1 Connectivity

1.1 3-connected graphs

Lemma 1. *Every 3-connected graph G where $G \neq K_4$ has an edge e such that G/e is 3-connected.*

Proof. Suppose no edge e exists. Then every edge G/xy contains a separator of at most 2 vertices. Since G is 3-connected, then the contracted vertex y is in S, and $|S| = 2$. Call the separator $S = \{v_{xy}, z\}$. Then any two vertices separated by S in G/xy is also separated by $T := \{x, y, z\}$ in G. Since no proper subset of T separates G, every vertex in T has a neighbour in every component C of $G - T$. Now choose the edge xy, vertex z and component C so that $|C|$ is small as possible. Pick a neighbour v of z in C. By assumption G/zv is not 3-connected, so there is a vertex w such that $\{z, v, w\}$ separates G. As x, y are adjacent, $G - \{z, v, w\}$ has a component D such that $D \cap \{x, y\} = \emptyset$. Then every neighbour of v in D lies in C so $D \cap C \neq \emptyset$ and so D is a proper subset of C. But this contradicts the minimality of C. $\square$

1.2 Menger's theorem

Theorem 2. *Let G be a graph and $A, B \subseteq V(G)$. Then the minimum sized $A - B$ separator in G is equal to the maximum number of disjoint $A - B$ paths in G.*

Let $k(G, A, B)$ be the size of the minimum separator of A and B. Clearly, the number of disjoint paths is at most $k = k(G, A, B)$. We wish to show that k paths exist.

Proof. Apply induction on $|E(G)|$. If G has no edges, then $|A \cap B| = k$, so there are k disjoint $A - B$ paths. Now assume G has an edge $e = xy$. Assume for the contrary that G has no k disjoint $A - B$ paths. Then neither does G/e. G/e has an $A - B$ separator Y with fewer than k vertices. The contracted vertex v_e is in Y otherwise $Y \subseteq V$ is an AB-separator in G. Then $X = Y \setminus \{v_e\} \cup \{x, y\}$ is an $A - B$ separator in G with exactly k vertices.

Now consider $G - e$. Since $x, y \in X$, every $A - X$ separator in $G - e$ is also an $A - B$ separator in G and hence contains at least k vertices. There are k disjoint $A - X$ paths in $G - e$, similarly there are k disjoint paths in $G - e$. As X separates A from B, these two path systems do not meet outside X and can be joined to form k disjoint $A - B$ paths. $\square$