DATA605: Fundamentals of Computational Mathematics
Assignment 10

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This power function will be used to perform repeated multiplication of our transition matrix.

```
power <- function(m, k) {
  r <- m
  for (i in 2:k) {
    r <- r %*% m
  }
  return(r)
}
```

Problem 1.

Smith is in jail and has 1 dollar; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with a probability .4 and loses A dollars with probability .6. Find the probability that he wins 8 dollars before losing all of his money if:

We will construct the transition matrix as a 9x9 where the row and column represents the number of dollars that Smith has. The first row, and column, represents 0 dollars and the 9th row represents 8 dollars, which is enough to post bail. If Smith has 0 dollars remaining, he will be unable to place more bets, so he has 100% chance of remaining with 0 dollars. Similarly, if he has 8 dollars, he has enough to post bail and doesn't need to make more bets. We see that this is an absorbing chain with final states either $0 or $8.

a. he bets 1 dollar each time (timid strategy)

When Smith has between 1 and 7 dollars, he will bet $1 and has a 40% chance of winning.

Here is the transitional matrix for the timid strategy.

```
t_timid = rbind(c(1,0,0,0,0,0,0,0,0),
                c(.6,0,.4,0,0,0,0,0,0),
                c(0,.6,0,.4,0,0,0,0,0),
                c(0,0,.6,0,.4,0,0,0,0),
                c(0,0,0,.6,0,.4,0,0,0),
                c(0,0,0,0,.6,0,.4,0,0),
                c(0,0,0,0,0,.6,0,.4,0),
                c(0,0,0,0,0,0,.6,0,.4),
                c(0,0,0,0,0,0,0,0,1))

t_timid
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
## [2,]  0.6  0.0  0.4  0.0  0.0  0.0  0.0  0.0  0.0
## [3,]  0.0  0.6  0.0  0.4  0.0  0.0  0.0  0.0  0.0
## [4,]  0.0  0.0  0.6  0.0  0.4  0.0  0.0  0.0  0.0
## [5,]  0.0  0.0  0.0  0.6  0.0  0.4  0.0  0.0  0.0
## [6,]  0.0  0.0  0.0  0.0  0.6  0.0  0.4  0.0  0.0
## [7,]  0.0  0.0  0.0  0.0  0.0  0.6  0.0  0.4  0.0
## [8,]  0.0  0.0  0.0  0.0  0.0  0.0  0.6  0.0  0.4
## [9,]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
```

We will repeatedly multiply the transition matrix to generate the probability of reaching each final state.

```
t_timid_10000 <- power(t_timid,10000)
t_timid_10000
```

```
##            [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]        [,9]
## [1,] 1.0000000    0    0    0    0    0    0    0 0.00000000
## [2,] 0.9796987    0    0    0    0    0    0    0 0.02030135
## [3,] 0.9492466    0    0    0    0    0    0    0 0.05075337
## [4,] 0.9035686    0    0    0    0    0    0    0 0.09643140
## [5,] 0.8350515    0    0    0    0    0    0    0 0.16494845
## [6,] 0.7322760    0    0    0    0    0    0    0 0.26772403
## [7,] 0.5781126    0    0    0    0    0    0    0 0.42188739
## [8,] 0.3468676    0    0    0    0    0    0    0 0.65313243
## [9,] 0.0000000    0    0    0    0    0    0    0 1.00000000
```

Since Smith starts with $1, he has a 2.03% chance of earning enough to post bail.

b. he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy)

Here is the transitional matrix for the bold strategy.

```
t_bold = rbind(c(1,0,0,0,0,0,0,0,0),
               c(.6,0,.4,0,0,0,0,0,0),
               c(.6,0,0,0,.4,0,0,0,0),
               c(.6,0,0,0,0,0,.4,0,0),
               c(.6,0,0,0,0,0,0,0,.4),
               c(0,0,.6,0,0,0,0,0,.4),
               c(0,0,0,0,.6,0,0,0,.4),
               c(0,0,0,0,0,0,.6,0,.4),
               c(0,0,0,0,0,0,0,0,1))

t_bold
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]  1.0    0  0.0    0  0.0    0  0.0    0  0.0
## [2,]  0.6    0  0.4    0  0.0    0  0.0    0  0.0
## [3,]  0.6    0  0.0    0  0.4    0  0.0    0  0.0
## [4,]  0.6    0  0.0    0  0.0    0  0.4    0  0.0
```

```
## [5,]  0.6    0  0.0    0  0.0    0  0.0    0  0.4
## [6,]  0.0    0  0.6    0  0.0    0  0.0    0  0.4
## [7,]  0.0    0  0.0    0  0.6    0  0.0    0  0.4
## [8,]  0.0    0  0.0    0  0.0    0  0.6    0  0.4
## [9,]  0.0    0  0.0    0  0.0    0  0.0    0  1.0
```

Looking at this matrix, it's interesting to note that it is impossible to reach the states of \$1, \$3, \$5, or \$7.

Again, we will repeatedly multiply the transition matrix to generate the probability of reaching each final state.

```
t_bold_10000 <- power(t_bold,10000)
t_bold_10000
```

```
##         [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]  [,9]
## [1,] 1.000    0    0    0    0    0    0    0 0.000
## [2,] 0.936    0    0    0    0    0    0    0 0.064
## [3,] 0.840    0    0    0    0    0    0    0 0.160
## [4,] 0.744    0    0    0    0    0    0    0 0.256
## [5,] 0.600    0    0    0    0    0    0    0 0.400
## [6,] 0.504    0    0    0    0    0    0    0 0.496
## [7,] 0.360    0    0    0    0    0    0    0 0.640
## [8,] 0.216    0    0    0    0    0    0    0 0.784
## [9,] 0.000    0    0    0    0    0    0    0 1.000
```

Since Smith starts with \$1, he has a 6.4% chance of earning enough to post bail.

It's interesting to note that in this case, starting with \$1, we only needed to raise the transition matrix to the third power, because if Smith lost any of his first 3 bets, he'd be at \$0, so the only way to reach \$8 is to win all 3 bets. Constructing a tree diagram for each possible starting amount, shows that 3 is the most needed for any amount. You can see that this is the case because the $t_{bold}^3 = t_{bold}^{10000}$.

```
t_bold_3 <- power(t_bold, 3)
t_bold_3
```

```
##         [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]  [,9]
## [1,] 1.000    0    0    0    0    0    0    0 0.000
## [2,] 0.936    0    0    0    0    0    0    0 0.064
## [3,] 0.840    0    0    0    0    0    0    0 0.160
## [4,] 0.744    0    0    0    0    0    0    0 0.256
## [5,] 0.600    0    0    0    0    0    0    0 0.400
## [6,] 0.504    0    0    0    0    0    0    0 0.496
## [7,] 0.360    0    0    0    0    0    0    0 0.640
## [8,] 0.216    0    0    0    0    0    0    0 0.784
## [9,] 0.000    0    0    0    0    0    0    0 1.000
```

Which strategy gives Smith the better chance of getting out of jail?

The bold strategy gives Smith the better chance. It's easy to see why that is the case because in the timid strategy he needs to win at least 7 bets, while in the bold strategy he only needs to win 3.