Interaction Cross Sections: Flux of the Initial state

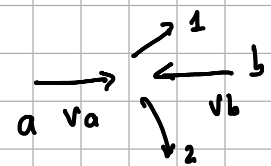
Flux: Number of particles crossing a unit area per unit time.

Beam of particles of type a, with flux ϕ_a crossing a region of space in which there are n_b particles per unit volume of type b.

$$r_b = \sigma \phi_a$$

Cross section is simply an expression of the underlying QM probability that an interaction will occur.

Scattering process $a+b \to 1+2$



rate: $\phi_a n_b V \sigma = (\vec{v_a}+\vec{v_b}) n_a n_b V \sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_a+v_b)} = \frac{(2\pi)^4}{(v_a+v_b)} \int |T_{fi}|^2 \delta(E_a+E_b-E_1-E_2)\, \delta^3(\vec{p_a}+\vec{p_b}-\vec{p_1}-\vec{p_2}) \frac{d^3p_1\, d^3p_2}{(2\pi)^3(2\pi)^3}$$

$$M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2}\, T_{fi}$$

$$\sigma = \frac{(2\pi)^{-2}}{4(v_a+v_b)E_1E_2} \int |M_{fi}|^2 \delta(E_a+E_b-E_1-E_2)\, \delta^3(\vec{p_a}+\vec{p_b}-\vec{p_1}-\vec{p_2}) \frac{d^3p_1\, d^3p_2}{2E_1\, 2E_2}$$

$F = 4E_aE_b(v_a+v_b)$ Lorentz invariant flux Factor.

$$F = 4E_aE_b\left(v_a+v_b\right) = 4E_aE_b\left(\frac{p_a}{E_a}+\frac{p_b}{E_b}\right) = 4(p_aE_b+p_bE_a)$$

$$F^2 = 16\left(E_a^2p_b^2 + E_b^2p_a^2 + 2E_aE_b\, p_a p_b\right)$$

$\to$ Colinear: $(p_a \cdot p_b)^2 = [(E_a, p_a)\cdot(E_b, -p_b)]^2 = (E_aE_b + p_ap_b)^2$

$$(p_a \cdot p_b)^2 = \left(E_a^2E_b^2 + 2E_aE_b\, p_ap_b + p_a^2p_b^2\right)$$

$$F^2 = 16\left(E_a^2 P_b^2 + E_b^2 P_a^2 + (P_a \cdot P_b)^2 - E_a^2 E_b^2 - P_a^2 P_b^2\right)$$

$$F^2 = 16\left((P_a \cdot P_b)^2 + E_a^2(P_b^2 - E_b^2) + P_a^2(E_b^2 - P_b^2)\right)$$

$$F^2 = 16\left((P_a \cdot P_b)^2 + (E_a^2 - P_a^2)(P_b^2 - E_b^2)\right) = 16\left((P_a \cdot P_b)^2 - (E_a^2 - P_a^2)(E_b^2 - P_b^2)\right)$$

So then:

$$\boxed{F = 4\left((P_a \cdot P_b)^2 - M_a M_b\right)^{1/2}}$$

Scattering in the CM frame:

$\vec{P}_a = -\vec{P}_b = P_i^*$ and $\vec{P}_1 = -\vec{P}_2 = P_f^*$

Center of mass $\sqrt{s} = (E_a^* + E_b^*)$

$$F = 4E_a^* E_b^* (v_a^* + v_b^*) = 4E_a^* E_b^* \left(\frac{P_i^*}{E_a^*} + \frac{P_i^*}{E_b^*}\right) = 4P_i^* \underbrace{(E_a^* + E_b^*)}_{\sqrt{s}}$$

Constraint $\vec{P}_a + \vec{P}_b = 0$ $\quad a + b \to 1 + 2$

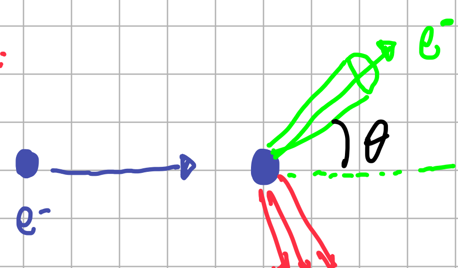
$$\sigma = \frac{(2\pi)^2}{4(v_a^* + v_b^*)E_a^* E_b^*} \int |M_{fi}|^2 \delta(E_a^* + E_b^* - E_1^* - E_2^*)\, \delta^3(\vec{P}_1 + \vec{P}_2) \frac{d^3\vec{P}_1}{2E_1} \frac{d^3\vec{P}_2}{2E_2}$$

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{4P_i^* \sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_1 - E_2)\, \delta^3(\vec{P}_1 + \vec{P}_2) \frac{d^3\vec{P}_1\, d^3\vec{P}_2}{2E_1\, 2E_2}$$

$$\boxed{\sigma = \frac{1}{(2\pi)^2} \frac{1}{4P_i^* \sqrt{s}} \frac{P_f^*}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*}$$

$$\sigma = \frac{1}{64\pi^2 s} \frac{P_f^*}{P_i^*} \int |M_{fi}|^2 d\Omega$$

Differential Cross section:



$e^- p \to e X$ where the proton breaks up.

Angular distribution of the scattered electron provides essential information about the fundamental physics of the interaction.

$$d\Omega = d(\cos\theta)\, d\phi$$

$$\frac{d\sigma}{d\Omega} = \frac{\#\text{ Particles scattered into } d\Omega \text{ per unit time per target particle}}{\text{Incident flux}}.$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega.$$

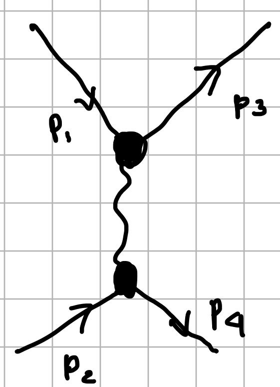
$$d\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|M_{fi}|^2 d\Omega^*.$$

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|M_{fi}|^2$$

$d\Omega^*$ in terms of the Mandelstam variables, specifically $t = p_1 - p_3$

For $e^- p \to e^- p$ scattering, t is a function of the initial and final state electron 4-momenta.

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$



$$t = (p_1^* - p_3^*)^2 = p_1^{*2} + p_3^{*2} - 2p_1^* p_3^*$$

$$t = m_1^{*2} + m_3^{*2} - 2(E_1^* E_3^* - \vec{p}_1 \cdot \vec{p}_3)$$

$$t = m_1^{*2} + m_3^{*2} - 2(E_1^* E_3^* - p_1^* p_3^* \cos\theta^*)$$

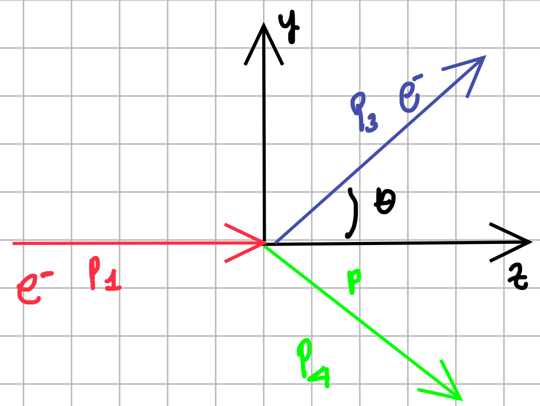
Free Parameter: $\to dt = 2p_1^* p_3^* d(\cos\theta^*) \to d(\cos\theta^*) = \dfrac{dt}{2p_1^* p_3^*}$

$$d\Omega = d(\cos\theta^*)\, d\phi^* = \frac{dt\, d\phi^*}{2p_1^* p_3^*}$$

$$d\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2 \frac{dt\, d\phi^*}{2 p_i^* p_f^*} = \frac{1}{128\pi^2 s} \frac{1}{p_i^*} |M_{fi}|^2 (2\pi)\, dt$$

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64 \pi s p_i^*} |M_{fi}|^2}$$

Lab Frame:



$$E^2 = p^2 + m^2$$

$$p_1 \approx (E_1, 0, 0, E_1)$$

$$p_2 = (m_p, 0, 0, 0)$$

$$p_3 \approx (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$

$$p_4 = (E_4, \vec{p}_4)$$

$$E_1 + m_p = E_p + E_3$$

Center of mass frame: $p_i^{2*} \approx \frac{1}{4s}[s - m_p^2]^2$

$$s = (p_1 + p_2)^2 = (p_1^2 + p_2^2 + 2 p_1 \cdot p_2) \approx m_p^2 + 2 E_1 m_p$$

Therefore $p_i^{2*} = \frac{1}{4[m_p^2 + 2E_1 m_p]} [2 E_1 m_p]^2 = \frac{E_1^2 m_p^2}{s}$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{2\pi\, d(\cos\theta)}$$

$$t = (p_1 - p_3)^2 = (E_1 - E_3, 0, -E_3 \sin\theta, E_1 - E_3 \cos\theta)$$

$$t = (E_1 - E_3)^2 - E_3^2 \sin^2\theta - (E_1 - E_3 \cos\theta)^2$$

$$t = (E_1 - E_3)^2 - E_3^2 \sin^2\theta - E_1^2 + 2 E_1 E_3 \cos\theta - E_3^2 \sin^2\theta$$

$$t = \cancel{E_1^2} - 2 E_1 E_3 + \cancel{E_3^2} - \cancel{E_3^2} - \cancel{E_1^2} + 2 E_1 E_3 \cos\theta$$

$$t = -2 E_1 E_3 (1 - \cos\theta)$$

$$p_1 + p_2 = p_3 + p_4 \qquad (p_1 - p_3)^2 = (p_2 - p_4)^4$$

$t = (p_2 - p_4)^2 = (m_p - E_4, -\vec{p}_4)^2 = (m_p - E_4)^2 - \vec{p}_4^{\,2}$

$t = m_p^2 - 2m_p E_4 + E_4^2 - \vec{p}_4^{\,2} = 2m_p^2 - 2m_p E_4 = -2m_p(E_4 - m_p)$

Energy Conservation $\left\{ \begin{array}{l} E_1 + m_p = E_3 + E_4 \\ \text{Then } E_4 = E_1 + m_p - E_3 \end{array} \right.$

So $t = -2m_p(E_1 + m_p - E_3 - m_p) = -2m_p(E_1 - E_3)$

then $-\cancel{2}E_1 E_3(1 - \cos\theta) = -\cancel{2}m_p(E_1 - E_3)$

$E_1 E_3 (1 - \cos\theta) = m_p E_1 - m_p E_3$

$E_3(E_1 - E_1 \cos\theta) + m_p E_3 = m_p E_1$

$$\boxed{E_3 = \frac{m_p E_1}{E_1(1-\cos\theta) + m_p}}$$

$$\frac{dt}{d(\cos\theta)} = 2m_p \frac{dE_3}{d(\cos\theta)} = 2m_p \left[-\frac{m_p E_1}{(E_1(1-\cos\theta) + m_p)^2} (-E_1) \right]$$

$$= 2 \frac{m_p^2 E_1^2}{(E_1(1-\cos\theta) + m_p)^2} = 2E_3^2$$

Finally $\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}(2E_3^2)\frac{1}{64\pi s p_i^2}|M_{fi}|^2 = \frac{E_3^2}{64\pi^2 s p_i^2}|M_{fi}|^2$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3^2}{E_1^2 m_p^2}\right)|M_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$