Hamiltonian density $\mathcal{H}$ by: $H = \int d^3x \, \mathcal{H}$

Lagrangian density $L = \int d^3x \, \mathcal{L}$

Conjugate momentum $\pi(x)$ in terms: $\pi(x) = \frac{\delta L}{\delta \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$

This implies that: $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0$$

For the EM potential $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(E^2 + B^2)$

when $V=0 \Rightarrow \mathcal{L} = \frac{1}{2}(E^2 + B^2) = \frac{1}{2}(\dot{A}^2 - B^2)$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = -\dot{A}$$

$$\mathcal{H} = \pi^i \dot{A}_i - \mathcal{L} = \frac{1}{2}(E^2 + B^2)$$

The Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial (\partial_\lambda A_\mu)} \right) = 0$

- Simplest Lagrangian: Scalar field assigns a scalar amplitude to each position in space time.: $\mathcal{L} = \frac{1}{2} \partial^\mu \partial_\mu \phi = \frac{1}{2} (\partial_\mu \phi)^2$

Or it can be expanded as: $\mathcal{L} = \underbrace{\frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi}_{\text{Kinetic energy}} \underbrace{- \frac{1}{2} \nabla \phi \cdot \nabla \phi}_{\text{Potential Energy.}}$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \frac{\partial}{\partial (\partial_\mu \phi)} \left[\frac{1}{2} \left(\partial_t^2 \phi - \partial_x^2 \phi - \partial_y^2 \phi - \partial_z^2 \phi \right) \right]$$

$$\frac{\partial}{\partial(\partial_\mu \phi)} = \partial_t \phi - \partial_x \phi - \partial_y \phi - \partial_z \phi = \partial^\mu \phi$$

Euler Lagrange $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \Rightarrow \partial_\mu \partial^\mu \phi = 0$

or in other words: $\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi = 0$ whose solutions are:

$$\phi(x,t) = \sum_p a_{\vec{p}}\, e^{-i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})}$$

Massive Scalar Fields:

By Choosing $U(\phi) = \phi^2$ Cuadratic potential with a minimum at $\phi = 0$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2 \phi^2}{2}$$

$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$ $\quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi$ and then the equation of motion is given by:

$$(\partial_\mu \partial^\mu \phi + m^2 \phi) = 0$$

Solutions $\phi(x,t) = a e^{-i(E_p t - \vec{p}\cdot\vec{x})}$ with $E_p^2 = p^2 + m^2$.

External Source:

Current source $J(x)$ $\quad \mathcal{L} = \frac{1}{2}[\partial_\mu \phi]^2 - \frac{1}{2} m^2 \phi^2 + J(x)\phi(x)$

. The equation of motion become:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi + J(x)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = J(x)$$

$$(\partial_\mu \partial^\mu + m^2)\phi = J(x).$$

→ The ϕ^4 theory:

How do we get particles to interact with other particles (or equivalently, fields interact with other fields?)

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4$$

λ = Interaction Strength.

• Another way to make particles interact:

Two different types of particles in the Universe ϕ_1 and ϕ_2

In a simple theory they have the same mass and will interact with themselves and each other via

$$\mathcal{U}(\phi_1, \phi_2) = g(\phi_1^2 + \phi_2^2) \qquad g = \text{interaction strength.}$$

The resulting Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2} m^2 \phi^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m^2 \phi_2^2 - g(\phi_1^2 + \phi_2^2)^2$$

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

We say that particles have an internal degree of freedom.

Consider $\psi, \psi^\dagger$ $\quad \psi = \frac{1}{\sqrt{2}}[\phi_1 + i\phi_2] \quad \psi^\dagger = \frac{1}{\sqrt{2}}[\phi_1 - i\phi_2]$

New Lagrangian: $\mathcal{L} = \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2$

Rotation $\psi \to \psi e^{i\alpha}$, $\psi^\dagger \to e^{-i\alpha} \psi^\dagger$ which express a $\mathcal{U}(1)$ symmetry.

Ex 1: Given $\mathcal{L} = \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}$

Obtain the equation of motion

$$\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} = \partial^\mu \phi \qquad \text{so} \qquad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1}$$

So using E.L equations: $(\partial_\mu \partial^\mu + m^2)\phi^2 + \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1} = 0$

Ex 2: OK

Ex 3: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m \phi_2^2 - g(\phi_1^2 + \phi_2^2)^2$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial^\mu \phi_1 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \right) = \partial_\mu \partial^\mu \phi_1 \quad \text{and Similarly for } \phi_2 : \partial_\mu \partial^\mu \phi_2$$

Now $$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2 \phi_1 - 2g(\phi_1^2 + \phi_2^2)\, 2\phi_1 = -m^2\phi_2 - 4g(\phi_1^2+\phi_2^2)\phi_1 .$$

Rearranging these eq:

$$\partial_\mu \partial^\mu \phi_1 + m^2 \phi_1 + 4g\phi_1(\phi_1^2+\phi_2^2) = 0$$
$$\partial_\mu \partial^\mu \phi_2 + m^2 \phi_2 + 4g\phi_1(\phi_1^2+\phi_2^2) = 0$$

Ex 4: $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow \partial^\mu \phi\, \partial_\mu \phi = \partial^0\phi\, \partial_0 \phi - \partial^1 \phi\, \partial_1 \phi - \partial^2\phi\,\partial_2\phi - \partial^3\phi\,\partial_3\phi$$
$$= \dot{\phi}^2 - (\nabla\phi)^2$$

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \qquad \mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

$$\mathcal{H} = \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2\right) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\boxed{\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\dot{\phi}^2}$$

Defining $$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad \Pi^0 = \pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi} \quad \blacksquare$$

QM Transformations:

$$\vec{x}' = T(\vec{a})\,\vec{x} = \vec{x} + \vec{a} \qquad \text{(translation)}$$

$$\vec{x}' = R(\theta)\,\vec{x} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{(Rotation)}$$

Transformations on Spacetime:

$$U(\vec{a})\,|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$$ active transformation where we move a particle to a new position

Properties: $\langle \psi(\vec{x}) | \psi(\vec{x}) \rangle = \langle \psi(\vec{x}+\vec{a}) | \psi(\vec{x}+\vec{a}) \rangle$

$$= \langle \psi(\vec{x}) | \mathcal{U}^{\dagger}(\vec{a}) \mathcal{U}(a) | \psi(\vec{x}) \rangle$$

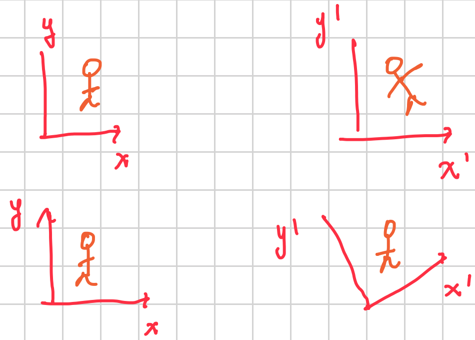
- Unitary $\hat{\mathcal{U}}^{\dagger}(\vec{a}) \hat{\mathcal{U}}(\vec{a}) = \mathbb{1}$
- $\hat{\mathcal{U}}(\vec{a}) \hat{\mathcal{U}}(\vec{b}) = \hat{\mathcal{U}}(\vec{a}+\vec{b})$
- Trivial property $\mathcal{U}(0) = \mathbb{1}$

Lie Group.

ACTIVE VS PASIVE:

Active

Pasive



$\psi(x+\delta a) = \psi(x) + \frac{d\psi}{dx}\delta a + \dots$ remembering $\hat{p} = -i\frac{d}{dx}$

$$\psi(x+\delta a) = (1 - ip\,\delta a)\,\psi(x)$$

Momentum is the Generator for the space translation:

$$\psi(x+a) = \lim_{N\to\infty}\left(1 - \frac{ipa}{N}\right)^N \psi(x)$$

$$\psi(x+a) = e^{i\hat{p}a}\psi(x)$$

Evolution operator: $\mathcal{U} = e^{-i\hat{p}\cdot\vec{a}}$

$$\psi(t+t_a) = \psi(t) + \frac{d\psi}{dt}\delta t = (1 - iH\delta t)\,\psi \Rightarrow \mathcal{U}(t_a) = e^{-i\hat{H}t_a}$$

$$\mathcal{U}(\theta)\,\mathcal{U}^{\dagger}(\theta) = \mathcal{U}(\theta)\left(\int dp\, |p\rangle\langle p|\right)\mathcal{U}^{\dagger}(\theta) = \int dp\, |R(\theta)p\rangle\langle R(\theta)p|$$

remember that $\vec{p}\,' = |R(\theta)\vec{p}\rangle$ then:

$$\int d\vec{p}\, |R(\theta)\vec{p}\rangle\langle R(\theta)\vec{p}| = \int d\vec{p}\,'\, |\vec{p}\,'\rangle\langle\vec{p}\,'| = \mathbb{1}.$$

Why $d\vec{p} = d\vec{p}\,'$ $\Rightarrow$ $\mathrm{Det}(R(\theta)) = 1$ $\Rightarrow$ So the Jacobian is unity.

→ Rotation of the Wave function: $\psi(\theta^z + \delta\theta^z) = \psi(\theta^z) + \frac{d\psi}{d\theta^z}\delta\theta^z + \dots$

$\hat{J} = -i\frac{d}{d\theta^z}$ so $\psi(\theta^z+\delta\theta^z) = (1 - i\hat{J}\,\delta\theta^z)\,\psi(\theta^z)$

$U = e^{-i\hat{\vec{J}}\cdot\vec{\theta}}$

Any rotation $R(\theta)$ can be represented by a square matrix $D(\theta)$

$$D(\theta) = e^{-i\vec{J}\cdot\vec{\theta}}$$

$$J^i = -\frac{1}{i}\frac{\partial D(\theta^i)}{\partial\theta^i}\Big|_{\theta^i=0}$$

Consider rotations about the z-axis. Trivial representation $D(\theta^z) = \mathbb{1}$

$$J^z = -\frac{1}{i}\frac{\partial D(\theta^z)}{\partial\theta^z}\Big|_{\theta^z=0} = 0$$

Similarly for J_x and J_y because a scalar can have no angular momentum.

Spin 1/2: $D(\theta^z) = \begin{bmatrix} e^{-i\theta^z/2} & 0 \\ 0 & e^{i\theta^z/2} \end{bmatrix}$

Hence $J^z = -\frac{1}{i}\frac{\partial D(\theta^i)}{\partial\theta^i}\Big|_{\theta^i=0} \Rightarrow \begin{bmatrix} -i/2 & 0 \\ 0 & +i/2 \end{bmatrix}\left(-\frac{1}{i}\right) = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$

For rotations about x-axis:

$$R(\theta^x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta^x & -\sin\theta^x \\ 0 & 0 & \sin\theta^x & \cos\theta^x \end{bmatrix}$$

$$J^x = -\frac{1}{i}\frac{\partial R(\theta^x)}{\partial\theta^x}\Big|_{\theta^x=0} = -\frac{1}{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Lorentz transformations:

Consider a boost of a four vector in the x-direction $x'^\mu = \Lambda(\beta^1)^\mu{}_\nu\, x^\nu$

$$\Lambda(\beta^1) = \begin{pmatrix} \gamma^1 & \beta^1\gamma^1 & 0 & 0 \\ \beta^1\gamma^1 & \gamma^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\gamma^i = \cosh\phi^i$ $\quad \tanh\phi^i = \beta^i$

$\gamma^i\beta^i = \sinh\phi^i$

ϕ^i = Rapidity

$$\Lambda(\phi^1) = \begin{pmatrix} \cosh\phi^1 & \sinh\phi^i & 0 & 0 \\ \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow D(\phi) = e^{iK\phi}$$

Generators: $$K^i = \frac{1}{i} \frac{\partial D(\phi^i)}{\partial \phi^i}\Big|_{\phi^i = 0}$$

$$[J^i, K^j] = i\varepsilon^{ijk} K^k$$

Lorentz Transformation $D(\theta, \phi) = e^{-i(\vec{J}\cdot\theta - \vec{K}\cdot\phi)}$