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1 Outline

In this lecture, we cover

- modeling with VaR,
- comparing risk measures,
- the newsvendor problem with various risk measures.

2 Modeling with VaR

Assume that we can model any constraint of the form $g(x, \xi_i) \leq b_i$. Based on this, let us try to model

$$\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \leq 0.$$

This is equivalent to

$$\min\left\{t : \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \leq t\right] > \alpha\right\} \leq 0.$$

We may rewrite this as

$$\begin{aligned} t &\leq 0 \\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \leq t\right] &> \alpha \end{aligned}$$

Without loss of generality, we can take $t = 0$ and just consider

$$\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \leq 0\right] > \alpha.$$

This is because $\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \leq 0\right]$ never decreases as t increases.

Therefore, $\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \leq 0$ is equivalent to a **chance constraint**.

$$\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \leq 0\right] > \alpha \quad \Leftrightarrow \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) > 0\right] \leq 1-\alpha.$$

To model this, we introduce binary variables $z_i \in \{0, 1\}$ for $i \in [N]$ for scenarios.

$$z_i = \begin{cases} 1, & \text{if } g(x,\xi_i) > 0 \\ 0, & \text{otherwise.} \end{cases}.$$

Basically, we add implications

$$z_i = 0 \quad \Rightarrow \quad g(x,\xi_i) \leq 0, \quad i \in [N]$$

This can be modelled with the big-M technique:

$$g(x,\xi_i) \leq M z_i, \quad i \in [N].$$

We need to ensure that the probability $g(x, \xi) > 0$ is no greater than $1 - \alpha$:

$$\sum_{i \in [N]} p_i z_i \leq 1 - \alpha.$$

In summary,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \mathrm{VaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) \leq 0 \\ & x \in \mathcal{X} \end{aligned}$$

is equivalent to

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x, \xi_i) \leq M z_i, \quad i \in [N] \\ & \sum_{i \in [N]} p_i z_i \leq 1 - \alpha \\ & x \in \mathcal{X}, \ z \in \{0, 1\}^N \end{aligned}$$

3 Which risk measure should we use?

3.1 Modeling aspects

Value at Risk (VaR) and Conditional Value at Risk (CVaR) can adjust the **degree of risk aversion** by adjusting the parameter α. The higher the value of α, the greater the risk aversion of the decision x. Note that

$$\begin{aligned} \lim_{\alpha \to 1} \mathrm{VaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) &= \max_{i \in [N]} g(x, \xi_i) \\ \lim_{\alpha \to 1} \mathrm{CVaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) &= \max_{i \in [N]} g(x, \xi_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\alpha \to 1} \mathrm{VaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) \leq 0 \quad &\Leftrightarrow \quad g(x, \xi_i) \leq 0 \quad \forall i \in [N] \\ \lim_{\alpha \to 1} \mathrm{CVaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) \leq 0 \quad &\Leftrightarrow \quad g(x, \xi_i) \leq 0 \quad \forall i \in [N] \end{aligned}$$

In contrast,

$$\begin{aligned} \lim_{\alpha \to 0} \mathrm{VaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) &= \min_{i \in [N]} g(x, \xi_i) \\ \lim_{\alpha \to 0} \mathrm{CVaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) &= \mathbb{E}_{\xi \sim \hat{P}_N} \left[g(x, \xi) \right]. \end{aligned}$$

Moreover, we always have

$$\mathrm{CVaR}_\alpha \left(g(x, \xi); \hat{P}_N \right) \geq \mathrm{VaR}_\alpha \left(g(x, \xi); \hat{P}_N \right).$$

The Conditional Value at Risk takes into account the **tail risk/tail probability** whereas the Value at Risk does not. Considering the tail risk is important for some applications, but it leads to more conservative solutions.

A guideline is as follows. CVaR is used when we have uncertain objectives or when computing VaR is too demanding. VaR is used when we have uncertain constraints.

3.2 Computational aspects

Expectation and worst-case value are relatively simple. Computing expectation requires a new function

$$\frac{1}{N}\sum_{i=1}^{N} g(x,\xi_i),$$

which is essentially the sum of functions $g(x,\xi_1),\ldots,g(x,\xi_N)$. The worst-case vlaue requires N constraints

$$g(x,\xi_i)\leq 0 \quad i\in[N].$$

CVaR requires adding new variables $t, r_1,\ldots,r_N$ that are all continuous and imposing $2N+1$ new constraints given by

$$\begin{aligned} t+\frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \leq 0 \\ r_i \geq 0 \quad \forall i\in[N] \\ t+r_i \geq g(x,\xi_i) \quad \forall i\in[N] \end{aligned}$$

VaR requires adding new binary variables $z_1,\ldots,z_N$ and $N+1$ new constraints given by

$$\begin{aligned} g(x,\xi_i)\leq Mz_i, \quad i\in[N] \\ \sum_{i\in[N]} p_i z_i \leq 1-\alpha \end{aligned}$$

Moreover, when using VaR and CVaR, we need to choose the parameter α.

4 Newsvendor problem

Recall the newsvendor problem. The newsvendor must choose a quantity $x\in\mathbb{Z}_+$ of newpapers to print on a particular day. The cost of printing a copy is $c>0$. Each copy is sold at price p, and we assume that $p>c$. We also assume that each copy of unsold newspapers must be discarded at cost $h>0$. The profit from printing x copies under customer demand ξ is given by

$$f(x,\xi)=p\cdot\min\{x,\xi\}-cx-h\cdot\max\{0,x-\xi\}.$$

The demand ξ is unknown, but we have historical data $\xi_1,\ldots,\xi_N$ that can be treated as scenarios. Note that

$$f(x,\xi)=\begin{cases} p\xi-cx-h(x-\xi), & \text{if } x\geq\xi, \\ px-cx & \text{if } x<\xi. \end{cases}$$

Moreover, $p\xi-cx-h(x-\xi)\leq px-cx$ if and only if $x\geq\xi$. Hence, it follows that

$$f(x,\xi)=\min\{(p+h)\xi-(c+h)x,\ (p-c)x\}.$$

Here, we want to **maximize** the profit. Although the demand ξ is unknown, we define risk measures based on the data set $\{\xi_1,\ldots,\xi_N\}$. Consider

$$\rho_f\left(x;\{\xi_i\}_{i\in[N]}\right).$$

Then we want to solve

$$\max_{x} \quad \rho_f\left(x; \{\xi_i\}_{i\in[N]}\right).$$

Previously, we considered risk measures for the constraint function, with the goal of achieving a low value. However, for the newsvendor problem, we want to achieve a high profit. Nevertheless, the developments for the constraint function extend to the maximizing objective.

4.1 Expectation

The expectation risk measure is given by

$$\rho_f\left(x; \{\xi_i\}_{i\in[N]}\right) = \frac{1}{N}\sum_{i=1}^{N} f(x,\xi_i).$$

Basically, we want to solve

$$\max \quad \frac{1}{N}\sum_{i=1}^{N} f(x,\xi_i).$$

This can be reformulated as

$$\begin{aligned} \max \quad & \frac{1}{N}\sum_{i=1}^{N} t_i \\ \text{s.t.} \quad & f(x,\xi_i) \geq t_i, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

Plugging in the formula $f(x,\xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$\begin{aligned} \min \quad & \frac{1}{N}\sum_{i=1}^{N} t_i \\ \text{s.t.} \quad & (p+h)\xi_i - (c+h)x \geq t_i, \quad \forall i \in [N] \\ & (p-c)x \geq t_i, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

4.2 Worst-case value

Next, we consider

$$\rho_f\left(x; \{\xi_i\}_{i\in[N]}\right) = \min_{i\in[N]} f(x,\xi_i).$$

Here, the worst-case value is defined as the **minimum** of $f(x,\xi_1),\ldots,f(x,\xi_N)$. Again, this is because we are trying to maximize the profit. Then we want to solve

$$\max_{x} \quad \min_{i\in[N]} f(x,\xi_i).$$

This can be reformulated as

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & f(x,\xi_i) \geq t, \quad \forall i \in [N], \\ & x \in \mathbb{Z}_+ \end{aligned}$$

Plugging in the formula $f(x,\xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & (p+h)\xi_i - (c+h)x \geq t, \quad \forall i \in [N] \\ & (p-c)x \geq t, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

4.3 Conditional Value at Risk

We may reformulate the problem $\max_x f(x,\xi)$ as

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & f(x,\xi) \geq v. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v - f(x,\xi) \leq 0. \end{aligned}$$

Then $v - f(x,\xi)$ can be viewed as a constraint function. Here, the Conditional Value at Risk at level $\alpha \in (0,1)$ defined as

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(v - f(x,\xi); \hat{P}_N\right) \quad = \quad \max \quad & \frac{1}{1-\alpha}\sum_{i=1}^{N} z_i \cdot (v - f(x,\xi_i)) \\ \text{s.t.} \quad & 0 \leq z_i \leq p_i \\ & \sum_{i=1}^{N} z_i = 1 - \alpha. \end{aligned}$$

By strong LP duality,

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(v - f(x,\xi); \hat{P}_N\right) \quad = \quad \min \quad & t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \\ \text{s.t.} \quad & r_i \geq 0, \quad i \in [N] \\ & t + r_i \geq v - f(x,\xi_i), \quad i \in [N]. \end{aligned}$$

Then

$$\mathrm{CVaR}_\alpha\left(v - f(x,\xi); \hat{P}_N\right) \leq 0$$

can be written as

$$\begin{aligned} t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \leq 0 \\ r_i \geq 0, \quad i \in [N] \\ t + r_i \geq v - f(x,\xi_i), \quad i \in [N]. \end{aligned}$$

Then we solve

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & t + \frac{1}{1-\alpha} \sum_{i=1}^{N} p_i r_i \leq 0 \\
& r_i \geq 0, \quad i \in [N] \\
& v - t - r_i \leq f(x, \xi_i), \quad i \in [N] \\
& x \in \mathbb{Z}_+.
\end{aligned}
$$

Plugging in the formula $f(x, \xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & t + \frac{1}{1-\alpha} \sum_{i=1}^{N} p_i r_i \leq 0 \\
& r_i \geq 0, \quad i \in [N] \\
& v - t - r_i \leq (p+h)\xi - (c+h)x, \quad i \in [N] \\
& v - t - r_i \leq (p-c)x, \quad i \in [N] \\
& x \in \mathbb{Z}_+.
\end{aligned}
$$