Outline

The first part of this lecture covers perfect matching and explains Hall's marriage theorem characterizing the existence of a perfect matching in a bipartite graph. Then the second part is about the vertex cover problem and König's theorem. We provide a combinatorial proof and an linear programming-based proof for König's theorem.

1 Perfect matching and Hall's marriage theorem

Given a graph $G = (V, E)$, not necessarily bipartite, a matching M in G is **perfect** if every vertex $v \in V$ is incident to an edge in M. In other words, every vertex is attached to a matching edge in a perfect matching. The first graph of Figure 4.1 is a perfect matching in a bipartite graph while the second one shows a perfect matching in a non-bipartite graph.

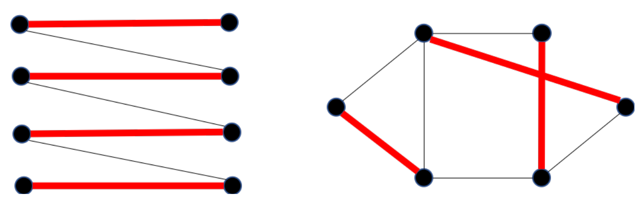

Figure 4.1: perfect matching examples

We can compute a perfect matching by solving an optimization problem described as follows. As in the previous section, we use x_e to indicate whether e is picked for our matching M or not. To guarantee that M is a perfect matching, we impose the constraint

$$\sum_{v \in V: uv \in E} x_v = 1$$

for any vertex $u \in V$. Given the edge weight vector $w \in \mathbb{R}^{|E|}$, a maximum weight perfect matching can be computed by the following integer linear program:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{v \in V: uv \in E} x_{uv} = 1 \quad \text{for all } u \in V, \\ & x_e \in \{0, 1\} \quad \text{for all } e \in E. \end{aligned} \tag{4.1}$$

A matching always exists as the empty set is trivially a matching, but a perfect matching does not always exsits even for a bipartite graph. Figure 4.2 is a bipartite graph that does not have a perfect matching. This means that the integer linear program (4.1) may not have a feasible solution. For bipartite graphs, we have a simple structural characterization for the existence of a perfect matching. The characterization is referred to as **Hall's marriage theorem**. Given a subset $S \subseteq V$, $N(S)$ denotes the set of vertices that are adjacent to any vertex in S.

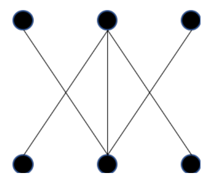

Figure 4.2: a bipartite graph that does not admit a perfect matching

Theorem 4.1 (Hall's marriage theorem)**.** *Let $G = (V, E)$ be a bipartite graph where V is partitioned into V_1 and V_2. Then G has a perfect matching if and only if $|N(S)| \geq |S|$ for any $S \subseteq V_1$.*

Proof. Since G is bipartite, $N(S) \subseteq V_2$ for any $S \subseteq V_1$. Suppose that $|N(S)| < |S|$ for some $S \subseteq V_1$. As the vertices in S can be matched to only the vertices in $N(S)$, $|N(S)| < |S|$ means that not all vertices of S can be matched. Hence, G has no perfect matching in this case.

Next suppose that G has no perfect matching. Let M be a maximum matching. Since M is not perfect, there is an M-exposed vertex r. Starting from r, we build an M-alternating tree using the alternating tree procedure. Since G has no M-augmenting path, the procedure ends up with an M-alternating tree rooted at r as illustrated in Figure 4.3 where U is the set of vertices at an odd level of the tree and W collects the vertices at an even level. Moreover, we proved that the vertices

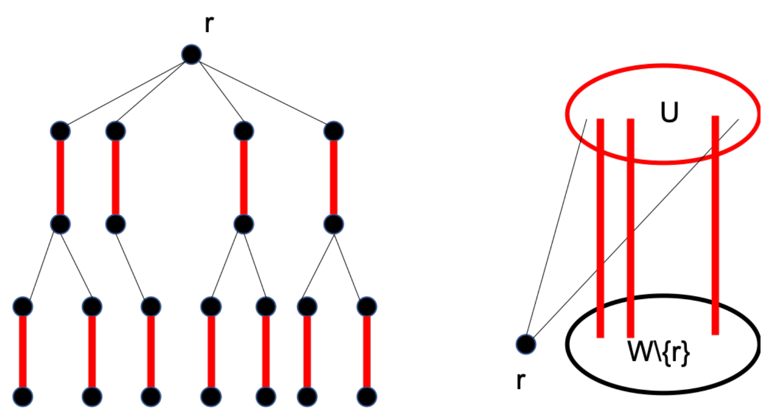

Figure 4.3: the M-alternating tree from an M-exposed vertex

in the set W are adjacent to none of the vertices in $V \setminus (W \cup U)$. This implies that $N(W) = U$. As $|W| = |U| + 1$, we have $|N(W)| < |W|$, as required. $\square$

The stable matching problem also admits a linear programming-based solution.

2 Vertex cover problem

So far, we have focused on the bipartite matching problem. Just for a moment, let us turn our attention to a different problem, yet it is closely related to bipartite matching. Given a graph $G = (V, E)$, a subset B of the vertex set V is called a **vertex cover** if for every edge $e \in E$, e has an endpoint in B. The **vertex cover problem** is to find a vertex cover with the minimum number of vertices. The following provides a bridge between the matching problem and the vertex cover problem.

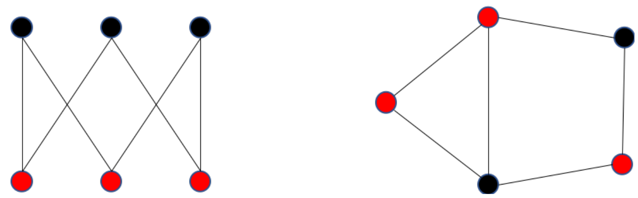

Figure 4.4: vertex cover examples

Proposition 4.2. *Let $G = (V, E)$ be a graph. Then the minimum size of a vertex cover for G is greater than or equal to the maximum size of a matching in G.*

Proof. Let M be a maximum matching of G. Note that the edges in M are pairwise vertex-disjoint. This means that any vertex cover B contains at least one endpoint of each edge in M, which implies that $|B| \geq |M|$. $\square$

For a bipartite graph, we can derive the folllowing stronger result.

Theorem 4.3 (König's theorem)**.** *Let $G = (V, E)$ be a bipartite graph. Then the minimum size of a vertex cover for G equals the maximum size of a matching in G.*

Proof. Remember the augmenting path algorithm for maximum bipartite matching and the alternating tree procedure to find an M-augmenting path. Suppose that M is a maximum matching in G. Then we know that G has no M-augmenting path. In this case, the alternating tree procedure ends up with a decomposition of the vertex set V into

$$V = (W_1 \cup V_1) \cup \cdots \cup (W_k \cup V_k)$$

for some k illustrated as in Figure 4.5. We proved that every edge $e \in E$ is incident to a vertex in

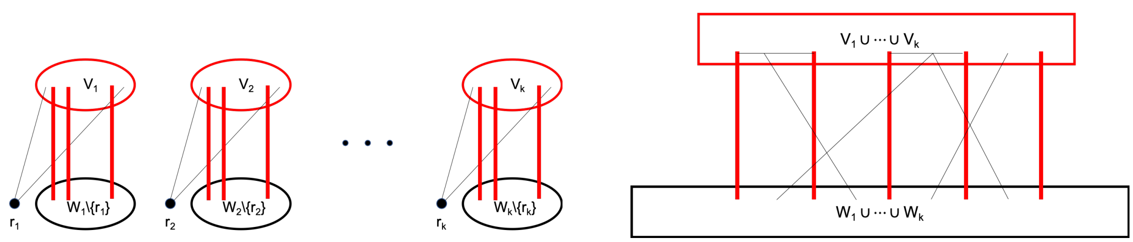

Figure 4.5: vertex set decomposition by the alternating tree procedure

$V_! \cup \cdots \cup V_k$. This means that $V_1 \cup \cdots \cup V_k$ is a vertex cover. Moreover, recall that $|M| = |V_1 \cup \cdots \cup V_k|$. By Proposition 4.2, it follows that $V_1 \cup \cdots \cup V_k$ is a minimum vertex cover and its size equals the maximum size of a matching, as required. $\square$

The proof of Theorem 4.3 suggests that the augmenting path algorithm with the alternating tree procedure not only gives us a maximum matching but also a minimum vertex cover. This means that the vertex cover problem can be solved in polynomial time, but the vertex cover problem for general graphs is known to be NP-hard.

3 Linear programming duality-based proof for König's theorem

As for the matching problem, vertex cover also admits an integer linear programming formulation. For each vertex $v \in V$, we use a variable y_v to indicate whether v is picked for our vertex cover B or not, i.e.,

$$y_v = \begin{cases} 1 & \text{if } v \text{ is included in vertex cover } B, \\ 0 & \text{otherwise.} \end{cases}$$

Then we may impose the condition that y corresponds to a vertex cover by setting

$$y_u + y_v \geq 1$$

for all $uv \in E$. Therefore, the vertex cover problem can be equivalently formulated as the following integer linear program:

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} y_v \\ \text{subject to} \quad & y_u + y_v \geq 1 \quad \text{for all } uv \in E, \\ & y_v \in \{0,1\} \quad \text{for all } v \in V. \end{aligned} \tag{4.2}$$

Proposition 4.4. *Let $G = (V, E)$ be a graph, not necessarily bipartite. Then solving the optimization problem (4.2) computes a minimum vertex cover for G.*

The LP relaxation of (4.2) is given by

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} y_v \\ \text{subject to} \quad & y_u + y_v \geq 1 \quad \text{for all } uv \in E, \\ & y_v \geq 0 \quad \text{for all } v \in V. \end{aligned} \tag{4.3}$$

Theorem 4.5. *Let $G = (V, E)$ be a bipartite graph. Then the LP relaxation (4.3) has an optimal solution y^* that satisfies $y^*_v \in \{0,1\}$ for all $v \in V$. Moreover, one can find a minimum vertex cover for G by solving the linear program (4.3).*

Proof. Let $\bar{y}$ be an optimal solution to (4.3). By the nonnegativity constraint, we have $\bar{y}_v \geq 0$ for all $v \in V$. If $\bar{y}_v > 1$ for some $v \in V$, then one may replace $\bar{y}_v$ with 1 to improve the objective while keeping feasibility. This means that $\bar{y}_v \leq 1$ for all $v \in V$ because $\bar{y}$ is an optimal solution.

Let the vertex V be partitioned into V_1 and V_2. Then we run the following procedure.

1. Pick a random threshold $\theta \in (0, 1)$ uniformly at random.
2. Take $U_1 = \{v \in V_1 : \bar{y}_v \geq \theta\}$ and $U_2 = \{v \in V_2 : \bar{y}_v \geq 1 - \theta\}$.
3. Define $y^* \in \{0,1\}^{|V|}$ as the incidence vector of $U_1 \cup U_2$.

Let $uv \in E$ with $u \in V_1$ and $v \in V_2$. Note that either $u \in U_1$ or $v \in V_1$ holds, for otherwise,

$\bar{y}_u + \bar{y}_v < \theta + (1-\theta) = 1$. This shows that $U_1 \cup U_2$ is a vertex cover. Note that

$$\begin{aligned}
\mathbb{E}_\theta\left[\sum_{v\in V} y_v^*\right] &= \sum_{v\in V_1} \mathbb{E}_\theta\left[y_v^*\right] + \sum_{v\in V_2} \mathbb{E}_\theta\left[y_v^*\right] \\
&= \sum_{v\in V_1} \mathbb{P}_\theta\left[\bar{y}_v \geq \theta\right] + \sum_{v\in V_2} \mathbb{P}_\theta\left[\bar{y}_v \geq 1-\theta\right] \\
&= \sum_{v\in V_1} \bar{y}_v + \sum_{v\in V_2} \bar{y}_v \\
&= \sum_{v\in V} \bar{y}_v
\end{aligned} \tag{4.4}$$

where the first equality is by the linearity of expectation, the second equality is by the definition of U_1 and U_2, and the third equality holds because θ is chosen uniformly at random.

Recall that y^* under any threshold θ corresponds to a vertex cover, so we have

$$\sum_{v\in V} y_v^* \geq \sum_{v\in V} \bar{y}_v.$$

Then it follows from (4.4) that for any threshold $\theta \in (0,1)$, $y^* \in \{0,1\}^{|V|}$ satisfies

$$\sum_{v\in V} y_v^* = \sum_{v\in V} \bar{y}_v.$$

This in turn implies that y^* for any choice of θ corresponds to a minimum vertex cover. $\square$

Consequently, the optimal value of (4.3) equals that of (4.2) when the graph G is bipartite. Moreover, the proof of Theorem 4.5 provides the following algorithm for computing a minimum vertex cover in a bipartite graph. The proof of Theorem 4.5 guarantees that Algorithm 1 returns a minimum vertex cover for a bipartite graph.

Algorithm 1 LP-based algorithm for minimum vertex cover

The bipartition $V_1 \cup V_2$ of the vertex set V
Solve the linear program (4.3) and get an optimal solution $\bar{y}$
Take $U_1 = \{v \in V_1 : \bar{y}_v \geq 1/2\}$ and $U_2 = \{v \in V_2 : \bar{y}_v \geq 1/2\}$
Return $U_1 \cup U_2$

Lastly, we conclude this lecture by describing an alternate proof for König's theorem stating that the minimum size of a vertex cover equals the maximum size of a matching in a bipartite graph. Recall that the optimal value of the linear program

$$\begin{aligned}
\text{maximize} \quad & \sum_{e\in E} w_e x_e \\
\text{subject to} \quad & \sum_{v\in V: uv\in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\
& x_e \geq 0 \quad \text{for all } e \in E
\end{aligned} \tag{4.5}$$

is equal to the maximum size of a matching. Moreover, we have just proved that the optimal value of the linear program (4.3) is equal to the minimum size of a vertex cover by Theorem 4.5. In fact, the linear program (4.3) is the **linear programming dual** of (4.5). Then by the **strong duality theorem for linear programming**,

$$\min\left\{\sum_{v\in V} y_v :\ y_u + y_v \geq 1 \quad \text{for all } uv \in E,\ y \in \mathbb{R}_+^{|V|}\right\}$$
$$= \max\left\{\sum_{e\in E} w_e x_e :\ \sum_{v\in V: uv\in E} x_{uv} \leq 1 \quad \text{for all } u \in V,\ x \in \mathbb{R}_+^{|E|}\right\}.$$

This leads us to the conclusion that the minimum size of a vertex cover equals the maximum size of a matching in a bipartite graph, which is König's theorem.