Lecture #24: Ellipsoid method and separation vs optimization June 8, 2023
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1 Outline

In this lecture, we study

- Optimization via separation of constraints,
- The ellipsoid method,
- The equivalence between optimization and separation.

2 Strong cutting planes and separation problems

Consinder a linear program of the form

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & \alpha^\top x \le \beta \quad \text{for } (\alpha, \beta) \in \mathcal{F} \\ & x \in \mathbb{R}^d \end{aligned} \tag{$LP(\mathcal{F})$}$$

Here, the first set of constraints is given by $Ax \le b$, while we have additional constraints that correspond to the family $\mathcal{F}$. We may apply the row generation framework to ($LP(\mathcal{F})$) by sequentially generating inequalities from the family $\mathcal{F}$.

Example 24.1. For the maximum weight matching problem, we have the following formulation.

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \le 1, \quad v \in V \\ & \sum_{e \in E(S)} x_e \le \frac{|S| - 1}{2}, \quad S \subseteq V \text{ odd} \\ & 0 \le x_e \le 1 \quad e \in E \end{aligned}$$

Here, the family of odd set inequalities

$$\sum_{e \in E(S)} x_e \le \frac{|S| - 1}{2} \quad S \subseteq V \text{ odd}$$

correspond to the family $\mathcal{F}$.

Example 24.2. For the traveling salesman problem, we consider the following linear program.

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 2, \quad v \in V \\ & 0 \le x_e \le 1, \quad e \in E \end{aligned}$$

Then we may generate subtour elimination inequalities, comb inequalities, path inequalities, clique inequalities, and ladder inequalities, etc.

The row generation framework proceeds as follows.

- **Initialization**

  $\mathcal{F}_0 = \emptyset$ and set $t = 1$.

- For $t = 1, \ldots, T$, we repeat the following procedure.

  (1) Solve the linear program

  $$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & \alpha^\top x \le \beta \quad \text{for } (\alpha, \beta) \in \mathcal{F}_{t-1} \\ & x \in \mathbb{R}^d, \end{aligned}$$

  and let x^t denote an optimal solution.

  (2) If $\alpha^\top x^t \le \beta$ for all $\mathcal{F}$, then x^t is optimal to ($LP(\mathcal{F})$).

  (3) Let $\mathcal{F}_t = \mathcal{F}_{t-1} \cup \left\{(\alpha^t, \beta^t)\right\}$ for some $(\alpha^t, \beta^t) \in \mathcal{F}$ such that ${\alpha^t}^\top x^t > \beta^t$.

  (4) Set $t \leftarrow t + 1$.

To run the procedure (2), we need to solve the so-called **separation problem** for $\mathcal{F}$. The problem is given as follows.

> Given a point $\bar{x} \in \mathbb{R}^d$ and a family $\mathcal{F}$ of linear inequalities, (i) show that $\bar{x}$ satisfies all inequalities in $\mathcal{F}$, or (ii) find $(\alpha, \beta) \in \mathcal{F}$ such that $\alpha^\top \bar{x} > \beta$.

The important result for today is the following theorem.

Theorem 24.3. *The separation problem for $\mathcal{F}$ can be solved in polynomial time if and only if the optimization problem ($LP(\mathcal{F})$) can be solved in polynomial time.*

This result is referred to as the **equivalence between optimization and separation**.

3 Ellipsoid algorithm and its consequences in combinatorial optimization

In this section, we introduce the ellipsoid algorithm. The problem that we consider is as follows.

> Given a polyhedron $P = \{x \in \mathbb{R}^d : Ax \le b\}$, (1) conclude that the interior of P is empty, or (2) find a point $\bar{x}$ contained in the interior of P.

This is a variant of the **feasibility problem**. The basic outline of the ellipsoid algorithm for the feasibility problem is as follows.

Theorem 24.4 (Kachyan)**.** *The ellipsoid algorithm (Algorithm 1) terminates with a correct answer if E_1 and T are properly chosen.*

Algorithm 1 Ellipsoid algorithm

Initialize a polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ and a sufficiently large ellipsoid E_1.
for $t = 1, \ldots, T$ **do**
  if the center x^t of ellipsoid E_t is in the interior of P **then**
    Stop and conclude that P contains x^t.
  else
    There exists some inequality $\alpha^\top x \leq \beta$ in the system $Ax \leq b$ such that $\alpha^\top x^t \geq \beta$.
    Let E_{t+1} be the smallest ellipsoid containing $E_t \cap \left\{x \in \mathbb{R}^d : \alpha^\top x \leq \beta\right\}$.
    $t \to t + 1$.
  end if
  Conclude that the interior of P is empty.
end for

In fact, Kachyan showed that one can choose E_1 and T so that their encoding sizes are polynomially bounded, in which case Algorithm 1 runs in polynomial time.

The important part is that the ellipsoid algorithm can be turned into a polynomial algorithm for the problem of optimizing a linear function over P. The idea is based on binary search. Basically, if we want to minimize a linear function $c^\top x$, then we consider

$$\left\{x \in \mathbb{R}^d : \ Ax \leq b, \ c^\top x \leq v\right\}$$

for varying v.

Example 24.5. We think about the minimum weight *st*-cut problem in an undirected graph $G = (V, E)$ with $s, t \in V$. Let $\mathcal{F}$ be the family of all *st*-paths in G. Then the following linear program provides a valid formulation for the minimum *st*-cut problem.

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in P} x_e \geq 1, \quad P \in \mathcal{F} \\ & 0 \leq x_e \leq 1, \quad e \in E. \end{aligned}$$

Theorem 24.6 (Fulkerson)**.** *The polytope defined by*

$$\left\{x \in [0,1]^E : \sum_{e \in P} x_e \geq 1, \ P \in \mathcal{F}\right\}$$

is integral.

Then the separation problem for the family $\mathcal{F}$ is equivalent to the shortest *st*-path problem where the edge weights are given by $x \in \mathbb{R}^E_+$.

Next we formally state the **equivalence between optimization and separation**. Let $P \subseteq \mathbb{R}^d$ be a rational polyhedron such that

$$P = \text{conv}\{v^1, \ldots, v^n\} + \text{cone}\{r^1, \ldots, r^\ell\}.$$

Then we say that $P \subseteq \mathbb{R}^d$ belongs to a **well-described family of rational polyhedra** if the length L of input needed to describe P satisfies $d \leq L$ and $\log D$ is bounded by a polynomial function of L, where D is the largest numerator or denominator of the rational vectors v^k, r^h for $k \in [n]$ and $h \in [\ell]$. Here, we care about the number D to bound the complexity of the ellipsoid method.

Example 24.7. Both $P = \{x \in \mathbb{R}^d : \ Ax \leq b\}$ and $P_I = \{x \in \mathbb{Z}^d : \ Ax \leq b\}$ where A, b are rational are well-defined. In addition, the *st*-cut polytope of a graph G (Note that the input in this case is the graph G).

1. **Separation Problem**
   Given a well-defined polyhedron $P \subseteq \mathbb{R}^d$ and $\bar{x} \in \mathbb{Q}^d$, either show that $\bar{x} \in P$ or find an inequality $\alpha^\top x \leq \beta$ satisfied by all $x \in P$ such that $\alpha^\top \bar{x} > \beta$.

2. **Optimization Problem**
   Given a well-defined polyhedron $P \subseteq \mathbb{R}^d$ and $c \in \mathbb{Q}^d$, find x^* such that $c^\top x^* = \max\{c^\top x : x \in P\}$ or show that $P = \emptyset$ or find a direction z in which P is unbounded.

Theorem 24.8. *For a well-defined polyhedron P, the separation can be solved in polynomial time if and only if the optimization problem can be solved in polynomial time.*