Introduction

11/3/24

Note All vector spaces are real and finite dimensional unless otherwise stated.

Definition 0.1 A bilinear form $\Omega : V \times V \to \mathbb{R}$ on a vector space V is a *linear symplectic form* if it is

(a) skew-symmetric, i.e. $\Omega(v, w) = -\Omega(w, v) \quad \forall\ v, w \in V$;

(b) non-degenerate, i.e. $\Omega(v, w) = 0 \quad \forall\ v \in V \implies w = 0$.

Example 0.2

(1) Consider[1] $V = \mathbb{R}^2, B = (e_1, e_2)$. Then[2] $[\Omega]_B^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a linear symplectic form on $\mathbb{R}^2$.

(2) Consider $V = \mathbb{R}^{2n}, B = (e_1, \dots, e_n, f_1, \dots, f_n)$. Then

$$[\Omega]_B^B = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

represents a linear symplectic form on $\mathbb{R}^{2n}$. Moreover, we have that

$$\begin{aligned} \Omega(e_1, e_j) &= 0, \\ \Omega(f_i, f_j) &= 0, \\ \Omega(e_i, f_j) &= \delta_{ij} = -\Omega(f_j, e_i). \end{aligned} \tag{0.1}$$

Definition 0.3 The pair (V, Ω) is a *(real) symplectic vector space*. If it has a basis $B = (e_1, \dots, e_n, f_1, \dots, f_n)$ satisfying the relations given by the equations in (0.1), we say that B is a *symplectic basis* of (V, Ω).

Remark 0.4 If B is an ordered basis of V, then $[\Omega]_B^B$ is antisymmetric and invertible.

[1] Whenever order is needed for a basis, I write them as ordered bases (i.e. between parenthesis) instead of unordered sets, and this is important to me!

[2] The notation $[\Gamma]_\alpha^\beta$ indicates the matrix representation of the bilinear form Γ which takes column vectors $[w]_\alpha$ from the right represented, which are with an ordered basis α, and row vectors $([v]_\beta)^T$ from the left, which are represented with an ordered basis β and transposed, and computes $\Gamma(v, w)$.

Example 0.5 Let W be a vector space with dual W^* and $V = W \oplus W^*$. Note there is an isomorphism of vector spaces $W \oplus W^* \xrightarrow{\sim} W^* \oplus W, (w, f) \mapsto (f, -w)$. Then,

$$\begin{gathered}\Omega : V \times V \to R,\\ ((w_1, f_1), (w_2, f_2)) \mapsto f_2(w_1) - f_1(w_2)\end{gathered}$$

is a linear symplectic form on V.

Lemma 0.6 If (V, Ω) is a symplectic vector space, then $dim(V) \equiv 0 \mod 2$.

Proof. If A represents Ω, then

$$\begin{aligned}\det(A) &= \det(-A^T)\\ &= \det(-A)\\ &= (-1)^{\dim(V)}\det(A).\end{aligned}$$

□

Definition 0.7 Let (V, Ω) be a symplectic vector space. A linear subspace $U \subseteq V$ is

(a) *symplectic* if $\Omega\mid_U$ is a linear symplectic form on U (non-degeneracy is sufficient);

(b) *isotropic* if $\Omega\mid_U= 0$;

(c) *coisotropic* if $\Omega(u, v) = 0 \quad \forall\ u \in U \implies v \in U$;

(d) *Lagrangian* if it is both isotropic and coisotropic.

Example 0.8 Let $V = \mathbb{R}^{2n}, B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$.

(1) $\langle\{e_1, f_1\}\rangle$ is a symplectic subspace which is neither isotropic nor coisotropic (thus neither Lagrangian).

(2) $\langle\{e_1, \ldots, e_n\}\rangle$ is a Lagrangian subspace which is not symplectic.

(3) $\{0\}$ and $\langle\{e_1, \ldots, \hat{e_k}, \ldots, e_n\}\rangle$, where $\hat{e_k}$ indicates the exclusion of the k-th vector of the canonical ordered basis from the set, are isotropic subspaces which are not coisotropic, where the first one is trivially symplectic.

(4) If $I, J \subseteq \{1, \ldots, n\}$ and $\langle\{e_i\}_{i\in I} \cup \{f_j\}_{j\in J}\rangle$ is

- symplectic if, and only if, $I = J$;
- isotropic if, and only if, $I \cap J = \varnothing$;
- coisotropic if, and only if, $I \cup J = \{1, \ldots, n\}$ [3];
- Lagrangian if, and only if, $I = J^c$.

[3] I'm pretty sure of this characterization. But keep this note until we get any kind of confirmation.

12/3/24

Remark 0.9 Let V be a real finite dimensional vector space with dual V^* and $\Omega : V \times V \to \mathbb{R}$ a bilinear form. Then, we have the linear isomorphism[4]

$$\begin{aligned} \Omega^\# = \tilde{\Omega} : V &\to V^*, \\ v &\mapsto \Omega(v, \cdot). \end{aligned}$$

Ω is symplectic if and only if

(1) $\Omega^\#$ is antiselfdual, i.e. $(\Omega^\#)^* = -\Omega^\#$, and

(2) $\Omega^\#$ is injective (or, equivalently, an isomorphism).

Definition 0.10 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. The *symplectic orthogonal* or *symplectic annihilator* of U is

$$U^\Omega := \{v \in V \mid \Omega(u, v) = 0 \quad \forall\, u \in U\}.$$

Proposition 0.11 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. Then,

$$\dim(U) + \dim(U^\Omega) = \dim(V) \quad \text{and} \quad (U^\Omega)^\Omega = U.$$

Proof. The first part follows from the fact that $\Omega^\#$ is an isomorphism. Moreover, the inclusion $U \subseteq (U^\Omega)^\Omega$ follows from Definition 0.10 and the equality follows by noticing their dimensions are equal. □

Remark 0.12 Let U be a linear subspace of a symplectic vector space (V, Ω). Then U is

- symplectic if, and only if $V = U \oplus U^\Omega$;
- isotropic if, and only if, $U \subseteq U^\Omega$;
- coisotropic if, and only if, $U \supseteq U^\Omega$;
- Lagrangian if, and only if, $U = U^\Omega$.

Exercise 0.13 Prove that

(1) U is symplectic if, and only if, U^Ω is symplectic;

(2) U is isotropic if, and only if, U^Ω is coisotropic;

(3) $(U \cap W)^\Omega = U^\Omega + W^\Omega$.

Proposition 0.14 A symplectic vector space (V, Ω) with $\dim(V) = 2n$ has a symplectic basis.

Proof. We will prove this by induction, taking as basis $\dim(V) = 2$. Let $e_1 \neq 0$. Then there exists $v \in V$ such that $\Omega(e_1, v) \neq 0$. By the Gram-Schmidt orthonormalization process, it follows that $\{e_1, \frac{v}{(e_1, v)}\}$ is a symplectic basis.

Assume the Proposition holds for $\dim(V) = 2n$. Let $v, v' \in V$ be such that $\Omega(v, v') \neq 0$. Then $(S := \langle v, v' \rangle, \Omega\mid_S)$ is a symplectic space of dimension 2. Note that $(S^\Omega, \Omega\mid_{S^\Omega})$ is a symplectic vector space of dimension $2n$. Since $V = S \oplus S^\Omega$, where both subspaces have symplectic bases due to the induction hypothesis, it follows that (V, Ω) has a symplectic basis. □

[4] I believe we need to ask for $v \neq 0$, although the professor didn't mention it.

Remark 0.15 If L is a Lagrangian subspace of a symplectic vector space (V, Ω), then $\dim(L) = \frac{\dim(V)}{2}$.

Proposition 0.16 (Lagrangian split) Let (V, Ω) be a symplectic vector space. Then, there exist Lagrangian subspaces L, L' such that $V = L \oplus L'$.

Proof.

Exercise 0.17 Write the proof; the idea is to show that you can find a maximal (with respect to dimension) isotropic subspace.

□

Remark 0.18 Recall that linear maps $\varphi : V \to V$ induce a map between bilinear forms via

$$\varphi^*(\Omega(v, v')) = \Omega(\varphi(v), \varphi(v')).$$

The Lagrangian split $V = L \oplus L'$ of Proposition 0.16 is canonical in the sense that there exists a canonical isomorphism $\varphi : V \to L \oplus L'$ such that $(\varphi^{-1})^*\Omega$ is the canonical symplectic form on $L \oplus L'$, i.e.

$$\Omega(v_1 + v_1', v_2 + v_2') = \Omega(v_1, v_2') - \Omega(v_2, v_1').$$

Definition 0.19 M is a *topological manifold* if it is a topological space such that

- $\forall\, p \in M$, there exists a neighborhood V of p that is homeomorphic to an open set in $\mathbb{R}^n$;
- it is Hausdorff, i.e. for any two points we can find a neighbourhood for each such that they are disjoint;
- it satisfies the second countability axiom, i.e. there exists a countable basis.

Example 0.20

(1) The torus $T^2 = S^1 \times S^1$, obtained via the identification

Add identification!

.

(2) The Klein bottle K^2, obtained via the identification $(x, y) \sim (x + 1, y) \sim (1 - x, y + 1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

(3) The projective plane, obtained via the identification $(x, y) \sim (x + 1, 1 - y) \sim (1 - x, y + 1)$ in $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$.

Remark 0.21 Recall that M is a differential manifold if it is a topological manifold of dimension n with an atlas, which is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that

(1) $U_{\alpha \in A} \varphi_\alpha(U_\alpha) = M$,

(2) $W = \varphi_\beta(U_\beta) \cap \varphi_\alpha(U_\alpha) \neq 0$, where $\varphi_\beta^{-1}\varphi_\alpha$ and $\varphi_\alpha\varphi_\beta^{-1}$ are of class C^∞,

(3) $U_{\alpha \in A} \varphi_\alpha(U_\alpha)$ is maximal.

Example 0.22

Add examples and complete notes for the last half hour of the class.

18/3/24

Remark 0.23 For every symplectic vector space (V, Ω) we already saw that we can build a symplectic basis (0.14). Taking each time two elements of that basis and doing the ortogonal we get the following split of V:

$$V = W_1 \oplus \cdots \oplus W_n,$$

where W_i are symplectic vector spaces with dimension 2.

Definition 0.24 Given two symplectic vetor spaces (V, Ω) and (V', Ω') we call a *symplectomorphism* a linear isomorphism $\phi : V \to V'$ such that $\phi^\star \Omega' = \Omega$, where $\phi^\star$ is the pullback.

We say that V and V' are symplectomorphic.

Proposition 0.25 The only global invariant is the dimension.

Example 0.26

(1) $\mathbb{R}^2$ and $\mathbb{R}^4$ can not be symplectiomorphic because a symplectomorphism is always a liner isomorphism over the vector space,

(2) Every symplectic vector space (V, Ω) of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical basis. In fact we can take the symplectic basis of V and than take the ismorphism that sends this basis to the canonical basis of $\mathbb{R}^{2n}$.

Remark 0.27 Being symplectomorphic is an equivalence relation. It is interesting to see and study the acting group that preserves the structure.

Definition 0.28 We call $Sp(V, \Omega) = Sp(2n, \mathbb{R})$ the group of symplectic automorphisms of (V, Ω) of dimension $2n$. It is given by the following subset of $\mathrm{Mat}_{2n\times 2n}$, $\left\{ A : A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) A = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \right\}$.

In fact if ϕ is a symplectic automorphism of $\mathbb{R}^{2n}$ with the canonical form and A is the associated matrix we get that:

$$\begin{aligned} v^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) u &= \Omega_0(u, v) \\ &= \phi^\star \Omega_0(u, v) \\ &= \Omega_0(\phi(u), \phi(v)) \\ &= (Av)^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) (Au) \\ &= v^T A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) Au, \end{aligned}$$

but this must be true for each u and each v, so A respects the condition that we have imposed.

Proposition 0.29 $Sp(2n, \mathbb{R})$ is a group with standard multiplication and inverse or quadratic matrices. Also the determinant of A is always 1.

Remark 0.30 Both GL and SO are groups of automorphisms, the difference is that SO preserves both the lenght of the vectors and the areas.

Symplectic Manifold

Definition 0.31 Given a differential manifold M we say that a 2–form ω is symplectic if it is

(a) closed, i.e. $d\omega = 0$;

(b) non-degenerate, i.e. $\omega(u, v) = 0 \quad \forall\, v \in M \implies u = 0$.

(c) for each p we have that ω_p is a symplectic form on T_pM.

We call the couple (M, ω) symplectic manifold.

Proposition 0.32 A symplectic manifold always has even dimension.

Example 0.33 $(\mathbb{R}^{2n}, \omega_{std})$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. $\omega = \sum_i dx_i \wedge dy_i$.
For each p, ω_p is given by: $\left((\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, \dots, (\frac{\partial}{\partial y_n})_p\right)$. It is symplectic on T_pM.
We can do the same considering $\mathbb{C}^n$ with $\omega = \frac{1}{2}\sum_i dz_i \wedge d\bar{z}_i$.

Example 0.34 Take now the sphere S^2 as a subset of $\mathbb{R}^3$. We know that T_pS^2 is given by $\{p\}^\perp$.
Now we define the symplectic form ω in the following way $\omega_p(u, v) =< p; v \times u >$. In fact it is closed is because it is a 2–form in a dimension 2 manifold and it is non degenerate becuase $v \times u$ always has the direction of p as they are in T_pS^2.
This ω is a volume form for S^2.

Exercise 0.35 The above definition is an implicit one; we can define the same form in coordinates taking the coordinates θ, z. In this case the form can be expressed as $\omega = d\theta \wedge dz$.
Also how will the form change if we take the sphere of radius R?

Definition 0.36 Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) we say that a *symplectomorphism* is a diffeomorphism g such that $g^\star\omega_2 = \omega_1$.
As before, the dimension is the only invariant, but now only in a local scope.

Add all the cotangent bundles!

Proposition 0.37 S^2 is the only sphere that can be symplectic, all the others have a trivial second order cohomology group, so they cannot have a symplectic form.

Proposition 0.38 All the symplectomorphisms preserve the areas but the converse is not true nor a banal proof. It was worthy of a Field's medal.

19/03/2024

Last class we considered a manifold X of dimension n and saw that

$$M = T^* := \{(x, \xi^*), x \in X, \xi^* \in T^*_x X\}$$

is a symplectic manifold of dimension 2n. If $(x_1, \dots, x_n)$ is a coordinate system on X, then $\{(dx_1)_x, \dots, (d_n)_x)\}$ is a basis for $T^*_x X$. Also, $\xi^* = \sum_{i=1}^n y_i (dx_i)_x$ and $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates on T^*X, with

$$\begin{aligned}\omega_{\text{can}} &= \sum_{i=1}^n dx_1 \wedge dy_1, \\ &= \lambda_{\text{can}} = \sum_{i=1}^n y_i dx_i, \\ &= -d\lambda_{\text{can}}.\end{aligned}$$

Part of the reminder is missing (she erased it too fast).

Let X_1, X_2 be differentiable manifolds of dimension n,

$$\begin{aligned} M_1 &= (T^*X_1, -d\lambda_1), \\ M_2 &= (T^*X_2, -d\lambda_2), \end{aligned}$$

where λ_i is the canonical Liouville term on X_i for $i \in \{1,2\}$. Let $f : X_1 \to X_2$ be a differomorphism. Then

$$\begin{aligned} f_{\mathbb{X}} : M_1 &\to M_2, \\ p_1 &\mapsto p_2, \end{aligned}$$

where $p_1 = (x_1, \xi_1^*)$ and $p_2 = (f(x_1), \xi_2^*)$, where $\xi_2^* \in T^*_{f(x_1)}X_2$. We have that

$$\begin{aligned} df : TX_1 &\to TX_2 \\ df_{x_1} : T_{x_1}X_1 &\to T_{f(x_1)}X_2, \\ (df_{x_1})^* : T^*_{f(x_1)}X_2 &\to T^*_{x_1}X_1. \end{aligned}$$

Since f is in particular a bijection then, defining $x_2 := f(x_1)$, we have that

$$\begin{aligned} df^{-1}_{x_1} : T_{x_2}X_2 &\to T_{x_1}X_1, \\ (df^{-1}_{x_1})^* : T^*_{x_1}X_1 &\to T^*_{x_2}X_2, \\ \xi_1^* &\mapsto (df^{-1}_{x_2})^*(\xi_1^*) =: \xi_2^*. \end{aligned}$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "f_{\mathbb{X}}"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"'] & X_2. \end{tikzcd} \tag{0.2}$$

Proposition 0.39 Let $f_{\mathbb{X}} : M_1 \to M_2$ be a diffeomorphism be such that $f_{\mathbb{X}}^*(\lambda_2) = \lambda_1$. Then $f_{\mathbb{X}}^*(-d\lambda_2) = -d\lambda_1$.

Proof. Let $p_1 = (x_1, \xi_1^*) \in T^*X_1$ and $p_2 = f_{\mathbb{X}}(p_1) = (f(x_1), \xi_2^*)$.

$$\begin{aligned} (f_{\mathbb{X}})^*((\lambda_2)_{p_2}) &= (f_{\mathbb{X}}^*)_{p_1}((d\pi_2)^*_{p_2}\xi_2^*) \\ &= d(f\pi_1)^*_{p_1}\xi_2^* \\ &= (d\pi_1)^*_{p_1} f_{\mathbb{X}}^*(\xi_2^*) \\ &= (d\pi_1)^*_{p_1}\xi_1^* \\ &= (\lambda_1)_{p_1}. \end{aligned}$$

□

Corollary 0.40 If X_1 and X_2 are diffeomorphic, then T^*X_1 and T^*X_2 are symplectomorphic.

Example 0.41 Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Then we have a product $M_1 \times M_2$ with projections

$$\begin{tikzcd}
 & M_1 \times M_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M_1 & & M_2.
\end{tikzcd}$$

Then, $\omega_{a,b} := ap_1^*\omega_1 + bp_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$ for all $a, b \in \mathbb{R}$. In fact,

$$\begin{aligned}
d\omega_{a,b} &= d(ap_1^*\omega_1 + bp_2^*\omega_2) \\
&= ad(p_1^*\omega_1) + bd(p_2^*\omega_2) \\
&= ap_1^*d\omega_1 + bp_2^*d\omega_2 \\
&= 0,
\end{aligned}$$

i.e. the closeness of $\omega_{a,b}$ is induced from that of ω_1 and ω_2. Similarly, the other properties of induced.

Example 0.42 $\mathbb{R}^{2n}, \mathbb{C}^n, S^2$. Consider $S^2 \times \mathbb{R}^2$ with the form $\omega = 2p_1^*\omega_1 - p_2^*\omega_2$. On S^2, $\omega_p(u, v) = \langle p, u \times v \rangle$, where $p = (x_1, x_2, x_3) \in S^2 \subseteq \mathbb{R}^3$ is such that $\sum_{i=1}^3 x_1^2 = 1$, and $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in T_pS^2 \simeq T_p^*S^2$. Explicitely,

$$\omega_p(u, v) = x_1(u_2v_3 - v_2u_3) + x_2(u_3v_1 - u_1v_3) + x_3(u_1v_2 - v_1u_2).$$

On the other hand, we know that $\omega_p = \sum_{i,j} a_{ij} dx_i \wedge dx_j$, where

$$a_{ij} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

In particular, $a_{ii} = 0$ for all i. We can thus calculate

$$\omega_p = x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3$$

Complete the notes for the last half hour of the class.