be any prizes and let $\alpha \in [0,1]$. If $\succsim$ is risk-averse then $[\alpha x + (1-\alpha)y] \succsim \alpha \cdot x \oplus (1-\alpha) \cdot y$, so that $v(\alpha x + (1-\alpha)y) \geq \alpha v(x) + (1-\alpha)v(y)$. That is, v is concave.

Now assume that v is concave. Then Jensen's inequality implies that $v\left(\sum_{z\in Z} p(z)z\right) \geq \sum_{z\in Z} p(z)v(z)$, so that $\left[\sum_{z\in Z} p(z)z\right] \succsim p$. Thus the individual is risk-averse.

x & y can be bundled

?

Problems

1. *Most likely prize.* An individual evaluates a lottery by the probability that the most likely prize is realized (independently of the identity of the prize). That is, for any lotteries p and q we have $p \succsim q$ if $\max_z p(z) \geq \max_z q(z)$. Such a preference relation is reasonable in a situation where the individual is indifferent between all prizes (e.g., the prizes are similar vacation destinations) and she can prepare herself for only one of the options (in contrast to Example 3.3, where she wants to prepare herself for all options and prefers a lottery with a smaller support).

   Show that if Z contains at least three elements, this preference relation is continuous but does not satisfy independence.

2. *A parent.* A parent has two children, A and B. The parent has in hand only one gift. She is indifferent between giving the gift to either child but prefers to toss a fair coin to determine which child obtains the gift over giving it to either of the children.

   Explain why the parent's preferences are not consistent with expected utility.

3. *Comparing the most likely prize.* An individual has in mind a preference relation $\succsim^*$ over the set of prizes. Whenever each of two lotteries has a single most likely prize she compares the lotteries by comparing the most likely prizes using $\succsim^*$. Assume Z contains at least three prizes. Does such a preference relation satisfy continuity or independence?

we have $b \succsim c$. It follows from the transitivity of the preference relation that $b \succ a$, contradicting $a \succsim b$.

Propositions 2.1 and 2.2 show that for a general choice problem, a choice function is rationalizable if and only it it satisfies property α. Notice that by contrast, Proposition 5.3 provides only a necessary condition for a demand function to be rationalized by a monotone preference relation, not a sufficient condition. We do not discuss a sufficient condition, called the strong axiom of revealed preference.

5.6 Properties of demand functions

A demand function describes the bundle chosen by a consumer as a function of the prices and the consumer's wealth. If we fix the price of good 2 and the consumer's wealth, the demand function describes how the bundle chosen by the consumer varies with the price of good 1. This relation between the price of good 1 and its demand is called the consumer's regular, or Marshallian, demand function for good 1 (given the price of good 2 and wealth). The relation between the price of good 2 and the demand for good 1 (given a price of good 1 and the level of wealth), is called the consumer's cross-demand function for good 2. And the relation between the consumer's wealth and her demand to good i (given the prices) is called the consumer's Engel function for good i.

Definition 5.8: Regular, cross-demand, and Engel functions

Let x be the demand function of a consumer.

- For any given price p_2^0 of good 2 and wealth w^0, the function x_1^* defined by $x_1^*(p_1) = x_1((p_1, p_2^0), w^0)$ is the consumer's *regular (or Marshallian) demand function for good 1 given* (p_2^0, w^0), and the function $\hat{x}_2$ defined by $\hat{x}_2(p_1) = x_2((p_1, p_2^0), w^0)$ is the consumer's *cross-demand function for good 2 given* (p_2^0, w^0).

- For any given prices (p_1^0, p_2^0), the function $\overline{x}_k$ defined by $\overline{x}_k(w) = x_k((p_1^0, p_2^0), w)$ is the consumer's *Engel function for good k given* (p_1^0, p_2^0).

Consider, for example, a consumer who spends the fraction α of her budget on good 1 and the rest on good 2, so that $x((p_1, p_2), w) = (\alpha w/p_1, (1-\alpha)w/p_2)$. The consumer's regular demand function for good 1 given p_2^0 and w^0 is given by $x_1^*(p_1) = \alpha w^0/p_1$ (and in particular does not depend on p_2^0), her cross-demand function for good 2 given p_2^0 and w^0 is the constant function $\hat{x}_2(p_1) =$

Output maximization

The producer prefers (a,y,π) to (a',y',π') if (*i*) $y > y'$ and $\pi \geq 0$, or (*ii*) $\pi \geq 0$ and $\pi' < 0$. Such a producer chooses (a,y) to maximize output subject to profit being nonnegative. The producer's preference for profit to be nonnegative may be due to the difficulty of surviving when she makes a loss.

P

The std.

Profit maximization

The producer prefers (a,y,π) to (a',y',π') if and only if $\pi > \pi'$. Such a producer chooses (a,y) to maximize profit.

Profit maximization with lower bar on output

For some given output $\overline{y}$, the producer prefers (a,y,π) to (a',y',π') if (*i*) $\pi > \pi'$ and $y \geq \overline{y}$, or (*ii*) $y \geq \overline{y}$ and $y' < \overline{y}$. Such a producer chooses (a,y) to maximize profit subject to producing at least $\overline{y}$. If $\overline{y}$ is small, the constraint does not bind. But if it is large, it constrains the amount of the input to at least the number $\overline{a}$ for which $f(\overline{a}) = \overline{y}$.

A cooperative

Assume that the producer is an organization that decides the number of its members and divides its profit equally among them. It employs only its own members. Each member contributes one unit of labor, so that the amount a of the input is the number of members of the cooperative. The cooperative aims to maximize the profit per member, π/a.

In this chapter we explore the implications of only the first two forms of preferences: output maximization and profit maximization.

6.2 Output maximization

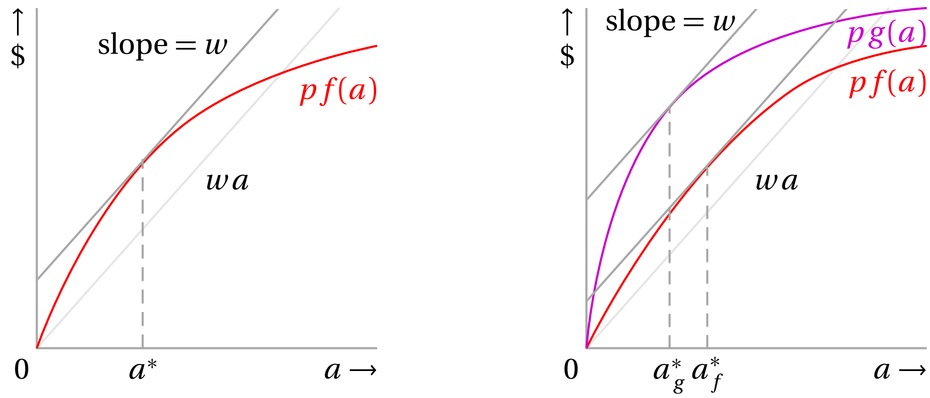
We start by considering a producer who aims to maximize the amount of output subject to not making a loss.

Definition 6.2: Output-maximizing producer

Given the prices p for output and w for input, an *output-maximizing producer with production function* f chooses the amount a of input to solve the problem

$$\max_a f(a) \text{ subject to } pf(a) - wa \geq 0.$$

Figure 6.1a illustrates such a producer's decision problem. The producer's profit is given by the difference between the red curve labelled $pf(a)$ and the line labelled wa. If, in addition to the assumptions we have made about the production function f, it is strictly concave, then either profit is negative for all



(a) The amount a^* of input chosen by a profit-maximizing producer.

(b) A possible effect of an improvement in technology for a profit-maximizing producer.

Figure 6.2 A profit-maximizing producer

6.3 Profit maximization

Producers are more commonly assumed to be profit-maximizers than output-maximizers.

Definition 6.3: Profit-maxmizing producer

Given the prices p for output and w for input, a *profit-maximizing producer with production function* f chooses the amount of input to solve the problem

$$\max_a pf(a) - wa.$$

Figure 6.2a illustrates such a producer's decision problem. If the production function is differentiable and strictly concave then a solution of the producer's problem is characterized as follows.

Proposition 6.3: Optimal input for profit-maximizing producer

If the production function f is differentiable and strictly concave then the amount of input chosen by a profit-maximizing producer with production function f facing the price w of input and the price p of output is

$$\begin{cases} 0 & \text{if } pf(a) - wa < 0 \text{ for all } a > 0 \\ a^* & \text{otherwise} \end{cases}$$

where a^* is the unique positive number for which $pf'(a^*) - w = 0$.

*P

Proof

The producer's profit when she chooses the amount a of input is $pf(a) - wa$. Given that f is strictly concave, this function is strictly concave in a. The result follows from the standard conditions for a maximizer of a differentiable function.

A change in the price of input or the price of output changes the amount of input chosen by a profit-maximizing producer in the same direction as it does for an output-maximizing producer.

Proposition 6.4: Comparative statics for profit-maximizing producer

An increase in the price of input or a decrease in the price of output causes the amount of input chosen by a profit-maximizing producer to decrease or remain the same.

Proof

Denote by $\alpha(w)$ the amount of input chosen by the producer when the input price is w. By definition,

$$pf(\alpha(w)) - w\alpha(w) \geq pf(a) - wa \text{ for all } a,$$

or

$$p[f(\alpha(w)) - f(a)] \geq w[\alpha(w) - a] \text{ for all } a.$$

In particular, for the prices w^1 and w^2 of the input,

$$p[f(\alpha(w^1)) - f(\alpha(w^2))] \geq w^1[\alpha(w^1) - \alpha(w^2)]$$

and

$$p[f(\alpha(w^2)) - f(\alpha(w^1))] \geq w^2[\alpha(w^2) - \alpha(w^1)].$$

Adding these inequalities yields

$$0 \geq (w^1 - w^2)(\alpha(w^1) - \alpha(w^2)).$$

Thus if $w^1 < w^2$ then $\alpha(w^1) \geq \alpha(w^2)$. A similar argument applies to changes in the price p of output.

Note that this proof does not use any property of the production function, so that the result in particular does not depend on the concavity of this function.

We refrain from expressing our opinion on the issue. We suggest only that you consider it taking into account how students of economics and other disciplines have responded to the question at the beginning of the chapter.

Did you decide to maximize profit and lay off 96 workers? Or did you decide to give up all profit and not lay off any worker? Or did you compromise and choose to lay off only 26 or 52 workers? Surely you did not lay off more than 96 workers, since doing so is worse than laying off 96 in terms of both the number of layoffs and profit.

When the question is posed to students in various disciplines, students of economics tend to lay off more workers than students in philosophy, law, mathematics, and business. It is not clear whether this effect is due to selection bias (students who choose to study economics are different from students in the other disciplines) or to indoctrination (studying material in which profit-maximization is assumed has an effect). In any case, even among students of economics, only about half choose the profit-maximizing option. So maybe profit-maximization is not the only goal of producers that we should investigate?

Problems

1. *Comparative statics.* Propositions 6.2 and 6.4 give comparative static results for a producer with a concave production function. Consider analogous results for a producer with a convex cost function.

   For an output maximizer and a profit maximizer, analyze diagrammatically the effect of (*a*) an increase in the price of output and (*b*) a technological change such that all marginal costs decrease.

2. *Two factories.* A producer can use two factories to produce output. The production functions for the factories are $f(a_1) = \sqrt{a_1}$ and $g(a_2) = \sqrt{a_2}$, where a_i is the amount of input used in factory i. The cost of a unit of input is 1 and the cost of activating a factory is $k > 0$. Calculate the producer's cost function.

3. *A producer with a cost of firing workers.* A producer uses one input, workers, to produce output according to a production function f. She has already hired a_0 workers. She can fire some or all of them, or hire more workers. The wage of a worker is w and the price of output is p. Compare the producer's behavior if she maximizes profit to her behavior if she also takes into account that firing workers causes her to feel as if she bears the cost $l > 0$ per fired worker.

7 Monopoly

7.1 Basic model

In the previous chapter we consider a producer who acts as if her behavior has no effect on the prices of the input or output. We argue that this assumption may be appropriate if the producer's quantities of inputs and output are small relative to the total volume of trade in the markets.

In this chapter we study several variants of a model that fits a very different situation, in which the producer of a single good is the only one serving a market. The variants differ in the type of options the producer can offer potential buyers. In the basic case, the producer can post a price per unit, and each buyer can purchase any amount of the good at that price. In other cases, the producer has other instruments like offering all consumers a set of price-quantity pairs. In each case, every potential buyer chooses the option she most prefers. The producer predicts correctly the buyers' responses and acts to advance her target (like maximizing profit or increasing production).

We allow for the possibility that the market has a number of segments, with distinct demand functions. Thus a specification of the market consists of two elements, (*i*) a demand function for each segment and (*ii*) a description of a producer, which includes her cost function and preferences.

Definition 7.1: Monopolistic market

A *monopolistic market* $\langle (q_i)_{i=1}^k, C, \succsim \rangle$ for a single good has the following components.

Demand

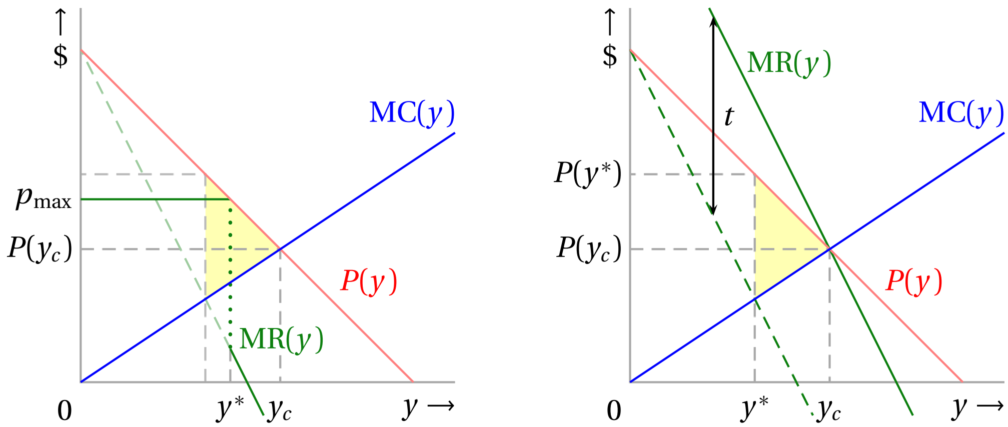
A collection $(q_i)_{i=1}^k$ of decreasing functions, where $q_i : \mathbb{R}_+ \to \mathbb{R}_+$. The function q_i, the *demand function in segment* i, associates with every price p_i for segment i the total amount $q_i(p_i)$ of the good demanded in that segment.

Producer

A single producer, called a *monopolist*, characterized by a cost function C that is continuous and convex and satisfies $C(0) = 0$, and a preference

Chapter of *Models in Microeconomic Theory* by Martin J. Osborne and Ariel Rubinstein. Version 2020.3.26 (s).

https://doi.org/10.11647/OBP.0211.07



(a) The output chosen by a monopolist who can set a price of at most $p_{\max}$.

(b) The output chosen by a monopolist who receives a subsidy of t per unit sold.

Figure 7.3 The effect of policies to change the output of a producer in a uniform-price monopolistic market.

7.2.2 Output-maximizing monopolist

As we discussed in the previous chapter, profit maximization is not the only possible target for a producer. Consider a monopolist who maximizes output subject to obtaining nonnegative profit. Such a monopolist produces the quantity y^* for which $\text{AC}(y^*) = P(y^*)$ (see Figure 7.4). This output is larger than the output y_c that maximizes the consumers' welfare $W(y)$.

7.3 Discriminatory monopoly $\not\propto$ P

We now consider a monopolistic market in which the producer can set different prices in different segments.

Definition 7.4: Discriminatory monopolistic market

A *discriminatory monopolistic market* is a monopolistic market in which the producer chooses a collection of prices, one for each segment of the market.

Note that the model assumes that the demand in each segment depends only on the price in that segment. In particular, this demand does not depend on the prices in other segments, so that we are assuming implicitly that consumers' demands are not affected by any feeling they may have that charging different prices to different groups is unfair.

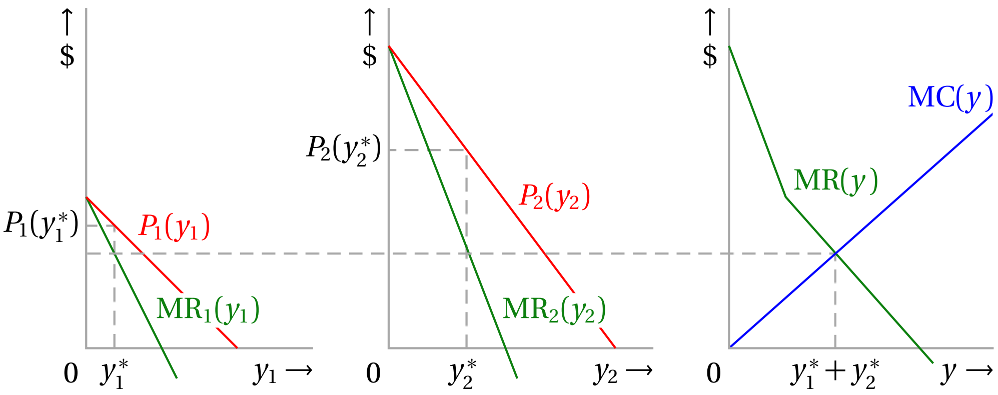


Figure 7.5 The outputs chosen by a profit-maximizing producer in each segment of a discriminatory monopolistic market. In segment i the output is y_i^* and the price is $P_i(y_i^*)$.

Two intuitions lie behind this result. First, the marginal revenues for all segments in which output is positive must be the same since otherwise the producer could increase her profit by moving some production from a segment with a low marginal revenue to one with a high marginal revenue. Second, if the marginal cost is higher than the common marginal revenue then the producer can increase her profit by reducing production, and if the marginal cost is smaller than the common marginal revenue she can increase her profit by increasing production.

The result is illustrated in Figure 7.5. The curve $\mathrm{MR}(y)$ is the horizontal sum of the MR_i curves. For any output y, $\mathrm{MR}(y)$ is the marginal revenue of the monopolist given that she allocates the output y optimally between the segments.

7.4 Implicit discrimination

In this section we assume that the monopolist is aware that the consumers have different demand functions, but cannot discriminate between them explicitly, either because she is prohibited from doing so or because she does not know who is who. We consider the possibility that she can offer an arbitrary set of pairs (q, m), where q is an amount of the good and m is the (total) price of purchasing q. She offers the same set to all consumers, each of whom is limited to choosing one member of the set or not buying the good at all.

Specifically, we consider a market for a good that can be consumed in any quantity between 0 and 1. Each consumer i is willing to pay up to $V^i(q)$ for the quantity q, where the function V^i is increasing and continuous, and $V^i(0) = 0$. A single producer (a monopolist) produces the good at no cost.

The monopolist offers a finite set of pairs (q, m), referred to as a menu. If consumer i chooses (q, m), then her utility is $V^i(q) - m$. Each consumer chooses

Condition I captures the idea that offers that are not accepted by any worker do not survive. Condition II requires that no offer that is accepted yields a loss to the employer who posts it. Condition III requires that no employer can post a new offer that is optimal for a worker and yields the employer a positive profit.

The notion of equilibrium differs from ones we analyze in earlier chapters in that it does not specify the choices made by specific participants. An equilibrium is a set of acceptable contracts in the market. The equilibrium is silent about who makes which offer.

We claim that the set $\{v\}$, consisting of the single wage offer v, is an equilibrium. Workers optimally choose it, as it is better than not accepting an offer; it yields zero profit to an employer; and any new offer is either not accepted (if it is less than v) or is accepted (if it is greater than v) but yields negative profit to the employer who posts it.

In fact, $\{v\}$ is the only equilibrium. Let W^* be an equilibrium. If $W^* = \varnothing$ then any offer w with $0 < w < v$ is optimal given $W^* \cup \{w\} = \{w\}$ and yields an employer who posts it a positive profit, violating III. By I, if $W^* \neq \varnothing$ then W^* consists of a single offer, say w^*. By II, $w^* \leq v$. If $w^* < v$, then an offer w with $w^* < w < v$ is optimal for a worker given $W^* \cup \{w\}$ and yields a positive profit, violating III.

14.2 Labor market with education

Imagine a labor market in which employers do not know, before hiring workers, how productive they will be, but do know their educational backgrounds. If education enhances productivity, we might expect employers to be willing to pay higher wages to more educated workers. We study a model in which education does *not* affect productivity, but productivity is negatively related to the cost of acquiring education: the more productive a worker, the less costly it is for her to acquire education. Under this assumption, we might expect employers to be willing to pay a high wage to a worker with a high level of eduction because they believe that acquiring such an education is worthwhile only for high productivity workers. The model we study investigates whether such a relation between wages and education exists in equilibrium.

In the model there are two types of worker, H and L. When employed, a type H worker creates output worth v_H and a type L worker creates output worth v_L, where $0 < v_L < v_H$. The proportion of workers of type L in the population is α_L and the proportion of workers of type H is α_H, with $\alpha_H + \alpha_L = 1$. No employer knows the type of any given worker before hiring her. Note $\gamma_H \neq \alpha_H$

If an employer offers a wage w and a worker who accepts the offer is of type H with probability γ_H and type L with probability γ_L then the employer obtains

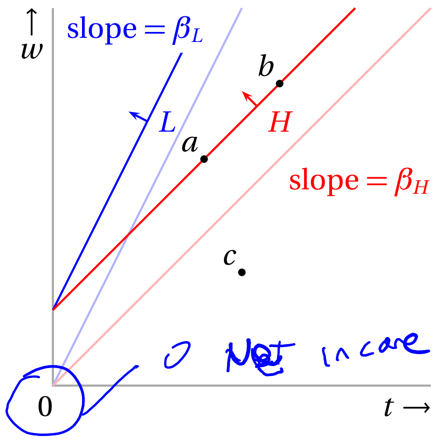


Figure 14.1 Iso-income lines for workers. Along each blue line the income of a worker of type L is constant, and along each red line the income of a worker of type H is constant. Income is higher along the darker lines. A type H worker prefers a to b because a requires less education and both contracts yield her the same income. If only c is offered, each type of worker prefers not to accept any offer, because c yields negative income.

$w - \beta_X t$. The preference relation $\succsim^X$ of a worker of type X over the set of contracts is lexicographic, giving first priority to larger income $w - \beta_X t$ and second priority to smaller values of the education requirement t.

For any set C of contracts the alternative that is *optimal given C for a worker of type X* ($= H, L$) is

Pref operator

$$\begin{cases} (t,w) \in C & \text{if } w - \beta_X t \geq 0 \text{ and } (t,w) \succsim^X (t',w') \text{ for all } (t',w') \in C \\ \phi & \text{if } w' - \beta_X t' < 0 \text{ for every } (t',w') \in C, \end{cases}$$

where ϕ means that the worker does not accept any contract.

Figure 14.1 shows iso-income lines for each type of worker. The blue lines belong to a type L worker; their slope is β_L. Each additional unit of education has to be compensated by an increase β_L in the wage to keep the income of such a worker the same. The red iso-income lines belong to a type H worker; their slope is β_H. Incomes for each type increase in a northwesterly direction: every worker prefers contracts with lower educational requirements and higher wages.

Given that the set of contracts offered is C, an employer who offers a contract $c = (t, w)$ expects a payoff that depends on the types of workers for whom c is optimal given C. If c is not optimal given C for any worker, the employer's payoff is zero; if c is optimal given C only for type X workers, her payoff is $v_X - w$, the profit from hiring a type X worker; and if c is optimal for all workers, her payoff is $\alpha_L v_L + \alpha_H v_H - w$.

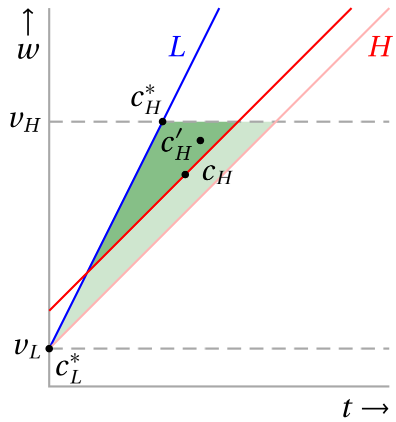


Figure 14.4 Step 5 of the proof of Proposition 14.1.

- Any contract $c = (t, w)$ that is optimal given $\{c^*_L, c^*_H, c\}$ for type L workers has $w > v_L$, so that if c is optimal only for type L workers it yields a negative profit.
- Any contract $c = (t, w)$ that is optimal given $\{c^*_L, c^*_H, c\}$ for both types of worker lies above the iso-income curve of a type H worker through c^*_H (the dark red line in Figure 14.5a), so that $w > m$; since $m > \overline{v}$ we have $w > \overline{v}$, so that c is not profitable.

If, on the other hand, the proportion of workers of type H is large enough that the average productivity in the population exceeds m, then an employer who adds the contract $(0, m)$ (or any other contract in the green triangle in Figure 14.5b) attracts workers of both types and obtains a positive profit. Thus in this case the set C^* of contracts is not an equilibrium.

Comment

The model is related to the "handicap principle" in biology. This principle provides an explanation for phenomena like the long horns of male deer. The male deer signals his unobserved fitness (biological value) by wasting resources on useless horns. The usefulness of the signal depends on the fact that spending resources on useless horns is less costly for fitter animals. In the economic story, a worker signals her unobserved quality by obtaining education, which has no effect on her productivity but is less costly for workers with high productivity.

Problems

1. *Quality certificate.* A market contains producers, each of whom can produce one unit of a good. The quality of the good produced by half of the producers