NENG 685
HW 5

Fall 2017
Due Dec. 4, 2017

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On homework:

- Well organized and documented work scores better. If I cannot figure out what is going on, then I am less likely to "intuit" what you intended, and the score will be reflective of this fact.
- If you work with anyone else, document what you worked on together.
- Show your work.
- Always clearly label plots (axis labels, a title, and a legend if applicable).
- Homework should be done "by hand" (i.e. not with a numerical program such as MATLAB, Python, or Wolfram Alpha) unless otherwise specified. You may use a numerical program to check your work.
- If you use a numerical program to solve a problem, submit the associated code, input, and output.
- ***I should not have to run your code to see your answers.*** The attached code is an additional form of feedback for me and a method to give partial credit. If you want full credit, then include the outputs (plots, tables, answers, etc.) in your write-up.

Do not write in the table to the right.

Problem	Points	Score
1	5	4
2	25	21.5
3	20	16
4	10	10
5	20	20
6	15	13
Total:	95	89.5

+5

-1pt: What is the necessary cond for convergence? Also, x-sec & spectral radius can be related but are not synonymous

1. (5 points) Discuss the significance of the spectral radius for the iterative solution of $\mathbf{A}\vec{x} = \vec{b}$, including how it is used to determine convergence and how it is related to rate of convergence. In highly scattering system such as 41 thermal upscattering groups, we can use their cross sections (spectral radius) to find how different iterative methods converge (fast? vs. slow?) when we define the iterative error: $\vec{e}^{(k)} = \vec{x}^{(k)} - \vec{x}$, we can determine convergence: $\vec{e}^{(k+1)} = \vec{P}\vec{e}^{(k)}$

$\lim_{k \to \infty} \|P^k\|^{1/k} = \rho(P)$

$\|P^k\| \approx \rho^k(P)$

spectral radius relates to convergence!

2. (25 points) We will use the following system of n equations:

$$\mathbf{A}\vec{x} \equiv \begin{pmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ \vdots \\ 100 \end{pmatrix} \equiv \vec{b}$$

Write a program to implement the

(a) Jacobi method : Code attached on Github

(b) Gauss Seidel method : 〃

(c) SOR method : 〃

-2pts: No run instructions found

for a matrix with n unknowns. Turn in your source code electronically; include instructions for how to run it, input files, etc. if necessary.

Solve the above system of equations with each program.

Use $\omega = 1.15$ for SOR; use $\vec{x}^{(0)} = \vec{0}$ and $n = 5$.

Print the solution vector from each method converged to an **absolute** tolerance of 10^{-6}.

sol. attached in the back

Indicate the final error and the number of iterations required to meet this tolerance for each method.

Sol attached in the back

3. (20 points) Use the programs you just wrote with the same matrix and using the same settings to answer the following.

-2pts: no abs error; -2pts: 1E-8 should increase num iterations; cannot tell from code where error is

(a) (10 points) How many iterations are required for each method to reach the stopping criterion (*relative error*):

$$\frac{||x^{(k+1)} - x^{(k)}||}{||x^{(k+1)}||} < \epsilon$$

	$\epsilon = 10^{-6}$	$\epsilon = 10^{-8}$
Jacobi	25 iterations	25 iter'
GS	14 "	14 "
SOR	9 "	9 "

for $\epsilon = 10^{-6}$ and $\epsilon = 10^{-8}$?
Also:

- For each method, how does the number of iterations using the absolute error (from the previous question) with $\epsilon = 10^{-6}$ compare to the relative error?
- Which method required the fewest iterations? SOR as seen in the table above.
- What do you observe about reaching a tighter convergence tolerance?

i.e.) For SOR, Convergence Tol = 0.1 → 3 iterations; con.tol. = 1E-10 → 14 iteration: tighter conTol, more iterati

(b) (10 points) Perform an experiment to determine ω_{opt} for SOR. Explain your procedure and include the results.

ω is the value that speeds up the convergence rate.
Whatever the ω value that gets me less # of iterations would be my ωopt.
1.102 ≤ ωopt ≤ 1.153, ω values in this range got me iteration # = 9 which is the smallest value that I could get h

4. (10 points) Harness your knowledge from your differential equations class to analytically solve the fixed-source diffusion equation (assuming D and Σ_a are constant):

$$-\underbrace{D}_{\text{const}}\frac{d}{dx}\frac{d\phi(x)}{dx} + \underbrace{\Sigma_a}_{\text{const}}\phi(x) = S(x)$$

Boundary Conditions: $\phi(\pm a) = 0$

in the following situations:

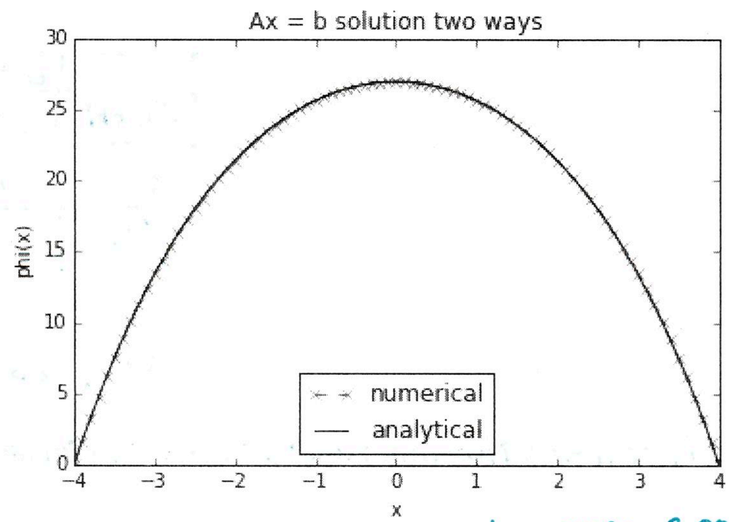
$S(x) = S_0$ (a constant) for $x \in [-a, a]$

Hint: You can deduce an additional boundary condition that may make things slightly simpler. Solution attached in the back

5. (20 points) Numerically solve the fixed-source diffusion equation as described in Question 4 using the finite *difference method* for discretization of the spatial variable and Gaussian elimination (a.k.a. the Thomas algorithm; note that you will have a tridiagonal system to solve) for solving the system of linear algebraic equations.

Use the following parameters: Code attached on Github.

- $a = 4$ cm,
- $D = 1$ cm,
- $\Sigma_a = 0.2$ cm^{-1},
- $S = 8$ n/(cm^3 s), and
- $h = 0.1$ cm.



• Max abs error: 0.0018335
• Max relative error: 9.8427E−5

Plot the solution from $x = -a$ to $x = a$. Compare your answer (in terms of max error) to your solution from Question 4.

6. (15 points) Investigate how well your numerical solution approximates the analytical solution by computing ϕ_i for various constant mesh sizes: h = *1 cm, 0.5 cm, 0.1 cm, 0.05 cm, 0.01 cm.*

-2pts: No order of convergence

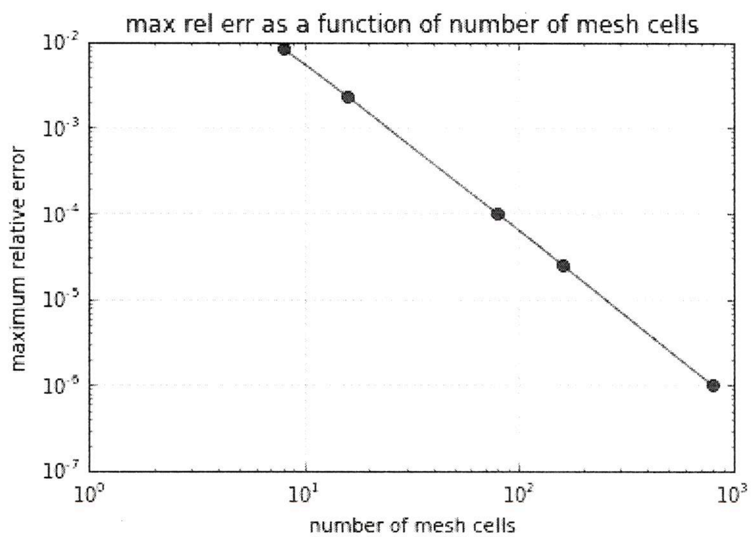
For each mesh length calculate the relative error between your numerical and analytical solutions. Plot the maximum relative error as a function of total number of meshes for each case. What can you conclude about the relationship between the maximum error and the total number of meshes? What is the order of convergence?

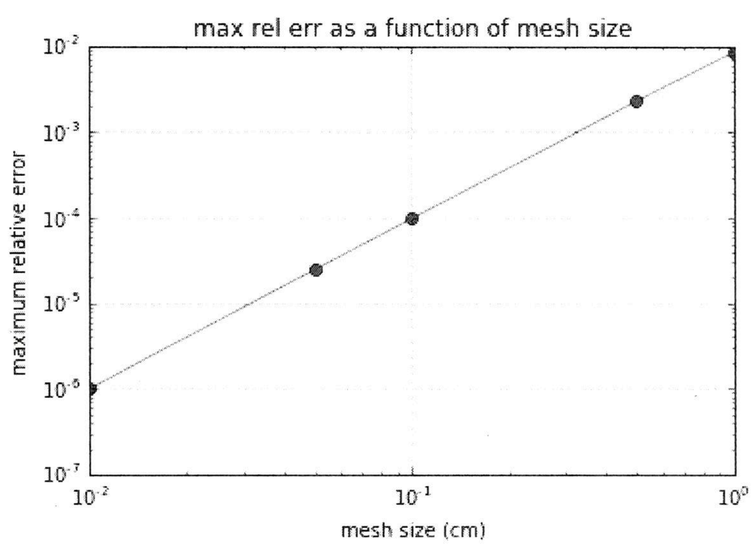
Rel error for each "h" calculated @ attachment
plots & results are attached in the back

BONUS (5 points): submit your code by providing read/clone access to an online version control repository where your code is stored (e.g. github or bitbucket).

NOTE: If you are unsure if your code is working properly you can check with me before submitting as that is a big part of this homework.

As seen in the plot 2, Max rel error vs number of mesh cells, we get less rel error (more accuracy to the solution) as the number of mesh cells increase.


max rel err as a function of number of mesh cells
maximum relative error
number of mesh cells
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{0}
10^{1}
10^{2}
10^{3}

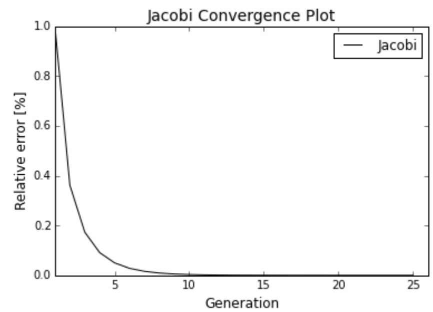

max rel err as a function of mesh size
maximum relative error
mesh size (cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-2}
10^{-1}
10^{0}

Problem 2

Jacobi result

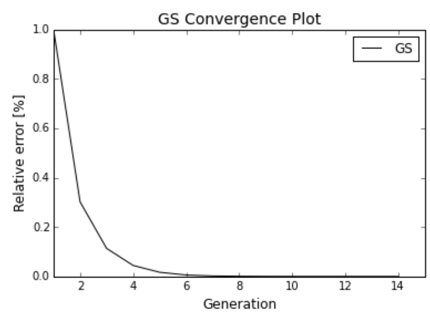
```
Graphite = 883.305065279
Heavy Water = 23023.5482681
Iron = 9.37872608635
The Jacobi Method took 25 iterations and the solution was:
[ 61.11105884  83.33323925  88.88878435  83.33323925  61.11105884]
Error:
[  4.18150316e-05   6.27225474e-05   8.36300633e-05   6.27225474e-05
   4.18150316e-05]
```

-1.5pts: errors should be single value



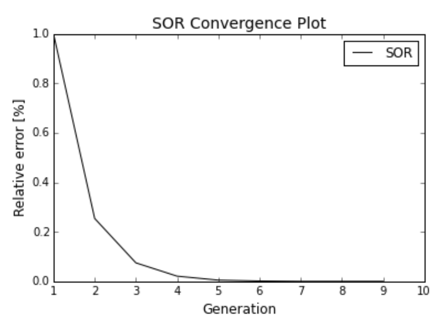
GS result

```
Graphite = 883.305065279
Heavy Water = 23023.5482681
Iron = 9.37872608635
The GS Method took 14 iterations and the solution was:
[ 61.11106233  83.33328455  88.88885637  83.33331707  61.11110569]
Error:
[  9.75677252e-05   9.75681798e-05   6.50456047e-05   3.25228276e-05
   1.08409425e-05]
```



SOR result

```
Graphite = 883.305065279
Heavy Water = 23023.5482681
Iron = 9.37872608635
The SOR Method took 9 iterations and the solution was:
[ 61.1110834   83.33330959  88.88889617  83.33333525  61.11111151]
Error:
[  8.52847512e-05   9.27671331e-05   8.71049210e-05  -9.59775673e-06
  -1.81054948e-06]
```



4 Diffusion equation.

$$-D\frac{d}{dx}\frac{d\phi}{dx}+\Sigma_a\phi=S_0 \rightarrow -D\frac{d^2\phi}{dx^2}+\Sigma_a\phi=S_0$$

$$\rightarrow \frac{d^2\phi}{dx^2}-\frac{\Sigma_a\phi}{D}=-\frac{S_0}{D}$$

Homogeneous solution: if $S_0=0=D$

$$\phi=Ae^{-x\sqrt{\frac{\Sigma_a}{D}}}+Be^{x\sqrt{\frac{\Sigma_a}{D}}}$$

particular solution:

$$\phi=-\frac{S_0}{D}\left(-\frac{D}{\Sigma_a}\right)=\frac{S_0}{\Sigma_a} \quad \therefore \frac{d^2\phi}{dx^2}=0$$

General solution:

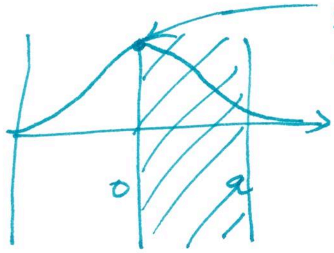
$$\phi(x)=Ae^{-x\sqrt{\frac{\Sigma_a}{D}}}+Be^{x\sqrt{\frac{\Sigma_a}{D}}}+\frac{S_0}{\Sigma_a}$$

Apply Boundary Conditions.

$$\phi(x=\pm a)=0 \xrightarrow{x=a} Ae^{-a\sqrt{\frac{\Sigma_a}{D}}}+Be^{a\sqrt{\frac{\Sigma_a}{D}}}+\frac{S_0}{\Sigma_a}=0$$

$$\xrightarrow{x=-a} Ae^{a\sqrt{\frac{\Sigma_a}{D}}}+Be^{-a\sqrt{\frac{\Sigma_a}{D}}}+\frac{S_0}{\Sigma_a}=0$$

By symmetry @ the B.Cs



$$\frac{d\phi}{dx}(x=0)=0$$

$$\frac{d\phi(x)}{dx}=-\sqrt{\frac{\Sigma_a}{D}}Ae^{-x\sqrt{\frac{\Sigma_a}{D}}}+\sqrt{\frac{\Sigma_a}{D}}Be^{x\sqrt{\frac{\Sigma_a}{D}}}$$

$$\frac{d\phi(x=0)}{dx}=-\sqrt{\frac{\Sigma_a}{D}}A+\sqrt{\frac{\Sigma_a}{D}}B=0$$

$$\rightarrow A=B$$

using this relationship,

$$\phi(x=a)=Ae^{-a\sqrt{\frac{\Sigma_a}{D}}}+Ae^{a\sqrt{\frac{\Sigma_a}{D}}}+\frac{S_0}{\Sigma_a}=0$$

$$\hookrightarrow A\left(e^{-a\sqrt{\frac{\Sigma_a}{D}}}+e^{a\sqrt{\frac{\Sigma_a}{D}}}\right)=-\frac{S_0}{\Sigma_a}$$

$$\hookrightarrow A=\frac{-S_0}{\Sigma_a\left(e^{-a\sqrt{\frac{\Sigma_a}{D}}}+e^{a\sqrt{\frac{\Sigma_a}{D}}}\right)}=B$$

Plug A, B into general solution,

$$\phi(x)=-\frac{S_0}{\Sigma_a}\frac{e^{-x\sqrt{\frac{\Sigma_a}{D}}}}{\left(e^{-a\sqrt{\frac{\Sigma_a}{D}}}+e^{a\sqrt{\frac{\Sigma_a}{D}}}\right)}-\frac{S_0}{\Sigma_a}\frac{e^{x\sqrt{\frac{\Sigma_a}{D}}}}{\left(e^{-a\sqrt{\frac{\Sigma_a}{D}}}+e^{a\sqrt{\frac{\Sigma_a}{D}}}\right)}+\frac{S_0}{\Sigma_a}$$

$$=\frac{S_0}{\Sigma_a}\left(1-\frac{\left(e^{-x\sqrt{\frac{\Sigma_a}{D}}}+e^{x\sqrt{\frac{\Sigma_a}{D}}}\right)}{\left(e^{-a\sqrt{\frac{\Sigma_a}{D}}}+e^{a\sqrt{\frac{\Sigma_a}{D}}}\right)}\right)=\frac{S_0}{\Sigma_a}\left(1-\frac{\left(e^{-x/L}+e^{x/L}\right)}{\left(e^{-a/L}+e^{a/L}\right)}\right) \quad \therefore L=\sqrt{\frac{D}{\Sigma_a}}$$

Problem 5 and 6,

```
('The Maxium Absolute Error is:', 0.0018335021879174462)
('The Maxium Relative Error is:', 9.84273962707844e-05)
('The Maxium Relative Error (while h=1cm) is:', 0.0084466017910521489)
('The Maxium Relative Error (while h=0.5cm) is:', 0.0023028644800499507)
('The Maxium Relative Error (while h=0.1cm) is:', 9.84273962707844e-05)
('The Maxium Relative Error (while h=0.1cm) is:', 2.480436558546421e-05)
('The Maxium Relative Error (while h=0.01cm) is:', 9.9848816345736518e-07)
```