Introduction

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Note All vector spaces are real and finite dimensional unless otherwise stated.

Definition 0.1 A bilinear form $\Omega : V \times V \to \mathbb{R}$ on a vector space V is a *linear symplectic form* if it is

(a) skew-symmetric, i.e. $\Omega(v, w) = -\Omega(w, v) \quad \forall\ v, w \in V$;

(b) non-degenerate, i.e. $\Omega(v, w) = 0 \quad \forall\ v \in V \implies w = 0$.

Example 0.2

(1) Consider[1] $V = \mathbb{R}^2, B = (e_1, e_2)$. Then[2] $[\Omega]_B^B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ represents a linear symplectic form on $\mathbb{R}^2$.

(2) Consider $V = \mathbb{R}^{2n}, B = (e_1, \dots, e_n, f_1, \dots, f_n)$. Then

$$[\Omega]_B^B = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

represents a linear symplectic form on $\mathbb{R}^{2n}$. Moreover, we have that

$$\begin{aligned} \Omega(e_1, e_j) &= 0, \\ \Omega(f_i, f_j) &= 0, \\ \Omega(e_i, f_j) &= \delta_{ij} = -\Omega(f_j, e_i). \end{aligned} \tag{0.1}$$

Definition 0.3 The pair (V, Ω) is a *(real) symplectic vector space.* If it has a basis $B = (e_1, \dots, e_n, f_1, \dots, f_n)$ satisfying the relations given by the equations in (0.1), we say that B is a *symplectic basis* of (V, Ω).

Remark 0.4 If B is an ordered basis of V, then $[\Omega]_B^B$ is antisymmetric and invertible.

Example 0.5 Let W be a vector space with dual W^* and $V = W \oplus W^*$. Note there is an isomorphism of vector spaces $W \oplus W^* \xrightarrow{\sim} W^* \oplus W, (w, f) \mapsto (f, -w)$. Then,

$$\begin{gathered} \Omega : V \times V \to R, \\ ((w_1, f_1), (w_2, f_2)) \mapsto f_2(w_1) - f_1(w_2) \end{gathered}$$

is a linear symplectic form on V.

[1] Whenever order is needed for a basis, I write them as ordered bases (i.e. between parenthesis) instead of unordered sets, and this is important to me!

[2] The notation $[\Gamma]_\alpha^\beta$ indicates the matrix representation of the bilinear form Γ which takes column vectors $[w]_\alpha$ from the right represented, which are with an ordered basis α, and row vectors $([v]_\beta)^T$ from the left, which are represented with an ordered basis β and transposed, and computes $\Gamma(v, w)$.

Lemma 0.6 If (V, Ω) is a symplectic vector space, then $dim(V) \equiv 0 \mod 2$.

Proof. If A represents Ω, then

$$\begin{aligned}\det(A) &= \det(-A^T)\\ &= \det(-A)\\ &= (-1)^{\dim(V)}\det(A).\end{aligned}$$

$\square$

Definition 0.7 Let (V, Ω) be a symplectic vector space. A linear subspace $U \subseteq V$ is

(a) *symplectic* if $\Omega\mid_U$ is a linear symplectic form on U (non-degeneracy is sufficient);

(b) *isotropic* if $\Omega\mid_U = 0$;

(c) *coisotropic* if $\Omega(u, v) = 0 \quad \forall\, u \in U \implies v \in U$;

(d) *Lagrangian* if it is both isotropic and coisotropic.

Example 0.8 Let $V = \mathbb{R}^{2n}$, $B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$.

(1) $\langle\{e_1, f_1\}\rangle$ is a symplectic subspace which is neither isotropic nor coisotropic (thus neither Lagrangian).

(2) $\langle\{e_1, \ldots, e_n\}\rangle$ is a Lagrangian subspace which is not symplectic.

(3) $\{0\}$ and $\langle\{e_1, \ldots, \hat{e_k}, \ldots, e_n\}\rangle$, where $\hat{e_k}$ indicates the exclusion of the k-th vector of the canonical ordered basis from the set, are isotropic subspaces which are not coisotropic, where the first one is trivially symplectic.

(4) If $I, J \subseteq \{1, \ldots, n\}$ and $\langle\{e_i\}_{i\in I} \cup \{f_j\}_{j\in J}\rangle$ is

- symplectic if, and only if, $I = J$;
- isotropic if, and only if, $I \cap J = \varnothing$;
- coisotropic if, and only if, $I \cup J = \{1, \ldots, n\}$ [3];
- Lagrangian if, and only if, $I = J^c$.

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Remark 0.9 Let V be a real finite dimensional vector space with dual V^* and $\Omega : V \times V \to \mathbb{R}$ a bilinear form. Then, we have the linear isomorphism[4]

$$\begin{aligned}\Omega^\# = \tilde{\Omega} : V &\to V^*,\\ v &\mapsto \Omega(v, \cdot).\end{aligned}$$

Ω is symplectic if and only if

(1) $\Omega^\#$ is antiselfdual, i.e. $(\Omega^\#)^* = -\Omega^\#$, and

[3] I'm pretty sure of this characterization. But keep this note until we get any kind of confirmation.

[4] I believe we need to ask for $v \neq 0$, although the professor didn't mention it.

(2) $\Omega^{\#}$ is injective (or, equivalently, an isomorphism).

Definition 0.10 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. The *symplectic orthogonal* or *symplectic annihilator* of U is

$$U^{\Omega} := \{v \in V \mid \Omega(u, v) = 0 \quad \forall\, u \in U\}.$$

Proposition 0.11 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. Then,

$$\dim(U) + \dim(U^{\Omega}) = \dim(V) \quad \text{and} \quad (U^{\Omega})^{\Omega} = U.$$

Proof. The first part follows from the fact that $\Omega^{\#}$ is an isomorphism. Moreover, the inclusion $U \subseteq (U^{\Omega})^{\Omega}$ follows from Definition 0.10 and the equality follows by noticing their dimensions are equal. $\square$

Remark 0.12 Let U be a linear subspace of a symplectic vector space (V, Ω). Then U is

- symplectic if, and only if $V = U \oplus U^{\Omega}$;
- isotropic if, and only if, $U \subseteq U^{\Omega}$;
- coisotropic if, and only if, $U \supseteq U^{\Omega}$;
- Lagrangian if, and only if, $U = U^{\Omega}$.

Exercise 0.13 Prove that

(1) U is symplectic if, and only if, U^{Ω} is symplectic;

(2) U is isotropic if, and only if, U^{Ω} is coisotropic;

(3) $(U \cap W)^{\Omega} = U^{\Omega} + W^{\Omega}$.

Proposition 0.14 A symplectic vector space (V, Ω) with $\dim(V) = 2n$ has a symplectic basis.

Proof. We will prove this by induction, taking as basis $\dim(V) = 2$. Let $e_1 \neq 0$. Then there exists $v \in V$ such that $\Omega(e_1, v) \neq 0$. By the Gram-Schmidt orthonormalization process, it follows that $\{e_1, \frac{v}{(e_1,v)}\}$ is a symplectic basis.

Assume the Proposition holds for $\dim(V) = 2n$. Let $v, v' \in V$ be such that $\Omega(v, v') \neq 0$. Then $(S := \langle v, v' \rangle, \Omega \mid_S)$ is a symplectic space of dimension 2. Note that $(S^{\Omega}, \Omega \mid_{S^{\Omega}})$ is a symplectic vector space of dimension $2n$. Since $V = S \oplus S^{\Omega}$, where both subspaces have symplectic bases due to the induction hypothesis, it follows that (V, Ω) has a symplectic basis. $\square$

Remark 0.15 If L is a Lagrangian subspace of a symplectic vector space (V, Ω), then $\dim(L) = \frac{\dim(V)}{2}$.

Proposition 0.16 (Lagrangian split) Let (V, Ω) be a symplectic vector space. Then, there exist Lagrangian subspaces L, L' such that $V = L \oplus L'$.

Proof.

Exercise 0.17 Write the proof; the idea is to show that you can find a maximal (with respect to dimension) isotropic subspace.

$\square$

Remark 0.18 Recall that linear maps $\varphi : V \to V$ induce a map between bilinear forms via

$$\varphi^*(\Omega(v, v')) = \Omega(\varphi(v), \varphi(v')).$$

The Lagrangian split $V = L \oplus L'$ of Proposition 0.16 is canonical in the sense that there exists a canonical isomorphism $\varphi : V \to L \oplus L'$ such that $(\varphi^{-1})^*\Omega$ is the canonical symplectic form on $L \oplus L'$, i.e.

$$\Omega(v_1 + v_1', v_2 + v_2') = \Omega(v_1, v_2') - \Omega(v_2, v_1').$$

Definition 0.19 M is a *topological manifold* if it is a topological space such that

- $\forall\, p \in M$, there exists a neighborhood V of p that is homeomorphic to an open set in $\mathbb{R}^n$;
- it is Hausdorff, i.e. for any two points we can find a neighbourhood for each such that they are disjoint;
- it satisfies the second countability axiom, i.e. there exists a countable basis.

Example 0.20

(1) The torus $T^2 = S^1 \times S^1$, obtained via the identification

Add identification!

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(2) The Klein bottle K^2, obtained via the identification $(x, y) \sim (x + 1, y) \sim (1 - x, y + 1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

(3) The projective plane, obtained via the identification $(x, y) \sim (x + 1, 1 - y) \sim (1 - x, y + 1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

Remark 0.21 Recall that M is a differential manifold if it is a topological manifold of dimension n with an atlas, which is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that

(1) $U_{\alpha \in A} \varphi_\alpha(U_\alpha) = M$,

(2) $W = \varphi_\beta(U_\beta) \cap \varphi_\alpha(U_\alpha) \neq 0$, where $\varphi_\beta^{-1}\varphi_\alpha$ and $\varphi_\alpha\varphi_\beta^{-1}$ are of class C^∞,

(3) $U_{\alpha \in A} \varphi_\alpha(U_\alpha)$ is maximal.

Example 0.22

Add examples and complete notes for the last half hour of the class.

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