

Reinforcement Learning for Infinite-Horizon Average-Reward MDPs with Multinomial Logistic Function Approximation

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Abstract

We study model-based reinforcement learning with non-linear function approximation where the transition function of the underlying Markov decision process (MDP) is given by a multinomial logistic (MNL) model. In this paper, we develop two algorithms for the infinite-horizon average reward setting. Our first algorithm UCRL2-MNL applies to the class of communicating MDPs and achieves an $\tilde{O}(dD\sqrt{T})$ regret, where d is the dimension of feature mapping, D is the diameter of the underlying MDP, and T is the horizon. The second algorithm OVIFH-MNL is computationally more efficient and applies to the more general class of weakly communicating MDPs, for which we show a regret guarantee of $\tilde{O}(d^{2/5}\text{sp}(v^*)T^{4/5})$ where $\text{sp}(v^*)$ is the span of the associated optimal bias function.

We also prove a lower bound of $\Omega(d\sqrt{DT})$ for learning communicating MDPs with MNL transitions of diameter at most D . Furthermore, we show a regret lower bound of $\Omega(dH^{3/2}\sqrt{K})$ for learning H -horizon episodic MDPs with MNL function approximation where K is the number of episodes, which improves upon the best-known lower bound for the finite-horizon setting.

Keywords: Reinforcement Learning, Multinomial Logistic Model, Infinite-Horizon Average-Reward MDP, Regret Analysis

1 Introduction

Function approximation schemes have been successful in modern reinforcement learning under the presence of large state and action spaces. Applications and domains where function approximation approaches have been deployed include Atari games (Mnih et al., 2015), Go (Silver et al., 2017), robotics (Kober et al., 2013), and autonomous driving (Yurtsever et al., 2020). Such empirical success has motivated a plethora of theoretical studies that establish provable guarantees for reinforcement learning with function approximation. The first line of theoretical work considers linear function approximation, such as linear Markov Decision Processes (MDPs) (Yang and Wang, 2019) and linear mixture MDPs (Modi et al., 2020) where the reward and transition functions are linear. While (nearly) minimax optimal algorithms have been developed for linear MDPs (He et al., 2023; Agarwal et al., 2023; Hu et al., 2022) and for linear mixture MDPs (Zhou et al., 2021), the linearity assumption is restrictive and rarely holds in practice. In particular, when a linear model is misspecified, those algorithms may suffer from linear regret (Jin et al., 2020).

Reinforcement learning with *general* function approximation has recently emerged as an alternative to the linear function approximation framework. The term *general* here means that it makes minimal structural assumptions about the family of functions taken for approximation. Some concepts that lead to conditions ensuring sample-efficient learning are the Bellman rank (Jiang et al., 2017), the eluder dimension (Wang et al., 2020), the Bellman eluder dimension (Jin et al., 2021), the bilinear class (Du et al., 2021), the decision-estimation coefficient (Foster et al., 2023), and the generalized eluder coefficient (Zhong et al., 2023). Recently, He et al. (2024) considered infinite-horizon average-reward MDP with general function approximation. However, algorithms for these frameworks require an oracle to query from some abstract function class. In practice, the oracle would correspond to solving an abstract optimization or regression problem. Furthermore, no regret lower bound has been identified for a general function approximation framework.

More concrete non-linear function approximation models have been proposed recently. Yang et al. (2020); Xu and Gu (2020); Fan et al. (2020) considered representing the Q function by an overparametrized neural network based on the neural tangent kernel. Wang et al. (2021) studied generalized linear models for approximating the Q function. Liu et al. (2022); Zhang et al. (2023) focused on the case where the Q function is smooth and lies in the Besov space or the Barron space, and they used a two-layer neural network to approximate the Q function. In contrast to these works, Hwang and Oh (2023) proposed a framework to represent the transition function by a multinomial logistic model.

Indeed, the multinomial logistic model can naturally represent state transition probabilities, providing a practical alternative to linear function approximation. The model is widely used for modeling multiple outcomes, such as multiclass classification (Bishop, 2006), news recommendations (Li et al., 2010, 2012), and assortment optimization (Caro and Gallien, 2007).

Hwang and Oh (2023) initiated the study of RL with the MNL function approximation framework. They developed a computationally efficient model-based algorithm, UCRL-MNL, and proved that the algorithm achieves an $\tilde{O}(dH^{3/2}\sqrt{T})$ regret bound where d is the dimension of the transition core, H is the horizon, and T is the total number of steps. While they did not provide a lower bound, they conjectured that the regret upper bound is the best possible. Recently, Cho et al. (2024) proposed computationally efficient sampling-based algorithms, Li et al. (2024) developed algorithms that improve the per-iteration complexity of UCRL-MNL based on online estimation of the transition core. Li et al. (2024) presented the first lower bound for this setting, given by $\Omega(dH\sqrt{T})$.

This paper contributes to the RL with MNL approximation literature with the following new theoretical results.

- We prove that there is a family of H -horizon episodic MDPs with MNL transitions for which any algorithm incurs a regret of $\Omega(dH^{3/2}\sqrt{T})$. This improves upon the lower bound of $\Omega(dH\sqrt{T})$ due to Li et al. (2024) by a factor of $O(\sqrt{H})$.
- We initiate the study of learning infinite-horizon average-reward MDPs with MNL function approximation. For the class of communicating MDPs with diameter at most D , we develop an extended value iteration-based algorithm, UCRL2-MNL, that guarantees a regret upper bound of $\tilde{O}(dD\sqrt{T})$.

- For the class of weakly communicating MDPs, we propose an episodic optimistic value iteration-based algorithm, **OVIFH-MNL**, that attains a regret upper bound of $\tilde{O}(d^{2/5}\text{sp}(v^*)T^{4/5})$ where $\text{sp}(v^*)$ denotes the span of the optimal associated bias function.
- We prove a lower bound of $\Omega(d\sqrt{DT})$ for learning infinite-horizon average-reward communicating MDPs with MNL transitions and diameter at most D .

2 Preliminaries and Problem Settings

Notations We use $\|x\|_2$ to denote the ℓ_2 -norm of a vector x . For a positive definite matrix $A \in \mathbb{R}^{d \times d}$ and a vector $x \in \mathbb{R}^d$, we denote by $\|x\|_A = \sqrt{x^\top A x}$ the weighted ℓ_2 -norm of x . Given a matrix A , $\|A\|_2$ denotes its spectral norm. For a symmetric matrix A , let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote its minimum and maximum eigenvalues, respectively. Let $\mathbf{1}\{\mathcal{E}\}$ be the indicator function of event \mathcal{E} . We say that a random variable $Y \in \mathbb{R}$ is R -sub-Gaussian if $\mathbb{E}[Y] = 0$ and $\mathbb{E}[\exp(sY)] \leq \exp(R^2 s^2/2)$ for any $s \in \mathbb{R}$. Let $\Delta(\mathcal{X})$ denote the family of probability measures on \mathcal{X} .

2.1 Infinite-Horizon Average-Reward MDP

We consider an infinite-horizon MDP specified by $M = (\mathcal{S}, \mathcal{A}, p, r)$, where \mathcal{S} is the state space, \mathcal{A} is the action space, $p(s' | s, a)$ denotes the transition probability of transitioning to state s' from state s after taking action a , and $r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ is the instantaneous reward function. Throughout this paper, we assume that both \mathcal{S} and \mathcal{A} are finite. A stationary policy $\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$ is given by $\pi(a | s)$ specifying the probability of taking action a at state s . When π is deterministic, i.e., for each $s \in \mathcal{S}$ there exists $a \in \mathcal{A}$ with $\pi(a | s) = 1$, we write that $a = \pi(s)$ with abuse of notation. Starting from an initial state s_1 , for each time step t , an algorithm \mathfrak{A} selects action a_t based on state s_t , and then s_{t+1} is drawn according to the transition function $p(\cdot | s_t, a_t)$. Then we consider the cumulative reward incurred over T times steps and the average reward defined as

$$R(M, \mathfrak{A}, s, T) = \sum_{t=1}^T r(s_t, a_t) \quad \text{and} \quad J(M, \mathfrak{A}, s) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[R(M, \mathfrak{A}, s, T)],$$

respectively. It is known that the average reward can be maximized by a deterministic stationary policy (see (Puterman, 2014)).

In this paper, following Jaksch et al. (2010), we focus on communicating MDPs which have a finite diameter. Here, the diameter is defined as follows. Given an MDP M and a policy π , let $T(s' | M, \pi, s)$ denotes the number of steps after which state s' is reached from state s for the first time. Then the diameter of M is defined as $D(M) = \max_{s \neq s' \in \mathcal{S}} \min_{\pi: \mathcal{S} \rightarrow \mathcal{A}} \mathbb{E}[T(s' | M, \pi, s)]$. For a communicating MDP M , it is known that the optimal average reward does not depend on the initial state s (Puterman, 2014), and therefore, there exists $J^*(M)$ such that

$$J^*(M) = J^*(M, s) := \max_{\mathfrak{A}} J(M, \mathfrak{A}, s).$$

Based on this, we consider the following notion of regret to assess the performance of any algorithm.

$$\text{Regret}(M, \mathfrak{A}, s, T) = T \cdot J^*(M) - R(M, \mathfrak{A}, s, T).$$

2.2 Multinomial Logistic Function Approximation

Despite being finite, the state space \mathcal{S} and the action space \mathcal{A} can be intractably large, in which case tabular model-based reinforcement learning algorithms suffer from a large regret. To remedy this, linear and linear mixture MDPs take some structural assumptions on the underlying MDP which lead to efficient learning. However, imposing linearity structures is indeed restrictive and limits the scope of practical applications. Inspired by this issue, we consider the recent framework of MNL function approximation proposed by [Hwang and Oh \(2023\)](#), which assumes that the transition function is given by a feature-based multinomial logistic model as follows. For each $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$, its associated feature vector $\varphi(s, a, s') \in \mathbb{R}^d$ is known, and the transition probability is given by

$$p(s' \mid s, a) := \frac{\exp(\varphi(s, a, s')^\top \theta^*)}{\sum_{s'' \in \mathcal{S}_{s,a}} \exp(\varphi(s, a, s'')^\top \theta^*)} \quad (1)$$

where $\theta^* \in \mathbb{R}^d$ is an unknown parameter, which we refer to as the transition core, and $\mathcal{S}_{s,a} := \{s' \in \mathcal{S} : \mathbb{P}(s' \mid s, a) > 0\}$ is the set of reachable states from s in one step after taking action a . Let $\mathcal{U} := \max_{(s,a) \in \mathcal{S} \times \mathcal{A}} |\mathcal{S}_{s,a}|$. The general intuition is that the ambient dimension d of the feature vectors and the parameter vector is small compared to the size of \mathcal{S} and that of \mathcal{A} . Moreover, it is often the case that $\mathcal{S}_{s,a}$ is small in comparison with \mathcal{S} .

3 Algorithms

We present two algorithms for learning infinite-horizon average reward MDPs with multinomial logistic function approximation. The first is UCRL2-MNL (Algorithm 1 in Section 3.2), named after UCRL2 of [Jaksch et al. \(2010\)](#). UCRL2-MNL runs extended value iteration for each episode as UCRL2, while it uses certain confidence sets, introduced in Section 3.1, designed to estimate the transition core of the underlying multinomial logistic transition model. UCRL2-MNL is also closely related to UCRL2-VTR by [Wu et al. \(2022\)](#) developed for linear mixture MDPs. The second algorithm is OVIFH-MNL (Algorithm 2 in Section 3.3), where OVIFH stands for optimistic value iteration over a finite horizon. The main idea behind it is to divide the horizon into fixed-length episodes and apply a finite-horizon episodic RL method with multinomial logistic approximation, which is similar in spirit to OLSVI.FH of [Wei et al. \(2021\)](#). Although UCRL2-MNL achieves a better regret guarantee than OVIFH-MNL, it is computationally more tractable than UCRL2-MNL.

3.1 Confidence Sets for the Transition Core

We may estimate the transition core θ^* via maximum likelihood estimation. To elaborate, we define the transition response variable $y_{t,s'} := \mathbf{1}\{s_{t+1} = s'\}$ for $t \in [T]$ and $s' \in \mathcal{S}_{s_t, a_t}$. Here, $y_{t,s'}$ basically corresponds to a sample from the multinomial distribution over \mathcal{S}_{s_t, a_t}

with probability $p(s'|s_t, a_t)$. Next, we introduce notation $p_t(s', \theta)$ to denote

$$p_t(s', \theta) = p(s' | s_t, a_t, \theta) \quad \text{where} \quad p(s' | s, a, \theta) := \frac{\exp(\varphi(s, a, s')^\top \theta)}{\sum_{s'' \in \mathcal{S}_{s,a}} \exp(\varphi(s, a, s'')^\top \theta)}$$

Then we have $p(s' | s, a, \theta^*) = p(s' | s, a)$ and $p_t(s', \theta^*) = p(s' | s_t, a_t)$. Then for each time step $t \in [T]$, the log-likelihood function and the ridge penalized maximum likelihood estimator are given by

$$\ell_t(\theta) = \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_{s_i, a_i}} y_{i,s'} \log p_i(s', \theta) \quad \text{and} \quad \hat{\theta}_t = \operatorname{argmax}_{\theta} \left\{ \ell_t(\theta) - \frac{\lambda}{2} \|\theta\|_2^2 \right\} \quad (2)$$

for some $\lambda > 0$, respectively. Before we construct confidence sets for the transition core θ^* , let us state some assumptions that hold throughout the paper.

Assumption 1 *There exist some positive constants L_φ, L_θ such that $\|\varphi(s, a, s')\|_2 \leq L_\varphi$ for all $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$ and $\|\theta^*\|_2 \leq L_\theta$.*

Assumption 2 *There exists $0 < \kappa < 1$ such that for all $t \in [T]$ and $s', s'' \in \mathcal{S}_{s_t, a_t}$, we have $\inf_{\theta \in \mathbb{R}^d} p_t(s', \theta) p_t(s'', \theta) \geq \kappa$.*

Assumption 1 makes our regret bounds scale-free for convenience and is indeed standard in contextual bandits and RL with function approximation. Moreover, Assumption 2 is also common in generalized linear contextual bandit literature (Filippi et al., 2010; Li et al., 2017; Oh and Iyengar, 2019; Kveton et al., 2020; Russac et al., 2020) and is taken for RL with MNL function approximation (Hwang and Oh, 2023; Li et al., 2024; Cho et al., 2024). Assumption 2 guarantees that the associated Fisher information matrix of the log-likelihood function in our setting is non-singular. The last assumption is as follows.

Assumption 3 *For every $(s, a) \in \mathcal{S} \times \mathcal{A}$, there exists $s' \in \mathcal{S}_{s,a}$ such that $\varphi(s, a, s') = 0$.*

In fact, we may impose Assumption 3 without loss of generality, by the following procedure. For a given pair $(s, a) \in \mathcal{S} \times \mathcal{A}$, we take an arbitrary $s' \in \mathcal{S}_{s,a}$ and replace $\varphi(s, a, s'')$ by $\varphi(s, a, s'') - \varphi(s, a, s')$. Note that $\|\varphi(s, a, s'') - \varphi(s, a, s')\|_2 \leq 2L_\varphi$ and the probability term $p(s' | s, a, \theta)$ remains the same. Therefore, up to doubling the parameter L_φ , Assumptions 1 and 2 remain valid even after the procedure to enforce Assumption 3.

Let us now define some confidence sets for the transition core. To simplify notations, we refer to $\varphi(s_t, a_t, s')$ by $\varphi_{t,s'}$ for each t and $s' \in \mathcal{S}_{s_t, a_t}$. Moreover, we define our gram matrix A_t for $t \in [T]$ as

$$A_t := \lambda I_d + \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_{s_i, a_i}} \varphi_{i,s'} \varphi_{i,s'}^\top \quad (3)$$

where λ is the same regularization parameter used in (2), I_d is the $d \times d$ identity matrix. Then for each $t \in [T]$, we construct and consider a confidence set \mathcal{C}_t for θ^* given by

$$\mathcal{C}_t := \left\{ \theta \in \mathbb{R}^d : \left\| \theta - \hat{\theta}_t \right\|_{A_t} \leq \beta_t \right\} \quad \text{where} \quad \beta_t = \frac{1}{\kappa} \sqrt{d \log \left(1 + \frac{t \mathcal{U} L_\varphi^2}{d \lambda} \right) + 2 \log \frac{1}{\delta}} + \frac{\sqrt{\lambda}}{\kappa} L_\theta \quad (4)$$

for some $\delta > 0$ where $\mathcal{U} = \max_{(s,a) \in \mathcal{S} \times \mathcal{A}} |\mathcal{S}_{s,a}|$. This idea of taking confidence ellipsoids goes back to the seminal work by Abbasi-yadkori et al. (2011) for linear contextual bandits and is also adopted for linear mixture MDPs (Zhou et al., 2021; Wu et al., 2022) and MDPs with MNL function approximation (Hwang and Oh, 2023).

Lemma 1 *Suppose that Assumptions 1–3 hold. For a given $\delta \in (0, 1)$, let \mathcal{C}_t be defined as in (4) for each $t \in [T]$. Then with probability at least $1 - \delta$, it holds that $\theta^* \in \mathcal{C}_t$ for all $t \in [T]$.*

Proof See Appendix A. ■

3.2 Extended Value Iteration-Based Algorithm

This section presents UCRL2-MNL. As UCRL2, the algorithm proceeds with multiple episodes. For the k th episode, we denote by t_k the first time step of episode k . Before episode k begins, we construct confidence set \mathcal{C}_{t_k} for estimating θ^* . Then, following UCRL2 and UCRL2-VTR, we run extended value iteration (EVI) described in Algorithm 0. By definition, any $\theta \in \mathbb{R}^d$

Algorithm 0 Extended Value Iteration (EVI(\mathcal{C}, ϵ))

Inputs: confidence set \mathcal{C} , a desired accuracy level ϵ

Initialize: $u^{(0)}(s) = 0$ for every $s \in \mathcal{S}$ and $i = 0$.

while $\max_{s \in \mathcal{S}} \{u^{(i+1)}(s) - u^{(i)}(s)\} - \min_{s \in \mathcal{S}} \{u^{(i+1)}(s) - u^{(i)}(s)\} > \epsilon$ **do**

Set $u^{(i+1)}(s) = \max_{a \in \mathcal{A}} \left\{ r(s, a) + \max_{\theta \in \mathcal{C}} \left\{ \sum_{s' \in \mathcal{S}_{s,a}} p(s' | s, a, \theta) u^{(i)}(s') \right\} \right\}$

Set $i = i + 1$

end while

Return $u^{(i)}(s)$ for $s \in \mathcal{S}$

induces a valid transition function. Moreover, by Lemma 1, we know that \mathcal{C}_{t_k} contains θ^* with high probability. Then it follows from (Jaksch et al., 2010, Theorem 7) that EVI with $\mathcal{C} = \mathcal{C}_{t_k}$ is guaranteed to converge (see also Appendix B.1). Then we denote by $u_k(s)$ the outcome of EVI for $s \in \mathcal{S}$.

Next we deduce a policy π_k for episode k based on the value function u_k and the confidence set \mathcal{C}_{t_k} . Given a value function u and a confidence set \mathcal{C} , we call a policy π the greedy policy of u over \mathcal{C} if

$$\pi(s) = \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r(s, a) + \max_{\theta \in \mathcal{C}} \left\{ \sum_{s' \in \mathcal{S}_{s,a}} p(s' | s, a, \theta) u(s') \right\} \right\}, \quad s \in \mathcal{S}. \quad (5)$$

For episode k , we take the greedy policy of u_k over \mathcal{C}_{t_k} for π_k . Then UCRL2-MNL applies policy π_k until the end of episode k . Then it switches to the next episode when the determinant of the gram matrix A_t doubles compared to the beginning of episode k . A small technical point to note is that instead of u_k , UCRL2-MNL takes a recentered value function w_k given by $u_k(s) - (\max_{s \in \mathcal{S}} u_k(s) + \min_{s \in \mathcal{S}} u_k(s)) / 2$ for $s \in \mathcal{S}$. While replacing u_k with w_k induces the same greedy policy π_k , the purpose of the recentering step is to control the size of

Algorithm 1 UCRL2-MNL

Input: feature map $\varphi : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow \mathbb{R}^d$, confidence level $\delta \in (0, 1)$, and parameters $\lambda, L_\varphi, L_\theta, \kappa, \mathcal{U}$

Initialize: $t = 1, \hat{\theta}_1 = 0, A_1 = \lambda I_d$, and observe the initial state $s_1 \in \mathcal{S}$

for episodes $k = 1, 2, \dots$, **do**

Set $t_k = t$

Set $u_k(s)$ as the output of $\text{EVI}(\mathcal{C}_{t_k}, \epsilon)$ for $s \in \mathcal{S}$ where \mathcal{C}_{t_k} is given as in (4)

Set $w_k(s) = u_k(s) - (\max_{s \in \mathcal{S}} u_k(s) + \min_{s \in \mathcal{S}} u_k(s)) / 2$ for $s \in \mathcal{S}$

Take policy π_k by setting $\pi_k(s)$ as in (5) with $u = w_k$ and $\mathcal{C} = \mathcal{C}_{t_k}$ for $s \in \mathcal{S}$

while $\det(A_t) \leq 2 \det(A_{t_k})$ **do**

Take action $a_t = \pi_k(s_t)$ and observe s_{t+1} sampled from $p(\cdot \mid s_t, a_t)$

Set $A_{t+1} = A_t + \sum_{s' \in \mathcal{S}_t} \varphi_{t,s'} \varphi_{t,s'}^\top$

Update $t = t + 1$

end while

end for

the value function. To explain this, we may argue that $\max_{s \in \mathcal{S}} u_k(s) - \min_{s \in \mathcal{S}} u_k(s) \leq D$ (see (Jaksch et al., 2010, Section 4.3.1) and Appendix B.1), which in turn implies that $-D/2 \leq u_k(s) \leq D/2$ for each $s \in \mathcal{S}$. Recentering value functions is also part of UCRL2-VTR for linear mixture MDPs (Wu et al., 2022).

Theorem 2 *Let M be a communicating MDP governed by the model (1), and let D denote the diameter of M . Setting $\lambda = L_\varphi^2$ and $\epsilon = 1/\sqrt{T}$, for any initial state s_1 , UCRL2-MNL guarantees that*

$$\text{Regret}(M, \text{UCRL2-MNL}, s_1, T) = \tilde{O}\left(\kappa^{-1} D d \sqrt{T} + \kappa^{-1} L_\varphi L_\theta D \sqrt{dT}\right)$$

with probability at least $1 - 2\delta$ where $\tilde{O}(\cdot)$ hides logarithmic factors of T, \mathcal{U} , and $1/\delta$.

Proof See Appendix B. ■

3.3 Finite-Horizon Optimistic Value Iteration

Although UCRL2-MNL achieves a near-optimal regret guarantee, it has some computational issues. First, the inner maximization part over the transition parameter θ in (5) for computing greedy policies is a non-convex optimization problem, though we have access to the explicit form of the multinomial logistic transition model $p(s' \mid s, a, \theta)$. We have the same issue when running extended value iteration as well. Second, the algorithm is limited to the class of communicating MDPs, although it does not require knowledge of the diameter of the underlying MDP. For the tabular setting, broader classes of MDPs can be handled, such as weakly communicating MDPs.

Our second algorithm is designed to resolve the aforementioned issues while sacrificing regret. OVIFH-MNL divides the horizon of T time steps into T/H episodes of equal length H . Then we apply a finite-horizon episodic RL framework with MNL function approximation

Algorithm 2 OVIFH-MNL

Input: feature map $\varphi : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow \mathbb{R}^d$, confidence level $\delta \in (0, 1)$, and parameters $\lambda, L_\varphi, L_\theta, \kappa, \mathcal{U}$

Initialize: $\hat{\theta}_1 = 0$, $A_1 = \lambda I_d$, and observe the initial state $s_1 \in \mathcal{S}$

for episodes $k = 1, 2, \dots, T/H$ **do**

 Set $\hat{Q}_{k,h}(s, a)$ for $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$ as in (6)

for steps $h = 1, \dots, H$ **do**

 Set $t = (k-1)H + h$

 Take action $a_t = \operatorname{argmax}_{a \in \mathcal{A}} \hat{Q}_{k,h}(s_t, a)$ and observe s_{t+1} sampled from $p(\cdot \mid s_t, a_t)$

end for

end for

over the episodes. Specifically, we take UCRL-MNL due to [Hwang and Oh \(2023\)](#), but one may take other algorithms. OVIFH-MNL applies to any MDPs satisfying the following form of Bellman optimality condition. There exist $J^*(M) \in \mathbb{R}$, $v^* : \mathcal{S} \rightarrow \mathbb{R}$, and $q^* : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ such that for all $(s, a) \in \mathcal{S} \times \mathcal{A}$,

$$J^*(M) + q^*(s, a) = r(s, a) + \sum_{s' \in \mathcal{S}_{s,a}} p(s' \mid s, a) v^*(s') \quad \text{and} \quad v^*(s) = \max_{a \in \mathcal{A}} q^*(s, a).$$

Under the Bellman optimality condition, the average reward $J^*(M, s)$ does not depend on the initial state s , and $J^*(M, s) = J^*(M)$ for any $s \in \mathcal{S}$ ([Bartlett and Tewari, 2009](#)). Moreover, the class of weakly communicating MDPs satisfies the condition (see ([Puterman, 2014](#))). There indeed exist other general classes of MDPs with which the condition holds ([Hernandez-Lerma, 2012](#), Section 3.3).

UCRL-MNL by [Hwang and Oh \(2023\)](#) runs with the following optimistic value iteration procedure. For episode k , step $h \in [H]$, and $(s, a) \in \mathcal{S} \times \mathcal{A}$, we construct optimistic value functions given by

$$\hat{Q}_{k,h}(s, a) := r(s, a) + \sum_{s' \in \mathcal{S}_{s,a}} p(s' \mid s, a, \hat{\theta}_{t_k}) \hat{V}_{k,h+1}(s') + 2H\beta_{t_k} \max_{s' \in \mathcal{S}_{s,a}} \|\phi(s, a, s')\|_{A_{t_k}^{-1}} \quad (6)$$

where $t_k = (k-1)H + 1$, $\hat{V}_{k,h}(s) := \min\{H, \max_{a \in \mathcal{A}} \hat{Q}_{k,h}(s, a)\}$, and $\hat{Q}_{k,H+1}(s, a) := 0$. Here, $\hat{\theta}_{t_k}$ and A_{t_k} are computed according to (2) and (3), respectively.

To analyze the performance of OVIFH-MNL, we consider the span of the optimal bias function v^* . Namely, the span of v^* is defined as $\operatorname{sp}(v^*) = \sup_{s, s' \in \mathcal{S}} |v^*(s) - v^*(s')|$. For a weakly communicating MDP, the corresponding $\operatorname{sp}(v^*)$ is bounded. In particular, for a communicating MDP with diameter D , we have $\operatorname{sp}(v^*) \leq D$. Note that OVIFH-MNL given by Algorithm 2 does not assume knowledge of $\operatorname{sp}(v^*)$.

Theorem 3 *Let M be an MDP governed by the model (1) satisfying the Bellman optimality condition with optimal bias function v^* . Setting $\lambda = L_\varphi^2$ and $H = \kappa^{2/5} d^{-2/5} T^{1/5}$, for any initial state s_1 , OVIFH-MNL guarantees that*

$$\operatorname{Regret}(M, \text{OVIFH-MNL}, s_1, T) = \tilde{O} \left(\kappa^{-2/5} \operatorname{sp}(v^*) d^{2/5} T^{4/5} + \kappa^{-2/5} L_\varphi L_\theta \operatorname{sp}(v^*) d^{-1/10} T^{4/5} \right)$$

with probability at least $1 - 2\delta$ where $\tilde{O}(\cdot)$ hides logarithmic factors of T , \mathcal{U} , and $1/\delta$.

Proof See Appendix C. ■

4 Regret Lower Bounds

In this section, we provide regret lower bounds for learning MDPs with multinomial logistic function approximation. Section 4.1 presents our lower bound for learning infinite-horizon average reward MDP with diameter at most D . In Section 4.2, we provide a lower bound for learning H -horizon episodic MDPs with distinct transition cores over the horizon.

4.1 Lower Bound for Learning Infinite-Horizon Average Reward MDPs

In this section, we prove a regret lower bound for learning communicating MDPs of diameter at most D . Our construction of the following hard-to-learn MDP is motivated by the instance proposed by Wu et al. (2022) for the linear mixture MDP case. There are two states x_0 and x_1 as in Figure 1. The action space is given by $\mathcal{A} = \{-1, 1\}^{d-1}$. Let

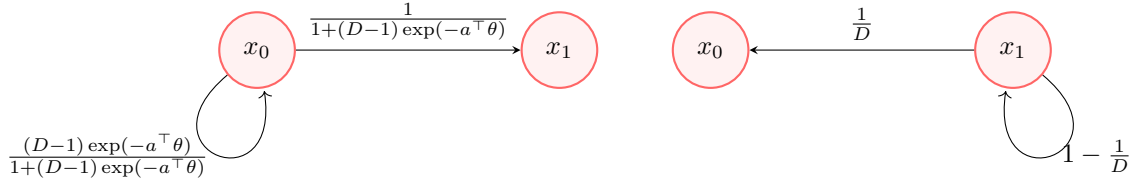


Figure 1: Illustration of the Hard-to-Learn MDP with MNL Transition Model

the reward function be given by $r(x_0, a) = 0$ and $r(x_1, a) = 1$ for any $a \in \mathcal{A}$. Then a higher stationary probability at state x_1 means a larger average reward. The feature vector is given by $\varphi(x_0, a, x_0) = (-\alpha a, \beta \log(D-1))$, $\varphi(x_0, a, x_1) = \varphi(x_1, a, x_0) = (0, 0)$, and $\varphi(x_1, a, x_1) = (0, \beta \log(D-1))$ with $\alpha = \sqrt{\bar{\Delta}/((d-1)(1+\bar{\Delta}))}$ and $\beta = \sqrt{1/(1+\bar{\Delta})}$. The transition core $\bar{\theta}$ is given by

$$\bar{\theta} = \left(\frac{\theta}{\alpha}, \frac{1}{\beta} \right) \quad \text{where} \quad \theta \in \left\{ -\frac{\bar{\Delta}}{d-1}, \frac{\bar{\Delta}}{d-1} \right\}^{d-1}, \quad \bar{\Delta} = \log \left(\frac{(1-\delta)(\delta+\Delta)}{\delta(1-\delta-\Delta)} \right),$$

and $\delta = 1/D$ and $\Delta = (d-1)/(45\sqrt{(2/5)DT \log 2})$. We denote this MDP by M_θ to indicate that it is parameterized by θ .

Theorem 4 Suppose that $d \geq 2$, $D \geq 101$, $T \geq 45(d-1)^2 D$. Then for any algorithm \mathfrak{A} , there exists an MDP M_θ described as in Figure 1 such that $L_\theta \leq 100/99$ and $L_\varphi \leq 1 + \log(D-1)$,

$$\mathbb{E}[\text{Regret}(M_\theta, \mathfrak{A}, x_0, T)] \geq \frac{1}{4050} d \sqrt{DT}$$

where the expectation is taken over the randomness generated by M_θ and \mathfrak{A} .

Proof See Appendix D. ■

Note that L_φ can grow logarithmically in D . Nevertheless, the upper bounds by Theorems 2

and 3 have linear dependence on L_φ . This means that our algorithms guarantee the same regret upper bounds on the hard-to-learn MDP up to additional logarithmic factors in D .

One of the main steps to derive the lower bound is to construct an upper bound on the gap between $p(x_1 | x_0, a, \bar{\theta})$ and $p(x_1 | x_0, a, \bar{\theta}')$ for $\bar{\theta} \neq \bar{\theta}'$. We use the mean value theorem to argue that the gap is bounded above by $c^\top(\theta - \theta')$ for some $c \in \mathbb{R}^{d-1}$. Based on this, we can build a close connection to the linear mixture MDP setting.

4.2 Lower Bound for Learning Finite-Horizon Episodic MDPs

To provide a regret lower bound on learning finite-horizon MDPs with MNL approximation, we consider an instance inspired by Zhou et al. (2021) illustrated as in Figure 2. There are

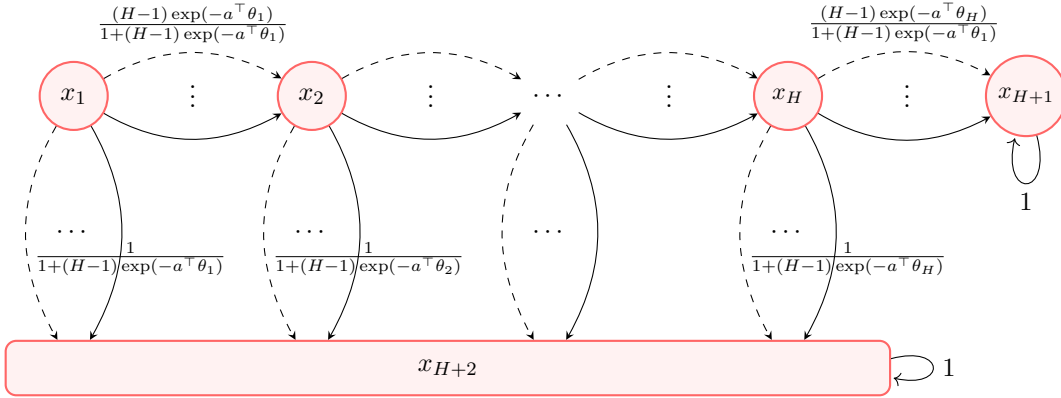


Figure 2: Illustration of the Hard Finite-Horizon MDP Instance

$H + 2$ states x_1, \dots, x_{H+2} where x_{H+1} and x_{H+2} are absorbing states. As in the previous section, we have action space $\mathcal{A} = \{-1, 1\}^{d-1}$. For any action $a \in \mathcal{A}^{d-1}$, the reward function is given by $f(x_i, a) = 1$ if $i = H + 2$ and $f(x_i, a) = 0$ if $i \neq H + 2$. The feature vector is given by $\varphi(x_h, a, x_{H+2}) = (0, 0)$ and $\varphi(x_h, a, x_{H+1}) = (-\alpha a, \beta \log(H - 1))$ for $h \in [H]$ with $\alpha = \sqrt{\bar{\Delta}/(1 + (d - 1)\bar{\Delta})}$ and $\beta = \sqrt{1/(1 + (d - 1)\bar{\Delta})}$. The transition core $\bar{\theta}_h$ for each step $h \in [H]$ is given by

$$\bar{\theta}_h = \left(\frac{\theta_h}{\alpha}, \frac{1}{\beta} \right) \quad \text{where} \quad \theta_h \in \{-\bar{\Delta}, \bar{\Delta}\}^{d-1}, \quad \bar{\Delta} = \frac{1}{d-1} \log \left(\frac{(1-\delta)(\delta + (d-1)\Delta)}{\delta(1-\delta - (d-1)\Delta)} \right),$$

and $\delta = 1/H$ and $\Delta = 1/(4\sqrt{2HK})$. Here, we denote this MDP by M_θ to indicate that it is parameterized by $\theta = \{\theta_h\}_{h=1}^H$.

Theorem 5 Suppose that $d \geq 2$, $H \geq 3$, $K \geq \{(d-1)^2 H/2, H^3(d-1)^2/32\}$. Then for any algorithm \mathfrak{A} , there exists an MDP M_θ described as in Figure 2 such that $L_\theta \leq 3/2$ and $L_\varphi \leq 1 + \log(H - 1)$,

$$\mathbb{E} [\text{Regret}(M_\theta, \mathfrak{A}, K)] \geq \frac{(d-1)H^{3/2}\sqrt{K}}{480\sqrt{2}}$$

where the expectation is taken over the randomness generated by M_θ and \mathfrak{A} .

Proof See Appendix E. ■

Recall that the lower bound provided by Li et al. (2024) is $\Omega(dH\sqrt{K\kappa^*})$ where κ^* is a constant satisfying $p_t(s', \theta^*)p_t(x'', \theta^*) \geq \kappa^*$ for all $t \in [T]$ and $s', s'' \in \mathcal{S}_{s_t, a_t}$. Hence, our lower bound from Theorem 5 improves the previous lower bound by a factor of $O(\sqrt{H/\kappa^*})$.

Notice that the instance M_θ has $L_\varphi \leq 1 + \log(H - 1)$. Nonetheless, the regret upper bound by Hwang and Oh (2023) grows linearly in L_φ , so the upper bound remains the same up to logarithmic factors in H .

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Appendix A. Concentration of the Transition Core

In this section, we prove Lemma 1. To prove the result, we need some tools from linear contextual bandit literature. The following lemma is due to [Abbasi-yadkori et al. \(2011\)](#), providing some self-normalized bound for vector-valued martingales.

Lemma 6 ([Abbasi-yadkori et al., 2011](#), Theorem 1). *Let $\{\mathcal{F}_t\}_{t=0}^\infty$ be a filtration, and let $\{\eta_t\}_{t=1}^\infty$ be a real-valued stochastic process such that η_t is \mathcal{F}_t -measurable and conditionally R -sub-Gaussian for some $R \geq 0$. Let $\{x_t\}_{t=1}^\infty$ be an \mathbb{R}^d -valued stochastic process such that x_t is \mathcal{F}_{t-1} -measurable. Assume that V is a $d \times d$ positive definite matrix. Then define $V_t = V + \sum_{s=1}^t x_s x_s^\top$ and $S_t = \sum_{s=1}^t \eta_s x_s$. Then, for any $\delta > 0$, with probability $1 - \delta$, for all $t \geq 0$ we have*

$$\|S_t\|_{V_t^{-1}}^2 \leq 2R^2 \log \left(\frac{\det(V_t)^{1/2} \det(V)^{-1/2}}{\delta} \right). \quad (7)$$

Note that in the right-hand side of (7), we have the terms V_t and V . For our analysis, we take $A_t = \lambda I_d + \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_{s_i, a_i}} \varphi_{i, s'} \varphi_{i, s'}^\top$ for V_t and λI_d for V . While $\det(\lambda I_d) = \lambda$, we need to provide a bound on $\det(A_t)$. For this task, we apply another lemma due to [Abbasi-yadkori et al. \(2011\)](#).

Lemma 7 ([Abbasi-yadkori et al., 2011](#), Lemma 10). *Suppose $x_1, \dots, x_t \in \mathbb{R}^d$ and $\|x_s\|_2 \leq L$ for any $1 \leq s \leq t$. Let $V_t = \lambda I_d + \sum_{i=1}^t x_i x_i^\top$ for some $\lambda > 0$. Then $\det(V_t)$ is increasing with respect to t and*

$$\det(V_t) \leq \left(\lambda + \frac{tL^2}{d} \right)^d.$$

Being equipped with Lemmas 6 and 7, we are ready to state our proof of Lemma 1. Recall that the log-likelihood function $\ell_t(\theta)$ is given by

$$\ell_t(\theta) = \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_{s_i, a_i}} y_{i, s'} \log p_i(s', \theta).$$

Throughout the appendix, let us use notation \mathcal{S}_t to denote \mathcal{S}_{s_t, a_t} for each t . Then its gradient is given by

$$\nabla_\theta (\ell_t(\theta)) = \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} (y_{i, s'} - p_i(s', \theta)) \varphi_{i, s'}.$$

Then we define $G_t(\theta)$ as

$$\begin{aligned} G_t(\theta) &:= (-\nabla_\theta (\ell_t(\theta)) + \lambda\theta) - (-\nabla_\theta (\ell_t(\theta^*)) + \lambda\theta^*) \\ &= \nabla_\theta (\ell_t(\theta^*)) - \nabla_\theta (\ell_t(\theta)) + \lambda(\theta - \theta^*) \\ &= \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} (p_i(s', \theta) - p_i(s', \theta^*)) \varphi_{i, s'} + \lambda(\theta - \theta^*). \end{aligned}$$

Moreover, since $\widehat{\theta}_t$ is the minimizer of $-\ell_t(\theta) + \lambda \|\theta\|_2^2/2$, it follows that $-\nabla_\theta(\ell_t(\widehat{\theta}_t)) + \lambda \widehat{\theta}_t = 0$ and thus

$$\begin{aligned} G_t(\widehat{\theta}_t) &= \left(-\nabla_\theta(\ell_t(\widehat{\theta}_t)) + \lambda \widehat{\theta}_t \right) - \left(-\nabla_\theta(\ell_t(\theta^*)) + \lambda \theta^* \right) \\ &= \nabla_\theta(\ell_t(\theta^*)) - \lambda \theta^* \\ &= \sum_{t=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \epsilon_{i,s'} \varphi_{i,s'} - \lambda \theta^* \end{aligned}$$

where $\epsilon_{i,s'} := y_{i,s'} - p_i(s', \theta^*)$. Next let us consider $G_t(\theta_1) - G_t(\theta_2)$ for arbitrary $\theta_1, \theta_2 \in \mathbb{R}^d$. By the mean value theorem, for any $\theta_1, \theta_2 \in \mathbb{R}^d$, there exists $\alpha \in [0, 1]$ such that $\vartheta := \alpha \theta_1 + (1 - \alpha) \theta_2$ satisfying

$$\begin{aligned} G_t(\theta_1) - G_t(\theta_2) &= \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} (p_i(s', \theta_1) - p_i(s', \theta_2)) \varphi_{i,s'} + \lambda(\theta_1 - \theta_2) \\ &= \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \left(\nabla_\theta(p_i(s', \vartheta))^\top (\theta_1 - \theta_2) \right) \varphi_{i,s'} + \lambda(\theta_1 - \theta_2). \end{aligned}$$

To characterize $\nabla_\theta(p_i(s', \theta))$, we consider the following. Assumption 3 implies that for each $i \in [T]$, there exists a state ς_i such that $\varphi_{i,\varsigma_i} = 0$. This implies that for any $s' \in \mathcal{S}_i$,

$$p_i(s', \theta) = \frac{\exp(\varphi_{i,s'}^\top \theta)}{\sum_{s'' \in \mathcal{S}_i} \exp(\varphi_{i,s''}^\top \theta)} = \frac{\exp(\varphi_{i,s'}^\top \theta)}{1 + \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \exp(\varphi_{i,s''}^\top \theta)}$$

For $j \in [d]$, we denote by $(\varphi_{i,s'})_j$ the j th coordinate of $\varphi_{i,s'}$ for $s' \in \mathcal{S}_i$. Note that for $j \in [d]$, we have

$$\begin{aligned} &\frac{\partial p_i(s', \theta)}{\partial \theta_j} \\ &= \frac{(\varphi_{i,s'})_j \exp(\varphi_{i,s'}^\top \theta) \left(1 + \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \exp(\varphi_{i,s''}^\top \theta) \right) - \exp(\varphi_{i,s'}^\top \theta) \left(\sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} (\varphi_{i,s''})_j \exp(\varphi_{i,s''}^\top \theta) \right)}{\left(1 + \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \exp(\varphi_{i,s''}^\top \theta) \right)^2} \\ &= (\varphi_{i,s'})_j p_i(s', \theta) - p_i(s', \theta) \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} (\varphi_{i,s''})_j p_i(s'', \theta) \end{aligned}$$

Then it holds that

$$\begin{aligned} \nabla_\theta(p_i(s', \theta)) &= p_i(s', \theta) \varphi_{i,s'} - p_i(s', \theta) \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s'', \theta) \varphi_{i,s''} \\ &= p_i(s', \theta) \varphi_{i,s'} - p_i(s', \theta) \sum_{s'' \in \mathcal{S}_i} p_i(s'', \theta) \varphi_{i,s''} \end{aligned} \tag{8}$$

where the second equality holds because $\varphi_{i,\varsigma_i} = 0$. This implies that

$$\begin{aligned}
 & G_t(\theta_1) - G_t(\theta_2) \\
 &= \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \left(\left(p_i(s', \vartheta) \varphi_{i,s'} - p_i(s', \vartheta) \sum_{s'' \in \mathcal{S}_i} p_i(s'', \vartheta) \varphi_{i,s''} \right)^\top (\theta_1 - \theta_2) \right) \varphi_{i,s'} + \lambda(\theta_1 - \theta_2) \\
 &= \sum_{i=1}^{t-1} (H_i + \lambda I_d) (\theta_1 - \theta_2)
 \end{aligned}$$

where

$$H_i := \sum_{s' \in \mathcal{S}_i} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top - \sum_{s' \in \mathcal{S}_i} \sum_{s'' \in \mathcal{S}_i} p_i(s', \vartheta) p_i(s'', \vartheta) \varphi_{i,s'} \varphi_{i,s''}^\top.$$

Next we argue that H_i is positive semidefinite. Note that $(x - y)(x - y)^\top = xx^\top + yy^\top - xy^\top - yx^\top \succeq 0$ where $A \succeq B$ means that $A - B$ is positive semidefinite. This implies that $xx^\top + yy^\top \succeq xy^\top + yx^\top$. Then consider

$$\begin{aligned}
 H_i &= \sum_{s' \in \mathcal{S}_i} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top - \frac{1}{2} \sum_{s' \in \mathcal{S}_i} \sum_{s'' \in \mathcal{S}_i} p_i(s', \vartheta) p_i(s'', \vartheta) \left(\varphi_{i,s'} \varphi_{i,s''}^\top + \varphi_{i,s''} \varphi_{i,s'}^\top \right) \\
 &= \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top - \frac{1}{2} \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) p_i(s'', \vartheta) \left(\varphi_{i,s'} \varphi_{i,s''}^\top + \varphi_{i,s''} \varphi_{i,s'}^\top \right) \\
 &\succeq \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top - \frac{1}{2} \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) p_i(s'', \vartheta) \left(\varphi_{i,s'} \varphi_{i,s'}^\top + \varphi_{i,s''} \varphi_{i,s''}^\top \right) \\
 &= \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top - \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s', \vartheta) p_i(s'', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top \\
 &= \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \left\{ 1 - \sum_{s'' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(s'', \vartheta) \right\} p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top \\
 &= \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} p_i(\varsigma_i, \vartheta) p_i(s', \vartheta) \varphi_{i,s'} \varphi_{i,s'}^\top \\
 &\succeq \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \kappa \varphi_{i,s'} \varphi_{i,s'}^\top
 \end{aligned}$$

where the first equality holds because $\varphi_{i,\varsigma_i} = 0$ and the last inequality is from Assumption 2. Hence H_i is positive semidefinite. Then for any $\theta_1 \neq \theta_2$, we have

$$\begin{aligned}
 (\theta_1 - \theta_2)^\top (G_t(\theta_1) - G_t(\theta_2)) &= (\theta_1 - \theta_2)^\top \left(\sum_{i=1}^{t-1} (H_i + \lambda I) \right) (\theta_1 - \theta_2) \\
 &\geq (\theta_1 - \theta_2)^\top \left(\sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \kappa \varphi_{i,s'} \varphi_{i,s'}^\top + \lambda I_d \right) (\theta_1 - \theta_2) \\
 &= (\theta_1 - \theta_2)^\top \kappa \left(\sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i \setminus \{\varsigma_i\}} \varphi_{i,s'} \varphi_{i,s'}^\top + \frac{\lambda}{\kappa} I_d \right) (\theta_1 - \theta_2) \\
 &\geq (\theta_1 - \theta_2)^\top (\kappa A_t) (\theta_1 - \theta_2) \\
 &> 0
 \end{aligned}$$

where the second last inequality holds because $\varphi_{i,\varsigma_i} = 0$ and $0 < \kappa < 1$ while the last inequality holds because A_t is positive definite for any $\lambda > 0$. This inequality implies that $G_t(\theta)$ is an injection from \mathbb{R}^d to \mathbb{R}^d , and therefore, the inverse mapping G^{-1} is well-defined. Recall that by definition, $G_t(\theta^*) = 0$. Then for any $\theta \in \mathbb{R}^d$, we have that

$$\begin{aligned}
 \|G_t(\theta)\|_{A_t^{-1}}^2 &= \|G_t(\theta) - G_t(\theta^*)\|_{A_t^{-1}}^2 \\
 &= (G_t(\theta) - G_t(\theta^*))^\top A_t^{-1} (G_t(\theta) - G_t(\theta^*)) \\
 &= (\theta - \theta^*)^\top \left(\sum_{i=1}^{t-1} H_i + \lambda I_d \right) A_t^{-1} \left(\sum_{i=1}^{t-1} H_i + \lambda I_d \right) (\theta - \theta^*) \\
 &\geq \kappa^2 (\theta - \theta^*)^\top A_t (\theta - \theta^*) \\
 &= \kappa^2 \|\theta - \theta^*\|_{A_t}^2
 \end{aligned}$$

where the inequality holds because $\sum_{i=1}^{t-1} H_i + \lambda I_d \succeq \kappa A_t$. Setting $\theta = \hat{\theta}_t$ in this inequality, we obtain

$$\kappa \|\hat{\theta}_t - \theta^*\|_{A_t} \leq \|G_t(\hat{\theta}_t)\|_{A_t^{-1}} = \left\| \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \epsilon_{i,s'} \varphi_{i,s'} - \lambda \theta^* \right\|_{A_t^{-1}} \leq \left\| \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \epsilon_{i,s'} \varphi_{i,s'} \right\|_{A_t^{-1}} + \lambda \|\theta^*\|_{A_t^{-1}}.$$

In the first term of the rightmost side of this inequality, $\epsilon_{i,s'}$ is 1-sub-Gaussian because $-1 \leq \epsilon_{i,s'} \leq 1$. Applying Lemma 6, we deduce that with probability at least $1 - \delta$, it holds that

$$\left\| \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \epsilon_{i,s'} \varphi_{i,s'} \right\|_{A_t^{-1}}^2 \leq 2 \cdot 1^2 \log \left(\frac{\det(A_t)^{1/2} \det(\lambda I_d)^{-1/2}}{\delta} \right).$$

We may bound the right-hand side by Lemma 7 as follows. Since $|\mathcal{S}_i| \leq \mathcal{U}$ for each i and $\|\varphi_{i,s'}\| \leq L_\varphi$ for every (i, s') due to Assumption 1, applying Lemma 7 gives us that

$$2 \cdot \log \left(\frac{\det(A_t)^{1/2} \det(\lambda I_d)^{-1/2}}{\delta} \right) \leq d \log \left(1 + \frac{t \mathcal{U} L_\varphi^2}{d \lambda} \right) + 2 \log \frac{1}{\delta}.$$

Moreover,

$$\|\theta^*\|_{A_t^{-1}}^2 = \theta^{*\top} A_t^{-1} \theta^* \leq \frac{1}{\lambda} \theta^{*\top} \theta^* = \frac{\|\theta^*\|_2^2}{\lambda} \leq \frac{L_\theta^2}{\lambda}.$$

Therefore, we have $\lambda \|\theta^*\|_{A_t^{-1}} \leq \sqrt{\lambda} L_\theta$. Consequently, with probability at least $1 - \delta$, it holds that for all $t \in [T]$,

$$\left\| \hat{\theta}_t - \theta^* \right\|_{A_t} \leq \frac{1}{\kappa} \sqrt{d \log \left(1 + \frac{t \mathcal{U} L_\varphi^2}{d \lambda} \right)} + 2 \log \frac{1}{\delta} + \frac{1}{\kappa} \sqrt{\lambda} L_\theta,$$

as required.

Appendix B. Performance Analysis of UCRL2-MNL

B.1 Convergence of Extended Value Iteration

The following lemma is a restatement of Theorem 7 and related results in Section 4.3.1 of (Jaksch et al., 2010) to suit our setting. Recall that the transition model $p(s' | s, a) = p(s' | s, a, \theta^*)$ where θ^* is the true transition core induces a communicating MDP with diameter D . Given a set $\mathcal{C} \subseteq \mathbb{R}^d$ of parameters θ and a value function $u : \mathcal{S} \rightarrow \mathbb{R}$, recall that a deterministic stationary policy $\pi_{u,\mathcal{C}} : \mathcal{S} \rightarrow \mathcal{A}$ is the greedy policy with respect to u over \mathcal{C} if

$$\pi_{u,\mathcal{C}}(s) = \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r(s, a) + \max_{\theta \in \mathcal{C}} \left\{ \sum_{s' \in \mathcal{S}_{s,a}} p(s' | s, a, \theta) u(s') \right\} \right\}, \quad s \in \mathcal{S}.$$

Moreover, let us define the associated transition function $p_{u,\mathcal{C}}$ follows. For $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$,

$$p_{u,\mathcal{C}}(s' | s, a) = p(s' | s, a, \theta_{u,\mathcal{C}}(s, a)) \quad \text{where} \quad \theta_{u,\mathcal{C}}(s, a) \in \operatorname{argmax}_{\theta \in \mathcal{C}} \left\{ \sum_{s' \in \mathcal{S}_{s,a}} p(s' | s, a, \theta) u(s') \right\}. \quad (9)$$

Next, let $M_{u,\mathcal{C}}$ be the MDP associated with the transition model $p_{u,\mathcal{C}}$, and let $J_{u,\mathcal{C}}$ be defined as

$$J_{u,\mathcal{C}} = \min_{s \in \mathcal{S}} J(M_{u,\mathcal{C}}, \pi_{u,\mathcal{C}}, s) = \min_{s \in \mathcal{S}} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} [R(M_{u,\mathcal{C}}, \pi_{u,\mathcal{C}}, s, T)] \right\}. \quad (10)$$

Lemma 8 (Jaksch et al., 2010, Theorem 7 and Section 4.3.1). *Let $\mathcal{C} \subseteq \mathbb{R}^d$ be some set containing θ^* . Then $\text{EVI}(\mathcal{C}, \epsilon)$ given by Algorithm 0 terminates with some i such that*

$$\max_{s \in \mathcal{S}} \left\{ u^{(i+1)}(s) - u^{(i)}(s) \right\} - \min_{s \in \mathcal{S}} \left\{ u^{(i+1)}(s) - u^{(i)}(s) \right\} < \epsilon.$$

Moreover, it holds that $J_{u^{(i)},\mathcal{C}} \geq J^*(M) - \epsilon$, $|u^{(i+1)}(s) - u^{(i)}(s) - J_{u^{(i)},\mathcal{C}}| \leq \epsilon$ for all $s \in \mathcal{S}$, and

$$\left| \left(J_{u^{(i)},\mathcal{C}} - r \left(s, \pi_{u^{(i)},\mathcal{C}}(s) \right) \right) - \left(\sum_{s' \in \mathcal{S}_{s, \pi_{u^{(i)},\mathcal{C}}(s)}} p_{u,\mathcal{C}}(s' | s, \pi_{u^{(i)},\mathcal{C}}(s)) u^{(i)}(s') - u^{(i)}(s) \right) \right| \leq \epsilon.$$

(Jaksch et al., 2010, Section 4.3.1) also proved that

$$\max_{s \in \mathcal{S}} u^{(i)}(s) - \min_{s \in \mathcal{S}} u^{(i)}(s) \leq D. \quad (11)$$

To make our paper self-contained, we include the argument here. Note that $u^{(i)}(s)$ is the expected cumulative reward over i steps under an optimal non-stationary policy starting from s . Suppose for the sake of contradiction that some s and s' satisfy $u^{(i)}(s) - D > u^{(i)}(s')$. Then we may argue that we can attain a better value of $u^{(i)}(s')$ by adopting the following policy. First, travel to state s as fast as possible, which takes at most D steps in expectation as the diameter is D . For the next step onwards, follow the optimal policy with initial state s . Since the reward for each step belongs to $[0, 1]$, the new policy will gain at least $u^{(i)}(s) - D$, contradicting the optimality of $u^{(i)}(s')$.

B.2 Proof of Theorem 2

Let K_T denote the total number of distinct episodes over the horizon of T time steps. For simplicity, we assume that the last time step of the last episode and that time step $T + 1$ is the beginning of the $(K_T + 1)$ th episode, i.e., $t_{K_T+1} = T + 1$. Note that

$$\text{Regret}(M, \text{UCRL2-MNL}, s, T) = T \cdot J^*(M) - R(M, \text{UCRL2-MNL}, s, T) = \sum_{t=1}^T (J^*(M) - r(s_t, a_t)).$$

Then it follows that

$$\text{Regret}(M, \text{UCRL2-MNL}, s, T) = \sum_{k=1}^{K_T} \text{Regret}_k \quad \text{where} \quad \text{Regret}_k = \sum_{t=t_k}^{t_{k+1}-1} (J^*(M) - r(s_t, a_t)).$$

Recall that the value function u_k is given by $u_k = u^{(i)}$ where $u^{(i)}$ is given by $\text{EVI}(\mathcal{C}_{t_k}, \epsilon)$ and that π_k is equivalent to the greedy policy with respect to u_k over \mathcal{C}_{t_k} . Let J_k denote the optimistic average reward $J_{u_k, \mathcal{C}_{t_k}}$ given as in (10). Since UCRL2-MNL applies π_k for episode k , we have $a_t = \pi_k(s_t)$. Then it follows from Lemma 8 that

$$\text{Regret}_k \leq \sum_{t=t_k}^{t_{k+1}-1} (J_k + \epsilon - r(s_t, a_t)) \leq \sum_{t=t_k}^{t_{k+1}-1} (u^{(i+1)}(s_t) - u_k(s_t) + 2\epsilon - r(s_t, a_t)).$$

To provide an upper bound on the right-hand side, by the stopping condition of $\text{EVI}(\mathcal{C}_{t_k}, \epsilon)$, we have

$$u^{(i+1)}(s_t) = r(s_t, a_t) + \sum_{s' \in \mathcal{S}_t} p_t(s', \tilde{\theta}_t) u_k(s')$$

where $\tilde{\theta}_t = \theta_{u_k, \mathcal{C}_{t_k}}(s_t, a_t)$ given as in (9). This implies that

$$\begin{aligned} \text{Regret}_k &\leq 2(t_{k+1} - t_k)\epsilon + \sum_{t=t_k}^{t_{k+1}-1} \left(\sum_{s' \in \mathcal{S}_t} p_t(s', \tilde{\theta}_t) u_k(s') - u_k(s_t) \right) \\ &= 2(t_{k+1} - t_k)\epsilon + \sum_{t=t_k}^{t_{k+1}-1} \left(\sum_{s' \in \mathcal{S}_t} p_t(s', \tilde{\theta}_t) w_k(s') - w_k(s_t) \right) \end{aligned}$$

where the second equality holds because $w_k(s)$ is obtained from $u_k(s)$ after subtracting a fixed constant. Then it follows that

$$\begin{aligned}
& \text{Regret}(M, \text{UCRL2-MNL}, s, T) \\
&= \underbrace{2 \sum_{k=1}^{K_T} (t_{k+1} - t_k) \epsilon}_{\text{Term 1}} + \underbrace{\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} (w_k(s_{t+1}) - w_k(s_t))}_{\text{Term 2}} + \underbrace{\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \left(\sum_{s' \in \mathcal{S}_t} p_t(s', \theta^*) w_k(s') - w_k(s_{t+1}) \right)}_{\text{Term 3}} \\
&\quad + \underbrace{\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \sum_{s' \in \mathcal{S}_t} \left(p_t(s', \tilde{\theta}_t) - p_t(s', \theta^*) \right) w_k(s')}_{\text{Term 4}}
\end{aligned}$$

First of all,

$$\text{Term 1} = 2(t_{K_T+1} - t_1) \epsilon = 2(T + 1 - 1) \epsilon = 2T \epsilon. \quad (12)$$

Second, note that

$$\text{Term 2} = \sum_{k=1}^{K_T} (w_k(s_{t_{k+1}}) - w_k(s_{t_k})) \leq \sum_{k=1}^{K_T} w_k(s_{t_{k+1}}) \leq \frac{D}{2} K_T \quad (13)$$

where the last inequality holds because $|w_k(s)| \leq D/2$ for each $s \in \mathcal{S}$. To provide an upper bound on the rightmost side of (13), we prove the following lemma.

Lemma 9 $K_T \leq d \log_2 (2 + 2TUL_\varphi^2/\lambda)$.

Proof Since $A_1 = \lambda I_d$, we have $\det A_1 = \lambda^d$. Furthermore,

$$\|A_T\|_2 = \left\| \lambda I_d + \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \sum_{s' \in \mathcal{S}_t} \varphi_{t,s'} \varphi_{t,s'}^\top \right\|_2 \leq \lambda + \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \sum_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_2^2 \leq \lambda + TUL_\varphi^2,$$

where the first inequality is by the triangle inequality and the second is due to Assumption 1. This implies that $\det(A_T) \leq (\lambda + TUL_\varphi^2)^d$. Therefore, we have

$$(\lambda + TUL_\varphi^2)^d \geq \det(A_T) \geq \det(A_{t_{K_T}}) \geq 2^{K_T-1} \det(A_{t_1}) = 2^{K_T-1} \lambda^d, \quad (14)$$

where the second inequality holds because $A_T \succeq A_{t_{K_T}}$ and the last holds due to $\det(A_{t_{k+1}}) \geq 2 \det(A_{t_k})$. Then it follows from (14) that $K_T \leq d \log_2 (2 + 2TUL_\varphi^2/\lambda)$, as required. \blacksquare

By Lemma 9 and (13), we have

$$\text{Term 2} \leq \frac{1}{2} D d \log_2 \left(2 + \frac{2TUL_\varphi^2}{\lambda} \right). \quad (15)$$

Providing upper bounds on Terms 3 and 4 is more involved. Let us state the following lemmas giving bounds on Terms 3 and 4, respectively. We defer their proofs to later in this section.

Lemma 10 *With probability at least $1 - \delta$, it holds that Term 3 $\leq D\sqrt{2T \log(1/\delta)}$.*

Proof See Appendix B.3 ■

Lemma 11 *Suppose that $\theta^* \in \mathcal{C}_t$ for all $t \in [T]$. Then*

$$\text{Term 4} \leq 4D\beta_T \sqrt{2dT \log \left(1 + \frac{T\mathcal{U}L_\varphi^2}{d\lambda} \right)}.$$

Proof See Appendix B.4 ■

Now we are ready to finalize our proof of Theorem 2. Lemmas 1 and 10 imply that with probability at least $1 - 2\delta$, it holds that $\theta^* \in \mathcal{C}_t$ for all $t \in [T]$ and Term 3 $\leq D\sqrt{2T \log(1/\delta)}$. Set $\epsilon = 1/\sqrt{T}$. Then it follows that

$$\begin{aligned} & \text{Regret}(M, \text{UCRL2-MNL}, s, T) \\ & \leq 2\sqrt{T} + \frac{Dd}{2} \log_2 \left(2 + \frac{2T\mathcal{U}L_\varphi^2}{\lambda} \right) + D\sqrt{2T \log \left(\frac{1}{\delta} \right)} + 2D\beta_{K_T} \sqrt{2dT \log \left(1 + \frac{T\mathcal{U}}{d\lambda} \right)} \\ & = 2\sqrt{T} + \frac{Dd}{2} \log_2 \left(2 + \frac{2T\mathcal{U}L_\varphi^2}{\lambda} \right) + D\sqrt{2T \log \left(\frac{1}{\delta} \right)} \\ & \quad + 4D \left(\frac{1}{\kappa} \sqrt{d \log \left(1 + \frac{T\mathcal{U}L_\varphi^2}{d\lambda} \right)} + 2 \log \frac{1}{\delta} + \frac{1}{\kappa} \sqrt{\lambda} L_\theta \right) \sqrt{2dT \log \left(1 + \frac{T\mathcal{U}L_\varphi^2}{d\lambda} \right)} \\ & = \tilde{\mathcal{O}} \left(\kappa^{-1} D d \sqrt{T} + \kappa^{-1} L_\varphi L_\theta D \sqrt{dT} \right) \end{aligned}$$

where $\tilde{\mathcal{O}}(\cdot)$ hides some logarithmic factors in T , \mathcal{U} , and $1/\delta$ as $d \geq 1$ and the last equality holds because λ can be set to L_φ^2 .

B.3 Upper Bound on the Regret Term 3

In this subsection, we prove Lemma 10. For $t \in [T]$, let $k(t)$ denote the index of the episode containing time slot t . Then take Y_t as $Y_t = \sum_{s' \in \mathcal{S}_t} p_t(s', \theta^*) w_{k(t)}(s') - w_{k(t)}(s_{t+1})$ for $t \in [T]$. For $t \in [T]$, let \mathcal{F}_t be the σ -algebra generated by the randomness up to time step t . Then we have $\mathbb{E}[Y_t | \mathcal{F}_t] = 0$, which means that Y_1, \dots, Y_T gives rise to a Martingale difference sequence. Then Term 3, which is essentially the summation of Y_1, \dots, Y_T , can be bounded by Azuma's inequality given as follows.

Lemma 12 (Azuma's inequality) *Let Y_1, \dots, Y_T be a martingale difference sequence with respect to a filtration $\mathcal{F}_1, \dots, \mathcal{F}_T$. Assume that $|Y_t| \leq B$ for $t \in [T]$. Then with probability at least $1 - \delta$, we have $\sum_{t=1}^T Y_t \leq B\sqrt{2T \log(1/\delta)}$.*

Since $|w_k(s)| \leq D/2$ for any episode k and $s \in \mathcal{S}$, we have

$$|Y_t| \leq \left| \sum_{s' \in \mathcal{S}_t} p_t(s', \theta^*) w_{k(t)}(s') \right| + |w_{k(t)}(s_{t+1})| \leq \frac{D}{2} \left| \sum_{s' \in \mathcal{S}_t} p_t(s', \theta^*) \right| + \frac{D}{2} = D.$$

Then it follows from Lemma 12 that with probability at least $1 - \delta$, Term 3 = $\sum_{t=1}^T Y_T \leq D\sqrt{2T \log(1/\delta)}$, as required.

B.4 Upper Bound on the Regret Term 4

To prove the desired upper bound on Term 4, we take an episode k and a time slot t in the episode. Let t_k denote the time index of the beginning of episode k . Then we consider

$$I_t := \sum_{s' \in \mathcal{S}_t} \left(p_t(s', \tilde{\theta}_t) - p_t(s', \theta^*) \right) w_k(s')$$

where $\tilde{\theta}_t = \theta_{u_k, \mathcal{C}_{t_k}}(s_t, a_t)$ given as in (9). Note that

$$I_t = \sum_{s' \in \mathcal{S}_t} \left(p_t(s', \tilde{\theta}_t) - p_t(s', \hat{\theta}_{t_k}) \right) w_k(s') + \sum_{s' \in \mathcal{S}_t} \left(p_t(s', \hat{\theta}_{t_k}) - p_t(s', \theta^*) \right) w_k(s')$$

where $\hat{\theta}_{t_k}$ is given as in (2). By the mean value theorem, there exists $\vartheta_1 = \alpha_1 \tilde{\theta}_t + (1 - \alpha_1) \hat{\theta}_{t_k}$ and $\vartheta_2 = \alpha_2 \hat{\theta}_{t_k} + (1 - \alpha_2) \theta^*$ for some $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$\begin{aligned} p_t(s', \tilde{\theta}_t) - p_t(s', \hat{\theta}_{t_k}) &= \nabla_{\theta}(p_t(s', \vartheta_1))^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}), \\ p_t(s', \hat{\theta}_{t_k}) - p_t(s', \theta^*) &= \nabla_{\theta}(p_t(s', \vartheta_2))^{\top} (\hat{\theta}_{t_k} - \theta^*) \end{aligned}$$

where we have from (8) that

$$\nabla_{\theta}(p_t(s', \vartheta)) = p_t(s', \vartheta) \varphi_{t, s'} - p_t(s', \theta) \sum_{s'' \in \mathcal{S}_t} p_t(s'', \vartheta) \varphi_{t, s''}.$$

Note that

$$\begin{aligned} & \left| \nabla_{\theta}(p_t(s', \vartheta_1))^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}) w_k(s') \right| \\ & \leq p_t(s', \vartheta_1) \left| \varphi_{t, s'}^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}) w_k(s') \right| + p_t(s', \theta) \sum_{s'' \in \mathcal{S}_t} p_t(s'', \vartheta_1) \left| \varphi_{t, s''}^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}) w_k(s') \right| \\ & \leq \frac{D}{2} p_t(s', \vartheta_1) \max_{s \in \mathcal{S}_t} \left| \varphi_{t, s}^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}) \right| + \frac{D}{2} p_t(s', \theta) \sum_{s'' \in \mathcal{S}_t} p_t(s'', \vartheta_1) \max_{s \in \mathcal{S}_t} \left| \varphi_{t, s}^{\top} (\tilde{\theta}_t - \hat{\theta}_{t_k}) \right| \\ & \leq D p_t(s', \vartheta_1) \max_{s \in \mathcal{S}_t} \|\varphi_{t, s}\|_{A_t^{-1}} \|\tilde{\theta}_t - \hat{\theta}_{t_k}\|_{A_t} \\ & \leq 2D \beta_{t_k} p_t(s', \vartheta_1) \max_{s \in \mathcal{S}_t} \|\varphi_{t, s}\|_{A_t^{-1}} \end{aligned}$$

where the second inequality is due to $|w_k(s')| \leq D/2$ for any $s' \in \mathcal{S}$, the third inequality follows from the Cauchy-Schwarz inequality and $\sum_{s'' \in \mathcal{S}_t} p_t(s'', \vartheta_1) = 1$, and the last inequality

holds because $\|\tilde{\theta}_t - \hat{\theta}_{t_k}\|_{A_t} \leq 2\|\tilde{\theta}_t - \hat{\theta}_{t_k}\|_{A_{t_k}}$ as $\det(A_t) \leq 2\det(A_{t_k})$ and $\tilde{\theta}_t \in \mathcal{C}_{t_k}$. Similarly, as $\theta^* \in \mathcal{C}_{t_k}$, we have

$$\left| \nabla_{\theta}(p_t(s', \vartheta_2))^{\top} (\hat{\theta}_{t_k} - \theta^*) w_k(s') \right| \leq 2D\beta_{t_k} p_t(s', \vartheta_2) \max_{s \in \mathcal{S}_t} \|\varphi_{t,s}\|_{A_t^{-1}}.$$

Since $\sum_{s' \in \mathcal{S}_t} p_t(s', \vartheta) = 1$ for any ϑ , it follows that

$$I_t \leq 4D\beta_{t_k} \max_{s \in \mathcal{S}_t} \|\varphi_{t,s}\|_{A_t^{-1}}.$$

This in turn implies that

$$\text{Term 4} = \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} I_t \leq 4D\beta_T \sum_{t=1}^T \max_{s \in \mathcal{S}_t} \|\varphi_{t,s}\|_{A_t^{-1}} \leq 4D\beta_T \sqrt{T \sum_{t=1}^T \max_{s \in \mathcal{S}_t} \|\varphi_{t,s}\|_{A_t^{-1}}^2} \quad (16)$$

where the second inequality is from the Cauchy-Schwarz inequality. Now, it remains to provide an upper bound on the rightmost side of (16). For this task, we apply the following lemma.

Lemma 13 *Suppose that $\|\varphi_{t,s'}\|_2 \leq L_{\varphi}$ for any $t \in [T]$ and $s' \in \mathcal{S}_t$. For $t \in [T]$, let $A_t = \lambda I_d + \sum_{i=1}^{t-1} \sum_{s' \in \mathcal{S}_i} \varphi_{i,s'} \varphi_{i,s'}^{\top}$. If $\lambda \geq L_{\varphi}^2$, then*

$$\sum_{t=1}^T \max_{s \in \mathcal{S}_t} \|\varphi_{t,s}\|_{A_t^{-1}}^2 \leq 2d \log \left(1 + \frac{TUL_{\varphi}^2}{d\lambda} \right).$$

Proof Let $t \in [T]$. Then $A_{t+1} = A_t + \sum_{s' \in \mathcal{S}_t} \varphi_{t,s'} \varphi_{t,s'}^{\top}$. Since A_t is positive definite, A_t^{-1} exists and $\det(A_{t+1}) = \det(A_t) \cdot \det(B_t)$ where $B_t = I_d + A_t^{-1/2} \sum_{s' \in \mathcal{S}_t} A_t^{-1/2}$. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of B_t . Then

$$\sum_{i=1}^d (\lambda_i - 1) = \text{tr}(B_t) - d = \text{tr} \left(A_t^{-1/2} \sum_{s' \in \mathcal{S}_t} A_t^{-1/2} \right) = \sum_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2 \geq \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2.$$

This implies that

$$\det(B_t) = \prod_{i=1}^d \lambda_i = \prod_{i=1}^d (1 + (\lambda_i - 1)) \geq 1 + \sum_{i=1}^d (\lambda_i - 1) \geq 1 + \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2.$$

Furthermore,

$$\det(A_{T+1}) \geq \det(A_T) \left(1 + \max_{s' \in \mathcal{S}_T} \|\varphi_{T,s'}\|_{A_T^{-1}}^2 \right) \geq \det(A_1) \prod_{t=1}^T \left(1 + \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2 \right).$$

Since $\lambda \geq L_{\varphi}^2$, for any $t \in [T]$ and $s' \in \mathcal{S}_t$,

$$\|\varphi_{t,s'}\|_{A_t^{-1}}^2 = \varphi_{t,s'}^{\top} A_t^{-1} \varphi_{t,s'} \leq \frac{1}{\lambda_{\min}(A_t)} \|\varphi_{t,s'}\|_2^2 \leq \frac{1}{\lambda} \|\varphi_{t,s'}\|_2^2 \leq 1.$$

Since $x \leq 2\log(1+x)$ for $x \in [0, 1]$, it follows that

$$\begin{aligned} \sum_{t=1}^T \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2 &\leq 2 \sum_{t=1}^T \log \left(1 + \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2 \right) \\ &= 2 \log \left(\prod_{t=1}^T \left(1 + \max_{s' \in \mathcal{S}_t} \|\varphi_{t,s'}\|_{A_t^{-1}}^2 \right) \right) \\ &\leq 2 \log \left(\frac{\det(A_{T+1})}{\det(A_1)} \right). \end{aligned}$$

Note that $\det(A_1) = \lambda^d$. Moreover, by Lemma 7, it follows that $\det(A_{T+1}) \leq (\lambda + T\mathcal{U}L_\varphi^2/d)^d$, which implies the desired upper bound. \blacksquare

Combining (16) and Lemma 13, we deduce that

$$\text{Term 4} \leq 4D\beta_T \sqrt{2dT \log \left(1 + \frac{T\mathcal{U}L_\varphi^2}{d\lambda} \right)},$$

as required.

Appendix C. Performance Analysis of OVIFH-MNL

Recall that OVIFH-MNL divides the learning process to T/H episodes of equal length H . Let us consider an episode that consists of H time steps and take a policy $\pi = \{\pi_h\}_{h=1}^H$ given by a collection of H functions, where $\pi_h : \mathcal{S} \rightarrow \mathcal{A}$ specifies a function over the action for each state $s_h \in \mathcal{S}$. Actions are sampled from the policy π_h at state s_h in stage h , $a_h \sim \pi_h(\cdot | s_h)$. The value function, $V_h^\pi(s) : \mathcal{S} \rightarrow \mathbb{R}$, gives the expected total reward under the policy π until the end of the episode, beginning from state s at stage h ,

$$V_h^\pi(s) := \mathbb{E}_\pi \left[\sum_{h'=h}^H r(s_{h'}, a_{h'}) \mid s_h = s \right].$$

The action-value function of policy π , $Q_h^\pi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ gives the expected total reward under the π until the end of the episode, starting from (s, a) at stage h .

$$Q_h^\pi(s, a) := \mathbb{E}_\pi \left[\sum_{h'=h}^H r(s_{h'}, a_{h'}) \mid s_h = s, a_h = a \right].$$

Let $\pi^* = \{\pi_h^*\}_{h=1}^H$ be an optimal policy that satisfies $V_h^{\pi^*}(s) \geq V_h^\pi(s)$ for all policies π and for all states $s \in \mathcal{S}$. We denote by $V_h^*(s) = V_h^{\pi^*}(s)$ and $Q_h^*(s, a) = Q_h^{\pi^*}(s, a)$ the optimal value function and the optimal action-value function, respectively.

Recall that the cumulative regret under OVIFH-MNL is given by

$$\text{Regret}(M, \text{OVIFH-MNL}, s, T) = T \cdot J^*(M) - R(M, \text{OVIFH-MNL}, s, T) = \sum_{t=1}^T (J^*(M) - r(s_t, a_t)).$$

For simplicity, we use notations $s_{k,h} := s_{(k-1)H+h}$ and $a_{k,h} := a_{(k-1)H+h}$. Here, $s_{k,1}$ is the initial state of the k th episode. We denote by $\pi^k = \{\pi_h^k\}_{h=1}^H$ the policy taken by OVIFH-MNL for the k th episode. Then it follows that

$$\begin{aligned} \text{Regret}(M, \text{OVIFH-MNL}, s, T) &= \sum_{k=1}^{T/H} \sum_{h=1}^H (J^*(M) - r(s_{k,h}, a_{k,h})) \\ &= \underbrace{\sum_{k=1}^{T/H} (HJ^*(M) - V_1^*(s_{k,1}))}_{R_1} + \underbrace{\sum_{k=1}^{T/H} (V_1^*(s_{k,1}) - V_1^{\pi^k}(s_{k,1}))}_{R_2} + \underbrace{\sum_{k=1}^{T/H} \left(V_1^{\pi^k}(s_{k,1}) - \sum_{h=1}^H r(s_{k,h}, a_{k,h}) \right)}_{R_3} \end{aligned}$$

For R_1 , we have the following lemma.

Lemma 14 (Wei et al., 2021, Lemma 13). $|HJ^*(M) - V_1^*(s_{k,1})| \leq \text{sp}(v^*)$.

Then it follows from Lemma 14 that $R_1 \leq T \cdot \text{sp}(v^*)/H$.

Next, note that R_2 is the cumulative regret under UCRL-MNL by Hwang and Oh (2023) over T/H episodes. The following lemma provides an upper bound on the regret incurred by UCRL-MNL.

Lemma 15 (Hwang and Oh, 2023, Theorem 1). *Suppose that Assumptions 1-3 hold. Setting $\lambda = L_\varphi^2$, UCRL-MNL guarantees that*

$$R_2 = \sum_{k=1}^{T/H} (V_1^*(s_{k,1}) - V_1^{\pi^k}(s_{k,1})) = \tilde{O} \left(\kappa^{-1} H^{3/2} d \sqrt{T} + \kappa^{-1} L_\varphi L_\theta H^{3/2} \sqrt{dT} \right)$$

with probability at least $1 - \delta$ where $\tilde{O}(\cdot)$ hides logarithmic factors of T , \mathcal{U} , and $1/\delta$.

For R_3 , taking Y_k as

$$Y_k = V_1^{\pi^k}(s_{k,1}) - \sum_{h=1}^H r(s_{k,h}, a_{k,h})$$

and \mathcal{F}_k as the σ -algebra by the randomness up to episode $k-1$, $\mathbb{E}[Y_k | \mathcal{F}_k] = 0$. This implies that $Y_1, \dots, Y_{T/H}$ give rise to a Martingale difference sequence. Moreover, $|Y_k| \leq H$. Then Lemma 12 implies that

$$R_3 = \sum_{k=1}^{T/H} Y_k \leq H \sqrt{\frac{2T}{H} \log \left(\frac{1}{\delta} \right)}$$

with probability at least $1 - \delta$.

Consequently, it follows that

$$\text{Regret}(M, \text{OVIFH-MNL}, s, T) = \tilde{O} \left(\frac{T}{H} \text{sp}(v^*) + \frac{1}{\kappa} H^{3/2} d \sqrt{T} + \frac{1}{\kappa} L_\varphi L_\theta H^{3/2} \sqrt{dT} + H^{1/2} \sqrt{2T \log \left(\frac{1}{\delta} \right)} \right)$$

holds with probability at least $1 - 2\delta$. Then we take $H = \kappa^{2/5} d^{-2/5} T^{1/5}$, and as a result,

$$\text{Regret}(M, \text{OVIFH-MNL}, s, T) = \tilde{O} \left(\kappa^{-2/5} \text{sp}(v^*) d^{2/5} T^{4/5} + \kappa^{-2/5} L_\varphi L_\theta \text{sp}(v^*) d^{-1/10} T^{4/5} \right)$$

holds with probability at least $1 - 2\delta$, as required.

Appendix D. Lower Bound Proof for the Infinite-Horizon Average-Reward Setting

Recall that the transition core $\bar{\theta}$ is given by

$$\bar{\theta} = \left(\frac{\theta}{\alpha}, \frac{1}{\beta} \right) \quad \text{where} \quad \theta \in \left\{ -\frac{\bar{\Delta}}{d-1}, \frac{\bar{\Delta}}{d-1} \right\}^{d-1}, \quad \bar{\Delta} = \log \left(\frac{(1-\delta)(\delta+\Delta)}{\delta(1-\delta-\Delta)} \right),$$

and $\delta = 1/D$ and $\Delta = (d-1)/(45\sqrt{(2/5)DT \log 2})$.

D.1 Linear Approximation of the Multinomial Logistic Model

Let us define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(x) = \frac{1}{1 + \frac{1-\delta}{\delta} \exp(-x)}$$

where $\delta = 1/D$. The derivative of f is given by

$$f'(x) = \frac{\frac{1-\delta}{\delta} \exp(-x)}{\left(1 + \frac{1-\delta}{\delta} \exp(-x)\right)^2} = f(x) - f(x)^2.$$

The following lemma bridges the multinomial logistic function x and a linear function based on the mean value theorem.

Lemma 16 *For any $x, y \in [-\bar{\Delta}, \bar{\Delta}]$ with $x \geq y$, we have*

$$0 \leq f(x) - f(y) \leq (\delta + \Delta)(x - y).$$

Proof By the mean value theorem, there exists $y \leq z \leq x$ such that $f(x) - f(y) = f'(z)(x - y)$. Note that $f'(z) = f(z) - f(z)^2 \leq f(z) \leq f(\bar{\Delta}) = \delta + \Delta$ where the last equality holds by our choice of $\bar{\Delta}$. \blacksquare

By our choice of feature vector φ and transition core $\bar{\theta} = (\theta/\alpha, 1/\beta)$, we have

$$p(x_1 | x_0, a) = \frac{1}{1 + (D-1) \exp(-a^\top \theta)} = f(a^\top \theta) \quad \text{and} \quad p(x_1 | x_0, a) = \delta = f(0).$$

D.2 Basic Properties of the Hard-to-Learn MDP

For simplicity, we introduce notation p_θ given by

$$p_\theta(x_j | x_i, a) := p(x_j | x_i, a, \bar{\theta})$$

for any $i, j \in \{0, 1\}$. Note that inducing a higher probability of transitioning to x_1 from x_0 results in a larger average reward. This means that the optimal policy to choose action a that maximizes $a^\top \theta$ so that $p(x_1 | x_0, a)$ is maximized. Then under the optimal policy, $a^\top \theta = \bar{\Delta}$. We denote by p^* the transition function under the optimal policy, so we have

$$p^*(x_1 | x_0, a) = f(\bar{\Delta}) = \delta + \Delta$$

where the second equality follows from our choice of $\bar{\Delta}$. Note that the expected travel time from state x_1 to state x_0 is $1/(\delta + \Delta)$ which is less than $1/\delta = D$, while the expected travel time from state x_0 to state x_1 is $1/\delta = D$. Hence, the diameter of our hard-to-learn MDP M_θ is D . Moreover, under the optimal policy, the stationary distribution over states x_0 and x_1 is given by

$$\mu = \left(\frac{\delta}{2\delta + \Delta}, \frac{\delta + \Delta}{2\delta + \Delta} \right).$$

As $r(x_0, a) = 0$ and $r(x_1, a) = 1$ for any $a \in \mathcal{A}$, it follows that the optimal average reward equals $J^*(M_\theta) = (\delta + \Delta)/(2\delta + \Delta)$.

Recall that δ and Δ are given by

$$\delta = \frac{1}{D} \quad \text{and} \quad \Delta = \frac{1}{45\sqrt{(2/5)\log 2}} \cdot \frac{(d-1)}{\sqrt{DT}},$$

respectively. The following lemma characterizes the sizes of parameters δ and Δ under the setting of our hard-to-learn MDP.

Lemma 17 *Suppose that $d \geq 2$, $D \geq 101$, $T \geq 45(d-1)^2D$. Then the following statements hold.*

$$100\Delta \leq \delta, \quad 2\delta + \Delta \leq 1, \quad \Delta \leq \delta(1 - \delta), \quad \frac{1}{\delta} \leq \left(\frac{3}{2} \cdot \frac{4}{5} \cdot \left(\frac{99}{101} \right)^4 - 1 \right) T.$$

Proof If $T \geq 45(d-1)^2D$, then $T \geq (100/15)^2(d-1)^2D$. Note that $\sqrt{(2/5)\log 2} > 1/3$. Then $100\Delta < (100/15)(d-1)/\sqrt{DT}$, and as $T \geq (100/15)^2(d-1)^2D$, we get that $100\Delta < 1/D = \delta$. Moreover, since $\delta \leq 1/3$, we also have that $2\delta + \Delta \leq 1$ and $\Delta \leq \delta(1 - \delta)$. Moreover, we know that

$$\frac{3}{2} \cdot \frac{4}{5} \cdot \left(\frac{99}{101} \right)^4 > \frac{11}{10}.$$

Since $T \geq 45(d-1)^2D \geq 10D = 10/\delta$, the last inequality holds. ■

The following lemma provides upper bounds on L_φ and L_θ .

Lemma 18 *For any $\bar{\theta} = (\theta/\alpha, 1/\beta)$, we have $\|\bar{\theta}\|_2 \leq 100/99$. Moreover, for any $a \in \mathcal{A}$ and $i, j \in \{0, 1\}$, $\|\varphi(x_i, a, x_j)\|_2 \leq 1 + \log(D-1)$.*

Proof Recall that $\alpha = \sqrt{\bar{\Delta}/((d-1)(1+\bar{\Delta}))}$ and $\beta = \sqrt{1/(1+\bar{\Delta})}$. Moreover,

$$\|\bar{\theta}\|_2^2 = \frac{\|\theta\|_2^2}{\alpha^2} + \frac{1}{\beta^2} = (1 + \bar{\Delta})^2.$$

Note that

$$\bar{\Delta} = \log \left(\frac{(1-\delta)(\delta+\Delta)}{\delta(1-\delta-\Delta)} \right) = \log \left(\frac{\Delta + \delta(1-\delta-\Delta)}{\delta(1-\delta-\Delta)} \right) \leq \frac{\Delta}{\delta(1-\delta-\Delta)}.$$

Then it follows from Lemma 17 that

$$\bar{\Delta} \leq \frac{1}{100} \cdot \frac{1}{1 - \frac{101}{100}\delta} = \frac{1}{100 - 101\delta} \leq \frac{1}{99},$$

which implies that $\|\bar{\theta}\|_2 \leq 1 + \bar{\Delta} \leq 100/99$. Moreover, for any $i, j \in \{0, 1\}$,

$$\|\varphi(x_i, a, x_j)\|^2 \leq \alpha^2 \|a\|_2^2 + \beta^2 (\log(D-1))^2 = \frac{\bar{\Delta}}{1 + \bar{\Delta}} + \frac{(\log(D-1))^2}{1 + \bar{\Delta}} \leq (1 + \log(D-1))^2,$$

as required. \blacksquare

D.3 Proof of Theorem 4

To provide a lower bound, it is sufficient to consider deterministic stationary policies (Auer et al., 2002). Let \mathfrak{A} be a deterministic policy. Then we refer to $\text{Regret}(M_\theta, \mathfrak{A}, x_0, T)$ as $\text{Regret}_\theta(T)$. Let \mathcal{P}_θ denote the distribution over \mathcal{S}^T where $s_1 = x_0$, a_t is determined by \mathfrak{A} , and s_{t+1} is sampled from $p_\theta(\cdot | s_t, a_t)$. Let \mathbb{E}_θ denote the expectation taken over \mathcal{P}_θ . Moreover, we define N_i for $i \in \{0, 1\}$ and N_0^a as the number of times x_i is visited for $i \in \{0, 1\}$ and the number of time steps in which state x_0 is visited and action a is chosen. We also define $N_0^\mathcal{V}$ for $\mathcal{V} \subseteq \mathcal{A}$ as the number of time steps in which state x_0 is visited and an action from the set \mathcal{V} is chosen.

Note that we have

$$\mathbb{E}_\theta [\text{Regret}_\theta(T)] = TJ^*(M_\theta) - \mathbb{E}_\theta \left[\sum_{t=1}^T r(s_t, a_t) \right] = TJ^*(M_\theta) - \mathbb{E}_\theta N_1.$$

Taking $\Theta = \{-\bar{\Delta}/(d-1), \bar{\Delta}/(d-1)\}^{d-1}$, it follows that

$$\frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_\theta [\text{Regret}_\theta(T)] = TJ^*(M_\theta) - \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_\theta N_1. \quad (17)$$

To provide an upper bound on $\mathbb{E}_\theta N_1$, we prove the following lemma that is analogous to (Wu et al., 2022, Lemma C.2).

Lemma 19 *Suppose that $2\delta + \Delta \leq 1$, $\Delta \leq \delta(1 - \delta)$, and*

$$\frac{1}{\delta} \leq \left(\frac{3}{2} \cdot \frac{4}{5} \cdot \left(\frac{99}{101} \right)^4 - 1 \right) T.$$

Then

$$\mathbb{E}_\theta N_1 \leq \frac{T}{2} + \frac{\delta + \Delta}{2\delta} \sum_{a \in \mathcal{A}} a^\top \theta \cdot \mathbb{E}_\theta N_0^a \quad \text{and} \quad \mathbb{E}_\theta N_0 \leq \left(\frac{99}{101} \right)^4 \cdot \frac{4}{5} T.$$

Proof See Lemma D.4. \blacksquare

Note that since $a \in \{-1, 1\}^{d-1}$,

$$(\delta + \Delta)a^\top \theta \leq (\delta + \Delta) \frac{\bar{\Delta}}{d-1} \sum_{j=1}^{d-1} \mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\}.$$

Moreover,

$$\bar{\Delta} = \log \left(\frac{(1-\delta)(\delta+\Delta)}{\delta(1-\delta-\Delta)} \right) = \log \left(\frac{\Delta + \delta(1-\delta-\Delta)}{\delta(1-\delta-\Delta)} \right) \leq \frac{\Delta}{\delta(1-\delta-\Delta)}$$

where the inequality holds because $1+x \leq \exp(x)$ for any $x \in \mathbb{R}$. Moreover, since $100\Delta \leq \delta$ and $D \geq 101$, we have $\delta \leq 1/101$ and

$$(\delta + \Delta)\bar{\Delta} \leq \frac{(\delta + \Delta)}{\delta(1-\delta-\Delta)} \leq \frac{101}{100} \cdot \frac{1}{1 - \frac{101}{100}\delta} \cdot \Delta \leq \frac{101}{99}\Delta. \quad (18)$$

Then it follows from Lemma 19 that

$$\begin{aligned} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_\theta N_1 &\leq \frac{T}{2} + \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \frac{\Delta}{\delta(d-1)} \sum_{a \in \mathcal{A}} \sum_{j=1}^{d-1} \mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} \frac{101\mathbb{E}_\theta N_0^a}{198} \\ &\leq \frac{T}{2} + \frac{101\Delta}{198\delta|\Theta|(d-1)} \sum_{j=1}^{d-1} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \mathbb{E}_\theta [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a]. \end{aligned} \quad (19)$$

For a given θ and a coordinate $j \in [d-1]$, we consider θ' that differs from θ only in the j th coordinate. Then we have

$$\begin{aligned} &\mathbb{E}_\theta [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] + \mathbb{E}_{\theta'} [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta'_j)\} N_0^a] \\ &= \mathbb{E}_{\theta'} N_0^a + \mathbb{E}_\theta [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] - \mathbb{E}_{\theta'} [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] \end{aligned}$$

because $\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} + \mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta'_j)\} = 1$. Summing up this equality for $\theta \in \Theta$ and $a \in \mathcal{A}$, we obtain

$$\begin{aligned} &2 \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \mathbb{E}_\theta [\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] \\ &= \sum_{\theta \in \Theta} \mathbb{E}_{\theta'} N_0 + \sum_{\theta \in \Theta} \left(\mathbb{E}_\theta \left[\sum_{a \in \mathcal{A}} \mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a \right] - \mathbb{E}_{\theta'} \left[\sum_{a \in \mathcal{A}} \mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a \right] \right) \\ &= \sum_{\theta \in \Theta} \mathbb{E}_{\theta'} N_0 + \sum_{\theta \in \Theta} \left(\mathbb{E}_\theta [N_0^{\mathcal{A}_j}] - \mathbb{E}_{\theta'} [N_0^{\mathcal{A}_j}] \right) \end{aligned}$$

where \mathcal{A}_j is the set of all actions a which satisfy $\mathbf{1}\{\text{sign}(a_j) = \text{sign}(\theta_j)\}$. Here, to provide an upper bound on the term $\mathbb{E}_\theta [N_0^{\mathcal{A}_j}] - \mathbb{E}_{\theta'} [N_0^{\mathcal{A}_j}]$, we apply the version of Pinsker's inequality due to [Jaksch et al. \(2010\)](#).

Lemma 20 (Jaksch et al., 2010, Equation (49)). *Let $s = \{s_1, \dots, s_T\} \in \mathcal{S}^T$ denote the sequence of the observed states from time step 1 to T . Then for any two distributions \mathcal{P}_1 and \mathcal{P}_2 over \mathcal{S}^T and any bounded function $f : \mathcal{S}^T \rightarrow [0, B]$, we have*

$$\mathbb{E}_{\mathcal{P}_1} f(s) - \mathbb{E}_{\mathcal{P}_2} f(s) \leq \sqrt{\log 2/2B} \sqrt{\text{KL}(\mathcal{P}_2 || \mathcal{P}_1)}$$

where $\text{KL}(\mathcal{P}_2 || \mathcal{P}_1)$ is the Kullback–Leibler divergence of \mathcal{P}_2 from \mathcal{P}_1 .

By Lemma 20, it holds that

$$2 \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \mathbb{E}_{\theta} [\mathbf{1} \{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] \leq \sum_{\theta \in \Theta} \mathbb{E}_{\theta'} N_0 + \sum_{\theta \in \Theta} \sqrt{\log 2/2T} \sqrt{\text{KL}(\mathcal{P}_{\theta'} || \mathcal{P}_{\theta})}.$$

Here, we need to provide an upper bound on the KL divergence term $\text{KL}(\mathcal{P}_{\theta'} || \mathcal{P}_{\theta})$. For this, we prove the following lemma which is analogous to (Wu et al., 2022, Lemma C.4).

Lemma 21 *Suppose that θ and θ' only differ in the j th coordinate and $100\Delta \leq \delta \leq 1/101$. Then we have the following bound for the KL divergence of $\mathcal{P}_{\theta'}$ from \mathcal{P}_{θ} .*

$$\text{KL}(\mathcal{P}_{\theta'} || \mathcal{P}_{\theta}) \leq \left(\frac{101}{99} \right)^2 \frac{16\Delta^2}{(d-1)^2\delta} \mathbb{E}_{\theta'} N_0$$

Proof See Lemma D.5. ■

By Lemma 21, we deduce that

$$\begin{aligned} 2 \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \mathbb{E}_{\theta} [\mathbf{1} \{\text{sign}(a_j) = \text{sign}(\theta_j)\} N_0^a] &\leq \sum_{\theta \in \Theta} \mathbb{E}_{\theta'} N_0 + \sum_{\theta \in \Theta} \frac{202}{99} \sqrt{2 \log 2} \frac{T\Delta}{(d-1)\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta'} N_0} \\ &\leq \sum_{\theta \in \Theta} \mathbb{E}_{\theta} N_0 + \sum_{\theta \in \Theta} \frac{202}{99} \sqrt{2 \log 2} \frac{T\Delta}{(d-1)\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta} N_0}. \end{aligned} \tag{20}$$

Combining (19) and (20), we deduce that

$$\begin{aligned} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_{\theta} N_1 &\leq \frac{T}{2} + \frac{101\Delta}{396\delta|\Theta|} \sum_{\theta \in \Theta} \left(\mathbb{E}_{\theta} N_0 + \frac{202}{99} \sqrt{2 \log 2} \frac{T\Delta}{(d-1)\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta} N_0} \right) \\ &\leq \frac{T}{2} + \frac{\Delta}{4\delta|\Theta|} \sum_{\theta \in \Theta} \left(\frac{4}{5}T + 2\sqrt{2 \log 2} \frac{T\Delta}{(d-1)\sqrt{\delta}} \frac{2\sqrt{T}}{\sqrt{5}} \right) \\ &\leq \frac{T}{2} + \frac{\Delta T}{5\delta} + \sqrt{\frac{2}{5} \log 2} \frac{\Delta^2 T^{3/2}}{(d-1)\delta^{3/2}} \end{aligned} \tag{21}$$

where the second inequality follows from Lemma 19. Furthermore, by (17)

$$\begin{aligned} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_{\theta} [\text{Regret}_{\theta}(T)] &\geq \frac{(\delta + \Delta)T}{2\delta + \Delta} - \frac{T}{2} - \frac{\Delta T}{5\delta} - \sqrt{\frac{2}{5} \log 2} \frac{\Delta^2 T^{3/2}}{(d-1)\delta^{3/2}} \\ &= \frac{\Delta(\delta - 2\Delta)T}{10\delta(2\delta + \Delta)} - \sqrt{\frac{2}{5} \log 2} \frac{\Delta^2 T^{3/2}}{(d-1)\delta^{3/2}} \\ &\geq \frac{2\Delta}{45\delta} T - \sqrt{\frac{2}{5} \log 2} \frac{\Delta^2 T^{3/2}}{(d-1)\delta^{3/2}} \end{aligned}$$

where the second inequality holds because $0 < 4\Delta \leq \delta$. Setting Δ as

$$\Delta = \frac{1}{45\sqrt{(2/5)\log 2}} \cdot \frac{(d-1)}{\sqrt{DT}},$$

the rightmost side equals

$$\frac{1}{2025\sqrt{(2/5)\log 2}}(d-1)\sqrt{DT}.$$

When $d \geq 2$, we have $2(d-1) \geq d$, so we get that

$$\frac{1}{|\Theta|} \sum_{\theta \in \Theta} \mathbb{E}_{\theta} [\text{Regret}_{\theta}(T)] \geq \frac{1}{4050} d\sqrt{DT},$$

as required.

D.4 Proof of Lemma 19

We have that

$$\begin{aligned} \mathbb{E}_{\theta} N_1 &= \sum_{t=2}^T \mathcal{P}_{\theta}(s_t = x_1) \\ &= \underbrace{\sum_{t=2}^T \mathcal{P}_{\theta}(s_t = x_1 \mid s_{t-1} = x_1) \mathcal{P}_{\theta}(s_{t-1} = x_1)}_{I_1} + \underbrace{\sum_{t=2}^T \mathcal{P}_{\theta}(s_t = x_1, s_{t-1} = x_0)}_{I_2}. \end{aligned} \quad (22)$$

For I_1 , note that $\mathcal{P}_{\theta}(s_t = x_1 \mid s_{t-1} = 1 - \delta)$ regardless of action a_{t-1} , so we have

$$I_1 = (1 - \delta) \sum_{t=2}^T \mathcal{P}_{\theta}(s_{t-1} = x_1) = (1 - \delta) \mathbb{E}_{\theta} N_1 - (1 - \delta) \mathcal{P}_{\theta}(s_T = x_1). \quad (23)$$

For I_2 , note that

$$\begin{aligned} I_2 &= \sum_{t=2}^T \sum_{a \in \mathcal{A}} \mathcal{P}_{\theta}(s_t = x_1 \mid s_{t-1} = x_0, a_{t-1} = a) \mathcal{P}_{\theta}(s_{t-1} = x_0, a_{t-1} = a) \\ &= \sum_{t=2}^T \sum_{a \in \mathcal{A}} f(a^{\top} \theta) \mathcal{P}_{\theta}(s_{t-1} = x_0, a_{t-1} = a) \\ &= \sum_{a \in \mathcal{A}} f(a^{\top} \theta) (\mathbb{E} N_0^a - \mathcal{P}_{\theta}(s_T = x_0, a_T = a)). \end{aligned} \quad (24)$$

Plugging (23) and (24) to (22), we deduce that

$$\begin{aligned} \mathbb{E}_{\theta} N_1 &= \sum_{a \in \mathcal{A}} \frac{f(a^{\top} \theta)}{\delta} \mathbb{E}_{\theta} N_0^a - \underbrace{\left(\frac{1 - \delta}{\delta} \mathcal{P}_{\theta}(s_T = x_1) + \sum_{a \in \mathcal{A}} \frac{f(a^{\top} \theta)}{\delta} \mathcal{P}_{\theta}(s_T = x_0, a_T = a) \right)}_{\psi_{\theta}} \\ &= \mathbb{E}_{\theta} N_0 + \frac{1}{\delta} \sum_{a \in \mathcal{A}} (f(a^{\top} \theta) - \delta) \mathbb{E}_{\theta} N_0^a - \psi_{\theta}. \end{aligned} \quad (25)$$

Since $T = \mathbb{E}_\theta N_0 + \mathbb{E}_\theta N_1$, it follows that

$$\mathbb{E}_\theta N_1 \leq \frac{T}{2} + \frac{1}{2\delta} \sum_{a \in \mathcal{A}} (f(a^\top \theta) - \delta) \mathbb{E}_\theta N_0^a. \quad (26)$$

Note that

$$f(a^\top \theta) - \delta = f(a^\top \theta) - f(0) \leq (\delta + \Delta) a^\top \theta$$

where the first inequality is from Lemma 16.

Next, for $\mathbb{E}_\theta N_0$, since $f(-\bar{\Delta}) \leq f(a^\top \theta) \leq f(\bar{\Delta}) = \delta + \Delta$, we have from (25) that

$$\begin{aligned} \mathbb{E}_\theta N_1 &\geq \left(1 + \frac{f(-\bar{\Delta}) - f(0)}{\delta}\right) \mathbb{E}_\theta N_0 - \frac{1 - \delta}{\delta} \mathcal{P}_\theta(x_T = x_1) - \frac{\delta + \Delta}{\delta} \mathcal{P}_\theta(s_T = x_0) \\ &\geq \left(1 + \frac{f(-\bar{\Delta}) - f(0)}{\delta}\right) \mathbb{E}_\theta N_0 - \frac{1 - \delta}{\delta} + \frac{1 - 2\delta - \Delta}{\delta} \mathcal{P}_\theta(s_T = x_0) \\ &\geq \left(1 + \frac{f(-\bar{\Delta}) - f(0)}{\delta}\right) \mathbb{E}_\theta N_0 - \frac{1 - \delta}{\delta} \end{aligned}$$

where the second inequality holds because $2\delta + \Delta \leq 1$. This implies that

$$\mathbb{E}_\theta N_0 \leq \frac{T + \frac{1-\delta}{\delta}}{2 - \frac{1}{\delta}(\delta - f(-\bar{\Delta}))}.$$

The following lemma provides a lower bound on $f(-\bar{\Delta})$.

Lemma 22 $f(-\bar{\Delta}) \geq \delta/2$ if and only if $\Delta \leq \delta(1 - \delta)$.

Proof $f(-\bar{\Delta}) \geq \delta/2$ if and only if $1 + \frac{1-\delta}{\delta} \exp(\bar{\Delta}) \leq 2/\delta$, which is equivalent to $\exp(-\bar{\Delta}) \geq (1-\delta)/(2-\delta)$. By plugging in the definition of $\bar{\Delta}$ to the inequality, we get that $f(-\bar{\Delta}) \geq \delta/2$ if and only if $\delta(1-\delta-\Delta)/((1-\delta)(\delta+\Delta)) \geq (1-\delta)/(2-\delta)$, which is equivalent to $\Delta \leq \delta(1-\delta)$. ■

By simple algebra, we may derive from $f(-\bar{\Delta}) \geq \delta/2$ that $2(\delta - f(-\bar{\Delta})) \leq \delta$ holds. Since we assumed that $\Delta \leq \delta(1 - \delta)$, it follows that

$$\mathbb{E}_\theta N_0 \leq \frac{T + \frac{1-\delta}{\delta}}{3/2} = \left(\frac{99}{101}\right)^4 \cdot \frac{4}{5} T$$

where the inequality holds because

$$\frac{1 - \delta}{\delta} \leq \frac{1}{\delta} \leq \left(\frac{3}{2} \cdot \frac{4}{5} \cdot \left(\frac{99}{101}\right)^4 - 1\right) T,$$

as required.

D.5 Proof of Lemma 21

First of all, we consider the following lemma.

Lemma 23 (Jaksch et al., 2010, Lemma 20). *Suppose $0 \leq \delta' \leq 1/2$ and $\epsilon' \leq 1 - 2\delta'$, then*

$$\delta' \log \frac{\delta'}{\delta' + \epsilon'} + (1 - \delta') \log \frac{(1 - \delta')}{1 - \delta' - \epsilon'} \leq \frac{2(\epsilon')^2}{\delta'}.$$

Let \mathbf{s}_t denote the sequence of states $\{s_1, \dots, s_t\}$ from time step 1 to T . By the Markovian property of MDP, we may decompose the KL divergence term of $\mathcal{P}_{\theta'}$ from \mathcal{P}_θ as follows.

$$\text{KL}(\mathcal{P}_{\theta'} \parallel \mathcal{P}_\theta) = \sum_{t=1}^{T-1} \text{KL}(\mathcal{P}_{\theta'}(s_{t+1} \mid \mathbf{s}_t) \parallel \mathcal{P}_\theta(s_{t+1} \mid \mathbf{s}_t))$$

where the KL divergence of $\mathcal{P}_{\theta'}(s_{t+1} \mid \mathbf{s}_t)$ from $\mathcal{P}_\theta(s_{t+1} \mid \mathbf{s}_t)$ is given by

$$\text{KL}(\mathcal{P}_{\theta'}(s_{t+1} \mid \mathbf{s}_t) \parallel \mathcal{P}_\theta(s_{t+1} \mid \mathbf{s}_t)) = \sum_{\mathbf{s}_{t+1} \in \mathcal{S}^{t+1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t+1}) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} \mid \mathbf{s}_t)}{\mathcal{P}_\theta(s_{t+1} \mid \mathbf{s}_t)}.$$

The right-hand side can be further decomposed as follows.

$$\begin{aligned} & \sum_{\mathbf{s}_{t+1} \in \mathcal{S}^{t+1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t+1}) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} \mid \mathbf{s}_t)}{\mathcal{P}_\theta(s_{t+1} \mid \mathbf{s}_t)} \\ &= \sum_{\mathbf{s}_t \in \mathcal{S}^t} \mathcal{P}_{\theta'}(\mathbf{s}_t) \sum_{x \in \mathcal{S}} \mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_t) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_t)}{\mathcal{P}_\theta(s_{t+1} = x \mid \mathbf{s}_t)} \\ &= \sum_{\mathbf{s}_{t-1} \in \mathcal{S}^{t-1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t-1}) \sum_{x' \in \mathcal{S}} \sum_{a \in \mathcal{A}} \mathcal{P}_{\theta'}(s_t = x', a_t = a \mid \mathbf{s}_{t-1}) \\ & \quad \times \underbrace{\sum_{x \in \mathcal{S}} \mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x', a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x', a_t = a)}{\mathcal{P}_\theta(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x', a_t = a)}}_{I_1}. \end{aligned}$$

Note that at state x_1 , the transition probability does not depend on the action taken and the underlying transition core. This implies that $\mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x', a_t = a) = \mathcal{P}_\theta(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x', a_t = a)$ for all θ, θ' . This means that if $x' = x_1$, we have $I_1 = 0$.

Then it holds that

$$\begin{aligned}
& \sum_{\mathbf{s}_{t+1} \in \mathcal{S}^{t+1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t+1}) \log \frac{\mathcal{P}_{\theta'}(\mathbf{s}_{t+1} \mid \mathbf{s}_t)}{\mathcal{P}_{\theta}(\mathbf{s}_{t+1} \mid \mathbf{s}_t)} \\
&= \sum_{\mathbf{s}_{t-1} \in \mathcal{S}^{t-1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t+1}) \sum_a \mathcal{P}_{\theta'}(s_t = x_0, a_t = a \mid \mathbf{s}_{t-1}) \\
&\quad \times \sum_{x \in \mathcal{S}} \mathcal{P}_{\theta'}(s_{t+1} = x \mid \mathbf{s}_{t-1}, s_t = x_0, a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = s \mid \mathbf{s}_{t-1}, s_t = x_0, a_t = a)}{\mathcal{P}_{\theta}(s_{t+1} = s \mid \mathbf{s}_{t-1}, s_t = x_0, a_t = a)} \\
&= \sum_a \mathcal{P}_{\theta'}(s_t = x_{0,1}, a_t = a) \\
&\quad \times \underbrace{\sum_{x \in \mathcal{S}} \mathcal{P}_{\theta'}(s_{t+1} = s \mid s_t = x_0, a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x \mid s_t = x_0, a_t = a)}{\mathcal{P}_{\theta}(s_{t+1} = x \mid s_t = x_0, a_t = a)}}_{I_2}.
\end{aligned}$$

To bound I_2 , we know that s_{t+1} follows the Bernoulli distribution over x_0 and x_1 with probability $1 - f(a^\top \theta')$ and $f(a^\top \theta')$. Then, we have

$$I_2 = \left(1 - f(a^\top \theta')\right) \log \frac{1 - f(a^\top \theta')}{1 - f(a^\top \theta)} + f(a^\top \theta') \log \frac{f(a^\top \theta')}{f(a^\top \theta)}.$$

Note that

$$\frac{1}{100} \geq \frac{101}{100} \delta \geq \delta + \Delta = f(\bar{\Delta}) \geq f(a^\top \theta') \geq f(-\bar{\Delta}) \geq \frac{\delta}{2}$$

where the first inequality is due to $\delta \leq 1/101$, the second holds because $100\Delta \leq \delta$, and the last inequality is by Lemma 22. Moreover, since $f(\bar{\Delta}) \leq 1/100$,

$$f(a^\top \theta) - f(a^\top \theta') \leq f(\bar{\Delta}) \leq \frac{1}{100} \leq 1 - f(\bar{\Delta}) \leq 1 - f(a^\top \theta').$$

Then we deduce that

$$I_2 \leq \frac{2(f(a^\top \theta') - f(a^\top \theta))^2}{f(a^\top \theta')} \leq \frac{16(\delta + \Delta)^2 \bar{\Delta}^2}{\delta(d-1)^2} \leq \left(\frac{101}{99}\right)^2 \frac{16\Delta^2}{\delta(d-1)^2}$$

where the first inequality is implied by Lemma 23 with $\delta' = f(a^\top \theta')$ and $\epsilon' = f(a^\top \theta) - f(a^\top \theta')$, the second inequality holds because of $f(a^\top \theta') \geq \delta/2$ and Lemma 16. Then

$$\begin{aligned}
\text{KL}(\mathcal{P}_{\theta'} \parallel \mathcal{P}_{\theta}) &= \sum_{t=1}^{T-1} \sum_{\mathbf{s}_{t+1} \in \mathcal{S}^{t+1}} \mathcal{P}_{\theta'}(\mathbf{s}_{t+1}) \log \frac{\mathcal{P}_{\theta'}(\mathbf{s}_{t+1} \mid \mathbf{s}_t)}{\mathcal{P}_{\theta}(\mathbf{s}_{t+1} \mid \mathbf{s}_t)} \\
&\leq \left(\frac{101}{99}\right)^2 \frac{16\Delta^2}{(d-1)^2 \delta} \sum_{t=1}^{T-1} \sum_a \mathcal{P}_{\theta'}(s_t = x_0 \mid a_t = a) \\
&= \left(\frac{101}{99}\right)^2 \frac{16\Delta^2}{(d-1)^2 \delta} \sum_{t=1}^{T-1} \mathcal{P}_{\theta'}(s_t = x_0) \\
&= \left(\frac{101}{99}\right)^2 \frac{16\Delta^2}{(d-1)^2 \delta} \mathbb{E}_{\theta'} N_0,
\end{aligned}$$

as required.

Appendix E. Lower Bound Proof for the Finite-Horizon Episodic Setting

Recall that the transition core $\bar{\theta}_h$ for each step $h \in [H]$ is given by

$$\bar{\theta}_h = \left(\frac{\theta_h}{\alpha}, \frac{1}{\beta} \right) \quad \text{where} \quad \theta_h \in \{-\bar{\Delta}, \bar{\Delta}\}^{d-1}, \quad \bar{\Delta} = \frac{1}{d-1} \log \left(\frac{(1-\delta)(\delta + (d-1)\Delta)}{\delta(1-\delta - (d-1)\Delta)} \right),$$

and $\delta = 1/H$ and $\Delta = 1/(4\sqrt{2HK})$.

E.1 Linear Approximation of the Multinomial Logistic Model

As before, we consider a multinomial logistic function given by $f : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(x) = \frac{1}{1 + \frac{1-\delta}{\delta} \exp(-x)}.$$

In contrast to the infinite-horizon average-reward case, we take $\delta = 1/H$ where H is the horizon of each episode. Recall that the derivative of f is given by

$$f'(x) = \frac{\frac{1-\delta}{\delta} \exp(-x)}{\left(1 + \frac{1-\delta}{\delta} \exp(-x)\right)^2} = f(x) - f(x)^2.$$

For simplicity, for $h \in [H]$, we use notation p_{θ_h} given by

$$p_{\theta_h}(x_i | x_h, a) := p(x_i | x_h, a, \bar{\theta}_h) = \begin{cases} f(a^\top \theta_h), & \text{if } i = H+2 \\ 1 - f(a^\top \theta_h), & \text{if } i = h+1. \end{cases}$$

Note that $-(d-1)\bar{\Delta} \leq a^\top \theta_h \leq (d-1)\bar{\Delta}$ for any $a \in \mathcal{A}$, which means that $f(-(d-1)\bar{\Delta}) \leq p_{\theta_h}(x_{H+2} | x_h, a) \leq f((d-1)\bar{\Delta})$. The following lemma is analogous to Lemma 16.

Lemma 24 *For any $x, y \in [-(d-1)\bar{\Delta}, (d-1)\bar{\Delta}]$ with $x \geq y$, we have*

$$0 \leq f(x) - f(y) \leq (\delta + (d-1)\Delta)(x - y).$$

Proof By the mean value theorem, there exists $y \leq z \leq x$ such that $f(x) - f(y) = f'(z)(x - y)$. Note that $f'(z) = f(z) - f(z)^2 \leq f(z) \leq f((d-1)\bar{\Delta}) = \delta + (d-1)\Delta$ where the last equality holds by our choice of $\bar{\Delta}$. \blacksquare

E.2 Basic Properties of the Hard Finite-Horizon Episodic MDP Instance

Recall that δ and Δ are given by

$$\delta = \frac{1}{D} \quad \text{and} \quad \Delta = \frac{1}{45\sqrt{(2/5)\log 2}} \cdot \frac{(d-1)}{\sqrt{DT}},$$

respectively. The following lemma characterizes the sizes of parameters δ and Δ under the setting of our hard-t0-learn MDP.

Lemma 25 Suppose that $T \geq H^3(d-1)^2/32$. Then $(d-1)\Delta \leq \delta/H$

Proof Note that $(d-1)\Delta \leq \delta/H$ if and only if $K \geq H^3(d-1)^2/32$. ■

The following lemma provides upper bounds on L_φ and L_θ . Moreover,

Lemma 26 Suppose that $H \geq 3$. For any $\bar{\theta} = (\theta/\alpha, 1/\beta)$, we have $\|\bar{\theta}\|_2 \leq 3/2$. Moreover, for any $a \in \mathcal{A}$ and $(i, j) \in \{(h, h+1) : h \in [H]\} \cup \{(h, H+2) : h \in [H]\}$, $\|\varphi(x_i, a, x_j)\|_2 \leq 1 + \log(H-1)$.

Proof Recall that $\alpha = \sqrt{\bar{\Delta}/(1+(d-1)\bar{\Delta})}$ and $\beta = \sqrt{1/(1+(d-1)\bar{\Delta})}$. Moreover,

$$\|\bar{\theta}\|_2^2 = \frac{\|\theta\|_2^2}{\alpha^2} + \frac{1}{\beta^2} = (1+(d-1)\bar{\Delta})^2.$$

Note that

$$(d-1)\bar{\Delta} = \log\left(\frac{(d-1)\Delta + \delta(1-\delta-(d-1)\Delta)}{\delta(1-\delta-(d-1)\Delta)}\right) \leq \frac{(d-1)\Delta}{\delta(1-\delta-(d-1)\Delta)}.$$

Since $(d-1)\Delta \leq \delta/H$ by Lemma 25, it follows that

$$(d-1)\bar{\Delta} \leq \frac{1}{H} \cdot \frac{1}{1 - \frac{H+1}{H}\delta} = \frac{1}{H - (H+1)/H} \leq \frac{1}{H-1},$$

which implies that $\|\bar{\theta}\|_2 \leq 1 + \bar{\Delta} \leq 3/2$. Moreover, for any $(i, j) \in \{(h, h+1) : h \in [H]\} \cup \{(h, H+2) : h \in [H]\}$,

$$\begin{aligned} \|\varphi(x_i, a, x_j)\|_2^2 &\leq \alpha^2 \|a\|_2^2 + \beta^2 (\log(H-1))^2 \\ &= \frac{(d-1)\bar{\Delta}}{1+(d-1)\bar{\Delta}} + \frac{(\log(H-1))^2}{1+(d-1)\bar{\Delta}} \\ &\leq (1 + \log(H-1))^2, \end{aligned}$$

as required. ■

E.3 Proof of Theorem 5

Let $\pi = \{\pi_h\}_{h=1}^H$ be a policy for the H -horizon MDP. Recall that the value function V_1^π under policy π is given by

$$V_1^\pi(x_1) = \mathbb{E}_{\theta, \pi} \left[\sum_{h=1}^H r(s_h, a_h) \mid s_1 = x_1 \right]$$

where the expectation is taken with respect to the distribution that has dependency on the transition core θ and the policy π . Let N_h denote the event that the process visits state x_h in step h and then enters x_{H+2} , i.e., $N_h = \{s_h = x_h, x_{h+1} = x_{H+2}\}$. Then we have that

$$V_1^\pi(x_1) = \sum_{h=1}^{H-1} (H-h) \mathbb{P}_{\theta, \pi}(N_h \mid s_1 = x_1).$$

Moreover, note that

$$\begin{aligned}
 & \mathbb{P}_{\theta,\pi}(s_{h+1} = x_{H+2} \mid s_h = x_h, s_1 = x_1) \\
 &= \sum_{a \in \mathcal{A}} \mathbb{P}_{\theta,\pi}(s_{h+1} = x_{H+2} \mid s_h = x_h, a_h = a) \mathbb{P}_{\theta,\pi}(a_h = a \mid s_h = x_h, s_1 = x_1) \\
 &= \sum_{a \in \mathcal{A}} f(a^\top \theta_h) \mathbb{P}_{\theta,\pi}(a_h = a \mid s_h = x_h, s_1 = x_1) \\
 &= \delta + \underbrace{\sum_{a \in \mathcal{A}} (f(a^\top \theta_h) - \delta) \mathbb{P}_{\theta,\pi}(a_h = a \mid s_h = x_h, s_1 = x_1)}_{a_h}.
 \end{aligned}$$

Then it follows that

$$\mathbb{P}_{\theta,\pi}(s_{h+1} = x_{h+1} \mid s_h = x_h, s_1 = x_1) = 1 - \delta - a_h,$$

which implies that

$$\mathbb{P}_{\theta,\pi}(N_h) = (\delta + a_h) \prod_{j=1}^{h-1} (1 - \delta - a_j).$$

Therefore, we deduce that

$$V_1^\pi(x_1) = \sum_{h=1}^H (H - h)(\delta + a_h) \prod_{j=1}^{h-1} (1 - \delta - a_j).$$

Note that the optimal policy $\pi^* = \{\pi_h^*\}_{h=1}^H$ deterministically chooses the action maximizing $a^\top \theta_h$ at each step h . Recall that the maximum value of $a^\top \theta_h$ is $(d-1)\bar{\Delta}$ for any h , and moreover, $f((d-1)\bar{\Delta}) = \delta + (d-1)\Delta$. Therefore, under the optimal policy,

$$\mathbb{P}_{\theta,\pi^*}(s_{h+1} = x_{H+2} \mid s_h = x_h, s_1 = x_1) = \delta + (d-1)\Delta$$

This further implies that the value function under the optimal policy is given by

$$V_1^*(x_1) = \sum_{h=1}^H (H - h)(\delta + (d-1)\Delta)(1 - \delta - (d-1)\Delta)^{h-1}.$$

Next, let us define S_i and T_i for $i \in [H]$ as follows.

$$S_i = \sum_{h=i}^H (H - h)(\delta + a_h) \prod_{j=i}^{h-1} (1 - \delta - a_j) \text{ and } T_i = \sum_{h=i}^H (H - h)(\delta + (d-1)\Delta)(1 - \delta - (d-1)\Delta)^{h-i}.$$

Following the induction argument of (Zhou et al., 2021, Equation (C.25)) we may deduce that

$$T_1 - S_1 = \sum_{h=1}^{H-1} ((d-1)\Delta - a_h)(H - h - T_{h+1}) \prod_{j=1}^{h-1} (1 - \delta - a_j).$$

Moreover, since $3(d-1)\Delta \leq \delta = 1/H$ and $H \geq 3$ by Lemma 25, it follows from (Zhou et al., 2021, Equations (C.26)) that $H - h - T_{h+1} \geq H/3$ for $h \leq H/2$. Moreover, as $a_j \leq (d-1)\Delta \leq \delta/3$, we have $\delta + a_j \leq 4\delta/3$. Since $H \geq 3$, it holds that

$$\prod_{j=1}^{h-1} (1 - \delta - a_j) \geq \left(1 - \frac{4\delta}{3}\right)^H \geq \frac{1}{3}.$$

Consequently, we deduce that

$$V_1^*(x_1) - V_1^\pi(x_1) = T_1 - S_1 \geq \frac{H}{10} \sum_{h=1}^{H/2} ((d-1)\Delta - a_h). \quad (27)$$

From the right-hand side of (27), we have that

$$(d-1)\Delta = \max_{a \in \mathcal{A}} \mu_h^\top a \quad \text{where} \quad \mu_h = \frac{\Delta}{\bar{\Delta}} \theta_h \in \{-\Delta, \Delta\}^{d-1}.$$

Moreover, note that

$$f(\theta_h^\top a) - \delta \leq (\delta + (d-1)\Delta) \theta_h^\top a = \frac{\bar{\Delta}(\delta + (d-1)\Delta)}{\Delta} \mu_h^\top a \leq \frac{\delta + (d-1)\Delta}{\delta(1 - \delta - (d-1)\Delta)} \mu_h^\top a$$

where the first inequality is due to Lemma 24 and the second inequality holds because

$$\bar{\Delta} = \frac{1}{d-1} \log \left(1 + \frac{(d-1)\Delta}{\delta(1 - \delta - (d-1)\Delta)} \right) \leq \frac{1}{d-1} \cdot \frac{(d-1)\Delta}{\delta(1 - \delta - (d-1)\Delta)} = \frac{\Delta}{\delta(1 - \delta - (d-1)\Delta)}.$$

Furthermore, as $(d-1)\Delta \leq \delta/H$ by Lemma 25, we have

$$\frac{\delta + (d-1)\Delta}{\delta(1 - \delta - (d-1)\Delta)} \leq \frac{(1 + 1/H)\delta}{\delta(1 - (1 + 1/H)\delta)} = \frac{H^2 + H}{H^2 - H - 1} = 1 + \frac{2H + 1}{H^2 - H - 1} \leq 1 + \frac{3}{H}$$

where the first inequality holds because $(d-1)\Delta \leq \delta/H$, the first equality holds due to $\delta = 1/H$, and the last inequality is by $H \geq 3$. Then it follows that

$$f(\theta_h^\top a) - \delta \leq \frac{\delta + (d-1)\Delta}{\delta(1 - \delta - (d-1)\Delta)} \mu_h^\top a \leq \mu_h^\top a + \frac{3}{H} \mu_h^\top a \leq \mu_h^\top a + \frac{3(d-1)\Delta}{H}$$

where the last inequality holds because $\mu_h \in \{-\Delta, \Delta\}^{d-1}$ and thus $\mu_h^\top a \leq (d-1)\Delta$. This in turn implies that

$$a_h \leq \frac{3(d-1)\Delta}{H} + \underbrace{\mu_h^\top \sum_{a \in \mathcal{A}} \mathbb{P}_{\theta, \pi}(a_h = a \mid s_h = x_h, s_1 = x_1) \cdot a}_{\bar{a}_h^\pi}$$

Based on (27), we get

$$V_1^*(x_1) - V_1^\pi(x_1) = T_1 - S_1 \geq \frac{H}{10} \sum_{h=1}^{H/2} \left(\max_{a \in \mathcal{A}} \mu_h^\top a - \mu_h^\top \bar{a}_h^\pi \right) - \frac{H(d-1)\Delta}{20}. \quad (28)$$

Let \mathfrak{A} be an algorithm that takes policy $\pi^k = \{\pi_h^k\}_{h=1}^H$ for episodes $k \in [K]$. Then we deduce from (28) that

$$\begin{aligned} \mathbb{E} [\text{Regret}(M_\theta, \mathfrak{A}, K)] &= \mathbb{E} \left[\sum_{k=1}^K \left(V_1^*(x_1) - V_1^{\pi^k}(x_1) \right) \right] \\ &\geq \frac{H}{10} \sum_{h=1}^{H/2} \underbrace{\mathbb{E} \left[\sum_{k=1}^K \left(\max_{a \in \mathcal{A}} \mu_h^\top a - \mu_h^\top \bar{a}_h^{\pi^k} \right) \right]}_{I_h(\theta, \pi)} - \frac{H(d-1)}{20} K \Delta. \end{aligned} \quad (29)$$

Here, we now argue that the term $I_h(\theta, \pi)$ corresponds to the regret under a bandit algorithm for a linear bandit problem. Let \mathcal{L}_{μ_h} denote the linear bandit problem parameterized by $\mu_h \in \{-\Delta, \Delta\}^{d-1}$ where the action set is $\mathcal{A} = \{-1, 1\}^{d-1}$ and the reward distribution for taking action $a \in \mathcal{A}$ is a Bernoulli distribution $B(\delta + \mu_h^\top a)$. Recall that $\bar{a}_h^{\pi^k}$ is given by

$$\bar{a}_h^{\pi^k} = \sum_{a \in \mathcal{A}} \mathbb{P}_{\theta, \pi^k}(a_h = a \mid s_h = x_h, s_1 = x_1) \cdot a.$$

Basically, \mathfrak{A} corresponds to a bandit algorithm that takes action $a \in \mathcal{A}$ with probability $\mathbb{P}_{\theta, \pi^k}(a_h = a \mid s_h = x_h, s_1 = x_1)$ in episode k . Let $a_h^{\pi^k}$ denote the random action taken by \mathfrak{A} . Then by linearity of expectation,

$$I_h(\theta, \pi) = \mathbb{E} \left[\sum_{k=1}^K \left(\max_{a \in \mathcal{A}} \mu_h^\top a - \mu_h^\top a_h^{\pi^k} \right) \right]$$

where the expectation is taken with respect to the randomness generated by \mathfrak{A} and which is the expected pseudo-regret under \mathfrak{A} . The following lemma provides a lower bound on the expected pseudo-regret for the particular linear bandit instance.

Lemma 27 (Zhou et al., 2021, Lemma C.8). *Suppose that $0 < \delta \leq 1/3$ and $K \geq (d-1)^2/(2\delta)$. Let $\Delta = 4\sqrt{2\delta/K}$ and consider the linear bandit problems \mathcal{L}_{μ_h} described above. Then for any bandit algorithm \mathfrak{A} , there exists a parameter $\mu_h^* \in \{-\Delta, \Delta\}^{d-1}$ such that the expected pseudo-regret of \mathfrak{A} over the first K steps on $\mathcal{L}_{\mu_h^*}$ is at least $(d-1)\sqrt{K\delta}/(8\sqrt{2})$.*

Applying Lemma 27 to (29), we deduce that

$$\begin{aligned} \mathbb{E} [\text{Regret}(M_\theta, \mathfrak{A}, K)] &\geq \frac{H^{3/2}(d-1)\sqrt{K}}{160\sqrt{2}} - \frac{H^{1/2}(d-1)\sqrt{K}}{80\sqrt{2}} \\ &\geq \frac{H^{3/2}(d-1)\sqrt{K}}{160\sqrt{2}} - \frac{H^{3/2}(d-1)\sqrt{K}}{240\sqrt{2}} \\ &= \frac{H^{3/2}(d-1)\sqrt{K}}{480\sqrt{2}} \end{aligned}$$

where the second inequality holds because $H \geq 3$.