Add characterization	2
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Add examples and complete notes for the last half hour of the class	Δ

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Note All vector spaces are real and finite dimensional unless otherwise stated.

Definition 0.1 A bilinear form $\Omega: V \times V \to \mathbb{R}$ on a vector space V is a linear symplectic form if it is

- (a) skew-symmetric, i.e. $\Omega(v, w) = -\Omega(w, v) \quad \forall \ v, w \in V;$
- (b) non-degenerate, i.e. $\Omega(v, w) = 0 \quad \forall v \in V \implies w = 0$.

Example 0.2

- (1) Consider $V = \mathbb{R}^2$, $B = (e_1, e_2)$. Then $\Omega_B^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a linear symplectic form on \mathbb{R}^2 .
- (2) Consider $V = \mathbb{R}^{2n}, B = (e_1, \dots, e_n, f_1, \dots, f_n)$. Then

$$[\Omega]_B^B = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

represents a linear symplectic form on \mathbb{R}^{2n} . Moreover, we have that

$$\Omega(e_1, e_j) = 0,
\Omega(f_i, f_j) = 0,
\Omega(e_i, f_j) = \delta_{ij} = -\Omega(f_j, e_i).$$
(0.1)

Definition 0.3 The pair (V, Ω) is a *(real) symplectic vector space*. If it has a basis $B = (e_1, \dots, e_n, f_1, \dots, f_n)$ satisfying the relations given by the equations in (0.1), we say that B is a *symplectic basis* of (V, Ω) .

Remark 0.4 If B is an ordered basis of V, then $[\Omega]_B^B$ is antisymmetric and invertible.

Example 0.5 Let W be a vector space with dual W^* and $V = W \oplus W^*$. Note there is an isomorphism of vector spaces $W \oplus W^* \xrightarrow{\sim} W^* \oplus W, (w, f) \mapsto (f, -w)$. Then,

$$\Omega: V \times V \to R,$$

 $((w_1, f_1), (w_2, f_2)) \mapsto f_2(w_1) - f_1(w_2)$

is a linear symplectic form on V.

¹Whenever order is needed for a basis, I write them as ordered bases (i.e. between parenthesis) instead of unordered sets, and this is important to me!

²The notation $[\Gamma]^{\beta}_{\alpha}$ indicates the matrix representation of the bilinear form Γ which takes column vectors $[w]_{\alpha}$ from the right represented, which are with an ordered basis α , and row vectors $([v]_{\beta})^T$ from the left, which are represented with an ordered basis β and transposed, and computes $\Gamma(v, w)$.

Lemma 0.6 If (V, Ω) is a symplectic vector space, then $2 \mid \dim(V)$.

Proof. If A represents Ω , then

$$det(A) = det(-A^{T})$$

$$= det(-A)$$

$$= (-1)^{\dim(V)} det(A).$$

Definition 0.7 Let (V,Ω) be a symplectic vector space. A linear subspace $U\subseteq V$ is

- (a) symplectic if $\Omega \mid_U$ is a linear symplectic form on U (non-degeneracy is sufficient);
- (b) isotropic if $\Omega \mid_U = 0$;
- (c) coisotropic if $\Omega(u, v) = 0 \quad \forall \ u \in U \implies v \in U$;
- (d) Lagrangian if it is both isotropic and coisotropic.

Example 0.8 Let $V = \mathbb{R}^{2n}, B = (e_1, \dots, e_n, f_1, \dots, f_n)$.

- (1) $\langle \{e_1, f_1\} \rangle$ is a symplectic subspace which is neither isotropic nor coisotropic (thus neither Lagrangian).
- (2) $\langle \{e_1, \ldots, e_n\} \rangle$ is a Lagrangian subspace which is not symplectic.
- (3) $\{0\}$ and $\langle \{e_1, \ldots, \hat{e_k}, \ldots, e_n\} \rangle$, where $\hat{e_k}$ indicates the exclusion of the k-th vector of the canonical ordered basis from the set, are isotropic subspaces which are not coisotropic, where the first one is trivially symplectic.
- (4) If $I, J \subseteq \{1, \dots, n\}$ and $\langle \{e_i\}_{i \in I} \cup \{f_i\}_{i \in J} \rangle$ is
 - symplectic if, and only if, I = J;
 - isotropic if, and only if, $I \cap J = \emptyset$;
 - ullet coisotropic if, and only if,

Add characterization.

• Lagrangian if, and only if, $I = J^c$.

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Remark 0.9 Let V be a real finite dimensional vector space with dual V^* and $\Omega: V \times V \to \mathbb{R}$ a bilinear form. Then, we have the linear isomorphism³

$$\Omega^{\#} = \tilde{\Omega} : V \to V^*,$$

$$v \mapsto \Omega(v, \cdot).$$

 Ω is symplectic if and only if

(1) $\Omega^{\#}$ is antiselfdual, i.e. $(\Omega^{\#})^* = -\Omega^{\#}$, and

³I believe we need to ask for $v \neq 0$, although the professor didn't mention it.

(2) $\Omega^{\#}$ is injective (or, equivalently, an isomorphism).

Definition 0.10 Let (V, Ω) be a symplectic vector space and $U \subseteq V$ a linear subspace. The *symplectic* orthogonal or symplectic annihilator of U is

$$U^{\Omega} := \{ v \in V \mid \Omega(u, v) = 0 \quad \forall \ u \in U \}.$$

Proposition 0.11 Let (V,Ω) be a symplectic vector space and $U\subseteq V$ a linear subspace. Then,

$$\dim(U) + \dim(U^{\Omega}) = \dim(V)$$
 and $(U^{\Omega})^{\Omega} = U$.

Proof. The first part follows from the fact that $\Omega^{\#}$ is an isomorphism. Moreover, the inclusion $U \subseteq (U^{\Omega})^{\Omega}$ follows from Definition 0.10 and the equality follows by noticing their dimensions are equal. \square

Remark 0.12 Let U be a linear subspace of a symplectic vector space (V,Ω) . Then U is

- symplectic if, and only if $V = U \oplus U^{\Omega}$;
- isotropic if, and only if, $U \subseteq U^{\Omega}$;
- coisotropic if, and only if, $U \supseteq U^{\Omega}$;
- Lagrangian if, and only if, $U = U^{\Omega}$.

Exercise 0.13 Prove that

- (1) U is symplectic if, and only if, U^{Ω} is symplectic;
- (2) U is isotropic if, and only if, U^{Ω} is coisotropic;
- $(3) \ (U \cap W)^{\Omega} = U^{\Omega} + W^{\Omega}.$

Proposition 0.14 A symplectic vector space (V,Ω) with $\dim(V)=2n$ has a symplectic basis.

Proof. We will prove this by induction, taking as basis $\dim(V) = 2$. Let $e_1 \neq 0$. Then there exists $v \in V$ such that $\Omega(e_1, v) \neq 0$. By the Gram-Schmidt orthonormalization process, it follows that $\{e_1, \frac{v}{(e_1, v)}\}$ is a symplectic basis.

Assume the Proposition holds for $\dim(V) = 2n$. Let $v, v' \in V$ be such that $\Omega(v, v') \neq 0$. Then $(S := \langle v, v' \rangle, \Omega \mid_S)$ is a symplectic space of dimension 2. Note that $(S^{\Omega}, \Omega \mid_{S^{\Omega}})$ is a symplectic vector space of dimension 2n. Since $V = S \oplus S^{\Omega}$, where both subspaces have symplectic bases due to the induction hypothesis, it follows that (V, Ω) has a symplectic basis.

Remark 0.15 If L is a Lagrangian subspace of a symplectic vector space (V, Ω) , then $\dim(L) = \frac{\dim(V)}{2}$.

Proposition 0.16 (Lagrangian split) Let (V, Ω) be a symplectic vector space. Then, there exist Lagrangian subspaces L, L' such that $V = L' \oplus L'$.

Proof.

Exercise 0.17 Write the proof; the idea is to show that you can find a maximal (with respect to dimension) isotropic subspace.

Remark 0.18 Recall that linear maps $\varphi: V \to V$ induce a map between bilinear forms via

$$\varphi^*(\Omega(v, v')) = \Omega(\varphi(v), \varphi(v')).$$

The Lagrangian split $V = L \oplus L'$ of Proposition 0.16 is canonical in the sense that there exists a canonical isomorphism $\varphi : V \to L \oplus L'$ such that $(\varphi^{-1})^*\Omega$ is the canonical symplectic form on $L \oplus L'$, i.e.

$$\Omega(v_1 + v_1', v_2 + v_2') = \Omega(v_1, v_2') - \Omega(v_2, v_1').$$

Definition 0.19 M is a topological manifold if it is a topological space such that

- $\forall p \in M$, there exists a neighborhood V of p that is homeomorphic to an open set in \mathbb{R}^n ;
- it is Hausdorff, i.e. for any two points we can find a neighbourhood for each such that they are disjoint;
- it satisfies the second countability axiom, i.e. there exists a countable basis.

Example 0.20

(1) The torus $T^2 = S^1 \times S^1$, obtained via the identification

Add identification!

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- (2) The Klien bottle K^2 , obtained via the identification $(x,y) \sim (x+1,y) \sim (1-x,y+1)$ in $[0,1] \times [0,1] \subseteq \mathbb{R}^2$.
- (3) The projective plane, obtained via the identification $(x,y) \sim (x+1,1-y) \sim (1-x,y+1)$ in $[0,1] \times [0,1] \subseteq \mathbb{R}^2$.

Remark 0.21 Recall that M is a differential manifold if it is a topological manifold of dimension n with an atlas, which is a collection of charts $\{U_{\alpha}, \varphi_{\alpha}\}_{{\alpha} \in A}$ such that

- $(1) \ U_{\alpha \in A} \varphi_{\alpha}(U_{\alpha}) = M,$
- (2) $W = \varphi_{\beta}(U_{\beta}) \cap \varphi_{\alpha}(U_{\alpha}) \neq 0$, where $\varphi_{\beta}^{-1}\varphi_{\alpha}$ and $\varphi_{\alpha}\varphi_{\beta}^{-1}$ are of class C^{∞} ,
- (3) $U_{\alpha \in A} \varphi_{\alpha}(U_{\alpha})$ is maximal.

Example 0.22

Add examples and complete notes for the last half hour of the class.

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