Introduction

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Note All vector spaces are real and finite dimensional unless otherwise stated.

Definition 0.1 A bilinear form $\Omega : V \times V \to \mathbb{R}$ on a vector space V is a *linear symplectic form* if it is

(a) skew-symmetric, i.e. $\Omega(v, w) = -\Omega(w, v) \quad \forall\, v, w \in V$;

(b) non-degenerate, i.e. $\Omega(v, w) = 0 \quad \forall\, v \in V \implies w = 0$.

Example 0.2

(1) Consider[1] $V = \mathbb{R}^2, B = (e_1, e_2)$. Then[2] $[\Omega]_B^B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ represents a linear symplectic form on $\mathbb{R}^2$.

(2) Consider $V = \mathbb{R}^{2n}, B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$. Then

$$[\Omega]_B^B = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

represents a linear symplectic form on $\mathbb{R}^{2n}$. Moreover, we have that

$$\begin{aligned} \Omega(e_1, e_j) &= 0, \\ \Omega(f_i, f_j) &= 0, \\ \Omega(e_i, f_j) &= \delta_{ij} = -\Omega(f_j, e_i). \end{aligned} \tag{0.1}$$

Definition 0.3 The pair (V, Ω) is a *(real) symplectic vector space*. If it has a basis $B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ satisfying the relations given by the equations in (0.1), we say that B is a *symplectic basis* of (V, Ω).

Remark 0.4 If B is an ordered basis of V, then $[\Omega]_B^B$ is antisymmetric and invertible.

[1]Whenever order is needed for a basis, I write them as ordered bases (i.e. between parenthesis) instead of unordered sets, and this is important to me!

[2]The notation $[\Gamma]_\alpha^\beta$ indicates the matrix representation of the bilinear form Γ which takes column vectors $[w]_\alpha$ from the right represented, which are with an ordered basis α, and row vectors $([v]_\beta)^T$ from the left, which are represented with an ordered basis β and transposed, and computes $\Gamma(v, w)$.

Example 0.5 Let W be a vector space with dual W^* and $V = W \oplus W^*$. Note there is an isomorphism of vector spaces $W \oplus W^* \xrightarrow{\sim} W^* \oplus W, (w, f) \mapsto (f, -w)$. Then,

$$\begin{gathered}\Omega : V \times V \to R,\\ ((w_1, f_1), (w_2, f_2)) \mapsto f_2(w_1) - f_1(w_2)\end{gathered}$$

is a linear symplectic form on V.

Lemma 0.6 If (V, Ω) is a symplectic vector space, then $dim(V) \equiv 0 \mod 2$.

Proof. If A represents Ω, then

$$\begin{aligned}\det(A) &= \det(-A^T)\\ &= \det(-A)\\ &= (-1)^{\dim(V)}\det(A).\end{aligned}$$

□

Definition 0.7 Let (V, Ω) be a symplectic vector space. A linear subspace $U \subseteq V$ is

(a) *symplectic* if $\Omega\mid_U$ is a linear symplectic form on U (non-degeneracy is sufficient);

(b) *isotropic* if $\Omega\mid_U = 0$;

(c) *coisotropic* if $\Omega(u, v) = 0 \quad \forall\, u \in U \implies v \in U$;

(d) *Lagrangian* if it is both isotropic and coisotropic.

Example 0.8 Let $V = \mathbb{R}^{2n}, B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$.

(1) $\langle\{e_1, f_1\}\rangle$ is a symplectic subspace which is neither isotropic nor coisotropic (thus neither Lagrangian).

(2) $\langle\{e_1, \ldots, e_n\}\rangle$ is a Lagrangian subspace which is not symplectic.

(3) $\{0\}$ and $\langle\{e_1, \ldots, \hat{e_k}, \ldots, e_n\}\rangle$, where $\hat{e_k}$ indicates the exclusion of the k-th vector of the canonical ordered basis from the set, are isotropic subspaces which are not coisotropic, where the first one is trivially symplectic.

(4) If $I, J \subseteq \{1, \ldots, n\}$ and $\langle\{e_i\}_{i\in I} \cup \{f_j\}_{j\in J}\rangle$ is

- symplectic if, and only if, $I = J$;
- isotropic if, and only if, $I \cap J = \varnothing$;
- coisotropic if, and only if, $I \cup J = \{1, \ldots, n\}$ [3];
- Lagrangian if, and only if, $I = J^c$.

[3] I'm pretty sure of this characterization. But keep this note until we get any kind of confirmation.

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Remark 0.9 Let V be a real finite dimensional vector space with dual V^* and $\Omega : V \times V \to \mathbb{R}$ a bilinear form. Then, we have the linear isomorphism[4]

$$\Omega^{\#} = \tilde{\Omega} : V \to V^*, \\ v \mapsto \Omega(v, \cdot).$$

Ω is symplectic if and only if

(1) $\Omega^{\#}$ is antiselfdual, i.e. $(\Omega^{\#})^* = -\Omega^{\#}$, and

(2) $\Omega^{\#}$ is injective (or, equivalently, an isomorphism).

Definition 0.10 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. The *symplectic orthogonal* or *symplectic annihilator* of U is

$$U^{\Omega} := \{v \in V \mid \Omega(u, v) = 0 \quad \forall\, u \in U\}.$$

Proposition 0.11 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. Then,

$$\dim(U) + \dim(U^{\Omega}) = \dim(V) \quad \text{and} \quad (U^{\Omega})^{\Omega} = U.$$

Proof. The first part follows from the fact that $\Omega^{\#}$ is an isomorphism. Moreover, the inclusion $U \subseteq (U^{\Omega})^{\Omega}$ follows from Definition 0.10 and the equality follows by noticing their dimensions are equal. $\square$

Remark 0.12 Let U be a linear subspace of a symplectic vector space (V, Ω). Then U is

- symplectic if, and only if $V = U \oplus U^{\Omega}$;
- isotropic if, and only if, $U \subseteq U^{\Omega}$;
- coisotropic if, and only if, $U \supseteq U^{\Omega}$;
- Lagrangian if, and only if, $U = U^{\Omega}$.

Exercise 0.13 Prove that

(1) U is symplectic if, and only if, U^{Ω} is symplectic;

(2) U is isotropic if, and only if, U^{Ω} is coisotropic;

(3) $(U \cap W)^{\Omega} = U^{\Omega} + W^{\Omega}$.

Proposition 0.14 A symplectic vector space (V, Ω) with $\dim(V) = 2n$ has a symplectic basis.

Proof. We will prove this by induction, taking as basis $\dim(V) = 2$. Let $e_1 \neq 0$. Then there exists $v \in V$ such that $\Omega(e_1, v) \neq 0$. By the Gram-Schmidt orthonormalization process, it follows that $\{e_1, \frac{v}{(e_1, v)}\}$ is a symplectic basis.

Assume the Proposition holds for $\dim(V) = 2n$. Let $v, v' \in V$ be such that $\Omega(v, v') \neq 0$. Then $(S := \langle v, v' \rangle, \Omega \mid_S)$ is a symplectic space of dimension 2. Note that $(S^{\Omega}, \Omega \mid_{S^{\Omega}})$ is a symplectic vector space of dimension $2n$. Since $V = S \oplus S^{\Omega}$, where both subspaces have symplectic bases due to the induction hypothesis, it follows that (V, Ω) has a symplectic basis. $\square$

[4] I believe we need to ask for $v \neq 0$, although the professor didn't mention it.

Remark 0.15 If L is a Lagrangian subspace of a symplectic vector space (V, Ω), then $\dim(L) = \frac{\dim(V)}{2}$.

Proposition 0.16 (Lagrangian split) Let (V, Ω) be a symplectic vector space. Then, there exist Lagrangian subspaces L, L' such that $V = L \oplus L'$.

Proof.

Exercise 0.17 Write the proof; the idea is to show that you can find a maximal (with respect to dimension) isotropic subspace.

□

Remark 0.18 Recall that linear maps $\varphi : V \to V$ induce a map between bilinear forms via

$$\varphi^*(\Omega(v, v')) = \Omega(\varphi(v), \varphi(v')).$$

The Lagrangian split $V = L \oplus L'$ of Proposition 0.16 is canonical in the sense that there exists a canonical isomorphism $\varphi : V \to L \oplus L'$ such that $(\varphi^{-1})^*\Omega$ is the canonical symplectic form on $L \oplus L'$, i.e.

$$\Omega(v_1 + v_1', v_2 + v_2') = \Omega(v_1, v_2') - \Omega(v_2, v_1').$$

Definition 0.19 M is a *topological manifold* if it is a topological space such that

- $\forall\, p \in M$, there exists a neighborhood V of p that is homeomorphic to an open set in $\mathbb{R}^n$;
- it is Hausdorff, i.e. for any two points we can find a neighbourhood for each such that they are disjoint;
- it satisfies the second countability axiom, i.e. there exists a countable basis.

Example 0.20

(1) The torus $T^2 = S^1 \times S^1$, obtained via the identification

Add identification!

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(2) The Klein bottle K^2, obtained via the identification $(x, y) \sim (x+1, y) \sim (1-x, y+1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

(3) The projective plane, obtained via the identification $(x, y) \sim (x+1, 1-y) \sim (1-x, y+1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

Remark 0.21 Recall that M is a differential manifold if it is a topological manifold of dimension n with an atlas, which is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that

(1) $U_{\alpha \in A} \varphi_\alpha(U_\alpha) = M$,

(2) $W = \varphi_\beta(U_\beta) \cap \varphi_\alpha(U_\alpha) \neq 0$, where $\varphi_\beta^{-1}\varphi_\alpha$ and $\varphi_\alpha\varphi_\beta^{-1}$ are of class C^∞,

(3) $U_{\alpha \in A} \varphi_\alpha(U_\alpha)$ is maximal.

Example 0.22

Add examples and complete notes for the last half hour of the class.

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Remark 0.23 For every symplectic vector space (V, Ω) we already saw that we can build a symplectic basis (0.14). Taking each time two elements of that basis and doing the ortogonal we get the following split of V:

$$V = W_1 \oplus \cdots \oplus W_n,$$

where W_i are symplectic vector spaces with dimension 2.

Definition 0.24 Given two symplectic vetor spaces (V, Ω) and (V', Ω') we call a *symplectomorphism* a linear isomorphism $\phi : V \to V'$ such that $\phi^\star \Omega' = \Omega$, where $\phi^\star$ is the pullback.

We say that V and V' are symplectomorphic.

Proposition 0.25 The only global invariant is the dimension.

Example 0.26

(1) $\mathbb{R}^2$ and $\mathbb{R}^4$ can not be symplectiomorphic because a symplectomorphism is always a liner isomorphism over the vector space,

(2) Every symplectic vector space (V, Ω) of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical basis. In fact we can take the symplectic basis of V and than take the ismorphism that sends this basis to the canonical basis of $\mathbb{R}^{2n}$.

Remark 0.27 Being symplectomorphic is an equivalence relation. It is interesting to see and study the acting group that preserves the structure.

Definition 0.28 We call $Sp(V, \Omega) = Sp(2n, \mathbb{R})$ the group of symplectic automorphisms of (V, Ω) of dimension $2n$. It is given by the following subset of $\text{Mat}_{2n\times 2n}$, $\left\{ A : A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) A = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \right\}$.

In fact if ϕ is a symplectic automorphism of $\mathbb{R}^{2n}$ with the canonical form and A is the associated matrix we get that:

$$\begin{aligned} v^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) u &= \Omega_0(u, v) \\ &= \phi^\star \Omega_0(u, v) \\ &= \Omega_0(\phi(u), \phi(v)) \\ &= (Av)^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) (Au) \\ &= v^T A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) Au, \end{aligned}$$

but this must be true for each u and each v, so A respects the condition that we have imposed.

Proposition 0.29 $Sp(2n, \mathbb{R})$ is a group with standard multiplication and inverse or quadratic matrices. Also the determinant of A is always 1.

Remark 0.30 Both GL and SO are groups of automorphisms, the difference is that SO preserves both the lenght of the vectors and the areas.

Definition 0.31 Given a differential manifold M we say that a 2–form ω is symplectic if it is

(a) closed, i.e. $d\omega = 0$;

(b) non-degenerate, i.e. $\omega(u, v) = 0 \quad \forall\, v \in M \implies u = 0$.

(c) for each p we have that ω_p is a symplectic form on T_pM.

We call the couple (M, ω) symplectic manifold.

Proposition 0.32 A symplectic manifold always has even dimension.

Example 0.33 $(\mathbb{R}^{2n}, \omega_{std})$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. $\omega = \sum_i dx_i \wedge dy_i$.

For each p, ω_p is given by: $\Big((\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, \dots, (\frac{\partial}{\partial y_n})_p\Big)$. It is symplectic on T_pM.

We can do the same considering $\mathbb{C}^n$ with $\omega = \frac{1}{2}\sum_i dz_i \wedge d\bar{z}_i$.

Example 0.34 Take now the sphere S^2 as a subset of $\mathbb{R}^3$. We know that T_pS^2 is given by $\{p\}^\perp$.

Now we define the symplectic form ω in the following way $\omega_p(u, v) =< p; v \times u >$. In fact it is closed is because it is a 2–form in a dimension 2 manifold and it is non degenerate becuase $v \times u$ always has the direction of p as they are in T_pS^2.

This ω is a volume form for S^2.

Exercise 0.35 The above definition is an implicit one; we can define the same form in coordinates taking the coordinates θ, z. In this case the form can be expressed as $\omega = d\theta \wedge dz$.

Also how will the form change if we take the sphere of radius R?

Definition 0.36 Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) we say that a *symplectomorphism* is a diffeomorphism g such that $g^\star\omega_2 = \omega_1$.

As before, the dimension is the only invariant, but now only in a local scope.

Add all the cotangent bundles!

Proposition 0.37 S^2 is the only sphere that can be symplectic, all the others have a trivial second order cohomology group, so they cannot have a symplectic form.

Proposition 0.38 All the symplectomorphisms preserve the areas but the converse is not true nor a banal proof. It was worthy of a Field's medal.

19/03/2024

Last class we considered a manifold X of dimension n and saw that

$$M = T^* := \{(x, \xi^*), x \in X, \xi^* \in T_x^*X\}$$

is a symplectic manifold of dimension 2n. If $(x_1, \dots, x_n)$ is a coordinate system on X, then $\{(dx_1)_x, \dots, (d_n)_x)\}$ is a basis for T_x^*X. Also, $\xi^* = \sum_{i=1}^n y_i(dx_i)_x$ and $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates on T^*X, with

$$\begin{aligned}\omega_{\text{can}} &= \sum_{i=1}^n dx_1 \wedge dy_1,\\ &= \lambda_{\text{can}} = \sum_{i=1}^n y_i dx_i,\\ &= -d\lambda_{\text{can}}.\end{aligned}$$

Part of the reminder is missing (she erased it too fast).

Let X_1, X_2 be differentiable manifolds of dimension n,

$$\begin{aligned} M_1 &= (T^*X_1, -d\lambda_1), \\ M_2 &= (T^*X_2, -d\lambda_2), \end{aligned}$$

where λ_i is the canonical Liouville term on X_i for $i \in \{1,2\}$. Let $f : X_1 \to X_2$ be a differomorphism. Then

$$\begin{aligned} f_{\mathbb{X}} : M_1 &\to M_2, \\ p_1 &\mapsto p_2, \end{aligned}$$

where $p_1 = (x_1, \xi_1^*)$ and $p_2 = (f(x_1), \xi_2^*)$, where $\xi_2^* \in T^*_{f(x_1)}X_2$. We have that

$$\begin{aligned} df &: TX_1 \to TX_2 \\ df_{x_1} &: T_{x_1}X_1 \to T_{f(x_1)}X_2, \\ (df_{x_1})^* &: T^*_{f(x_1)}X_2 \to T^*_{x_1}X_1. \end{aligned}$$

Since f is in particular a bijection then, defining $x_2 := f(x_1)$, we have that

$$\begin{aligned} df_{x_1}^{-1} &: T_{x_2}X_2 \to T_{x_1}X_1, \\ (df_{x_1}^{-1})^* &: T^*_{x_1}X_1 \to T^*_{x_2}X_2, \\ &\quad \xi_1^* \mapsto (df_{x_2}^{-1})^*(\xi_1^*) =: \xi_2^*. \end{aligned}$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "f_{\mathbb{X}}"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"'] & X_2. \end{tikzcd} \tag{0.2}$$

Proposition 0.39 Let $f_{\mathbb{X}} : M_1 \to M_2$ be a diffeomorphism be such that $f_{\mathbb{X}}^*(\lambda_2) = \lambda_1$. Then $f_{\mathbb{X}}^*(-d\lambda_2) = -d\lambda_1$.

Proof. Let $p_1 = (x_1, \xi_1^*) \in T^*X_1$ and $p_2 = f_{\mathbb{X}}(p_1) = (f(x_1), \xi_2^*)$.

$$\begin{aligned} (f_{\mathbb{X}})^*((\lambda_2)_{p_2}) &= (f_{\mathbb{X}}^*)_{p_1}((d\pi_2)^*_{p_2}\xi_2^*) \\ &= d(f\pi_1)^*_{p_1}\xi_2^* \\ &= (d\pi_1)^*_{p_1} f_{\mathbb{X}}^*(\xi_2^*) \\ &= (d\pi_1)^*_{p_1}\xi_1^* \\ &= (\lambda_1)_{p_1}. \end{aligned}$$

□

Corollary 0.40 If X_1 and X_2 are diffeomorphic, then T^*X_1 and T^*X_2 are symplectomorphic.

Example 0.41 Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Then we have a product $M_1 \times M_2$ with projections

$$\begin{tikzcd} & M_1 \times M_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M_1 & & M_2. \end{tikzcd}$$

Then, $\omega_{a,b} := ap_1^*\omega_1 + bp_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$ for all $a, b \in \mathbb{R}$. In fact,

$$\begin{aligned} d\omega_{a,b} &= d(ap_1^*\omega_1 + bp_2^*\omega_2) \\ &= ad(p_1^*\omega_1) + bd(p_2^*\omega_2) \\ &= ap_1^*d\omega_1 + bp_2^*d\omega_2 \\ &= 0, \end{aligned}$$

i.e. the closeness of $\omega_{a,b}$ is induced from that of ω_1 and ω_2. Similarly, the other properties of induced.

Example 0.42 $\mathbb{R}^{2n}, \mathbb{C}^n, S^2$. Consider $S^2 \times \mathbb{R}^2$ with the form $\omega = 2p_1^*\omega_1 - p_2^*\omega_2$. On S^2, $\omega_p(u, v) = \langle p, u \times v\rangle$, where $p = (x_1, x_2, x_3) \in S^2 \subseteq \mathbb{R}^3$ is such that $\sum_{i=1}^3 x_1^2 = 1$, and $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in T_pS^2 \simeq T_p^*S^2$. Explicitely,

$$\omega_p(u, v) = x_1(u_2v_3 - v_2u_3) + x_2(u_3v_1 - u_1v_3) + x_3(u_1v_2 - v_1u_2).$$

On the other hand, we know that $\omega_p = \sum_{i,j} a_{ij} dx_i \wedge dx_j$, where

$$a_{ij} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

In particular, $a_{ii} = 0$ for all i. We can thus calculate

$$\omega_p = x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3$$