Tensors :

i) Tensors - (p,q)-tensor $\Rightarrow$ $H^{\mu\nu}{}_{\lambda\rho}{}^{\sigma}$

Remember that V^{μ} V_{μ} $\quad V_{\mu}V^{\mu}$ = invariant

$\underbrace{r \quad \tilde{r}^T}_{SO(1,3)} \Rightarrow \tilde{r}^T r$ = invariant

$V_{\mu}T^{\mu\nu} \longrightarrow V_{\mu'}T^{\mu'\nu'}$

Some tensors can be represented by matrices $V^{\mu} = \begin{bmatrix} \\ \end{bmatrix}$ $V_{\mu} = \begin{bmatrix} \quad \end{bmatrix}$

$T^{\mu\nu}$ two vector indices
$T^{\mu}{}_{\nu}$ one vector index and one dual index
$T_{\mu\nu}$ two dual indices.

- $V^{\mu} \to V^{\mu'} = \Lambda^{\mu'}{}_{\mu}V^{\mu}$
  $V_{\mu} \to V_{\mu'} = \Lambda_{\mu'}{}^{\mu}V_{\mu}$

- The metric tensor : $g_{\mu\nu}V^{\mu} = V_{\nu}$
  $g^{\mu\nu}V_{\nu} = V^{\mu}$ $\Rightarrow g^{\mu\nu} = (g_{\mu\nu})^{-1}$

$g_{\mu\nu} \to g_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu} = g^{\mu\nu}$ in special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Trace $\eta_{\mu\nu}\eta^{\mu\nu} = 4$

- Transformations $\Lambda^{\mu'}{}_{\mu}$, $\Lambda_{\mu'}{}^{\mu} = (\Lambda^{\mu'}{}_{\mu}{}^{-1})^T$

Index notation and matrices :

2D : $dx^{\mu} = \begin{pmatrix} dx \\ dy \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta\, dx + \sin\theta\, dx \\ -\sin\theta\, dx + \cos\theta\, dy \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$

$\Lambda^{1'}{}_{1} = \cos\theta$, $\Lambda^{1'}{}_{2} = \sin\theta$, $\Lambda^{2'}{}_{1} = -\sin\theta$, $\Lambda^{2'}{}_{2} = \cos\theta$

$\Lambda^{\text{row}}{}_{\text{column}}$

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, dx^\mu = \Lambda^{\mu'}{}_1\, dx + \Lambda^{\mu'}{}_2\, dy$$

$$dx^{1'} = \Lambda^{1'}{}_1\, dx + \Lambda^{1'}{}_2\, dy \quad \text{and} \quad dx^{2'} = \Lambda^{2'}{}_1\, dx + \Lambda^{2'}{}_2\, dy$$

- Note that:

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, dx^\mu = dx^\mu \Lambda^{\mu'}{}_{\mu}$$

- $\eta_{\mu\nu} = \eta_{\nu\mu}$ only with the metric in general $H_{\mu\nu} \neq H_{\nu\mu}$

Some important vectors:

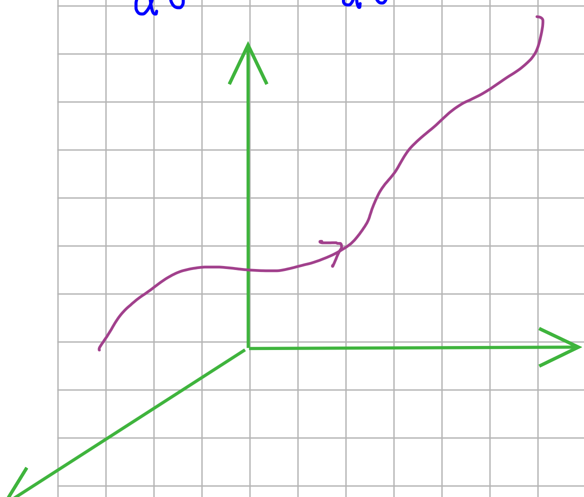
$$\partial_\mu = \left(\partial/c\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z \right) \qquad (0,1)\text{-tensor or dual tensor}$$

$$\partial_\mu \to \partial_{\mu'} = \Lambda^{\mu}{}_{\mu'}\, \partial_\mu \qquad \begin{cases} \text{4 Gradient}: & \partial_\mu \phi \to (0,1)\text{-tensor} \\ \text{4 Divergence}: & \partial_\mu A^\mu \to (0,0) \text{ tensor.} \end{cases}$$

To naively generalize to 4D: $d\vec{r} \to dx^\mu = \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix}$ (1,0) tensor

What is $\dfrac{dx^\mu}{dt}$?

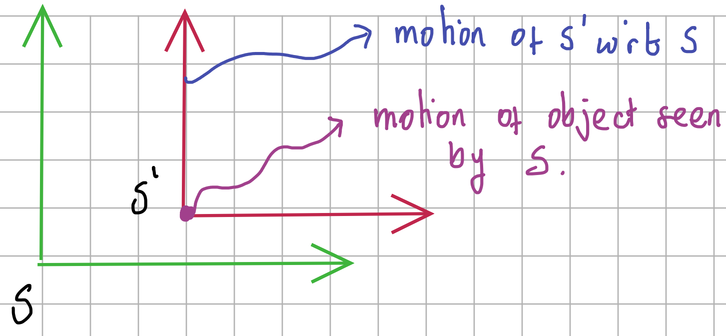
$$\frac{dx^\mu}{dt} \to \frac{dx^{\mu'}}{dt} = \Lambda^{\mu'}{}_{\mu} \frac{dx^\mu}{dt}$$ But this one does not transform as a tensor.



$$\frac{d}{dt} \Rightarrow \frac{d}{d\tau} \quad \text{where} \quad \tau = \int \sqrt{-ds^2}$$

$$ds^2 = -c^2dt + dx^2 + dy^2 + dz^2 = \text{Invariant.}$$

Now $$u^\mu = c\,\frac{dx^\mu}{d\tau}.$$



As seen in S:

$$U^0 = c^2\, dt/d\tau$$
$$U^1 = c\, dx/d\tau$$
$$U^2 = c\, dy/d\tau$$
$$U^3 = c\, dz/d\tau.$$

For example: $U^1 = c \frac{dx}{dt}\frac{dt}{d\tau} = c V_x$ Something.

$$-ds^2 = d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \rightarrow \frac{d\tau^2}{dt^2} = c^2 - v^2$$

$$\frac{dt^2}{d\tau^2} = \frac{1}{c^2 - v^2} = \frac{1}{c^2(1-\beta^2)} = \frac{\gamma^2}{c^2} \quad \text{and} \quad \frac{dt}{d\tau} = \frac{\gamma}{c}.$$

In this case $U^\mu = \begin{bmatrix} \gamma c \\ \gamma \vec{v} \end{bmatrix}$

- Momentum: $p^\mu = m U^\mu = \begin{pmatrix} mc\gamma \\ m\gamma\vec{v} \end{pmatrix} \Rightarrow p^\mu_{rest} = \begin{pmatrix} mc \\ 0 \end{pmatrix}$

For $\frac{v}{c} \ll 1$ $\quad \gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots \Rightarrow p^\mu = \begin{pmatrix} \frac{1}{c}(mc^2 + \frac{1}{2}mv^2 + \dots) \\ mv \end{pmatrix}$

and $p^\mu = \begin{pmatrix} \frac{1}{c} \text{ Energy relativistic} \\ \vec{P}\text{relativistic} \end{pmatrix}$ $\quad E_{rest} = mc^2$

Since p^μ is a vector, $p_\mu p^\mu$ is a scalar.

$$p_\mu = (-m\gamma c, m\gamma\vec{v}).$$

In particular $p_\mu p^\mu = -m^2c^2 = -\frac{E^2}{c^2} + p^2$, and finally we get:

$$E^2 - p^2c^2 = m^2c^4$$

$$p_\mu p^\mu \begin{cases} <0 & m^2>0 \\ =0 & m^2=0 \\ >0 & m^2<0. \end{cases}$$