Mixture Stretch

1. __________ teaspoons Ming's recipe for sweet tea calls for 4 teaspoons of sugar. If Ming wants to make the tea 25% less sweet, how much less sugar should he use?

2. __________ Carla is mixing cherry, grape and lime candies in a bowl. Since her favorite flavor is cherry, she wants $\frac{2}{5}$ of the candies to be cherry. Since her least favorite flavor is lime, she wants $\frac{1}{4}$ of the candies to be lime. What fraction of the candies will be grape? Express your answer as a common fraction.

3. __________ A paving company makes concrete by adding water to a mix that is 1 part cement, 3 parts sand and 3 parts aggregate (stone). What fraction of this mix is aggregate? Express your answer as a common fraction.

4. __________ ounces A beef stew recipe calls for 12 ounces of beef, 4 ounces of carrots, 7 ounces of potatoes, 4 ounces of peas and 5 ounces of beef stock. Given that there are 16 ounces in a pound, how many ounces of potatoes are needed to make 4 pounds of this stew?

5. __________ % Jin adds 1 gallon of a water-and-bleach mixture that is 4% bleach to 2 gallons of a water-and-bleach mixture that is 10% bleach. What percent of the final mixture is bleach?

6. $ __________

Cashews cost $2.36 per pound, almonds cost $1.48 per pound and peanuts cost $0.98 per pound. To make a 20% profit, how much should Myrna charge per pound for a mixture that is 1 part cashews, 1 part almonds and 2 parts peanuts?

7. __________ gallons Manny's cleaning supply store receives a mixture of 80% detergent and 20% water in 15-gallon buckets. Manny would like a mixture of 60% detergent and 40% water in 5-gallon buckets. To make this, he combines some 80/20 mixture with some pure water in each 5-gallon bucket. How many gallons of pure water does Manny add to each 5-gallon bucket? Express your answer as a decimal to the nearest hundredth.

8. __________ buckets Based on the information in problem 7, how many 5-gallon buckets of 60/40 solution can Manny make from one 15-gallon bucket of 80/20 solution?

9. __________ quarts

Dara is mixing her own paint color, using 3 parts green paint to 2 parts blue to 1 part white. Given that there are 4 quarts in a gallon, if she needs 3 gallons of her paint, how many quarts of white paint should she buy?

10. __________ g/cm^3 To make a sand sculpture, Arthur used 2 cm^3 of red sand with a density of 4 g/cm^3, 7 cm^3 of yellow sand with a density of 5 g/cm^3 and 5 cm^3 of brown sand with a density of 6 g/cm^3. What is the average density of this sculpture in grams per cubic centimeter? Express your answer as a decimal to the nearest tenth.

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Statistics Stretch

11. ______ A class of 28 students had a mean score of 72 on a math test. After the teacher realized that one of the questions had an alternative correct answer, he gave 4 points each to the 7 students who had given the alternative answer. What is the new mean test score?

12. ______ pages

stem	leaf
1	27 52 82
2	25 57 63 82 97
3	37 51 68 75

1|27 = 127 pages

The stem-and-leaf plot shows the number of pages in each book that Kalem read last summer. How many pages did Kalem read last summer?

13. ______ Based on the information in problem 12, what portion of the pages that Kalem read were in books having more than 275 pages? Express your answer as a common fraction.

14. ______ % A cafeteria offers apples, oranges and bananas with lunch. A student may take at most one of each fruit. Of the 61 students who got fruit with lunch, 5 students got only an apple and 7 got only an orange. Of the 16 students who got an apple and an orange, the 17 who got an orange and a banana, and the 20 who got an apple and a banana, 6 got all three fruits. What portion of the fruit taken by the 61 students were bananas? Express your answer to the nearest whole percent.

15. ______

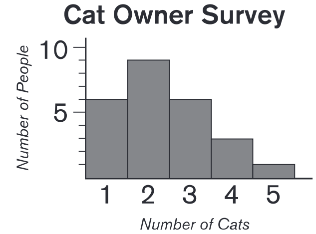
Chess Club Membership

	6th	7th	8th
Beginners	1	?	5
Advanced	2	?	6

The table shows the grade and skill level of members of the chess club. If half of the members are eighth graders and one-third of the beginners are seventh-graders, what fraction of chess club members are advanced chess players? Express your answer as a common fraction.

16. ______ If A, M and R represent the arithmetic mean, median and range of the set {13, 16, 18, 23, 25, 28, 30, 31}, what is the value of $A + M - R$?

Cat Owner Survey



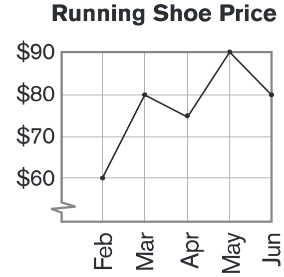
17. ______ cats This graph shows results of a survey of 25 cat owners. What is the mean number of cats per person surveyed? Express your answer as a decimal to the nearest hundredth.

18. ______ A total of 44 Mathletes competed in a MATHCOUNTS competition. The mean score for all the competitors was 28. The mean score for all competitors *except* the 16 highest scorers was 20. What was the mean score for the 16 highest scorers?

19. ______ % Bryce orders 6 bats, 60 baseballs and 8 gloves. If each bat costs \$29.95, a pack of 12 baseballs costs \$39.95 and a glove costs \$69.95, what portion of the total cost of this order is for the gloves? Express your answer to the nearest whole percent.

20. ______ % The graph shows the price for a popular running shoe over five months. What is the absolute difference of the percent change in price from February to April and the percent change in price from April to June? Express your answer to the nearest tenth of a percent.

Running Shoe Price



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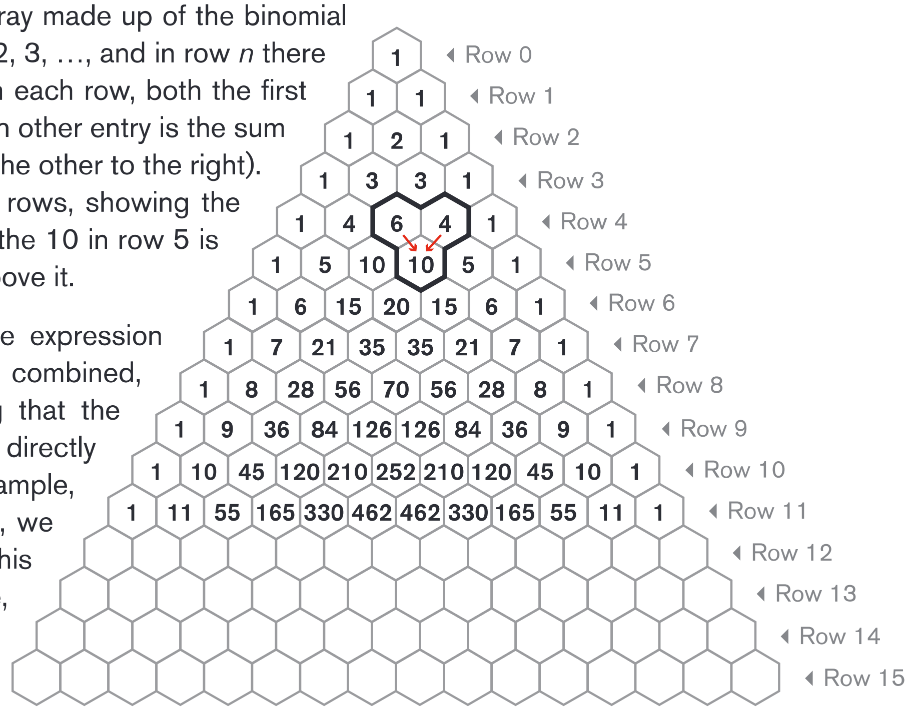
Pascal's Triangle Stretch

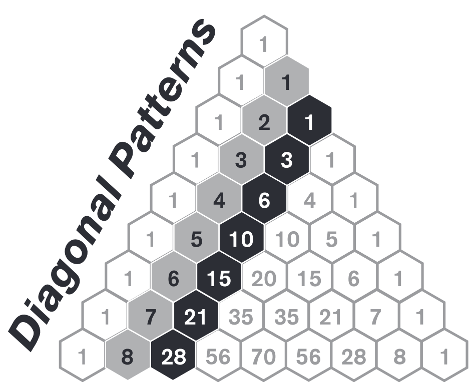
Pascal's triangle is a famous triangular array made up of the binomial coefficients. The rows are numbered 0, 1, 2, 3, …, and in row n there are $n + 1$ entries, numbered 0, 1, …, n. In each row, both the first entry (entry 0) and the last entry are 1. Each other entry is the sum of the two entries above it (one to the left, the other to the right). Here we have a Pascal's triangle with 16 rows, showing the entries for rows 0 through 11. Notice that the 10 in row 5 is the sum of the 6 and 4 in row 4, directly above it.



The binomial theorem says that when the expression $(x + y)^n$ is expanded and like terms are combined, the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$, meaning that the coefficients in this expansion can be read directly from row n in Pascal's triangle. For example, consider $(x + y)^4$. Expanding this binomial, we get $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. In this case, $n = 4$. Referring to Pascal's triangle, we see that row 4 does indeed give us the coefficients of the terms in this expansion: 1, 4, 6, 4 and 1.

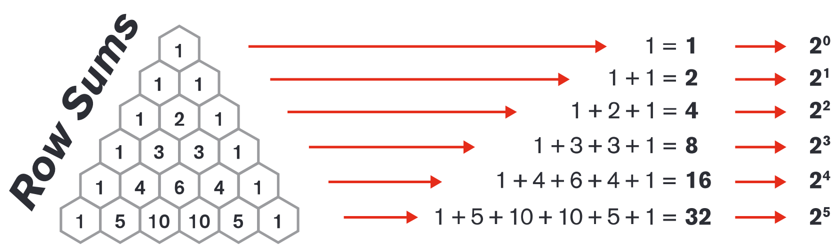
In general, the kth entry in row n is the binomial coefficient $\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$, the number of combinations of n objects taken k at a time. Given five objects, to find the number of ways to choose three of them, or $\binom{5}{3}$, we locate entry 3 of row 5 in Pascal's triangle and see that there are **10** ways.

Below are a few interesting properties of Pascal's triangle.

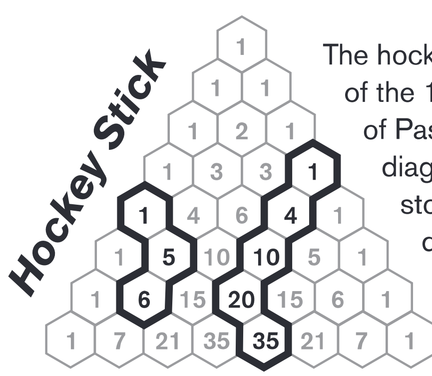


The gray diagonal contains the counting numbers: 1, 2, 3, 4, 5, ….

The black diagonal contains the triangular numbers: 1, 3, 6, 10, 15, …. In fact, entry 2 in row $(n + 1)$ is the nth triangular number.



The sum of the entries in the nth row of Pascal's triangle is equal to 2^n.



The hockey stick identity: Start at any of the 1s on the left or right side of Pascal's triangle. Sum entries diagonally in a straight line, and stop at any time. The next entry down diagonally in the opposite direction will equal that sum.

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Solve the following problems, using what you've learned about Pascal's triangle. It may be helpful to fill in some of the missing entries in the Pascal's triangle on the previous page.

21.__________ What is the greatest entry in row 15 of Pascal's triangle?

22.______ entries How many of the entries in row 14 of Pascal's triangle are even?

23.__________ What is the sum of the entries in row 12 of Pascal's triangle?

24.______ choices A sports team of eight players must choose three starting players. How many different choices of three starters are there if the order in which they are chosen does not matter?

25.__________ What is the sum of the first 10 triangular numbers?

26.__________ When the expression $(x + 2)^8$ is expanded and like terms are combined, what is the coefficient of x^3?

27.__________ When the expression $(2x + y)^4$ is expanded and like terms are combined, what is the sum of the coefficients?

28.__________ When the expression $(x + 1)^{2022}$ is expanded and like terms are combined, the term with the greatest coefficient can be expressed as ax^b. What is the value of b?

29.__________ A fair coin is flipped four times. What is the probability that it lands heads up at least as many times as it lands tails up? Express your answer as a common fraction.

30.______ times Only the number 1 appears in Pascal's triangle more times than the number 3003 appears. How many times does 3003 appear in Pascal's triangle?

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Systems of Equations Stretch

For problems 1-3, solve each of the following systems of equations. Describe the method you used. What other methods could you have used to solve each system? Express each answer as an ordered pair. Express any non-integer value as a common fraction.

1. $2x + 3y = 11$
   $3x - y = 11$

2. $\frac{2}{x} + \frac{3}{y} = 11$
   $\frac{3}{x} - \frac{1}{y} = 11$

3. $2x^2 + 3y^2 = 11$
   $3x^2 - y^2 = 11$

In problems 4 and 5, for what value of k does the linear system have no solution? Express any non-integer value as a common fraction.

4. $kx + 3y = 11$
   $3x - y = 11$

5. $2x + ky = 11$
   $3x - y = 11$

6. Given that $2a + b = 19$, $2c + d = 37$ and $b + d = 24$, what is the value of $a + b + c + d$?

7. Given that $a + b = 29$ and $ab = 204$, what is the value of $a^2 + b^2$?

8. Solve each of the following linear systems. Express your answer as an ordered pair.

a. $5x + 6y = 7$
   $8x + 9y = 10$

b. $x + 2y = 3$
   $4x + 5y = 6$

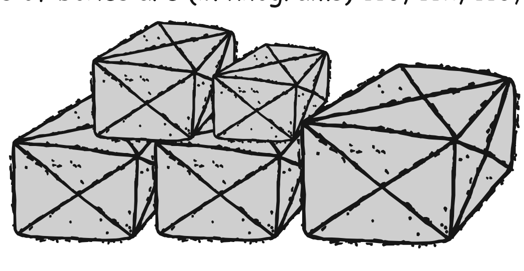
Many problems can be solved using a linear system of equations. For problems 9 and 10, use linear systems of equations to help you solve.

9a. For each equation below (i, ii, iii and iv), does there exist a solution (x, y) with positive integers x and y? In each case, either find all pairs of integers satisfying the equation or explain why none exist. Hint for the first equation: $x^2 - y^2 = (x + y)(x - y)$, so $x + y = 12$ and $x - y = 4$ is one case to check.

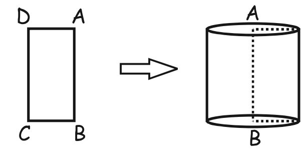
i. $x^2 - y^2 = 48$ ii. $x^2 - y^2 = 23$ iii. $x^2 - y^2 = 45$ iv. $x^2 - y^2 = 90$

b. In general, for what type of integers, n, does $x^2 - y^2 = n$ have at least one solution?

10. A shipping clerk wishes to determine the weights of each of five boxes. Each box weighs a different integer amount less than 100 kg. Unfortunately the only scales available measure weights in excess of 100 kg. The clerk therefore decides to weigh the boxes in pairs so that each box is weighed with every other box. The weights for the 10 pairs of boxes are (in kilograms) 110, 112, 113, 114, 115, 116, 117, 118, 120 and 121. From this information the clerk can determine the weight of each box. What are the weights of each of the five boxes?

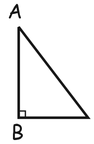


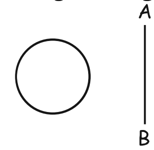
Solids Stretch



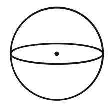
If you revolve rectangle ABCD about side AB, you will generate a cylinder.
Line AB is the axis of revolution for this solid of revolution.

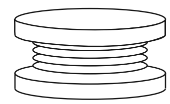
For questions 1 and 2, describe the solid of revolution generated by revolving the given figure 360° about segment AB.

1. ____________ 

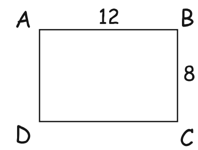
2. ____________ 

For questions 3 and 4, what plane figure will generate the given solid when revolved 360° about an axis of revolution? Draw or describe the figure and indicate the axis of revolution.

3. ____________ 

4. ____________ 

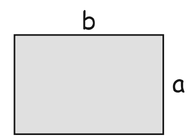
5. If you revolve this rectangle about side AB or side BC, you form a cylinder.



a. Revolving around which of these two segments produces a cylinder with a larger volume? _________

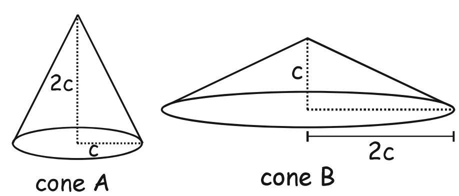
b. Revolving around which of these two segments produces a cylinder with a larger total surface area? ________

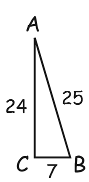
6. _________ Consider this shaded rectangle with b > a. Let V_a denote the volume of the cylinder that is created when the shaded rectangle is revolved around side a, and let V_b denote the volume of the cylinder that is created when the shaded rectangle is revolved around side b. What is the value of the ratio $\frac{V_b}{V_a}$? Express your answer as a common fraction in terms of a and b.



7. _________ Let S_a denote the total surface area of the cylinder that is created when the shaded rectangle used in question 6 is revolved around side a, and let S_b denote the total surface area of the cylinder that is created when the shaded rectangle is revolved around side b. What is the value of the ratio $\frac{S_b}{S_a}$? Express your answer as a common fraction in terms of a and b.

8. _________ These cones can be formed by revolving a right triangle about its two legs. What is the ratio of the volume of cone A to the volume of cone B? Express your answer as a common fraction.



9. 

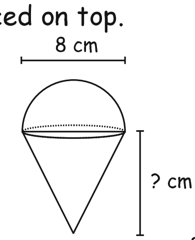
The side lengths in the figure are given. What is the volume of the solid that results from revolving the triangle 360° around . . .

a) side AC? Express your answer in terms of π.__________ units³

b) side BC? Express your answer in terms of π. __________ units³

c) side AB? Express your answer as a common fraction in terms of π.__________ units³

10. An ice cream cone is packed full of ice cream, and a hemisphere of ice cream is placed on top.



a. What plane figure will generate this solid when revolved 360° about an axis of revolution? Draw or describe the figure and indicate the axis of revolution. ______

b. If the volume of ice cream inside the cone is the same as the volume of ice cream outside the cone, how many centimeters is the height of the cone? __________ cm

Sum and Product SUPER Stretch

1. __________ What is the sum of the solutions of $6x^2 + 5x - 4 = 0$? Express your answer as a common fraction.

2. __________ A quadratic equation of the form $x^2 + kx + m = 0$ has solutions $x = 3 + 2\sqrt{2}$ and $x = 3 - 2\sqrt{2}$. What is the value of $k + m$?

3. __________ What is the sum of the reciprocals of the solutions of $4x^2 - 13x + 3 = 0$? Express your answer as a common fraction.

4. __________ If r and s are the solutions of $2x^2 + 9x + 3 = 0$, what is the value of $r^2 + s^2$? Express your answer as a common fraction.

5. __________ If r and s are the solutions of $x^2 + 6x - 2 = 0$, what is the value of $r^3 + s^3$?

6. (,) The solutions of $x^2 + bx + c = 0$ are each 5 more than the solutions of $x^2 + 7x + 3 = 0$. What are the values of b and c? Express your answer as an ordered pair (b, c).

7. (, ,) A cubic equation of the form $x^3 + bx^2 + cx + d = 0$ has solutions $x = 3$, $x = 4$ and $x = 5$. What are the values of b, c and d? Express your answer as an ordered triple (b, c, d).

8. __________ What is the sum of the reciprocals of the solutions of $x^3 - 3x^2 - 13x + 15 = 0$? Express your answer as a common fraction.

9. __________ What is the sum of the squares of the solutions of $x^3 - 15x^2 + 66x - 80 = 0$?

10. __________ The solutions of $x^3 - 63x^2 + cx - 1728 = 0$ form a geometric sequence. What is the value of c?

Right Triangles Stretch

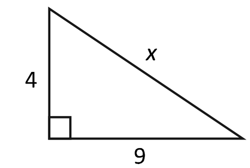
For problems 271-276, the Pythagorean Theorem or a knowledge of Pythagorean triples can be used to determine the value of *x* in each figure. For each figure, provide the exact value of *x* (as an integer or in simplest radical form). If *x* is not an integer value, also provide the value of *x* expressed as a decimal to the nearest tenth.

271. ____________ units

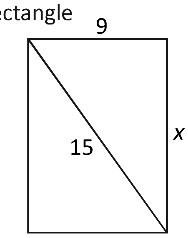
272. ____________ units

273. ____________ units

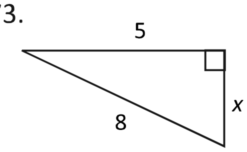
271.



272. rectangle



273.

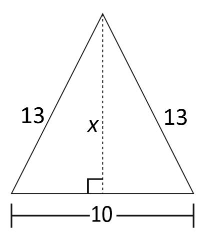


274. ____________ units

275. ____________ units

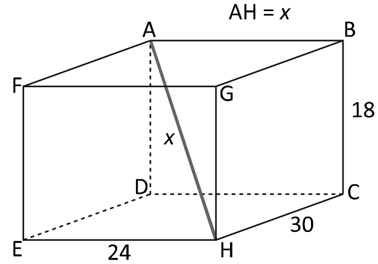
276. ____________ units

274. isosceles triangle

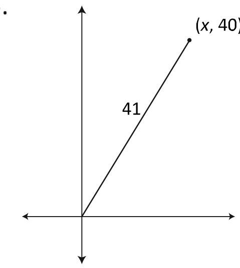


275. right rectangular prism

AH = x



276.



For problems 277-280, a knowledge of 30-60-90 or 45-45-90 triangles can be used to determine the value of *x* in each figure. For each figure, provide the exact value of *x* (as an integer or in simplest radical form). If *x* is not an integer value, also provide the value of *x* expressed as a decimal to the nearest tenth.

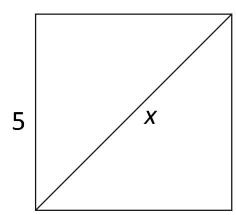
277. ____________ units

278. ____________ units

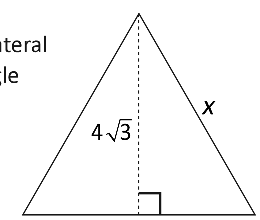
279. ____________ units

280. ____________ units

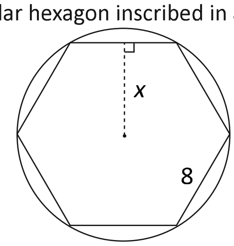
277. square



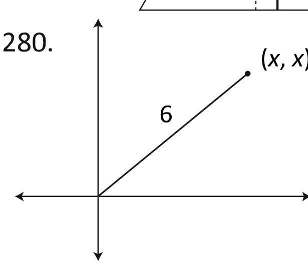
278. equilateral triangle



279. regular hexagon inscribed in a circle



280.



For problems 281-282, a knowledge of the properties of similar triangles and right triangles can be used to determine the value of *x* and *y* in each figure. If *x* and/or *y* is not an integer value, provide the value expressed as a decimal to the nearest tenth.

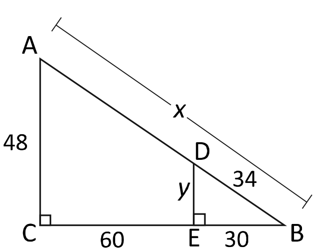
281. x = ________ units

y = ________ units

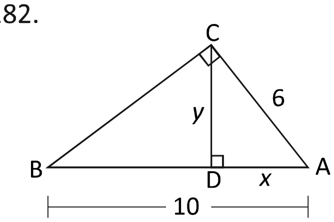
282. x = ________ units

y = ________ units

281.



282.



Sequences Stretch

283. ______________ What is the 100th term of the arithmetic sequence 3, 11, 19, 27, …?

284. ______________ What is the sum of the first 100 terms of the arithmetic sequence 3, 11, 19, 27, …?

285. ______________ What is the 10th term of the geometric sequence 729, 243, 81, 27, …? Express your answer as a common fraction.

286. ______________ The 1st and 18th terms of an arithmetic sequence are 4 and 8.25, respectively. What is the 35th term of the sequence? Express your answer as a decimal to the nearest tenth.

287. ______________ The first three terms of an arithmetic sequence are p, $2p + 6$ and $5p – 12$. What is the 4th term of this sequence?

288. ______________ All terms in a geometric sequence are positive integers, and the first three terms are n, $n + 3$ and $2n + 6$. What is the 4th term of this sequence?

289. ______________ The 3rd term of an arithmetic sequence is 17, and the 9th term is 83. What is the 1st term?

290. ______________ The 2nd term of a geometric sequence is 24, and the 5th term is 81. What is the 1st term?

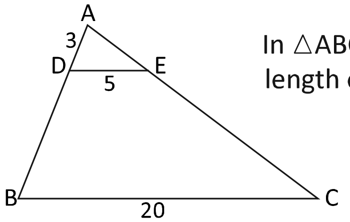
291. ______________ The 6th term of an arithmetic sequence is 24. What is the sum of the 5th and 7th terms?

292. ____________ cells The number of bacterial cells within a Petri dish doubles every hour. If there are 8 cells in the dish at the end of the 2nd hour, how many cells will be in the dish at the end of the 8th hour?

Similarity Stretch

Two geometric figures are similar if all of their corresponding angles are congruent and all of their corresponding sides are proportional. This means that the figures have the exact same shape but not necessarily the same size. For two triangles to be similar, it is sufficient to know that two pairs of corresponding angles are congruent.

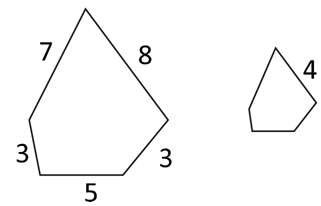
293. ________ units



In $\triangle ABC$, $\overline{DE}$ is parallel to $\overline{BC}$, AD = 3, DE = 5 and BC = 20. What is the length of $\overline{BD}$?

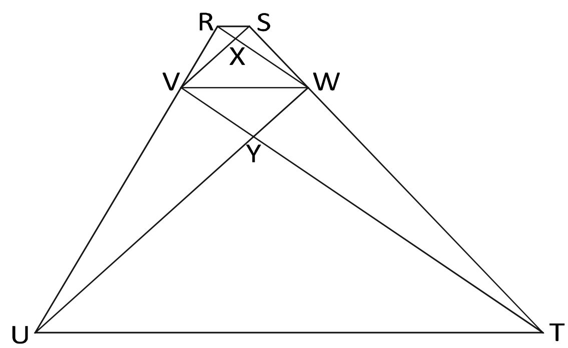
294. ________ units

The two pentagons shown here are similar, with the side of length 4 in the smaller pentagon corresponding to the side of length 8 in the larger pentagon, and with the indicated lengths given. What is the perimeter of the smaller pentagon?

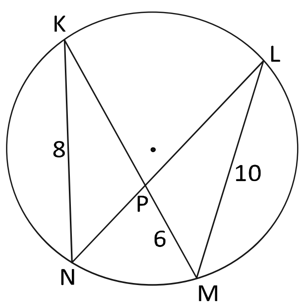


295. ________ units

In the figure below, $\overline{RS}$, $\overline{VW}$ and $\overline{UT}$ are parallel. If RS = 3, VW = 12, UT = 48 and XW = 10, what is the length of $\overline{YT}$?



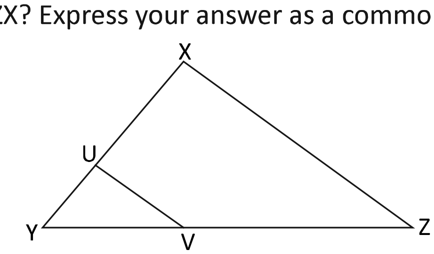
296. ________ units



Chords KM and NL of the circle shown intersect at point P. If KN = 8, PM = 6 and LM = 10, what is the length of $\overline{PN}$? Express your answer as a decimal to the nearest tenth.

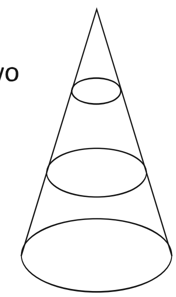
297. __________

Given that $\overline{UV}$ and $\overline{XZ}$ are parallel and that YV = 3 and YZ = 5, what is the ratio of the area of ΔUYV to the area of trapezoid UVZX? Express your answer as a common fraction.

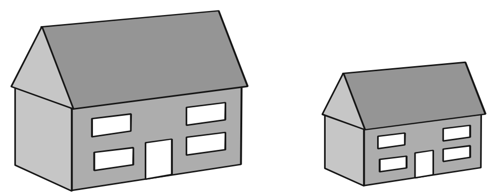


298. ________units² Trapezoid ABCD has right angles at A and D, and diagonals AC and BD intersect at point E. The area of ΔABE is 25 units², and the area of ΔDEC is 49 units². If AD = 6, what is the area of trapezoid ABCD?

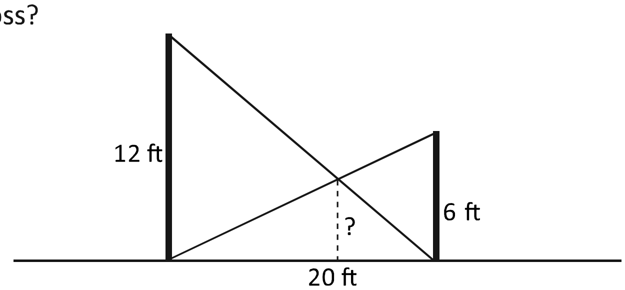
299. __________ In the figure shown, the largest cone has been divided into a smaller cone and two frustums by two planes that trisect the altitude of the original cone. What is the ratio of the volume of the smaller frustum to the volume of the larger frustum? Express your answer as a common fraction.



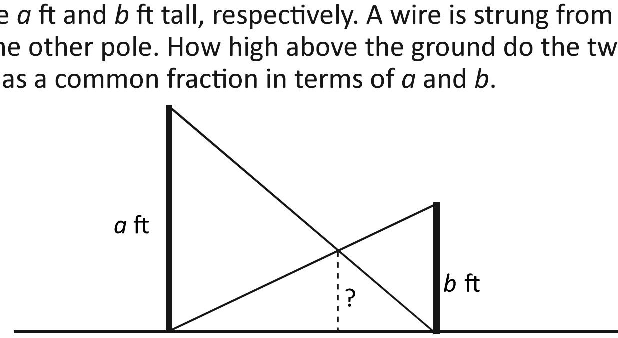
300. _________cm² The two toy houses shown here are similar. If the volume of the larger one is 1000 cm³ and the volume of the smaller one is 216 cm³, what is the surface area of the smaller house if the larger one has a surface area of 400 cm²?



301. __________ft Two vertical poles with heights 6 ft and 12 ft, respectively, are placed 20 ft apart. A wire is strung from the top of each pole to the base of the other pole. How high above the ground do the two wires cross?



302. __________ft Two vertical poles are *a* ft and *b* ft tall, respectively. A wire is strung from the top of each pole to the base of the other pole. How high above the ground do the two wires cross? Express your answer as a common fraction in terms of *a* and *b*.



Functions Stretch

A *function* maps a set of input values (*domain*) to a set of output values (*range*) in such a way that each input value maps to exactly one output value. The following exercises explore various types of functions through the use of equations, tables and graphs.

271. __________ What is the value of $f(-2)$ if $f(x) = x^2 + x + 5$?

272. __________ What is the value of x if $f(x) = 4$ for $f(x) = 3x - 14$?

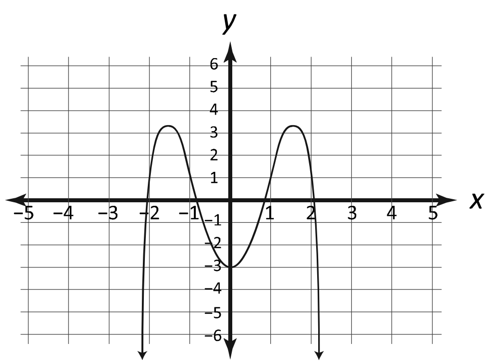
The table shown represents the function $h(x)$.

273. __________ According to the table, what is the value of $h(2)$?

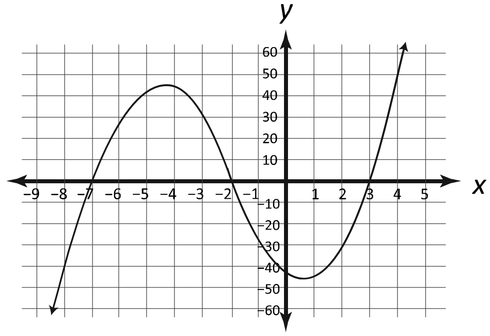
274. __________ What is the value of x if $h(x) = 58$?

x	***h(x)***
−1	3
0	10
1	7
2	6
3	19
4	58

275. __________ Based on the graph of $f(x) = -x^4 + 5x^2 - 3$, shown here, what is the value of $f(0)$?

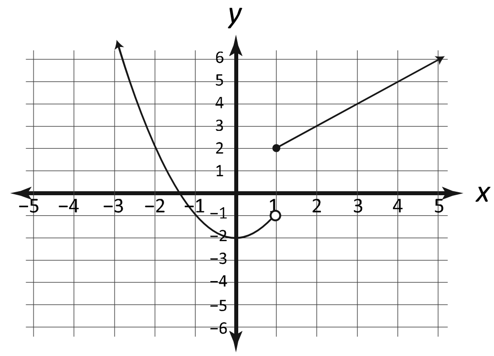


276. __________ Based on the graph of $f(x) = x^3 + 6x^2 - 13x - 42$, shown here, what is the sum of all x-values for which $f(x) = 0$?



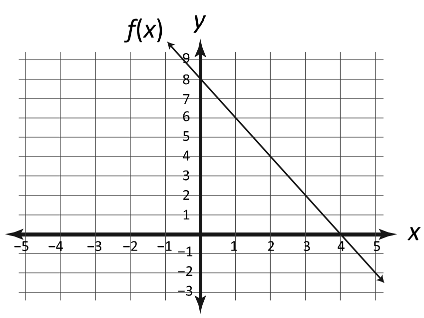
277. __________ A function that is defined by more than one equation, each with its own domain, is called a *piecewise function*. What is the value of $f(1)$ for the piecewise function defined and graphed as shown?

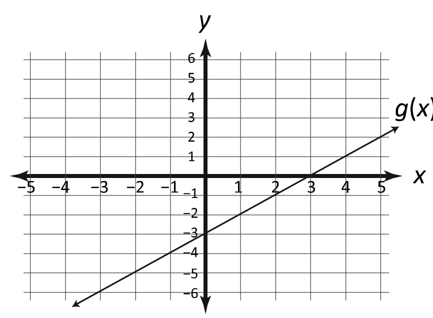
$$f(x)=\begin{cases} x^2-2, & x<1 \\ x+1, & x\geq 1 \end{cases}$$



278. __________ The composition of the function f with the function g is defined as $f(g(x))$. The domain of $f(g(x))$ is the set of all x-values such that x is in the domain of g, and $g(x)$ is in the domain of f. If $f(x) = 4x - 6$ and $g(x) = 2x + 1$, what is the value of $f(g(2))$?

279. __________ The functions f and g are graphed below. If $g(5) = r$ and $f(r) = s$, what is the value of s?





280. a = __________ What are the values of a and b that complete the table, representing the linear function $k(x)$?

b = __________

x	***k(x)***
−3	−5.5
0	*b*
3	9.5
a	17

Work Stretch

281. ____________ minutes Working alone, Alice can paint a room in 1 hour. Bob can paint the room in 2 hours alone. How many minutes will it take them to paint the room together?

282. ____________ minutes James is trying to learn a new yoga posture. Reading about it from a book, he would take 30 minutes to learn the posture. Working with an instructor doubles his rate of learning. How many total minutes will it take James to learn the posture if his instructor arrives after he has studied from the book for 10 minutes?

283. ____________ minutes A hose could fill a small pool in 50 minutes if the pool did not leak. Alas, the pool leaks at a steady rate that can drain it completely in 300 minutes. How many minutes will it take the hose to fill the leaky pool?

284. ______:______ pm A vineyard's grapes can be harvested by 10 workers in 5 hours. If one worker starts the harvest at noon, and another worker joins the harvest each hour on the hour, at what time will the harvest be completed?

285. ______:______ pm Alfonso can write 100 practice problems for the math team in 20 hours, Beauregard can write the same number of problems in 30 hours, and Clyde can write that number of problems in 40 hours. Working together, at what time will the three of them finish writing 100 practice problems if Alfonso starts at noon, Beauregard joins him at 1 pm and Clyde joins them at 2 pm?

286. ____________ minutes Vincent can process 50 orders in 2 hours working alone. When Leela is in the room, Vincent works at twice his normal speed. When Fry is in the room, Vincent works at half his normal speed. If Vincent works alone for 10 minutes, then with Leela for 10 minutes, then alone for 10 minutes, then with Fry for 10 minutes, then alone for 10 minutes, and this pattern continues (alone, with Leela, alone, with Fry), how many minutes will it take to process 50 orders?

287. ____________ minutes Larry and Curly are trying to fill a sandbox with sand. Working alone, Larry can fill an empty sandbox in 4 hours, and Curly can do the same job in 5 hours. Moe is trying to empty the sandbox. Working alone, Moe can empty a full sandbox in 6 hours. If the sandbox is half full at the time Larry and Curly begin filling the sandbox and Moe begins emptying it, how many minutes will it take for the sandbox to be filled? Express your answer to the nearest whole number.

288. ____________ hours Danielle and Jennifer can do a job in 2 hours working together. Danielle could do it in 3 hours alone. How many hours would it take Jennifer to do the job alone?

289. ____________ hours One hose can fill a pool twice as quickly as another, smaller hose. If the two hoses together can fill the pool in 6 hours, how many hours would it take the smaller hose alone to fill the pool?

290. ____________ housekeepers The Hilbert Lodge has a housekeeping staff of ten. Working alone, one housekeeper can clean all of the rooms in the lodge in 4 hours. A different housekeeper can clean all of the rooms in 5 hours, and still another takes 6 hours to clean all the rooms, working alone. Working alone, each of the remaining seven housekeepers can clean all the rooms in 7, 8, 9, 10, 11, 12 and 13 hours, respectively. What is the minimum number of housekeepers needed to clean all of the rooms in Hilbert Lodge in exactly 2 hours?

Coordinate Geometry Stretch

Coordinate geometry is where algebra intersects geometry. The characteristics of equations are made visual by plotting ordered pairs on a Cartesian coordinate plane. The mathematician and philosopher René Descartes is given credit for creating the idea of using a coordinate plane.

Some of the problems that follow involve using the slope, $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, and a point, (x_1, y_1), or points on a line to determine the equation of the line. The equation of a line can be written in the following forms:

Standard (general) form	Ax + By = C, where A, B and C are integers
Slope-intercept form	$y = mx + b$, where b is the y-intercept
Point-slope form	$y - y_1 = m(x - x_1)$
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the x- and y-intercepts, respectively
Two-point form	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

The preceding forms may be of use when solving the following problems, but all equation answers should be in slope-intercept form. Any non-integer values should be expressed as common fractions unless otherwise indicated.

291. (____ , ____) What is the midpoint of the line segment whose endpoints are (−4, −2) and (6, −5)? Express your answer as an ordered pair.

292. __________ Two lines are parallel if they have the same slope. What is the equation of a line parallel to $y = 2x - 5$ that passes through the point (3, 5)?

293. __________ Two lines are perpendicular if the product of their slopes is −1. In other words, their slopes are opposite inverses (or negative reciprocals). What is the equation of the line perpendicular to $y = \frac{2}{3}x - \frac{1}{3}$ at the point (−4, −3)?

294. __________ What is the equation of a line with x-intercept (−3, 0) and y-intercept (0, 6)?

295. __________ A line has a slope of $\frac{1}{2}$ and a y-intercept of (0, −3). That line intersects the line $2x + 3y = -2$ at a point. What is the sum of the coordinates of the point?

296. __________ A circle can be reprsented by an equation in the form $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) are the coordinates of the center of the circle, and r is the radius of the circle. The graph of $x^2 + y^2 = 25$ is a circle with center (0, 0) and a radius of 5 units ($r^2 = 25$, from the equation). The graph of the line $y = x - 7$ intersects this circle at two points. What is the sum of the x-coordinates of these two points?

297. __________ The point (−2, 10) lies on the circle $(x - 3)^2 + (y + 2)^2 = 169$. What is the equation of the line tangent to that circle at that point?

298. __________ The two circles, $x^2 + y^2 = 25$ and $(x - 7)^2 + (y - 7)^2 = 25$, have a common chord. What is the equation of the line containing that chord?

299. ________ units Based on the Pythagorean Theorem, the distance d between two points (x_1, y_1) and (x_2, y_2) in the coordinate plane is $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. What is the length of the common chord in question 298? Express your answer in simplest radical form.

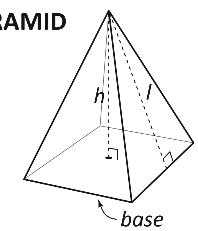
300. __________ Triangle ABC has vertices A(3, 2), B(−2, 1) and C(6, −5). What is the equation of the line containing the altitude from vertex A to side BC?

Surface Area & Volume Stretch

This activity involves determining the surface area (SA) and volume (V) of various geometric solids. For the purposes of these exercises, all solids are assumed to be right (the height is perpendicular to the base at its center). Below is an example of each solid along with the formulas for determing its surface area and volume.

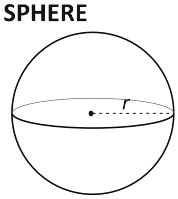
B = base area P = base perimeter h = height l = slant height r = radius

PYRAMID



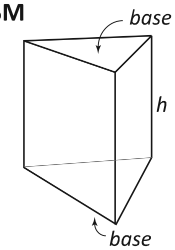
SA = $B + \frac{1}{2}Pl$
V = $\frac{1}{3}Bh$

SPHERE



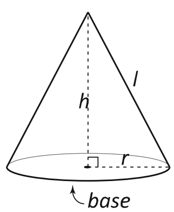
SA = $4\pi r^2$
V = $\frac{4}{3}\pi r^3$

PRISM



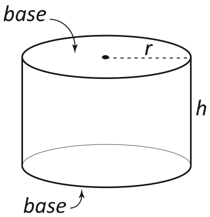
SA = $2B + Ph$
V = Bh

CONE: *Pyramid with circular base*



SA = $\pi r^2 + \pi rl$
V = $\frac{1}{3}\pi r^2 h$

CYLINDER: *Prism with circular bases*



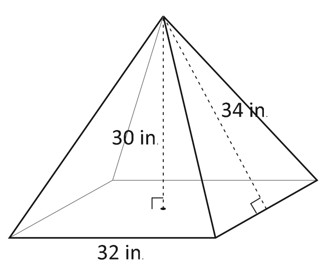
SA = $2\pi r^2 + 2\pi rh$
V = $\pi r^2 h$

For 271 and 272, find the surface area and volume of the geometric solid.

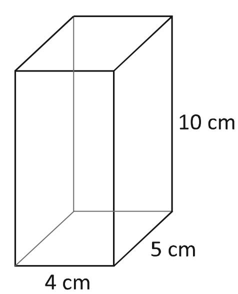
271. SA = ____________ in^2
V = ____________ in^3

272. SA = ____________ cm^2
V = ____________ cm^3

271. square pyramid



272. rectangular prism



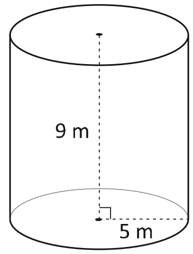
For 273-275, find the surface area and volume of the geometric solid. Express your answer in terms of π.

273. SA = ____________ m^2
V = ____________ m^3

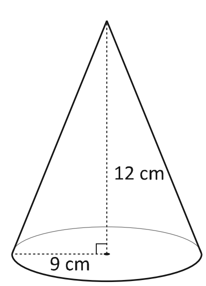
274. SA = ____________ cm^2
V = ____________ cm^3

275. SA = ____________ ft^2
V = ____________ ft^3

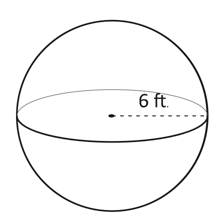
273. cylinder



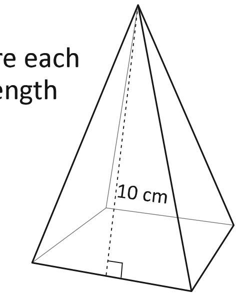
274. cone

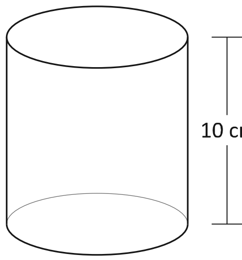


275. sphere



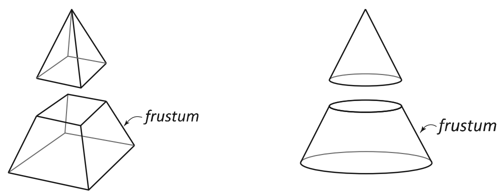
276. ____________ cm A right square pyramid has lateral faces with slant heights that are each 10 cm. If the surface area of this pyramid is 96 cm^2, what is the length of one of the edges of the base?



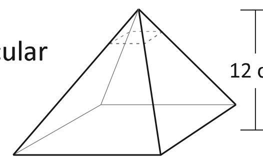


277. ____________ cm^2 What is the surface area, in square centimeters, of a cylinder with volume 250π cm^3 and height 10 cm? Express your answer in terms of π.

The frustum of a cone or a pyramid is that part of the solid left when the top portion is cut off by a plane parallel to its base.



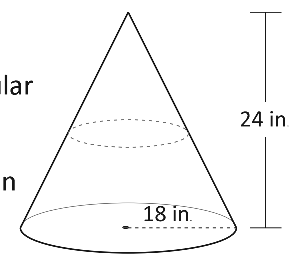
278. A pyramid with height 12 cm has a square base with area 64 cm^2. A plane perpendicular to the height intersects the pyramid 3 cm from its apex.



a. ____________ cm^3 What is the volume of the resulting frustum?

b. ____________ cm^2 What is the surface area of the frustum? Express your answer in simplest radical form.

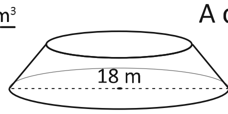
279. A cone with a height of 24 inches has a base with radius 18 inches. A plane perpendicular to the height intersects the cone halfway between its apex and base.



a. ____________ in^3 What is the volume of the resulting frustum? Express your answer in terms of π.

b. ____________ in^2 What is the surface area of the frustum? Express your answer in terms of π.

280. ____________ m^3

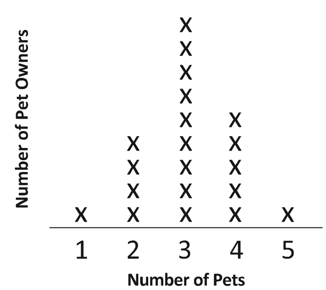


A cone of height 9 m was cut parallel to its base at 3 m above its base. If the base of the original cone had diameter 18 m, what is the volume of the resulting frustum? Express your answer in terms of π.

Data & Statistics Stretch

281. ________ pets As part of a survey, 20 pet owners indicated the total number of pets they currently own, and the results are displayed in the line plot shown. What is the mean of the median and the mode of the data?

Total Number of Pets per Owner



282. ________ mi/h The maximum speed of the fastest roller coaster at 20 different amusement parks is shown in this stem-and-leaf plot, where 10|7 represents 107 mi/h. What is the absolute difference between the mean and the median of the data? Express your answer as a decimal to the nearest tenth.

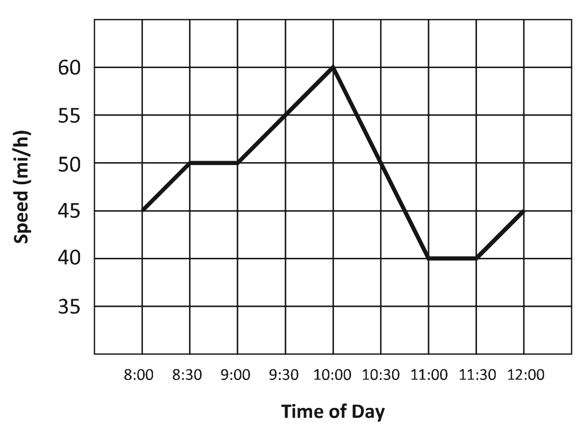
Maximum Speed of 20 Roller Coasters (mi/h)

Stem	Leaf
9	0 2 3 5 6
10	0 0 7
11	6 9
12	0 8
13	0 5 6 9
14	1 3 8
15	0

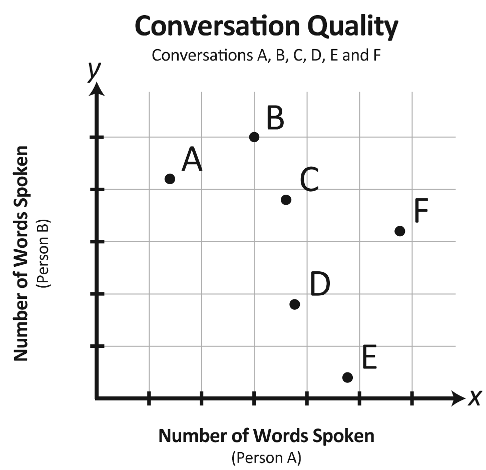
283. ________ mi/h The line graph shows one driver's speed, in miles per hour, from 8:00 a.m. to 12:00 noon. Based on the graph, what was the driver's average speed from 10:30 to 11:30? Express your answer as a decimal to the nearest tenth.

Driver's Speed

8:00 a.m. to 12:00 noon



284. point______ The quality of a conversation between two people can be described by the ratio of the number of words spoken by each person. The closer the ratio is to 1, the higher the quality. Each point in the graph shown represents a two-person conversation, with the number of words spoken by one person on the *x*-axis and the number of words spoken by the other person on the *y*-axis. If both axes use the same scale, which point in the graph represents the conversation with the highest quality?



285. __________ If the mean of five values is 27, what is the sum of the five values?

286. __________ When each of five numbers is doubled, the mean of the five new numbers is 60. What was the mean of the five original numbers?

287. __________ A list of 20 numbers has a mean of 37. When two numbers are removed from the list, the new mean is 38. What is the mean of the two numbers that were removed?

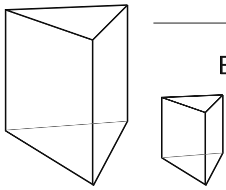
288. __________ The mean, median and unique mode of six positive integers are 8, 7 and 3, respectively. What is the maximum possible value for the range of the six numbers?

289. __________ The mean of three consecutive terms in an arithmetic sequence is 10, and the mean of their squares is 394. What is the largest of the three original terms?

290. __________ Six different positive integers add to 66. If one of them is the mean and another is the range, what is the largest possible number in the set?

Geometric Proportions Stretch

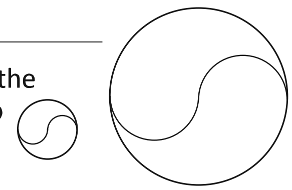
291. __________ smaller prisms



Each edge of the smaller triangular prism shown is $\frac{1}{2}$ the corresponding edge length of the larger triangular prism. How many of the smaller prisms combined have a total volume equal to the volume of the larger prism?

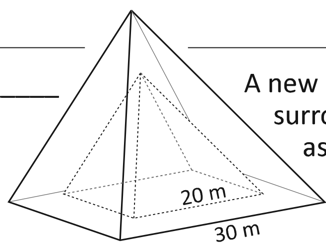
292. __________ A plane, parallel to the bases, slices the smaller prism $\frac{3}{4}$ of the way from one base to the other, dividing it into two smaller prisms. Of the three prisms, what fraction of the volume of the largest prism is the volume of the smallest prism? Express your answer as a common fraction.

293. __________ For the two similar circles shown, the area of the large circle is nine times the area of the small circle. What is the ratio of the radius of the small circle to the radius of the large circle? Express your answer as a common fraction.



294. __________ cm If the S-curve in the large circle has length 3.12 cm, what is the length of the S-curve in the small circle? Express your answer as a decimal to the nearest hundredth.

295. __________

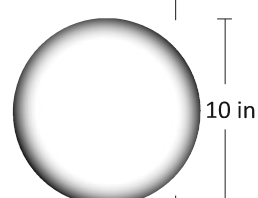


A new solid pyramid with a square base of side length 30 m will be constructed surrounding an existing solid pyramid with a square base of side length 20 m, as shown. If the existing and new pyramids are similar, what is the ratio of the total volume of the new pyramid to the volume of the old pyramid? Express your answer as a common fraction.

296. __________ years It took 16 years to completely build the original pyramid. Adding to the original pyramid, at the same volume-per-year rate, in how many years will construction of the new, larger pyramid be completed?

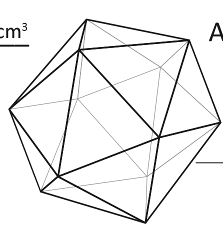
297. __________ in^3 A glassblower starts with a solid glass sphere that is 2 inches in diameter. What is the volume of the glass sphere? Express your answer as a decimal to the nearest hundredth.

298. __________ The glassblower will heat the glass sphere and blow air into it to create a hollow sphere 10 inches in diameter of uniform thickness. What is the ratio between the surface area of the original sphere and that of the new, hollow sphere? Express your answer as a common fraction.



299. __________ In the finished hollow sphere, what is the ratio of the volume of glass to the volume of enclosed air? Express your answer as a common fraction.

300. __________ cm^3



An icosahedron sculpture is being installed at a state fair. On the architect's model of the sculpture, each of the 20 equilateral triangular faces has area 2.85 cm^2. The actual sculpture has a total surface area of 4617 cm^2. If the volume of the model is 34 cm^3, what is the volume of the actual sculpture?

Logic Stretch

271. ________ Celia, Desi and Everett are each wearing a hat that displays a different whole number from 1 to 9, inclusive. Each number cannot be seen by the person wearing it, but that number is visible to the other two individuals. Everett says, “The sum of the numbers I see is 6.” Celia says, “The product of the numbers I see is 10.” What is the sum of the numbers that Everett could possibly have on his hat?

272. ________ people In a survey, 30 people reported that they enjoy some combination of walking, hiking and jogging. The number who enjoy only walking, the number who enjoy only hiking and the number who enjoy only jogging are all equal. Likewise, the number who enjoy only walking and hiking, the number who enjoy only walking and jogging and the number who enjoy only hiking and jogging are equal. In addition, the survey showed that half as many people enjoy exactly two of these activities as those who enjoy only one activity. If three people enjoy all three activities, how many people enjoy jogging?

273. ________

$$\begin{array}{r} \diamond\ \square\ \bigcirc \\ -\quad \bigcirc\ \diamond \\ \hline \bigcirc\ \square \end{array}$$

In the subtraction problem shown, the shapes $\diamond$, $\square$ and $\bigcirc$ each represent a different digit. What is the value of $\square\diamond \div \bigcirc$?

274. Box ________ Three identical boxes contain tennis balls, baseballs or both. A label is affixed to each box. The three labels correctly describe the three boxes, but none of the labels is on the correct box. Box 1 is labeled “Tennis Balls.” Box 2 is labeled “Baseballs.” Box 3 is labeled “Tennis Balls & Baseballs.” Devon reaches into Box 3 and pulls out a baseball. Which box contains only tennis balls?

275. Page ________ Drew purchased a used 50-page book at the book fair. Drew later realized that the book, in which left-hand pages contained even page numbers and right-hand pages contained odd page numbers, did not contain all 50 pages. The sum of the page numbers on the pages that Drew's book did contain was 1242. What is the greatest page number that could be on a page missing from Drew's book?

276.__________ In the addition problem shown, each letter stands for a different digit. If T = 3, what is the value of the four-digit number MATH?

$$
\begin{array}{cccc}
 & \mathbf{G} & \mathbf{E} & \mathbf{T} \\
+ & \mathbf{T} & \mathbf{H} & \mathbf{E} \\
\hline
\mathbf{M} & \mathbf{A} & \mathbf{T} & \mathbf{H}
\end{array}
$$

277.______ seconds Starting at the lower landing of a staircase, Porscha goes up the steps by repeating a three-step sequence: moving two steps up and then moving one step down. Starting at the upper landing of the same staircase, Micah goes down the steps by repeating a different three-step sequence: moving two steps down and then moving one step up. After simultaneously moving to their first steps, Porscha and Micah both move to another step every 3 seconds. To go from the upper landing to the lower landing of the staircase involves a net movement of 12 steps. How many seconds after moving to the starting steps will Porscha and Micah reach the same step?

278.__________ If the six-digit number 3D6,D92 is divisible by 11, what is the value of D?

279.__________ A special deck of cards contains cards numbered 1 through 4 for each of four suits. Each of the 16 cards has a club, diamond, heart or spade on one side and the number 1, 2, 3 or 4 on the other side. After a dealer mixed up the cards, three were selected at random. What is the probability that of these three randomly selected cards, displayed here, one of the cards showing the number 2 has a heart printed on the other side? Express your answer as a common fraction.

280.__________ The units digit of a three-digit number, ABC, is moved to the left of the remaining two digits to make a new three-digit number, CAB. If CAB − ABC = 162, what is the sum of the least and greatest possible values of ABC?

Solving Inequalities Stretch

QUICK REVIEW OF INEQUALITY PROPERTIES

For any numbers *a*, *b* and *c*,

- if $a > b$, then $a + c > b + c$ and $a - c > b - c$. (applies to >, ≥, < and ≤)
- if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. (applies to >, ≥, < and ≤)
- if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$. (applies to >, ≥, < and ≤)
- if $|a| < b$, then $a < b$ and $a > -b$. (applies to < and ≤)
- if $|a| > b$, then $a > b$ or $a < -b$. (applies to > and ≥)

Solve each inequality, and graph the solution on the number line provided.

281.__________ $3 - \frac{x}{3} \le 5$

282.__________ $3 - \frac{x}{3} \ge -5$

283.__________ $\left|3 - \frac{x}{3}\right| \le 5$

284.__________ $3 - \frac{x}{3} < x - 5$

285.__________ $3 - \frac{x}{3} > 5 - x$

286.__________ $\left|3 - \frac{x}{3}\right| < x - 5$

287.__________ $x^2 \le 25$

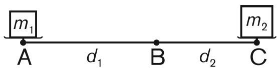
288.__________ $x^2 \ge 25$

289.__________ $x^2 + 4x - 4 > -8$

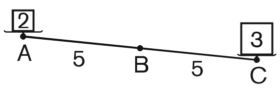
290.__________ $x^2 + 4x - 4 > -7$

Mass Point Geometry Stretch

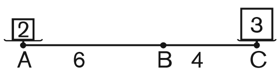
Center of Mass



Consider a seesaw, with a fulcrum at B, that has objects at A and C. As shown, the object at A has a mass of m_1, and its distance from B is d_1. The object at C has mass m_2, and its distance from B is d_2. These examples show how the position of the fulcrum determines whether the seesaw is balanced. The mass at B is $m_1 + m_2$, the sum of the masses at A and C.

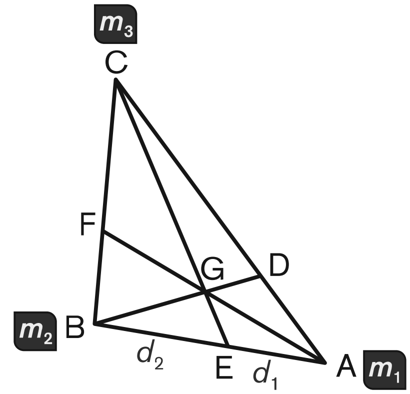


In this first example, B is positioned so that $d_1 = d_2 = 5$. Notice that $m_1 \times d_1 = 2 \times 5 = 10$ and $m_2 \times d_2 = 3 \times 5 = 15$. Although the objects at A and C are equidistant from B, the object at C is lower than the object at A because $m_1 \times d_1 < m_2 \times d_2$.



In this example, B is positioned so that $d_1 = 6$ and $d_2 = 4$. Here, $m_1 \times d_1 = 2 \times 6 = 12$ and $m_2 \times d_2 = 3 \times 4 = 12$. This time, the seesaw is balanced because $m_1 \times d_1 = m_2 \times d_2$. In this case, the position of B is known as the *center of mass*.

A *cevian* is a line segment that joins a vertex of a triangle with a point on the opposite side. Mass point geometry is a technique used to solve problems involving triangles and intersecting cevians by applying center of mass principles. Because triangle ABC, shown here, has cevians AF, BD and CE that intersect at point G, we can apply the center of mass principles presented. For example, side AB is balanced on point E when $m_1 \times d_1 = m_2 \times d_2$.



A *mass point*, denoted *m*P, consists of point P and its associated mass, *m*. Assume point G is the center of mass on which the entire triangle balances. Then the mass at G is the sum of the masses at the endpoints for each cevian and *m*G = *m*A + *m*F = *m*B + *m*D = *m*C + *m*E.

Suppose BF:CF = 3:4 and AD:CD = 2:5, and we are asked to determine the ratios AE:BE, AG:FG and BG:DG.

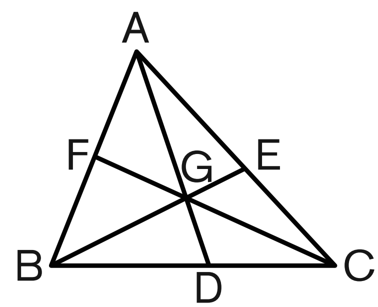
Start by finding *m*B and *m*C for side BC, which is balanced on point F. We know $m_2 \times 3 = m_3 \times 4$. We can let $m_2 = 4$ and $m_3 = 3$, so 4B + 3C = (4 + 3)F = 7F.

Next, find *m*A for side AC, which is balanced on point D. We know $m_1 \times 2 = m_3 \times 5$. Since $m_3 = 3$, it follows that $m_1 \times 2 = 3 \times 5$ and $m_1 = 15/2$. Rather than having mass point (15/2)A, we can multiply 4B, (15/2)A, 3C and 7F by 2 to get the following mass points: 8B, 15A, 6C and 14F. Now the mass at each point is of integer value.

Now, there is enough information to find *m*D and *m*E, since 15A + 6C = (15+ 6)D = 21D and 15A + 8B = (8 + 15)E = 23E. Therefore, given mass points 15A and 8B, it follows that side AB is balanced on point E when AE:BE = 8:15. In addition, given mass points 21D and 14F, we see that cevians AF and BD both are balanced on point G when AG:FG = 14:15 and BG:DG =21:8.

Solve the following problems by using mass point geometry. Express ratio answers as common fractions.

Triangle ABC, shown here, has cevians AD, BE and CF intersecting at point G, with AF:BF = 3:2 and BD:CD = 5:3.

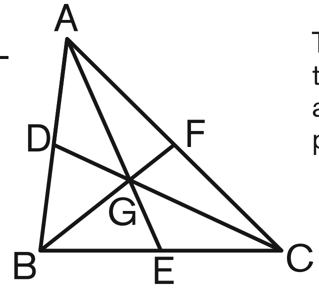


291.__________ What is the ratio of AE to CE?

292.__________ What is the ratio of BG to EG?

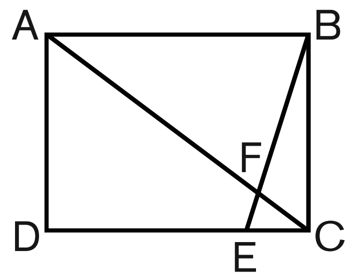
293.__________ What is the ratio of DG to AG?

294.__________

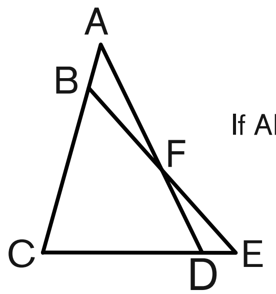


The medians of a triangle intersect at a point in the interior of the triangle as shown. What is the ratio of the lengths of the shorter and longer segments into which each median is divided at the point of intersection?

295.__________ In rectangle ABCD, point E is on side DC such that BC = 8, BE = 10 and AC = 17. If segments AC and BE intersect at F, what is the ratio of the area of triangle CFE to the area of triangle AFB?

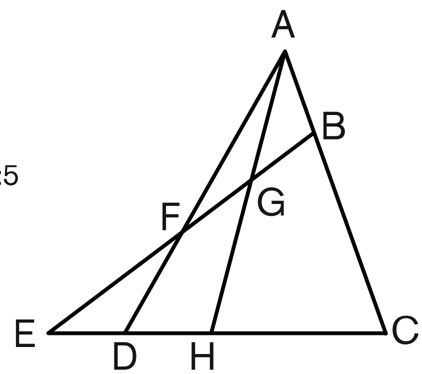


296.__________

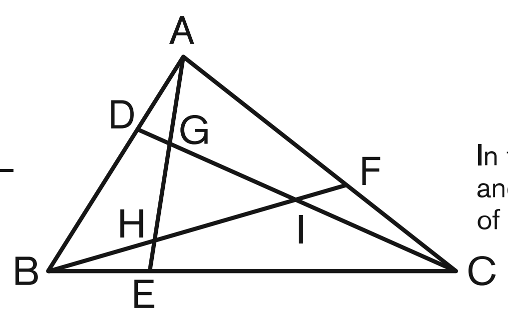


If AB:BC = 1:4 and AF:DF = 5:4, what is the ratio of DE to CD?

297.__________ For integers *x*, *y* and *z*, if AB:BC = 1:4, AG:GH = 3:5 and AF:DF = 5:4, then CH:DH:DE = x:y:z. What is the value of $x + y + z$?

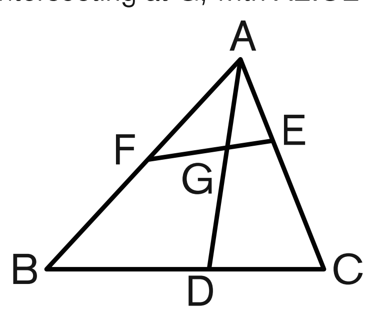


298.__________



In triangle ABC, AD:BD = 1:2, BE:EC = 1:3 and AF:CF = 3:2. What is the ratio of the area of triangle GHI to the area of triangle ABC?

Triangle ABC, shown here, has cevian AD and transversal EF intersecting at G, with AE:CE = 1:2, AF:BF = 5:4 and BD:CD = 3:2.



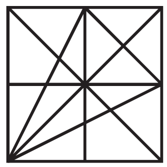
299.__________ What is the ratio of AG to DG?

300.__________ What is the ratio of EG to FG?

Counting Stretch

221. ________ Hazel wrote the integers 1 through 321 on the board. How many total digits did she write?

222. ________ triangles



How many triangles of any size are in this figure?

223. ________ ways In how many ways can one knife, one fork and one spoon be distributed, in any order, to three people, if each person is given 0, 1, 2 or 3 utensils?

224. ________ ways Using pennies, nickels, dimes and quarters, how many ways can you make 67 cents?

225. ________ scores In the game Fortrix, a player can earn 3, 7 or 11 points on a turn. How many different scores are possible for a single player after six turns?

226. ________ integers How many 3-digit integers are divisible by both 5 and 17?

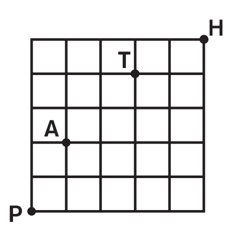
227. ________ integers How many positive integers less than 40 are relatively prime to both 7 and 10?

228. ________ palin-dromes How many palindromes are between 9 and 1009?

229. ________ paths In the 3×3 grid shown, a path can begin in any cell and can pass through a cell more than once. How many such paths spell ROTOR?

R	O	R
O	T	O
R	O	R

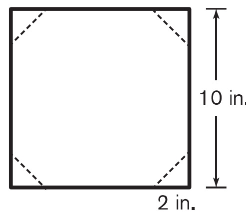
230. ________ paths



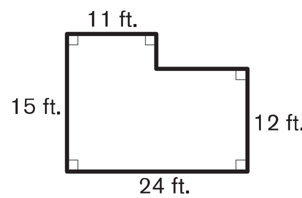
Moving only up and right, how many paths from P to H pass through A and T?

Area Stretch

231.________% Norm has a square sheet of paper with 10-inch sides. Along each side, he makes a mark 2 inches from each corner. He then draws a line segment connecting the two marks near each corner. Finally, he cuts along each line segment, removing a triangle from each corner of the square and creating an octagon. What percentage of the area of the square is the area of the octagon?



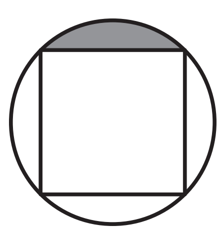
232.________ft² The figure shows an office floor plan. How many square feet does this office occupy?



233.________m² A running track consists of two parallel straight segments, each 100 meters long, connected by two semicircular stretches, each with inner diameter 50 meters. What is the total area enclosed by the running track? Express your answer to the nearest hundred.

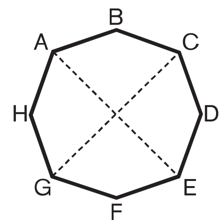
234.________units² What is the greatest possible area of a concave pentagon in the coordinate plane with vertices $(-2, 0)$, $(2, 0)$, $(2, 10)$, $(0, 6)$ and $(-2, 10)$?

235.________units² A square is inscribed in a circle of radius 4 units. The square divides the interior of the circle into five regions, four of which lie outside the square. What is the area of the shaded region? Express your answer in terms of π.

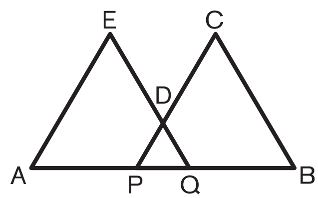


236.________ in² Amy marks two points A and B that are 4 inches apart. She draws one circle that has segment AB as a diameter. She then draws a larger circle, which overlaps the first circle, such that the arc from A to B along its circumference is a quarter-circle. What is the total area covered by the two circles? Express your answer in terms of π.

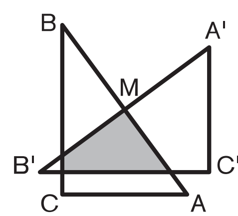
237.________ units² In convex octagon ABCDEFGH, shown here, each side has length 6 units, and diagonals AE and CG have length 16 units. If the octagon is symmetric across both diagonals AE and CG, what is its area? Express your answer in simplest radical form.



238.________ units² In this figure, AE = EQ = BC = CP = 10 units, and AQ = BP = 12 units. The points A, P, Q and B are collinear. If the perimeter of the concave pentagon ABCDE is 52 units, what is its area? Express your answer as a common fraction.



239.________ units² Right triangle ABC with AC = 3 units, BC = 4 units and AB = 5 units is rotated 90° counterclockwise about M, the midpoint of side AB, to create a new right triangle A'B'C'. What is the area of the shaded region where triangles ABC and A'B'C' overlap? Express your answer as a common fraction.



240.________ units In right triangle ABC, ∠C is a right angle, AC = 10 units and BC = 24 units. If a point X is located inside triangle ABC so that the distance from X to side AB is twice the distance from X to side AC, and the distance from X to side AC is twice the distance from X to side BC, what is the distance from X to side AB? Express your answer as a common fraction.

Modular Arithmetic Stretch

Modular arithmetic is a system of integer arithmetic that enables us obtain information and draw conclusions about large quantities and calculations. It would be extremely helpful, for instance, when asked to find the units digit of 2^{2015} if we didn't really have to calculate the value of the expression to get that information. Modular arithmetic allows us to do just that!

THE BASICS:

The simplest example of modular arithmetic is commonly referred to as "clock arithmetic." Suppose it is 3 o'clock now and I want to know what time it will be in 145 hours. We could count from 3 o'clock for 145 consecutive hours. We certainly wouldn't be expected to count 145 hours starting with 3 o'clock. Suppose we did counting the hours from 3 o'clock. What happens when we get to 12 o'clock? We continue counting but begin a new 12-hour cycle. Instead of counting 145 hours, we can just see how many of these 12-hour cycles we'd go through counting 145 hours. More importantly, we need to determine how many hours would remain after making it through the last full 12-hour cycle.

In this example, the value 12 is called the ***modulus*** and what is left over is called the remainder. In this case, we can determine fairly quickly that there are 12 full 12-hour cycles in 145 hours, with a remainder of 1 hour (since $12 \times 12 = 144$ and $145 - 144 = 1$).

Standard arithmetic: $145 = 12 \times 12 + 1$

Modular arithmetic we write: $145 \equiv 1 \pmod{12}$ *Read **"145 is congruent to 1 modulo 12"***

The remainder of 1 tells me that it will be the same time 145 hours after 3 o'clock that it will be 1 hour after 3 o'clock. And that time is 4 o'clock.

Here's another example of modular arithmetic. Suppose today is Tuesday. What day of the week will it be 417 days from now? Since the days of the week are on a 7-day repeating cycle, the modulus here is 7. If we divide 417 by 7, we get

Standard arithmetic: $417 = 59 \times 7 + 4$

Modular arithmetic we write: $417 \equiv 4 \pmod{7}$

Thus, 417 days from Tuesday will be the same day of the week as 4 days from Tuesday, Saturday.

TRY THESE

241.__________ If the current month is July, what month will it be in 152 months?

242.__________ a.m. / p.m. If the time is currently 8 a.m., what time will it be in 255 hours? *Circle a.m. or p.m. in answer blank.*

243.__________ m Jennie goes out every morning and jogs on the school track. The track is 400 meters around. If Jennie runs 5310 meters then how far will she be from where she started once she finished her run?

Modular Addition: What is the remainder when 9813 + 7762 + 11252 is divided by 10?

$$9813 + 7762 + 11252 = (981 \times 10 + 3) + (776 \times 10 + 2) + (1125 \times 10 + 2)$$
$$= (981 + 776 + 1125) \times 10 + (3 + 2 + 2)$$

Since we are only interested in the remainder, we need only focus on the last part. We see that the remainder is 3 + 2 + 2 = **7.** Written in modular arithmetic notation it would look like this:

$$9813 + 7761 + 11252 \equiv 3 + 2 + 2 \equiv 7 \pmod{10}$$

Modular Multiplication: What is the remainder whsen 9813×7762 is divided by 10?

$$9813 \times 7762 = (981 \times 10 + 3) \times (776 \times 10 + 2)$$
$$= (981 \times 776 \times 10^2) + (981 \times 2 \times 10) + (776 \times 3 \times 10) + (3 \times 2)$$

The first three terms are multiples of 10, and once again last term is the remainder $3 \times 2 =$ **6.** Written in modular arithmetic notation would look like this:

$$9813 \times 7762 \equiv 3 \times 2 \equiv 6 \pmod{10}$$

More Mod Shortcuts: There are many useful applications of modular arithmetic. Here are just a few more.

- Consider the powers of 3: $3^0 = 1$; $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; $3^5 = 243$; $3^6 = 729$
  Notice that the units digits are repeated every four powers of 3, so the modulus is 4. Repeating units digits correspond to remainders 1, 2, 3 and 0.
- Suppose you want the unit digit of 3^{53}. First, we note that $53 \equiv 1 \pmod 4$ since the remainder 1 corresponds to units digit 3, thus, the expansion of 3^{53} has a units digit of 3.
- The smallest number that has remainder 1 when divided by 2 and 3 is 7. Why?
  $$1 \equiv 7 \pmod 2 \text{ and } 1 \equiv 7 \pmod 3$$

MODULAR ARITHMETIC PRACTICE

244.__________ What is the last digit of 2^{2015}?

245.__________ What is the value of 122×71 modulo 11?

246.__________ What is the remainder when $5981 \times 8162 \times 476$ is divided by 5?

247.__________ Jon has 29 boxes of donuts with 51 donuts in each box. He wants to divide them into groups of a dozen each. Once he groups them again, how many donuts will be left over?

248.__________ What is the least integer greater than 6 that leaves a remainder of 6 when it is divided by 7 and by 11?

249.__________ When organizing her pencils, Faith notices that when she puts them in groups of 3, 4, 5, or 6, she always has exactly one pencil left over. If Faith has between 10 and 100 pencils, how many pencils does she have?

250.__________ When organizing her pens, Faith notices that when she puts them in groups of 3, 4, 5, or 6, she is always one pen short of being able to make full groups. If Faith has between 10 and 100 pens, how many pens does she have?

Fractions Stretch

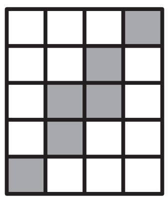
Solve the following problems. Express any non-integer answer as a common fraction.

221.________ What fraction of 100 is 25?

222.________ What fraction of $\frac{3}{8}$ is $\frac{9}{16}$?

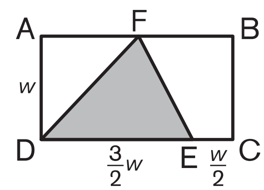
223.________ What is the value of $\sqrt{\frac{3}{11} \div \frac{11}{12}}$?

224.________



What fractional part of this grid of 20 unit squares is shaded?

225.________ What fraction of the area of rectangle ABCD is the area of inscribed triangle DEF?



226.________ On a number line, what common fraction is $\frac{3}{4}$ of the way from $\frac{1}{2}$ to $\frac{3}{4}$?

227.________ What is the reciprocal of $\frac{1}{2+\frac{1}{3}}$?

228.________ What common fraction is equal to $0.7\overline{5}$?

229.________ If $\dfrac{1}{\dfrac{1}{\frac{1}{n}+\frac{1}{3}}+\dfrac{1}{\frac{1}{3}+\frac{1}{n}}} = \dfrac{5}{12}$, what is the value of n?

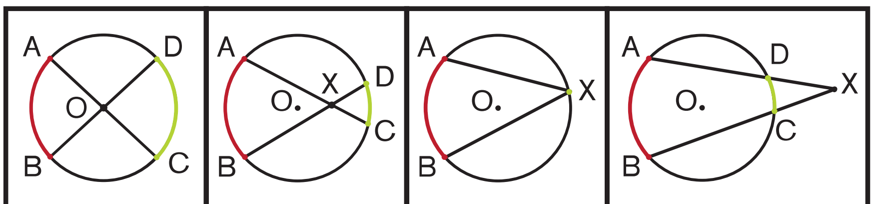
230.________ If $\frac{2x}{x-3} - 2 = \frac{4}{x+2}$, what is the value of x?

Angles and Arcs Stretch

Secant	a line that intersects the circle at two points
Chord	a line segment whose endpoints are two points on the circle
Tangent	a coplanar line that intersects the circle at a single point of tangency
Central Angle	an angle with its vertex at the center of the circle
Inscribed Angle	an angle with its vertex on the circle and whose sides are chords of the circle
Major Arc	an arc of the circle with measure greater than or equal to 180°
Minor Arc	an arc of the circle with measure less than 180°

Angle and Arc Measures

In the figures below, observe how the degree measure of ∠AXB decreases as the distance between the vertex of the angle and the center of the circle increases.



$m\overset{\frown}{AB} = 80°$	$m\overset{\frown}{AB} = 80°$	$m\overset{\frown}{AB} = 80°$	$m\overset{\frown}{AB} = 80°$
$m\overset{\frown}{CD} = 80°$	$m\overset{\frown}{CD} = 40°$	$m\overset{\frown}{CD} = 0°$	$m\overset{\frown}{CD} = 40°$
$m\angle AOB = 80°$	$m\angle AXB = 60°$	$m\angle AXB = 40°$	$m\angle AXB = 20°$
Figure I	Figure II	Figure III	Figure IV

- In Figure I, angles AOB and COD are central angles of circle O that intercept arcs AB and CD, respectively. The degree measure of a central angle and the arc it intercepts are equal.

$$m\angle AOB = m\overset{\frown}{AB} \text{ and } m\angle COD = m\overset{\frown}{CD}$$

- In Figure II, vertical angles AXB and CXD, formed by the intersection of chords AC and BD inside circle O, intercept arcs AB and CD, respectively. The degree measure of vertical angles formed by two chords intersecting inside a circle is half the sum of the measures of their intercepted arcs.

$$m\angle AXB = m\angle CXD = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$$

- In Figure III, ∠AXB is inscribed in circle O. The degree measure of an inscribed angle is half the measure of the intercepted arc.

$$m\angle AXB = \frac{1}{2}m\overset{\frown}{AB}$$

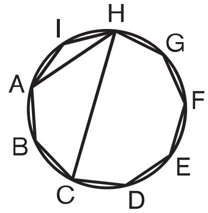
- In Figure IV, ∠AXB, formed by the intersection of two secants at point X outside of circle O, intercepts arcs AB and CD. The degree measure of an angle formed by two secants, two tangents or a secant and a tanget is half the difference of the measures of its intercepted arcs.

$$m\angle AXB = \frac{1}{2}(m\overset{\frown}{AB} - m\overset{\frown}{CD})$$

It may appear that there are four different formulas for calculating the four types of angles. But in each case, the measure of the angle in question is, essentially, the average of the measures of the intercepted arcs. In Figure IV, note that, with respect to ∠AXB, $\overset{\frown}{AB}$ appears concave, while $\overset{\frown}{CD}$ appears convex. So the measure of ∠AXB can be thought of as the average of 80° and −40°.

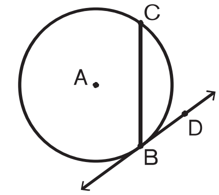
Solve the following problems by using what you've learned about angles and arcs. Express any non-integer value as a decimal to the nearest tenth.

231.________ °

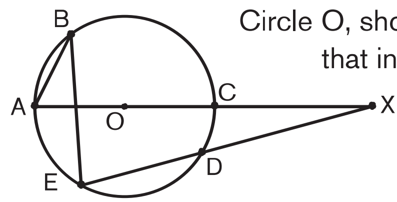


Regular nonagon ABCDEFGHI is inscribed in a circle, as shown. What is $m\angle AHC$?

232.________ ° In circle A, shown here, $\overleftrightarrow{BD}$ is tangent to the circle at B, and major $\widehat{BC}$ has measure 230°. What is $m\angle CBD$?

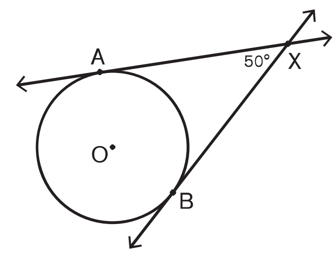


233.________ °



Circle O, shown here with chords AB and BE, has secants AC and DE that intersect at X. If $m\angle ABE = 35°$ and $m\angle AXE = 15°$, what is the measure of $\widehat{CD}$?

234.________ ° In this figure, lines AX and BX are tangent to circle O at A and B, respectively. If $m\angle AXB = 50°$, what is the measure of major $\widehat{AB}$?



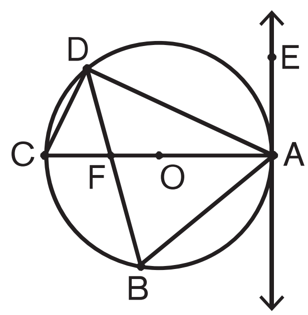
Use the figure at the right for questions 235 through 238.

235.________ ° What is $m\angle ABD$?

236.________ ° What is $m\widehat{AB}$?

237.________ ° What is $m\angle BAE$?

238.________ ° What is $m\angle CFD$?



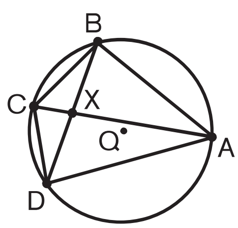
$\overleftrightarrow{AE}$ is tangent to circle O

$\overleftrightarrow{AE} \perp \overline{AC}$

$m\angle BDC = 40°$

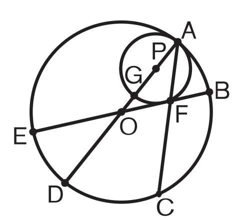
$m\widehat{AD} = 125°$

239.________ °



Quadrilateral ABCD is inscribed in circle Q, as shown, with diagonals intersecting at X. If $m\widehat{AB} = 110°$, $m\widehat{BC} = 60°$ and AB = BD, what is $m\angle CXD$?

240.________ ° Circle P is internally tangent to circle O at A, as shown. $\overline{AC}$ and $\overline{BE}$ intersect at F, which is also the point of tangency between $\overline{BE}$ and circle P. $\overline{AD}$ and $\overline{BE}$ are diameters of circle O, and $\overline{AG}$ is a diameter of circle P. If $m\widehat{CD} = 50°$, what is the measure of minor $\widehat{BC}$?



Bases Stretch

The **base 10** number system, the number system we are most familiar with, uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Numerals with these digits in the ones, tens, hundreds and higher places express specific numerical quantities. In base 10, the number 245, for example, is composed of 2 hundreds, 4 tens and 5 ones. That is, $2(10^2) + 4(10^1) + 5(10^0) = 200 + 40 + 5 = 245$.

A ***base b*** number system uses the digits 0, 1, ..., $b - 1$. Numerical quantities are expressed with these digits in the b^0, b^1, b^2 and higher places. In base b, if $b \geq 6$, the numeral 245_b represents the number $2(b^2) + 4(b^1) + 5(b^0)$. In base 8, for example, $245_8 = 2(8^2) + 4(8^1) + 5(8^0) = 2(64) + 4(8) + 5(1) = 128 + 32 + 5 = 165$.

Bases greater than 10 use letters to represent the digits greater than 9. For example, the 12 digits used in base 12 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A and B. The numeral 10 in base 12 has 1 twelve and 0 ones. That is, $10_{12} = 1(12^1) + 0(12^0) = 1(12) + 0(1) = 12 + 0 = 12$.

Practice Problems

What is the representation of each of the following in base 10?

241.__________ 24_9

242.__________ 24_8

243.__________ 24_7

What is the representation of 24 in each of the following bases?

244.__________ base 9

245.__________ base 8

246.__________ base 7

Now try these.

247.__________ What is the representation of 4991 in base 12?

248.__________ What is the representation of $3BB_{12}$ in base 6?

249.__________ If $523_b = 262$, what is the value of b?

250.__________ If $441_b = n^2$ and $351_b = (n - 2)^2$, for some $b < 10$, what is the value of n?

Probability Stretch

1. ________ % Petra randomly selects a card from a standard deck of 52 playing cards. What is the percent probability that the card shows a red number greater than 6? Express your answer to the nearest hundredth.

2. ________ Max has eight identical cups. Each cup contains a different combination of nickels, dimes and quarters, each totaling 45 cents. Max randomly selects a cup. What is the probability that the cup he selects contains at least three dimes? Express your answer as a common fraction.

3. ________ A bag contains five chips numbered 2 through 6. Danya draws chips from the bag one at a time and sets them aside. After each draw, she totals the numbers on all the chips she has already drawn. What is the probability that at any point in this process her total will equal 10? Express your answer as a decimal to the nearest tenth.

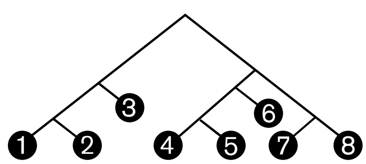
4. ________ A drawer contains five socks: two green and three blue. What is the probability that two socks pulled out of the drawer at random will match? Express your answer as a common fraction.

5. ________ A penny, a nickel and a dime are flipped. What is the probability that at least two coins land heads up and one of them is the nickel? Express your answer as a common fraction.

6. ________ % When the circuit containing blinking lights A and B is turned on, lights A and B blink together. Then A blinks once every 5 seconds and B blinks once every 11 seconds. Lindsey looks at the two lights just in time to see A blink alone. What is the percent probability that the next light to blink will be A blinking alone?

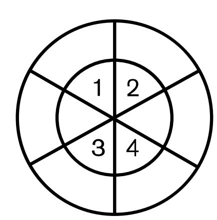
7. ________ % What is the percent probability that a randomly selected multiple of 3 less than or equal to 3000 is also a multiple of 5?

8. ________ Starting at the top and selecting paths randomly as you move downward, what is the probability of ending at an odd number? Express your answer as a common fraction.



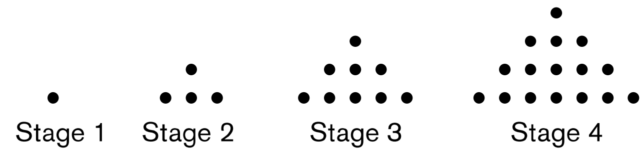
9. ________ A five-digit number is made by randomly ordering the digits 1, 2, 3, 4 and 5. What is the probability that this number is divisible by 4? Express your answer as a common fraction.

10. ________ Pierre throws darts that land randomly in the dartboard shown here. The dartboard is a circle of radius 2 units, with an inner circle of radius 1 unit. Both circles are divided into six congruent sectors. What is the probability that a dart Pierre throws will land in one of the four inner numbered sectors? Express your answer as a decimal to the nearest hundredth.



Patterns Stretch

11. ________ dots The first four stages of a dot pattern are shown. How many more dots are in the figure at Stage 47 than in the figure at Stage 27?



12. ________ The first three terms of a sequence are 1, 2 and 3. Each subsequent term is the sum of the three previous terms. What is the 11th term of this sequence?

13. ________ What is the sum of the terms in the arithmetic series 2 + 5 + 8 + 11 + 14 + ... + 89 + 92?

14. ________ Three consecutive terms in an arithmetic sequence are x, $2x + 11$ and $4x - 3$. What is the constant difference between consecutive terms in this sequence?

15. ________ What is the sum of the terms in the geometric series 1 + 4 + 16 + ... + 1024?

16. ________ What is the sum of the first 51 consecutive odd positive integers?

17. ________ What is the sum of the terms in the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$?

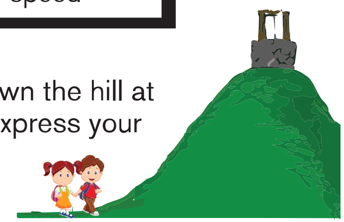
18. ________ What is the sum of the terms in the infinite series $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$? Express your answer as a common fraction.

19. ________ Let $f(x) = 2x + 3$ and $f^2(x) = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$. If $f^5(x) = ax + b$, what is the value of $a + b$?

20. ________ degrees The degree measures of the interior angles of a quadrilateral form a geometric sequence whose terms have integer values and are all integer multiples of the first term. What is the largest possible degree measure of an angle in this quadrilateral?

Travel Stretch

speed = $\frac{\text{distance}}{\text{time}}$	distance = speed × time	time = $\frac{\text{distance}}{\text{speed}}$



21. ________ mi/h Jack and Jill travel up a hill at a speed of 2 mi/h. They travel back down the hill at a speed of 4 mi/h. What is their average speed for the entire trip? Express your answer as a mixed number.

22. ____ : ____ p.m. At 2:20 p.m., Jack is at the top of the hill and starts walking down at the exact same time that Jill, who is at the bottom of the hill, starts walking up. If they maintain the same uphill and downhill speeds from the previous problem, and the distance from the bottom to the top of the hill is 1.5 miles, at what time will Jack and Jill meet?

23. ________ yards When Jack and Jill meet, as described in the previous problem, how many yards will they be from the bottom of the hill?

24. ________ minutes Alysha's average speed when walking from home to the market is 5 mi/h, and it takes her 21 minutes longer than when she drives to the market. If Alysha drives to the market, along the same route, at an average speed that is eight times her average walking speed, how many minutes does it take her to drive from home to the market?

25. ________ miles Based on problem 24, how many miles does Alysha travel to get from home to the market?

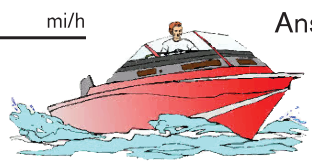
26. ________ minutes

Jana begins jogging along a path and, 5 minutes later, Zhao begins riding his bicycle along the same path, which has a length of 2 miles. Zhao rides his bicycle at a speed of 10 mi/h, and Jana's jogging speed is 6 mi/h. If they both begin at one end of the path and end at the other, how many minutes after Zhao reaches the end of the path will Jana reach the end of the path?

27. ________ minutes Based on problem 26, how many minutes after Zhao begins riding will he catch up with Jana? Express your answer as a mixed number.

28. ________ miles Again, based on problem 26, how many miles will Jana have traveled when Zhao catches up with her? Express your answer as a mixed number.

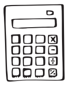
29. ________ mi/h



Ansel left the dock in his motorboat, traveled 10 miles, and then returned to the dock along the same route. On the return trip, Ansel was traveling against the current of the river, and his average speed relative to the water was 20 mi/h. If the round-trip took Ansel 64 minutes, what is the speed of the river's current?

30. ________ Based on problem 29, what fraction of Ansel's total travel time was spent traveling upstream? Express your answer as a common fraction.

Measurement Stretch



1. ________ units

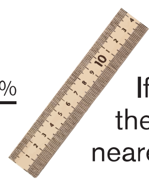
Merri places weights of 6 units and 28 units on the right side of a balance and weights of 3 units and 19 units on the left side. If she adds an object to the left side that makes the balance level, how many units does the object weigh?

2. ________ tacks The weight of a small clip is $\frac{2}{3}$ the weight of a large clip. If 2 tacks weigh the same amount as a large clip, how many tacks weigh the same amount as 12 small clips?

3. ________ Dems On the planet Klem, 1 Bem plus 7 Dems equals 4 Pems, and 2 Bems plus 1 Dem equals 1 Pem. How many Dems equal 7 Bems?

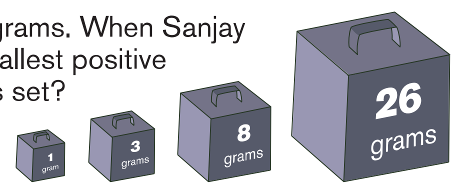
4. ________ meters If a race car is traveling at 99 mi/h, how many meters does it travel in a second, given that 0.305 meter = 1 foot? Express your answer as a decimal to the nearest tenth.

5. ________ %



If the results when reading a measuring stick can be off by at most 1 cm, what is the maximum percent error when 24 cm is measured? Express your answer to the nearest tenth.

6. ________ grams Vijay gives Sanjay a set of four weights of 1, 3, 8 and 26 grams. When Sanjay places weights on either side of a balance, what is the smallest positive integer number of grams that he **cannot** measure with this set?

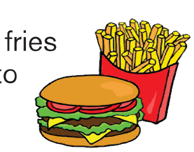


7. ________ gallons If Clem has 2 cups, 7 pints, 8 quarts and 11 half-gallons of lemonade, how many total gallons of lemonade does she have? Express your answer as a mixed number.

8. ________ Klegs If 2 Blams equal 15 Droms and 5 Droms equal 28 Klegs, how many Klegs are in a Blam?

9. ________ What is the ratio of 1 ounce to 1000 grams, given that 1 pound equals 454 grams? Express your answer as a decimal to the nearest thousandth.

10. ________ times If one order of fries and five burgers cost twice as much as three orders of fries and two burgers, how many times as much does a burger cost compared to one order of fries?



Expected Value Stretch

If the outcomes of random variable X have values $x_1, x_2, x_3, \ldots, x_n$ and the probabilities of these outcomes occurring are $p_1, p_2, p_3, \ldots, p_n$, respectively, then the **expected value** of the outcome is the sum of the products of the probability of each outcome and the value of that outcome.

$$E(X) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots + p_n x_n$$

11. __________ An unfair six-sided die with faces labeled 1, 2, 3, 5, 8 and 13 is rolled. The table lists the probability of the die landing with each number showing on the top face. The expected value of the roll is the sum of the products of each face value and its corresponding probability of being rolled. What is the expected value when the die is rolled? Express your answer as a mixed number.

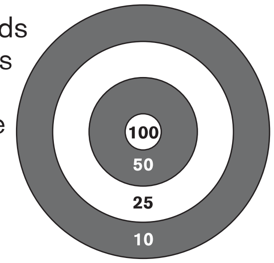
Top Face Value	Probability
1	$\frac{1}{3}$
2	$\frac{1}{15}$
3	$\frac{1}{6}$
5	$\frac{1}{5}$
8	$\frac{2}{15}$
13	$\frac{1}{10}$

12. $ __________ Terry plays a game with prizes of 5, 10, 15 and 20 dollars. The graph shows each possible prize amount and its corresponding probability. The expected value of her prize is the sum of the products of each prize and the probability of winning that prize. What is the expected value of Terry's prize?

13. __________ A fair 10-sided die with one face labeled 1, two faces labeled 2, three faces labeled 3 and four faces labeled 4 is rolled. What is the expected value when this die is rolled?

14. __________ cm^2 Ana has a bowl containing two square tiles, one with side length 2 cm and the other with side length 3 cm. She randomly chooses a tile from the bowl. The expected value of the area of the chosen tile is the sum of the products of each tile's area and its corresponding probability of being chosen. If the probability of choosing a particular tile is proportional to its area, what is the expected value of the area of the tile Ana chooses? Express your answer as a common fraction.

15.________ points For the dartboard shown, the number of points scored when a dart lands in each region is indicated. The innermost circle of the board has radius 1 inch, and each subsequent circle has a radius 2 inches greater than the previous circle. Kane throws a dart that lands randomly somewhere on the board. What is the expected value of the number of points he scores? Express your answer as a decimal to the nearest tenth.



16.________ Gwen randomly draws a card from a deck of 40 cards numbered 1 through 40. What is the expected value of the number on the card she draws? Express your answer as a decimal to the nearest tenth.

17.________ faces Luke paints each face of a 5 × 5 × 5 cube red. He then cuts the cube into 125 unit cubes and randomly chooses a single unit cube. What is the expected value of the number of painted faces on this unit cube? Express your answer as a decimal to the nearest tenth.

A property of *E* is that it is a linear function of the random variable. So, for random variables *X* and *Y*, the expected value of the sum of random variables equals the sum of their expected values.

$$E(X + Y) = E(X) + E(Y)$$

18.________ points In each round of a particular game, Dinara can win at most one point. If she has a 70% chance of winning a point in each round, what is the expected value of Dinara's total score after three rounds? Express your answer as a decimal to the nearest tenth.

19.________ Jo and her four friends each secretly pick a random integer from −5 to 5, inclusive. What is the expected value of the sum of the five chosen numbers?

20.________ jelly beans Allen randomly distributes 1000 jelly beans into 10 jars lined up in a row from left to right. What is the expected value of the number of jelly beans in the leftmost jar?

Transformations Stretch

21. ________ units A point $P(-3, 2)$ is translated right 4 units to its image P'. The point P' is then translated up 3 units to its image P''. What is the distance from P to P''?

22. ________ units A segment has endpoints $A(0, 0)$ and $B(-3, 4)$. Point C is the image of point B translated down 4 units and left 3 units. What is the perimeter of $\triangle ABC$?

23. (,) A point $Q(-3, 4)$ is reflected across the x-axis, and then the image Q' is reflected across the line $x = 2$. What are the coordinates of the image Q''? Express your answer as an ordered pair.

24. ________ A point $S(1, 6)$ is reflected across the line $x - 2y = -6$. What is the sum of the coordinates of the image S'?

25. (,) What are the coordinates of the image of point $D(-5, -3)$ when it is rotated 90 degrees clockwise about the origin? Express your answer as an ordered pair.

26. (,) What are the coordinates of the image of the point $E(3, -1)$ when it is rotated 90 degrees counterclockwise about the point $F(5, 4)$? Express your answer as an ordered pair.

27. ________ A segment with endpoints $G(-2, 3)$ and $H(4, 7)$ is dilated by a scale factor of $\frac{2}{3}$ with center of dilation $(0, 0)$. What is the sum of all the coordinates of G' and H'?

28. ________ Point $J(4, 8)$ is dilated by a scale factor of $\frac{3}{2}$ with center of dilation $K(2, 2)$. What is the product of the coordinates of J'?

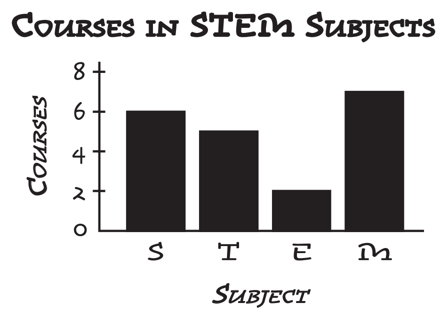
29. ________ units² A point $L(-2, 4)$ is rotated 90 degrees clockwise about the point $M(3, 2)$. Point N is the image of L' dilated by a scale factor of $\frac{3}{2}$ with center of dilation M. What is the area of $\triangle LMN$? Express your answer as a common fraction.

30. ________ units A point $R(-5, 3)$ is reflected across the line $y = x - 2$, and then the image R' is rotated 90 degrees clockwise about the origin. What is the distance from R to R''? Express your answer in simplest radical form.

Ratios Stretch

Definition

A *ratio* is the comparison of two quantities by division.



This graph shows the number of courses in **S**cience, **T**echnology, **E**ngineering and **M**ath offered at a particular school. According to the graph, the ratio of Engineering to Technology courses is 2 to 5, also written E:T = 2:5. The ratio E:T is a comparison of two of the four "parts" that combine to make up the "whole" group of STEM courses. Some ratios compare a "part" to the "whole." Ratios are often written in the form of a fraction, decimal or percent. According to the graph, this school offers a total of 20 STEM courses. So, the ratio of Math courses to STEM courses is 7 to 20. In other words, Math courses account for $\frac{7}{20} = 0.35 = 35\%$ of the STEM courses at this school.

1. __________ Dave's digital library contains 25 fiction books and 15 nonfiction books. What is the ratio of nonfiction books to fiction books in Dave's digital library? Express your answer as a common fraction.

2. __________ For a 30-60-90 right triangle, what is the ratio of the length of the longer leg to the length of the hypotenuse? Express your answer as a common fraction in simplest radical form.

3. __________ A jar contains seven blue marbles and eight green marbles. Ming adds four yellow marbles and five blue marbles to the jar. What is the ratio of green marbles to non-green marbles in the jar? Express your answer as a common fraction.

Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio 6:8:11. Use this information to solve problems 4 through 7.

4. __________ What fraction of the wishes were granted to Belle? Express your answer as a common fraction.

5. __________ % What percent of the wishes granted by Fairy Godmother were *not* granted to Aurora?

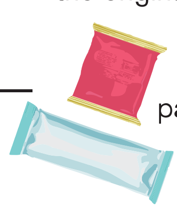
6. __________ % What is the absolute difference between the percents of wishes Fairy Godmother has granted to Aurora and to Cindi?

7. __________ wishes Fairy Godmother has granted at least 20 wishes each to Aurora, Belle and Cindi. What is the least possible number of wishes that she has granted to Cindi?

8. __________ % Jennie put 38 mL of water in a cylinder with a capacity of 60 mL. If she increases the volume of water in the cylinder by 50%, what percent of the cylinder will contain water?

9. __________ % A shop owner increased the price of a jacket by 17%. What percent of the new price is the original price of the jacket? Express your answer to the nearest tenth.

10. __________ New packaging for fruit snacks contains 10% less weight than the original packaging. If the new package costs 15% more than the original package, by what fraction did the unit price increase? Express your answer as a common fraction.



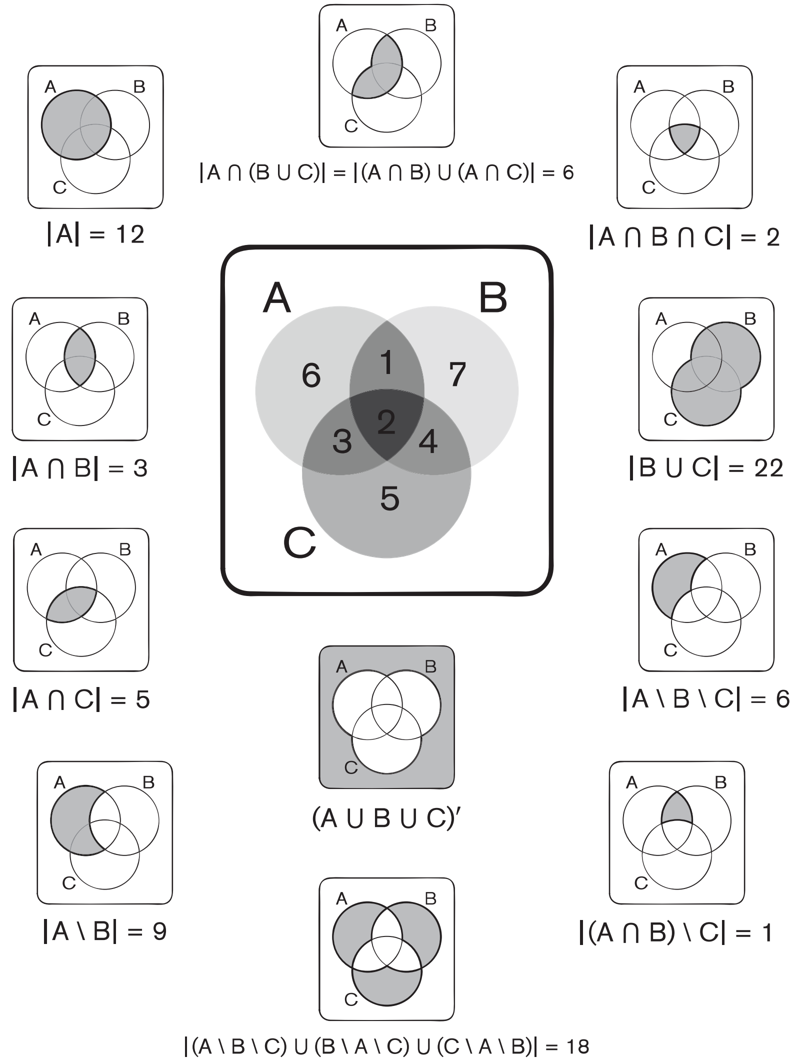
Venn Diagrams Stretch

Set Theory Review

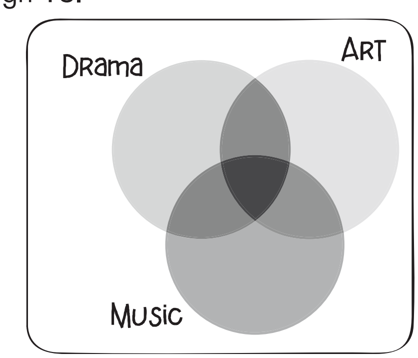
A **set** is a collection of objects or elements, called **members.** Consider two sets A and B.

- The **intersection** of A and B, denoted A ∩ B, is the set of elements that are in both A and B.
- The **union** of A and B, denoted A ∪ B, is the set of elements in A or in B or in both.
- The **universal set U** is the set of all possible elements.
- The **relative complement** of A, denoted B \ A, is the set of all elements in B but not in A.
- The **complement** of A, denoted A′, is the set of all elements not in A, in other words, U \ A.
- The **cardinality** of A, denoted |A|, is the number of elements in set A.

A **Venn diagram** is a useful tool for comparison. It helps us visualize the relationships between two or more sets. The Venn diagrams shown compare sets A, B and C.



At Mesa Performing Arts Center, 30 students take courses in one or more of the drama, music and art departments. Five students take courses in exactly one department. Of these students, twice as many take drama courses as take music courses. Five students take courses in exactly two departments. Of these students, twice as many take drama and music courses as take music and art courses. Use the provided Venn diagram to organize this information, and then answer questions 11 through 13.

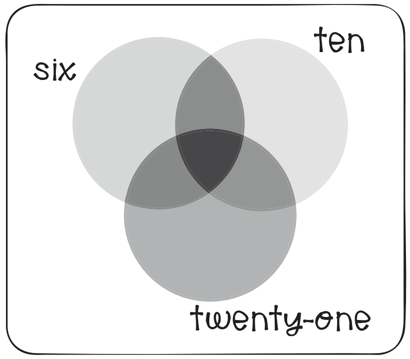


11. ________ students How many students take only art courses?

12. ________ students How many students take courses in all three departments?

13. ________ students How many students take courses in art or music but not both?

The integers from 1 to 630, inclusive, are tested for divisibility by 6, 10 and 21. Use the provided Venn diagram to help determine the cardinality of various sets that contain multiples of 6, 10 and 21, and then answer questions 14 through 16.

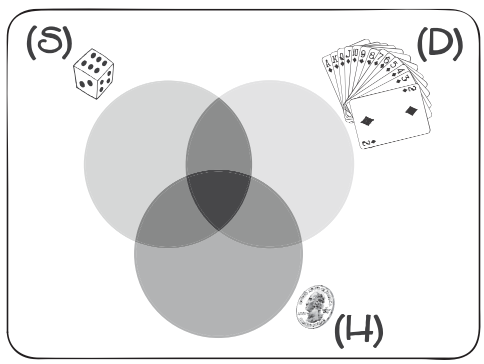


14.________ integers How many of these integers are divisible by both 6 and 10 but not by 21?

15.________ integers How many of these integers are divisible by 6 but not by either 10 or 21?

16.________ integers How many of these integers are not divisible by any of 6, 10 or 21?

A fair coin is flipped, a standard six-sided die is rolled and a card is randomly selected from a standard deck of 52 playing cards. The Venn diagram shown can be used to organize the numbers of outcomes that include flipping heads (H), rolling a 6 (S) and/or selecting a diamond card (D). Use your answers to questions 17 through 19 to fill in this diagram.



17.________ ways How many ways are there to flip heads, roll a 6 and select a diamond card?

Hint: This is the value of $|H \cap S \cap D|$.

18.________ ways Since there are 1 × 1 × 52 = 52 ways to roll a 6 and flip heads, how many ways are there to roll a 6 and flip heads but not select a diamond card?

Hint: Use $|S \cap H| = 52$ to find the value of $|(S \cap H) \setminus D|$.

19.________ ways Since there are 1 × 6 × 52 = 312 ways to flips heads and 13 × 1 × 6 = 78 ways to select a diamond card and flip heads, how many ways are there to flip heads but not select a diamond card and not roll a 6?

Hint: To start, use $|D \cap H| = 78$ to find the value of $|(D \cap H) \setminus S|$. Remember that $|H| = 312$.

20.________ What is the probability of rolling a 6 and not flipping heads ***or*** flipping heads and not selecting a diamond? Express your answer as a common fraction.

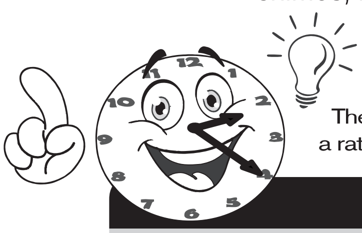
Clocks Stretch

Problems in this stretch involve 12-hour digital and analog clocks, and all time answers should be expressed to the nearest minute, unless otherwise stated.

21. ____:____ a.m. What time will it be 47 minutes after 7:37 a.m.?

22. ____:____ a.m. What time was it 43 minutes before 9:32 a.m.?

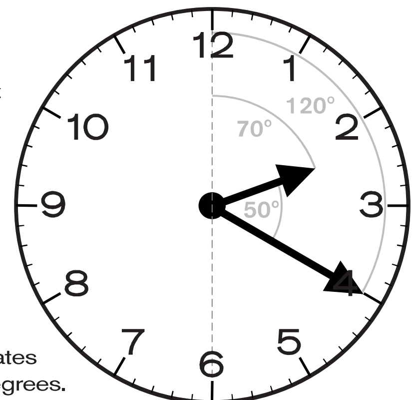
23. ________ seconds A certain clock sounds one chime at 1 o'clock, two chimes at 2 o'clock, three chimes at 3 o'clock, and so on. If this clock behaves in this manner every hour, on the hour so that each chime lasts one second and there is a one-second pause between consecutive chimes, how many seconds long are the chimes that sound at 11 o'clock?



The angle formed by the hour hand and minute hand changes (±) at a rate of $6 - \frac{1}{2} = 11/2 = 5\frac{1}{2}$ degrees each minute.

Rates of Rotation

Minute Hand	Hour Hand
$\frac{360 \text{ degrees}}{1 \text{ hour}} = \frac{6 \text{ degrees}}{1 \text{ minute}} = \frac{1 \text{ degree}}{1/6 \text{ minute}}$	$\frac{30 \text{ degrees}}{1 \text{ hour}} = \frac{1/2 \text{ degree}}{1 \text{ minute}} = \frac{1 \text{ degree}}{2 \text{ minutes}}$



In the example shown, the time is 2:20. In 2 hours 20 minutes, the hour hand rotates clockwise 70 degrees. In 20 minutes, the minute hand rotates clockwise 120 degrees. The degree measure of the acute angle formed by the clock hands is 120 − 70 = 50 degrees.

24. ________ degrees What is the degree measure of the acute angle formed by the hour and minute hands at 2:16?

25. ________ degrees What is the absolute difference in the degree measures of the smaller angles formed by the hour and minute hands at 2:08 and 8:02?

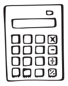
26. ____:____ After 5:30, when is the next time that the hour and minute hands are aligned so that the angle formed measures 0 degrees?

27. ________ minutes After 4 o'clock, how many minutes elapse between the first and second times that the hour and minute hands form a 38-degree angle? Express your answer as a mixed number.

28. ________ minutes After 3:24, how many minutes have elapsed the first time that the angle formed by the hour and minute hands is twice the measure of the angle formed by the hands at 3:24? Express your answer as a mixed number.

29. ________ What fraction of the times displayed on a digital clock contain the digit 5? Express your answer as a common fraction.

30. ________ times A 24-hour digital clock displays times from 00:00 to 23:59. How many of the times displayed on this clock contain the digit 2?



Arithmetic Mean Stretch

In the following problems, the terms *mean* and *average* refer to the arithmetic mean, unless otherwise stated.

1. ________ points Simone's average score for seven tests is 82 points. If she scores 90 points on her eighth test, what is the average of all eight test scores?

2. ________ points The average of Geoffrey's first four test scores is 72 points. If Geoffrey's fifth test score is 15 points more than the average of his first four test scores, what is the average of all five test scores?

3. ________ The mean of four numbers is 18. If one of the four numbers is removed, the mean of the three remaining numbers is 17. What is the value of the number that was removed?

4. ________ points Abel, Bilal and Cara played a game of Scrabble. The average of the points scored by Abel and Bilal was 261 points, while the average number of points scored by Abel, Bilal and Cara was 269 points. How many points did Cara score?

5. ________ grams Sage has 12 pennies and 8 nickels. The average mass of Sage's coins is 3.5 grams. If the average mass of Sage's pennies is 2.5 grams, what is the average mass of Sage's nickels?

6. ________ ounces

Ruby mailed three packages to a friend. The mean weight of the packages was 85 ounces. When Ruby sent a fourth package, the mean weight increased by 2 ounces. How many ounces did the final package weigh?

7. ________ pounds Six cocker spaniels have a total weight of 192 pounds. Five golden retrievers have an average weight of 71 pounds. What is the average weight of all 11 dogs? Express your answer as a decimal to the nearest tenth.

8. ________ pages Gil read 21 pages of a book on Monday, 34 pages on Tuesday, 17 pages on Wednesday, and 12 pages on Thursday. On Friday, Gil read 5 pages more than the mean number of pages read on the first four days. How many pages did Gil read in all?

9. ________ points Sam scored 50 points on the first of six Spanish exams in the semester. If Sam's goal is to have an average exam score of 90 points at the end of the semester, how many points will Sam need to score, on average, on the remaining five exams?

10. ________ points After taking three of the four exams in history class, Srinivasa has an average exam score of 66 points. If the fourth exam counts twice as much as the other exams, what is the fewest points Srinivasa can score on the fourth exam to pass the course with an overall exam average of at least 70 points?

Circles Stretch

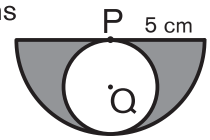
11. ________ mm What is the circumference of a circle that has an area of 289π mm^2? Express your answer in terms of π.

12. ________ The area of a circle with a diameter of $2\sqrt{13}$ feet equals $b\pi$ ft^2. What is the value of b?

13. ________ times If the diameter of a circle is multiplied by four, the area of the new circle is how many times that of the original circle?

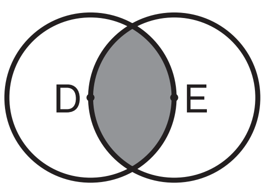
14. ________ The circumference of a certain circle with radius r cm is equal to the perimeter of a certain square with side length s cm. What is the ratio of r to s? Express your answer as a decimal to the nearest tenth.

15. ________ cm^2 Semicircle P has radius 5 cm, and circle Q is the largest possible circle that can be inscribed in semicircle P. What is the combined area of the shaded regions that are in the interior of semicircle P but exterior to circle Q? Express your answer as a common fraction in terms of π.



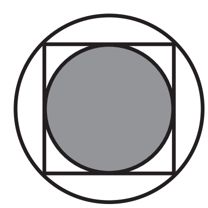
16. ________ inches Adele draws a circle with radius 10 inches. Bernie draws a circle that has four times the area of Adele's circle. What is the diameter of Bernie's circle?

17. ________

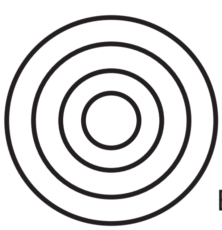


Circle D intersects the center of circle E, and circle E intersects the center of circle D. The radius of each circle is 6 cm. The area of the shaded region where the circles overlap can be expressed in simplest radical form as $a\pi + b\sqrt{c}$ cm^2. What is the value of $a + b + c$?

18. ________ mm The figure shows a circle inscribed in a square that is inscribed in a circle. If the larger circle has area 256π mm^2, what is the radius of the smaller, shaded circle? Express your answer in simplest radical form.



19. ________



Tomas shoots an arrow that lands in a random location on the target shown. The radius of the center circle is 1 foot, and the radius of each successively larger circle is 1 foot greater than that of the previous circle. What is the probability that Tomas's arrow will land in the center circle? Express your answer as a common fraction.

20. ________ units^2 A circle of radius 17 units has its center in the second quadrant. The circle intersects the y-axis at (0, 0) and $(0, 17\sqrt{2})$. What is the area of the region of the circle that lies in the third quadrant? Express your answer as a decimal to the nearest tenth.

Stars and Bars Stretch

Stars and bars, also known as balls and urns or sticks and stones, is a technique for finding the number of ways to distribute indistinguishable items (stars, balls, stones) among distinguishable containers (like urns) or distinguishable groups (separated by bars and sticks). The number of separators used is always one less than the number of containers or groups. Here are two examples of the stars and bars technique.

Here are arrangements of stars and bars that show four ways to distribute 8 pennies among 3 piggy banks.

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th></tr>
<tr><td>* * * * * * * * | |</td><td>8</td><td>0</td><td>0</td></tr>
<tr><td>* * * | * * | * * *</td><td>3</td><td>2</td><td>3</td></tr>
<tr><td>* * * | | * * * * *</td><td>3</td><td>0</td><td>5</td></tr>
<tr><td>| * * * * * * * * |</td><td>0</td><td>8</td><td>0</td></tr>
</table>

In how many ways can 8 pennies be distributed among 3 piggy banks?

The number of arrangements of 8 stars and 3 − 1 = 2 bars can be computed from the formula for permutations of items of different types. There are 8 + 2 = 10 items, 8 stars and 2 bars, and the formula tells us that the number of permutations is $\frac{(8+2)!}{8!2!} = \frac{(10 \times 9)8!}{8!(2 \times 1)} = 5 \times 9 = 45$.

The formula for combinations can also be applied here, viewing the problem as choosing the locations for the 2 bars among the 8 + 2 = 10 spots for stars or bars, leading to exactly the same computation.

$$\binom{s+b}{b} = \frac{(s+b)!}{s!b!}$$

In how many ways can a group of 10 balloons be made using red, orange, yellow, blue and purple balloons if there must be at least one balloon of each color?

We need only count the number of ways 5 of the balloons are colored since 5 have predetermined colors. The number of ways to choose 5 − 1 = 4 of the 5 + 4 = 9 items to be bars is $\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{(9 \times 8 \times 7 \times 6)5!}{(4 \times 3 \times 2 \times 1)5!} = 9 \times 7 \times 2 = 126$.

Use the stars and bars technique to solve the following problems.

21.________ ways In how many ways can 9 yellow marbles be divided among 4 distinguishable cups?

22.________ assort-ments A tray contains a dozen each of 3 kinds of cookies. How many different assortments of 7 cookies can Devon select from the tray?

23.________ assort-ments Baca's Bakery sells chocolate, vanilla, pumpkin and carrot cake muffins. How many different assortments of 10 muffins can Baca make?

24.________ assort-ments How many different assortments of pennies, nickels, dimes and quarters can Ashley's coin holder contain if it has 15 coins total?

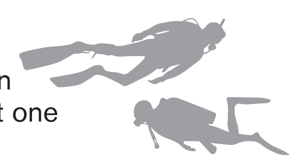
25.________ ways

At the flower shop, Maggie is making a bouquet from asters, dahlias, irises, roses and chrysanthemums. How many ways can Maggie choose 9 flowers for her bouquet if it should contain at least one of each type of flower?

26.________ combi-nations Carla selects 5 fruit-flavored candies from a bowl containing 6 apple, 5 banana and 4 cherry candies. How many possible combinations of candies can Carla select?

27.________ se-quences Lynn has 3 cats and a row of 4 cat beds. Each cat bed can hold one or two cats, and each cat is in a bed. Listing the number of cats in each bed from left to right, how many unique sequences are there?

28.________ ways Marvin randomly places 7 Scuba Steve action figures and 6 Diving Dan action figures in 4 distinguishable boxes. If each box must have at least one of each type of action figure, how many ways can Marvin do this?



29.________ ways Ms. Grow has 10 identical MATHCOUNTS pencils to distribute to 5 Mathletes. Two brothers, Minhtet and Linnhtet, must get the same number of pencils. If not every Mathlete is required to get a pencil, in how many ways can Ms. Grow distribute the pencils?

30.________ integers How many four-digit integers are there with a digit sum of 31?