Propagators and Green Functions:

Amplitudes for a given process: Particle starts at point y at a time t_y and ends up at point x at time t_x:

$$\langle x(t_x) | y(t_y) \rangle \qquad \text{(Propagator)}$$

Propagators for single particles: They are the green functions of the equation of motion for a particle.

→ Green function: $\hat{L}\, x(t) = f(t)$: The green function $G(t,u)$ of the linear operator L is defined by:

$$\hat{L}\, G(t,u) = \delta(t-u) \qquad (2)$$

Example:

$$m \frac{d^2}{dt^2} x + k x = f(t)$$

Linear operator $\hat{L} = m\frac{d^2}{dx^2} + k$, let's take $f(t) = \int_0^\infty du\, f(u)\, \delta(t-u)$

For one Delta: $\left[m\frac{d^2}{dx^2} + k \right] G(t,u) = \delta(t-u)$

Complete solution: $x(t) = \int_0^\infty du\, G(t,u)\, f(u)$

Let's see: $\hat{L}\, x(t) = \int du\, \hat{L}\, G(t,u)\, f(u)$

$\hat{L}\, x(t) = \int du\, \delta(t-u)\, f(u)$

Then: $\boxed{\hat{L}\, x(t) = f(t)}$

Two variables: Time and position of the Delta function.

Example: Green Function for $\hat{L} = \vec{\nabla}^2$

$\nabla^2 \phi = -\rho/\varepsilon_0$. For a point located at u:

$$\varepsilon_0 \nabla^2 \phi = -\delta^{(3)}(x-u)$$

Potential: $\phi(x) = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\vec{x}-\vec{u}|}$ so $G(x,u) = \frac{-1}{4\pi|\vec{x}-\vec{u}|}$

In general the Green function for $\hat{L} = (\vec{\nabla}^2 + \vec{k}^2)$:

$$(\nabla^2 + k^2) G_k(\vec{x},\vec{u}) = \delta^3(\vec{x}-\vec{u})$$

$$\boxed{G_k(\vec{x},\vec{u}) = -\frac{e^{ik(|\vec{x}-\vec{u}|)}}{4\pi|\vec{x}-\vec{u}|}}$$

Propagators in QM: Schrödinger equation: $\hat{H}\phi(x,t) = i\frac{\partial \phi}{\partial t}$

$\Rightarrow \phi(x,t) = \int dy\, G^+(x,t_x,y,t_y)\,\phi(y,t_y)$ Green function takes a wave function at some time and place and evolves it to another time and place.

Green function propagates the particle from spacetime point (y,t_y) to (x,t_x) and we constraint $t_x > t_y$ $G^+ = \theta(t_x - t_y) G$.

$$\rightarrow G^+(x,t_x,y,t_y) = \theta(t_x-t_y)\langle x(t_x)|y(t_y)\rangle \quad (18)$$

→ **Example:**

$$G^+(x,t_x,y,t_y) = \theta(t_x-t_y)\langle x|\hat{U}(t_x-t_y)|y\rangle$$
$$= \theta(t_x-t_y)\langle x|e^{-iH}|y\rangle$$
$$= \theta(t_x-t_y)\sum_n \langle x|e^{-iH}|n\rangle\langle n|y\rangle \qquad \langle x|n\rangle = \phi_n(x)$$

$$G^+(x,t_x,y,t_y) = \theta(t_x-t_y)\cdot\sum_n \phi_n(x)\,\phi_n^*(y)\,e^{-iE_n(t_x-t_y)}$$

- **Definition of G^+**

$$\left[\hat{H}_x - i\frac{\partial}{\partial t_x}\right] G^+(x,t_x,y,t_y) = -i\delta(x-y)\,\delta(t_x-t_y) = -i\delta^{(2)}(x-y)$$

The Green function:

$$G^+(x,t_x,y,t_y) = \theta(t_x-t_y)\langle x(t_x)|y(t_y)\rangle =$$
$$\theta(t_x-t_y)\sum_n \phi_n(x)\,\phi_n^*(y)\,e^{-iE_n(t_x-t_y)}$$

$$\left[\hat{H}_x - i\frac{\partial}{\partial t}\right] \theta(t_x - t_y) \sum_n \phi_n(x)\, \phi_n^*(y)\, e^{-iE_n(t_x - t_y)}$$

$$i\frac{\partial}{\partial t_x} G^+ = i\delta(t_x - t_y) \sum_n \phi_n(x)\, \phi_n^*(y)\, e^{-iE_n(t_x - t_y)}$$

$$+ \theta(t_x - t_y) \sum_n E_n\, \phi_n(x)\, \phi_n^*(y)\, e^{-iE_n(t_x - t_y)}$$

Remember now: $\hat{H}_x \phi_n(x) = E_n \phi_n(x)$

$$\hat{H}_x G^+ = \theta(t_x - t_y) \sum_n E_n\, \phi_n(x)\, \phi_n^*(y)\, e^{-iE_n(t_x - t_y)}$$

$$\left[\hat{H}_x - i\frac{\partial}{\partial t}\right] G^+ = i\delta(t_x - t_y) \sum_n \phi_n(x)\, \phi_n^*(y)\, e^{-iE_n(t_x - t_y)}$$

$$= i\delta(t_x - t_y)\, \delta(x - y)$$

Example: A non-relativistic free particle $\hat{H} = \hat{p}^2/2m$ with eigenfunctions $\phi(x) = \frac{1}{\sqrt{L}} e^{ipx}$ and eigenvalues $E_p = \frac{p^2}{2m}$:

$$G^+(x, t_x, y, t_y) = \theta(t_x - t_y)\, L \int \frac{dp}{2\pi}\, \phi_p(x)\, \phi_p^*(y)\, e^{-iE_p(t_x - t_y)}$$

$$G^+ = \theta(t_x - t_y)\, L \int \frac{dp}{2\pi}\, \phi_p(x)\, \phi_p^*(y)\, e^{-\frac{p^2}{2m}(t_x - t_y)}$$

$$\int_{-\infty}^{\infty} dx\, \exp\left(-\frac{ax^2}{2} + bx\right) = \sqrt{\frac{2\pi}{a}}\ \frac{b^2}{2a}$$

$$G^+(x, t_x, y, t_y) = \theta(t_x - t_y) \sqrt{\frac{m}{2\pi i (t_x - t_y)}}\, e^{\frac{im(x-t)^2}{2(t_x - t_y)}}$$

- $$G^+(x, y, E) = \sum_n \frac{i\phi_n(x)\phi_n^*(y)}{E - E_n} \qquad (30)$$

Field Propagator:

Interacting system in its ground state $|\Omega\rangle$

New particle is introduced (created) at spacetime point $(y^0, \vec{y})$. Then we remove the particle at spacetime point $(x^0, \vec{x})$

$$G^+(x, y) = \langle \Omega | \begin{pmatrix} \text{Particle annihilated} \\ \text{at } (x^0, \vec{x}) \end{pmatrix} \begin{pmatrix} \text{Particle created} \\ \text{at } (y^0, \vec{y}) \end{pmatrix} | \Omega \rangle$$

Probability amplitude that the system is still in its ground state after we create a particle at y and later annihilate it at x.

$$G^+(x,y) = \theta(x^0-y^0)\langle\Omega|\hat{\phi}(x)\hat{\phi}^+(y)|\Omega\rangle \quad (17.2)$$

$\rightarrow \hat{\phi}(t,\vec{x}) = e^{iHt}\hat{\phi}(x)e^{-iHt}$

$$G^+(x,y) = \langle\Omega|e^{i\hat{H}x^0}\hat{\phi}(x)e^{-i\hat{H}(x^0-y^0)}\hat{\phi}^+(y)e^{-i\hat{H}y^0}|\Omega\rangle$$

- $e^{-i\hat{H}y^0}|\Omega\rangle$ is the state $|\Omega\rangle$ evolved to a time y^0
- $\hat{\phi}^+(y)e^{-i\hat{H}y^0}|\Omega\rangle$ is that state with a particle added at time y^0 at position y.

Wick time-ordering symbol T:

$$T\hat{\phi}(x^0)\hat{\phi}(y^0) = \begin{cases}\hat{\phi}(x^0)\hat{\phi}(y^0) & x^0>y^0\\ \hat{\phi}(y^0)\hat{\phi}(x^0) & x^0<y^0\end{cases}$$

- Earliest on the right, latest on the left.

$$G(x,y) = \langle\Omega|T\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle$$
$$= \theta(x^0-y^0)\underbrace{\langle\Omega|\hat{\phi}(x)\hat{\phi}^+(y)|\Omega\rangle}_{\text{Creates particle at } y \text{ and annihilates at } x} + \theta(x^0-y^0)\underbrace{\langle\Omega|\hat{\phi}^+(y)\hat{\phi}(x)|\Omega\rangle}_{\text{Creates antiparticle at } x \text{ annihilates at } y.}$$

Field creation operator for complex scalar field

$$\psi^+(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{1/2}}\left(\hat{a}_p^+e^{ip\cdot x} + \hat{b}_p e^{-ip\cdot x}\right)$$

$$\phi^+(y)|0\rangle = \int\frac{d^3p}{N}\left(\hat{a}_p^+|0\rangle e^{ip\cdot y} + \hat{b}_p|0\rangle e^{-ip\cdot y}\right)$$

Since $\hat{b}_p|0\rangle = 0$

$$\phi^+(y)|0\rangle = \int\frac{d^3p}{N}|\vec{p}\rangle e^{ip\cdot y}$$

$$\rightarrow \langle 0|\hat{\phi}(x) = \int\frac{d^3q}{N}\langle\vec{q}|e^{-iq\cdot x}$$

Sandwiching

$$\langle 0|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle = \int \frac{d^3q\, d^3p}{N^2} \langle \vec{q}|e^{-iq.x} e^{ip.y}|\vec{p}\rangle$$

$$= \int \frac{d^3q\, d^3p}{N^2} e^{-iq.x+ip.y} \underbrace{\langle \vec{q}|\vec{p}\rangle}_{\delta(\vec{q}-\vec{p})}$$

$$\langle 0|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle = \int \frac{d^3q}{N^2} e^{-ip(x-y)}$$

Finally:

$$\langle 0|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3(2E_p)} e^{-ip.(x-y)}$$

Particle being created at $(y^0, \vec{y})$ and propagating to $(x^0, \vec{x})$ where it is annihilated at the later time.

Reverse order: $\hat{\phi}(x)|0\rangle = \int \frac{d^3p}{N} \left(a_p|0\rangle e^{-ip.x} + b_p^{\dagger}|0\rangle e^{ip.x} \right)$

antiparticle creation part contributes since $\hat{a}_p|0\rangle = 0$

$$\hat{\phi}(x)|0\rangle = \int \frac{d^3p}{N} |\vec{p}\rangle e^{ip.x}$$

$x \to y$ and $p \to q$

$$\langle 0|\hat{\phi}^{\dagger}(y) = \int \frac{d^3q}{N} \langle q| e^{-iq.y}$$

$$\langle 0|\hat{\phi}^{\dagger}(y)\hat{\phi}(x)|0\rangle = \int \frac{d^3q\, d^3p}{N^2} \underbrace{\langle \vec{q}|\vec{p}\rangle}_{\delta(\vec{q}-\vec{p})} e^{-iq.y+ip.x}$$

$$\langle 0|\hat{\phi}^{\dagger}(y)\hat{\phi}(x)|0\rangle = \int \frac{d^3p}{N^2} e^{ip.(x-y)}$$

Then the Feynman propagator

$$\Delta(x,y) = \langle 0|T\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\Theta(x^0-y^0) e^{-ip(x-y)} + \Theta(y^0-x^0) e^{ip(x-y)} \right]$$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

If all b's are zero then $f(z_0)$ is analytic at $z = z_0$

Coefficient b_1 is called the residue of $f(z)$ at $z = z_0$

$$f(z) = \frac{\alpha}{z-\beta}$$

Residue $b_1 = \alpha$
Simple pole at $z = \beta$.

Non relativistic retarded, free electron propagator is Given by:

$$\tilde{G}_0^{+}(E) = \frac{i}{E - E_p + i\epsilon}$$

First order pole at $E_p - i\epsilon$

- Feynman propagator for the free scalar field:

$$\tilde{\Delta}(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\tilde{\Delta}(p) = \frac{1}{2E_p}\left[\frac{i}{(p^0) - E_p + i\epsilon} - \frac{i}{(p^0) + E_p - i\epsilon}\right]$$

One pole at $E_p - i\epsilon$ (Particle)
and $-E_p + i\epsilon$ (antiparticle)

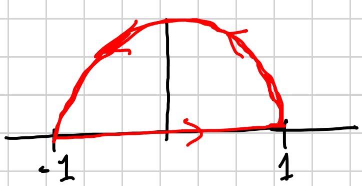
You can find a Residue $R(z_0)$ at the pole z_0 by writen a Laurent Series

$$R(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

Residue of $\tilde{G}(E) = \frac{iZ}{E - E_p + i\epsilon}$

$$\lim_{E \to E_p - i\epsilon} \tilde{G}(E) \; (E - E_p + i\epsilon) \frac{iZ}{E - E_p + i\epsilon} = iZ$$

- $\oint_c dz \, z^2$



Take $z = re^{i\theta}$

- $(\text{straight line}) = \displaystyle\int_{r=-1}^{1} dr \, r^2 = \frac{r^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$
- $(\text{Semi Circle}) = \displaystyle\int i r e^{i\theta} d\theta \; r^2 e^{2i\theta} = i\int_0^{\pi} r^3 e^{3i\theta} d\theta = \frac{i e^{3i\theta}}{3i}\Big|_0^{\pi} = \frac{e^{3i\pi} - 1}{3}$

- Semicircle = $-\frac{2}{3}$

Then (line + semicircle) = 0.

- If $f(z)$ is analytic on and inside C then: $\oint_C dz\, f(z) = 0$
- If $f(z)$ is analytic on and inside a simply closed curve C, and the point a is inside C, then the value of $f(a)$ is given by:

$$f(a) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z-a}$$

- If $f(z)$ has singularities at points z_i, then, for a closed curve enclosing these points we have:

$$\oint dz\, f(z) = 2\pi i \sum_i \left(\text{Res at } f(z_i) \atop \text{inside } C \right)$$

$$G_0^+(t-t') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{i e^{-iE(t-t')}}{E - E_p + i\epsilon}$$

limits along the real axis to $\pm\infty$ the semicircular path gets larger and larger

This will make a large negative imaginary contribution to $E \rightarrow -i|\eta|$

and $e^{-|\eta|(t-t')}$ if $t-t' > 0 \rightarrow$ contribution gets smaller

Pole at: $E_p - i\epsilon$

$$\oint \frac{dE}{2\pi} \frac{i e^{-iE(t-t')}}{E - E_p + i\epsilon} = -2\pi i \left(\text{Res at } E = E_p - i\epsilon \right)$$

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) = (E - E_p + i\epsilon) \frac{i e^{-i(E_p - i\epsilon)(t-t')}}{E - E_p + i\epsilon}$$

$$R(E_p - i\epsilon) = i e^{-iE_p(t-t')} e^{-\epsilon(t-t')}$$

$$G^+(t-t') = e^{-iE_p(t-t')} e^{-\epsilon(t-t')} = \theta(t-t') e^{-iE_p(t-t')}$$