1 Planar graphs

1.1 Kuratowski-Wagner theorem

Wagner [2] prove the following:

Theorem 1 ([2])**.** *A graph G is planar if and only if G is K_5-minor-free and $K_{3,3}$-minor-free.*

The forwards direction is easy. K_5 is nonplanar and $K_{3,3}$ is nonplanar (from Euler's theorem).

Kuratowski [1] proved a refinement. A graph H is a *topological minor* of a graph G if a graph subdivision of H is isomorphic to a subgraph of G. Topological minors are a type of graph minor.

Theorem 2 ([1])**.** *A graph G is planar if and only if G does not have K_5 as a topological minor or $K_{3,3}$ as a topological minor.*

It is easy to show that topological minors are minors.

Lemma 3. *Let G, H be graphs. If H is a minor of G and $\Delta(H) \leq 3$, then H is a topological minor of G.*

Proof. Let K be a minimal model of H in G. Then the branch sets of K are subdivisions of $K_{1,3}$ and touch each model exactly once. Then H is a model in G. ☐

Lemma 4. *A graph G contains K_5 or $K_{3,3}$ as a minor if and only if G contains K_5 or $K_{3,3}$ as a topological minor.*

Proof. Topological minor implies minor, so that is the backwards direction. $K_{3,3}$ has degree ≤ 3 and therefore if $K_{3,3}$ is a minor of G $K_{3,3}$ is a topological minor of G.

Suppose $K_5 \leq G$. Then either K_5 is a topological minor of G or $K_{3,3}$ is a minor of G. Let K be a minimal model of K_5 in G. Every branch set of K is a tree in G. Between any two branch sets K has exactly one edge. Take the tree induced by V_x, $x \in K$. If every branch set of K is a subdivision of the star $K_{1,3}$, then this is a subdivision of K_5 in G. Otherwise, there is a branch set which contains two vertices of degree 3. The branch sets that the vertices are adjacent to are split evenly between these two vertices as the branch set is a tree. Then contracting that branch set to two vertices and every other branch set to a single vertex yields a $K_{3,3}$ topological minor. ☐

Lemma 5. *Every 3-connected graph G without K_5 or $K_{3,3}$ as a minor is planar.*

Proof. Proof by induction. If $|V(G)| = 4$, then $G = K_4$, and any subgraph of K_4 is planar. Now suppose $|V(G)| > 4$. Since G is 3-connected, G has an edge xy such that G/xy is 3-connected. Then G/xy has no K_5 or $K_{3,3}$ minor, so prove
G/xy can be drawn on the plane. Call the plane graph $\tilde{G}$. Let f be the face of $\tilde{G} - v_{xy}$ containing v_{xy} and let C be the boundary of f. Let $X := N_G(x) \setminus \{y\}$,

let $Y := N_G(y) \setminus \{x\}$. Then $X \cup Y$ is in C. Now we draw $G - y$ by replacing v_{xy} with x. Our aim is to add back on y. Since $\tilde{G}$ is 3-connected, $\tilde{G} - v_{xy}$ is 2-connected, so C is a cycle. Let $x_1, \dots, x_k$ be the enumeration around C of the vertices in X and let $P_i := x_i, \dots, x_{i+1}$ be the subpaths on C between. Then $Y \subseteq V(P_i)$.

Suppose not. If y has a neighbour y' in the interior of P_i for some i and another neighbour y'' in $C - P_i$, separated by $x' = x_i$ and x_{i+1}. If $Y \subseteq X$ and $|Y \cap X| \leq 2$, then y has exactly two neighbours y', y'' on C not in the same P_i, so y' and y'' are separated by two vertices x', x'' in X. Then x, y', y'' and y, x', x'' are the two ends of a subdivision of $K_{3,3}$. Then the other case is that y, x have three common neighbours on C. Then this forms a subvivision of K_5.

Then we can draw y on the arc between x and P_i where $Y \subseteq P_i$. □

Lemma 6. *Every graph G has a tree-decomposition $(T, (B_x)_x)$ of adhesion 2 where every torso is either $K_{\leq 3}$ or a 3-connected graph. Moreover, if K is not a minor of G then K is not a minor of any torso.*

Proof. □

simple proof, use induction

Suppose G has a tree-decomposition of this kind above. Then glue together the graph to form the embedding of G on a surface.

1.2 Exercises

Every graph can be embedded in $\mathbb{R}^3$ with all edges straight. By analysis, we can place all vertices so that no four vertices lie on a flat plane.

Planar graphs are a minor-closed class. Let G be a planar graph. Subgraphs of G are obviously planar graphs. Now let uv be an edge. Then taking a deformation retraction of uv to a point preserves the topology. Therefore, edge contractions of G are also planar. Then every planar graph is a minor of a grid. Let G be a planar graph. Then let G_n be the $n \times n$ grid, where $n = |V(G)|^2$. Then we can delete vertices and contract edges of G_n to form G.

References

[1] Casimir Kuratowski. "Sur Le Problème Des Courbes Gauches En Topologie". In: *Fundamenta Mathematicae* 15 (1930), pp. 271–283. ISSN: 0016-2736, 1730-6329. DOI: 10.4064/fm-15-1-271-283.

[2] K. Wagner. "Über eine Eigenschaft der ebenen Komplexe". In: *Mathematische Annalen* 114.1 (Dec. 1, 1937), pp. 570–590. ISSN: 1432-1807. DOI: 10.1007/BF01594196.