Understanding Analysis - Chapter 1 Notes

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1 The Real Numbers

1.3 The Axiom of Completeness

What is $\mathbb{R}$? The author talking about challenges around providing precise definitions, and at some point one has to draw an arbitrary line and accept that as a starting point. Detailing a bit of the history, saying that it was an intuitive understanding of $\mathbb{R}$ that really led the way, followed by methods for rigorously constructing $\mathbb{R}$ from the set of rational numbers Q.

1.3.1 An Initial Definition for R

$\mathbb{R}$ is an extension of Q, meaning that every element in $\mathbb{R}$ has an additive inverse and every nonzero element has a multiplicative inverse. $\mathbb{R}$ is a *field*, where addition and multiplication are commutative, associative, and the distributive property holds. This gives us algebra and logical orderings, such as "If $a < b$ and $c > 0$, then $ac < bc$". Finally, we need a way of insisting that $\mathbb{R}$ does not contain the gaps in its number line that Q contain.

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound.*

1.3.2 Least Upper Bounds and Greatest Lower Bounds

Beginning with definitions.

Definition 1. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A. Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 2. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A.

(ii) if b is any upper bound for A, then $s \leq b$.

Least upper bound also referred to as the *supremum* of the set A, also $s = \text{lub}A$. This text will use $s = \sup A$. $s = \inf A$ will be used to denote lower bound.

Okay so the upper and lower bounds are just the highest and lowest elements in the set, because, for highest: $a \leq b$ for all $a \in A$ and all $b \in \mathbb{R}$.

Oh he goes on to show how this intuition isn't always true.

Example 1.

$$A = \left\{ \frac{1}{n} : n \in N \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}.$$

The set A is bounded above and below. The upper bound is 1. The lower bound is more difficult... it would be $\frac{1}{\infty}$ or 0.

A lesson to note here is that the sup and inf of a set are not always elements of that set.

Definition 3. A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Example 2. To further illustrate the point between bounds and maxima / minima, consider the open interval:

$$(0, 2) = \{x \in \mathbb{R} : 0 < x < 2\},$$

and the closed interval

$$[0, 2] = \{x \in \mathbb{R} : 0 \leq x \leq 2\}.$$

Both of these sets are bounded in both directions, but only one set (the closed interval) has a maximum. There is no element in the open interval that is the maximum of the set.

Axiom of Completeness asserts that every nonempty bounded set has a least upper bound.

An axiom is meant to be a statement that's so clear or intuitive that it can be accepted on its face and needs no proof.

Example 3. Consider the set:

$$S = \{r \in Q : r^2 < 2\}$$

There are plenty of possible upper bounds, anything $b \geq 2$ will do. But we can't find a least upper bound where $b \in Q$, because the least upper bound should be $r = \sqrt{2}$, which is irrational.

Example 4. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Then $\sup(c + A) = c + \sup A$.

Need to verify the two aspects of least upper bound definition, namely that s is an upper bound for A, and that if b is any upper bound for A, then $s \leq b$.

(Not looking at the explanation). I guess we need to show that $c + \sup A$ qualifies as an upper bound of $c + A$. So $c + \sup A \geq a + c$ for all $a \in A$. I can propose some $s = \sup A$, such that $s \geq a$ for all $a \in A$, meaning it's necessarily true that $s + c \geq a + c$. I think that proves the first component of the definition.

Next is to show that for any $b \geq A + c$ that $s \leq b$. (looking at explanation) then $b - c \geq A$, so $b - c$ is an upper bound on A. Because s is the least upper bound, we can write $s \leq b - c$, which can be rewritten as $s + c \leq b$, thus proving part two.

Lemma 1. Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof. The idea is given that s is an upper bound on set A, it is the least upper bound if and only if any number less than s is in set A and is not an upper bound.

Prove it forwards, meaning that if $s = \sup A$ then for some arbitrarily chosen $\epsilon > 0$, there should be some $a \in A$ where $s - \epsilon < a$. We can note how $s - a < s$, and if $s = \sup A$, then anything less than s is not an upper bound, and there should be some $a \in A$ where $a > s - \epsilon$ (because otherwise $s - \epsilon$ would be an upper bound).

Proving it backwards. Which means that if there exists some $a \in A$ which, for every choice $\epsilon > 0$, satisfies $s - \epsilon < a$, then $s = \sup A$. I can observe that $s < a + \epsilon$.

(Their solution). Assume s is an upper bound with the property that no matter how $\epsilon > 0$ is chosen, $s - \epsilon$ is no longer an upper bound for A. Notice that what this implies is that if b is any number less than s, then b is not an upper bound. (Just let $\epsilon = s - b$). To prove that $s = \sup A$, we must verify part (ii) of the definition (which is that if b is any upper bound, that $s \leq b$.) Because we have just argued that any number smaller than s cannot be an upper bound, it follows that if b is some other upper bound for A, then $s \leq b$.

Struggling to follow this one, but okay.

□

Exercises

1. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

   Definition 4. i is the greatest upper bound of set $A \subseteq \mathbb{R}$ if:

   i. i is a lower bound on set A

   ii. For any other lower bound b on set A, $i \geq b$.

   (b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

   Lemma 2. Assume $i \in \mathbb{R}$ is a lower bound on set $A \subseteq \mathbb{R}$. $i = \inf A$ if and only if for every choice of $\epsilon > 0$, $i + \epsilon \geq a$ for some $a \in A$.

   Proof. Okay so let's prove it forwards. Which is to say if $i = \inf A$, then for every $\epsilon > 0$, $i + \epsilon \geq a$ for some $a \in A$. If $i = \inf A$, then $i + a > i$. Since any number plus i can't be a lower bound if i is the greatest lower bound, then we know that if any number is added to i it will no longer be a lower bound and there should be some $a \in A$ where $i + \epsilon \geq a$.

   Proving it backwards. Assume $i \in \mathbb{R}$ is a lower bound on set $A \subseteq \mathbb{R}$. If for every choice of $\epsilon > 0$, $i + \epsilon \geq a$ for some $a \in A$, then $i = \inf A$. Intuitively, it's clear that, if i is a lower bound, and any nudge in the positive direction causes i to no longer be a lower bound, then i must be the greatest lower bound. I can also propose an arbitrary lower bound b. Part (ii) of the definition requires that $i \geq b$. Then I guess this is self evident because anything greater than i would no longer be a lower bound... ? But I struggle with that step because $i = \inf A$ is the conclusion we're trying to prove here, but the solution kind of relies on that as a premise to support the claim $i \geq b$. Puzzling.

   □

2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.
I think the only thing that satisfies this is a set with one element (eg, $B = \{3\}$. Such that $\inf B = 3 = \sup B$.

(b) A finite set that contains its infimum but not its supremum.

Hmmm. Maybe a set with one closed interval and one open? $A = [0, 4)$.

Okay the answer is that this is impossible because finite sets contain their upper and lower bounds.

(c) A bounded subset of Q that contains its supremum but not its infimum.

My intuition is to answer this one similarly to the last, which suggests that it's probably wrong. The only question is whether the set with an open interval is technically rational or not, because you can't really pick out a number on the rational number line to end the set.

Let $B = \{r \in Q | 1 < r \leq 2\}$ we have $\inf B = 1 \notin B$ and $\sup B = 2 \in B$.

3. (a) Let A be nonempty and bounded below, and define $B = \{b \in R : b \text{is a lower bound for} A\}$. Show that $\sup B = \inf A$.

I'm struggling with this one because in order for $\sup B = \inf A$ we'd need to show that A and B perfectly overlap at their upper and lower bounds, and I can't see that from the definitions.

The solution just appeals to the definition and says it's self evident. But all we know is that some element of B, b, is a lower bound of A. We could have $A \cap B \neq \emptyset$ meaning that $\sup B > \inf A$. Or $A \cap B = \emptyset$ such that $\sup B < \inf A$. I reject the question.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

It seems like it should just be the inverse of the definition. Every non empty set of real numbers that is bounded below has a greatest lower bound. Even some set set $A \subseteq (3, 5]$, the greatest lower bound is is irrational but is still a real number.

4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for sup $(A_1 \cup A_2)$. Extend this to sup $(\bigcup_{k=1}^{n} A_k)$.

$$\sup\ (A_1 \cup A_2) = \max(\sup(A_1),\ \sup(A_2))$$
$$\sup\ \left(\bigcup_{k=1}^{n} A_k\right) = \sup\ \{\sup\ A_k | k = 1, \dots, n\}$$

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

No, because set $\bigcup_{k=1}^{\infty} A_k$ might be unbounded, for example with $A_n = [n, n+1]$.

Skipping ahead.

1.4 Consequences of Completeness

Theorem 1. (Nested Interval Property). For each $n \in N$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

In human terms. You start with some interval $I_n = [a_n, b_n]$ in which there is some element x. Now you propose infinite subset intervals of I_n just like closing in on themselves. The idea is to show that, even in those infinitely closing intervals, there is some $a_n \leq x \leq b_n$ such that there exists some intersection of all of those intervals $\bigcap_{k=1}^{\infty} I_n \neq \emptyset$.