Tensors:

i) Tensors - (p,q)-tensor $\Rightarrow H^{\mu\nu}{}_{\lambda\rho}{}^{\sigma}$

Remember that $\underbrace{V^\mu \quad V_\mu}$, $\underbrace{r \quad \tilde{r}^T}_{SO(1,3)}$ $\Rightarrow$ $V_\mu V^\mu =$ invariant, $\tilde{r}^T r =$ invariant

$$V_\mu T^{\mu\nu} \longrightarrow V_{\mu'} T^{\mu'\nu'}$$

Some tensors can be represented by matrices $V^\mu = [\;]$ $V_\mu = [\qquad]$

$T^{\mu\nu}$ two vector indices
$T^\mu{}_\nu$ one vector index and one dual index
$T_{\mu\nu}$ two dual indices.

- $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$
  $V_\mu \to V_{\mu'} = \Lambda_{\mu'}{}^\mu V_\mu$

- The metric tensor: $g_{\mu\nu} V^\mu = V_\nu$, $g^{\mu\nu} V_\nu = V^\mu$ $\Rightarrow g^{\mu\nu} = (g_{\mu\nu})^{-1}$

$g_{\mu\nu} \to g_{\mu'\nu'} = \Lambda_{\mu'}{}^\mu \Lambda_{\nu'}{}^\nu = g^{\mu\nu}$ in special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Trace $\eta_{\mu\nu}\eta^{\mu\nu} = 4$

- Transformations $\Lambda^{\mu'}{}_\mu$, $\Lambda_{\mu'}{}^\mu = (\Lambda^{\mu'}{}_\mu{}^{-1})^T$

Index notation and matrices:

2D: $dx^\mu = \begin{pmatrix} dx \\ dy \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta\, dx + \sin\theta\, dx \\ -\sin\theta\, dx + \cos\theta\, dy \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$

$\Lambda^{1'}{}_1 = \cos\theta$, $\Lambda^{1'}{}_2 = \sin\theta$, $\Lambda^{2'}{}_1 = -\sin\theta$, $\Lambda^{2'}{}_2 = \cos\theta$

$\Lambda^{\text{row}}{}_{\text{column}}$

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_\mu \, dx^\mu = \Lambda^{\mu'}{}_1 \, dx + \Lambda^{\mu'}{}_2 \, dy$$

$$dx^{1'} = \Lambda^{1'}{}_1 \, dx + \Lambda^{1'}{}_2 \, dy \quad \text{and} \quad dx^{2'} = \Lambda^{2'}{}_1 \, dx + \Lambda^{2'}{}_2 \, dy$$

- Note that:

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_\mu \, dx^\mu = dx^\mu \, \Lambda^{\mu'}{}_\mu$$

- $\eta_{\mu\nu} = \eta_{\nu\mu}$ only with the metric in general $H_{\mu\nu} \neq H_{\nu\mu}$

Some important vectors:

$$\partial_\mu = \left(\partial/c\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z \right) \qquad (0,1)\text{-tensor or dual tensor}$$

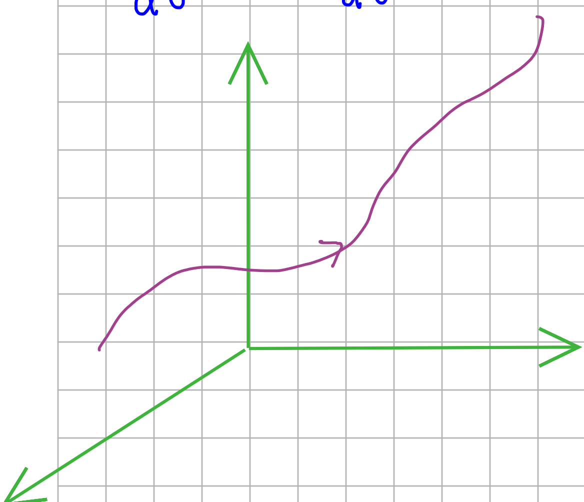
$$\partial_\mu \to \partial_{\mu'} = \Lambda^{\mu}{}_{\mu'} \, \partial_\mu \qquad \begin{cases} 4 \text{ Gradient}: & \partial_\mu \phi \to (0,1)\text{-tensor} \\ 4 \text{ Divergence}: & \partial_\mu A^\mu \to (0,0) \text{ tensor.} \end{cases}$$

To naively generalize to 4D: $d\vec{r} \to dx^\mu = \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix}$ $(1,0)$ tensor

What is $\dfrac{dx^\mu}{dt}$?

$$\frac{dx^\mu}{dt} \longrightarrow \frac{dx^{\mu}}{dt} = \Lambda^{\mu}{}_\mu \frac{dx^\mu}{dt}$$

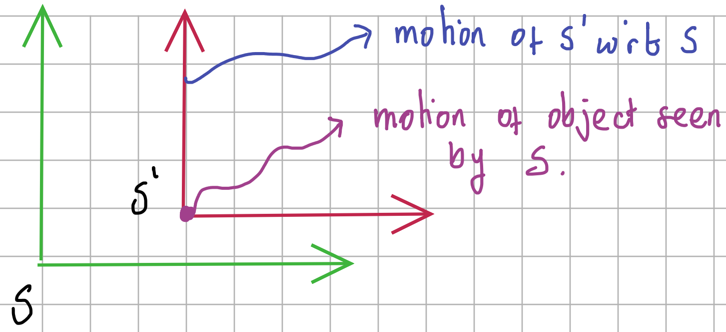
But this one does not transform as a tensor.



$$\frac{d}{dt} \Rightarrow \frac{d}{d\tau} \quad \text{where} \quad \tau = \int \sqrt{-ds^2}$$

$$ds^2 = -c^2 dt + dx^2 + dy^2 + dz^2 = \text{Invariant.}$$

Now $$u^\mu = c \frac{dx^\mu}{d\tau}.$$



As seen in S:

$$U^0 = c^2\, dt/d\tau$$
$$U^1 = c\, dx/d\tau$$
$$U^2 = c\, dy/d\tau$$
$$U^3 = c\, dz/d\tau.$$

For example: $U^1 = c\,\frac{dx}{dt}\frac{dt}{d\tau} = c\,V_x$ Something.

$$-ds^2 = d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \rightarrow \frac{d\tau^2}{dt^2} = c^2 - v^2$$

$$\frac{dt^2}{d\tau^2} = \frac{1}{c^2-v^2} = \frac{1}{c^2(1-\beta^2)} = \frac{\gamma^2}{c^2} \quad \text{and} \quad \frac{dt}{d\tau} = \frac{\gamma}{c}.$$

In this case $U^\mu = \begin{bmatrix} \gamma c \\ \gamma \vec{v} \end{bmatrix}$

- **Momentum:** $p^\mu = m U^\mu = \begin{pmatrix} mc\gamma \\ m\gamma\vec{v} \end{pmatrix} \Rightarrow p^\mu_{rest} = \begin{pmatrix} mc \\ 0 \end{pmatrix}$

For $\frac{v}{c} \ll 1$ $\quad \gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots \Rightarrow p^\mu = \begin{pmatrix} \frac{1}{c}(mc^2 + \frac{1}{2}mv^2 + \dots) \\ mv \end{pmatrix}$

and $p^\mu = \begin{pmatrix} \frac{1}{c}\ \text{Energy relativistic} \\ \vec{P}\ \text{relativistic} \end{pmatrix}$ $\quad E_{rest} = mc^2$

Since p^μ is a vector, $p_\mu p^\mu$ is a scalar.

$$p_\mu = (-m\gamma c,\ m\gamma\vec{v}).$$

In particular $p_\mu p^\mu = -m^2c^2 = \frac{E^2}{c^2} + p^2$, and finally we get:

$$E^2 - p^2c^2 = m^2c^4$$

$$p_\mu p^\mu \begin{cases} <0 & m^2>0 \\ =0 & m^2=0 \\ >0 & m^2<0. \end{cases}$$

Collisions :

$$p_i^\mu = p_f^\mu$$

- **Scattering Matrix :** $S = \lim_{t\to\infty} T\left[\exp\left(i\int_{-\infty}^{\infty} dt'\, H_I(t')\right)\right]$

- Wick : $T[\phi(x_1)\phi(x_2)] = :\phi(x_1)\phi(x_2): + \phi(x_1)\phi(x_2)$

Remember that:

- $\phi(x_1)\phi(x_2) = i D_F(x_1 - x_2) = \int \frac{dp^4}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} e^{-ip(x_1-x_2)}$

- $\psi(x_1)\overline{\psi}(x_2) = i S_F(x_1 - x_2) = i\int \frac{d^4p}{(2\pi)^4} \frac{p^\mu \gamma_\mu + m\mathbb{1}}{p^2 - m^2 + i\epsilon}$

- $A_\mu(x_1) A_\nu(x_2) = i D^F_{\mu\nu}(x_1 - x_2) = \int_p \frac{-i(\eta_{\mu\nu} - (3-1) p_\mu p_\nu / p^2)}{p^2 + i\epsilon}$

QFT (Quantum electrodynamics):

→ Fermions $(\overline{\psi}, \psi)$ $\qquad \mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$

If we propose the changes: $\psi \to \psi' = e^{i\theta}\psi$
$\overline{\psi} \to \overline{\psi} = e^{-i\theta}\psi$

$g = e^{i\theta} \in U(1)$ (Circle rotations)

Using this in the Lagrangian: $\mathcal{L}' = i\overline{\psi}'\gamma^\mu \partial_\mu \psi' - m\overline{\psi}'\psi$

$$\mathcal{L}' = i e^{-i\theta} \overline{\psi}\gamma^\mu \partial_\mu e^{i\theta}\psi - m e^{-i\theta}\overline{\psi} e^{i\theta}\psi$$

$$\mathcal{L} = i\psi\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

- **Noether theorem :** Global symmetry → Quantity conserved.

Conserved current: $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi + T^\mu{}_\nu \Delta x$

- $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\dot\psi)}$

$\psi' = e^{i\theta}\psi = (1+i\theta)\psi = \psi + i\theta\psi$

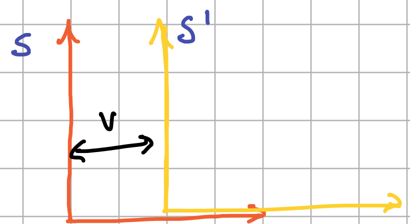
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar\psi\gamma^\mu \Longrightarrow J^\mu = i\bar\psi\gamma^\mu\psi$$

Conserved Current: $J^\mu = ie\bar\psi\gamma^\mu\psi$.

Local: $\psi \to e^{i\theta(x)}\psi \; ; \; \bar\psi \to e^{-i\theta(x)}\bar\psi$

Relativity:

- The laws of physics are equally valid in all inertial reference frames.



$t = t' = 0$
when $x = x' = 0$

Some event occurs at position (x, y, z) and time t in S

What are the space time coordinates (x', y', z') and t' of this same event in S'?

Lorentz Transformations:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

- If two events occur at the same time in S, but at different locations, then they do not occur at the same time in S'.

If
$$t_A' = \gamma\left(t_A - \frac{v}{c^2}x_A\right)$$
$$t_B' = \gamma\left(t_B - \frac{v}{c^2}x_B\right)$$

$$t_A = \gamma\left(t_A' + \frac{v}{c^2}x_A'\right)$$
$$t_B = \gamma\left(t_B' + \frac{v}{c^2}x_B'\right)$$

If $t_A = t_B$

$$t_A' = -\frac{v}{c^2} x_A' + t_B + \frac{v}{c^2} x_B'$$

$$t_A' = t_B + \frac{v}{c^2}(x_B' - x_A') = t_B + \frac{v}{c^2}\left([\gamma(x_B - v t_B)] - \gamma[x_A - v t_A]\right)$$

$$\boxed{t_A' = t_B' - \frac{\gamma v}{c^2}(x_B - x_A)}$$

Events that are simultaneous in one inertial system are not simultaneous in others.

Lorentz contractions: Suppose a stick lies on the x' axis, at rest in S'. What is the length as measured in S?

$x_{Left} = 0$, right end $x' = \gamma(x - vt) \Rightarrow \gamma x = L' \Rightarrow x = L'/\gamma$

Length of the stick is $L = L'/\gamma$ in S.

Since $\gamma \geq 1 \Rightarrow$ A moving object is shortened by a factor γ, as compared with its length in the system which is at rest.

Time dilation: Suppose an interval in S' $t' \in [0, T']$
How long is this period as measured in S?

$$t = \gamma\left(t' + \frac{v}{c^2} x'\right) \rightarrow \text{it occurs at } x' = 0$$

$$t = \gamma\left(T' + \frac{v}{c^2}(0)\right) \Rightarrow t = \gamma T' \Rightarrow T = \gamma T'$$

Moving clocks run slow.

In particle physics: A moving particle lasts longer than it would at rest.

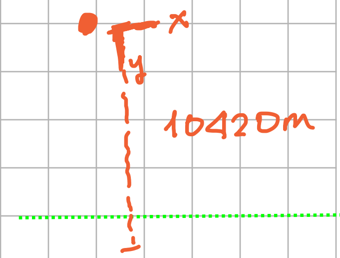
Cosmic ray muons are produced high in the atmosphere (at 8000m, say) and travel toward the earth at very nearly the speed of light (0.998c)

Given the lifetime of the muon (2.2×10^{-6} s), how far would it go before disintegrating? (Classical view)

$$x = v \cdot t = 658\text{m}.$$

Using relativistic physics:

$$x' = \gamma (x - vt) \quad , \quad \gamma^2 = \frac{1}{1 - v^2/c^2}$$



$$x = \gamma (x' + vt)$$

$$x = \frac{1}{\left(1 - (0.998)^2\right)^{1/2}} \left(0.998c \left(2.2 \times 10^{-6}\,s\right)\right) = 10420\,m.$$

- Velocity addition: Suppose a particle is moving in the x direction at speed u' with respect to S'. What is the speed, u, with respect to S?

$$\Delta x = \gamma (\Delta x' + v \Delta t') \quad \text{and} \quad \Delta t = \gamma [\Delta t' + v/c^2 \, \Delta x']$$

so

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v \Delta t'}{\Delta t' + v/c^2 \Delta x'} = \frac{(\Delta x'/\Delta t') + v}{1 + v/c^2 (\Delta x'/\Delta t')}$$

$$u = \frac{u' + v}{1 + v/c^2 u'} \Rightarrow \boxed{u = \frac{u' + v}{1 + v u'/c^2}}$$

If $u' = c \Rightarrow u = \frac{(c + v)\, c^2}{c^2 + cv} = \frac{c^2 (c + v)}{c (c + v)} = c$

$\boxed{u = c}$ The speed of light is the same in all inertial systems.

$$a^2 \equiv a \cdot a = (a^0)^2 - \mathbf{a}^2 \begin{cases} a^2 > 0 & a^\mu \text{ timelike} \\ a^2 < 0 & a^\mu \text{ spacelike} \\ a^2 = 0 & a^\mu \text{ lightlike} \end{cases}$$

Energy and momentum:

Proper time $d\tau = \frac{dt}{\gamma}$

Particle physics → Laboratory, Particle

Usually we work with the proper time because τ is invariant.

- Velocity of the particle (wrt Laboratory) : Distance it travels measured in lab frame divided by the time it takes:

$$\vec{v} = \frac{d\vec{x}}{dt}$$

$\Rightarrow$ Proper velocity: Distance traveled (lab frame) divided by proper time

$$\eta \equiv \frac{d\vec{x}}{d\tau}. \qquad \vec{\eta} = \gamma \vec{v}$$

$$\eta = \frac{d\vec{x}}{dt/\gamma} = \gamma \vec{v}$$

Proper velocity is part of the four vector :

$$\eta^{\mu} = \frac{dx^{\mu}}{d\tau} \qquad x^{\mu} = (ct, \vec{x})$$

$$\eta^{\mu} = \gamma \frac{dx^{\mu}}{dt} = \gamma (c, \vec{v})$$ and $$\eta_{\mu}\eta^{\mu} = \gamma^2(c^2 - v_x^2 - v_y^2 - v_z^2)$$
$$= \gamma^2 c^2 (1 - v^2/c^2) = c^2.$$

$$p^{\mu} = m\eta^{\mu}$$

Spatial components: $\vec{p} = m\gamma\vec{v}$ (Relativistic momentum)

Temporal component: $p^0 = \gamma m c$

$\Rightarrow$ Relativistic energy: $E = \gamma m c^2$

Momentum and energy together make up a four vector

$$p^{\mu} = \left(\frac{E}{c}, \vec{p}\right)$$

$$p^{\mu} p_{\mu} = \frac{E^2}{c^2} - p^2 = m^2 c^2$$

Collisions: In relativistic collisions, energy and momentum are always Conserved.

$$p_A^{\mu} + p_B^{\mu} = p_C^{\mu} + p_D^{\mu}$$

$\Rightarrow$ $\pi^0 \rightarrow \gamma + \gamma$ initial mass is 135 GeV, but final mass is zero.

REST ENERGY $\rightarrow$ KINETIC ENERGY.

Problem 1: Two lumps of clay, each of mass m, collide head-on at 3/5 c. They stick together

What is the mass M of the composite lump?

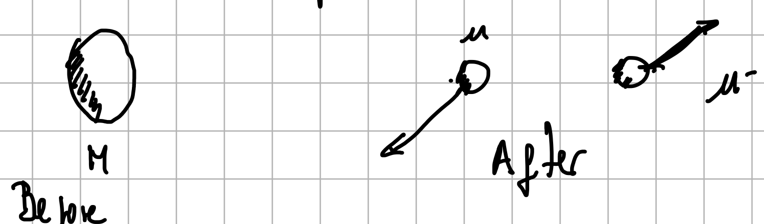
$E_1 + E_2 = E_M$, $\vec{p}_1 + \vec{p}_2 = \vec{p}_T$ $\vec{p}_1 = -\vec{p}_2$

$$p_1^{\mu} + p_2^{\mu} = p_f \Rightarrow \left(\frac{E_1}{c}, \vec{p}_1\right) \cdot \left(\frac{E_2}{c}, -\vec{p}_1\right) = \frac{E_1 E_2}{c^2} + p_1^2 = Mc^2$$

$$2E_m = Mc^2 \rightarrow Mc^2 = \frac{2mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}(2mc^2)$$

$$\boxed{M = \frac{5}{2}m}$$

Problem 3 A pion at rest decays into a muon plus a neutrino
What is the speed of the muon?



Momentum : $\vec{P}_\pi = \vec{P}_\mu + \vec{P}_\nu$ $\vec{P}_\pi = 0$ $\vec{P}_\mu = -\vec{P}_\nu$

$E_\pi = E_\mu + E_\nu$ Muon and Neutrino fly back to back. with equal and opposite momenta.

Suggestion: To get the energy of a particle, when you know its momentum use:

$$E^2 - p^2c^2 = m^2c^4$$

$$E_\pi = m_\pi c^2$$

$$E_\mu = c\sqrt{m_\mu^2 c^2 + p_\mu^2}$$

But since $\vec{p}_\mu = -\vec{p}_\nu$

$$E_\nu = |\vec{p}_\nu| c = c|\vec{p}_\mu|$$

$$E_\pi = E_\mu + E_\nu$$

$$m_\pi c^2 = c\sqrt{m_\mu^2 c^2 + p_\mu^2} + c|\vec{p}_\mu|$$

$$(m_\pi c - p_\mu)^2 = m_\mu^2 c^2 + p_\mu^2$$

$$m_\pi^2 c^2 - 2 m_\pi c p_\mu + \cancel{p_\mu^2} = m_\mu^2 c^2 + \cancel{p_\mu^2}$$

$$p_\mu = \frac{m_\pi^2 c^2 - m_\mu^2 c^2}{2 m_\pi c} = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi} c$$

Pion energy:

$$E_\mu = c\sqrt{m_\mu^2 c^2 + \frac{(m_\pi^2 - m_\mu^2)^2}{4 m_\pi^2} c^2} = \frac{(m_\mu^2 + m_\pi^2)}{2 m_\pi} c^2$$

$$\frac{p}{E} = \frac{v}{c^2} \Rightarrow v = \frac{p}{E} c^2$$

$$\boxed{v = \frac{m_\pi^2 - m_\mu^2}{m_\mu^2 + m_\pi^2}}$$