# Relational Type Theory (All Proofs)

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Abstract—This paper introduces Relational Type Theory (RelTT), a new approach to type theory with extensionality principles, based on a relational semantics for types. The type constructs of the theory are those of System F plus relational composition, converse, and promotion of application of a term to a relation. A concise realizability semantics is presented for these types. The paper shows how a number of constructions of traditional interest in type theory are possible in RelTT, including  $\eta$ -laws for basic types, inductive types with their induction principles, and positive-recursive types. A crucial role is played by a lemma called Identity Inclusion, which refines the Identity Extension property familiar from the semantics of parametric polymorphism. The paper concludes with a type system for RelTT, paving the way for implementation.

#### I. INTRODUCTION

Modern constructive type theories have long wish lists of features, from inductive and coinductive types, to type-specific extensionality principles, quotient types, higher-order datatypes, and more. In tension with this, there are excellent reasons to seek to keep the core type theory small and trustworty. This has been done, in different ways, for Lean [1] and recently Coq [2]. Both those systems implement (variants of) the Calculus of Inductive Constructions, which lacks type-specific extensionality principles.

The present paper proposes Relational Type Theory (ReITT) for deriving expressive type constructs, with type-specific extensionality principles, from a formally small core theory. The approach followed is, to the authors' knowledge, novel: ReITT is based a semantics for types as binary relations on untyped terms. For example, the semantics for a function type  $R \to R'$  makes it the set of pairs of terms  $(t_1, t_2)$  that jointly map inputs related by the meaning of R to outputs related by the meaning of R' (the semantics familiar from the field of logical relations). The notion that a term "is" a function is expressed only by saying that it is related to itself at function type. So relations between terms are the primary concern of the theory, and expression of program behavior in isolation (i.e., traditional typing) is secondary.

This commitment to relational semantics leads us in an unexplored direction: we extend the set of type constructs with relational constructs. We may use these to express asymmetric relations, which are crucial for developing reasoning principles, like induction principles, within the theory. Interestingly, dependent types are unnecessary for this. The relational semantics already gives us a form of dependency which is all

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terms t ::= x \mid \lambda x. t \mid t t'
types R ::= X \mid R \rightarrow R' \mid \forall X. R \mid R^{\circ} \mid R \cdot R' \mid t
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Fig. 1. Syntax for relational types (X ranges over type variables)

we need for inductive reasoning about terms. So RelTT is an extension, with relational type constructs, of System F, not the Calculus of Constructions. Avoiding dependent types notably simplifies the semantics. The power of System F is needed because the terms that the theory (relationally) types are those of pure lambda calculus, so we adopt impredicative lambda encodings to represent inductive types.

The contributions of the paper are:

- The syntax and semantics of relational types (Section II)
- Basic properties of the semantics, crucially including  $\beta\eta$ -closure (Section III)
- Interesting derived type forms and examples (Section IV), and basic type-specific extensionality principles (Section V).
- Classes of types whose interpretations are proved to be, respectively, symmetric (Section VI) and transitive (Section VII). The proof of transitivity crucially relies on a novel theorem dubbed Identity Inclusion (Lemma 32). The intricate proof of this makes use of duality between types where all quantifiers occur only positively (∀<sup>+</sup> types), and ones where they occur only negatively.
- Derivation of induction principles from types for Churchencodings (Section X). This covers any inductive type definable by a type scheme which is positive and, due to a critical use of Identity Inclusion, ∀<sup>+</sup>. Positive-recursive types are also derived (Section XI).
- A proof system called RelPf (Section VIII) and type system RelTy (Section XII), which are proven sound with respect to the semantics, and are intended as the starting point for implementation of RelTT as a proof assistant.

We will reference lemmas and theorems by name, with the theorem number following in braces (e.g., " $\beta\eta$ -Closure {2}"). We will label assumptions and proven local facts with numbers (e.g., "(1)"), and goals with capital letters (e.g., "(A)").

# II. RELATIONAL TYPES AND THEIR SEMANTICS

The syntax of relational types R is given in Figure 1. Terms t are those of pure untyped lambda calculus. Relational type

$$t \begin{bmatrix} r_1 \rightarrow r_2 \end{bmatrix} t' = \forall a . \forall a' . a \begin{bmatrix} r_1 \end{bmatrix} a' \rightarrow t a \begin{bmatrix} r_2 \end{bmatrix} t' a'$$
  

$$t \begin{bmatrix} r \\ \end{bmatrix} t' = t' \begin{bmatrix} r \end{bmatrix} t$$
  

$$t \begin{bmatrix} r_1 \cdot r_2 \end{bmatrix} t' = \exists t'' . t \begin{bmatrix} r_1 \end{bmatrix} t'' \wedge t'' \begin{bmatrix} r_2 \end{bmatrix} t'$$

Fig. 2. Semantics for relational types; relational operators  $\rightarrow$ ,  $^{\cup}$ , and  $\cdot$ 

constructs include those of System F, plus  $R^{\cup}$  for converse of a relation,  $R \cdot R'$  for composition of relations, and promotion of terms t to relations, to be explained shortly. Usual parsing precedences from type theory are followed; additionally,  $R^{\cup}$ binds most tightly,  $R \cdot R'$  second most tightly, and the other constructs after these. We also follow the usual convention that distinct meta-variables ranging over variables denote distinct variables (so x and y denote different variables), and treat terms and types up to  $\alpha$ -equivalence. Capture-avoiding substitution of R for X in R' is denoted [R/X]R' (similarly [t/x]t' for terms). The set of free variables of any syntactic entity e is denoted FV(e). The obvious definitions of these syntactic notions are omitted.

**Definition 1.** A relation r on terms of pure untyped  $\lambda$ -calculus is  $\beta\eta$ -closed iff  $t_1$  [r]  $t_2$ ,  $t_1' =_{\beta\eta} t_1$ , and  $t_2' =_{\beta\eta} t_2$  imply  $t'_1$  [r]  $t'_2$ . Write  $\mathcal{R}$  for the set of all such relations, and use meta-variable r to range over R

The relational semantics of types is defined in Figure 2, where environment  $\gamma$  is a function mapping a finite set of type variables to elements of R. We use infix notation for application of a relation to a pair of terms and, following [3], we sometimes put square brackets around the relation for readability; for example, in the three equations at the bottom of the figure. In those equations, operators like "→" on the righthand sides have their standard meaning in the background meta-logic.

The interpretation  $[\![R]\!]_{\gamma}$  is defined iff  $\gamma$  is defined for all free type variables of R. When referencing  $[\![R]\!]_{\gamma}$  in theorems, we assume it is defined. The semantics extends  $\gamma$  from type variables to arbitrary types. Promotion of term t is the graph of the meta-level operation mapping t' to t t'. Many examples are below.

While the semantics of universal types quantifies (at the meta-level) over all relations in  $\mathcal{R}$ , we will restrict ourselves in all examples below to instantiating such quantifiers only with definable relations (i.e., ones of the form  $[R]_{\gamma}$ ). In Sections VIII and XII below, we will consider deductive systems for RelTT where this restriction will be enforced.

#### III. BASIC PROPERTIES

**Lemma 2** ( $\beta\eta$ -Closure).  $[R]_{\gamma} \in \mathcal{R}$ .

*Proof.* The proof is by induction on R. Suppose  $t_1 =_{\beta\eta} t_1'$ and  $t_2 =_{\beta \eta} t_2'$ , assume (1)  $t_1 \llbracket R \rrbracket_{\gamma} t_2$ , and show  $t_1' \llbracket R \rrbracket_{\gamma} t_2'$ . Case  $X: \gamma(X) \in \mathcal{R}$  by specification of  $\gamma$ .

<u>Case  $R \to R'$ :</u> assume (2)  $a [R]_{\gamma} a'$ , and show  $t'_1$   $a \llbracket R' \rrbracket_{\gamma} t'_2 a'$ . From (1) and (2) we have  $t_1$   $a \llbracket R' \rrbracket_{\gamma} t_2 a'$ . From this, the IH gives us the required conclusion, as  $t_1 \ a =_{\beta \eta} t'_1 \ a \text{ and } t_2 \ a' =_{\beta \eta} t'_2 \ a'.$ 

<u>Case  $\forall X.R$ :</u> assume  $r \in \mathcal{R}$ , and show  $t_1' [\![R]\!]_{\gamma[X \mapsto r]} t_2'$ . By (1), we have  $t_1$   $[\![R]\!]_{\gamma[X\mapsto r]}$   $t_2$ , from which the IH then yields the desired conclusion.

<u>Case  $R^{\cup}$ :</u> this follows by the IH (using symmetry of  $=_{\beta\eta}$ ). Case  $R \cdot R'$ : from (1), there exists t such that  $t_1 \ [\![R]\!]_{\gamma} \ t$  and  $t \ \llbracket R' \rrbracket_{\gamma} \ t_2$ . By the IH,  $t'_1 \ \llbracket R \rrbracket_{\gamma} \ t$  and  $t \ \llbracket R' \rrbracket_{\gamma} \ t'_2$ . These imply the desired conclusion.

<u>Case t:</u> from (1), we have t  $t_1 = \beta \eta$   $t_2$ ; t  $t'_1 = \beta \eta$   $t'_2$  then follows.

Lemma 3 (Symmetry Properties).

1)  $[(\forall X.R)^{\smile}]_{\gamma} = [\forall X.R^{\smile}]_{\gamma}$ 

2) 
$$[(R_1 \to R_2)^{\circ}]_{\gamma} = [R_1^{\circ} \to R_2^{\circ}]_{\gamma}$$

3) 
$$[(R_1 \cdot R_2)^{\cup}]_{\gamma} = [R_2^{\cup} \cdot R_1^{\cup}]_{\gamma}$$

*Proof.* (1): assume  $t \ [(\forall X.R)^{\cup}]_{\gamma} \ t'$ , and hence  $t' \ \llbracket (\forall \overline{X}.R) \rrbracket_{\gamma} \ t$ . For any  $r \in \mathcal{R}$ ,  $t' \ \llbracket R \rrbracket_{\gamma \lceil X \mapsto r \rceil} \ t$ , hence  $t [R^{\cup}]_{\gamma[X\mapsto r]} t'$ . From this,  $t [\forall X. R^{\cup}]_{\gamma} t'$  as required. Conversely, assume  $t \ [\![\forall X. R^{\cup}]\!]_{\gamma} \ t'$ , and  $r \in \mathcal{R}$ . Then  $t [R^{\cup}]_{\gamma[X\mapsto r]} t'$ , hence  $t' [R]_{\gamma[X\mapsto r]} t$ . From this,  $t' \ \llbracket \forall X.R \rrbracket_{\gamma} \ t$ , hence the required  $t \ \llbracket (\forall X.R) \cup \rrbracket_{\gamma} \ t'$ .

(2): Assume  $t [(R_1 \rightarrow R_2)^{\cup}]_{\gamma} t'$  and  $a [R_1^{\cup}]_{\gamma} a'$ . From these,  $t' [(R_1 \rightarrow R_2)]_{\gamma} t$  and  $a' [R_1]_{\gamma} a$ , which yield t' a'  $[\![R_1]\!]_{\gamma}$  t a. Thus, t a  $[\![R_1^{\cup}]\!]_{\gamma}$  t' a' as required. Conversely, assume  $t [R_1^{\cup} \to R_2^{\cup}]_{\gamma} t'$  and  $a [R_1]_{\gamma} a'$ . From the latter, a'  $\llbracket R_1^{\smile} \rrbracket_{\gamma}$  a, so t a'  $\llbracket R_2^{\smile} \rrbracket_{\gamma}$  t' a. From this, t' a  $\llbracket R_2 \rrbracket_{\gamma}$  t a', as required.

(3): assume  $t [[(R_1 \cdot R_2)^{\cup}]]_{\gamma} t'$ , hence  $t' [[R_1 \cdot R_2]]_{\gamma} t$ . So there exists t'' with  $t' [R_1]_{\gamma} t''$  and  $t'' [R_2]_{\gamma} t$ . From these,  $t'' [R_1^{\cup}]_{\gamma} t'$  and  $t [R_2^{\cup}]_{\gamma} t''$ ; thus,  $t [R_2^{\cup} \cdot R_1^{\cup}]_{\gamma} t'$ . Conversely, assume  $t \ \llbracket R_2^{\cup} \cdot R_1^{\cup} \rrbracket_{\gamma} \ t'$ . So there exists t'' with  $t \ \llbracket R_2^{\cup} \rrbracket_{\gamma} \ t''$ and  $t'' [R_1^{\circ}]_{\gamma} t'$ . From these,  $t'' [R_2]_{\gamma} t$  and  $t' [R_1]_{\gamma} t''$ . So  $t [(R_1 \cdot R_2) \cup ]_{\gamma} t'$ .

Lemma 4 (Deapplication).

1) 
$$t_1 [\![t \cdot R]\!]_{\gamma} t_2 = t \ t_1 [\![R]\!]_{\gamma} t_2$$
  
2)  $t_1 [\![R \cdot t^{\smile}]\!]_{\gamma} t_2 = t_1 [\![R]\!]_{\gamma} t \ t_2$ 

*Proof.* For the first fact: first, assume  $t_1 [t \cdot R]_{\gamma} t_2$ . The semantics gives t' such that (1)  $t_1 \llbracket t \rrbracket_{\gamma} t'$  and (2)  $t' \llbracket R \rrbracket_{\gamma} t_2$ . But (1) is equivalent to t  $t_1 = \beta \eta t'$ . Applying  $\beta \eta$ -Closure  $\{2\}$ ,  $t \ t_1 \ \llbracket R \rrbracket_{\gamma} \ t_2$  as required. Next, assume  $t \ t_1 \ \llbracket R \rrbracket_{\gamma} \ t_2$ . Then there is a t', namely t  $t_1$ , such that  $t_1 [\![t]\!]_{\gamma} t'$  and  $t' [\![R]\!]_{\gamma} t_2$ . Hence  $t_1 \ [\![t \cdot R]\!]_{\gamma} \ t_2$  as required.

For the second: assuming  $t_1 [R \cdot t^{\cup}]_{\gamma} t_2$ , the semantics gives t' such that (1)  $t_1$   $\llbracket R \rrbracket_\gamma$   $\bar{t'}$  and (2) t'  $\llbracket t^{\smile} \rrbracket_\gamma$   $t_2$ . But (2) is equivalent to t  $t_2 =_{\beta\eta} t'$ . Applying  $\beta\eta$ -Closure  $\{2\}$ ,

$$\begin{array}{lll} I & := & \lambda \, x. \, x \\ K & := & \lambda \, x. \, \lambda \, y. \, x \\ t \circ t' & := & \lambda \, x. \, t \, \left( t' \, \, x \right) \end{array}$$

Fig. 3. Some standard definitions and notations for terms, used below

 $t_1 \llbracket R \rrbracket_{\gamma} \ t \ t_2$  as required. Next, assume  $t_1 \llbracket R \rrbracket_{\gamma} \ t \ t_2$ . Then there is a t', namely t  $t_2$ , such that  $t_1 [\![R]\!]_{\gamma} t'$  and  $t' [\![t^{\cup}]\!]_{\gamma} t_2$ . Hence  $t_1 [\![R \cdot t^{\smile}]\!]_{\gamma} t_2$  as required.

We make use of a few definitions for terms in Figure 3.

Lemma 5 (Relational Laws).

1) 
$$[R_1 \cdot (R_2 \cdot R_3)]_{\gamma} = [(R_1 \cdot R_2) \cdot R_3]_{\gamma}$$

2) 
$$[\![(R^{\smile})^{\smile}]\!]_{\gamma} = [\![R]\!]_{\gamma}$$
3) 
$$[\![R \cdot I]\!]_{\gamma} = [\![I \cdot R]\!]_{\gamma} = [\![R]\!]_{\gamma}$$

*Proof.* (1) follows from the semantics of  $\cdot$  as relational composition, (2) from the semantics of  $\cup$  as relational converse, and (3) from Deapplication {4} (applying also  $\beta\eta$ -Closure  $\{2\}$ ).

We may observe that Symmetry Properties {3} part (3) and Relational Laws {5} validate the complement- and union-free axioms of the Calculus of Relations (RelTT omits complement and union) [4].

Lemma 6 (Interpretation Over Substitution).

$$[\![R/X]R']\!]_{\gamma}=[\![R']\!]_{\gamma[X\mapsto [\![R]\!]_{\gamma}]}$$

*Proof.* The proof is by induction on R'. Let  $\gamma'$  denote  $\gamma[X \mapsto$  $[R]_{\gamma}$ .

Case X:

$$[[R/X]X]_{\gamma} = [R]_{\gamma} = [X]_{\gamma'}$$

Case Y:

$$[[R/X]Y]_{\gamma} = \gamma(Y) = [Y]_{\gamma'}$$

Case  $R_1 \rightarrow R_2$ :

$$\begin{aligned}
& [[R/X](R_1 \to R_2)]_{\gamma} = \\
& [[R/X]R_1]_{\gamma} \to [[R/X]R_2]_{\gamma} = \\
& [R_1]_{\gamma'} \to [R_2]_{\gamma'} = \\
& [R_1 \to R_2]_{\gamma'}
\end{aligned}$$

Case  $\forall X. R_1$ :

$$\begin{aligned} & [[R/X] \forall X. R_1]_{\gamma} = \\ & \bigcap_{r \in \mathcal{R}} [[R/X] R_1]_{\gamma[X \mapsto r]} \\ & \bigcap_{r \in \mathcal{R}} [R_1]_{\gamma'[X \mapsto r]} \\ & [\forall X. R_1]_{\gamma'} \end{aligned}$$

Case  $R_1^{\circ}$ :

$$\begin{split} & [ [R/X](R_1^{\smile}) ]_{\gamma} = \\ & [ [R/X]R_1]_{\gamma^{\prime}}^{\smile} = \\ & [R_1]_{\gamma^{\prime}}^{\smile} = \\ & [R_1^{\smile}]_{\gamma^{\prime}} \end{split}$$

Case 
$$R_1 \cdot R_2$$
:

Case  $\hat{t}$ :

$$\llbracket [R/X]\hat{t} \rrbracket_{\gamma} = \llbracket \hat{t} \rrbracket_{\gamma} = \llbracket \hat{t} \rrbracket_{\gamma'}$$

Lemma 7 (Environment Extension).

1) If  $X \notin FV(R)$ , then

$$[\![R]\!]_{\gamma[X\mapsto r]} = [\![R]\!]_{\gamma} = [\![X/Y]R]\!]_{\gamma[X\mapsto \gamma(Y)]}$$

2) If R is closed, then  $[R]_{\gamma} = [R]_{\gamma'}$ .

*Proof.* The first fact is by an obvious induction on R. The second follows by iterating the first one to shrink  $\gamma$  to the empty environment, and then build it back up to  $\gamma'$  (recall that environments map a finite set of type variables).

## IV. BASIC EXAMPLES AND DEFINITIONS

**Lemma 8** (Identity).  $I [X \rightarrow X]_{\gamma} I$ 

*Proof.* Assume (1)  $t [\gamma(X)] t'$  and show  $I t [\gamma(X)] I t'$ . But this follows from (1) by  $\beta\eta$ -Closure {2}.

**Definition 9.** 

$$1) \quad [t]R := (K \ t) \cdot R$$

2) 
$$R[t] := R \cdot (K \ t)^{\cup}$$

We can express within the theory the property of being related to term t by R with the relational types [t]R and R[t]. In particular, this gives us a form of internalized typing: for example, we may use the type  $[I] \forall X. X \rightarrow X[I]$  to express the property that I has the expected polymorphic type. These notations are to be parsed with highest precedence.

**Lemma 10** (Internalized Typing).

1) 
$$t_1 [\![t]R]\!]_{\gamma} t_2 = t [\![R]\!]_{\gamma} t_2$$

2) 
$$t_1 [\![R[t]]\!]_{\gamma} t_2 = t_1 [\![R]\!]_{\gamma} t$$

*Proof.* For (1), use  $\beta\eta$ -Closure {2}:

$$(t_1 \ \llbracket [t]R \rrbracket_{\gamma} \ t_2) = (K \ t \ t_1 \ \llbracket R \rrbracket_{\gamma} \ t_2) = (t \ \llbracket R \rrbracket_{\gamma} \ t_2)$$

For (2), use  $\beta\eta$ -Closure {2} and also Deapplication {4}:

$$(t_1 \ \llbracket R[t] \rrbracket_{\gamma} \ t_2) = (t_1 \ \llbracket R \rrbracket_{\gamma} \ K \ t \ t_2) = (t_1 \ \llbracket R \rrbracket_{\gamma} \ t)$$

The following operations are reminiscent of conjugation in group theory:

**Definition 11.**  $t_1.R.t_2 := t_1 \cdot R \cdot t_2^{\cup}$ 

**Definition 12.** t \* R := t.R.t

Lemma 13 (Conjugation).

1) 
$$t_1 [t.R.t']_{\gamma} t_2 = t t_1 [R]_{\gamma} t' t_2.$$

2) 
$$t_1 \ [\![t * R]\!]_{\gamma} \ t_2 = t \ t_1 \ [\![R]\!]_{\gamma} \ t \ t_2.$$

*Proof.* Apply Deapplication {4}.

We may internalize inclusion of relations as a type, using term promotions:

**Definition 14.** 
$$R \subseteq R' := (K \ I) * (R \rightarrow R')$$

**Lemma 15** (Subset). 
$$t_1 [R \subseteq R']_{\gamma} t_2 \text{ iff } [R]_{\gamma} \subseteq [R']_{\gamma}$$
.

Proof. Making use of Conjugation {13}, deduce

$$\begin{array}{l} t_1 \ \llbracket R \subseteq R' \rrbracket_{\gamma} \ t_2 = \\ K \ I \ t_1 \ \llbracket R \rightarrow R' \rrbracket_{\gamma} \ K \ I \ t_2 = \\ I \ \llbracket R \rightarrow R' \rrbracket_{\gamma} \ I \end{array}$$

The semantics (Figure 2) states that this latter relational typing is true in environment  $\gamma$  iff for all  $(x, x') \in [\![R]\!]_{\gamma}$ ,  $(I \ x, I \ x') \in [\![R']\!]_{\gamma}$ , which by  $\beta\eta$ -Closure  $\{2\}$  is equivalent to  $(x, x') \in [\![R']\!]_{\gamma}$ .

Term promotions also enable us to derive implicit products [5]. In traditional type theories, implicit products are used to express quantifications without corresponding  $\lambda$ -abstractions in the subject. One may think of them as describing specificational (or "ghost") inputs to functions. In ReITT, we express this by stating that the subject has a function type but erases its input; i.e., it is of the form Kt for some t.

**Definition 16.** 
$$R \Rightarrow R' := K * (R \rightarrow R')$$

Note in the following theorem the essential feature of implicit products: we conclude by relating (with R') just  $t_1$  and  $t_2$ , not their applications to x and x' respectively.

**Lemma 17** (Implicit Product).  $t_1 \ \llbracket R \Rightarrow R' \rrbracket_{\gamma} \ t_2 \ \textit{iff for all} \ (x,x') \in \llbracket R \rrbracket, \ t_1 \ \llbracket R' \rrbracket_{\gamma} \ t_2.$ 

Proof.

$$\begin{array}{l} t_1 \ \llbracket R \Rightarrow R' \rrbracket_{\gamma} \ t_2 = \\ K \ t_1 \ \llbracket R \rightarrow R' \rrbracket_{\gamma} \ K \ t_2 \end{array}$$

And the latter holds iff for all  $(x, x') \in [\![R]\!]$ ,  $K \ t_1 \ x \ [\![R']\!]_{\gamma} \ K \ t_2 \ x'$ . By  $\beta \eta$ -Closure  $\{2\}$ , this is equivalent to  $t_1 \ [\![R']\!]_{\gamma} \ t_2$ .

Finally, using internalized inclusion, we may neatly express equality of relations as a type:

**Definition 18.** 
$$R \doteq R' := (R \subseteq R') \cdot (R' \subseteq R)$$

**Lemma 19** (Relational Equality).  $t_1 [\![R \doteq R']\!]_{\gamma} t_2 \text{ iff } [\![R]\!]_{\gamma} = [\![R']\!]_{\gamma}.$ 

*Proof.* First, suppose  $t_1 [\![R \doteq R']\!]_{\gamma} t_2$ . Then by semantics of composition, there exists some t such that

- $t_1 \llbracket R \subseteq R' \rrbracket_{\gamma} t$ , and
- $t \llbracket R \subseteq R' \rrbracket_{\gamma} t_2$ .

Applying Subset {15}, these facts are equivalent to

- $[R]_{\gamma} \subseteq [R']_{\gamma}$ , and
- $[R']_{\gamma} \subseteq [R]_{\gamma}$ .

This proves the two relations are equal.

Next, suppose  $[\![R]\!]_{\gamma} = [\![R']\!]_{\gamma}$ . Then similarly, applying Subset  $\{15\}$ , we may arbitrarily choose I for t to satisfy

- $t_1 [R \subseteq R']_{\gamma} t$ , and
- $t \llbracket R \subseteq R' \rrbracket_{\gamma} t_2$ .

which suffices, again by the semantics of composition.  $\Box$ 

**Lemma 20** (Substitutivity Of Relational Equality). If  $t_1 [\![R \doteq R']\!]_{\gamma} t_2$ , then  $[\![R/X]R'']\!]_{\gamma} = [\![R'/X]R'']\!]_{\gamma}$ 

*Proof.* The proof is by induction on R'', making use of Environment Extension  $\{7\}$  as we induct on the bodies of universal types (in extended environments). We omit the details, as all cases are obvious thanks to the compositionality of the semantics (Figure 2).

#### V. EXTENSIONALITY PRINCIPLES

We prove a few examples of standard type-specific extensionality principles.

**Lemma 21** ( $\eta$ -Unit). If  $t \ [\![\forall X. X \to X]\!]_{\gamma} t'$ , then  $t \ [\![\forall X. X \to X]\!]_{\gamma} I$ .

*Proof.* Assume (1)  $t \llbracket \forall X. X \to X \rrbracket_{\gamma} t'$ . Next, assume  $r \in \mathcal{R}$  with  $y \llbracket r \rrbracket y'$ . Instantiate (1) with  $\llbracket X \llbracket y' \rrbracket \rrbracket_{X \mapsto r}$  (note this is a definable relation) to get

$$t [\![X \to X]\!]_{\gamma[X \mapsto [\![X[y']]\!]_{X \mapsto r}]} t'$$

Simplifying using Interpretation Over Substitution  $\{6\}$  and also Environment Extension  $\{7\}$ , this gives us

$$t [X[y'] \to X[y']]_{\gamma[X \mapsto r]} t'$$

We may apply this to y[X[y']]y' which we have from (1) by Internalized Typing  $\{10\}$ . This application yields

$$ty [X[y']]_{\gamma[X\mapsto r]} t'y'$$

Again applying Internalized Typing  $\{10\}$ , this gives us ty[r]y', as required.

Definition 22.

$$\begin{array}{lll} R\times R' & := & \forall\, X.\,(R\to R'\to X)\to X\\ pair & := & \lambda\,x.\,\lambda\,y.\,\lambda\,c.\,c\,x\,y\\ (t,t') & := & pair\,t\,t'\\ t.1 & := & t\,\lambda\,x.\,\lambda\,y.\,x\\ t.2 & := & t\,\lambda\,x.\,\lambda\,y.\,y \end{array}$$

**Lemma 23** (Surjective Pairing). If  $t [R \times R']_{\gamma} t'$ , then

$$(t.1, t.2) [R \times R']_{\gamma} t'$$

*Proof.* Assume (1)  $t[R \times R']t'$ . Then assume  $r \in \mathcal{R}$  and (2)  $c[R \to R' \to X]_{\gamma[X \mapsto r]}c'$ , and show

$$pair(t.1)(t.2)c[r]t'c'$$
(A)

Instantiate (1) with  $[\![\lambda x. x c \cdot X]\!]_{[X \mapsto r]}$  (note this is a definable relation). Then (A) follows from

$$pair [\![R \to R' \to X]\!]_{\gamma[X \mapsto [\![\lambda \, x. \, x \, c \cdot X]\!]_{\lceil X \mapsto r \rceil}]} c' \qquad (B)$$

Let us apply Environment Extension  $\{7\}$  implicitly to simplify environments. To prove (B), assume (3)  $r_1 \llbracket R \rrbracket_\gamma r_1'$  and (4)  $r_2 \llbracket R' \rrbracket_\gamma r_2'$ , and show

$$pair \ r_1 \ r_2 [\![ \lambda \ x. \ x \ c \cdot X ]\!]_{[X \mapsto r]} \ c' \ r_1' \ r_2'$$

By Deapplication {4}, this is equivalent to

pair 
$$r_1 r_2 c[r] c' r'_1 r'_2$$

By  $\beta\eta$ -Closure {2}, this is equivalent to

$$c r_1 r_2[r] c' r'_1 r'_2$$

This follows from (2), (3), and (4) by the semantics.

## VI. SYMMETRIC TYPES

**Definition 24.** Call a type symmetric iff it does not use  $R \cdot R'$ , and every occurrence of a promotion of a term t either has  $t = \beta \eta$  I or occurs as t in subexpressions of the form t \* R. Use S as a metavariable for symmetric types.

**Definition 25.**  $\gamma^{\cup}(X) = (\gamma(X))^{\cup}$ ; i.e., the converse of relation  $\gamma(X)$ .

Lemma 26.  $(\gamma^{\cup})^{\cup} = \gamma$ .

**Theorem 27** (Symmetric types).  $[S]_{\gamma} = [S^{\cup}]_{\gamma \cup}$ 

*Proof.* The proof is by induction on S.

$$\underline{\operatorname{Case}\ X \colon} \, [\![X]\!]_{\gamma} = \gamma(X) = (\gamma^{\smile})^{\smile}(X) = [\![X^{\smile}]\!]_{\gamma^{\smile}}.$$

<u>Case  $S \to S'$ :</u> assume  $t \ \llbracket S \to S' \rrbracket_{\gamma} \ t'$ . To show  $t' \ \llbracket S \to S' \rrbracket_{\gamma^{\cup}} \ t$ , assume  $a \ \llbracket S \rrbracket_{\gamma^{\cup}} \ a'$ . By the IH,  $a' \ \llbracket S \rrbracket_{\gamma} \ a$ , so  $t \ a' \ \llbracket S' \rrbracket_{\gamma} \ t'$  a. By the IH again,  $t' \ a \ \llbracket S' \rrbracket_{\gamma^{\cup}} \ t \ a'$ , as required. Conversely, assume  $t \ \llbracket S \to S' \rrbracket_{\gamma^{\cup}} \ t'$ , and assume  $a \ \llbracket S \rrbracket_{\gamma} \ a'$ . By the IH,  $a' \ \llbracket S \rrbracket_{\gamma^{\cup}} \ a$ , so  $t \ a' \ \llbracket S' \rrbracket_{\gamma^{\cup}} \ t'$  a. By the IH again,  $t' \ a \ \llbracket S' \rrbracket_{\gamma} \ t'$  a', as required.

<u>Case  $\hat{t} * S$ :</u> assume  $t \ [\![\hat{t} * S]\!]_{\gamma} \ t'$ . By Conjugation {13}, this is equivalent to  $\hat{t} \ t \ [\![S]\!]_{\gamma} \ \hat{t} \ t'$ . By the IH,  $\hat{t} \ t' \ [\![S]\!]_{\gamma} \ \hat{t} \ t$ , which is then similarly equivalent to the desired typing. The converse follows similarly, applying Symmetry Properties {3} and Relational Laws {5}.

Case  $\hat{t} =_{\beta\eta} I$ :

$$(t \ \lVert \hat{t} \rVert_{\gamma} \ t') = (\hat{t} \ t =_{\beta\eta} t') = (t =_{\beta\eta} t') = (t =_{\beta\eta} \hat{t} \ t') = (t' \ \lVert \hat{t} \rVert_{\gamma} \ t)$$

## VII. TRANSITIVE TYPES

**Definition 28.** Use metavariable p to range over the set  $\{-,+\}$  of polarities.  $\bar{p}$  denotes the other polarity from p.

The following notion extends a similar one due to Krivine [6, Section 8.5], put also to good use in other works like [7].

**Definition 29**  $(\forall^p)$ . Define a property  $\forall^p$  of types inductively by the following clauses. Type variables X are  $\forall^p$ . If R is  $\forall^{\bar{p}}$  and R' is  $\forall^p$ , then  $R \to R'$  is  $\forall^p$ . If R is  $\forall^+$  then so is  $\forall X$ . R. If R is  $\forall^p$ , then so is  $R^{\circ}$ . If  $t = \beta_{\eta} I$ , then the promotion of

t to a type is  $\forall^p$ . Note that  $\forall^p$  types are symmetric types (Definition 24). We let P range over  $\forall^+$  types, and N over  $\forall^-$  types.

Recall the following fact from classical lambda calculus (e.g., Chapter 7 of [8]).

**Lemma 30** (Zeta). If  $t = \beta_{\eta} t' x$  and  $x \notin FV(t t')$ , then  $t = \beta_{\eta} t'$ .

*Proof.* From the assumption, deduce  $\lambda x.t \ x =_{\beta\eta} \lambda x.t' \ x$ . The sides of this equation are  $\eta$ -equal to t and t', respectively.

**Definition 31.** Let e denote the environment where e(X) is the relation  $=_{\beta\eta}$ , for all type variables X.

As discussed further in Section XIII, RelTT by design does not satisfy Identity Extension (a property proposed originally by Reynolds [9]). The following is a partial refinement:

**Theorem 32** (Identity Inclusion).

- 1)  $\llbracket P \rrbracket_e \subseteq =_{\beta\eta}$ .
- $2) =_{\beta\eta} \subseteq [\![N]\!]_e.$

*Proof.* Proceed by induction on the assumption of R in  $\forall^p$ . Case  $X \in \forall^p$ :  $[\![X]\!]_e = e(X)$ , which is  $=_{\beta\eta}$ .

<u>Case  $R \to R' \in \forall^+$ :</u> assume (1)  $t \ [R \to R']_e \ t'$ . By Zeta {30}, it suffices to prove  $t \ x =_{\beta\eta} t' \ x$ . Since  $R \in \forall^-$ , the IH applies to  $x =_{\beta\eta} x$  to yield  $x \ [R]_e \ x$ . Combining this with (1) gives  $t \ x \ [R']_e \ t' \ x$ . Then by the IH,  $t \ x =_{\beta\eta} t' \ x$ , as required.

<u>Case  $R \to R' \in \forall \bar{}$ </u>: assume (1)  $t =_{\beta\eta} t'$  and (2)  $a [\![R]\!]_e a'$ , and show  $t \ a [\![R']\!]_e \ t' \ a'$ . Since  $R \in \forall^+$ , the IH applies to (2) yielding  $a =_{\beta\eta} a'$ . Combining this with (1) gives  $t \ a =_{\beta\eta} t' \ a'$ , from which the IH yields  $t \ a [\![R']\!]_e \ t' \ a'$ .

<u>Case  $\forall X. R \in \forall^+$ :</u> assume (1)  $t \ [\![ \forall X. R ]\!]_e t'$ , and show  $t =_{\beta\eta} t'$ . From (1), we have  $t \ [\![ R ]\!]_{e[X\mapsto =_{\beta\eta}]} t'$ . By the IH, this yields  $t =_{\beta\eta} t'$ , as required.

<u>Case  $R^{\cup} \in \forall^{+}$ :</u> assume  $t \ [\![R^{\cup}]\!]_e \ t'$ , which implies  $t' \ [\![R]\!]_e \ t$ . By the IH,  $t' =_{\beta\eta} t$ , hence  $t =_{\beta\eta} t'$  as required.

<u>Case  $R^{\circ} \in \forall \bar{}$ </u>: assume  $t = \beta \eta t'$ , hence  $t' = \beta \eta t$ . By the IH,  $t' [\![R]\!]_e t$ , which equals the required  $t [\![R^{\circ}]\!]_e t'$ .

Case  $\hat{t} \in \forall^p$ :  $[\![\hat{t}]\!]_e$  is then just  $=_{\beta\eta}$ .

Using the terminology of [10], Identity Inclusion  $\{32\}$  identifies  $\forall^+$  types as *extensive* (they are included in the equality relation), and  $\forall^-$  types as *parametric* (the equality relation is included in them).

**Lemma 33** (Transitivity For  $\forall^+$ -Types).  $I \llbracket P \cdot P \rightarrow P \rrbracket_e I$ .

*Proof.* Assume (1) x  $[\![R]\!]_e$  y and (2) y  $[\![R]\!]_e$  z, and show x  $[\![R]\!]_e$  z. By Identity Inclusion {32}, (1) implies  $x = \beta_\eta y$ . From this and (2),  $\beta\eta$ -Closure {2} yields the desired conclusion.

**Corollary 34** ( $\forall^+$  Per). If R is  $\forall^+$  and closed, then  $[\![R]\!]_{\gamma}$  is a partial equivalence relation (i.e., symmetric and transitive; abbreviated per).

*Proof.* Since R is closed,  $[\![R]\!]_{\gamma} = [\![R]\!]_e$  by Environment Extension  $\{7\}$ . Transitivity for  $\forall^+$ -Types  $\{33\}$  then implies transitivity. Symmetry follows from Symmetric Types  $\{27\}$ , since  $\forall^+$  types are symmetric types (Definition 24).

**Definition 35** (simple transitive types). *Simple transitive types* T *are defined by the following grammar:* 

$$T \; ::= \; P \mid P \rightarrow T \mid N \rightarrow T \mid t * T$$

**Lemma 36** (transitivity for simple transitive types).  $I \ [\![ T \cdot T \to T ]\!]_e I$ 

*Proof.* The proof is by induction on T, in each case assuming (1)  $x \llbracket T \rrbracket_{\gamma} y$  and (2)  $y \llbracket T \rrbracket_{\gamma} z$ .

Case P: Transitivity for  $\forall^+$ -Types {33}.

<u>Case  $P \to T$ :</u> assume (3)  $a \ \llbracket P \rrbracket_e \ a'$ . By Symmetric Types  $\{27\}$ ,  $a' \ \llbracket P \rrbracket_e \ a$  (as  $e^{\cup} = e$ ). By Transitivity for  $\forall^+$ -Types  $\{33\}$ , this can be combined with (3) to obtain  $a \ \llbracket P \rrbracket_e \ a$ . Using this with (1),  $x \ a \ \llbracket T \rrbracket_e \ y \ a$ . Using (3) with (2),  $y \ a \ \llbracket T \rrbracket_e \ z \ a'$ . By the induction hypothesis,  $x \ a \ \llbracket T \rrbracket_e \ z \ a'$  as required.

<u>Case  $N \to T$ :</u> assume (3)  $a [N]_e a'$ . By Identity Inclusion {32}, since N is  $\forall^-$ ,  $a [N]_e a$  (since  $a =_{\beta\eta} a$ ). Using this with (1),  $x \ a [T]_e \ y \ a$ . Then as in the previous case, we obtain  $y \ a [T]_e \ z \ a'$  using (3) with (2), and the required  $x \ a [T]_e \ z \ a'$  by the induction hypothesis.

<u>Case  $\hat{t} * T$ :</u> by Conjugation {13}, it suffices to show  $\hat{t} x [T]_{\gamma} \hat{t} z$ . This follows by the IH from assumptions (1) and (2), since these are equivalent to  $\hat{t} x [T]_{\gamma} \hat{t} y$  and  $\hat{t} y [T]_{\gamma} \hat{t} z$  by Conjugation {13}.

#### VIII. A RELATIONAL PROOF SYSTEM

Figure 6 presents a proof system, RelPf, for judgments of the form  $\Gamma \vdash t$  [R] t'. (Here, the square brackets are part of the syntax for the judgment; in our meta-language, we are using them for application of a mathematical relation.) RelPf Soundness  $\{40\}$  below shows that this system is sound with respect to the semantics of Figure 2 (extended for contexts). In Section XII, we will develop a type theory based on RelPf, but introduce the proof system here because the fragment for System F types will be useful in Section X on inductive types. A few details:

• typing contexts  $\Gamma$  are described by the grammar

$$\Gamma ::= \cdot | \Gamma, t[R]t'$$

We may elide  $\cdot$  in examples.

- There is an introduction and elimination rule for each connective.
- The introduction rule for term promotions is the axiom  $\Gamma \vdash t [t'] t' t$ . This states that t is related to t' t by the relation (i.e., term promotion) t'.
- The rule allowing to change the sides of the relational typing to  $\beta\eta$ -equal terms is called *conversion*. While  $\beta\eta$ -equality is undecidable in general, we may view the side conditions on conversion as license for an implementation to check reductions to as deep a finite depth as desired. So we view reduction as being implicitly bounded in

$$\frac{\overline{\Gamma \vdash tt [Bool]ff}}{\overline{\Gamma \vdash tt [R \to R \to R]ff}} \frac{\overline{\Gamma \vdash x [R] x'}}{\overline{\Gamma \vdash tt x [R \to R]ff x' y'}} \frac{\overline{\Gamma \vdash y [R] y'}}{\overline{\Gamma \vdash tt x y [R]ff x' y'}} \frac{\underline{\Gamma \vdash tt x y [R]ff x' y'}}{\overline{\Gamma \vdash x [R] y'}}$$

Fig. 4. Derivation of True Different From False {38}. The final inference is by the conversion rule, noting  $tt \, x \, y =_{\beta\eta} x$  and  $ft \, x' \, y' =_{\beta\eta} y'$ 

$$\begin{array}{ll} \underline{x:T\in\Gamma}\\ \overline{\Gamma\vdash x:T} & \underline{\Gamma,x:T\vdash t:T'}\\ \overline{\Gamma\vdash \lambda x.t:T\to T'} \\ \\ \underline{\Gamma\vdash t:T'\to T\quad \Gamma\vdash t':T'}\\ \overline{\Gamma\vdash t:\forall X.T} & \underline{\Gamma\vdash t:T\quad X\notin FV(\Gamma)}\\ \\ \underline{\Gamma\vdash t:\forall X.T}\\ \overline{\Gamma\vdash t:[T'/X]T} \end{array}$$

Fig. 5. Typing rules for Curry-style System F

applications of this rule, making type-checking decidable. We do not formalize bounded reduction.

Here is an example in RelPf, deriving a form of inconsistency from an assumption that different constructors of an inductive type are equal. It states that if tt and ff are equal as booleans, then any relation R is trivial in the sense that  $R = dom(R) \times ran(R)$ .

## **Definition 37.**

$$\begin{array}{lll} \textit{Bool} & := & \forall \, X. \, X \to X \to X \\ \textit{tt} & := & \lambda \, x. \, \lambda \, y. \, x \\ \textit{ff} & := & \lambda \, x. \, \lambda \, y. \, y \end{array}$$

**Lemma 38** (True Different From False). For any type R, let  $\Gamma$  be a context with the following assumptions:

- 1) *tt* [*Bool*] *ff*
- 2) x[R]x'
- 3) y[R]y'

Then  $\Gamma \vdash x [R] y'$ .

*Proof.* A derivation is in Figure 4. 
$$\square$$

Turning now to meta-theory: let  $\sigma$  range over term substitutions (finite functions from term variables to terms). Denote capture-avoiding application of a substitution  $\sigma$  to a term t as  $\sigma$  t. Apply substitutions  $\sigma$  to types R by applying them to all terms contained in R. Now we will define an interpretation of contexts  $\Gamma$  as sets of substitutions satisfying the contexts constraints.

**Definition 39.**  $[\![\Gamma]\!]_{\gamma}$  is defined by recursion on  $\Gamma$ :

$$\begin{array}{lcl} \sigma \in \llbracket \Gamma, t \left[ R \right] t' \rrbracket_{\gamma} & = & \sigma \in \llbracket \Gamma \rrbracket_{\gamma} \ \land \ \sigma t \, \llbracket \sigma R \rrbracket_{\gamma} \, \sigma t' \\ \sigma \in \llbracket \cdot \rrbracket_{\gamma} & = & \mathit{True} \end{array}$$

$$\frac{t\left[R\right]t'\in\Gamma}{\Gamma\vdash t\left[R\right]t'} \qquad \frac{\Gamma,x\left[R\right]x'\vdash t\left[R'\right]t'\quad (*)}{\Gamma\vdash \lambda x.t\left[R\to R'\right]\lambda x'.t'} \qquad \frac{\Gamma\vdash t\left[R\to R'\right]t'\quad \Gamma\vdash t_1\left[R\right]t_2}{\Gamma\vdash tt_1\left[R'\right]t't_2}$$

$$\frac{\Gamma\vdash t\left[\forall X.R'\right]t'}{\Gamma\vdash t\left[\left[R/X\right]R'\right]t'} \qquad \frac{\Gamma\vdash t\left[R\right]t'\quad X\notin FV(\Gamma)}{\Gamma\vdash t\left[\forall X.R\right]t'} \qquad \frac{\Gamma\vdash t\left[R\right]t_2\quad t_1=_{\beta\eta}t'_1\quad t_2=_{\beta\eta}t'_2}{\Gamma\vdash t'_1\left[R\right]t'_2}$$

$$\frac{\Gamma\vdash t'\left[R\right]t}{\Gamma\vdash t\left[R^{\circ}\right]t'} \qquad \frac{\Gamma\vdash t\left[R^{\circ}\right]t'}{\Gamma\vdash t'\left[R\right]t} \qquad \frac{\Gamma\vdash t\left[t'\right]t't}{\Gamma\vdash t\left[t'\right]t't} \qquad \frac{\Gamma\vdash t\left[t''\right]t'\quad \Gamma\vdash \left[t'/x\right]t_1\left[R\right]\left[t'/x\right]t_2}{\Gamma\vdash \left[t''\right]t'_1\left[R\right]\left[t''\right]t_2}$$

$$\frac{\Gamma\vdash t\left[R\cdot R'\right]t'\quad \Gamma,t\left[R\right]x,x\left[R'\right]t'\vdash t_1\left[R''\right]t_2\quad (**)}{\Gamma\vdash t\left[R^{\circ}\right]t'} \qquad \frac{\Gamma\vdash t\left[R\right]t''\quad \Gamma\vdash t''\left[R'\right]t'}{\Gamma\vdash t\left[R\cdot R'\right]t'}$$

Side condition (\*) is  $x \notin FV(\Gamma, R, R')$ . Side condition (\*\*) is  $x \notin FV(\Gamma, t_1, t_2, t, t', R, R', R'')$ .

Fig. 6. Proof system for relational typing.

**Theorem 40** (RelPf Soundness). Suppose  $\gamma$  is defined on all free type variables of  $\Gamma$  and R. If  $\Gamma \vdash t [R] t'$ , and  $\sigma \in \llbracket \Gamma \rrbracket_{\gamma}$ , then  $\sigma$   $t \llbracket \sigma R \rrbracket_{\gamma} \sigma$  t'.

*Proof.* The proof is by induction on the RelPf derivation. In each case we assume arbitrary  $\sigma \in [\![\Gamma]\!]_\gamma$ . Case:

$$\frac{t[R]t' \in \Gamma}{\Gamma \vdash t[R]t'}$$

From  $t[R]t' \in \Gamma$  we obtain the desired  $\sigma t \llbracket \sigma R \rrbracket_{\gamma} \sigma t'$  from the semantics of contexts.

Case:

$$\frac{\Gamma, x [R] x' \vdash t [R'] t' \quad (*)}{\Gamma \vdash \lambda x. t [R \to R'] \lambda x'. t'}$$

Assume arbitrary  $t_1$  and  $t_2$  with (1)  $t_1 \llbracket \sigma R \rrbracket_{\gamma} t_2$ . Let  $\sigma'$  denote  $\sigma[x \mapsto t_1, x' \mapsto t_2]$ . By the IH,

$$\sigma' t \llbracket \sigma' R' \rrbracket_{\gamma} \sigma' t'$$

By side condition (\*),  $\sigma' R' = \sigma R$ . Applying then  $\beta \eta$ -Closure  $\{2\}$ , we have

$$(\sigma \lambda x. t) t_1 \llbracket \sigma R' \rrbracket_{\gamma} (\sigma \lambda x'. t') t_2$$

By the semantics of arrow types, the fact that this holds for all  $t_1$  and  $t_2$  satisfying (1) implies the desired  $\sigma \lambda x.t \llbracket \sigma (R \to R') \rrbracket_{\gamma} \sigma \lambda x'.t'$ .

Case:

$$\frac{\Gamma \vdash t \left[R \to R'\right] t' \quad \Gamma \vdash t_1 \left[R\right] t_2}{\Gamma \vdash t t_1 \left[R'\right] t' t_2}$$

By the IH,  $\sigma t \llbracket R \to R' \rrbracket_{\gamma} \sigma t'$  and  $\sigma t_1 \llbracket R \rrbracket_{\gamma} \sigma t_2$ . The semantics of arrow types then gives the desired  $\sigma(t t_1) \llbracket R' \rrbracket_{\gamma} \sigma(t' t_2)$ .

Case:

$$\frac{\Gamma \vdash t \left[ \forall X. R' \right] t'}{\Gamma \vdash t \left[ \left[ R/X \right] R' \right] t'}$$

By the IH, we have (1)  $\sigma t \llbracket \sigma \forall X. R' \rrbracket_{\gamma} \sigma t'$ . By the condition on  $\gamma$ ,  $\llbracket R \rrbracket_{\gamma}$  is defined, and we use it to instantiate (1). This gives

$$\sigma t \llbracket \sigma R' \rrbracket_{\gamma \llbracket X \mapsto \llbracket R \rrbracket_{\alpha} \rrbracket} \sigma t'$$

By Interpretation Over Substitution {6}, this implies the desired

$$\sigma t [\![ \sigma [R/X] R' ]\!]_{\gamma} \sigma t'$$

Case:

$$\frac{\Gamma \vdash t \left[R\right] t' \quad X \notin \mathit{FV}(\Gamma)}{\Gamma \vdash t \left[\forall \, X.\, R\right] t'}$$

Assume arbitrary  $r \in \mathcal{R}$ . Then by the IH,  $\sigma t \llbracket \sigma R \rrbracket_{\gamma[X \mapsto r]} \sigma' t'$ . The desired  $\sigma t \llbracket \sigma \forall X. R \rrbracket_{\gamma} \sigma' t'$  then follows by the semantics of universal quantification.

Case:

$$\frac{\Gamma \vdash t_1 \begin{bmatrix} R \end{bmatrix} t_2 \quad t_1 =_{\beta \eta} t'_1 \quad t_2 =_{\beta \eta} t'_2}{\Gamma \vdash t'_1 \begin{bmatrix} R \end{bmatrix} t'_2}$$

This case follows easily by the IH and  $\beta\eta$ -Closure {2}. Case:

$$\frac{\Gamma \vdash t' [R] t}{\Gamma \vdash t [R^{\cup}] t'}$$

By the IH,  $\sigma t'[\![R]\!]_{\gamma} \sigma t$ . By the semantics of converse, this implies the required  $\sigma t[\![R^{\cup}]\!]_{\gamma} \sigma t'$ .

Case:

$$\frac{\Gamma \vdash t [R^{\cup}] t'}{\Gamma \vdash t' [R] t}$$

By the IH,  $\sigma t \llbracket R^{\smile} \rrbracket_{\gamma} \sigma t'$ . By the semantics of converse, this implies the required  $\sigma t' \llbracket R \rrbracket_{\gamma} \sigma t$ .

Case:

$$\overline{\Gamma \vdash t \lceil t' \rceil t' \ t}$$

The desired conclusion is equivalent to  $\sigma(t't) =_{\beta\eta} \sigma(t't)$ , which holds.

Case:

$$\frac{\Gamma \vdash t [t''] t' \quad \Gamma \vdash [t'/x]t_1 [R] [t'/x]t_2}{\Gamma \vdash [t'' t/x]t_1 [R] [t'' t/x]t_2}$$

By the IH, we have

- $\sigma(t''t) =_{\beta\eta} \sigma t'$
- $\sigma[t'/x]t_1 \llbracket \sigma R \rrbracket_{\gamma} \sigma[t'/x]t_2$

Using basic properties of  $\beta\eta$ -equivalence and substitution, these facts imply the desired

$$\sigma [t'' t/x]t_1 [\![ \sigma R ]\!]_{\gamma} \sigma [t'' t/x]t_2$$

Case:

$$\frac{\Gamma \vdash t \begin{bmatrix} R \cdot R' \end{bmatrix} t' \quad \Gamma, t \begin{bmatrix} R \end{bmatrix} x, x \begin{bmatrix} R' \end{bmatrix} t' \vdash t_1 \begin{bmatrix} R'' \end{bmatrix} t_2 \quad (**)}{\Gamma \vdash t_1 \begin{bmatrix} R'' \end{bmatrix} t_2}$$

By the IH and semantics for composition we have that there exists t'' such that

- (1)  $\sigma t \llbracket \sigma R \rrbracket_{\gamma} t''$
- (2)  $t'' \llbracket \sigma R' \rrbracket_{\gamma} \sigma t'$

Let  $\sigma'$  denote  $\sigma[x \mapsto t'']$ . Using (1) and (2), we may prove that  $\sigma'$  is in the interpretation of the context in the right premise of the inference. Side condition (\*\*) is used to deduce that  $\sigma'$  satisfies the two constraints added to  $\Gamma$  in that context, from (1) and (2) (where only  $\sigma$  appears). Then by the IH and (\*\*), we have the required

$$\sigma t_1 \llbracket \sigma R'' \rrbracket_{\gamma} \sigma t_2$$

Case:

$$\frac{\Gamma \vdash t \left[R\right] t'' \quad \Gamma \vdash t'' \left[R'\right] t'}{\Gamma \vdash t \left[R \cdot R'\right] t'}$$

By the IH, we have

- $\sigma t \llbracket \sigma R \rrbracket_{\gamma} \sigma t''$
- $\sigma t'' \llbracket \sigma R' \rrbracket_{\gamma} \sigma t'$

These imply the desired  $\sigma t \left[\!\!\left[\sigma\left(R\cdot R'\right)\right]\!\!\right]_{\gamma} \sigma t'$  by the semantics of composition.

#### IX. EMBEDDING SYSTEM F

Similar to the Abstraction Theorem of Reynolds [9], we may prove that each term typable in System F is related to itself by the relational interpretation of its type. Figure 5 recalls the typing rules of Curry-style System F (also known as  $\lambda 2$ -Curry [11]). We consider the set of types of System F a subset of the set of relational types (Figure 1). We first show that typing derivations in System F can be translated to RelTT in the obvious way. Then we may appeal to RelTT Soundness  $\{40\}$ .

**Definition 41.** Partition the set of variables by an injection  $\dot{-}$ . Assume t does not contain any variables of the form  $\dot{x}$  with  $x \in FV(t)$ . Then let  $\dot{t}$  be the term where every variable x (free or bound) is renamed to  $\dot{x}$ .

**Definition 42.** Define  $\lceil - \rceil$  recursively on typing contexts  $\Gamma$  of System F by:

$$\begin{array}{cccc} \ulcorner \cdot \urcorner & = & \cdot \\ \ulcorner \Gamma, x : T \urcorner & = & \ulcorner \Gamma \urcorner, x \left[ T \right] \dot{x} \end{array}$$

**Theorem 43** (Soundness Of System F). If  $\Gamma \vdash t : T$  (in System F), then  $\Gamma \vdash t [T] \dot{t}$  (in RelPf), assuming  $\dot{t}$  is defined.

*Proof.* The proof is by induction on the typing derivation in System F.

Case:

$$\frac{x:T\in\Gamma}{\Gamma\vdash x:T}$$

From  $x: T \in \Gamma$  we derive  $x[T]\dot{x} \in \Gamma$ , and conclude using the assumption rule of RelPf.

Case:

$$\frac{\Gamma, x: T \vdash t: T'}{\Gamma \vdash \lambda \, x. \, t: T \to T'}$$

By the IH, we have

$$[\Gamma], x[T]\dot{x} \vdash t[T']\dot{t}$$

From this, use arrow introduction (of RelPf) to derive the desired

$$\lceil \Gamma \rceil \vdash \lambda x. t [T \to T'] \lambda \dot{x}. \dot{t}$$

Case:

$$\frac{\Gamma \vdash t: T' \to T \quad \Gamma \vdash t': T'}{\Gamma \vdash t \ t': T}$$

By the IH we have

$$\lceil \Gamma \rceil \vdash t [T' \to T] \dot{t}$$

$$\lceil \Gamma \rceil \vdash t' [T'] \dot{t'}$$

Use arrow elimination (of RelPf) to deduce the desired

$$\lceil \Gamma \rceil \vdash t \, t' \, \lceil T \rceil \, \dot{t} \, \dot{t}'$$

Case:

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X. T}$$

By the IH, we have  ${}^{\mathsf{r}}\Gamma^{\mathsf{r}} \vdash t\,[T]\,\dot{t}$ . Apply forall introduction (of RelPf) to conclude the desired  ${}^{\mathsf{r}}\Gamma^{\mathsf{r}} \vdash t\,[\forall\,X.\,T]\,\dot{t}$  Case:

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : [T'/X]T}$$

By the IH, we have  $\lceil \Gamma \rceil \vdash t \ [\forall X. T] \dot{t}$ . Apply forall elimination (of RelPf) to conclude the desired  $\lceil \Gamma \rceil \vdash t \ [[T'/X]T] \dot{t}$ .

**Corollary 44** (Soundness Of System F For Closed Terms). *If*  $\cdot \vdash t : T$  (in System F), then  $t \llbracket T \rrbracket_{\gamma} t$ .

*Proof.* Use Soundness of System F  $\{43\}$  (noting that  $t = \dot{t}$  since t closed), and then RelTT Soundness  $\{40\}$ .

Below we will also need this basic syntactic property:

**Proposition 45** (Weakening for System F). *If*  $\Gamma_1, \Gamma_2 \vdash t : T$ , then  $\Gamma_1, x : R, \Gamma_2 \vdash t : T$  where x is not declared in  $\Gamma_1, \Gamma_2$ .

## X. INDUCTIVE TYPES

Following a relational, and functorial, generalization of [10], this section shows how to derive a relational form of induction within RelTT. For this section, except as noted in Section X-A, let R be a type of System F, possibly containing specified variable X free. Under the usual requirement of positivity, we prove equal the following two relational types, where in the second one, we make use of our notation for internalized typing (Definition 9):

#### **Definition 46.**

- $D_{param} := \forall X. (R \rightarrow X) \rightarrow X$
- $D_{ind} := \forall X. ([in_{X.R}] (R \to X) [in_{X.R}]) \Rightarrow X$

 $in_{X,R}$  represents the constructors of the inductive datatype in a standard way, and is defined below (Definition 54).

#### A. Variable Polarity and Monotonicity

The first step to proving equality of  $D_{param}$  and  $D_{ind}$  is to extend the usual notion of a type variable's occurring free only positively or only negatively, to relational types (recall Definition 28 for polarities p). For inductive types, our results hold only for  $\forall^+$  types of System F. For positive-recursive types, however (Section XI), our derivation works for any relational type R. So we begin by defining when a variable occurs only with polarity p ( $X \in P$ ) generally for any relational type R:

**Definition 47.** Define  $X \in {}^{p}R$  inductively by the clauses:

- $X \in {}^+ X$
- $X \in {}^p Y$
- $X \in {}^{p} (R \to R')$  iff  $X \in {}^{\bar{p}} R$  and  $X \in {}^{p} R'$
- $X \in \mathcal{P} \ \forall Y. R \ iff \ X \in \mathcal{P} \ R$
- $X \in {}^{p} (R \cdot R')$  iff  $X \in {}^{p} R$  and  $X \in {}^{p} R'$
- $X \in {}^{p}(R^{\cup})$  iff  $X \in {}^{p}R$
- $X \in \mathcal{P} t$

(The intention is that  $X \in {}^+R$  means X occurs only positively in R, and  $X \in {}^-R$  only negatively.) The following form of monotonicity then holds for any relational type. The statement of the lemma using a polarity meta-variable p consolidates many dual cases in the proof (cf. [12]).

**Lemma 48** (Monotonicity). Suppose  $r_+$  and  $r_-$  are in  $\mathcal{R}$ , with  $r_+ \subseteq r_-$ . If  $X \in {}^p R$ , then  $[\![R]\!]_{\gamma[X \mapsto r_n]} \subseteq [\![R]\!]_{\gamma[X \mapsto r_{\bar{n}}]}$ .

*Proof.* The proof is by induction on  $X \in {}^{p}R$ , assuming (1)  $r_{+} \subseteq r_{-}$  and (2)  $t_{1} \llbracket R \rrbracket_{\gamma[X \mapsto r_{n}]} t_{2}$ .

Case  $X \in {}^+ X$ : by (1).

Case  $X \in {}^p Y$ : by (2), as  $[\![Y]\!]_{\gamma[X \mapsto r_p]} = [\![Y]\!]_{\gamma} = [\![Y]\!]_{\gamma[X \mapsto r_{\bar{p}}]}$ . Case  $X \in {}^p (R_1 \to R_2)$ : assume (3)  $t_a [\![R_1]\!]_{\gamma[X \mapsto r_p]} t_b$ . From this, the IH for  $R_1$  gives  $t_a [\![R_1]\!]_{\gamma[X \mapsto r_p]} t_b$  (instantiating the quantified polarity in the IH with  $\bar{p}$ ). Combine this with (2) to obtain  $t_1 t_a [\![R_2]\!]_{\gamma[X \mapsto r_p]} t_2 t_b$ . From this, the IH for  $R_2$  gives  $t_1 t_a [\![R_2]\!]_{\gamma[X \mapsto r_{\bar{p}}]} t_2 t_b$ , as required.

Case  $X \in {}^{p} \forall Y. R'$ : assume  $r \in \mathcal{R}$ , and instantiate (2) with r. Then apply the IH to obtain the required  $t_1 [\![R']\!]_{\gamma[X \mapsto r_{\bar{p}}, Y \mapsto r]} t_2$ .

Case  $X \in {}^p(R_1 \cdot R_2)$ : (2) implies that there exists t such that  $\overline{t_1} \ \llbracket R_1 \rrbracket_{\gamma[X \mapsto r_P]} \ t$  and  $t \ \llbracket R_2 \rrbracket_{\gamma[X \mapsto r_P]} \ t_2$ . Applying the IH, we obtain  $t_1 \ \llbracket R_1 \rrbracket_{\gamma[X \mapsto r_{\bar{P}}]} \ t$  and  $t \ \llbracket R_2 \rrbracket_{\gamma[X \mapsto r_{\bar{P}}]} \ t_2$ , which suffices.

Case  $X \in {}^p(R_a^{\cup})$ : (2) implies  $t_2 [\![R_a]\!]_{\gamma[X \mapsto r_p]} t_1$ . From this, the IH gives  $t_2 [\![R_a]\!]_{\gamma[X \mapsto r_{\bar{p}}]} t_1$ , which suffices.

П

$$\underline{\text{Case } X \in {}^{p} t: \text{ by (2), as } \llbracket t \rrbracket_{\gamma[X \mapsto r_{p}]}^{r_{j}} = \llbracket t \rrbracket_{\gamma} = \llbracket t \rrbracket_{\gamma[X \mapsto r_{\bar{p}}]}.$$

## B. Fmap, Fold, and In

Following a standard approach to derivation of inductive types (cf. [13]), we will define operations  $fmap_{X,R}$ , fold, and finally  $in_{X,R}$ , and prove relational typings about them. Because we will be considering terms related to themselves, it is convenient to introduce notation t::r:

**Definition 49.** 
$$(t :: r) := t [r] t$$

**Definition 50.** Define a term  $fmap_{X,R}$  by recursion on types R of System F (also, recall Figure 3):

$$\begin{array}{lll} \mathit{fmap}_{X,X} & = & I \\ \mathit{fmap}_{X,Y} & = & K \ I \\ \mathit{fmap}_{X,R \to R'} & = & \lambda \ f. \lambda \ a. \mathit{fmap}_{X,R'} \ f \ \circ a \circ \mathit{fmap}_{X,R} \ f \\ \mathit{fmap}_{X,\forall \, Y. \, R} & = & \lambda \ f. \mathit{fmap}_{X,R} \ f \end{array}$$

Note that as we treat expressions up to  $\alpha$ -equivalence, we do not need a case for  $fmap_{X,\forall\;X.\;R}$ , as this will be handled as  $fmap_{X,\forall\;Y.\;[Y/X]R}$ .

**Lemma 51** (Fmap (System F)). Suppose  $X_+$  and  $X_-$  are type variables. Suppose  $X_p \notin FV(R)$ , for all p. If  $X \in {}^p R$ , then in System F we have

$$\cdot \vdash fmap_{X|R} : (X_+ \to X_-) \to [X_p/X]R \to [X_{\bar{p}}/X]R$$

*Proof.* The proof is by induction on  $X \in {}^{p}R$ , implicitly applying Weakening for System F {45}.

Case  $X \in {}^+ X$ : the goal is

$$\cdot \vdash I : (X_+ \to X_-) \to (X_+ \to X_-)$$

which is derivable.

Case  $X \in {}^{p} Y$ : the goal is

$$\cdot \vdash K \ I : (X_+ \to X_-) \to (Y \to Y)$$

which is derivable.

Case  $X \in {}^{p}(R_1 \to R_2)$ : let  $\Gamma$  be the context

$$f: (X_+ \to X_-), a: [X_p/X](R_1 \to R_2), x: [X_{\bar{p}}/X]R_1$$

Using the typing rules of System F, it suffices to show

$$\Gamma \vdash fmap_{X,R_2} f (a (fmap_{X,R_1} f x)) : [X_{\bar{p}}/X]R_2$$

By the IH, since  $X \in \overline{p}$   $R_1$ , we have

$$\cdot \vdash fmap_{X,R_1} : (X_+ \to X_-) \to ([X_{\bar{p}}/X]R_1 \to [X_p/X]R_1$$

Hence we may derive

$$\cdot \vdash (fmap_{X,R_2} \ f \ x) : [X_p/X]R_1$$

and then  $\cdot \vdash a$  (fmap<sub>X,R2</sub> f(x)):  $[X_p/X]R_2$ . From this, using the IH with  $X \in {}^pR_2$ , we obtain the desired goal.

Case  $X \in {}^{p} \forall Y. R$ : by the IH we have

$$\cdot \vdash \mathit{fmap}_{X,R} : (X_+ \to X_-) \to [X_p/X]R \to [X_{\bar{p}}/X]R$$

From this we obtain

$$f:(X_+ \to X_-) \vdash fmap_{X,R} \ f:[X_p/X]R \to [X_{\bar{p}}/X]R$$

Applying ∀-introduction, we get

$$f:(X_+ \to X_-) \vdash \operatorname{fmap}_{X,R} f: \forall Y. [X_p/X]R \to [X_{\bar{p}}/X]R$$

Applying  $\rightarrow$ -introduction gives the desired conclusion (note we needed the  $\eta$ -expanded definition of  $fmap_{X,\forall Y,R}$ ).

**Definition 52** (Fold). *fold* :=  $\lambda a. \lambda x. x a$ 

**Lemma 53** (Fold). Let X be possibly free in R. Then in System F:

$$\cdot \vdash fold : \forall X. (R \to X) \to D_{param} \to X$$

*Proof.* Let  $\Gamma$  be the context  $a:R\to X, x:D_{param}$ . It suffices to prove  $\Gamma\vdash x\ a:X$ . Instantiating the type variable in  $D_{param}$  with X, we obtain

$$\Gamma \vdash x : (R \to X) \to X$$

So applying x to a indeed has type X in context  $\Gamma$ .

**Definition 54.**  $in_{X,R} := \lambda x. \lambda a. a (fmap_{X,R} (fold \ a) \ x)$ 

**Lemma 55** (In For  $D_{param}$  (System F)). If  $X \in {}^+R$ , then in System F we have

$$\cdot \vdash in_{X,R} : [D_{param}/X]R \to D_{param}$$

*Proof.* Let  $\Gamma$  be the context  $x: [D_{param}/X]R, a: R \to X$ . Applying typing rules of System F, it suffices to show

$$\Gamma \vdash a (fmap_{X|R} (fold \ a) \ x) : X$$

This holds if  $\Gamma \vdash fmap_{X,R}$  (fold a) x:R. Using the assumption that  $X \in {}^+R$ , instantiate Fmap (System F) {51} with  $D_{param}$  for  $X_+$  and X for  $X_-$  to obtain:

$$\cdot \vdash fmap_{X|R} : (D_{param} \to X) \to [D_{param}/X]R \to R$$

The desired typing follows using  $\Gamma \vdash fold \ a : D_{param} \to X$ , which holds by Fold  $\{53\}$ .

**Lemma 56** (In For  $D_{param}$  (RelTT)). If  $X \in {}^{+}R$ , then

$$in_{X,R} :: [[D_{param}/X]R \to D_{param}]_{\gamma}$$

*Proof.* Apply Soundness Of System F For Closed Terms  $\{44\}$  to In For  $D_{param}$  (System F)  $\{55\}$ .

We can prove a similar lemma about  $in_{X,R}$  and  $D_{ind}$ , but since  $D_{ind}$  is not a System F type we cannot use Soundness Of System F  $\{43\}$ . We first need:

Lemma 57 ( $D_{ind}$  Containment).

If 
$$in_{X,R} :: [R \to X]_{\gamma[X \mapsto r]}$$
, then  $[D_{ind}]_{\gamma} \subseteq r$ .

*Proof.* Call the hypothesis of the lemma (1), and suppose also (2)  $t [D_{ind}]_{\gamma} t'$ . We must show t [r] t'. Instantiating X in  $D_{ind}$  with r, by Implicit Product  $\{17\}$  (1) indeed implies t [r] t'.

**Lemma 58** (In for  $D_{ind}$  (RelTT)). If  $X \in {}^+R$ , then

$$in_{X,R} :: \llbracket [D_{ind}/X]R \to D_{ind} \rrbracket_{\gamma}$$

*Proof.* Assume (1)  $t \|[D_{ind}/X]R\|_{\gamma} t'$  and show

$$in_{X,R} t [D_{ind}]_{\gamma} in_{X,R} t'$$

Unfolding the definition of  $D_{ind}$  and applying Internalized Typing  $\{10\}$  and Implicit Product  $\{17\}$ , it suffices to assume  $r \in \mathcal{R}$  with

$$in_{X,R} [R \to X]_{\gamma[X \mapsto r]} in_{X,R}$$
 (2)

and show

$$in_{X,R} t [r] in_{X,R} t'$$

This will follow from (2) if we can show (A) t  $[\![R]\!]_{\gamma[X\mapsto r]}$  t'. To derive this, first instantiate Monotonicity  $\{48\}$  with  $D_{ind}$  for  $X_+$  and r for  $X_-$ . That tells us that if (B)  $[\![D_{ind}]\!]_{\gamma} \subseteq r$ , then also (applying Interpretation Over Substitution  $\{6\}$ )

$$[[D_{ind}/X]R]]_{\gamma} \subseteq [[R]]_{\gamma[X \mapsto r]}$$

This together with (1) proves (A). And (B) follows from (2) by  $D_{ind}$  Containment  $\{57\}$ .

C. Reflection

Next, we prove a property known as *reflection* (cf. [14]). For the specific case of natural numbers, a similar result is Proposition 14 of [10]. Recall the definitions of *fold* and *in* from Section X-B.

**Definition 59.** rebuild<sub>X,R</sub> := fold  $in_{X,R}$ 

**Lemma 60** (Reflection). If  $X \in {}^+R$ , then

$$rebuild_{X,R} [D_{param} \rightarrow D_{param}]_{\gamma} I$$

Before we can prove this, we need:

**Lemma 61** (Fmap Fold). Suppose  $Y \notin FV(R)$ . Let  $r_+ = [\![f \cdot X]\!]_{\gamma}$  and  $r_- = \gamma(X)$ . If  $X \in {}^pR$ , then, letting  $\gamma' = \gamma[Y \mapsto r_p, X \mapsto r_{\bar{p}}]$ , we have

$$fmap_{X,R} f \llbracket [Y/X]R \rightarrow R \rrbracket_{\gamma'} I$$

*Proof.* The proof is by induction on the derivation of  $X \in {}^{p} R$ . We simplify implicitly using  $\beta \eta$ -Closure  $\{2\}$ .

Case  $X \in {}^+ X$ : since  $fmap_{X,X} = I$ , the goal becomes

$$I f [r_+ \rightarrow r_-] I$$

So assume  $t_1$   $[r_+]$   $t_2$ , which is equivalent (by Deapplication  $\{4\}$ ) to (1) f  $t_1$   $[\gamma(X)]$   $t_2$ ; and show

$$I f t_1 [\gamma(X)] t_2$$

but this simplifies to (1).

Case  $X \in \mathcal{P} Z$ : since  $fmap_{X,Z} = K I$ , the goal becomes

$$K \ I \ f \ [Z \to Z]_{\gamma'} \ I$$

Further simplifying, it becomes

$$I [\![Z \to Z]\!]_{\gamma'} I$$

which holds obviously (Identity  $\{8\}$ ). Since  $Y \notin FV(R)$  by assumption, this concludes the variable cases.

Case  $X \in {}^{p}(R_1 \to R_2)$ : the goal becomes

$$\lambda a. (fmap_{X,R_2} f) \circ a \circ (fmap_{X,R_1} f) [[Y/X]R \to R]_{\gamma'} I$$

So assume (1)  $a \llbracket [Y/X](R_1 \to R_2) \rrbracket_{\gamma'} a'$ , and show

$$(fmap_{X,R_2} f) \circ a \circ (fmap_{X,R_1} f) [\![R]\!]_{\gamma'} a'$$

Next, assume (2)  $b [R_1]_{\gamma'} b'$ , and show

$$fmap_{X,R_2} f (a (fmap_{X,R_1} f b) \llbracket R_2 \rrbracket_{\gamma'} a' b'$$

Since  $X \in {}^{p} R_2$ , this follows by the IH from

$$a \ (fmap_{X,R_1} \ f \ b) \ [[Y/X]R_2]_{\gamma'} \ a' \ b'$$

In turn, this follows by (1) from

$$fmap_{X,R_1} f b \llbracket [Y/X]R_1 \rrbracket_{\gamma'} b'$$

Since  $X \in \overline{P}$   $R_1$ , this follows by the IH from (2). Case  $X \in \overline{P}$   $\forall Z. R'$ : the goal becomes

$$fmap_{X|R'} f \llbracket [Y/X]R \rightarrow R \rrbracket_{\gamma'} I$$

So assume (1)  $a \llbracket \forall Z. \lceil Y/X \rceil R' \rrbracket_{\gamma'} a'$ , and show

$$fmap_{X,R'} f a \llbracket \forall Z.R' \rrbracket_{\gamma'} a'$$

For this, assume  $r' \in \mathcal{R}$ , and show

$$fmap_{X,R'} f a [R']_{\gamma'[Z \mapsto r']} a'$$

Since  $X \in {}^{p}R'$ , this follows by the IH from

$$a \llbracket [Y/X]R' \rrbracket_{\gamma'[Z \mapsto r']} a'$$

But this follows by instantiating (1) with r'.

We may now return to:

*Proof of Reflection* {60}. Assuming (1)  $t [D_{param}]_{\gamma} t'$ , it suffices (applying  $\beta\eta$ -Closure {2}) to show

$$t \ in_{X,R} \ [\![D_{param}]\!]_{\gamma} \ t'$$

For this, assume  $r \in \mathcal{R}$  and (2)  $a \ [\![R \to X]\!]_{\gamma[X \mapsto r]} \ a',$  and show

$$t in_{X,R} a [r] t' a'$$
 (A

The key idea (generalizing Wadler's Proposition 14 already mentioned) is to instantiate (1) with the asymmetric relation

[fold 
$$a \cdot X$$
][ $X \mapsto r$ ]

Let us call this  $r_a$ . (A) will follow from that instantiation if we can prove

$$in_{X,R} [\![R \to X]\!]_{\gamma[X \mapsto r_a]} a'$$

So assume (3)  $t_1 [\![R]\!]_{\gamma[X\mapsto r_a]} t_2$ , and show

$$in_{X,R} t_1 [r_a] a' t_2$$

This follows, by Deapplication  $\{4\}$  and  $\beta\eta$ -Closure  $\{2\}$ , from

$$in_{X,R} t_1 a [r] a' t_2$$

Further applying  $\beta\eta$ -Closure {2}, this follows from

$$a (fmap_{X,R} (fold \ a) \ t_1) [r] \ a' \ t_2$$

By (2), this follows from

$$(fmap_{X,R} (fold \ a) \ t_1) \ \llbracket R \rrbracket_{\gamma \lceil X \mapsto r \rceil} \ t_2$$

which follows from (3) by Fmap Fold  $\{61\}$ , applying also Environment Extension  $\{7\}$  to get the contexts and types in the required form; and using  $X \in {}^+R$ .

D. Equating  $D_{param}$  and  $D_{ind}$ 

**Theorem 62** (Inductive Types). Suppose  $FV(R) = \{X\}$  and  $X \in {}^+R$  .

i. 
$$t [D_{ind} \subseteq D_{param}]_{\gamma} t'$$

ii. If R is 
$$\forall^+$$
, then  $t [D_{param} \subseteq D_{ind}]_{\gamma} t'$ 

iii. If R is 
$$\forall^+$$
, then  $t [D_{ind} \doteq D_{param}]_{\gamma} t'$ 

Proof. Recall the definitions:

$$\begin{array}{lll} D_{param} & := & \forall \ X. \ (R \to X) \to X \\ D_{ind} & := & \forall \ X. \ ([in_{X,R}] \ (R \to X) \ [in_{X,R}]) \Rightarrow X \end{array}$$

For this proof, let us apply Subset  $\{15\}$  implicitly. (iii) follows from (i) and (ii). To show (i), assume  $t [\![D_{ind}]\!]_{\gamma} t'$ , and instantiate X in this assumption with  $D_{param}$ . This implies the required  $t [\![D_{param}]\!]_{\gamma} t'$ , as long as (applying Interpretation Over Substitution  $\{6\}$ )

$$in_{X,R} \, \llbracket [D_{param}/X]R \rightarrow D_{param} \rrbracket_{\gamma} \, in_{X,R}$$

But this is exactly In For  $D_{param}$  {56}.

To show (ii), assume (1)  $t [D_{param}]_{\gamma} t'$ , and instantiate X in this assumption with  $D_{ind}$  to get

$$t \left[ \left( \left[ D_{ind} / X \right] R \to D_{ind} \right) \to D_{ind} \right]_{\gamma} t'$$

(Here we again applied Interpretation Over Substitution  $\{6\}$ .) From this and In For  $D_{ind}$   $\{58\}$ , we obtain (2)

$$t in_{X,R} [D_{ind}]_{\gamma} t' in_{X,R}$$

This is close to what we want. Applying Reflection  $\{60\}$  to (1), we obtain

$$t in_{X,R} [\![D_{param}]\!]_{\gamma} t'$$

Since FV(R) = X,  $D_{param}$  is closed, so we may change  $\gamma$  to e here and in (1), by Environment Extension  $\{7\}$ . Then since R is  $\forall^+$ ,  $D_{param}$  is also, and we can apply Identity Inclusion  $\{32\}$  to get:

$$t in_{X,R} =_{\beta \eta} t'$$
  
$$t =_{\beta \eta} t'$$

Using these facts with  $\beta\eta$ -Closure  $\{2\}$ , we may simplify (2) to the desired  $t [D_{ind}]_{\gamma} t'$ .

In light of this result, we denote  $D_{param}$  for particular X and R as  $D_{X,R}$ , and freely change between it and  $D_{ind}$  as long as R is  $\forall^+$ .

## E. Example: Nat

In this section, we consider the basic example of natural numbers. To express this type using the parameter R of  $D_{X,R}$ , we first need some standard types (namely A+B and 1) and associated term definitions: for A+B, constructors *inl* and *inr*, and eliminator  $\langle n, m \rangle$ ; and for 1, constructor *unit*.

## Definition 63.

$$\begin{array}{lll} A+B &:= & \forall \, X. \, (A \to X) \to (B \to X) \to X \\ 1 &:= & \forall \, X. \, X \to X \\ inl &:= & \lambda \, a. \, \lambda \, x. \, \lambda \, y. \, x \, \, a \\ inr &:= & \lambda \, b. \, \lambda \, x. \, \lambda \, y. \, y \, \, b \\ \langle n,m \rangle &:= & \lambda \, c. \, c \, \, n \, \, m \\ unit &:= & I \end{array}$$

Now we define *Nat* and its constructors as expected, with addition as an example operation:

#### Definition 64.

Nat := 
$$D_{X,1+X}$$
  
zero :=  $in_{X,1+X}$  (inl unit)  
succ :=  $in_{X,1+X} \circ inr$   
add :=  $\lambda n. \lambda m. n \langle m. succ \rangle$ 

Thanks to Soundness of System F For Closed Terms  $\{44\}$  and the usual System F typings of the above term definitions (including In For  $D_{param}$  (System F)  $\{55\}$ ), we have the following relational typings:

# Lemma 65 (Nat Operations).

zero :: 
$$[Nat]_{\gamma}$$
  
succ ::  $[Nat \rightarrow Nat]_{\gamma}$   
add ::  $[Nat \rightarrow Nat \rightarrow Nat]_{\gamma}$ 

Following a very similar development as for Inductive Types  $\{62\}$ , we may also equate A+B and 1 with inductive variants:

# Definition 66.

$$\begin{array}{rcl} A +_i B &:= & \forall \, X. \, [\mathit{inl}] \, (A \to X) \, [\mathit{inl}] \Rightarrow \\ & & [\mathit{inr}] \, (B \to X) \, [\mathit{inr}] \Rightarrow X \\ 1_i &:= & \forall \, X. \, [\mathit{unit}] \, X [\mathit{unit}] \Rightarrow X \end{array}$$

Recall the notation  $R \doteq R'$  (Definition 18).

## Proposition 67.

$$t_1 \ [\![A+B \doteq A +_i B]\!]_{\gamma} \ t_2$$
  
 $t_1 \ [\![1 \doteq 1_i]\!]_{\gamma} \ t_2$ 

Finally, let us prove a basic inductive property of *add*, as an example.

# Lemma 68.

$$\lambda n. add \ n \ zero \ [Nat \rightarrow Nat]_{\gamma} I$$

*Proof.* For (i): Assume (1)  $n [Nat]_{\gamma} n'$ , and show

add 
$$n$$
 zero  $[Nat]_{\gamma} n'$  (A

Applying Inductive Types  $\{62\}$  to (1) allows us to reason inductively; we instantiate the type variable X in  $D_{ind}$  with the interpretation of

$$r := \lambda n. add \ n \ zero \cdot Nat$$

We must show this is preserved by  $in_{X,1+X}$ ; that is

$$in_{X,1+X} :: [(1+r) \to r]_{\gamma}$$
 (B)

By Deapplication  $\{4\}$  this suffices for (A). For (B), assume (2)  $v [1 + r]_{\gamma} v'$ , and show

$$in_{X,1+X}v \llbracket r \rrbracket_{\gamma} in_{X,1+X}v'$$

Switch to the inductive view of 1+r in (2), and induct using the interpretation of

$$r' := in_{X,1+X} * r$$

By Deapplication {4}, this is sufficient for (B). We must prove

- inl unit ::  $[r']_{\gamma}$
- $inr :: [r' \rightarrow r']_{\gamma}$

Unfolding definitions of r' and r using Deapplication  $\{4\}$ , we confirm the following using  $\beta\eta$ -Closure  $\{2\}$  and *Nat* Operations  $\{65\}$ 

- $add (in_{X,1+X} (inl \ unit)) \ zero \ [Nat]_{\gamma} (in_{X,1+X} (inl \ unit))$
- $add\ (in_{X,1+X}\ (inr\ x))\ zero\ \llbracket Nat \rrbracket_{\gamma}\ (in_{X,1+X}\ (inr\ x'))$  from  $add\ x\ zero\ \llbracket Nat \rrbracket_{\gamma}\ x'$

#### F. Discussion

Wadler proves a result similar to Inductive Types {62} for the special case of the natural numbers, in Section 5 of [10]. He shows, as a theorem of a second-order logic, that being related by the relational interpretation of Natparam is the same as being equal natural numbers that satisfy a predicate of unary induction. The result here is more general, covering any inductive datatype defined by a positive type scheme R. The equivalence is expressed not in a second-order logic, but in RelTT. So the proof is in terms only of binary relations, including a binary-relational form of induction (instead of using unary induction). Another technical difference is that the proof here relies on Identity Inclusion {32}. This does not show up in Wadler's proof, but only because he considers just the simple example of natural numbers, with the type  $\forall X. (X \rightarrow X) \rightarrow X \rightarrow X$ . One may confirm that a categorical version, as we consider here, would require an analogous property for the proof of his Proposition 14 [10].

Thanks to Inductive Types  $\{62\}$ , we can transport properties between the denotations of  $D_{ind}$  and  $D_{param}$ . For a simple example:

**Lemma 69.** Suppose R is  $\forall^+$ . Then  $\llbracket D_{ind} \rrbracket$  is a per.

*Proof.* If R is  $\forall^+$ , then so is  $D_{param}$ , and hence  $[\![D_{param}]\!]_{\gamma}$  is a per by  $\forall^+$  Per {34}. This implies  $D_{ind}$  is also a per, by Inductive Types {62}.

Proving this lemma directly is not hard, but using Inductive Types {62}, unnecessary. Richer examples are enabled thanks (A) to Substitutivity Of Relational Equality {20}.

# XI. POSITIVE-RECURSIVE TYPES

A very useful type form from standard type theory is the recursive type  $rec\ X$ . R, where X is bound in R, and X occurs only positively in R. The type should be isomorphic to its unfolding  $[rec\ X.\ R/X]R$ , where we desire that the functions witnessing the isomorphism are identity functions. (This form of recursive type can be seen as unifying the standardly distinguished isorecursive and equirecursive.) This section shows how a relational version of this type can be derived in ReITT. The development is a (nontrivial) adaptation of ideas from [15], to our relational setting. It is built on the derivations of subset type and implicit product from Section III, and makes crucial use of Montonicity  $\{48\}$ . Let us assume that type R may contain type variable X free.

**Definition 70.**  $rec X. R := \forall X. (R \subseteq X) \Rightarrow X$ 

**Lemma 71** (Rec Body). If  $[\![R]\!]_{\gamma[X\mapsto r]}\subseteq r$ , then  $[\![rec\ X.\ R]\!]_{\gamma}\subseteq r$ .

*Proof.* Assume (1)  $t_1 \ [\![rec\ X.\ R]\!]_{\gamma} \ t_2$ , and instantiate this with r, to obtain

$$t_1 \ \llbracket (R \subseteq X) \Rightarrow X \rrbracket_{\gamma \lceil X \mapsto r \rceil} \ t_2$$

From this, applying Subset  $\{15\}$  and Implicit Product  $\{17\}$ , we have the desired  $t_1$  r  $t_2$ , as long as  $[\![R]\!]_{\gamma[X\mapsto r]}\subseteq r$ . But the latter is a condition of the lemma.

**Lemma 72** (Rec Fold). If  $X \in {}^+R$ , then  $t_1 \llbracket [rec \ X. \ R/X] R \subseteq rec \ X. \ R \rrbracket_{\gamma} t_2$ .

Proof. By Subset {15}, suffices show it  $\subseteq$  $\llbracket rec X. R \rrbracket_{\gamma}.$ So  $[rec X. R/X]R]_{\gamma}$ assume (1) $t \llbracket [rec X. R/X]R \rrbracket_{\gamma} t'$ , and show  $t \llbracket rec X. R \rrbracket_{\gamma} t'$ . Applying the semantics, Implicit Product {17}, and Subset {15}, it suffices to assume  $r \in \mathcal{R}$  and (2)  $[\![R]\!]_{\gamma[X \mapsto r]} \subseteq r$ , and show  $t_1$  [r]  $t_2$ . Applying Interpretation Over Substitution {6} to (1), we have (3)  $t \ \llbracket R \rrbracket_{\gamma[X \mapsto \llbracket rec \ X. \ R \rrbracket_{\gamma}]} t'$ . By  $Rec \ Body \ \{71\}$ with (2),  $[rec X. R]_{\gamma} \subseteq r$ . By Monotonicity {48}, (3) implies  $t[R]_{\gamma[X\mapsto r_{\gamma}]}$  t'. Combining this with (2), we obtain the desired t [r] t'.

**Lemma 73** (Rec Unfold). If  $X \in {}^+R$ , then  $t_1 \llbracket rec X. R \subseteq [rec X. R/X]R \rrbracket_{\gamma} t_2$ .

*Proof.* By Subset  $\{15\}$ , it suffices to show  $[\![rec\ X.\ R]\!]_{\gamma}\subseteq [\![rec\ X.\ R/X]\!]R]\!]_{\gamma}$ . So assume (1) t  $[\![rec\ X.\ R]\!]_{\gamma}$  t' and show t  $[\![rec\ X.\ R/X]\!]R]\!]_{\gamma}$  t'. Instantiate (1) with  $[\![rec\ X.\ R/X]\!]R]\!]_{\gamma}$  to obtain

$$t \ [\![ (R \subseteq X) \Rightarrow X ]\!]_{\gamma[X \mapsto [\![ rec \ X. \ R/X ]\!] R]\!]_{\gamma}]} t'$$

Applying Interpretation Over Substitution  $\{6\}$ , this is equivalent to

$$t \ \llbracket ( \llbracket (\llbracket \operatorname{rec} X.R/X \rrbracket R/X \rrbracket R / X \rrbracket R) \Rightarrow \\ \llbracket \operatorname{rec} X.R/X \rrbracket R ) \Rightarrow \\ \llbracket \operatorname{rec} X.R/X \rrbracket R \rrbracket_{\gamma} \ t'$$

By Implicit Product  $\{17\}$  and Subset  $\{15\}$ , this implies the desired typing as long as

$$\llbracket \llbracket [rec \, X.\, R/X]R/X]R \rrbracket_{\gamma} \subseteq \llbracket [rec \, X.\, R/X]R \rrbracket_{\gamma}$$

But this follows by Monotonicity  $\{48\}$  (since  $X \in {}^+R$ ) from

$$\llbracket [rec X. R/X]R \rrbracket_{\gamma} \subseteq \llbracket rec X. R \rrbracket_{\gamma}$$

And this follows (by Subset  $\{15\}$ ) directly from Rec Fold  $\{72\}$ .

**Theorem 74** (Recursive Types). If  $X \in {}^+$  R, then  $t_1 [rec X. R \doteq [rec X. R/X]R]_{\gamma} t_2$ 

*Proof.* Using Relational Equality  $\{19\}$ , this follows from *Rec* Fold  $\{72\}$  and *Rec* Unfold  $\{73\}$ 

#### XII. A RELATIONAL TYPE SYSTEM

Having considered now some of the expressive power of RelTT, in its ability to derive types which are often taken as primitive – for example, inductive types are derived here, but primitive for the Calculus of Inductive Constructions [16] – let us turn to the question of an implementable type system for RelTT. We follow the approach suggested by the Curry-Howard correspondence, to to devise a system of proof terms for derivations in RelPf.

Figure 7 gives the syntax for contexts  $\Gamma$  and proof terms p of RelTy, together with an erasure function mapping these back to pure  $\lambda$ -calculus. Proof terms (p,p') and  $\pi \, p - x.u.v.p'$  are used for composition; the  $\pi$ -term is like an existential elimination. Erasure will indeed treat proofs of relational typings by compositions as pairs (Definition 22). The typing rules for RelTy are given in Figure 8.

Given a context  $\Gamma$  and a proof term p, the rules may be read bottom-up as an algorithm to compute the relational typing  $t\left[T\right]t'$  (if any) proved by the proof term. Proofs are organized in natural-deduction style: each type construct has introduction and elimination forms. For example, the introduction form for an identity  $t\left[t'\right]t'$  is  $t\{t,t'\}$ . The elimination is more complicated, unfortunately, as we must describe substitution, using a proven identity  $t\left[t''\right]t'$ , into the terms in some other relational typing. The syntax for the elimination form uses a "guide"  $\{x.t_1,t_2\}$  to give a mechanism for locating instances of t in the left and right terms of the relational typing, to be rewritten to t'. The variable x in terms  $t_1$  and  $t_2$  marks these locations.

By design, RelTy exactly follows the structure of RelPf. Define ,  $\Gamma$  , by

$$\begin{array}{ccc} \cdot & = & \cdot \\ \cdot \Gamma, x : t \left[ R \right] t' & = & \cdot \Gamma, t \left[ R \right] t' \end{array}$$

This maps RelTy contexts to RelPf contexts. A reverse mapping  $\langle \Gamma \rangle$  can be defined as  $\langle \Gamma \rangle_k$  where k is the length of  $\Gamma$ , and the helper function is defined as follows, using a canonical ordering  $x_1, x_2, \ldots$  for assumption variables:

$$\begin{array}{lcl} \langle \cdot \rangle_k & = & \cdot \\ \langle \Gamma, t \left[ R \right] t' \rangle_k & = & \langle \Gamma \rangle_{k-1}, x_k : t \left[ R \right] t' \end{array}$$

Theorem 75 (RelTy-RelPf Isomorphism).

- i. If  $\Gamma \vdash p : t \lceil R \rceil t'$  in RelTy, then  $\lceil \Gamma \rceil \vdash t \lceil R \rceil t'$  in RelPf.
- ii. If  $\Gamma \vdash t[R]t'$  in RelPf, then there exists p such that  $\langle \Gamma \rangle \vdash p : t[R]t'$  in RelTy.

*Proof.* For (i): because RelTy just expands RelPf with proof terms, the proof amounts to erasing all proof terms (including assumptions u in contexts) from RelTy derivations. For (ii): by design, RelTy has proof-term constructs corresponding to all proof rules of RelPf, so the proof amounts to recursively adding in those terms.

If we project even further, we can map from RelTy to System F. Recall the definition of pairs (Definition 22), which are used in projecting composition.

**Definition 76.** Define |R| recursively by:

$$\begin{array}{rcl} |X| & = & X \\ |R \rightarrow R'| & = & |R| \rightarrow |R'| \\ |\forall X. R| & = & \forall X. |R| \\ |R^{\cup}| & = & |R| \\ |R \cdot R'| & = & |R| \times |R'| \\ |t| & = & \forall X. X \rightarrow X \end{array}$$

Extend this to contexts by recursively defining  $|\Gamma|$ :

$$\begin{array}{lcl} |\cdot| & = & \cdot \\ |\Gamma, u:t\left[R\right]t'| & = & |\Gamma|, u:|R| \end{array}$$

**Theorem 77** (RelTy Projection). If  $\Gamma \vdash p : t[R] \ t'$  then  $|\Gamma| \vdash |p| : |R|$  in System F.

*Proof.* The proof is by induction on the assumed RelTy derivation.

Case:

$$\frac{x:t[R]t'\in\Gamma}{\Gamma\vdash x:t[R]t'}$$

From  $x:t\left[R\right]t'\in\Gamma$  we get  $x:|R|\in\Gamma$  and hence the desired conclusion.

Case:

$$\frac{\Gamma, u: x \left[R\right] x' \vdash p: t \left[R'\right] t' \quad (*)}{\Gamma \vdash \lambda \, u: R. \, p: \lambda \, x. \, t \left[R \to R'\right] \lambda \, x'. \, t'}$$

By the IH we have  $|\Gamma|, u: |R| \vdash p: |R'|$ , from which we deduce the desired  $|\Gamma| \vdash \lambda u. |p|: |R \to R'|$ . Case:

 $\frac{\Gamma \vdash p_1 : t_1 [R \to R'] t_1' \quad \Gamma \vdash p_2 : t_2 [R] t_2'}{\Gamma \vdash p_1 p_2 : t_1 t_2 [R'] t_1' t_2'}$ 

By the IH we have  $|\Gamma| \vdash |p_1| : |R \to R'|$  and  $|\Gamma| \vdash |p_2| : |R|$ , from which we deduce the desired  $|\Gamma| \vdash |p_1| p_2| : |R'|$ . Case:

$$\frac{\Gamma \vdash p: t \left[ \forall X. \, R' \right] t'}{\Gamma \vdash p\{R\}: t \left[ \left[ R/X \right] R' \right] t'}$$

By the IH we have  $|\Gamma| \vdash |p| : \forall X. |R'|$ , from which the desired  $|\Gamma| \vdash |p| : |[R/X]R'|$  follows.

Case:

$$\frac{\Gamma \vdash p : t [R] t' \quad X \notin FV(\Gamma)}{\Gamma \vdash \Lambda X. p : t [\forall X. R] t'}$$

By the IH we have  $|\Gamma| \vdash |p| : |R|$ , from which the desired  $|\Gamma| \vdash |p| : \forall X. |R|$  follows.

Case:

$$\frac{\Gamma \vdash p: t_1 \begin{bmatrix} R \end{bmatrix} t_2 \quad t_1 =_{\beta\eta} t_1' \quad t_2 =_{\beta\eta} t_2'}{\Gamma \vdash t_1' \triangleleft p \blacktriangleright t_2': t_1' \begin{bmatrix} R \end{bmatrix} t_2'}$$

The erasure of  $t_1' \triangleleft p \triangleright t_2'$  is |p|, so the desired conclusion is just  $|\Gamma| \vdash |p| : |R|$ , which we have by the IH.

Case:

$$\frac{\Gamma \vdash p : t [R^{\cup}] t'}{\Gamma \vdash \cup_{e} p : t' [R] t}$$

Similar to the previous case.

Case:

$$\frac{\Gamma \vdash p : t \left[R\right] t'}{\Gamma \vdash \cup_{i} \, p : t' \left[R^{\cup}\right] t}$$

Similar to the previous case.

Case:

$$\overline{\Gamma \vdash \iota\{t, t'\} : t \lceil t' \rceil t' t}$$

 $|\iota\{t,t'\}|$  is I and erasure of the term promotion t' is  $\forall X.X \rightarrow X$ . So this inference translates to the familiar typing of the identity function in System F.

Case:

$$\frac{\Gamma \vdash p : t [t''] t' \quad \Gamma \vdash p' : [t'' t/x] t_1 [R] [t'' t/x] t_2}{\Gamma \vdash \rho \{x.t_1, t_2\} \ p - p' : [t'/x] t_1 [R] [t'/x] t_2}$$

By the IH we have  $|\Gamma| \vdash |p'| : |R|$ . Since the  $\rho$ -proof erases to just the erasure of its leftmost subproof, this suffices for the desired conclusion.

Case:

$$\frac{\Gamma \vdash p : t [R \cdot R'] t' \quad (**)}{\Gamma, u : t [R] x, v : x [R'] t' \vdash p' : t_1 [R''] t_2}{\Gamma \vdash \pi p - x.u.v.p' : t_1 [R''] t_2}$$

By the IH we have  $|\Gamma| \vdash |p| : |R| \times |R'|$  and  $|\Gamma|, u : |R|, v : |R'| \vdash |p'| : |R''|$ . By the definition of product types in System F, from these derivations we may easily establish  $|\Gamma| \vdash |p| \lambda u. \lambda v. |p'| : |R''|$ , which suffices since  $|\pi p - x.u.v.p'| = |p| \lambda u. \lambda v. |p'|$ .

Case:

$$\frac{\Gamma \vdash p : t[R]t'' \quad \Gamma \vdash p' : t''[R']t'}{\Gamma \vdash (p, p') : t[R \cdot R']t'}$$

By the IH we have  $|\Gamma| \vdash |p| : |R|$  and  $|\Gamma| \vdash |p'| : |R'|$ . With these we may deduce  $|\Gamma| \vdash (|p|, |p'|) : |R| \times |R'|$  by the definition of product types in System F.

This result is interesting, because it shows that any valid RelTy proof term proves a property of its own erasure:

**Proposition 78** (RelTy Self). *If*  $\Gamma \vdash p : t[R]t'$ , then  $\Gamma \vdash p : |p|[R]|p|$ .

$$\begin{array}{llll} \Gamma & & ::= & \cdot \mid \Gamma, u:t \, [R] \, t' \\ \\ proof \ terms \ p & & := & u \mid \lambda \, u:T. \, p \mid p \ p' \mid \\ & p \{T\} \mid \Lambda \, X. \, p \mid \\ & t \blacktriangleleft p \blacktriangleright t' \mid \\ & \cup_i p \mid \cup_e p \mid \\ & \iota \{t,t'\} \mid \rho \{x.t_1,t_2\} \ p - p' \mid \\ & (p,p') \mid \pi \, p - x.u.v.p' \\ \\ \end{matrix}$$

$$\begin{array}{lll} |u| & = & u \\ |\lambda \, u:T. \, p| & = & \lambda \, u. \, p \\ |p \ p' \mid & = & |p| \mid p' \mid \\ |p \{T\} \mid & = & |p| \\ |\Lambda \, X. \, p| & = & |p| \\ |\Lambda \, X. \, p| & = & |p| \\ |t \blacktriangleleft p \blacktriangleright t' \mid & = & |p| \\ |\cup_e p \mid & = & |p| \\ |\cup_e p \mid & = & |p| \\ |\iota \{t,t'\} \mid & = & I \\ |\rho \{x.t_1,t_2\} \ p - p' \mid & = & |p' \mid \\ |(p,p') \mid & = & (|p|,|p'|) \\ |\pi \, p - x.u.v.p' \mid & = & |p| \lambda \, u. \, \lambda \, v. \, |p'| \end{array}$$

Fig. 7. Syntax for proof terms of RelTy, and erasure to pure  $\lambda$ -calculus

*Proof sketch.* From the assumed RelTy derivation we get to  $\Gamma \vdash |p|[R]|p|$  using RelTy Projection  $\{77\}$  and Soundness of System F  $\{43\}$ . We need then just a somewhat more informative version of part (ii) of RelTy-RelPf Isomorphism  $\{75\}$ , which maps RelPf derivations to particular proof terms p (not just showing that some such p exists) in correspondence with the RelPf derivations.

# XIII. RELATED WORK

ReITT's semantics (Figure 2) is a relational realizability semantics, where realizers are terms of untyped lambda calculus (cf. [17], [18]). Relational semantics for types has been studied extensively in the context of logical relations; see Chapter 8 of [19]. An influential branch of this work was initiated by Reynolds, on what is now called parametricity [9]. [20] frames some recent results, using categorical semantics.

[21] proposes a similar realizabiliity semantics, for the Calculus of Constructions plus an extensional equality type. The major difference is that in RelTT, we propose a notation for asymmetric relations, which is lacking in [21]. Instead, constructions based on the semantics are done at the metalevel (where asymmetric relations can be described). Indeed, the denotable relations of [21] are partial equivalences – albeit of a modified form due to basing the semantics on "zig-zag complete" relations. In contrast, we have seen above some families of types whose denotations are partial equivalences (unmodified) in RelTT. But by design, not all types denote partial equivalences in RelTT, since reasoning about terms generally involves asymmetric relations; an important example we saw is Reflection {60}.

Observational Type Theory (OTT) is an approach to typespecific extensionality principles in an intensional dependent type theory, based on a primitive heterogeneous equality type and associated operators [22]. RelTT is similar in deriving extensionality principles, but more radical in design: where OTT extends a traditional (i.e., unary) type theory including W-types with an extensional form of equality, RelTT takes a binary view of all types, and does not use dependent types at all. The resulting system is hence formally quite a bit simpler.

Unlike [9] and subsequent works like [23]), RelTT lacks Identity Extension. This property states that when free type variables are interpreted by identity relations, the relational meaning of a type T is the identity relation on the unary (or "object") interpretation of T. This is a very strong property, showing that the object interpretation of types gives canonical forms for the equivalence defined by the relational interpretation of types. But it rules out expression of asymmetric relations as types. RelTT preserves this possibility, at the cost of weakening Identity Extension to Identity Inclusion  $\{32\}$ .

In [24], Plotkin and Abadi introduce a second-order logic for reasoning about (typable) terms of System F by quantification over relations, and using a parametricity axiom. In contrast, RelTT uses relational types to express relations in a more compact way. A parametricity axiom would not make sense here, for there is no separate notion of unary typing from which relational typing could be stated to follow. The only typings are relational.

RelTT may be compared with previous work of Stump et al. on Cedille [25], [26], [27]. Both systems aim at a minimalistic extension of a small pure type system as a foundation for type theory. Cedille extends the Curry-style Calculus of Constructions with dependent intersections, implicit products, and an equality type over untyped terms. RelTT extends System F with three relational operators based on a relational semantics. While the systems are roughly equivalent in formal complexity – with RelTT having the simplifying advantage of eschewing dependent types – RelTT delivers type-specific extensionality principles, which Cedille lacks.

[28] considers how parametricity results can be embedded in constructive type theory, by elaborating types into corresponding theorems in the logic of so-called "reflective" pure type systems. Subsequent work built an extended PTS internalizing these theorems [29]. These papers consider fairly rich Churchstyle lambda calculi, in contrast to the more compact Currystyle calculus of RelTT.

Finally, RelTT may be compared with Homotopy Type Theory (HoTT), a line too active in recent years to summarize here [30]. Both theories support functional extensionality. The two approaches have different origins: logical relations and parametricity for RelTT, homotopy theory and higher category theory for HoTT. A major point of difference is univalence: while RelTT allows one to express and derive relational equalities within the theory, these are based on semantic inclusions, not isomorphisms (as in univalence). Thus, transporting results between isomorphic types as done in HoTT is not (in an obvious way) directly possible in RelTT. Another point of

$$\frac{x:t\left[R\right]t'\in\Gamma}{\Gamma\vdash x:t\left[R\right]t'} \qquad \frac{\Gamma,p:x\left[R\right]x'\vdash p:t\left[R'\right]t'\quad (*)}{\Gamma\vdash \lambda x:R.p:\lambda x.t\left[R\to R'\right]\lambda x'.t'} \qquad \frac{\Gamma\vdash p_1:t_1\left[R\to R'\right]t'_1\quad \Gamma\vdash p_2:t_2\left[R\right]t'_2}{\Gamma\vdash p_1:p_2:t_1t_2\left[R'\right]t'_1t'_2} \\ \frac{\Gamma\vdash p:t\left[\forall X.R'\right]t'}{\Gamma\vdash p_1R\}:t\left[\left[R/X\right]R'\right]t'} \qquad \frac{\Gamma\vdash p:t\left[R\right]t'\quad X\notin FV(\Gamma)}{\Gamma\vdash \Lambda X.p:t\left[\forall X.R\right]t'} \qquad \frac{\Gamma\vdash p:t_1\left[R\right]t_2\quad t_1=_{\beta\eta}t'_1\quad t_2=_{\beta\eta}t'_2}{\Gamma\vdash t'_1\blacktriangleleft p \blacktriangleright t'_2:t'_1\left[R\right]t'_2} \\ \frac{\Gamma\vdash p:t\left[R^{\circlearrowright}\right]t'}{\Gamma\vdash \cup_e p:t'\left[R\right]t} \qquad \frac{\Gamma\vdash p:t\left[R\right]t'}{\Gamma\vdash \cup_i p:t'\left[R^{\circlearrowright}\right]t} \qquad \frac{\Gamma\vdash p:t\left[t''\right]t'}{\Gamma\vdash \iota\{t,t'\}:t\left[t''\right]t'} \qquad \frac{\Gamma\vdash p:t\left[t'''\right]t'}{\Gamma\vdash \rho\{x.t_1,t_2\}\; p-p':\left[t'/x\right]t_1\left[R\right]\left[t''t/x\right]t_2} \\ \frac{\Gamma\vdash p:t\left[R\cdot R'\right]t'}{\Gamma\vdash \pi\; p-x.u.v.p':t_1\left[R''\right]t_2} \qquad \frac{\Gamma\vdash p:t\left[R'''\right]t'}{\Gamma\vdash (p,p'):t\left[R\cdot R'\right]t'} \qquad \frac{\Gamma\vdash p:t\left[R\right]t''}{\Gamma\vdash (p,p'):t\left[R\cdot R'\right]t'} \\ \frac{\Gamma\vdash p:t\left[R\right]t''}{\Gamma\vdash (p,p'):t\left[R\cdot R'\right]t'} \qquad \frac{\Gamma\vdash p:t\left[R\right]t''}{\Gamma\vdash (p,p'):t\left[R\cdot R'\right]t'} \qquad \frac{\Gamma\vdash p:t\left[R\right]t''}{\Gamma\vdash (p,p'):t\left[R\cdot R'\right]t'}$$

Side condition (\*) is  $x \notin FV(\Gamma, R, R')$ . Side condition (\*\*) is  $x \notin FV(\Gamma, t_1, t_2, t, t', R, R', R'')$ .

Fig. 8. RelTy typing rules

comparison is the compactness of the theory. ReITT is based on a very compact semantics for a small number of relational type forms. In contrast, systems like, for one notable example, Cubical Agda, are based on a larger array of primitives [31]. Whereas the free theorems provided by parametricity allows proofs to be transported to observationally equivalent terms, HOTT uses explicit equivalences between terms for this purpose. Only very recent work has considered how to combine these two complementary approaches inside of univalent type theories [32].

## XIV. CONCLUSION AND FUTURE WORK

Based on a binary relational semantics, ReITT is a new minimalistic extensional type theory, where inductive and positive-recursive types are derivable. The theory does not have dependent types, and indeed, an indirect conclusion of the paper is that type theory does not require dependent types for reasoning about programs. Just passing from the traditional unary semantics to a binary-relational one opens the possibility for formal (extensional) reasoning about programs. Future work includes direct support for existential types, for deriving coinductive types; the standard double-negation encoding of existentials is problematic due to the requirement of forall-positivity for Identity Inclusion {32}.

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