- $\hat{a}|n\rangle = K|n-1\rangle$ where K is a constant so: $\langle n|\hat{a}^{+}a|n\rangle =$ $\langle n|\hat{a}^{+}\hat{a}|n\rangle = |K|^2 \langle n-1|n-1\rangle = |K|^2$

$\hat{a}^{+}\hat{a} = \hat{n}$ and Hence $\langle n|\hat{a}^{+}\hat{a}|n\rangle = \langle n|\hat{n}|n\rangle = n$

So $K = \sqrt{n}$

- $a^{+}|n\rangle = c|n+1\rangle$ and $\langle n|\hat{a}\hat{a}^{+}|n\rangle = c^2 \langle n+1|n+1\rangle = |c|^2$

$[\hat{a}, \hat{a}^{+}] = 1 \qquad \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1 \Rightarrow \hat{a}\hat{a}^{+} = 1 + \hat{a}^{+}\hat{a}$

Then: $\langle n|\hat{a}\hat{a}^{+}|n\rangle = \langle n|n\rangle + \langle n|\hat{a}^{+}\hat{a}|n\rangle = 1 + n$, hence

$C = \sqrt{n+1}$

In summary: $a|n\rangle = \sqrt{n}\,|n-1\rangle$
$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a^{+}|0\rangle = 1$, $\quad \hat{a}^{+}|1\rangle = \sqrt{2}\,|2\rangle \Rightarrow |2\rangle = \dfrac{a^{+}}{\sqrt{2}}|1\rangle = \dfrac{(a^{+})^2}{\sqrt{2}}|0\rangle$

- $a^{+}|2\rangle = \sqrt{3}\,|3\rangle \Rightarrow |3\rangle = \dfrac{a^{+}|2\rangle}{\sqrt{3}}$ but $|2\rangle = \dfrac{(a^{+})^2}{\sqrt{2}}|0\rangle$

then: $|3\rangle = \dfrac{(a^{+})^3}{\sqrt{2}\sqrt{3}}|0\rangle$

In general $|n\rangle = \dfrac{(a^{+})^n}{\sqrt{n!}}|0\rangle$

Consider a Hamiltonian $\hat{H} = \sum_{K=1}^{N} \hat{H}_K$ and $\hat{H}_K = \dfrac{P_K^2}{2m} + \dfrac{1}{2} m_K \omega_K^2 x_K^2$

$a_K^{+}|n_1, n_2, \dots, n_r, \dots\rangle \propto |n_1, n_2, \dots, n_K+1, \dots\rangle$

$\hat{a}_K|n_1, n_2, \dots, n_r, \dots\rangle \propto |n_1, n_2, \dots, n_{K-1}, \dots\rangle$

$\Rightarrow$ Commutation relations: $[\hat{a}_n, \hat{a}_q] = 0$

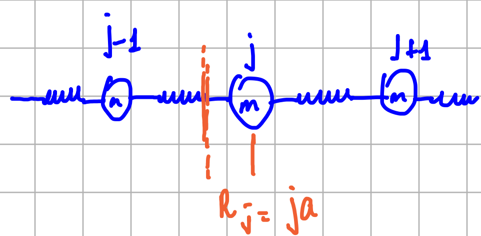
$[\hat{a}_K^{+}, \hat{a}_q^{+}] = 0$

$[\hat{a}_K, \hat{a}_q^{+}] = \delta_{Kq}$.

$$\hat{H} = \sum_{k=1}^{N} \hbar\omega_k \left(\hat{a}_k \hat{a}_k + 1/2 \right)$$ and the vacuum state: $|0,0,\ldots,0\rangle$

and also: $$|n_1, n_2, \ldots, n_N\rangle = \frac{1}{\sqrt{n_1!\, n_2! \ldots n_N!}} (a_1^{+})^{n_1} (a_2^{+})^{n_2} \ldots (a_N^{+})^{n_N} |0,0,\ldots,0\rangle$$

Phonons:



$$\hat{H} = \sum_j \frac{\hat{p}_j^2}{2m} + \frac{1}{2} K \left(\hat{x}_{j+1} - x_j \right)^2$$

To evaluate $$\frac{1}{N} \sum_j \sum_{kq} \tilde{p}_k \tilde{p}_q e^{i(k+q)ja}$$

Perform the spatial sum $$\sum_j e^{i(k-k')ja} = \sum_{pq} \tilde{p}_k \tilde{p}_q \delta_{k,-q}.$$

$$x_j = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja} \quad \text{and} \quad p_j = \frac{1}{\sqrt{N}} \sum_k \tilde{p}_k e^{ikja}$$

And equivalently: $$\tilde{x}_k = \frac{1}{\sqrt{N}} \sum_j x_j e^{-ikja} \quad \text{and} \quad \tilde{p}_k = \frac{1}{N} \sum_j p_j e^{-ikja}$$

Remember that $$\sum_j e^{i(k-k')ja} = N \delta_{kk'} \quad \text{then} \quad \sum_j e^{ikja} = N \delta_{k0}$$

$$[x_j, p_{j'}] = i\hbar \delta_{jj'}$$

$$[\tilde{x}_k, \tilde{p}_{k'}] = \frac{1}{N} \sum_{jj'} [x_j, p_{j'}] e^{-ikja} e^{-ik'j'a} = \frac{i\hbar}{N} \sum_j e^{-i(k+k')j}$$

Taking $k' = -k'$

$$[\tilde{x}_k, \tilde{p}_{k'}] = \frac{i\hbar}{N} \sum_j e^{-i(k-(-k'))j} = i\hbar \delta_{k,-k'}$$

$$\hat{H} = \sum_j \frac{p_j^2}{2m} + \frac{1}{2} K \left(\hat{x}_{j+1} - x_j \right)^2$$

$$\sum_j p_j^2 = \frac{1}{N}\sum_j \left(\sum_k \tilde{p}_k e^{ikja} \right)\left(\sum_{k'} \tilde{p}_{k'} e^{ik'ja} \right)$$

$$\sum_j p_j^2 = \frac{1}{N}\sum_j \left(\sum_{kk'} \tilde{p}_k \tilde{p}_{k'} e^{i(k-(-k'))ja} \right)$$ and using the Dirac's Delta.

$$= \frac{1}{N}\sum_{kk'} \hat{p}_k \hat{p}_{k'} \left(\sum_j e^{i(k-(-k'))ja} \right) = \frac{1}{N}\sum_{kk'} \hat{p}_k \hat{p}_k \, N \delta_{k,-k'}$$

$$\sum_j p_j^2 = \sum_k \tilde{p}_k \tilde{p}_{-k'}$$

$$x_j = \frac{1}{\sqrt{N}}\sum_k \tilde{x}_k e^{ikja}$$

$$\sum_j (x_{j+1} - x_j)^2 = \frac{1}{N}\sum_j \sum_k \sum_{k'} \left(\tilde{x}_k e^{ik(j+1)a} - \tilde{x}_{k'} e^{ik'(j+1)a} \right)^2$$

$$= \frac{1}{N}\sum_j \sum_{kk'} \Big(\tilde{x}_k \hat{x}_{k'} e^{ik(j+1)a} e^{ik(j+1)a} - \tilde{x}_k \tilde{x}_{k'} e^{ik(j+1)a} e^{ik'(j+1)a}$$
$$- \tilde{x}_{k'} x_k e^{ik'(j+1)a} e^{ik(j+1)a} + \tilde{x}_{k'} \tilde{x}_{k'} e^{ik'(j+1)a} e^{ik'(j+1)a} \Big)$$

$$= \frac{1}{N}\sum_j \sum_{kk'} \tilde{x}_k \tilde{x}_{k'} e^{i(k+k')ja} \left(e^{ika} - 1 \right)\left(e^{ik'a} - 1 \right)$$

$$= \sum_k \tilde{x}_k \hat{x}_{-k} \left(4\sin^2\left(\frac{ka}{2} \right) \right)$$

Thus: $$\hat{H} = \sum_k \left[\frac{1}{2m} \hat{p}_k \hat{p}_{-k} + \frac{1}{2} m\omega_k^2 \hat{x}_k \hat{x}_{-k} \right]$$

➔ Appropiate linear combination of operators using fourier sums : Field operators that create and annihilate particles in spatial locations. Thus the operator $\hat{\psi}^+(x)$ is defined as

$$\hat{\psi}^+(x) = \frac{1}{\sqrt{V}} \sum_p \hat{a}_p^+ e^{-ip\cdot x}$$

Creates a particle at position x, while $\hat{a}_p^+$ creates a particle in a state with three-momentum $\vec{p}$.

$\tilde{\psi}_\alpha(p) = \langle \vec{p} | \alpha \rangle$ $\qquad$ $\langle \vec{p} | \alpha \rangle = \int d^3x \, \langle \vec{p} | \vec{x} \rangle \langle \vec{x} | \alpha \rangle$

then: $\tilde{\psi}_\alpha(p) = \frac{1}{\sqrt{V}} \int d^3x \, e^{-ip.x} \psi_\alpha(x)$

implies that $\langle \vec{p} | \vec{x} \rangle = \frac{1}{\sqrt{V}} e^{-i\vec{p}.\vec{x}}$

The inverse transform $\psi_\alpha(\vec{x}) = \frac{1}{\sqrt{V}} \sum_p e^{i\vec{p}.\vec{x}} \tilde{\psi}_\alpha(\vec{p})$.

We also have: $\int d^3x \, e^{-i\vec{p}.\vec{x}} = V \delta_{p,0}$.

and $\frac{1}{V} \sum_p e^{i\vec{p}.\vec{x}} = \delta^3(\vec{x})$.

- Let us examine the state $|\Psi\rangle = \hat{\psi}^+ |0\rangle$

$$|\Psi\rangle = \hat{\psi}^+(x) |0\rangle = \frac{1}{\sqrt{V}} \sum_p e^{-i\vec{p}.\vec{x}} \hat{a}_p^+ |0\rangle$$

use $\hat{n}_p = \hat{a}_p^+ a_p$. Consider $\sum_q \hat{n}_q |\Psi\rangle = \frac{1}{\sqrt{V}} \sum_{pq} e^{-i\vec{p}.\vec{x}} \hat{n}_q \hat{a}_p^+ |0\rangle$

$\sum_{pq} e^{-i\vec{p}.\vec{x}} \hat{a}_q^+ a_q \hat{a}_p^+ |0\rangle$ $\qquad$ $\langle 0 | \hat{a}_q \hat{a}_p^+ |0\rangle = \delta_{qp}$

$$\sum_q \hat{n}_q |\Psi\rangle = |\Psi\rangle$$

Now define $|\vec{p}\rangle = \hat{a}_p^+ |0\rangle$ and consider

$$\langle y | \Psi \rangle = \langle y | \psi^+(x) |0\rangle = \frac{1}{\sqrt{V}} \sum_p e^{-i\vec{p}.\vec{x}} \langle y | \hat{a}^+_p |0\rangle$$

$$\langle y | \Psi \rangle = \frac{1}{\sqrt{V}} \sum_p e^{-i\vec{p}.\vec{x}} \langle \vec{y} | \vec{p} \rangle$$

Remember that $\langle \vec{p} | \vec{x} \rangle = \frac{1}{\sqrt{V}} e^{-i\vec{p}.\vec{x}}$

So then $\langle y|\Psi\rangle = \frac{1}{V}\sum_p e^{-i\vec{p}\cdot(x\ y)}$ using $\frac{1}{V}\sum_p e^{i\vec{p}\cdot\vec{x}} = \delta^3(x)$

$$\langle y|\Psi\rangle = \delta^{(3)}(\vec{x}-\vec{y})$$

$$\hat{\psi}^\dagger(x) = \frac{1}{\sqrt{V}}\sum_p \hat{a}_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \qquad \hat{\psi}(x) = \frac{1}{\sqrt{V}}\sum_p \hat{a}_p e^{i\vec{p}\cdot\vec{x}}$$

$$\frac{1}{V}\sum_p e^{i\vec{p}\cdot\vec{x}} = \delta^{(3)}(\vec{x})$$

$$[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \frac{1}{V}\sum_{pp'} \hat{a}_p e^{i\vec{p}\cdot\vec{x}} \hat{a}_{p'}^\dagger e^{-i\vec{p}'\cdot\vec{y}} - \frac{1}{V}\sum_{pp'} \hat{a}_{p'}^\dagger e^{-i\vec{p}'\cdot\vec{y}} \hat{a}_p e^{i\vec{p}\cdot\vec{x}}$$

$$= \frac{1}{V}\sum_{pp'} e^{-i\vec{p}'\cdot\vec{y}} e^{i\vec{p}\cdot\vec{x}} \underbrace{[\hat{a}_p, \hat{a}^\dagger_{p'}]}_{\delta_{pp'}} = \underbrace{\frac{1}{V}\sum_p e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}_{\delta^{(3)}(\vec{x}-\vec{y})}$$

So finally we obtain $[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta^{(3)}(\vec{x}-\vec{y})$ ∎