Tensors:

i) Tensors - (p, q)-tensor $\Rightarrow H^{\mu\nu}{}_{\lambda\rho}{}^{\sigma}$

Remember that $V^\mu \quad V_\mu \qquad V_\mu V^\mu =$ invariant

$\underbrace{r \quad \tilde{r}^T}_{SO(1,3)} \Rightarrow \tilde{r}^T r =$ invariant

$V_\mu T^{\mu\nu} \longrightarrow V_{\mu'} T^{\mu'\nu'}$

Some tensors can be represented by matrices $V^\mu = [\ ]$ $V_\mu = [\quad]$

$T^{\mu\nu}$ two vector indices
$T^\mu{}_\nu$ one vector index and one dual index
$T_{\mu\nu}$ two dual indices.

- $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$
  $V_\mu \to V_{\mu'} = \Lambda_{\mu'}{}^\mu V_\mu$

- The metric tensor: $g_{\mu\nu} V^\mu = V_\nu$
  $g^{\mu\nu} V_\nu = V^\mu \quad \Rightarrow g^{\mu\nu} = (g_{\mu\nu})^{-1}$

$g_{\mu\nu} \to g_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda_{\nu'}{}^{\nu} = g^{\mu\nu}$ in special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Trace $\eta_{\mu\nu} \eta^{\mu\nu} = 4$

- Transformations $\Lambda^{\mu'}{}_\mu, \quad \Lambda_{\mu'}{}^\mu = (\Lambda^{\mu'}{}_\mu{}^{-1})^T$

Index notation and matrices:

2D: $dx^\mu = \begin{pmatrix} dx \\ dy \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta\, dx + \sin\theta\, dx \\ -\sin\theta\, dx + \cos\theta\, dy \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$

$\Lambda^{1'}{}_1 = \cos\theta, \quad \Lambda^{1'}{}_2 = \sin\theta, \quad \Lambda^{2'}{}_1 = -\sin\theta, \quad \Lambda^{2'}{}_2 = \cos\theta$

$\Lambda^{\text{row}}{}_{\text{column}}$

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, dx^\mu = \Lambda^{\mu'}{}_{1}\, dx + \Lambda^{\mu'}{}_{2}\, dy$$

$$dx^{1'} = \Lambda^{1'}{}_{1}\, dx + \Lambda^{1'}{}_{2}\, dy \quad \text{and} \quad dx^{2'} = \Lambda^{2'}{}_{1}\, dx + \Lambda^{2'}{}_{2}\, dy$$

- Note that:

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, dx^\mu = dx^\mu\, \Lambda^{\mu'}{}_{\mu}$$

- $\eta_{\mu\nu} = \eta_{\nu\mu}$ only with the metric in general $H_{\mu\nu} \neq H_{\nu\mu}$

Some important vectors:

$$\partial_\mu = \left(\partial/c\partial t,\ \partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\right) \qquad (0,1)\text{-tensor or dual tensor}$$

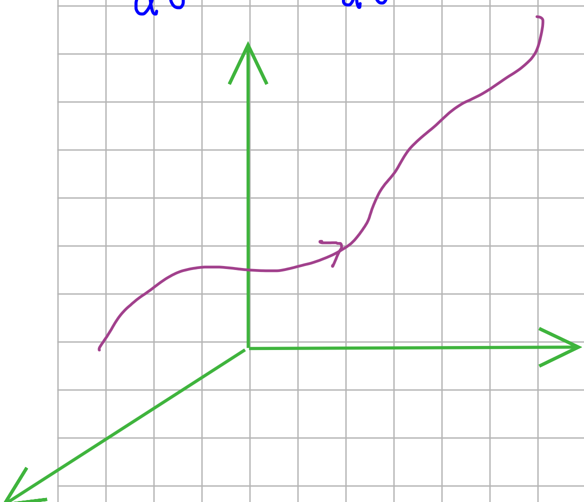
$$\partial_\mu \to \partial_{\mu'} = \Lambda^{\mu}{}_{\mu'}\, \partial_\mu \qquad \begin{cases} \text{4 Gradient: } \partial_\mu \phi \to (0,1)\text{-tensor} \\ \text{4 Divergence: } \partial_\mu A^\mu \to (0,0) \text{ tensor.} \end{cases}$$

To naively generalize to 4D: $d\vec{r} \to dx^\mu = \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix}$ (1,0) tensor

What is $\dfrac{dx^\mu}{dt}$?

$$\frac{dx^\mu}{dt} \to \frac{dx^{\mu'}}{dt} = \Lambda^{\mu'}{}_{\mu}\, \frac{dx^\mu}{dt}$$

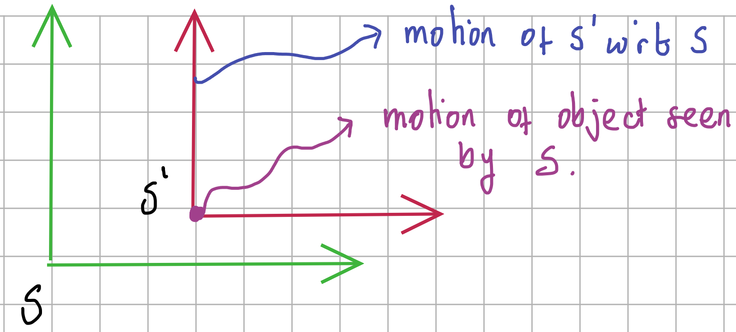
But this one does not transform as a tensor.



$$\frac{d}{dt} \Rightarrow \frac{d}{d\tau} \quad \text{where} \quad \tau = \int \sqrt{-ds^2}$$

$$ds^2 = -c^2 dt + dx^2 + dy^2 + dz^2 = \text{Invariant.}$$

Now $U^\mu = c\,\dfrac{dx^\mu}{d\tau}$.



As seen in S:

$U^0 = c^2\, dt/d\tau$
$U^1 = c\, dx/d\tau$
$U^2 = c\, dy/d\tau$
$U^3 = c\, dz/d\tau.$

For example: $U^1 = c\,\frac{dx}{dt}\frac{dt}{d\tau} = c\,V_x$ Something.

$-ds^2 = d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \rightarrow \frac{d\tau^2}{dt^2} = c^2 - v^2$

$\frac{dt^2}{d\tau^2} = \frac{1}{c^2 - v^2} = \frac{1}{c^2(1-\beta^2)} = \frac{\gamma^2}{c^2}$ and $\frac{dt}{d\tau} = \frac{\gamma}{c}.$

In this case $U^\mu = \begin{bmatrix} \gamma c \\ \gamma \vec{v} \end{bmatrix}$

- Momentum: $p^\mu = m\,U^\mu = \begin{pmatrix} mc\gamma \\ m\gamma\vec{v} \end{pmatrix} \Rightarrow p^\mu_{rest} = \begin{pmatrix} mc \\ 0 \end{pmatrix}$

For $\frac{v}{c} \ll 1$ $\quad \gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots \Rightarrow p^\mu = \begin{pmatrix} \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2 + \dots\right) \\ mv \end{pmatrix}$

and $p^\mu = \begin{pmatrix} \frac{1}{c}\ \text{Energy relativistic} \\ \vec{P}\ \text{relativistic} \end{pmatrix}$ $\qquad E_{rest} = mc^2$

Since p^μ is a vector, $p_\mu p^\mu$ is a scalar.

$p_\mu = (-m\gamma c, m\gamma\vec{v}).$

In particular $p_\mu p^\mu = -m^2c^2 = -\frac{E^2}{c^2} + p^2$, and finally we get:

$$E^2 - p^2c^2 = m^2c^4$$

$$p_\mu p^\mu \begin{cases} <0 & m^2>0 \\ =0 & m^2=0 \\ >0 & m^2<0. \end{cases}$$

Collisions :

$$p_i^\mu = p_f^\mu$$

- **Scattering Matrix :** $S = \lim_{t\to\infty} T\left[\exp\left(i\int_{-\infty}^{\infty} dt' \, H_I(t')\right)\right]$

- Wick : $T[\phi(x_1)\,\phi(x_2)] = :\phi(x_1)\,\phi(x_2): + \phi(x_1)\,\phi(x_2)$

Remember that:

- $\phi(x_1)\,\phi(x_2) = i\,D_F(x_1 - x_2) = \int \frac{d p^4}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} e^{-ip(x_1 - x_2)}$

- $\psi(x_1)\,\overline{\psi}(x_2) = i\,S_F(x_1 - x_2) = i\int \frac{d^4 p}{(2\pi)^4} \frac{p^\mu \gamma_\mu + m\mathbb{1}}{p^2 - m^2 + i\epsilon}$

- $A_\mu(x_1)\,A_\nu(x_2) = i\,D^F_{\mu\nu}(x_1 - x_2) = \int_p \frac{-i(\eta_{\mu\nu} - (3-1)\,p_\mu p_\nu/p^2)}{p^2 + i\epsilon}$

QFT (Quantum electrodynamics):

$\rightarrow$ Fermions $(\overline{\psi}, \psi)$ $\qquad \mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$

If we propose the changes: $\psi \rightarrow \psi' = e^{i\theta}\psi$

$\overline{\psi} \rightarrow \overline{\psi} = e^{-i\theta}\psi$

$g = e^{i\theta} \in U(1)$ (Circle rotations)

Using this in the Lagrangian: $\mathcal{L}' = i\overline{\psi}'\gamma^\mu \partial_\mu \psi' - m\overline{\psi}'\psi$

$$\mathcal{L}' = i\,e^{-i\theta}\,\overline{\psi}\gamma^\mu \partial_\mu e^{i\theta}\psi - m\,e^{-i\theta}\,\overline{\psi}\,e^{i\theta}\psi$$

$$\mathcal{L} = i\psi\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

- **Noether theorem :** Global symmetry $\rightarrow$ Quantity conserved.

Conserved current: $J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \psi)}\Delta\psi + T^\mu{}_\nu\,\Delta x$

- $\pi(x) = \dfrac{\partial \mathcal{L}}{\partial(\dot{\psi})}$

$$\psi' = e^{i\theta}\psi = (1+i\theta)\psi = \psi + i\theta\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu \implies J^\mu = i\bar{\psi}\gamma^\mu\psi$$

Conserved Current: $J^\mu = ie\bar{\psi}\gamma^\mu\psi$.

Local: $\psi \to e^{i\theta(x)}\psi$; $\bar{\psi} \to e^{-i\theta(x)}\bar{\psi}$