

# Introduction to linear regression

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The Human Freedom Index is a report that attempts to summarize the idea of “freedom” through a bunch of different variables for many countries around the globe. It serves as a rough objective measure for the relationships between the different types of freedom - whether it’s political, religious, economical or personal freedom - and other social and economic circumstances. The Human Freedom Index is an annually co-published report by the Cato Institute, the Fraser Institute, and the Liberales Institut at the Friedrich Naumann Foundation for Freedom.

In this lab, you’ll be analyzing data from Human Freedom Index reports from 2008-2016. Your aim will be to summarize a few of the relationships within the data both graphically and numerically in order to find which variables can help tell a story about freedom.

## Getting Started

### Load packages

In this lab, you will explore and visualize the data using the **tidyverse** suite of packages. The data can be found in the companion package for OpenIntro resources, **openintro**.

Let’s load the packages.

```
library(tidyverse)
library(openintro)
data('hfi', package='openintro')
```

### The data

The data we’re working with is in the openintro package and it’s called **hfi**, short for Human Freedom Index.

1. What are the dimensions of the dataset?

```
dim(hfi)
```

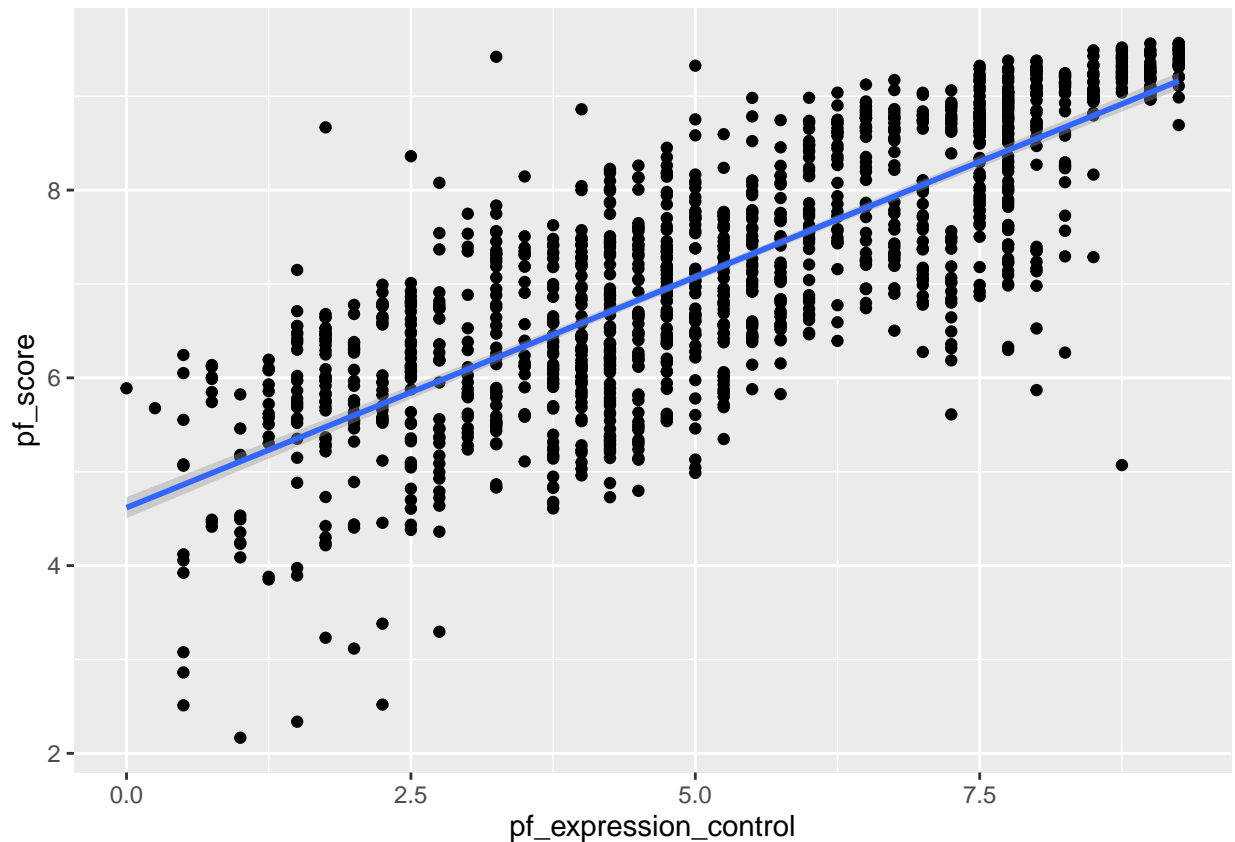
```
## [1] 1458 123
```

There are 1458 observations with 123 variables.

2. What type of plot would you use to display the relationship between the personal freedom score, **pf\_score**, and one of the other numerical variables? Plot this relationship using the variable **pf\_expression\_control** as the predictor. Does the relationship look linear? If you knew a country’s **pf\_expression\_control**, or its score out of 10, with 0 being the most, of political pressures and controls on media content, would you be comfortable using a linear model to predict the personal freedom score?

I would use a scatter plot to display the relationship between the personal freedom score and the expression control variables.

```
hfi %>%  
  ggplot(aes(pf_expression_control, pf_score)) +  
  geom_point() +  
  geom_smooth(method = lm)
```



The relationship could be linear and may be a good model to predict the personal freedom score. If the relationship looks linear, we can quantify the strength of the relationship with the correlation coefficient.

```
hfi %>%  
  summarise(cor(pf_expression_control, pf_score, use = "complete.obs"))  
  
## # A tibble: 1 x 1  
##   'cor(pf_expression_control, pf_score, use = "complete.obs")'  
##                                     <dbl>  
## 1                                     0.796
```

Here, we set the `use` argument to “complete.obs” since there are some observations of NA.

## Sum of squared residuals

In this section, you will use an interactive function to investigate what we mean by “sum of squared residuals”. You will need to run this function in your console, not in your markdown document. Running the function also requires that the `hfi` dataset is loaded in your environment.

Think back to the way that we described the distribution of a single variable. Recall that we discussed characteristics such as center, spread, and shape. It's also useful to be able to describe the relationship of two numerical variables, such as `pf_expression_control` and `pf_score` above.

3. Looking at your plot from the previous exercise, describe the relationship between these two variables. Make sure to discuss the form, direction, and strength of the relationship as well as any unusual observations.

With a correlation of .796, there is a strong positive relationship between expression control and the personal freedom score. As expression control increases, so do the personal freedom score. There are a few outliers, such as the observation with expression control value about 8 and personal freedom score of about 5.

Just as you've used the mean and standard deviation to summarize a single variable, you can summarize the relationship between these two variables by finding the line that best follows their association. Use the following interactive function to select the line that you think does the best job of going through the cloud of points.

```
# This will only work interactively (i.e. will not show in the knitted document)
hfi <- hfi %>% filter(complete.cases(pf_expression_control, pf_score))
DATA606::plot_ss(x = hfi$pf_expression_control, y = hfi$pf_score)
```

After running this command, you'll be prompted to click two points on the plot to define a line. Once you've done that, the line you specified will be shown in black and the residuals in blue. Note that there are 30 residuals, one for each of the 30 observations. Recall that the residuals are the difference between the observed values and the values predicted by the line:

$$e_i = y_i - \hat{y}_i$$

The most common way to do linear regression is to select the line that minimizes the sum of squared residuals. To visualize the squared residuals, you can rerun the plot command and add the argument `showSquares = TRUE`.

```
DATA606::plot_ss(x = hfi$pf_expression_control, y = hfi$pf_score, showSquares = TRUE)
```

Note that the output from the `plot_ss` function provides you with the slope and intercept of your line as well as the sum of squares.

4. Using `plot_ss`, choose a line that does a good job of minimizing the sum of squares. Run the function several times. What was the smallest sum of squares that you got? How does it compare to your neighbors?

After several attempts, the smallest sum of squares I was able to obtain was 974.

## The linear model

It is rather cumbersome to try to get the correct least squares line, i.e. the line that minimizes the sum of squared residuals, through trial and error. Instead, you can use the `lm` function in R to fit the linear model (a.k.a. regression line).

```
m1 <- lm(pf_score ~ pf_expression_control, data = hfi)
```

The first argument in the function `lm` is a formula that takes the form  $y \sim x$ . Here it can be read that we want to make a linear model of `pf_score` as a function of `pf_expression_control`. The second argument specifies that R should look in the `hfi` data frame to find the two variables.

The output of `lm` is an object that contains all of the information we need about the linear model that was just fit. We can access this information using the summary function.

```
summary(m1)
```

```
##
## Call:
## lm(formula = pf_score ~ pf_expression_control, data = hfi)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -3.8467 -0.5704  0.1452  0.6066  3.2060
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)      4.61707    0.05745   80.36  <2e-16 ***
## pf_expression_control 0.49143    0.01006   48.85  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8318 on 1376 degrees of freedom
## (80 observations deleted due to missingness)
## Multiple R-squared:  0.6342, Adjusted R-squared:  0.634
## F-statistic: 2386 on 1 and 1376 DF, p-value: < 2.2e-16
```

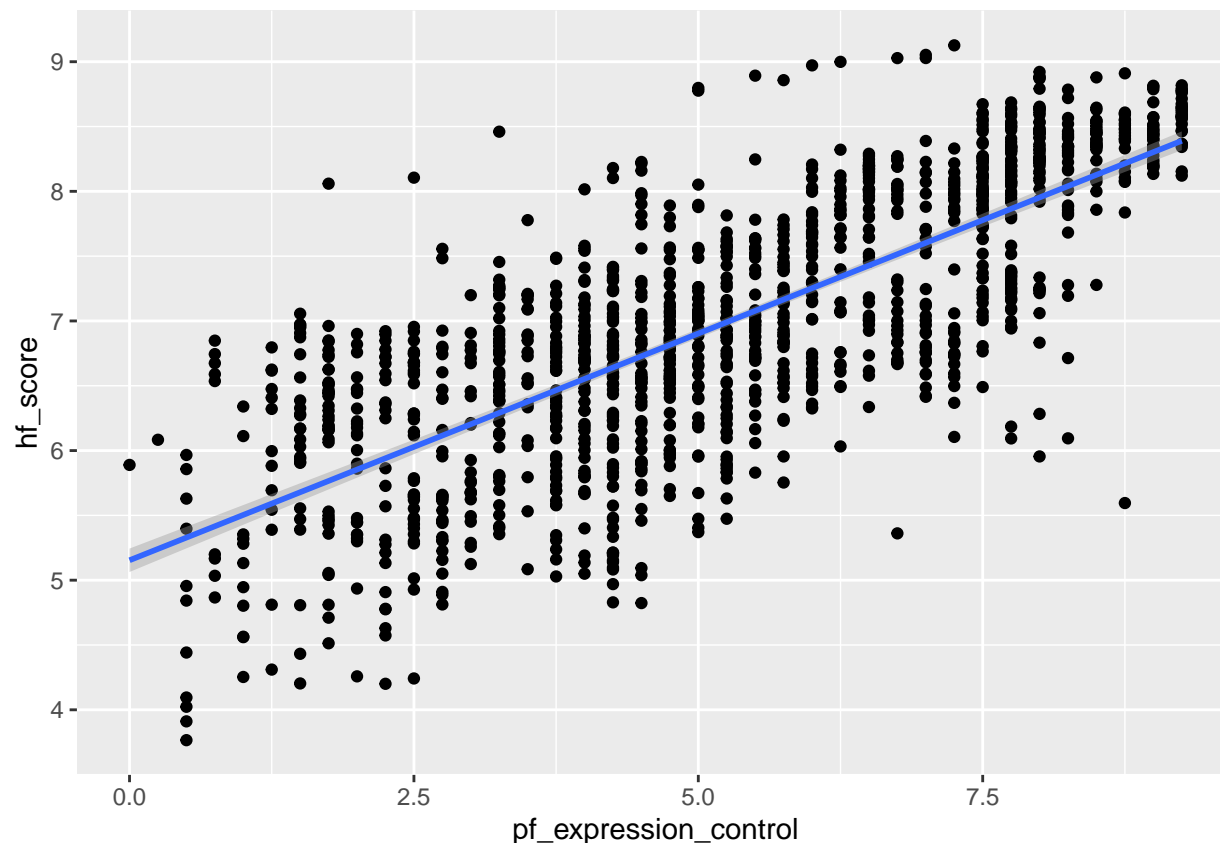
Let's consider this output piece by piece. First, the formula used to describe the model is shown at the top. After the formula you find the five-number summary of the residuals. The "Coefficients" table shown next is key; its first column displays the linear model's y-intercept and the coefficient of `at_bats`. With this table, we can write down the least squares regression line for the linear model:

$$\hat{y} = 4.61707 + 0.49143 \times pf\_expression\_control$$

One last piece of information we will discuss from the summary output is the Multiple R-squared, or more simply,  $R^2$ . The  $R^2$  value represents the proportion of variability in the response variable that is explained by the explanatory variable. For this model, 63.42% of the variability in personal freedom is explained by expression control.

5. Fit a new model that uses `pf_expression_control` to predict `hf_score`, or the total human freedom score. Using the estimates from the R output, write the equation of the regression line. What does the slope tell us in the context of the relationship between human freedom and the amount of political pressure on media content?

```
hfi %>%
  ggplot(aes(pf_expression_control, hf_score)) +
  geom_point() +
  geom_smooth(method = lm)
```



```
m2 <- lm(hf_score ~ pf_expression_control, data = hfi)
```

```
summary(m2)
```

```
##
## Call:
## lm(formula = hf_score ~ pf_expression_control, data = hfi)
##
## Residuals:
```

	Min	1Q	Median	3Q	Max
	-2.6198	-0.4908	0.1031	0.4703	2.2933

```
##
## Coefficients:
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	5.153687	0.046070	111.87	<2e-16 ***
pf_expression_control	0.349862	0.008067	43.37	<2e-16 ***

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.667 on 1376 degrees of freedom
## (80 observations deleted due to missingness)
## Multiple R-squared:  0.5775, Adjusted R-squared:  0.5772
## F-statistic: 1881 on 1 and 1376 DF, p-value: < 2.2e-16
```

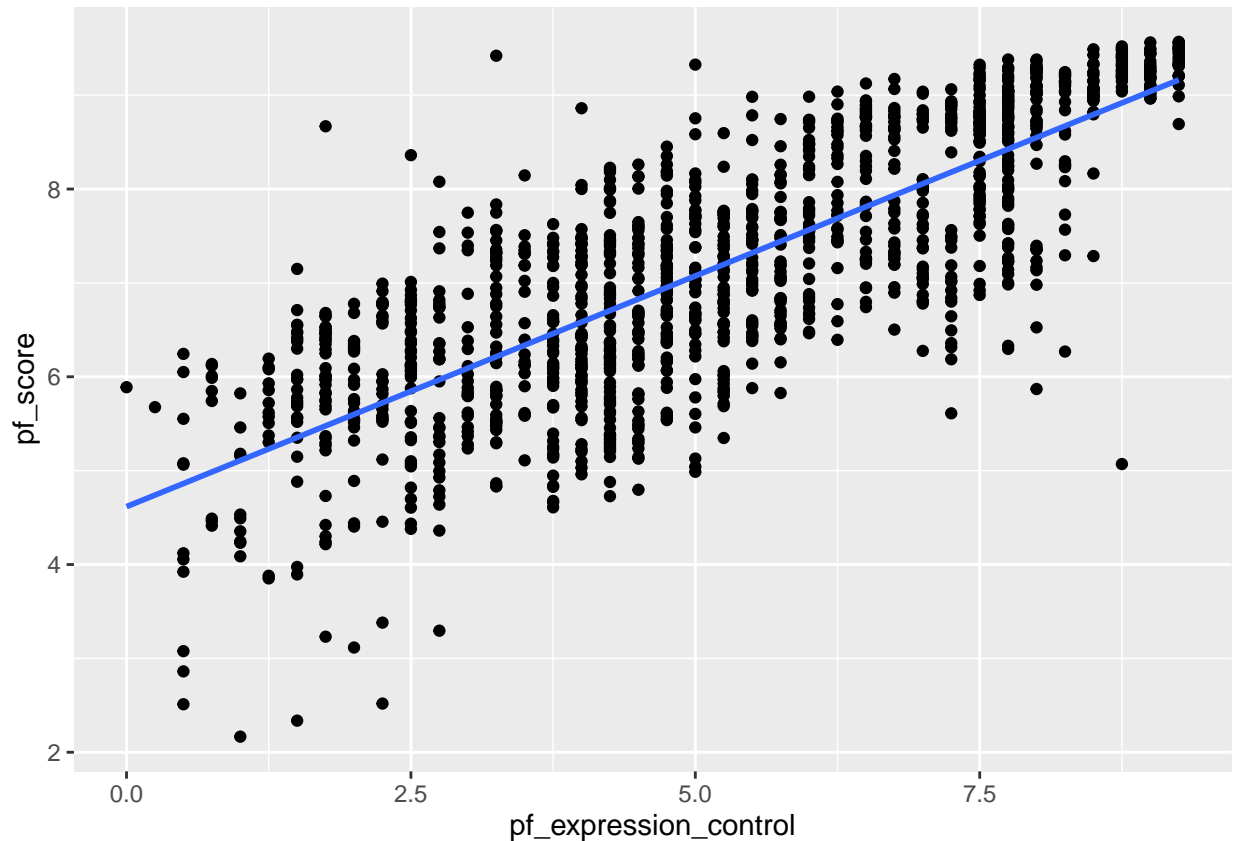
$$\hat{y} = 5.153687 + .349862 \times pf\_expression\_control$$

For each unit of increased expression control, the human freedom score increases by .35.

## Prediction and prediction errors

Let's create a scatterplot with the least squares line for `m1` laid on top.

```
ggplot(data = hfi, aes(x = pf_expression_control, y = pf_score)) +  
  geom_point() +  
  stat_smooth(method = "lm", se = FALSE)
```



Here, we are literally adding a layer on top of our plot. `geom_smooth` creates the line by fitting a linear model. It can also show us the standard error `se` associated with our line, but we'll suppress that for now.

This line can be used to predict  $y$  at any value of  $x$ . When predictions are made for values of  $x$  that are beyond the range of the observed data, it is referred to as *extrapolation* and is not usually recommended. However, predictions made within the range of the data are more reliable. They're also used to compute the residuals.

6. If someone saw the least squares regression line and not the actual data, how would they predict a country's personal freedom school for one with a 6.7 rating for `pf_expression_control`? Is this an overestimate or an underestimate, and by how much? In other words, what is the residual for this prediction?

```
prediction <- 4.61707 + 0.49143 * 6.75
```

```
residuals <- hfi %>%
  filter(pf_expression_control == 6.75) %>%
  select(pf_expression_control, pf_score) %>%
  mutate(residual = pf_score - prediction)

sum(residuals$residual)
```

```
## [1] 2.451148
```

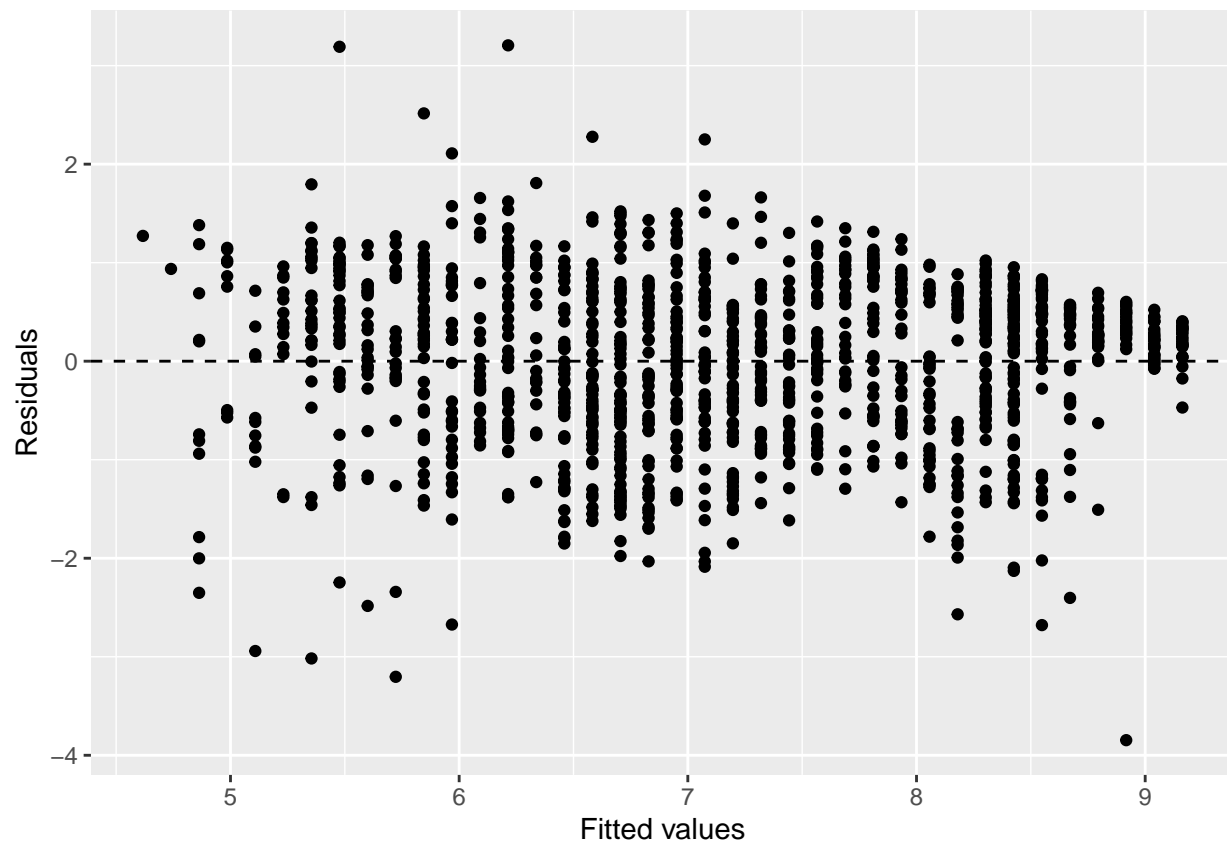
There were no observations for a expression control score of 6.7, but there were 34 observations for 6.75. The linear model would predict a personal freedom score of 7.9342225 for a expression control score of 6.75. The sum of residuals for the 34 observations is 2.45, so the model is underestimating the personal freedom score when the expression control is 6.75.

## Model diagnostics

To assess whether the linear model is reliable, we need to check for (1) linearity, (2) nearly normal residuals, and (3) constant variability.

**Linearity:** You already checked if the relationship between `pf_score` and 'pf\_expression\_control' is linear using a scatterplot. We should also verify this condition with a plot of the residuals vs. fitted (predicted) values.

```
ggplot(data = m1, aes(x = .fitted, y = .resid)) +
  geom_point() +
  geom_hline(yintercept = 0, linetype = "dashed") +
  xlab("Fitted values") +
  ylab("Residuals")
```



Notice here that `m1` can also serve as a data set because stored within it are the fitted values ( $\hat{y}$ ) and the residuals. Also note that we're getting fancy with the code here. After creating the scatterplot on the first layer (first line of code), we overlay a horizontal dashed line at  $y = 0$  (to help us check whether residuals are distributed around 0), and we also rename the axis labels to be more informative.

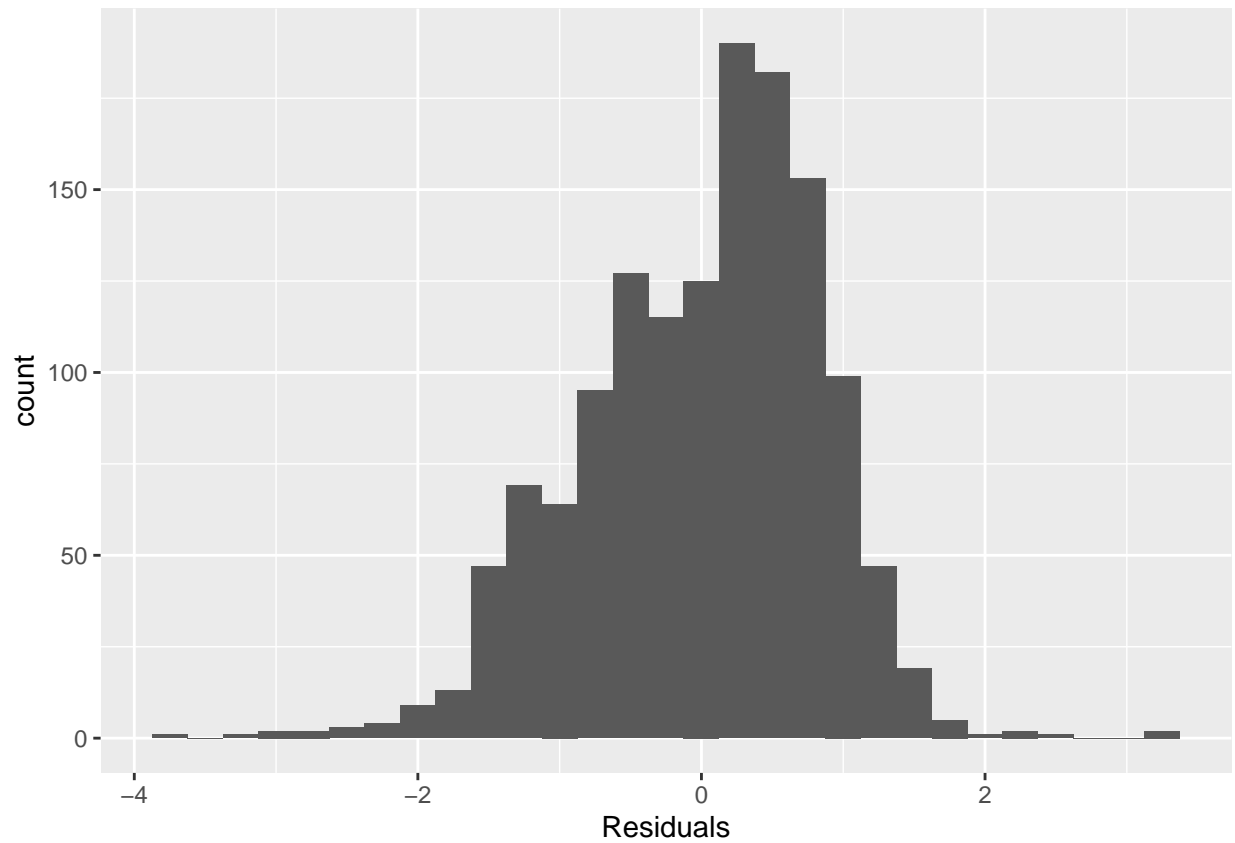
7. Is there any apparent pattern in the residuals plot? What does this indicate about the linearity of the relationship between the two variables?

There doesn't appear to be a pattern within the residuals. If there were a pattern, we could conclude that the linear model chosen was not a good fit for the data set.

**Nearly normal residuals:** To check this condition, we can look at a histogram

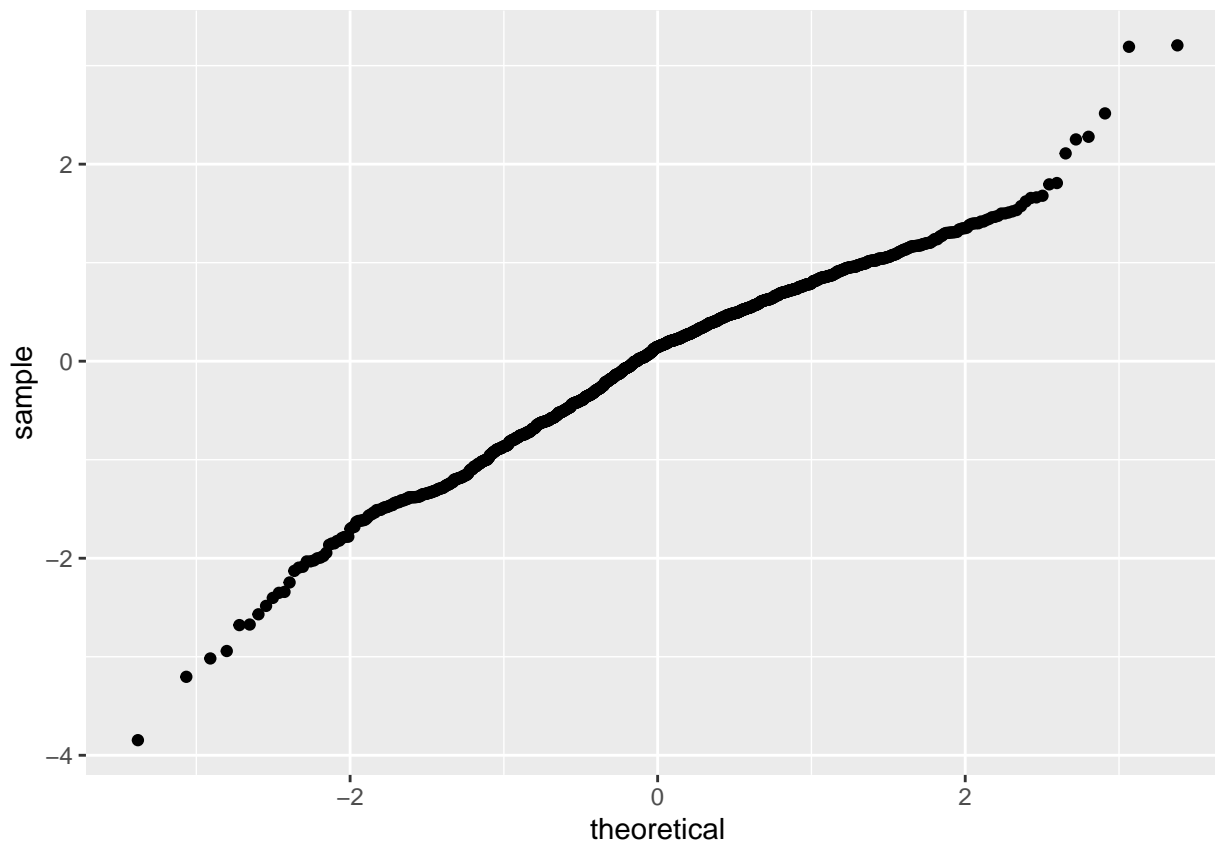
```
ggplot(data = m1, aes(x = .resid)) +
  geom_histogram(binwidth = .25) +
  xlab("Residuals")
```





or a normal probability plot of the residuals.

```
ggplot(data = m1, aes(sample = .resid)) +  
  stat_qq()
```



Note that the syntax for making a normal probability plot is a bit different than what you're used to seeing: we set `sample` equal to the residuals instead of `x`, and we set a statistical method `qq`, which stands for “quantile-quantile”, another name commonly used for normal probability plots.

8. Based on the histogram and the normal probability plot, does the nearly normal residuals condition appear to be met?

Yes, the histogram and normal probability plot both appear to indicate that the residuals are distributed nearly normal.

#### Constant variability:

9. Based on the residuals vs. fitted plot, does the constant variability condition appear to be met?

Yes, there doesn't appear to be a correlation between the fitted values and the residuals.

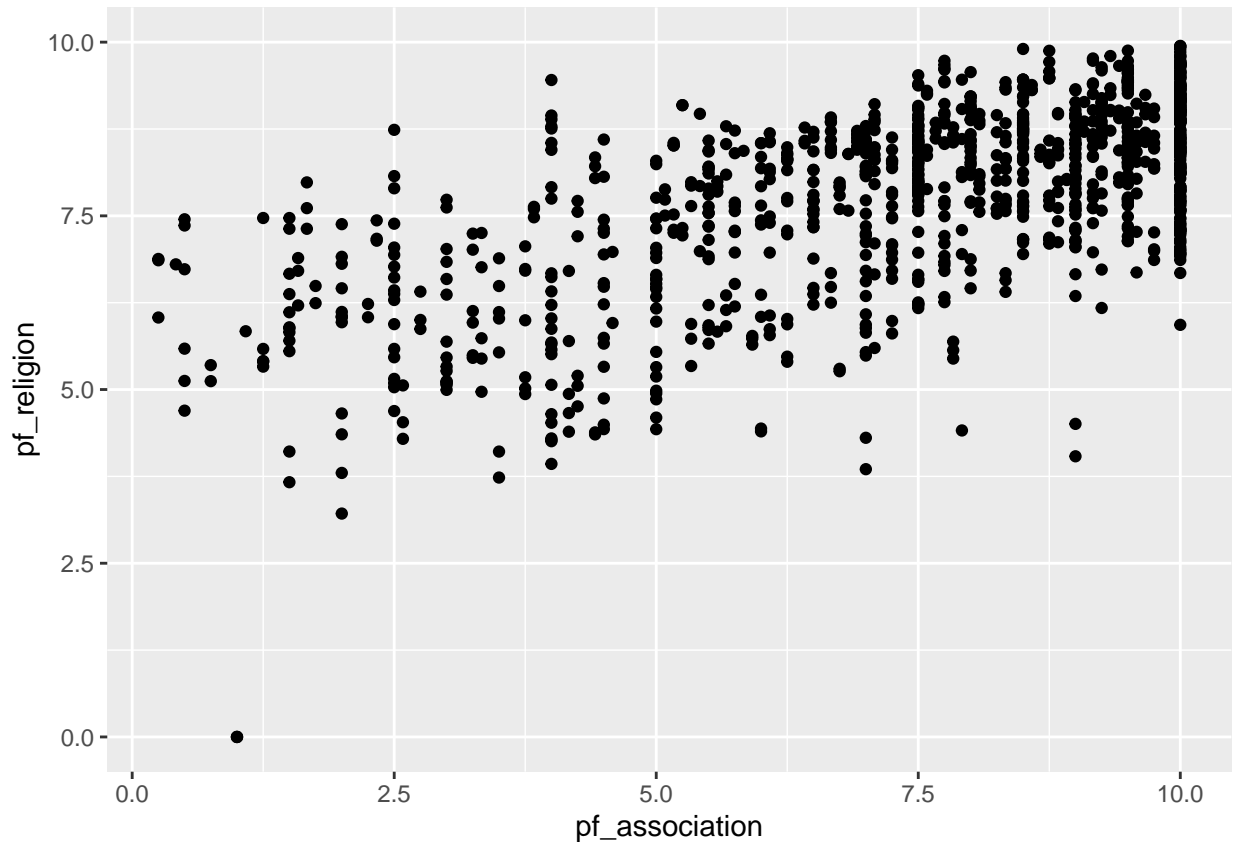
---

### More Practice

- Choose another freedom variable and a variable you think would strongly correlate with it.. Produce a scatterplot of the two variables and fit a linear model. At a glance, does there seem to be a linear relationship?

Comparing freedom to peacefully assemble and associate with freedom of religion.

```
hfi %>%  
  ggplot(aes(pf_association, pf_religion)) +  
  geom_point()
```



There does appear to be a linear relationship between the two variables.

- How does this relationship compare to the relationship between `pf_expression_control` and `pf_score`? Use the  $R^2$  values from the two model summaries to compare. Does your independent variable seem to predict your dependent one better? Why or why not?

```
m3 <- lm(pf_religion ~ pf_association, data = hfi)
```

```
summary(m3)
```

```
##  
## Call:  
## lm(formula = pf_religion ~ pf_association, data = hfi)  
##  
## Residuals:  
##      Min       1Q   Median       3Q      Max   
## -5.5230 -0.6077  0.1735  0.7458  2.8684   
##  
## Coefficients:
```

```
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)      5.1684      0.0979   52.79  <2e-16 ***
## pf_association    0.3546      0.0125   28.36  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.044 on 1127 degrees of freedom
## (329 observations deleted due to missingness)
## Multiple R-squared:  0.4164, Adjusted R-squared:  0.4159
## F-statistic: 804.2 on 1 and 1127 DF,  p-value: < 2.2e-16
```

The  $R^2$  value for the relationship for `pf_association` and `pf_religion` is significantly less than the  $R^2$  value for the relationship between `pf_expression_control` and `pf_score` which indicates that the relationship is not as strong.

- What's one freedom relationship you were most surprised about and why? Display the model diagnostics for the regression model analyzing this relationship.

Constructed the  $R^2$  for all combinations of relationships to identify surprising results.

```
r <- tibble()

for (col1 in 5:122) {
  for (col2 in (col1+1):123) {
    col1_name <- names(hfi)[col1]
    col2_name <- names(hfi)[col2]

    try({
      m <- lm(as.formula(paste(col1_name, '~', col2_name, sep='')), data = hfi)
      rsquared <- summary(m)$r.squared

      r <- r %>%
        rbind(tibble_row(col1 = col1, col1_name = col1_name, col2 = col2, col2_name = col2_name, rsquared = rsquared))
    })
  }
}
```

```
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
## Error in lm.fit(x, y, offset = offset, singular.ok = singular.ok, ...) :
## 0 (non-NA) cases
```

```
r <- r %>%
  arrange(desc(rsquared))

head(r,12)
```

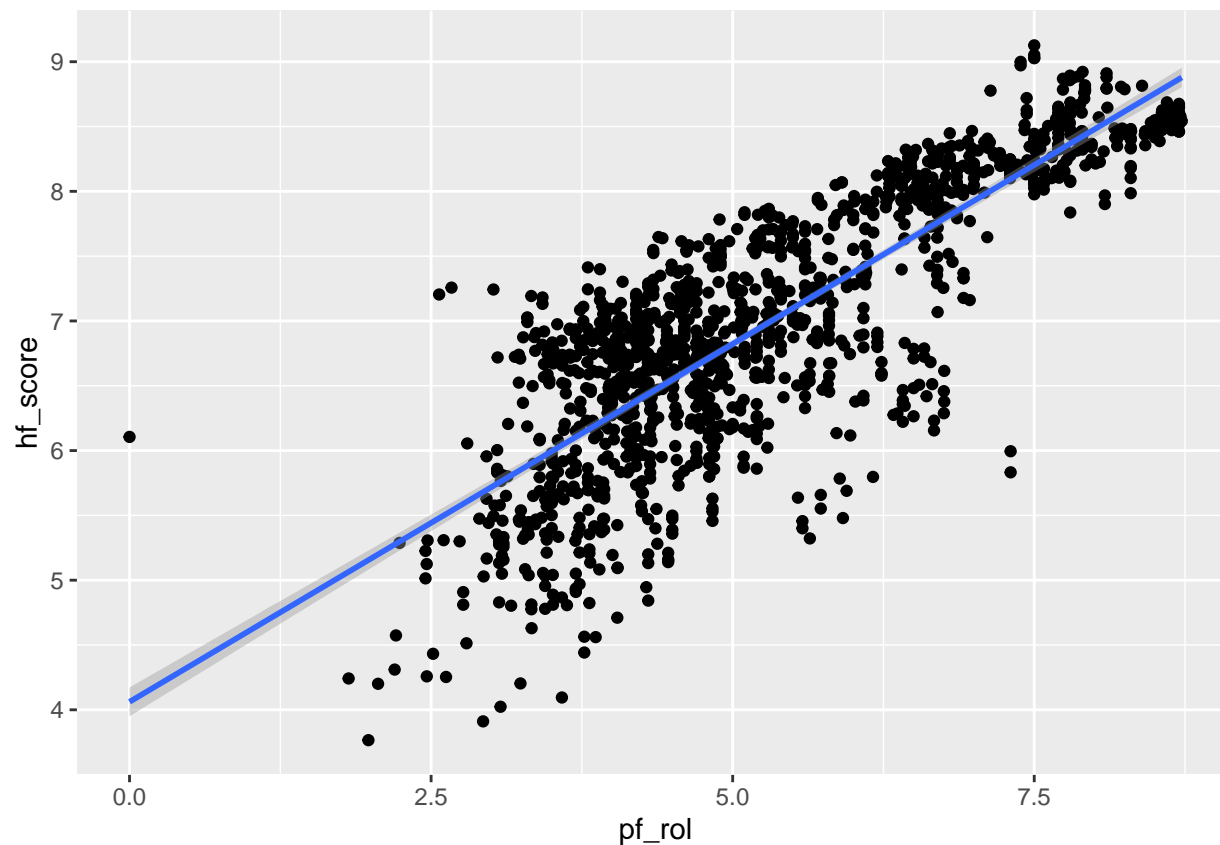
```
## # A tibble: 12 x 5
##   col1 col1_name          col2 col2_name          rsquared
##   <int> <chr>              <int> <chr>              <dbl>
## 1    62 pf_score          63 pf_rank            0.973
## 2   121 hf_score          122 hf_rank            0.969
## 3    19 pf_ss_women_inheritance_daughters 20 pf_ss_women_inheritan~ 0.958
## 4    18 pf_ss_women_inheritance_widows    20 pf_ss_women_inheritan~ 0.958
## 5   122 hf_rank          123 hf_quartile          0.930
## 6    54 pf_identity_parental_marriage      56 pf_identity_parental    0.927
## 7    55 pf_identity_parental_divorce      56 pf_identity_parental    0.916
## 8   119 ef_score          120 ef_rank            0.901
## 9    92 ef_trade_regulatory_compliance     93 ef_trade_regulatory    0.894
## 10   63 pf_rank          122 hf_rank            0.891
## 11   62 pf_score          121 hf_score            0.889
## 12  121 hf_score          123 hf_quartile          0.886
```

```
tail(r,12)
```

```
## # A tibble: 12 x 5
##   col1 col1_name          col2 col2_name          rsquared
##   <int> <chr>              <int> <chr>              <dbl>
## 1    31 pf_religion_restrictions          93 ef_trade_regulatory    2.39e-7
## 2    27 pf_religion_estop_establish        94 ef_trade_black        2.24e-7
## 3    17 pf_ss_women_missing              77 ef_legal_restrictions 2.22e-7
## 4    17 pf_ss_women_missing              91 ef_trade_regulatory_no~ 1.81e-7
## 5    97 ef_trade_movement_visit          104 ef_regulation_labor_mi~ 1.50e-7
## 6    16 pf_ss_women_fgm                  45 pf_expression_killed   8.29e-8
## 7    14 pf_ss_disappearances_injuries     27 pf_religion_estop_esta~ 7.39e-8
## 8    31 pf_religion_restrictions          110 ef_regulation_labor   6.37e-8
## 9    53 pf_identity_legal                 101 ef_regulation_credit_p~ 4.06e-8
## 10   16 pf_ss_women_fgm                   17 pf_ss_women_missing    2.63e-8
## 11   69 ef_government_tax                 87 ef_trade_tariffs_reven~ 5.70e-9
## 12   11 pf_ss_disappearances_violent      41 pf_association_sport_e~ 1.92e-9
```

The rule of law statistic is highly related to the human freedom score with  $R^2 = .677$ .

```
hfi %>%
  ggplot(aes(pf_rol,hf_score)) +
  geom_point() +
  geom_smooth(method = lm)
```



```
m4 <- lm(pf_rol ~ hf_score, data = hfi)
```

```
summary(m4)
```

```
##
## Call:
## lm(formula = pf_rol ~ hf_score, data = hfi)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -4.2197 -0.5834 -0.0184  0.5596  3.4199
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -3.27328    0.16127  -20.30  <2e-16 ***
## hf_score      1.22728    0.02282   53.79  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8685 on 1376 degrees of freedom
## (80 observations deleted due to missingness)
## Multiple R-squared:  0.6777, Adjusted R-squared:  0.6775
## F-statistic: 2893 on 1 and 1376 DF, p-value: < 2.2e-16
```