DATA605: Fundamentals of Computational Mathematics

Assignment 12

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Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

We can calculate the Taylor Series expansion by evaluating the first several derivatives of $f(x)$ to identify a pattern for the n-th derivative.

Problem 1

$$f(x) = \frac{1}{(1-x)}$$

Evaluating the first few derivatives, we can determine a general rule for the n-th derivative.

$$\begin{aligned} f(x) &= \frac{1}{1-x} & \Rightarrow f(0) = 1 \\ f'(x) &= (1-x)^{-2} & \Rightarrow f'(0) = 1 \\ f''(x) &= 2(1-x)^{-3} & \Rightarrow f''(0) = 2 \\ f'''(x) &= 6(1-x)^{-4} & \Rightarrow f'''(0) = 6 \\ f^{(n)}(x) &= n!(1-x)^{-(n+1)} & \Rightarrow f^{(n)}(0) = n! \end{aligned}$$

This allows us to generalize the series to:

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n$$

The series clearly diverges when $|x| >= 1$, so the interval of convergence is $(-1, 1)$.

Problem 2

$$f(x) = e^x$$

Evaluating the first few derivatives, we can determine a general rule for the n-th derivative.

$$
\begin{aligned}
f(x) &= e^x & \Rightarrow f(0) &= 1 \\
f'(x) &= e^x & \Rightarrow f'(0) &= 1 \\
f''(x) &= e^x & \Rightarrow f''(0) &= 1 \\
f'''(x) &= e^x & \Rightarrow f'''(0) &= 1 \\
f^{(n)}(x) &= e^x & \Rightarrow f^{(n)}(0) &= 1
\end{aligned}
$$

This allows us to generalize the series to:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using the ratio test to determine the interval of convergence for the series.

$$
\begin{aligned}
\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!}|/|\frac{x^n}{n!}| &= |\frac{x}{n+1}| \\
&= \frac{1}{n+1}|x| \\
&= 0|x|
\end{aligned}
$$

The series will converge for all values of x.

Problem 3

$$f(x) = \ln(1+x)$$

Evaluating the first few derivatives, we can determine a general rule for the n-th derivative.

$$
\begin{aligned}
f(x) &= \ln(1+x) & \Rightarrow f(0) &= 0 \\
f'(x) &= (1+x)^{-1} & \Rightarrow f'(0) &= 1 \\
f''(x) &= -1(1+x)^{-2} & \Rightarrow f''(0) &= -1 \\
f'''(x) &= -2(1+x)^{-3} & \Rightarrow f'''(0) &= -2 \\
f^{(n)}(x) &= (-1)^n(n-1)!(1+x)^{-n} & \Rightarrow f^{(n)}(0) &= (-1)^n(n-1)!
\end{aligned}
$$

This allows us to generalize the series to:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

Using the ratio test to determine the interval of convergence for the series.

$$
\begin{aligned}
\lim_{n\to\infty} |\frac{(-x)^{n+1}}{(n+1)}|/|\frac{(-x)^n}{n}| &= |\frac{(-x)n}{n+1}| \\
&= \frac{n}{n+1}|x| \\
&= 1|x|
\end{aligned}
$$

The ratio test requires that $|x| < 1$ for the series to converge.

Problem 4

$$f(x) = x^{(1/2)}$$

The function $f(x) = \sqrt{x}$ is not differentiable at $x = 0$ because the function is not defined for values of $x < 0$.