# 1 Outline

In this lecture, we cover

- the two-stage optimization framework,
- two-stage optimization models with various risk measures.

# 2 Two-stage optimization models

Let us describe general two-stage optimization models. The workflow proceeds as follows.

1. Implement the first-stage decision $x_1$, e.g., fixed testing center locations.
2. Observe information $\xi$.
3. Implement the second-stage decision $x_2$ based on $x_1$ and $\xi$, e.g., mobile testing center locations and test case allocations.

Here, we use the following terminologies.

- $x_1$ is called the **here-and-now decision** since they must be executed up-front before observing $\xi$.
- $x_2$ is called the **wait-and-see decision** or the **recourse decision** since they can be executed after information $\xi$ is revealed.

Note that the realized information $\xi$ can **change the course of action**.

Let $Q(x_1, \xi)$ be the cost of the first-stage decision $x_1$ associated with information $\xi$, given that the second-stage decision $x_2$ is chosen optimally with respect to $x_1$ and $\xi$. Formally, $Q(x_1, \xi)$ is given by

$$\begin{aligned} Q(x_1, \xi) \quad := \quad \min \quad & c_2(\xi)^\top x_2 \\ \text{s.t.} \quad & A_2(\xi)x_1 + B_2(\xi)x_2 \geq b_2(\xi). \end{aligned}$$

Again, the second-decision decision optimizes the second-stage problem that is specified after the first-stage decision is made and the information $\xi$ is realized. Here, the objective vector $c_2(\xi)$, constraint matrices $A_2(\xi), B_2(\xi)$, and the right-hand side vector $b_2(\xi)$ depend on the information $\xi$. Assuming that $x_2$ is always chosen as an optimal second-stage decision, $Q(x_1, \xi)$ encodes the value of the first-stage decision $x_1$. Then, we choose $x_1$ by minimizing the overall **expected cost**.

$$\begin{aligned} \min \quad & c_1^\top x_1 + \mathbb{E}\left[Q(x_1, \xi)\right] \\ \text{s.t.} \quad & A_1 x_1 \geq b_1. \end{aligned}$$

Here, we may use other risk measures instead of expectation.

Then the next question is, how do we solve this? As before, we assume that we are given $N$ scenarios about the information $\xi$. We are given

$$\xi_1, \ldots, \xi_N.$$

Assume that

$$\mathbb{P}\left[\xi = \xi_i\right] = p_i, \quad i \in [N]$$

with $\sum_{i\in[N]} p_i = 1$. Then

$$\mathbb{E}\left[Q(x_1, \xi)\right] = \sum_{i\in[N]} p_i Q(x_1, \xi_i).$$

Plugging this to the two-stage optimization model, we deduce

$$\begin{aligned} \min \quad & c_1^\top x_1 + \sum_{i\in[N]} p_i Q(x_1, \xi_i) \\ \text{s.t.} \quad & A_1 x_1 \geq b_1. \end{aligned}$$

By adding some auxiliary variables, we can rewrite the optimization model as

$$\begin{aligned} \min \quad & c_1^\top x_1 + \sum_{i\in[N]} p_i t_i \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & Q(x_1, \xi_i) \leq t_i, \quad i \in [N]. \end{aligned}$$

Next, we can handle constraint

$$Q(x_1, \xi_i) \leq t_i$$

by the procedure called **lifting**. In fact, we have already used the procedure without specifying the terminology. Recall that $Q(x_1, \xi_i)$ is given by the second stage optimization problem with $\xi = \xi_i$.

$$\begin{aligned} Q(x_1, \xi_i) \quad = \quad \min \quad & c_2(\xi)^\top x_2^i \\ \text{s.t.} \quad & A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi). \end{aligned}$$

Here, we used variable $x_2^i$ to indicate that $Q(x_1, \xi_i)$ corresponds to scenario $i$. Note that the minimum value of $c_2(\xi)^\top x_2^i$ over $x_2^i$ satisfying $A_2(\xi)x_1 + B_2(\xi)x_2^i \geq b_2(\xi)$ is less than or equal to $t_i$ if and only if **there exists** some $x_2^i$ satisfying $A_2(\xi)x_1 + B_2(\xi)x_2^i \geq b_2(\xi)$ such that $c_2(\xi)^\top x_2^i \leq t_i$. Then constraint $Q(x_1, \xi_i) \leq t_i$ is equivalent to the condition that there exists $x_2^i$ such that

$$\begin{gathered} c_2(\xi)^\top x_2^i \leq t_i \\ A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi). \end{gathered}$$

Therefore, we obtain the following formulation.

$$\begin{aligned} \min \quad & c_1^\top x_1 + \sum_{i\in[N]} p_i t_i \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & c_2(\xi)^\top x_2^i \leq t_i, \quad i \in [N] \\ & A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

In fact, it is not necessary to use the auxiliary variables $t_i$ for $i \in [N]$. We can simply write

$$\begin{aligned} \min \quad & c_1^\top x_1 + \sum_{i \in [N]} p_i c_2(\xi)^\top x_2^i \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

# 3 Two-stage optimization models with different risk measures

For the second-stage value, we considered the expectation of function $Q(x, \xi)$. In general, we may consider

$$\begin{aligned} \min \quad & c_1^\top x_1 + \rho\left(Q(x_1, \xi_1), \ldots, Q(x_1, \xi_N)\right) \\ \text{s.t.} \quad & A_1 x_1 \geq b_1. \end{aligned}$$

where $\rho : \mathbb{R}^N \to \mathbb{R}$ is some risk measure.

## 3.1 Worst-case value

Consider the case when

$$\rho\left(Q(x_1, \xi_1), \ldots, Q(x_1, \xi_N)\right) = \max\left\{Q(x_1, \xi_1), \ldots, Q(x_1, \xi_N)\right\}.$$

Then the optimization model is given by

$$\begin{aligned} \min \quad & c_1^\top x_1 + t \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & \max\left\{Q(x_1, \xi_1), \ldots, Q(x_1, \xi_N)\right\} \leq t, \quad i \in [N]. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \min \quad & c_1^\top x_1 + t \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & Q(x_1, \xi_i) \leq t, \quad i \in [N]. \end{aligned}$$

Then, by the lifting procedure,

$$\begin{aligned} \min \quad & c_1^\top x_1 + t \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & c_2(\xi)^\top x_2^i \leq t, \quad i \in [N] \\ & A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

## 3.2 Conditional-value at risk

Next, we consider

$$\begin{aligned} \min \quad & c_1^\top x_1 + \mathrm{CVaR}_\alpha\left(Q(x_1, \xi); \hat{P}_N\right) \\ \text{s.t.} \quad & A_1 x_1 \geq b_1. \end{aligned}$$

The model is equivalent to

$$\begin{aligned} \min \quad & c_1^\top x_1 + v \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & \mathrm{CVaR}_\alpha\left(Q(x_1, \xi); \hat{P}_N\right) \leq v. \end{aligned}$$

We may rewrite constraint $\mathrm{CVaR}_\alpha\left(Q(x_1, \xi); \hat{P}_N\right) \leq v$ as

$$\mathrm{CVaR}_\alpha\left(Q(x_1, \xi) - v; \hat{P}_N\right) \leq 0.$$

Recall that $\mathrm{CVaR}_\alpha\left(Q(x_1, \xi) - v; \hat{P}_N\right) \leq 0$ is equivalent to the constraints

$$\begin{aligned} t + \frac{1}{1-\alpha} \sum_{i \in [N]} p_i r_i \leq 0 \\ r \geq 0 \\ t + r_i \geq Q(x_1, \xi_i) - v, \quad i \in [N]. \end{aligned}$$

Furthermore,

$$Q(x_1, \xi_i) \leq v + t + r_i$$

can be rewritten as

$$\begin{aligned} c_2(\xi)^\top x_2^i \leq v + t + r_i, \quad i \in [N] \\ A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

Therefore, we can replace

$$\mathrm{CVaR}_\alpha\left(Q(x_1, \xi); \hat{P}_N\right) \leq v$$

by

$$\begin{aligned} t + \frac{1}{1-\alpha} \sum_{i \in [N]} p_i r_i \leq 0 \\ r \geq 0 \\ c_2(\xi)^\top x_2^i \leq v + t + r_i, \quad i \in [N] \\ A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

Therefore, the final equivalent reformulation is

$$\begin{aligned} \min \quad & c_1^\top x_1 + v \\ \text{s.t.} \quad & A_1 x_1 \geq b_1 \\ & t + \frac{1}{1-\alpha} \sum_{i \in [N]} p_i r_i \leq 0 \\ & r \geq 0 \\ & c_2(\xi)^\top x_2^i \leq v + t + r_i, \quad i \in [N] \\ & A_2(\xi) x_1 + B_2(\xi) x_2^i \geq b_2(\xi), \quad i \in [N]. \end{aligned}$$

### 3.3 Value at risk

Next, we consider

$$\begin{aligned}\min\quad & c_1^\top x_1 + \mathrm{VaR}_\alpha\left(Q(x_1,\xi);\hat{P}_N\right)\\ \text{s.t.}\quad & A_1x_1 \geq b_1.\end{aligned}$$

The model is equivalent to

$$\begin{aligned}\min\quad & c_1^\top x_1 + v\\ \text{s.t.}\quad & A_1x_1 \geq b_1\\ & \mathrm{VaR}_\alpha\left(Q(x_1,\xi)-v;\hat{P}_N\right) \leq 0.\end{aligned}$$

Recall that $\mathrm{VaR}_\alpha\left(Q(x_1,\xi)-v;\hat{P}_N\right) \leq 0$ is equivalent to the constraints

$$\begin{aligned}Q(x_1,\xi_i)-v \leq Mz_i,\quad i\in[N]\\ \sum_{i\in[N]} p_iz_i \leq 1-\alpha\\ z\in\{0,1\}^N.\end{aligned}$$

Furthermore,

$$Q(x_1,\xi_i) \leq v + Mz_i$$

can be rewritten as

$$\begin{aligned}c_2(\xi)^\top x_2^i \leq v + Mz_i,\quad i\in[N]\\ A_2(\xi)x_1 + B_2(\xi)x_2^i \geq b_2(\xi),\quad i\in[N].\end{aligned}$$

Therefore, the final equivalent reformulation is

$$\begin{aligned}\min\quad & c_1^\top x_1 + v\\ \text{s.t.}\quad & A_1x_1 \geq b_1\\ & \sum_{i\in[N]} p_iz_i \leq 1-\alpha\\ & z\in\{0,1\}^N\\ & c_2(\xi)^\top x_2^i \leq v + Mz_i,\quad i\in[N]\\ & A_2(\xi)x_1 + B_2(\xi)x_2^i \geq b_2(\xi),\quad i\in[N].\end{aligned}$$