1 Connectivity

1.1 Menger's theorem

1.2 3-connected graphs

Lemma 1. *Every* 3*-connected graph* G *where* $G \neq K_4$ *has an edge* e *such that* G/e *is* 3*-connected.*

Proof. Suppose no edge e exists. Then every edge G/xy contains a separator of at most 2 vertices. Since G is 3-connected, then the contracted vertex y is in S, and $|S| = 2$. Call the separator $S = \{v_{xy}, z\}$. Then any two vertices separated by S in G/xy is also separated by $T := \{x, y, z\}$ in G. Since no proper subset of T separates G, every vertex in T has a neighbour in every component C of $G - T$. Now choose the edge xy, vertex z and component C so that $|C|$ is small as possible. Pick a neighbour v of z in C. By assumption G/zv is not 3-connected, so there is a vertex w such that $\{z, v, w\}$ separates G. As x, y are adjacent, $G - \{z, v, w\}$ has a component D such that $D \cap \{x, y\} = \emptyset$. Then every neighbour of v in D lies in C so $D \cap C \neq \emptyset$ and so D is a proper subset of C. But this contradicts the minimality of C. $\square$