DATA605: Fundamentals of Computational Mathematics

Assignment 8

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Problem 1.

A company buys 100 light bulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out?

```
bulbs <- 100
expected_lifetime <- 1000
rate <- 1/expected_lifetime
trials <- 50000

results <- array(dim = trials)
for (trial in 1:trials) {
  results[trial] <- min(rexp(bulbs, rate))
}

mean(results)
```

```
## [1] 9.983354
```

Problem 2.

Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density

$$f_Z(z) = (\frac{1}{2})\lambda e^{-\lambda|z|}$$

.

$$f_{X_1}(x) = f_{X_2}(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$f_{X_1}(x_1) * f_{X_2}(x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$$

Since $Z = X_1 - X_2$, $x_1 = z + x_2$. $\lambda^2 e^{-\lambda((z+x_2)+x_2)} = \lambda^2 e^{-\lambda(z+2x_2)}$.

If $z \geq 0$, $x_2 \geq 0$, so $\int_0^\infty \lambda^2 e^{-\lambda(z+2x_2)} = \frac{\lambda}{2} e^{-\lambda z}$.

Since X_1 and X_2 are independent, the distribution of $X_1 - X_2$ is the same as $X_2 - X_1$. This means that $f_Z(z) = f_Z(-z)$, so

$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}$$

Problem 3.

Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 100/3$. Using Chebyshev's Inequality, find an upper bound for the following probabilities.

> **Theorem 8.3 (Chebyshev Inequality)** Let X be a continuous random variable with density function $f(x)$. Suppose X has a finite expected value $\mu = E(X)$ and finite variance $\sigma^2 = V(X)$. Then for any positive number $\epsilon > 0$ we have
>
> $$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$
>
> .

a.

$P(|X - 10| \geq 2)$.

```
V = 100/3
e = 2
V/e^2
```

```
## [1] 8.333333
```

b.

$P(|X - 10| \geq 5)$.

```
V = 100/3
e = 5
V/e^2
```

```
## [1] 1.333333
```

c.

$P(|X - 10| \geq 9)$.

```
V = 100/3
e = 9
V/e^2
```

```
## [1] 0.4115226
```

d.

$P(|X - 10| \geq 20)$.

```r
V = 100/3
e = 20
V/e^2
```

```
## [1] 0.08333333
```