Applications of LP-Duality

Max-Flow-Min-Cut Theorem Revisit, von Neumann’s Minimax Theorem

Strong Duality Theorem

- Theorem [Strong Duality Theorem]. Let $\mathbf{x}^*$ be the optimal solution to (a) and $\mathbf{y}^*$ be the optimal solution to (b), then $\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & A^\top \mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{aligned} \qquad \text{(b)}$$

Primal feasible

Primal OPT = Dual OPT

Dual feasible

Part I: Max-Flow-Min-Cut Theorem Revisited

The Maximum Flow Problem

- The maximum flow problem can be formulated by a linear program.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{u:(s,u)\in E} f_{su} \\
\text{subject to} \quad & 0 \le f_{uv} \le c_{uv} && \forall (u,v) \in E \\
& \sum_{v:(v,u)\in E} f_{vu} = \sum_{w:(v,w)\in E} f_{uw} && \forall u \in V \setminus \{s,t\}
\end{aligned}
$$

Let's Write It in Standard Form

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{u:(s,u)\in E} f_{su} & \\
\text{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \\
& \displaystyle\sum_{v:(v,u)\in E} f_{vu} - \sum_{w:(u,w)\in E} f_{uw} \leq 0 & \forall u \in V \setminus \{s,t\} \\
& \displaystyle -\sum_{v:(v,u)\in E} f_{vu} + \sum_{w:(u,w)\in E} f_{uw} \leq 0 & \forall u \in V \setminus \{s,t\} \\
& f_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

We also make it easier

$$\text{maximize} \quad \sum_{u:(s,u)\in E} f_{su}$$

$$\text{subject to} \quad f_{uv} \le c_{uv} \qquad \forall (u,v) \in E \qquad \to \boldsymbol{y_{uv}}$$

$$\sum_{v:(v,u)\in E} f_{vu} - \sum_{w:(u,w)\in E} f_{uw} = 0 \qquad \forall u \in V \setminus \{s,t\} \qquad \to \boldsymbol{z_u}$$

$$f_{uv} \ge 0 \qquad \forall (u,v) \in E$$

Compute Its Dual Program

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

- We aim to show the LP above describes the min-cut problem.
- Let $\mathrm{OPT}_{\mathrm{dual}}$ be its optimal objective value. We need to show $\mathrm{OPT}_{\mathrm{dual}}$ is the size of the min-cut.

Some Intuitions

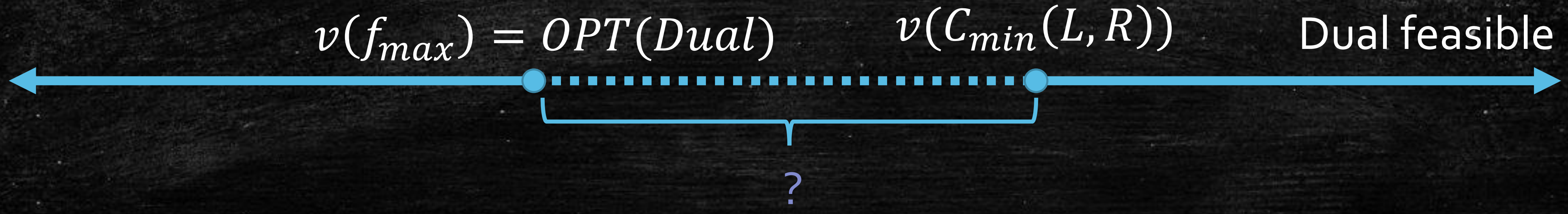
$$
\begin{aligned}
\text{minimize} \quad & \sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} \quad & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E
\end{aligned}
$$

- y_{uv} describes if edge (u,v) is cut:
$$y_{uv} = \begin{cases} 1 & \text{if } (u,v) \text{ is cut} \\ 0 & \text{otherwise} \end{cases}$$
- z_u describes u's "side":
$$z_u = \begin{cases} 1 & \text{if } u \text{ is on the } s-\text{side} \\ 0 & \text{if } u \text{ is on the } t-\text{side} \end{cases}$$

Do you have any ideas?

Strong Duality

- A straight-forward idea
- Max Flow: $v(f_{max}) = OPT(Primal) = OPT(Dual)$
- Min Cut: $v\left(C_{min}\ (L, R)\right) = OPT(Dual)$?
- Is that straight-forward?
- Min Cut: need integer solution!
- $OPT(Dual) \geq OPT(Cut\ IP) \geq v(C_{min}(L, R))$



Strong LP-Duality $\Longrightarrow$ Max-Flow-Min-Cut

Use Strong Duality Theorem to prove max-flow-min-cut theorem:

- Step 1: Write down the LP for max-flow problem.
- Step 2: Show that the dual program describes the fractional version of the min-cut problem.
- Step 3: Show that the dual program always have integral optimum.
  - So that the dual optimum is indeed the size of min-cut.
- Step 4: apply Strong Duality Theorem to show max-flow = min-cut

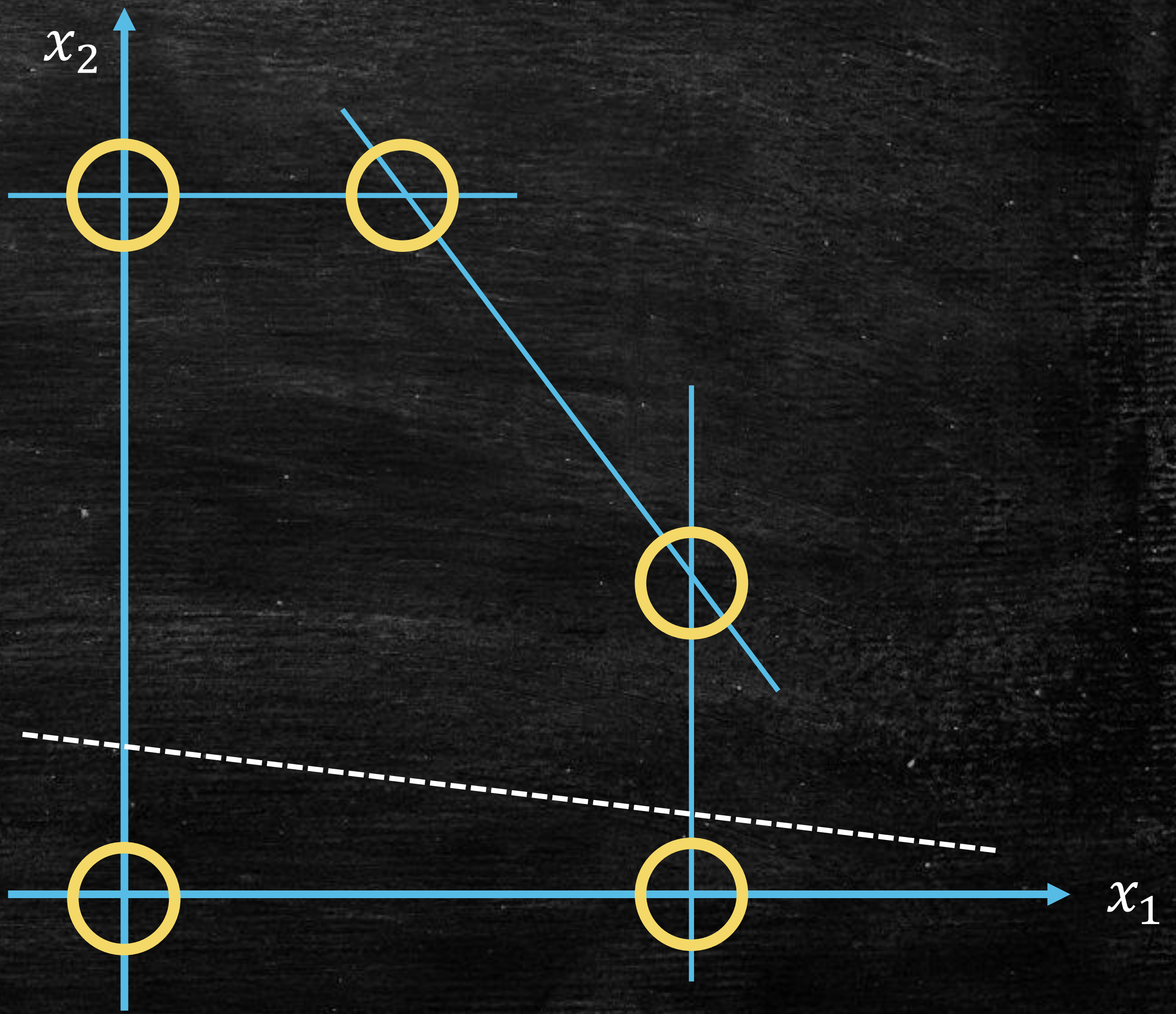
A General Question

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- When the LP has optimal **integral solution**?
- Thinking about the simplex method?

A simple observation

- If all vertices are integral,
- Then the optimal point must be integral.
- Next question: when will it be true?
- They are some solutions of



Totally Unimodular Matrix

- **Definition.** A matrix A is totally unimodular if every square submatrix has determinant 0, 1 or -1.
- **Theorem.** If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then the polytope $P = \{\mathbf{x}: A\mathbf{x} \leq \mathbf{b}\}$ has integer vertices.
- A Proof Sketch.
  - If $\mathbf{v} \in \mathbb{R}^n$ is a vertex of P. Then there exists an invertible square submatrix A' of A such that $A'\mathbf{v} = \mathbf{b}'$ for some sub-vector $\mathbf{b}'$ of $\mathbf{b}$.
  - By Cramer's Rule, we have $v_i = \frac{\det(A'_i|\mathbf{b}')}{\det(A'_i)}$, where $(A'_i|\mathbf{b}')$ is the matrix with i-th column replaced by $\mathbf{b}'$.
  - $\det(A'_i) = \pm 1$ and $\det(A'_i|\mathbf{b}') \in \mathbb{Z}$. Thus, $\mathbf{v}$ is integral.

Corollary on Integrality of LP

- **Theorem.** If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then the polytope $P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ has integer vertices.
- Since there always exists optimum at a vertex of the feasible region of LP, we have the following corollary.
- **Corollary.** If A is unimodular, then the optimal solution to LP (a) is integral when $\mathbf{b}$ is integral, and the optimal solution to LP (b) is integral when $\mathbf{c}$ is integral.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & A^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \qquad \text{(b)}$$

Proving Integrality of y_{uv}, z_u

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
 & y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
 & y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
 & y_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

- Now, we show that the matrix describing the first three rows of the constraints is totally unimodular.

Small Trick

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E \\
& z_u \geq 0 & \forall u \in V - \{s,t\}
\end{array}
$$

- Add the constraint does not affect the solution.

Proving Integrality of y_{uv}, z_u

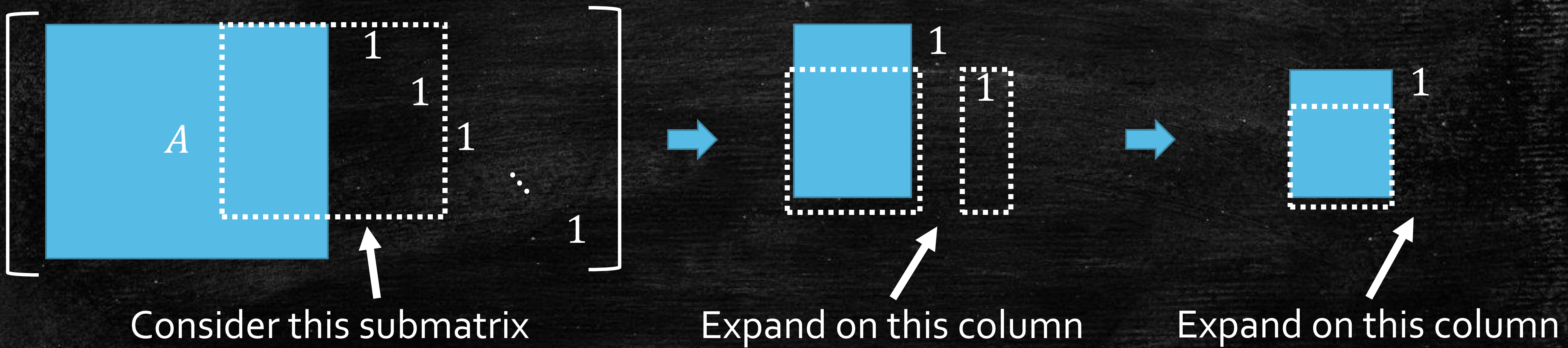
- The matrix can be written below:

$$
\begin{array}{cccc}
 & \overbrace{\hphantom{|E|\times|E| \text{ identity matrix}}}^{|E|} & \overbrace{\begin{array}{cc} \quad u\quad & \quad v\quad \end{array}}^{|V|-2} & \\
|E| \left\{ \vphantom{\begin{array}{c}1\\1\\1\\1\end{array}} \right. & \left[\begin{array}{c} \\ |E|\times|E| \\ \text{identity} \\ \text{matrix} \\ \\ \end{array} \right. & \left. \begin{array}{cc} 1 & \\ -1 & 1 \\ & \\ & -1 \end{array} \right] & \begin{array}{c} (s,u) \\ (u,v) \\ \\ (v,t) \end{array} \\
 & Y & Z &
\end{array}
$$

- Let the matrix be $[Y \;\; Z]$. Y is the identity matrix. We only need to show Z is totally unimodular.

Some Simple Observations

- If A is totally unimodular, then so are $A^\top, [I \;\; A], [A \;\; I], \begin{bmatrix} I \\ A \end{bmatrix}$, and $\begin{bmatrix} A \\ I \end{bmatrix}$. If any of $A^\top, [I \;\; A], [A \;\; I], \begin{bmatrix} I \\ A \end{bmatrix}$, and $\begin{bmatrix} A \\ I \end{bmatrix}$ is totally unimodular, then so is A.
- Proof. Just expand the determinant and you will see it…
- The determinant of $[A \quad I]$ equals to ± 1 times the determinant of some square submatrix of A.



Proving Integrality of y_{uv}, z_u

- The matrix can be written below:

$$
\begin{array}{ccc}
 & \overbrace{\begin{array}{cc} \quad u \quad & \quad v \quad \end{array}}^{|V|} & \\
|E| \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \left[\begin{array}{cc} 1 & \\ -1 & 1 \\ & \\ & -1 \end{array} \right] & \begin{array}{c} (s,u) \\ (u,v) \\ \\ (v,t) \end{array} \\
 & Z &
\end{array}
$$

- Let the matrix be $[Y \;\; Z]$. Y is the identity matrix. We only need to show Z is totally unimodular.

Proving Z is totally unimodular by Induction...

- Base Step: Each cell of Z belongs to $\{0, 1, -1\}$.
- Inductive Step: Suppose every $k \times k$ submatrix of Z has determinant belongs to $\{0, 1, -1\}$. Consider any $(k+1) \times (k+1)$ submatrix Z'.
- Case 1: If a row of Z' is all-zero, then $\det(Z') = 0$.
- Case 2: If a row of Z' contains only one non-zero entry, then $\det(Z')$ equals to ± 1 times the determinant of a $k \times k$ submatrix. $\det(Z') \in \{0, 1, -1\}$ by induction hypothesis.
- Case 3: If every row of Z' has two non-zero entries (one of them is -1 and the other is 1), then $\det(Z') = 0$:
  - Adding all the column vectors, we get a zero vector.

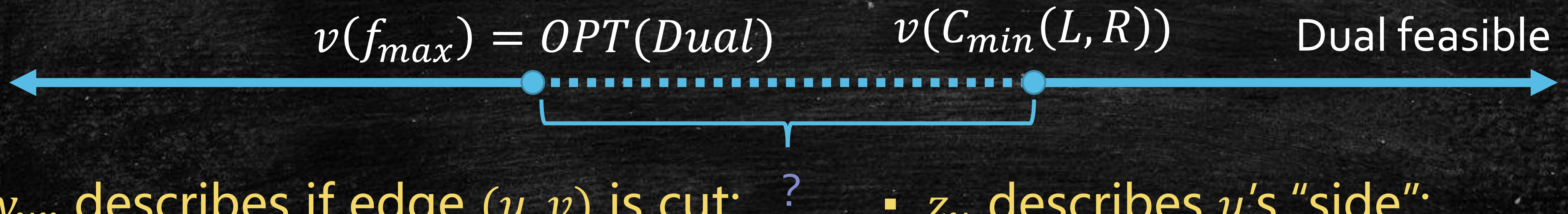
Proving Integrality of y_{uv}, z_u

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} \quad & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E
\end{aligned}
$$

- Now, we conclude that there exists an optimal solution with $y_{uv}, z_u \in \mathbb{Z}$.

Collect what we have.

- $OPT(Dual) \geq OPT(Primal) = v(f_{max})$
- $OPT(Dual)$ can be achieved by integral y^* and z^*.
- Are we done?
- We need to show $OPT(Dual)$ can be achieved by a cut!



- y_{uv} describes if edge (u,v) is cut:

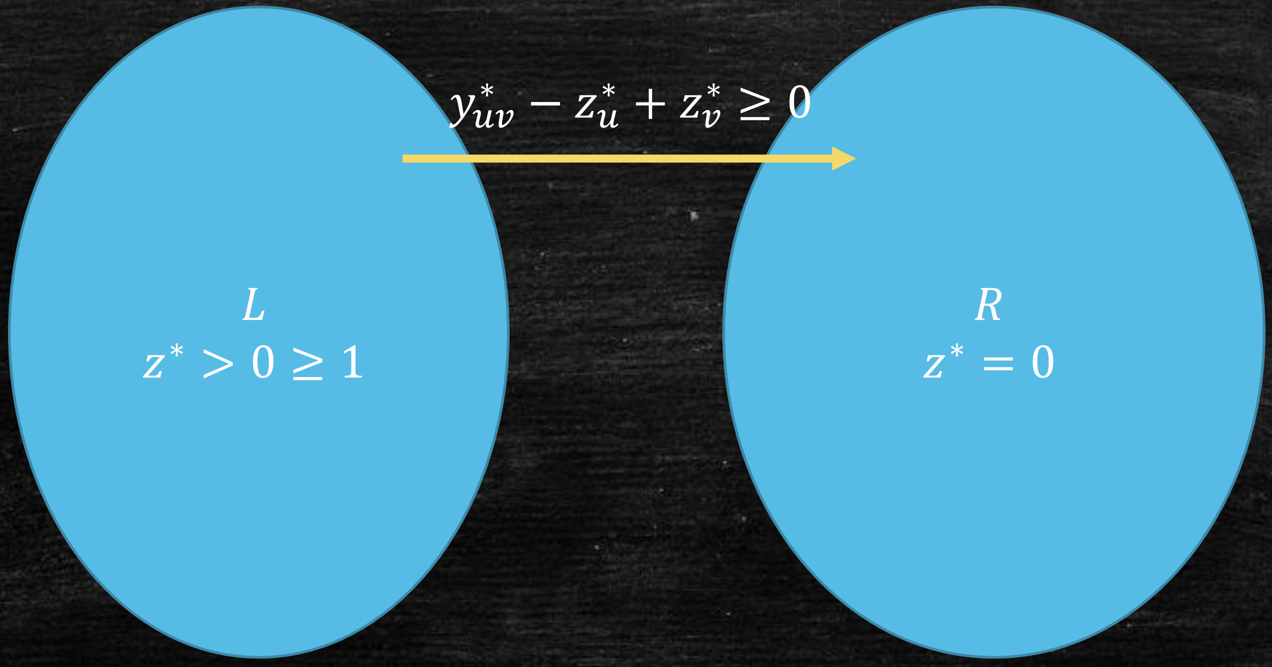
$$y_{uv} = \begin{cases} 1 & \text{if } (u,v) \text{ is cut} \\ 0 & \text{otherwise} \end{cases}$$

- z_u describes u's "side":

$$z_u = \begin{cases} 1 & \text{if } u \text{ is on the } s-\text{side} \\ 0 & \text{if } u \text{ is on the } t-\text{side} \end{cases}$$

Some Intuitions

- Let $L = \{v \mid z_v^* > 0\} + \{s\}$, and consider the cut $\{L, R = V - L\}$.
  - We make $s \in L$ and $t \notin R$.



$$y_{uv}^* \geq z_u^* - z_v^* \geq 1 - 0 \geq 1$$

We are done!

$$\text{minimize} \quad \sum_{(u,v)\in E} c_{uv} y_{uv}$$

- We find the cut $c(L,R)$.
  - $C(L,R) = \sum_{(u,v)\in out(L)} c(u,v) \le \sum_{(u,v)\in out(L)} y_{uv}^* \, c(u,v) \le OPT(Dual)$
- Also, because $OPT(Dual) \ge C(L,R)$
  - $C(L,R)$ is a choice of feasible dual.
- We have:
- $\text{Min}Cut = C(L,R) = OPT(Dual) = OPT(Primal) = MaxFlow$

A Framework for Proving Theorems Using Strong Duality

- Write down the primal and dual LPs.
- Justify that the primal and dual LPs describe the corresponding problems.
- If the problem described is discrete, prove that the corresponding LP always gives integral solution.
  - Total Unimodularity
- Apply strong duality theorem.

Revisiting Integrality Theorem for Max-Flow

- **Theorem.** If the capacities are all integers, then there exists an integral maximum flow.
- We have seen that "A" in the LP is totally unimodular
  - For dual program, we have proved $A^\top$ is totally unimodular.
- If all c_{uv} are integers, then vector "$\mathbf{b}$" in the LP is integral, and the LP has an integral optimal solution.

$$\text{maximize} \quad \sum_{u:(s,u)\in E} f_{su}$$

$$\text{subject to} \quad f_{uv} \le c_{uv}$$

$$\sum_{v:(v,u)\in E} f_{vu} - \sum_{w:(u,w)\in E} f_{uw} \le 0$$

$$-\sum_{v:(v,u)\in E} f_{vu} + \sum_{w:(u,w)\in E} f_{uw} \le 0$$

$$f_{uv} \ge 0$$

Exercise: Kőnig's Theorem

Maximum Bipartite Matching = Minimum Vertex Cover

Part II: von Neumann's Minimax Theorem

Zero-Sum Game

- Two players: A and B
- Each player has a set of actions that (s)he can play.
  - Set of actions A can play: $\mathbf{a} = \{a_1, a_2, \dots, a_m\}$
  - Set of actions B can play: $\mathbf{b} = \{b_1, b_2, \dots, b_n\}$
- For each pair of actions (a_i, b_j) that two players play, a utility is assigned to each player: $u_A(a_i, b_j), u_B(a_i, b_j)$.
- A game is a zero-sum game if $\forall x_i, y_j: u_A(a_i, b_j) + u_B(a_i, b_j) = 0$.
- Payoff Matrix $G \in \mathbb{R}^{m \times n}$, where $G_{i,j}$ is the utility gain for A, or the utility loss for B, when (a_i, b_j) is played.

Example

- The payoff matrix for the Rock-Scissors-Paper game:

		Player B		
		Rock	Scissors	Paper
Player A	Rock	0	1	-1
	Scissors	-1	0	1
	Paper	1	-1	0

Strategy

- Set of actions A can play: $\mathbf{a} = \{a_1, a_2, \dots, a_m\}$
- A strategy for A is a probability distribution of $\mathbf{x}$.
- A pure strategy specifies one of $a_1, a_2, \dots, a_m$ with probability 1.
  - In other words, a pure strategy is an action.
- Otherwise, it is a mixed strategy.
  - In other words, a mixed strategy specify at least two actions with non-zero probability.
- Fix A's strategy, the best response for B is the strategy that maximizes B's utility.

Rock-Scissors-Paper Example

- A plays $(R, S, P) = (1, 0, 0)$:
  - It is a pure strategy that always plays "rock".
  - The best response for B is $(0, 0, 1)$, with utility 1.
- A plays $(R, S, P) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$:
  - It is a mixed strategy.
  - The best response for B is $(0, 0, 1)$, with expected utility
$$\frac{1}{2} \times 1 + \frac{1}{4} \times -1 + \frac{1}{4} \times 0 = \frac{1}{4}.$$
- A plays $(R, S, P) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:
  - It is a mixed strategy.
  - Any strategy for B, pure or mixed, is a best response, with expected utility 0.

Expected Utility

- Let $\mathbf{x} = \{x_1, \dots, x_m\}$ and $\mathbf{y} = \{y_1, \dots, y_n\}$ be the strategies played by the two players.
- The expected utility for Player A is
$$U_A(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top G \mathbf{y} = \sum_{i,j} G_{i,j} x_i y_j$$
- The expected utility for Player B is
$$U_B(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^\top G \mathbf{y} = -\sum_{i,j} G_{i,j} x_i y_j$$

Does it matter who chooses strategy first?

Rock-Scissors-Paper: $G = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

- Suppose A chooses a strategy first.
  - Given that B will always play the best response
  - The optimal strategy for A is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
  - Expected utility for both players is 0
- Suppose B chooses a strategy first.
  - Similar analysis, expected utility for both players is 0
- Same outcome regardless who chooses strategy first.
- Does it always hold for any zero-sum game?
- Yes! This is von Neumann's Minimax Theorem.

Minimax Theorem

- Suppose A chooses strategy first. Knowing that B will play the best response, A will choose an optimal strategy $\mathbf{x}$ that maximizes his/her utility:

$$\underbrace{\max_{\mathbf{x}} \overbrace{\min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j}^{B \text{ plays the best response given } A\text{'s strategy } \mathbf{x}.}}_{\text{Given } B \text{ plays the best response, } A \text{ choose a strategy maximizing the utility.}}$$

- Suppose B chooses strategy first. Similarly, the utility for A is

$$\min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j$$

Minimax Theorem

- Minimax Theorem:

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j = \min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j$$

- Who chooses strategy first doesn't matter!

Pure Strategy Best Response

- **Lemma.** Fix A's strategy $\mathbf{x} = \{x_1, \ldots, x_m\}$, there exists a best response for B that is a pure strategy.
- Proof. Let $\mathbf{y} = \{y_1, \ldots, y_n\}$ be B's strategy.
- The utility for B is given by

$$-y_1 \sum_{i=1}^{m} G_{i,1} x_i - y_2 \sum_{i=1}^{m} G_{i,2} x_i - \cdots - y_n \sum_{i=1}^{m} G_{i,n} x_i$$

- Clearly, this is maximized if we set $y_i = 1$ where y_i has smallest coefficient.

LP formulation

- The lemma implies

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j = \max_{\mathbf{x}} \min_{j=1,\ldots,n} \sum_{i} G_{i,j} x_i$$

- Let z be the utility for Player A. The following LP formulates the max-min expression:

$$\begin{aligned}
\text{maximize} \quad & z \\
\text{subject to} \quad & \textstyle\sum_i G_{i,j} x_i \geq z \qquad \forall j = 1, \ldots, n \\
& x_1 + \cdots + x_m = 1 \\
& x_1, \ldots, x_m \geq 0
\end{aligned}$$

Its dual program is…

$$
\begin{aligned}
&\text{maximize} && z && \\
&\text{subject to} && z - \textstyle\sum_i G_{i,j} x_i \le 0 && \forall j = 1, \dots, n \rightarrow \boldsymbol{y_1 \sim y_n} \\
& && x_1 + \cdots + x_m = 1 && \rightarrow \boldsymbol{w} \\
& && x_1, \dots, x_m \ge 0 &&
\end{aligned}
$$

- Dual Objective: w
- Constraints
  - For z: we do not have $z \ge 0$ so, $z's$ coefficient should be exactly 1.
  - For x_i: The coefficient should at least 0.

Simplify it, we get...

$$\begin{aligned} \text{minimize} \quad & w \\ \text{subject to} \quad & \textstyle\sum_j G_{i,j} y_i \le w \qquad\qquad \forall i = 1, \ldots, m \\ & y_1 + \cdots + y_n = 1 \\ & y_1, \ldots, y_n \ge 0 \end{aligned}$$

- This is exactly

$$\min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j = \min_{\mathbf{y}} \max_{i=1,\ldots,m} \sum_{i,j} G_{i,j} y_j$$

- Strong duality theorem $\Rightarrow$ Minimax Theorem.