DATA605: Fundamentals of Computational Mathematics

Assignment 3

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Problem Set 1

1. What is the rank of the matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

The rank of a matrix is the maximum number of linearly independent columns of A, which can be found by counting the number of pivot columns in the reduced row echelon form of the matrix.

```
library(pracma)
A <- matrix(c(1,2,3,4,-1,0,1,3,0,1,-2,1,5,4,-2,-3), 4, 4, byrow = TRUE)
rref(A)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

The $rref(A)$ has 4 pivot columns, so the rank of A is 4.

2. Given an mxn matrix where $m > n$, what can be the maximum rank? The minimum rank, assuming that the matrix is non-zero?

The maximum rank for a matrix is the smaller of the number of columns or the number of rows, since $m > n$, the maximum rank is n. If the matrix is non-zero, the rank must be at least 1, which is achieved if the columns are all multiples of each other.

3. What is the rank of matrix B?

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

Just looking at B, you can tell that the rank is 1 because the columns are multiples of each other, $C_2 = 2C_1$ and $C_3 = C_1$.

Problem Set 2

Compute the eigenvalues and eigenvectors of the matrix A. You'll need to show your work. You'll need to write out the characteristic polynomial and show your solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

```r
A = matrix(c(1,2,3,0,4,5,0,0,6),3,3,byrow = TRUE)
```

$$\begin{aligned} Ax &= \lambda x && (1) \\ 0 &= \lambda x - Ax && (2) \\ 0 &= \lambda I_n x - Ax && (3) \\ 0 &= (\lambda I_n - A)x && (4) \end{aligned}$$

This has solutions when $det(\lambda I - A) = 0$.

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{vmatrix} &= (\lambda - 1) \begin{vmatrix} \lambda - 4 & -5 \\ 0 & \lambda - 6 \end{vmatrix} - 0 \begin{vmatrix} -2 & -3 \\ 0 & \lambda - 6 \end{vmatrix} + 0 \begin{vmatrix} -2 & -3 \\ \lambda - 4 & -5 \end{vmatrix} && (5) \\ &= (\lambda - 1)((\lambda - 4)(\lambda - 6) - (-5 * 0)) && (6) \\ &= (\lambda - 1)(\lambda - 4)(\lambda - 6) && (7) \end{aligned}$$

The characteristic polynomial is $\rho_A(x) = (x - 1)(x - 4)(x - 6)$.

For $\lambda = 1$

```r
lambda = 1
rref(lambda*diag(3)-A)
```

```
##      [,1] [,2] [,3]
## [1,]    0    1    0
## [2,]    0    0    1
## [3,]    0    0    0
```

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$v_2 = 0$
$v_3 = 0$

$$\xi_A(1) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

For $\lambda = 4$

```r
lambda = 4
rref(lambda*diag(3)-A)
```

```
##      [,1]       [,2] [,3]
## [1,]    1 -0.6666667    0
## [2,]    0  0.0000000    1
## [3,]    0  0.0000000    0
```

$$\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$v_1 - \frac{2}{3} v_2 = 0; v_1 = \frac{2}{3} v_2$
$v_3 = 0$

$$\xi_A(4) = \left\langle \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\rangle$$

For $\lambda = 6$

```r
lambda = 6
rref(lambda*diag(3)-A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0 -1.6
## [2,]    0    1 -2.5
## [3,]    0    0  0.0
```

$$\begin{bmatrix} 1 & 0 & -\frac{8}{5} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$v_1 - \frac{8}{5} v_3 = 0; v_1 = \frac{8}{5} v_3$
$v_2 - \frac{5}{2} v_3 = 0; v_2 = \frac{5}{2} v_3$

$$\xi_A(6) = \left\langle \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix} \right\rangle$$