- Add identification! . . . 4
- Add examples and complete notes for the last half hour of the class. . . . 4
- Add all the cotangent bundles! . . . 6
- Part of the reminder is missing (she erased it too fast). . . . 7
- Complete the notes for the last half hour of the class. . . . 8
- She also defined the integral curve and the lie derivative, I hope that she reprise them in the next lessons because she went really fast . . . 10
- Add example of the symplectic form given by a torus. . . . 10
- Finish this proof and complete the notes for the last half hour of the class. . . . 12
- Fill in notes for this Monday. . . . 12
- Complete notes from second half of the class. . . . 13
- I'm not really sure that it is an example . . . 14


# Introduction

## 11/3/24

**Note** All vector spaces are real and finite dimensional unless otherwise stated.

**Definition 0.1** A bilinear form $\Omega : V \times V \to \mathbb{R}$ on a vector space $V$ is a *linear symplectic form* if it is

(a) skew-symmetric, i.e. $\Omega(v, w) = -\Omega(w, v) \quad \forall\, v, w \in V$;

(b) non-degenerate, i.e. $\Omega(v, w) = 0 \quad \forall\, v \in V \implies w = 0$.

**Example 0.2**

(1) Consider[1] $V = \mathbb{R}^2$, $B = (e_1, e_2)$. Then[2] $[\Omega]_B^B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ represents a linear symplectic form on $\mathbb{R}^2$.

(2) Consider $V = \mathbb{R}^{2n}$, $B = (e_1, \dots, e_n, f_1, \dots, f_n)$. Then

$$[\Omega]_B^B = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

represents a linear symplectic form on $\mathbb{R}^{2n}$. Moreover, we have that

$$\begin{aligned} \Omega(e_1, e_j) &= 0, \\ \Omega(f_i, f_j) &= 0, \\ \Omega(e_i, f_j) &= \delta_{ij} = -\Omega(f_j, e_i). \end{aligned} \tag{0.1}$$

[1] Whenever order is needed for a basis, I write them as ordered bases (i.e. between parenthesis) instead of unordered sets, and this is important to me!

[2] The notation $[\Gamma]_\alpha^\beta$ indicates the matrix representation of the bilinear form $\Gamma$ which takes column vectors $[w]_\alpha$ from the right represented, which are with an ordered basis $\alpha$, and row vectors $([v]_\beta)^T$ from the left, which are represented with an ordered basis $\beta$ and transposed, and computes $\Gamma(v, w)$.

**Definition 0.3** The pair $(V, \Omega)$ is a *(real) symplectic vector space.* If it has a basis $B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ satisfying the relations given by the equations in (0.1), we say that $B$ is a *symplectic basis* of $(V, \Omega)$.

**Remark 0.4** If $B$ is an ordered basis of $V$, then $[\Omega]_B^B$ is antisymmetric and invertible.

**Example 0.5** Let $W$ be a vector space with dual $W^*$ and $V = W \oplus W^*$. Note there is an isomorphism of vector spaces $W \oplus W^* \xrightarrow{\sim} W^* \oplus W, (w, f) \mapsto (f, -w)$. Then,

$$\begin{gathered}\Omega : V \times V \to R, \\ ((w_1, f_1), (w_2, f_2)) \mapsto f_2(w_1) - f_1(w_2)\end{gathered}$$

is a linear symplectic form on $V$.

**Lemma 0.6** If $(V, \Omega)$ is a symplectic vector space, then $dim(V) \equiv 0 \mod 2$.

*Proof.* If $A$ represents $\Omega$, then

$$\begin{aligned}\det(A) &= \det(-A^T) \\ &= \det(-A) \\ &= (-1)^{\dim(V)}\det(A).\end{aligned}$$

□

**Definition 0.7** Let $(V, \Omega)$ be a symplectic vector space. A linear subspace $U \subseteq V$ is

(a) *symplectic* if $\Omega \mid_U$ is a linear symplectic form on $U$ (non-degeneracy is sufficient);

(b) *isotropic* if $\Omega \mid_U = 0$;

(c) *coisotropic* if $\Omega(u, v) = 0 \quad \forall\, u \in U \implies v \in U$;

(d) *Lagrangian* if it is both isotropic and coisotropic.

**Example 0.8** Let $V = \mathbb{R}^{2n}, B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$.

(1) $\langle\{e_1, f_1\}\rangle$ is a symplectic subspace which is neither isotropic nor coisotropic (thus neither Lagrangian).

(2) $\langle\{e_1, \ldots, e_n\}\rangle$ is a Lagrangian subspace which is not symplectic.

(3) $\{0\}$ and $\langle\{e_1, \ldots, \hat{e_k}, \ldots, e_n\}\rangle$, where $\hat{e_k}$ indicates the exclusion of the $k$-th vector of the canonical ordered basis from the set, are isotropic subspaces which are not coisotropic, where the first one is trivially symplectic.

(4) If $I, J \subseteq \{1, \ldots, n\}$ and $\langle\{e_i\}_{i \in I} \cup \{f_j\}_{j \in J}\rangle$ is

- symplectic if, and only if, $I = J$;
- isotropic if, and only if, $I \cap J = \varnothing$;
- coisotropic if, and only if, $I \cup J = \{1, \ldots, n\}$ [3];
- Lagrangian if, and only if, $I = J^c$.

[3] I'm pretty sure of this characterization. But keep this note until we get any kind of confirmation.

## 12/3/24

**Remark 0.9** Let $V$ be a real finite dimensional vector space with dual $V^*$ and $\Omega : V \times V \to \mathbb{R}$ a bilinear form. Then, we have the linear isomorphism[4]

$$\begin{aligned} \Omega^\# = \tilde{\Omega} : V &\to V^*, \\ v &\mapsto \Omega(v, \cdot). \end{aligned}$$

$\Omega$ is symplectic if and only if

(1) $\Omega^\#$ is antiselfdual, i.e. $(\Omega^\#)^* = -\Omega^\#$, and

(2) $\Omega^\#$ is injective (or, equivalently, an isomorphism).

**Definition 0.10** Let $(V, \Omega)$ be a symplectic vector space and $U \subseteq V$ a linear subspace. The *symplectic orthogonal* or *symplectic annihilator* of $U$ is

$$U^\Omega := \{v \in V \mid \Omega(u, v) = 0 \quad \forall\, u \in U\}.$$

**Proposition 0.11** Let $(V, \Omega)$ be a symplectic vector space and $U \subseteq V$ a linear subspace. Then,

$$\dim(U) + \dim(U^\Omega) = \dim(V) \quad \text{and} \quad (U^\Omega)^\Omega = U.$$

*Proof.* The first part follows from the fact that $\Omega^\#$ is an isomorphism. Moreover, the inclusion $U \subseteq (U^\Omega)^\Omega$ follows from Definition 0.10 and the equality follows by noticing their dimensions are equal. $\square$

**Remark 0.12** Let $U$ be a linear subspace of a symplectic vector space $(V, \Omega)$. Then $U$ is

- symplectic if, and only if $V = U \oplus U^\Omega$;
- isotropic if, and only if, $U \subseteq U^\Omega$;
- coisotropic if, and only if, $U \supseteq U^\Omega$;
- Lagrangian if, and only if, $U = U^\Omega$.

**Exercise 0.13** Prove that

(1) $U$ is symplectic if, and only if, $U^\Omega$ is symplectic;

(2) $U$ is isotropic if, and only if, $U^\Omega$ is coisotropic;

(3) $(U \cap W)^\Omega = U^\Omega + W^\Omega$.

**Proposition 0.14** A symplectic vector space $(V, \Omega)$ with $\dim(V) = 2n$ has a symplectic basis.

*Proof.* We will prove this by induction, taking as basis $\dim(V) = 2$. Let $e_1 \neq 0$. Then there exists $v \in V$ such that $\Omega(e_1, v) \neq 0$. By the Gram-Schmidt orthonormalization process, it follows that $\{e_1, \frac{v}{(e_1, v)}\}$ is a symplectic basis.

Assume the Proposition holds for $\dim(V) = 2n$. Let $v, v' \in V$ be such that $\Omega(v, v') \neq 0$. Then $(S := \langle v, v' \rangle, \Omega\mid_S)$ is a symplectic space of dimension 2. Note that $(S^\Omega, \Omega\mid_{S^\Omega})$ is a symplectic vector space of dimension $2n$. Since $V = S \oplus S^\Omega$, where both subspaces have symplectic bases due to the induction hypothesis, it follows that $(V, \Omega)$ has a symplectic basis. $\square$

[4] I believe we need to ask for $v \neq 0$, although the professor didn't mention it.

**Remark 0.15** If $L$ is a Lagrangian subspace of a symplectic vector space $(V, \Omega)$, then $\dim(L) = \frac{\dim(V)}{2}$.

**Proposition 0.16** (Lagrangian split) Let $(V, \Omega)$ be a symplectic vector space. Then, there exist Lagrangian subspaces $L, L'$ such that $V = L \oplus L'$.

*Proof.*

**Exercise 0.17** Write the proof; the idea is to show that you can find a maximal (with respect to dimension) isotropic subspace.

□

**Remark 0.18** Recall that linear maps $\varphi : V \to V$ induce a map between bilinear forms via

$$\varphi^*(\Omega(v, v')) = \Omega(\varphi(v), \varphi(v')).$$

The Lagrangian split $V = L \oplus L'$ of Proposition 0.16 is canonical in the sense that there exists a canonical isomorphism $\varphi : V \to L \oplus L'$ such that $(\varphi^{-1})^*\Omega$ is the canonical symplectic form on $L \oplus L'$, i.e.

$$\Omega(v_1 + v_1', v_2 + v_2') = \Omega(v_1, v_2') - \Omega(v_2, v_1').$$

**Definition 0.19** $M$ is a *topological manifold* if it is a topological space such that

- $\forall\, p \in M$, there exists a neighborhood $V$ of $p$ that is homeomorphic to an open set in $\mathbb{R}^n$;
- it is Hausdorff, i.e. for any two points we can find a neighbourhood for each such that they are disjoint;
- it satisfies the second countability axiom, i.e. there exists a countable basis.

**Example 0.20**

(1) The torus $T^2 = S^1 \times S^1$, obtained via the identification

Add identification!

.

(2) The Klein bottle $K^2$, obtained via the identification $(x, y) \sim (x+1, y) \sim (1-x, y+1)$ in $[0,1] \times [0,1] \subseteq \mathbb{R}^2$.

(3) The projective plane, obtained via the identification $(x, y) \sim (x+1, 1-y) \sim (1-x, y+1)$ in $[0,1] \times [0,1] \subseteq \mathbb{R}^2$.

**Remark 0.21** Recall that $M$ is a differential manifold if it is a topological manifold of dimension $n$ with an atlas, which is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that

(1) $U_{\alpha \in A} \varphi_\alpha(U_\alpha) = M$,

(2) $W = \varphi_\beta(U_\beta) \cap \varphi_\alpha(U_\alpha) \neq 0$, where $\varphi_\beta^{-1}\varphi_\alpha$ and $\varphi_\alpha\varphi_\beta^{-1}$ are of class $C^\infty$,

(3) $U_{\alpha \in A} \varphi_\alpha(U_\alpha)$ is maximal.

**Example 0.22**

Add examples and complete notes for the last half hour of the class.

## 18/3/24

**Remark 0.23** For every symplectic vector space $(V, \Omega)$ we already saw that we can build a symplectic basis (0.14). Taking each time two elements of that basis and doing the ortogonal we get the following split of $V$:

$$V = W_1 \oplus \cdots \oplus W_n,$$

where $W_i$ are symplectic vector spaces with dimension 2.

**Definition 0.24** Given two symplectic vetor spaces $(V, \Omega)$ and $(V', \Omega')$ we call a *symplectomorphism* a linear isomorphism $\phi : V \to V'$ such that $\phi^\star \Omega' = \Omega$, where $\phi^\star$ is the pullback.

We say that $V$ and $V'$ are symplectomorphic.

**Proposition 0.25** The only global invariant is the dimension.

**Example 0.26**

(1) $\mathbb{R}^2$ and $\mathbb{R}^4$ can not be symplectiomorphic because a symplectomorphism is always a liner isomorphism over the vector space,

(2) Every symplectic vector space $(V, \Omega)$ of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical basis. In fact we can take the symplectic basis of $V$ and than take the ismorphism that sends this basis to the canonical basis of $\mathbb{R}^{2n}$.

**Remark 0.27** Being symplectomorphic is an equivalence relation. It is interesting to see and study the acting group that preserves the structure.

**Definition 0.28** We call $Sp(V, \Omega) = Sp(2n, \mathbb{R})$ the group of symplectic automorphisms of $(V, \Omega)$ of dimension $2n$. It is given by the following subset of $\mathrm{Mat}_{2n\times 2n}$, $\left\{ A : A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) A = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \right\}$.

In fact if $\phi$ is a symplectic automorphism of $\mathbb{R}^{2n}$ with the canonical form and $A$ is the associated matrix we get that:

$$\begin{aligned} v^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) u &= \Omega_0(u, v) \\ &= \phi^\star \Omega_0(u, v) \\ &= \Omega_0(\phi(u), \phi(v)) \\ &= (Av)^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) (Au) \\ &= v^T A^T \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) Au, \end{aligned}$$

but this must be true for each $u$ and each $v$, so $A$ respects the condition that we have imposed.

**Proposition 0.29** $Sp(2n, \mathbb{R})$ is a group with standard multiplication and inverse or quadratic matrices. Also the determinant of $A$ is always 1.

**Remark 0.30** Both $GL$ and $SO$ are groups of automorphisms, the difference is that $SO$ preserves both the lenght of the vectors and the areas.

# Symplectic Manifold

**Definition 0.31** Given a differential manifold $M$ we say that a 2–form $\omega$ is symplectic if it is

(a) closed, i.e. $d\omega = 0$;

(b) non-degenerate, i.e. $\omega(u, v) = 0 \quad \forall\, v \in M \implies u = 0$.

(c) for each $p$ we have that $\omega_p$ is a symplectic form on $T_pM$.

We call the couple $(M, \omega)$ symplectic manifold.

**Proposition 0.32** A symplectic manifold always has even dimension.

**Example 0.33** $(\mathbb{R}^{2n}, \omega_{std})$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. $\omega = \sum_i dx_i \wedge dy_i$.

For each $p$, $\omega_p$ is given by: $\left((\frac{\partial}{\partial x_1})_p, \ldots, (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, \ldots, (\frac{\partial}{\partial y_n})_p\right)$. It is symplectic on $T_pM$.

We can do the same considering $\mathbb{C}^n$ with $\omega = \frac{1}{2}\sum_i dz_i \wedge d\bar{z}_i$.

**Example 0.34** Take now the sphere $S^2$ as a subset of $\mathbb{R}^3$. We know that $T_pS^2$ is given by $\{p\}^\perp$.

Now we define the symplectic form $\omega$ in the following way $\omega_p(u, v) =< p; v \times u >$. In fact it is closed is because it is a 2–form in a dimension 2 manifold and it is non degenerate becuase $v \times u$ always has the direction of $p$ as they are in $T_pS^2$.

This $\omega$ is a volume form for $S^2$.

**Exercise 0.35** The above definition is an implicit one; we can define the same form in coordinates taking the coordinates $\theta, z$. In this case the form can be expressed as $\omega = d\theta \wedge dz$.

Also how will the form change if we take the sphere of radius $R$?

**Definition 0.36** Given two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ we say that a *symplectomorphism* is a diffeomorphism $g$ such that $g^\star\omega_2 = \omega_1$.

As before, the dimension is the only invariant, but now only in a local scope.

Add all the cotangent bundles!

**Proposition 0.37** $S^2$ is the only sphere that can be symplectic, all the others have a trivial second order cohomology group, so they cannot have a symplectic form.

**Proposition 0.38** All the symplectomorphisms preserve the areas but the converse is not true nor a banal proof. It was worthy of a Field's medal.

## 19/03/2024

Last class we considered a manifold $X$ of dimension $n$ and saw that

$$M = T^* := \{(x, \xi^*), x \in X, \xi^* \in T_x^*X\}$$

is a symplectic manifold of dimension 2n. If $(x_1, \ldots, x_n)$ is a coordinate system on $X$, then $\{(dx_1)_x, \ldots, (d_n)_x)\}$ is a basis for $T_x^*X$. Also, $\xi^* = \sum_{i=1}^n y_i(dx_i)_x$ and $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are coordinates on $T^*X$, with

$$\begin{aligned}\omega_{\text{can}} &= \sum_{i=1}^n dx_1 \wedge dy_1,\\ &= \lambda_{\text{can}} = \sum_{i=1}^n y_i dx_i,\\ &= -d\lambda_{\text{can}}.\end{aligned}$$

Part of the reminder is missing (she erased it too fast).

Let $X_1, X_2$ be differentiable manifolds of dimension $n$,

$$\begin{aligned} M_1 &= (T^*X_1, -d\lambda_1), \\ M_2 &= (T^*X_2, -d\lambda_2), \end{aligned}$$

where $\lambda_i$ is the canonical Liouville term on $X_i$ for $i \in \{1, 2\}$. Let $f : X_1 \to X_2$ be a differomorphism. Then

$$\begin{aligned} f_{\mathbb{X}} : M_1 &\to M_2, \\ p_1 &\mapsto p_2, \end{aligned}$$

where $p_1 = (x_1, \xi_1^*)$ and $p_2 = (f(x_1), \xi_2^*)$, where $\xi_2^* \in T^*_{f(x_1)}X_2$. We have that

$$\begin{aligned} df : TX_1 &\to TX_2 \\ df_{x_1} : T_{x_1}X_1 &\to T_{f(x_1)}X_2, \\ (df_{x_1})^* : T^*_{f(x_1)}X_2 &\to T^*_{x_1}X_1. \end{aligned}$$

Since $f$ is in particular a bijection then, defining $x_2 := f(x_1)$, we have that

$$\begin{aligned} df^{-1}_{x_1} : T_{x_2}X_2 &\to T_{x_1}X_1, \\ (df^{-1}_{x_1})^* : T^*_{x_1}X_1 &\to T^*_{x_2}X_2, \\ \xi_1^* &\mapsto (df^{-1}_{x_2})^*(\xi_1^*) =: \xi_2^*. \end{aligned}$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "f_{\mathbb{X}}"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"'] & X_2. \end{tikzcd} \tag{0.2}$$

**Proposition 0.39** Let $f_{\mathbb{X}} : M_1 \to M_2$ be a diffeomorphism be such that $f_{\mathbb{X}}^*(\lambda_2) = \lambda_1$. Then $f_{\mathbb{X}}^*(-d\lambda_2) = -d\lambda_1$.

*Proof.* Let $p_1 = (x_1, \xi_1^*) \in T^*X_1$ and $p_2 = f_{\mathbb{X}}(p_1) = (f(x_1), \xi_2^*)$.

$$\begin{aligned} (f_{\mathbb{X}})^*((\lambda_2)_{p_2}) &= (f_{\mathbb{X}}^*)_{p_1}((d\pi_2)^*_{p_2}\xi_2^*) \\ &= d(f\pi_1)^*_{p_1}\xi_2^* \\ &= (d\pi_1)^*_{p_1} f_{\mathbb{X}}^*(\xi_2^*) \\ &= (d\pi_1)^*_{p_1}\xi_1^* \\ &= (\lambda_1)_{p_1}. \end{aligned}$$

□

**Corollary 0.40** If $X_1$ and $X_2$ are diffeomorphic, then $T^*X_1$ and $T^*X_2$ are symplectomorphic.

**Example 0.41** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds. Then we have a product $M_1 \times M_2$ with projections

$$\begin{tikzcd}
& M_1 \times M_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M_1 & & M_2.
\end{tikzcd}$$

Then, $\omega_{a,b} := ap_1^*\omega_1 + bp_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$ for all $a, b \in \mathbb{R}$. In fact,

$$\begin{aligned}
d\omega_{a,b} &= d(ap_1^*\omega_1 + bp_2^*\omega_2) \\
&= ad(p_1^*\omega_1) + bd(p_2^*\omega_2) \\
&= ap_1^*d\omega_1 + bp_2^*d\omega_2 \\
&= 0,
\end{aligned}$$

i.e. the closeness of $\omega_{a,b}$ is induced from that of $\omega_1$ and $\omega_2$. Similarly, the other properties of induced.

**Example 0.42** $\mathbb{R}^{2n}, \mathbb{C}^n, S^2$. Consider $S^2 \times \mathbb{R}^2$ with the form $\omega = 2p_1^*\omega_1 - p_2^*\omega_2$. On $S^2$, $\omega_p(u, v) = \langle p, u \times v \rangle$, where $p = (x_1, x_2, x_3) \in S^2 \subseteq \mathbb{R}^3$ is such that $\sum_{i=1}^3 x_1^2 = 1$, and $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in T_pS^2 \simeq T_p^*S^2$. Explicitely,

$$\omega_p(u, v) = x_1(u_2v_3 - v_2u_3) + x_2(u_3v_1 - u_1v_3) + x_3(u_1v_2 - v_1u_2).$$

On the other hand, we know that $\omega_p = \sum_{i,j} a_{ij} dx_i \wedge dx_j$, where

$$a_{ij} = \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

In particular, $a_{ii} = 0$ for all $i$. We can thus calculate

$$\omega_p = x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3$$

Complete the notes for the last half hour of the class.

## 25/03/2024

**Remark 0.43** Given two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ we can endorse the product $M_1 \times M_2$ with a symplectic form $\omega_{ab}$ of the form:

$$\omega_{ab} = ap_1^*\omega_1 + bp_2^*\omega_2,$$

where $p_i$ is the projection over $M_i$ and $a, b \in \mathbb{R} \setminus \{0\}$.

In fact fow each $u = (u_1, u_2), v = (v_1, v_2)$ we have that:

$$\begin{aligned}
\omega_{a,b}(u, v) &= ap_1^*\omega_1(u, v) + bp_2^*\omega_2(u, v) \\
&= a\omega_1(u_1, v_1) + b\omega_2(u_2, v_2),
\end{aligned}$$

but if we suppose that for each $v$ we have $= 0$ than choosing $v = (v_1, 0)$ we get $u_1 = 0$ for the non–degeneracy of $\omega_1$ and vice versa for $u_2$ but if and only if $a \neq 0$ and $b \neq 0$.

**Definition 0.44** Given a symplectic manifold $(M, \omega)$ and a submanifold $W$, we say that $W$ is symplectic if and only if $\omega_{|W}$ is a symplectic form, and then $\dim W$ is even.

**Remark 0.45** This condition pass to the tangent space, $T_pW$ si a symplectic subvector space of $T_pM$ for each point $p$.

**Definition 0.46** Given a symplectic manifold $(M, \omega)$ and a submanifold $L$, we say that $L$ is lagrangian if and only if one of the following three equivalent characterization is true:

(a) for each point $p$ $T_pL \subset T_pM$ is lagrangian as vector space;

(b) $\omega_{|L} \equiv 0$ and $\dim L = \frac{\dim M}{2}$;

(c) $i^*\omega \equiv 0$ and $\dim L = \frac{\dim M}{2}$.

**Exercise 0.47** Take $\mathbb{R}^{2n}$ with form $\omega = \sum_{i \neq j} dx_i \wedge dx_j$ then this is not a symplectic manifold because $\omega$ degenerates.

**Example 0.48** Take the cotangent bundle $T^\star M$ with the canonical form, what are its lagrangian submanifold?

We can take $\mathbb{R}^{2n}$ as $T^\star \mathbb{R}^n$ and the standard form is eqivalent to the Liouville ones, the $-$ is only because the order of the element in the basis is used different inside the two forms.

In general we have, as lagrangian submanifolds:

(1) the zero section $\{(m, 0)\}$;

(2) the fibers $T_xM$;

(3) some other sections, in particular all the ones such that the relative 1–form $\mu$ is a closed one, so $d\mu = 0$.

**Example 0.49** Take now the sphere $S^2$ with the standard form. Its lagrangian submanifolds are given by the equator, the meridians and the parallels. For this last one we have to see them like the earth, so line that connects the north pole to the south pole, they will fix $\theta$.

The symplectic structure breaks something in the symmetry of the sphere, only the meridians are lagrangian, while all the other geodetics are not.

**Proposition 0.50** Take a diffeomorphism $\phi : m_1 \to M_2$, it is a symplectomorphism if and only if the graph $\Gamma_\phi$ is a lagrangian submanifold inside $(M_1 \times M_2, \omega_{1,-1})$.

*Proof.* Let be $\gamma : M_1 \to M_1 \times M_2$ the map that sends $x$ into $(x, \phi(x))$. now we have:

$$\begin{aligned}
\Gamma_\phi \text{ is lagrangian} &\iff \gamma^*\omega_{1,-1} = 0 \\
&\iff \gamma^* p_1^* \omega_1 - \gamma^* p_2^* \omega_2 = 0 \\
&\iff (p_1 \circ \gamma)^* \omega_1 - (p_2 \circ \gamma)^* \omega_2 = 0 \\
&\iff \forall x \in M_1 \ \omega_1(x) - \omega_2(\phi(x)) = 0 \\
&\iff \omega_1 = \phi^* \omega_2 \\
&\iff \phi \text{ is a symplectomorphism.}
\end{aligned}$$

□

**Example 0.51** Take $\{x^0\} \times M_2 = W_2$ inside $M_1 \times M_2$, it is lagrangian? First of all we ahve to suppose $\dim M_1 = \dim M_2$ but it is not sufficient. For example $\mathbb{R}^2 \times \mathbb{R}^2$ with form $\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2$, if we take $\{(x_1^0, x_2^0)\} \times \mathbb{R}^2$ we get a symplectic submanifold.

**Example 0.52** Take $M$ as the product of $n$ copies of $S^2$ end use $E_1$ to denote the equator, as form we take $\omega = \sum_i a_1 p_i^* \omega_i$ where $\omega_i$ is the standard form on the $i$–th sphere.

Each sphere is a symplectic submanifold, once we fix any point on the other, the same is true for each families of sphere, fixing a pint in each sphere that we want to eliminate.

If we take the product of all the equators we get a lagrangian submanifold. Taking only some equators, like two, and fixing a point in the other sphere will give us a submanifold that is nor symplectic nor lagrangian for a dimension problem.

**Definition 0.53** Given a manifold $M$ and a function $\psi : M \times \mathbb{R} \to M$ we can define the family $\{\psi_t\}$ as $\psi_t(p) = \psi(p, t)$. We will call it an isotopy if they are all diffeomorphisms and $\psi_0$ is the identity over $M$.

From an isotopy we get a family of vector field fiven by $\frac{d\psi_t}{dt} = X_t \circ \psi_t$. Also if $M$ is compact, then we can define a family of vector field $X_t$ with compact support and find an isotopy that respects the above condition.

If $X_t = X$ we get the flow of $X$.

She also defined the integral curve and the lie derivative, I hope that she reprise them in the next lessons because she went really fast

## 25/3/24

If $S \subseteq \mathbb{R}^3$ is an orientable surface then $\omega \in \Omega^2(M)$ is the volume form. $\omega$ is symplectic in $M$.

**Example 0.54**

Add example of the symplectic form given by a torus.

**Theorem 0.55** Let $M$ be a symplectic manifold and $\{\psi_t\}$ an isotopy. Let $\omega_t, t \in \mathbb{R}$ be a family of $d$-forms that depends mostly on $t$ Then

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\mathcal{L}_{x_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

*Proof.* Not given. □

**Theorem 0.56** (Moser) Let $\omega_t$ be a family of symplectic forms such that $\frac{d}{dt}\omega_t = d\omega_t$ (note that $\frac{d}{dt}[\omega_t] = 0$). Then there exists a family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ such that $\psi_t^*\omega_t = \omega_0$.

*Proof.* Not given. □

**Example 0.57** Consider the compact manifold $S^2$ with the symplectic forms $\omega_0 = d\theta \wedge dz$ and $\omega_1 = rd\theta \wedge dz$. Then $[\omega_0] = [\omega_1]$. If we define

$$\begin{aligned} \omega_t &= (1-t)d\theta \wedge dz + trd\theta \wedge dz \\ &= (1-t+tr)d\theta \wedge dz \end{aligned}$$

for every $t \in [0, 1]$. We want a a family of functions $\psi_t : S^2 \to S^2$ such that $\psi_t^*\omega_t = \omega_0$, i.e.

$$\psi_t^*\omega_t(u, v) = \omega_0(d\psi_1(u), d\psi_t(v)),$$

for all $t \in [0, 1]$.

**Lemma 0.58** Let $M^{2n}$ be a compact manifold. Lat $\omega_0, \omega_1$ be symplectic forms on $M$ such that $[\omega_0] = [\omega_1]$. Assume that the 2-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for all $t \in [0, 1]$. Then, there exists an isotopy $\{\psi_t\}$ such that $\psi_t^*\omega_t = \omega_0$ for every $t \in [0, 1]$.

*Proof.* Not given. □

**Theorem 0.59** (Darboux) Every symplectic form on $M^{2n}$ can be deformed to the extended symplectic form $\omega_0$ on $\mathbb{R}^{2n}$. Equivalently, if $(M^{2n}, \omega)$ is a symplectic manifold, then every point $p \in M$ one can define a local chart centered at $p$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

*Proof.* Not given (for now?). □

**Remark 0.60** It follows from Theorem 0.59 that the only **local** symplectic invariant is the dimension.

**Proposition 0.61** Let $Q$ be a submanifold of a symplectic manifold $M$ and let $\omega_0, \omega_1$ be two symplectic forms on $M$ such that $\omega_0 \mid_Q = \omega_1 \mid_Q$. Then there exist two neighbourhoods $N_0, N_1$ of $Q$ such that there exists a differomorphism $\psi : N_0 \to N_1$ such that $\psi_0 = \mathrm{Id}$ and $\psi^* \omega_1 = \omega_0$.

## Complex structures

**Definition 0.62**

(a) A *complex structure* on a vector space $V$ is a linear operator $J : V \to V$ such that $J^2 = -\mathrm{Id}$.

(b) A complex structure $J$ on symplectic a vector space $(V, \Omega)$ is *compatible* with $\Omega$ if

$$G(u, v) := \Omega(u, Jv) \quad \forall\, u, v \in V$$

is an inner product in $V$.

**Example 0.63** For $(\mathbb{R}^2, \omega = dz \wedge \overline{d}z)$, $J : \mathbb{C} \to \mathbb{C}, u \mapsto iu$ is a compatible complex structure.

**Note** A complex structure $J$ is compatible with a symplectic a symplectic structure $\Omega$ on a symplectic vector space $V$ if, and only if, $J$ is a symplectomorphism. This follows from the fact that, for all $u, v \in V$,

$$\begin{aligned} J^*\Omega(u, v) &= \Omega(Ju, Jv) \\ &= G(Ju, v) \\ &= G(v, Ju) \\ &= \Omega(v, JJu) \\ &= \Omega(v, -u) \\ &= \Omega(u, v). \end{aligned}$$

**Proposition 0.64** Let $(V, \Omega)$ be a symplectic vector space. Then there exists a complex structure $J$ compatible with $\Omega$. Moreover, given an inner product $\langle \cdot, - \rangle$ on $V$ we can construct a complex structure $J$ on $V$ that is compatible with $\Omega$.

*Proof.* Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a symplectic basis. Define $J_{e_i} = f_i$, $J_{f_i} = -e_i$ and the maps

$$\begin{aligned} \phi_1 : V &\to V^* \\ v &\mapsto \Omega(v, -), \end{aligned}$$

$$\begin{aligned} \phi_2 : V &\to V^* \\ v &\mapsto \langle v, - \rangle, \end{aligned}$$

$$\begin{aligned} A : V &\to V, \\ v &\mapsto (\phi_2^{-1} \phi_1) v. \end{aligned}$$

Note that, for all $u, v \in V$,

$$\begin{aligned}
\langle Au, v\rangle &= \langle (\phi_2^{-1}\phi_1)(u), v\rangle \\
&= \Big(\phi_2\big((\phi_2^{-1}\phi_1)(u)\big)\Big)(v) \\
&= (\phi_1(u))(v) \\
&= \Omega(u, v);
\end{aligned}$$

$$\begin{aligned}
\langle A^*u, v\rangle &= \langle u, Av\rangle \\
&= \langle Av, u\rangle && \text{(Since our vector space is real.)} \\
&= \Omega(v, u) \\
&= -\Omega(u, v) \\
&= -\langle Au, v\rangle \\
&= \langle -Au, v\rangle.
\end{aligned}$$

Thus, $A^* = -A$ and $(A^*)^* = A$.

Finish this proof and complete the notes for the last half hour of the class.

$\square$

## 8/4/24

Fill in notes for this Monday.

## 9/4/24

**Definition 0.65** Let $(M, \omega)$ be a symplectic manifold. The *Poisson bracket* of $f, g \in \mathbb{C}^\infty(M, \mathbb{R})$ is

$$\{f, g\} := \omega(X_f, X_g).$$

In particular, $X_{\{f,g\}} = -[X_f, X_g]$.

**Theorem 0.66** The Poisson bracket satisfies the Jacobi identity.

*Proof.* $\square$

**Definition 0.67** A *Poisson algebra* $(P, \{\cdot, -\})$ is a commutative associative algebra equipped with a bracket $\{\cdot, -\}$ that satisfies Leibniz's rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

**Example 0.68** If $(M, \omega)$ is a symplectic manifold, then $(C^\infty(M, \mathbb{R}), \{\cdot, -\})$ is a Poisson algebra. With respect to the map

$$\begin{aligned}
C^\infty(M, \mathbb{R}) &\to \chi(M), \\
H &\mapsto X_H,
\end{aligned}$$

the Poisson bracket $\{\cdot, -\}$ "transforms into" $-[\cdot, -]$, as noted in Definition 0.65.

**Definition 0.69** A *Hamiltonian system* is a triple $(M, \omega, H)$ where

1. $(M, \omega)$ is a symplectic manifold,
2. $H \in C^\infty(M, \mathbb{R})$, called the *Hamiltonian function*.

**Example 0.70** $(S^2, \omega, H(0, z) = z)$ is a Hamiltonian system.

**Proposition 0.71** Let $(M, \omega, H)$ be a Hamiltonian system. Then $\{f, H\} = 0$ if, and only if, $f$ is constant along integral curves of $X_H$.

*Proof.* Note that

$$\begin{aligned}
\frac{d}{dt}(f\phi_H^t) &= \frac{d}{dt}((\phi_H^t)^* f) \\
&= (\phi_H^t)^* \mathcal{L}_{X_H} f \\
&= (\phi_H^t)^* df(X_H) \\
&= (\phi_H^t)^* (t_{X_f}\omega(X_H)) \\
&= (\phi_H^t)^* \omega(X_f, X_H) \\
&= (\phi_H^t)^* \{f, H\}.
\end{aligned}$$

□

**Definition 0.72** Let $(M, \omega, H)$ be a Hamiltonian system.

(a) A *first integral* of the Hamiltonian system is a function $f$ such that $\{f, H\} = 0$.

(b) Functions $f_1, \ldots, f_n \in C^\infty(M)$ are *independent* if $(df_1)_p, \ldots, (df_n)_p$ are linearly independent for all $p$ in a open dense subset of $M$.

(c) $(M, \omega, H)$ is (*completely*) *integrable* if it has $\frac{\dim(M)}{2}$ independent first integrals $f_1 = H, f_2, \ldots, f_n$ such that $\{f_i, f_j\} = 0$ for all $i, j \in \{1, \ldots, n\}$.

**Note** Let $(M, \omega, H)$ be an integrable Hamiltonian system with first integrals $f_1 = H, f_2, \ldots, f_n$, $f = (f_1, \ldots, f_n)$ and $f$ be a regular value for $f$. Then $f^{-1}(c)$ is a Lagrangian submanifold of $M$.

Complete notes from second half of the class.

## 16/04/2024

**Example 0.73** Let $M = \mathbb{CP}^m = \mathbb{P}(\mathbb{C}^{n+1})$, if we take $n = 1$ we get $\mathbb{C}^2 \cong \mathbb{R}^4$ and we can take the sphere $S^3$ inside. It is not symplectic but it is contained in $\mathbb{R}^4$ that is symplectic. On the sphere we consider the standard equivalence relation for getting the projective space as a quotient, in this manner the equivalence class of a point is given by two antipodal points on the sphere.

Now if we see the inclusion of the sphere and its projection, both with the standard symplectic form, we get the Fubini–Study 2–form that it is symplectic.

**Theorem 0.74** (Morden–Werenstain reduction theorem) $\omega_{FS}$ is symplectic and $i^\star\omega_{std} = p^\star\omega_{ps}$, where $i$ is the inclusion of the sphere, $p$ is the projection map given by the quotient and the other form are the standard ones.

*Sketch.* We take a point $z \in S^{2n-1}$, then the tangent space $T_z S^{2n-1} = T_{p(z)}\mathbb{P}^{n-1} \oplus \langle z \rangle$. So $di^\star\omega_{std} = dp^\star\omega_{ps}$, but $d$ commutes and we get $d\omega_{FS} = 0$. □

## Integrable System

**Definition 0.75** Let $(M, \omega, H)$ and Hamiltonian system, we will call *first integral* a function $f \in C^\infty(M, \mathbb{R})$ such that the Poisson bracket with $H$ are 0: $\{f, H\} = 0 = \omega(X_f, X_H)$.

**Definition 0.76** An Hamiltonian system is called *completely integrable* if exist $n$ function, where the dimension of $M$ is $2n$, such that one of this is equal to $H$ and they are all indipendent, so $\{f_i, f_j\} = 0$.

We can define a local action of $\mathbb{R}^n$ over $M$ as $t \cdot p = \phi_n^{t_n} \circ \circ \phi_1^{t_1}(p)$ where $\phi_i$ is the flow of $X_{f_i}$.

**Proposition 0.77** Thanks to the fact that the elements commutes we have that the action is locally free.

**Example 0.78**

I'm not really sure that it is an example

We want to study the $C$–connected components of a regular level set. As first observation we see that $\mathbb{R}^2$ acts on $C$ with discrete stabilizer $\{0\} \times 2^k$.

Let $(M, \omega, H)$ a completely integrable Hamiltonian system and let $f = (f_1, \dots, f_n)$. Take now a point $q$ that is a regular value for $f$ and $U$ a open neighbourhood of $q$ composed only by regular value. It exists $V$ such that $f(\overline{V}) \subset U$ compact. If we denote with $g$ the restricition of $f$ to $V$ we get that it has value in $U \times F_q$ where $F_q$ is a compact connected component of $ff^{-1}(q)$ if $U$ and $V$ are sufficientely small.

## 17/04/2024

## 30/04/2024

**Definition 0.79** A *Lie group* is a group $G$ and a manifold such that the multiplication and the inverse are smooth maps.

**Example 0.80** The following are Lie groups[5]:

1. $(\mathbb{R}^n, +)$,
2. $(\mathbb{R} \setminus \{0\}, \cdot)$,
3. $(\mathbb{R}_+, \cdot)$,
4. $(S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \cdot)$,
5. $\mathrm{GL}(n, \mathbb{R})$, where the smooth structure is induced when considering $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$;
6. $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) \mid A(\overline{A})^t = \mathrm{Id}\} = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \mid |\alpha|^2 + |\beta|^2 = 1, \alpha\beta \in \mathbb{C} \right\}$;
7. $\mathrm{SL}(n, \mathbb{R})$;
8. $\mathrm{O}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid AA^t = \mathrm{Id}\}$;
9. $\mathrm{SO}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid AA^t = \mathrm{Id}, \det(A) = 1\}$;
10. $\mathrm{U}(n, \mathbb{C}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid AA^t = \mathrm{Id}\}$;
11. $\mathrm{SU}(n, \mathbb{C}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid AA^t = \mathrm{Id}, \det(A) = 1\}$;
12. $\mathrm{Sp}(2n, \mathbb{R}) = \{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{R}) \mid M^t \Omega M = \Omega\}$, where $\Omega = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$.

[5]For sets of matrices, the group operation is the corresponding matrix multiplication.

**Definition 0.81** Let $G$ be a Lie group. A subgroup $H \leq G$ is a *Lie subgroup* if it is also a submanifold of $G$.

**Note** Any closed subgroup of a Lie group is a Lie subgroup.