Episode

Sorting Lower Bound

Sorting Algorithms

- $\Theta(n^2)$
  - Selection Sort
  - Bubble Sort
  - Insertion Sort
- $\Theta(n \log n)$
  - Quick Sort
  - Merge Sort
  - Balance Tree
- Guide Question
  - **Can we do better?**

Spaghetti Sort

$O(n)$

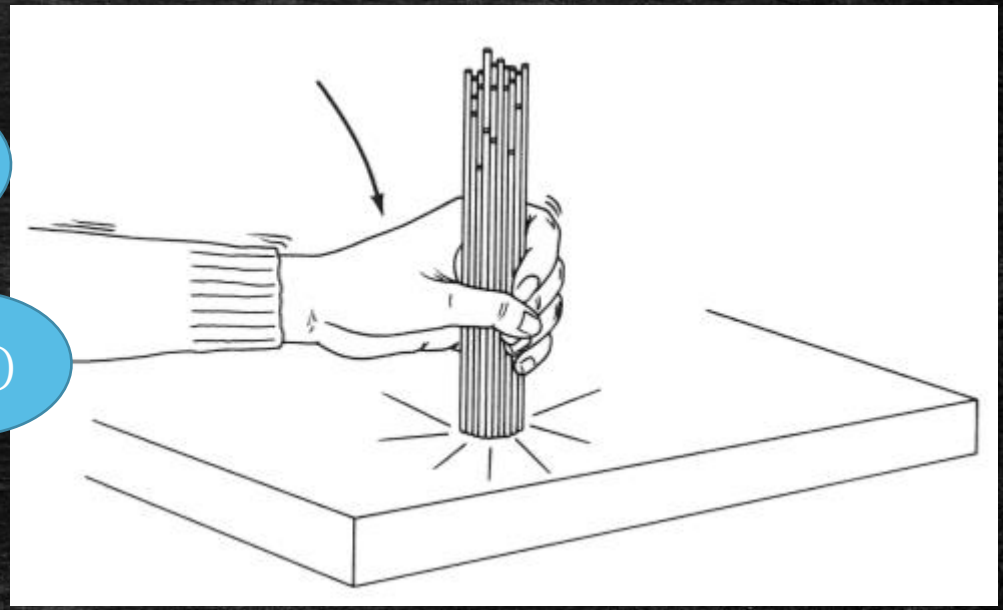
- For each number, **break** a piece of spaghetti whose length is that number.

$O(1)$

- Take all spaghetti and **push** them against the table!

$O(n)$

- Keep **touching** and **removing** spaghetti from the top by your other hand.



Is it what we want?

- Need to think
  - What can we do?
  - What can not we do?

Is it what we want?

- Need to think
  - ~~What can we do?~~
  - ~~What can not we do?~~
  - What can computer do?
  - What can not computer do?
- A proper Computation Model!
  - Allowed Operations: Comparison
  - Not allowed Operations: Break spaghetti, push spaghetti, touch spaghetti.

Comparison-based Sorting

- Only allowed operation: **comparison**.
- Can do
  - $Compare(a, b)$, answer $a > b$ or $b \geq a$.
- Can not do
  - Ask what a is, what b is.
- Examples
  - Merge Sort
  - Insertion Sort
  - Quick Sort
  - ……

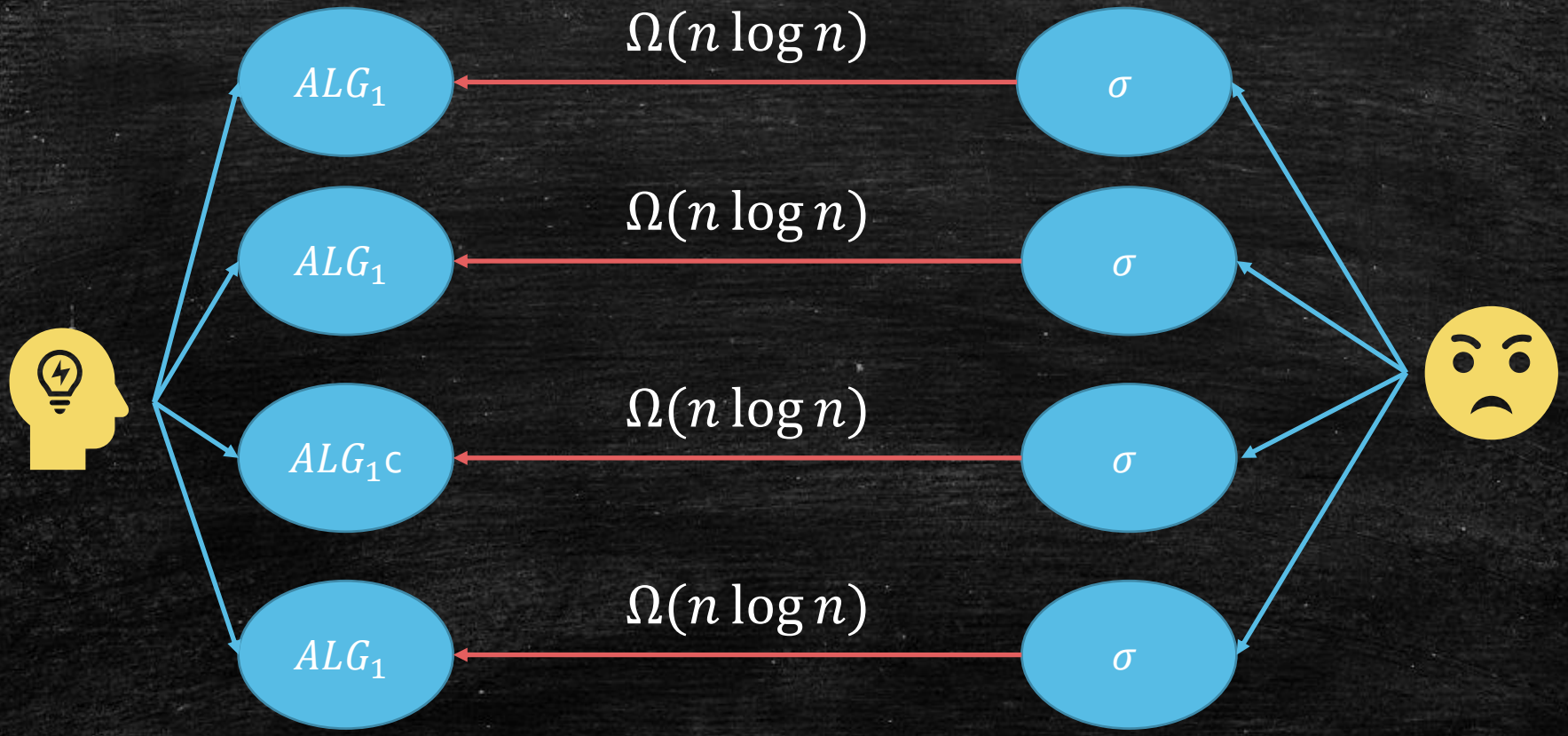
Recall Merge Sort

- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C

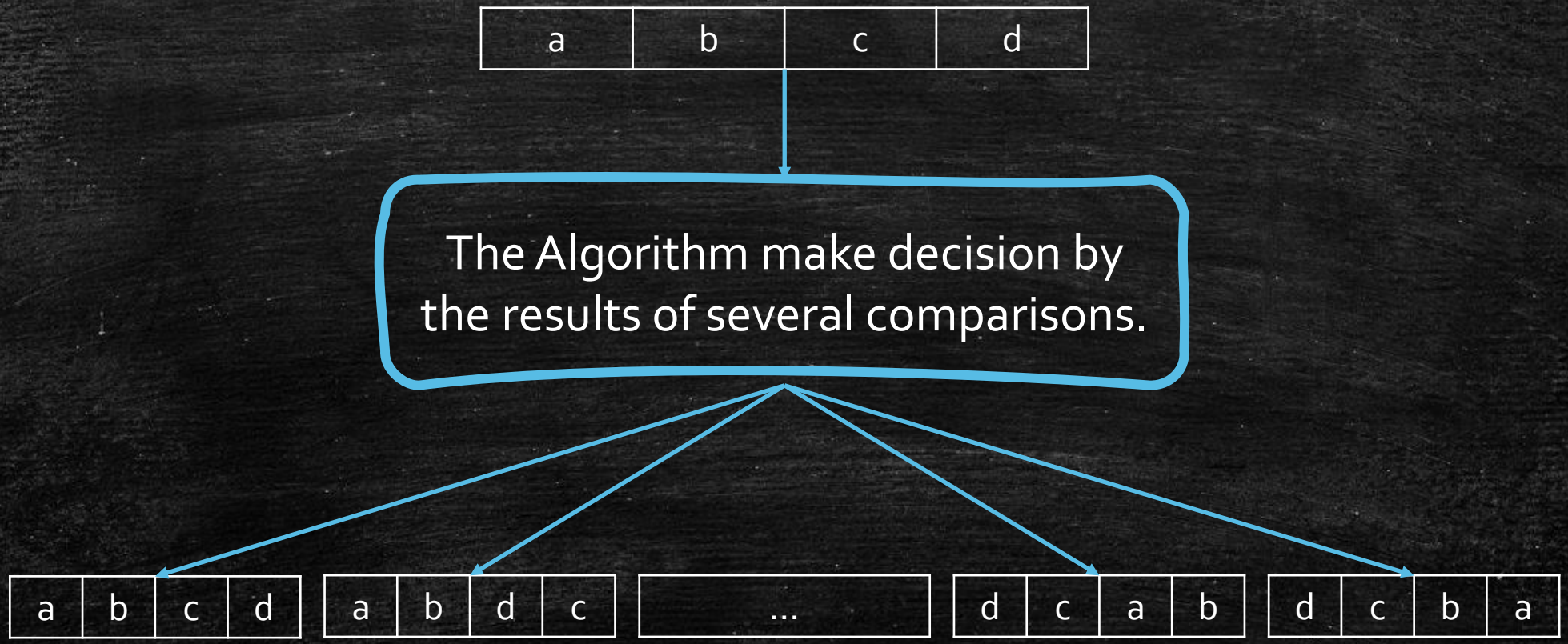
How to prove lower bounds?

- Some Algorithm A want to say: A's time complexity is something better than $O(n \log n)$.
  - Prove: For all input σ with size n, $T_A(\sigma) \leq T(n) = O(g(n))$.
  - $g(n) = o(n \log n)$, like $g(n) = n$.
- Disprove it!
- I want to say: For all Algorithm A, there is some input show that A can not do better than $O(n \log n)$!
  - Prove: Exists input σ with size n, $T_A(\sigma) \geq T(n) = \Omega(n \log n)$.
- Play as an **adversary**!

Play as An **Adversary**!

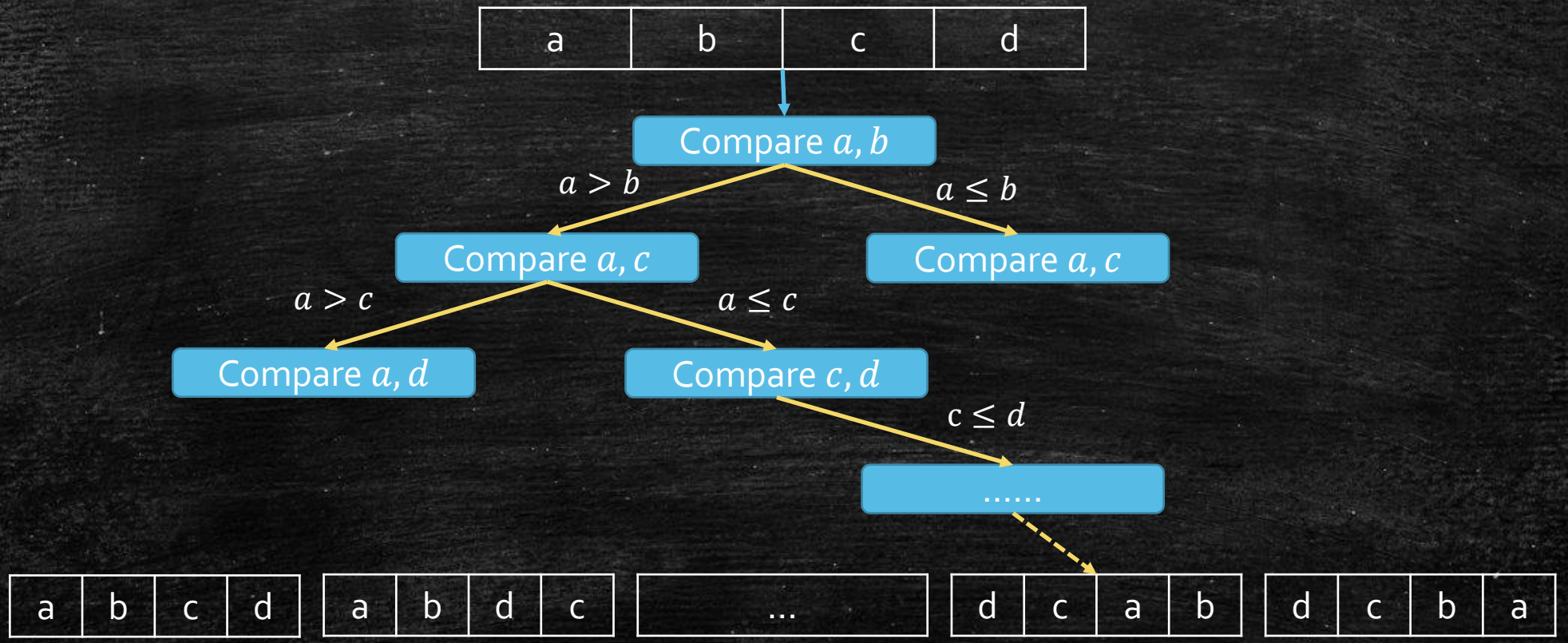


Comparison-based Sorting



- How many possible ordering?

Comparison-based Sorting



- How many possible ordering? --- $n!$

Comparison-based Sorting: Time Complexity

1. Comparison-based Sorting forms a binary tree.
2. It should have at least $n!$ leaves.

   Adversary: the **longest** path from root to leaves.
4. Best deterministic algorithm make the tree shallowest.
5. Shallowest tree is a balanced tree.
6. Height: $\log(n!)$

Last step!

- The lemma we have
  - Any deterministic comparison-based algorithms must take $\log(n!)$ steps to sort an array in the worst case.
- The theorem we want
  - Any deterministic algorithm comparison-based algorithms must take $\Omega(n \log n)$ time.

Proof

$$\log(n!) = \log 1 + \log 2 + \log 3 + \cdots + \log n$$
$$\geq \log\left(\frac{n}{2} + 1\right) + \log\left(\frac{n}{2} + 2\right) + \cdots + \log n$$
$$\geq \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}\log n - \frac{n}{2}\log 2$$
$$= \Omega(n \log n)$$

Good News!

Merge Sort is the Optimal Deterministic Comparison-based Algorithm

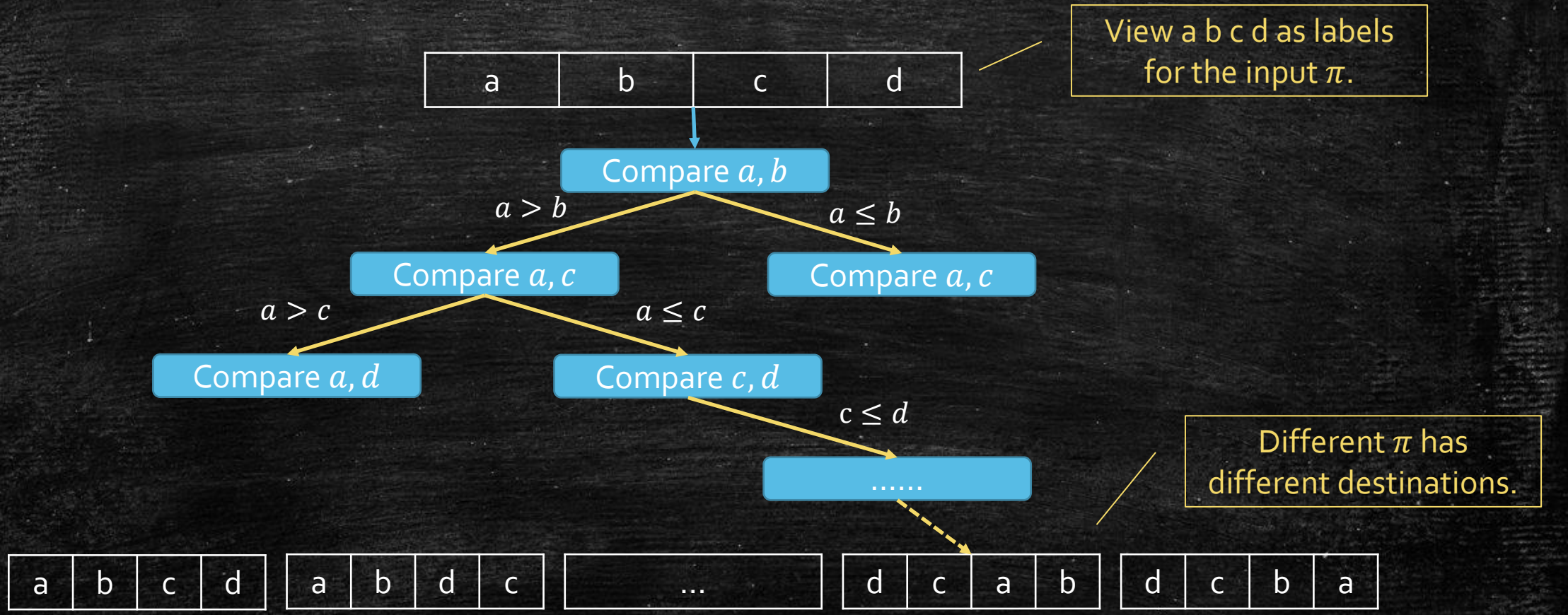
What about random algorithms?

Before that……

Average Case for Deterministic Algorithms

- **Input**
  - n integers.
- **Shuffle**
  - Uniformly random a permutation $\pi \in \Pi$.
- **Sort**
  - Run the deterministic algorithm on π.
- Average Comparisons
  - $E[T(n)] = \frac{1}{n!}\sum_{\pi \in \Pi}(T(\pi))$

Comparison-based Sorting

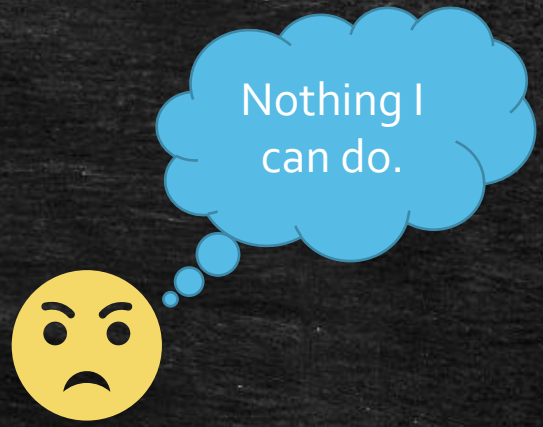


- How many possible ordering? --- $n!$

How to relate the Comparisons to the tree?

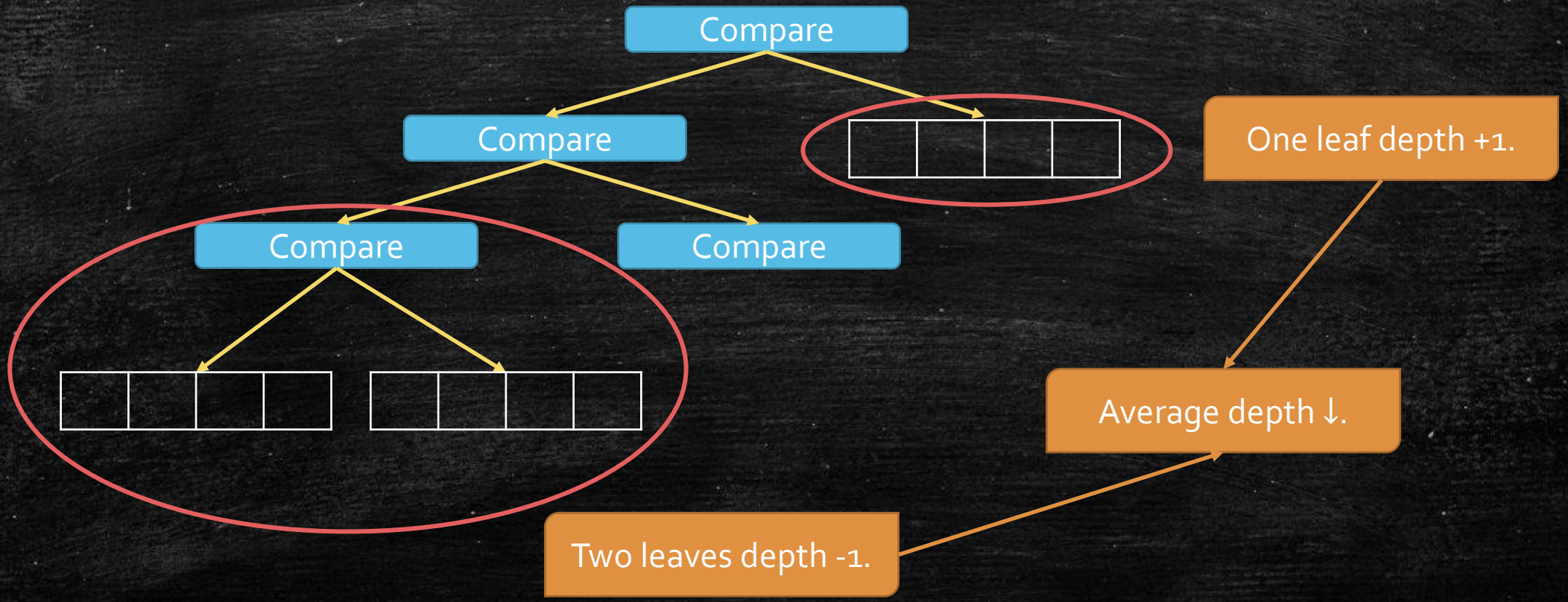
- Worst Case Comparisons
  - Longest path from root to leaves.
  - Balanced tree: The **best** we can do.

- Average Comparisons
  - Average length of paths from root to leaves.
  - Balanced tree: Still the **best**?



Balanced tree: Still the best?

- Unbalanced tree: What if we try to make it balanced?



Balanced tree is still the best!

Best Average Comparisons: $\log n! = \Omega(n \log n)$

The Input-ALG Game

- Two Players
  - One give Input Permutation.
  - One give ALG.

	ALG_1	ALG_2	ALG_3	ALG_4
Input = π_1	$\log n!$	$\log n!$	1	$\log n!$
Input = π_2	1	n^2	$\log n!$	$\log n!$
Input = π_3	n^2	$\log n!$	1	$\log n!$
Input = π_4	$\log n!$	1	$\log n!$	$\log n!$

Deterministic Lower Bound: Worst Case

- For Each ALG
  - A row has cost $\geq \log n!$

	ALG_1	ALG_2	ALG_3	ALG_4
Input = π_1	$\log n!$	$\log n!$	1	$\log n!$
Input = π_2	1	n^2	$\log n!$	$\log n!$
Input = π_3	n^2	$\log n!$	1	$\log n!$
Input = π_4	$\log n!$	1	$\log n!$	$\log n!$

Deterministic Lower Bound: Average Case

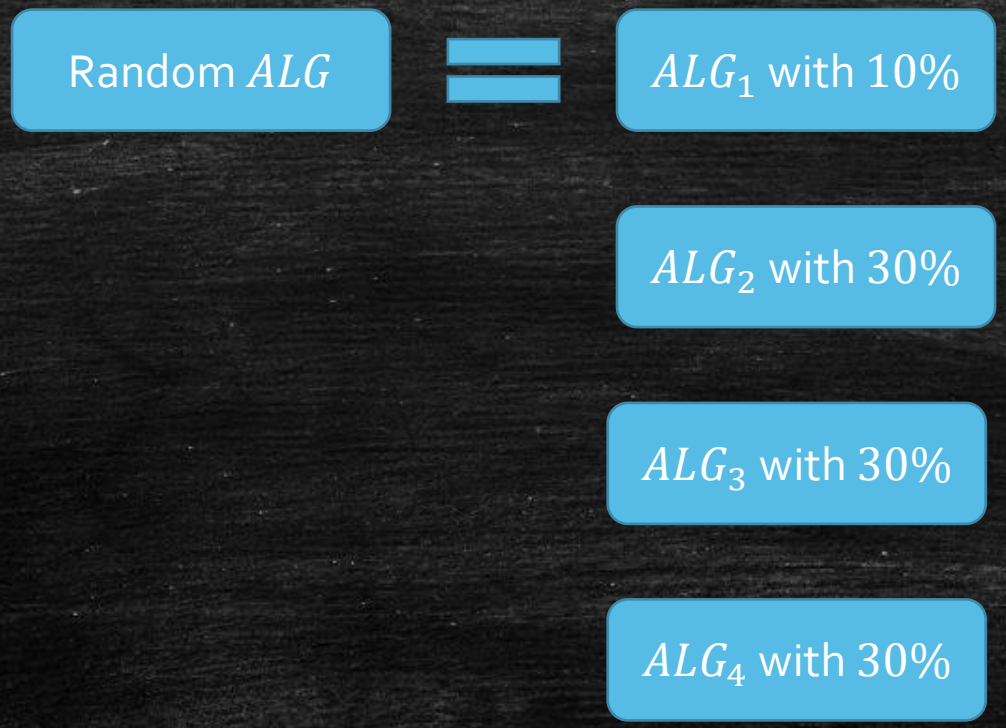
- For Each ALG
  - The row $(\frac{1}{4}\pi_1, \frac{1}{4}\pi_2, \frac{1}{4}\pi_3, \frac{1}{4}\pi_4)$ has cost $\geq \log n!$

	ALG_1	ALG_2	ALG_3	ALG_4
Input = π_1	$\log n!$	$\log n!$	1	$\log n!$
Input = π_2	1	n^2	$\log n!$	$\log n!$
Input = π_3	n^2	$\log n!$	1	$\log n!$
Input = π_4	$\log n!$	1	$\log n!$	$\log n!$
Input = $(\frac{1}{4}\pi_1, \frac{1}{4}\pi_2, \frac{1}{4}\pi_3, \frac{1}{4}\pi_4)$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$

What is a randomized algorithm?

- Randomized is a **distribution** of all deterministic algorithms.
  - Randomized Algorithm is a deterministic algorithms with a random bit tape.
  - Fix a sample "s" of the random bit tape → An original deterministic algorithm ALG_s.

What is a randomized algorithm?



Expected Running Time

- Randomized is a **distribution** of all deterministic algorithms.
  - Randomized Algorithm is a deterministic algorithms with a random bit tape.
  - Fix a sample "s" of the random bit tape → An original deterministic algorithm ALG_s.
- The expected running time of Randomized Algorithms
  - Input π
  - $E_s[T_{ALG_s}(\pi)]$
  - Consider the worst case!

Randomized Lower Bound: Average Case

- For the column $(p_1 ALG_1, p_2 ALG_2, p_3 ALG_3, p_4 ALG_4)$
  - The row $(\frac{1}{4}\pi_1, \frac{1}{4}\pi_2, \frac{1}{4}\pi_3, \frac{1}{4}\pi_4)$ has cost $\geq \log n!$

	ALG_1	ALG_2	ALG_3	ALG_4	$(p_1 A_1, p_2 A_2, p_3 A_3, p_4 A_4)$
Input = π_1	$\log n!$	$\log n!$	1	$\log n!$	
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Input = π_4	$\log n!$	1	$\log n!$	$\log n!$	
Input =$(\frac{1}{4}\pi_1, \frac{1}{4}\pi_2, \frac{1}{4}\pi_3, \frac{1}{4}\pi_4)$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$	

Randomized Lower Bound: Average Case

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Randomized Lower Bound: Average Case

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Input = π_3	n^2	$\log n!$	1	$\log n!$	$\geq \log n!$(the worst one)
Input = π_4	$\log n!$	1	$\log n!$	$\log n!$	
Input =$(\frac{1}{4}\pi_1, \frac{1}{4}\pi_2, \frac{1}{4}\pi_3, \frac{1}{4}\pi_4)$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$	$\geq \log n!$

Conclusion

- For any randomized algorithm, there is an input π, such that $E\big(T(\pi)\big) \geq \log n! = \Omega(n \log n)$

More Questions

- Do we have **linear time** sorting algorithms that is not comparison-based?
  - Yes! But with some restriction.