Understanding Analysis - Chapter 1 Notes

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1 The Real Numbers

1.3 The Axiom of Completeness

What is $\mathbb{R}$? The author talking about challenges around providing precise definitions, and at some point one has to draw an arbitrary line and accept that as a starting point. Detailing a bit of the history, saying that it was an intuitive understanding of $\mathbb{R}$ that really led the way, followed by methods for rigorously constructing $\mathbb{R}$ from the set of rational numbers Q.

1.3.1 An Initial Definition for R

$\mathbb{R}$ is an extension of Q, meaning that every element in $\mathbb{R}$ has an additive inverse and every nonzero element has a multiplicative inverse. $\mathbb{R}$ is a *field*, where addition and multiplication are commutative, associative, and the distributive property holds. This gives us algebra and logical orderings, such as "If $a < b$ and $c > 0$, then $ac < bc$". Finally, we need a way of insisting that $\mathbb{R}$ does not contain the gaps in its number line that Q contain.

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound.*

1.3.2 Least Upper Bounds and Greatest Lower Bounds

Beginning with definitions.

Definition 1. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A. Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 2. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A.

(ii) if b is any upper bound for A, then $s \leq b$.

Least upper bound also referred to as the *supremum* of the set A, also $s = \text{lub}A$. This text will use $s = \text{sup}A$. $s = \text{inf}A$ will be used to denote lower bound.

Okay so the upper and lower bounds are just the highest and lowest elements in the set, because, for highest: $a \leq b$ for all $a \in A$ and all $b \in \mathbb{R}$.

Oh he goes on to show how this intuition isn't always true.

Example 1.

$$A = \left\{ \frac{1}{n} : n \in N \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}.$$

The set A is bounded above and below. The upper bound is 1. The lower bound is more difficult... it would be $\frac{1}{\infty}$ or 0.

A lesson to note here is that the sup and inf of a set are not always elements of that set.

Definition 3. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Example 2. To further illustrate the point between bounds and maxima / minima, consider the open interval:

$$(0, 2) = \{x \in \mathbb{R} : 0 < x < 2\},$$

and the closed interval

$$[0, 2] = \{x \in \mathbb{R} : 0 \leq x \leq 2\}.$$

Both of these sets are bounded in both directions, but only one set (the closed interval) has a maximum. There is no element in the open interval that is the maximum of the set.

Axiom of Completeness asserts that every nonempty bounded set has a least upper bound.

An axiom is meant to be a statement that's so clear or intuitive that it can be accepted on its face and needs no proof.

Left off around p. 17.