MSD-space: Visualizing the Inner-Workings of TOPSIS Aggregations

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Abstract

TOPSIS is a popular approach to creating rankings of alternatives characterized by multiple criteria. Over the decades, numerous versions and modifications of the method have been proposed. Nevertheless, the core of TOPSIS, based on calculating and aggregating distances to ideal and anti-ideal alternatives, remains unchanged. This paper aims to describe the inner algebraic aspects of this core, revealing important dependencies between the calculated distances and the mean and standard deviation of the alternative. To visualize the effect of these dependencies on different TOPSIS aggregations, we introduce a new space based on the mean (M) and standard deviation (SD), called MSD-space. MSD-space is a practical tool for comparing aggregations and visualizing the effects that changes to the values of criteria can have on the resulting ratings of alternatives. The advantage of MSD-space is that it can always be successfully illustrated in a plane regardless of the number of criteria describing the alternatives. Using two case studies, we show how MSD-space can help visually compare aggregation functions and formulate improvement actions for selected alternatives. The revealed inner-workings of TOPSIS can be considered a step towards increasing the explainability of TOPSIS itself as well as other multi-criteria ranking methods.

Variety is the spice of life

A proverb

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1. Introduction

Introduced in the early 80'ties, TOPSIS (Hwang and Yoon, 1981) is a method designed to create rankings of alternatives described by multiple criteria. Out of the numerous approaches to the process of creating rankings, TOPSIS chooses one that is based on distances, namely: calculating distances from a predetermined (e.g. ideal) alternative to all considered alternatives produces non-negative real values that describe these alternatives. Now, because any set of real values naturally renders a linear pre-order, a ranking of the considered alternatives is thus created.

Despite various versions and modifications, the core of TOPSIS adopts this approach from its very beginnings and aptly calculates values that constitute combinations of two distances: a distance to a predefined ideal alternative (a cost-type criterion) and a distance to a predefined anti-ideal alternative (a gain-type criterion), effectively producing rankings that are simultaneously influenced by these two distances.

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The original TOPSIS method, with its various extensions as well as its numerous applications, has been widely used in the literature. The following brief survey cites, however, only those papers that directly address the algebraic notions used within the method and its potential extensions. For a much broader survey of TOPSIS-based methodologies and applications, see, e.g., the review of Behzadian et al. (2012).

The research of Opricovic and Tzeng (2004) presents an analysis of TOPSIS and VIKOR, which are two methods based on aggregation functions that represent closeness to a reference solution or solutions. Their analysis focuses on the study of the impact of different normalization procedures and different aggregation functions on the final ranking. Similarly, Zavadskas et al. (2006) describe a methodology for measuring the accuracy of the relative significance of the alternatives as a function of the criteria values, and analyze the influence of a normalization method on the final rankings.

Another method based on a concept similar to that of TOPSIS is the Relative Ratio method (Li, 2009). In this method, differences between alternatives and the ideal solution as well as the anti-ideal solution are estimated and the ranking is created so that it reflects a balance between the shortest distance from the ideal solution and the farthest distance from the anti-ideal solution.

Analogous research is reported in the study performed by Kuo (2017), which introduces a new relative closeness to an ideal solution based on two weights. These weights (that of benefit criteria and that of cost criteria) balance the ideal and anti-ideal shares of the alternative's relative distance. An extension of this approach can be found in the paper by Abootalebi et al. (2019), which shows how the balance of weights may be obtained by solving a temporary problem, thus producing unique weights and thereby unique final ranking.

Other interesting issues relating to TOPSIS, including its combinations with other methods, its variations, and adaptations, are described in, e.g., works by Yu et al. (2015); Chen (2019); Tian et al. (2018); Yoon and Kim (2017); Zielniewicz (2017); Nădăban et al. (2016); Fan and Feng (2009); Kahraman et al. (2007). Among these papers, Yu et al. (2015) and Chen (2019) discuss the variation of values describing the underlying objects, referring to this variation as coordination. To this end, they devise a measure of coordination and modify the classic aggregation function of TOPSIS to incorporate this measure, effectively constructing new versions of the method called Coordinated TOPSIS. Interestingly, in this paper we demonstrate that the original aggregation functions in TOPSIS already incorporate the level of variation. Over and above that we systematically illustrate how this level influences the final results of the method.

As shown by the referenced works, current studies on TOPSIS mainly focus on normalization/weighting procedures and practical applications. However, to the best of our knowledge, no studies attempted to formally describe the systematic relations between the properties of alternatives and the results of TOPSIS aggregations. As a result, aggregation functions are currently compared on a use case basis rather than generally, i.e., with respect to the space of all possible alternatives. Finally, no approaches exist that are capable of visualizing such general, dataset-independent properties of TOPSIS aggregations.

In this paper, we formalize and visualize the inner-workings of TOPSIS by describing aggregations using the mean and standard deviation of each alternative. This allows us to propose a dataset-independent way of analyzing TOPSIS aggregations. The detailed contributions of this paper are as follows:

- In Section 2, we formalize the TOPSIS procedure from the viewpoint of all possible alternative representations. We define the criterion space, utility space, and discuss how ideal/anti-ideal points are represented in each space. We also highlight the properties of distance calculation in those spaces and formally define three 'classic' TOPSIS aggregation functions. Finally, we underline the limitations of the criterion and utility spaces for visual analyses.

- In Section 3, we reveal the mechanics of classic TOPSIS by describing dependencies between the distances to the ideal/anti-ideal points and the mean and standard deviation of an alternative. For this purpose, we introduce the IA-MSD property and MSD-space that, contrary to the criterion and utility space, can be represented in two dimensions regardless of the number of analyzed criteria.

- In Section 4, we introduce a 2D visualization of MSD-space and show how the IA-MSD property makes it possible to express and compare various aggregation functions using the means and standard deviations of alternatives. This reveals the workings of the aggregations and allows the decision makers to choose them in an informed manner. We also show the preference-related interplay of the mean and standard deviation that influences the final ranking of alternatives under a given aggregation function.

- In Section 5, we interpret different aggregation functions in two practical ranking scenarios. We show how MSD-space can be used to: inform decision makers about the properties of a given dataset, highlight consequences of using particular aggregation functions, and potentially suggest actions that will improve a given alternative's ranking position.
- In Section 6, we summarize the paper and draw lines of future research.

2. Formalizing TOPSIS

Typical papers on TOPSIS do not introduce complicated denotation systems to explain how it works. This is because those papers usually deal with a predefined, finite family of m objects (*alternatives*) and a predefined, finite family of n attributes (*criteria*). If so, then the data may amply be represented in a $m \times n$ matrix of values, usually referred to as the *decision matrix* in the TOPSIS terminology. An exemplary decision matrix $\mathbf{X}$ is depicted in Figure 1A. It contains four alternatives (students) described by three criteria (final grades obtained from subjects).

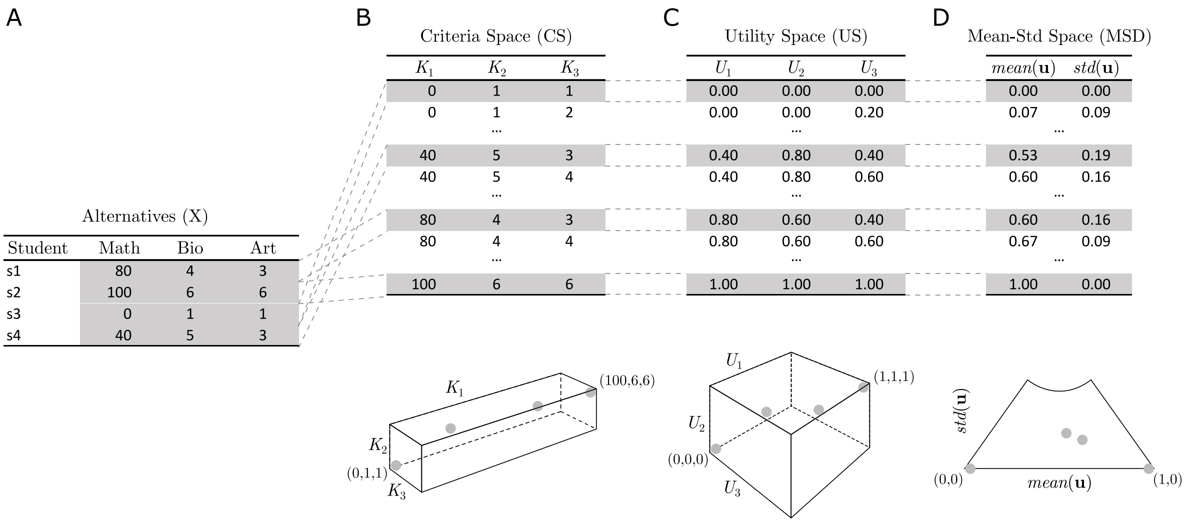

Figure 1: Running example and schematic representation of alternatives in different spaces. (A) Decision matrix with four alternatives (students) described by three criteria (grades). (B) The selected alternatives (students) represented as a subset of the criteria space, i.e., the space of all possible alternatives within the given criteria. (C) Alternatives represented in utility space, the re-scaled equivalent of criteria space. (D) Alternatives represented in MSD-space.

What is attempted in this paper differs from typical TOPSIS-related studies in that, given a set of n criteria, we will not only consider a particular set of m alternatives, but examine the general properties of all possible ones. This is why we will consider sets of all possible alternative representations defining them as three consecutive spaces, as depicted in Figure 1: *criteria space* (domain space of the original alternatives), *utility space* (re-scaled criteria space), and *MSD-space* (newly proposed space). We note that the idea of analyzing all possible alternative representations is inspired by approaches developed for visual-based inspection of general properties of machine learning measures Brzezinski et al. (2018, 2017); Susmaga and Szczech (2015a,b).

Notions formally defined in the following sections have accompanying proofs. To make the presented methodology easier to follow, in the main text we state the main properties of the defined spaces but leave their formal proofs for the Appendices (available as online supplementary materials).

2.1. MCDA Preliminaries

Consider a set of real-world objects, which are to be further processed. In our running example, visualized in Figure 1, the set of objects will consist of students. In Multi-Criteria Decision Analysis (MCDA) processing the objects usually requires taking into account multiple aspects of the objects and usually resolves itself to:

- choosing the most preferred objects,
- assigning objects to preference classes,
- ranking objects from the most preferred to the least preferred.

For more details on methods, models and software in MCDA see e.g., Keeney and Raiffa (1976); Bouyssou et al. (2000); Belton and Stewart (2002); Ishizaka and Nemery (2013); Bisdorff et al. (2015); Greco et al. (2016); Cinelli et al. (2022). In MCDA the objects are commonly referred to as *alternatives*. The set of all possible alternatives will be denoted by $\mathbb{A}$.

To allow any kind of processing, the alternatives must be assigned some predefined representations, which actually undergo the processing. Those representations are created with a predefined set of attributes, e.g., grades the students obtained in different subjects. An attribute is thus a function that assigns a given object a particular value from a predefined set of values, referred to as the domain of this attribute. Because objects are usually described with multiple attributes, their individual descriptions are organized into vectors.

Additionally, MCDA employs very special attributes, namely attributes with domains ordered according to a preference relation, referred to as *criteria*. Originally weakly monotonic forms of orderings have been gradually extended to include other, non-monotonic forms, see e.g., Ghaderi et al. (2017); Kadzinski et al. (2020). However, these later developments were not considered by earlier methods like TOPSIS. Therefore, in this paper, it is consistently assumed that all attributes are criteria, the domains of which are real-valued intervals with weakly monotonic form (either non-decreasing or non-increasing) of ordering according to preference. The set of all those possible criteria will be denoted by $\mathbb{K}$.

2.2. The General Procedure of TOPSIS

TOPSIS (Hwang and Yoon, 1981) uses multi-criteria descriptions of alternatives to rank them from the most preferred to the least preferred. Its procedure can be expressed in the form of the following three main steps:

1. **prepare the representations** of alternatives in terms of criteria, i.e. the decision matrix; this step also usually includes the normalization of criteria into a common scale and using preference information given by the decision maker in the form of weights of criteria;
2. **determine two reference points**: ideal and anti-ideal, and calculate distances between each representation to one or to both of those points;
3. **rank the alternatives** according to an assumed function that aggregates distances between the alternatives and the reference points.

TOPSIS assumes that the criteria describing the alternatives should be either expressed on an interval or ratio scale, or easily convertible to such. Providing numeric-scale criteria facilitates calculating distances between alternatives and the ideal/anti-ideal points, which are the input for the aggregation functions.

There are three 'classic' aggregation functions in TOPSIS. They are founded on: the distance to the ideal point, the distance to the anti-ideal point, or both of them used simultaneously. The most typical aggregation in TOPSIS, often referred to as the 'relative distance', is expressed as the distance to the anti-ideal point divided by the sum of the two distances. Notice that the aggregation founded on the distance to the anti-ideal point is naturally maximized. The same concerns the relative distance. On the other hand, the aggregation founded on the distance to the ideal point is minimized. To unify the interpretation of all aggregations used, further in the paper we will reverse the aggregation founded on the distance to the ideal point.

The following subsections formalize each of the three main steps of TOPSIS. The notation presented here will be later used to define MSD-space and its properties.

2.2.1. Preparing representations of alternatives

The first main step of TOPSIS is usually problem-specific and involves creating a representation of real-world objects in the form of a decision matrix $\mathbf{X}$ (Figure 1A). Here, we will focus on how the decision matrix can be defined as a subset of the criterion space and then re-scaled (transformed) into the utility space.

If a criterion $\mathcal{K} \in \mathbb{K}$, then its domain is a real-valued interval $\mathcal{V} = [v_{min}, v_{max}]$. We note that TOPSIS requires that the lower (v_{min}) and upper (v_{max}) bounds are finite, to make distance computation possible. Of course the values of v_{min} and v_{max} may be different for each criterion. Additionally, according to the monotonicity type of the ordering of its domain, criteria may differ in their preference types. We will denote the least preferred value as v_* and the most preferred value as v^*. In particular, criterion $\mathcal{K} \in \mathbb{K}$ is of type 'gain' when its domain is $\mathcal{V} = [v_{min}, v_{max}] = [v_*, v^*]$, v^* is preferred over v_*, and the preference of $v \in \mathcal{V}$ does not decrease with the increase of v. Analogously, criterion is of type 'cost' when its domain is $\mathcal{V} = [v_{min}, v_{max}] = [v^*, v_*]$, v^* is preferred over v_*, and the preference of $v \in \mathcal{V}$ does not increase with the increase of v.

In our running example, the set of four students $\mathbf{X}$ (Figure 1A) is actually a finite subset of an infinite criterion space CS (Figure 1B), where each possible alternative is described by particular values on criteria $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \in \mathbb{K}$. The domains of the criteria are as follows: $\mathcal{V}_{\mathcal{K}_1} = [0, 100]$, $\mathcal{V}_{\mathcal{K}_2} = [1, 6]$, $\mathcal{V}_{\mathcal{K}_3} = [1, 6]$. We note that the bounds for the criteria are predefined ('expert-driven approach') and not taken from the dataset ('data-driven approach'), i.e., the bounds would be the same even if the v_{min} and v_{max} values were not present in the dataset. The fact that the criteria bounds need to be established can be regarded as a limitation of TOPSIS itself, since bounds might not always be as naturally determined as in the case of school grades or percentages. In such situations, the bounds have to be set by the expert somewhat arbitrarily, and the results will depend partly on the chosen lower and upper bound.

Unfortunately, simultaneous analyses of sets of criteria with different intervals and different types are not very convenient. Additionally, in cases where the analyses perform some kind of criteria combining, it is in fact required that all evaluations are expressed on the same scale in order for the aggregation to be meaningful. Therefore, we introduce a simple min-max re-scaling of the criteria, which transforms the criteria using function $\mathcal{U} : \mathcal{V} \to [0, 1]$. Precisely, given:

- a domain $\mathcal{V} = [v_{min}, v_{max}] = [v_*, v^*]$ of a criterion $\mathcal{K} \in \mathbb{K}$ of type 'gain', the re-scaling function $\mathcal{U}$ associated with $\mathcal{K}$ is defined as $\mathcal{U}(v) = \frac{v - v_*}{v^* - v_*}$ for $v \in \mathcal{V}$,
- a domain $\mathcal{V} = [v_{min}, v_{max}] = [v^*, v_*]$ of a criterion $\mathcal{K} \in \mathbb{K}$ of type 'cost', the re-scaling function $\mathcal{U}$ associated with $\mathcal{K}$ is defined as $\mathcal{U}(v) = \frac{v - v^*}{v_* - v^*}$ for $v \in \mathcal{V}$.

Notice that the sole objective of the re-scaling function $\mathcal{U}(\cdot)$ is to unify the intervals and the preference types of all criteria. Whereas the original criteria may have different intervals as domains and different types, the re-scaled criteria will all have the same interval ($[0, 1]$) and the same type ('gain'), which simplifies our analyses without reducing their generality. We note that the used [0,1] re-scaling is independent of the type of normalization potentially used by decision makers to prepare the decision matrix. For the purposes of our analyses, we will assume no additional normalization and no criteria weighting; the effect of criteria weighting on MSD-space will be the topic of a follow-up paper.

Observe that the usefulness of the earlier mentioned simplification of the analyses in US may now be demonstrated with regard to the weak dominance relation between two alternatives: if $\mathbf{u}_1 \in US$ is an image of $E_1 \in CS$ and $\mathbf{u}_2 \in US$ is an image of $E_2 \in CS$, then the statement 'E_1 weakly dominates E_2' is equivalent to $\mathbf{u}_1 \geq \mathbf{u}_2$ (each element of $\mathbf{u}_1$ is greater or equal to the corresponding element $\mathbf{u}_2$). This is thus independent of the types of all criteria used in the comparison because re-scaled criteria in US are always of the 'gain' type. No equally simple statement may in general be formulated in CS.

In our running example, the criterion space CS of three criteria (Figure 1B) is transformed by min-max criteria re-scaling to the utility space US (Figure 1C). The utility space has thus also three criteria, each being of type 'gain' and ranging in $[0, 1]$. Both CS and US can contain an infinite number of alternatives, whereas our set of four students constitutes merely a finite subset; while this subset in included in CS, its image under the min-max criteria re-scaling operation is included in US. The criterion space has the form of a cuboid with the particular alternatives (students) located somewhere within it. The utility space, on the other hand, has the form of a cube, again with the images of the actual alternatives (students) in it.

It is important to stress that all the considerations contained in this paper apply to the entirety of US, in particular, to each element of this space. Thus, they are independent of a particular decision matrix, each of which is represented by only a finite subset of US.

2.2.2. Determination of the ideal/anti-ideal points and distance calculation

To formally determine the ideal and anti-ideal points within the utility space let us first consider their representation in the criterion space. Assume $\mathscr{K}$, $|\mathscr{K}| = n \geq 1$, be a set of criteria selected from $\mathbb{K}$. The criterion space CS is the set of all possible vectors $[v_1, v_2, ..., v_n]$ such that $v_j \in \mathcal{V}_j$ is the domain of criterion $\mathcal{K}_j \in \mathscr{K}$. The criterion space CS is thus an n-dimensional hypercuboid $\mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_n$ with 2^n vertices of the form $[s_1, s_2, ..., s_n]$, where $s_j \in \{v_{j*}, v_j^*\}$.

In particular, CS contains two vertices:

- $[v_1^*, v_2^*, ..., v_n^*]$, further denoted by I and referred to as the *ideal point*;
- $[v_{1*}, v_{2*}, ..., v_{n*}]$, further denoted by A and referred to as the *anti-ideal point*.

Recalling our running example in Figure 1, the ideal point I would represent any student with highest possible grades from each subject. More precisely, an ideal student is characterized by $v_1^* = 100$, $v_2^* = 6$ and $v_3^* = 6$, as such are the maximal grades on the particular subjects (criteria). Analogously, for an anti-ideal student $v_{1*} = 0$, $v_{2*} = 1$ and $v_{3*} = 1$, as such are the minimal grades on the particular criteria.

Now, let us formally move from the criterion space to the utility space and determine the ideal and anti-ideal points there. Given $\mathscr{K}$, $|\mathscr{K}| = n \geq 1$, the set of criteria selected from $\mathbb{K}$, consider the utility space US, i.e., the set of all possible vectors $[u_1, u_2, ..., u_n]$. The utility space US is an n-dimensional hypercube $[0, 1] \times [0, 1] \times ... \times [0, 1]$ with 2^n vertices of the form $[z_1, z_2, ..., z_n]$, where $z_j \in \{0, 1\}$.

It should be kept in mind that while the original shape of US is that of a hypercube, it may be changed with criteria weights. These weights, which in fact constitute preferential information of the decision maker, are used to differentiate the influence of the criteria on the final results of TOPSIS. The shape of the arising, weighted version of US, in which all further operations of TOPSIS are performed, generalizes thus to a hypercuboid, with the special case of the hypercube obtained for all weights equal to one. More precisely, every case when weights are equal to a pre-defined positive constant would also result in a hypercube. Notice that because the values of weights have formally different origin than the descriptions of alternatives, applying weights to US and thus producing the weighted version of US may be viewed as introducing a form of preferential bias.

As opposed to the special case of weights equal to one, the general case of varying weights is troublesome for some reasons. First of all, it alters the shape of the weighted US from the hypercube to a hypercuboid. As it turns out, it also alters the shape of the so-called MSD-space (defined and thoroughly analysed below). To avoid elaborating on the numerous and somewhat cumbersome details of the altered shape of MSD-space, this paper focuses exclusively on the special case, by assuming all weights to be equal to one. This ensures the weighted US to be identical to US, and thus to have the shape of the hypercube. This appreciably facilitates visualizations of the MSD-space. Additionally, it simplifies demonstrating and describing the fundamental properties of the MSD-space.[1] Last but not least, it allows to scrutinize alternatives in their preferentially unbiased forms.

Coming back to the ideal and the anti-ideal points in US recall that for each $E \in CS$ there exists $\mathbf{u} \in US$ such that $\mathbf{u}$ is the image of E under the re-scaling transformation. More precisely, if $E = [v_1, v_2, ..., v_n] \in CS$, then $[\mathcal{U}_1(v_1), \mathcal{U}_2(v_2), ..., \mathcal{U}_n(v_n)] \in US$. In particular, US as a hypercube contains vectors $\mathbf{1} = [1, 1..., 1]$ and $\mathbf{0} = [0, 0, ..., 0]$, which are the respective images of I and A. Recalling our running example in Figure 1, the image of the ideal point in the utility space is $\mathbf{1} = [1, 1, 1]$ and the image of the anti-ideal point is $\mathbf{0} = [0, 0, 0]$.

[1] Demonstrations and descriptions of these properties in the general case would require introducing some additional geometric concepts, which merit a separate follow-up paper.

Having determined **1** and **0**, TOPSIS proceeds with calculating the distances between each representation to one or both of those points. The paper sticks to the most commonly used distance measure, i.e., the Euclidean distance. Given vectors $\mathbf{a} = [a_1, a_2, ..., a_n]$, $\mathbf{b} = [b_1, b_2, ..., b_n]$, the Euclidean distance between them is defined as $\delta_2(\mathbf{a}, \mathbf{b}) = \sqrt{\sum_{j=1}^{n} |a_j - b_j|^2}$.

Notice that the maximal Euclidean distance in US extends between vectors: **1** and **0** and equals $\sqrt{n}$, which makes it dependent on n. To make this maximal distance n-independent, we define the so-called re-scaled Euclidean distance as $\delta_2^{01}(\mathbf{a}, \mathbf{b}) = \frac{\delta_2(\mathbf{a},\mathbf{b})}{\sqrt{n}}$. In result, given any n and any $\mathbf{a}, \mathbf{b} \in US$: $\delta_2(\mathbf{a}, \mathbf{b}) \in [0, \sqrt{n}]$, but $\delta_2^{01}(\mathbf{a}, \mathbf{b}) \in [0, 1]$.

Figure 2 presents the utility space US in a 2D and 3D scenario, i.e., when the number of criteria is $n = 2$ and $n = 3$. Naturally, cases when $n > 3$ cannot be visualized directly. The ideal and anti-ideal points are marked in each US as **1** and **0**, respectively. In each scenario, the utility space contains a particular alternative, represented by a point in the space. Figures 2A and 2C depict the coordinates of that alternative, whereas Figures 2B and 2D showcase the Euclidean distances of the alternative to **1** and **0**.

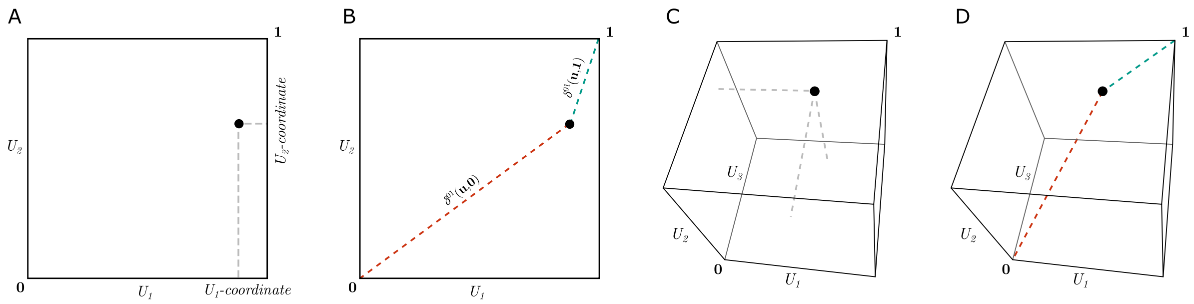

Figure 2: Exemplary coordinates and distances in the utility space for $n = 2$ (A, B) and $n = 3$ (C, D)

Having determined the ideal and anti-ideal reference points, and calculated distances between each representation to those points, TOPSIS moves on to ranking the alternatives according to an assumed aggregation function, examples of which are defined in the next section.

2.2.3. Ranking alternatives according to an aggregation

Recall that the three classic aggregations used in TOPSIS are founded on: the distance to the ideal point, the distance to the anti-ideal point, or both of them used simultaneously, as is the case of the relative distance. The three considered aggregations are denoted by I, A and R standing for the distance to the ideal point, distance to the anti-ideal point and the relative distance, respectively. When expressed in terms of $\delta_2^{01}(\mathbf{u}, \mathbf{1})$ and $\delta_2^{01}(\mathbf{u}, \mathbf{0})$, they are defined as follows:

$$\mathsf{I}(\mathbf{u}) = 1 - \delta_2^{01}(\mathbf{u}, \mathbf{1}),$$
$$\mathsf{A}(\mathbf{u}) = \delta_2^{01}(\mathbf{u}, \mathbf{0}),$$
$$\mathsf{R}(\mathbf{u}) = \frac{\delta_2^{01}(\mathbf{u}, \mathbf{0})}{\delta_2^{01}(\mathbf{u}, \mathbf{1}) + \delta_2^{01}(\mathbf{u}, \mathbf{0})},$$

where $\mathbf{u} \in US$ is the image of the representation $E \in CS$ of an alternative from $\mathbb{A}$.

Notice the reversal of the aggregation founded on the distance to the ideal point: $1 - \delta_2^{01}(\mathbf{u}, \mathbf{1})$ instead of $\delta_2^{01}(\mathbf{u}, \mathbf{1})$, introduced in order to unify the interpretation of all aggregations as functions to be maximized. It must also be emphasized that although out of the three aggregations only one (i.e. $\mathsf{R}(\mathbf{u})$) is commonly used, it draws on the other two, sharing with them some of its key properties. This is why in this paper all three aggregations will be considered and scrutinized in parallel.

3. The IA-MSD property and MSD-space

One of the inconveniences of the utility space US as well as the criterion space CS is that they cannot be visualized when the number of criteria exceeds three. As a result, for $n > 3$ different aggregations of TOPSIS cannot be subject to visual-based analyses and comparisons in those spaces. However, having formalized the algebraic aspects of classic TOPSIS aggregations, we can describe important dependencies between the distances of alternatives to the ideal point I and anti-ideal point A and two fundamental features of the alternatives: their mean (M) and standard deviation (SD). Based on the interplay between the above mentioned elements in US, we introduce a new two-dimensional space, called MSD-space. It is based on the mean and standard deviation and can depict the aggregations in a plane regardless of the number of criteria describing the alternatives. This section formalizes the dependencies between the distances and the mean and deviation into the IA-MSD property, introduces the MSD-space, and shows how aggregation functions can be compared thanks to MSD-space.

3.1. *The IA-MSD Property in the Utility Space*

Given a representation of an alternative in utility space $\mathbf{u} \in US$, let:

$$\begin{aligned} sum(\mathbf{u}) &= \sum_{j=1}^{n} u_j, \\ mean(\mathbf{u}) &= \frac{sum(\mathbf{u})}{n}, \\ var(\mathbf{u}) &= \frac{\|\mathbf{u} - \overline{\mathbf{u}}\|_2^2}{n}, \\ std(\mathbf{u}) &= \sqrt{var(\mathbf{u})}. \end{aligned}$$

Additionally, if $\overline{u} = mean(\mathbf{u})$, then $\overline{\mathbf{u}} = [\overline{u}, \overline{u}, ..., \overline{u}] \in US$ is a vector of means of $\mathbf{u}$.

Since for every $\mathbf{u} \in US$ vectors $\overline{\mathbf{u}} - \mathbf{0}$ and $\mathbf{u} - \overline{\mathbf{u}}$ as well as vectors $\mathbf{u} - \overline{\mathbf{u}}$ and $\mathbf{1} - \overline{\mathbf{u}}$ are orthogonal ($(\overline{\mathbf{u}} - \mathbf{0}) \perp (\mathbf{u} - \overline{\mathbf{u}})$ as well as $(\mathbf{1} - \overline{\mathbf{u}}) \perp (\mathbf{u} - \overline{\mathbf{u}})$), they may be subjected to the Pythagorean theorem. This characteristic of US allows us to formulate what we will refer to as the *IA-MSD property*.

Let us first, however, present the formal justification of the orthogonality. We shall demonstrate $(\overline{\mathbf{u}} - \mathbf{0}) \perp (\mathbf{u} - \overline{\mathbf{u}})$, as $(\mathbf{1} - \overline{\mathbf{u}}) \perp (\mathbf{u} - \overline{\mathbf{u}})$ is analogous owing to the collinearity of $\overline{\mathbf{u}} - \mathbf{0}$ and $\overline{\mathbf{u}} - \mathbf{1}$. Recall that column vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal when $\mathbf{a}^T\mathbf{b} = 0$. Notice that this means that $\mathbf{a} = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$, or $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ while $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. Applying the orthogonality condition to $\mathbf{a} = \overline{\mathbf{u}} - \mathbf{0}$ and $\mathbf{b} = \mathbf{u} - \overline{\mathbf{u}}$, one gets: $(\overline{\mathbf{u}} - \mathbf{0})^T(\mathbf{u} - \overline{\mathbf{u}}) = \overline{\mathbf{u}}^T(\mathbf{u} - \overline{\mathbf{u}}) = \overline{\mathbf{u}}^T\mathbf{u} - \overline{\mathbf{u}}^T\overline{\mathbf{u}} = \sum_{j=1}^{n} \overline{u} \cdot u_j - \sum_{j=1}^{n} \overline{u} \cdot \overline{u} = \overline{u} \sum_{j=1}^{n} u_j - \overline{u} \cdot \overline{u} \sum_{j=1}^{n} 1 = \overline{u} \sum_{j=1}^{n} u_j - \overline{u} \cdot \overline{u} \cdot n$. Now, using the sum-mean equivalence and the sum of ones: $(\overline{\mathbf{u}} - \mathbf{0})^T(\mathbf{u} - \overline{\mathbf{u}}) = \overline{u} \sum_{j=1}^{n} u_j - \overline{u} \cdot \overline{u} \cdot n = \overline{u} \cdot \overline{u} \cdot n - \overline{u} \cdot \overline{u} \cdot n = 0$. Thus, we can conclude the orthogonality: $(\overline{\mathbf{u}} - \mathbf{0}) \perp (\mathbf{u} - \overline{\mathbf{u}})$.

Let us also illustrate these orthogonalities with an example. Let $n = 2$, which implies a two-dimensional US, and consider an exemplary $\mathbf{u} = [0.75, 0.25] \in US$ (see Figure 3). This vector is used to form three other vectors: $\overline{\mathbf{u}} - \mathbf{0}$, $\overline{\mathbf{u}} - \mathbf{1}$ and $\mathbf{u} - \overline{\mathbf{u}}$, out of which $\mathbf{u} - \overline{\mathbf{u}}$ will be shown to be orthogonal both to $\overline{\mathbf{u}} - \mathbf{0}$ as well as to $\overline{\mathbf{u}} - \mathbf{1}$. Because $mean(\mathbf{u}) = mean([0.75, 0.25]) = 0.5$ (i.e. $\overline{u} = 0.5$), vector $\overline{\mathbf{u}} = [\overline{u}, \overline{u}] = [0.50, 0.50]^T$. Now, one arrives at $\overline{\mathbf{u}} - \mathbf{0} = [0.50, 0.50]^T - [0.00, 0.00]^T = [0.50, 0.50]^T$ and $\mathbf{u} - \overline{\mathbf{u}} = [0.75, 0.25]^T - [0.50, 0.50]^T = [0.25, -0.25]^T$ (notice that in Figure 3 this vector is depicted as originating from [0.5, 0.5] instead of [0.0, 0.0]). And since $[0.50, 0.50][0.25, -0.25]^T = 0$, these vectors are orthogonal ($(\overline{\mathbf{u}} - \mathbf{0}) \perp (\mathbf{u} - \overline{\mathbf{u}})$). Simultaneously, the following arises: $\overline{\mathbf{u}} - \mathbf{1} = [0.50, 0.50]^T - [1.00, 1.00]^T = [-0.50, -0.50]^T$. This vector can also be clearly seen to be orthogonal to $[0.25, -0.25]^T$, because $[-0.50, -0.50][0.25, -0.25]^T = 0$. Thus $(\overline{\mathbf{u}} - \mathbf{1}) \perp (\mathbf{u} - \overline{\mathbf{u}})$.

Now, notice that the lengths of the considered vectors may be determined as:

- $\delta_2^{01}(\overline{\mathbf{u}}, \mathbf{0}) = mean(\mathbf{u})$,
- $\delta_2^{01}(\overline{\mathbf{u}}, \mathbf{1}) = 1 - mean(\mathbf{u})$,

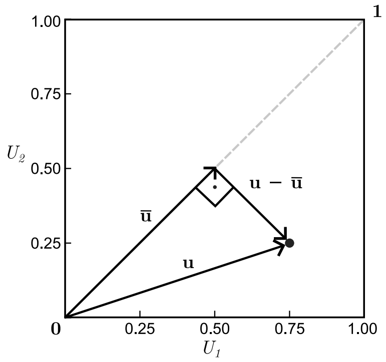

Figure 3: An exemplary illustration of the vector orthogonality property in US.

- $\delta_2^{01}(\mathbf{u}, \overline{\mathbf{u}}) = std(\mathbf{u})$.

Given the above-mentioned orthogonality of vectors, their lengths may be subjected to the Pythagorean theorem[2]. This is because, save for some straightforward exceptions, segments **0**–$\overline{\mathbf{u}}$, $\overline{\mathbf{u}}$–**u** and **u**–**0** happen to form one right triangle, while segments **1**–$\overline{\mathbf{u}}$, $\overline{\mathbf{u}}$–**u** and **u**–**1** happen to form another right triangle (see Figure 3, as well as Figures 4A and 4B). The only exceptions include situations when **u** has the form $[x, x, ..., x]$ for some $x \in [0, 1]$, in which case $mean(\mathbf{u}) \equiv \overline{u} = x$ and $\mathbf{u} = \overline{\mathbf{u}}$, so $\delta_2^{01}(\mathbf{u}, \overline{\mathbf{u}}) = 0$. Then the two right triangles are degenerated to mere segments: **0**–$\overline{\mathbf{u}}$ and **1**–$\overline{\mathbf{u}}$. Nevertheless, even in the case of a degenerated triangle, the Pythagorean theorem still holds true. If c is zero, then the hypotenuse a is identical to the base b, so $a^2 = b^2 + c^2 = b^2 + 0^2 = b^2$, which holds true as $a = b$.

All these considerations allow the following formulation of what will be referred to as the IA-MSD property.

Definition 1 (IA-MSD Property)**.**

$$\delta_2^{01}(\mathbf{u}, \mathbf{0}) = \sqrt{mean(\mathbf{u})^2 + std(\mathbf{u})^2},$$
$$\delta_2^{01}(\mathbf{u}, \mathbf{1}) = \sqrt{(1 - mean(\mathbf{u}))^2 + std(\mathbf{u})^2}.$$

Although IA-MSD property is n-independent, in US it may easily be illustrated only for $n = 2$ (Figure 4A) and $n = 3$ (Figure 4B). In the following section, we will show how the n-independence of the IA-MSD property can be used to create an n-independent 2D visualization of TOPSIS aggregations.

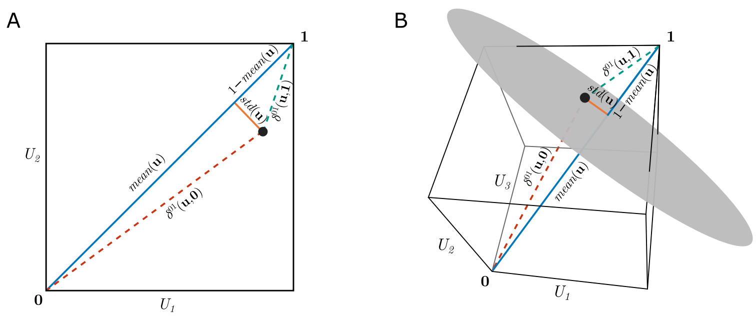

Figure 4: An exemplary illustration of the IA-MSD property in US.

[2]Recall that in a right triangle with a hypotenuse of size a, the base of size b and the height of size c: $a^2 = b^2 + c^2$, or $a = \sqrt{b^2 + c^2}$.

3.2. The MSD-space

The interesting dependency between the distances of an alternative to the ideal and anti-ideal points, inspired us to introduce a new space called *MSD-space* that uses mean (M) and standard deviation (SD) as its coordinates.

Definition 2 (MSD-space)**.**

$$\text{MSD-space} = \{[mean(\mathbf{u}), std(\mathbf{u})] | \mathbf{u} \in US\}$$

The MSD-space can be visualized as a 2D space wherein the mean (M) of each alternative is represented on the x-axis and the standard deviation (SD) of an alternative on the y-axis. Since MSD-space is based on the $[0, 1]$-scaled utility space, the maximum values of M and SD are bounded. In other words, for a given number of criteria, there is only a limited range of means and standard deviations an alternative can have. As a result, one can depict the boundaries (shape) of MSD-space, which depends on the number of analyzed criteria. Figure 5 shows two exemplary visualizations of the MSD-space for $n = 4$ and $n = 5$. As can be noticed, the number of criteria affects the number of vertices that can be defined in MSD-space and the maximum value of SD an alternative can obtain for each M. The introduced MSD-space is thoroughly studied analytically in Appendix A. In particular, this appendix formally describes the vertices, diagonals, lower and upper 'perimeters' of the MSD-space and its further properties, including the symmetry of the space as well as shape-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$.

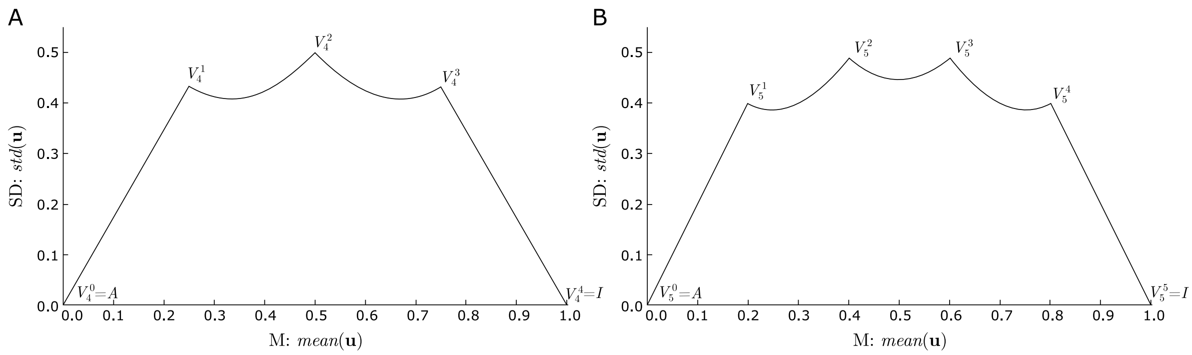

Figure 5: Visualizations of the MSD-space for (A) $n = 4$ and (B) $n = 5$. Each visualization has $n + 1$ vertices V_n^i for $i \in \{0, 1, ..., n\}$, n diagonals $V_n^i \leftrightarrow V_n^{i+1}$ for $i \in \{0, 1, ..., n-1\}$ as well as the $V_n^0 \leftrightarrow V_n^n$ diagonal. Each V_n^i represents the set of all possible vectors $\mathbf{u} = [u_1, u_2, ..., u_n]$ such that $u_j \in \{0, 1\}$ and $sum(\mathbf{u}) = i$ (this means that V_n^i consists of exactly i ones and $n - i$ zeros). See Appendix A for the formal definitions of the vertices and diagonals in MSD-space.

It should be stressed that the IA-MSD property holds in the MSD-space, where, again, it constitutes a direct application of the Pythagorean theorem to two right triangles; in this case

$$\overset{\triangle}{AMu} = ((0, 0), (mean(\mathbf{u}), 0), (mean(\mathbf{u}), std(\mathbf{u}))),$$
$$\overset{\triangle}{IMu} = ((1, 0), (mean(\mathbf{u}), 0), (mean(\mathbf{u}), std(\mathbf{u}))).$$

This is because MSD-space constitutes a very particular ('rotational') projection of US into two dimensions, namely one which retains the IA-MSD property.

Interestingly enough, the property may now be always successfully illustrated in 2D because, as opposed to US, the MSD-space is by definition two-dimensional (or, in the special case of $n = 1$, one-dimensional). Figure 6 illustrates exactly this phenomenon. In this case $n = 3$, which means that a three dimensional space would be actually required to illustrate the property in US. Handling three (or more) dimensions is, however, not required in MSD-space, as a point with two coordinates ($mean(\mathbf{u})$, $std(\mathbf{u})$) is an image of vector $\mathbf{u}$ of an arbitrary size (three in this case) from US. The IA-MSD property applies in both of those spaces. Needless to say, all our further considerations and visualizations will involve the MSD-space.

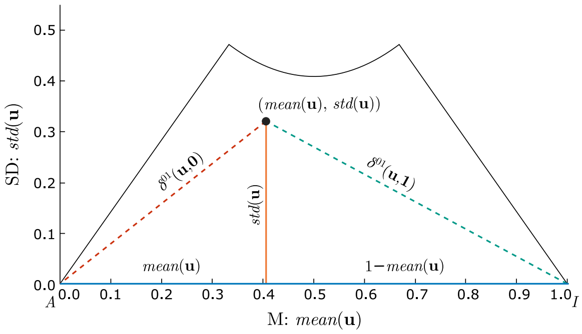

Figure 6: The IA-MSD property in the MSD-space for $n = 3$. Same property applies to any n, and could be analogously visualized, e.g. for the MSD-spaces in Figure 5.

4. Visualizing TOPSIS Aggregations in MSD-space

4.1. Consequences of the IA-MSD Property

Given any $\mathbf{u} \in US$, which is the image of the representation $E \in CS$ of some alternatives from $\mathbb{A}$, the IA-MSD property makes it possible to express all the aggregations with $mean(\mathbf{u})$ and $std(\mathbf{u})$:

$$
\begin{aligned}
\mathsf{I}(\mathbf{u}) &= 1 - \delta_2^{01}(\mathbf{u}, \mathbf{1}) \\
&= 1 - \sqrt{(1 - mean(\mathbf{u}))^2 + std(\mathbf{u})^2}, \\
\mathsf{A}(\mathbf{u}) &= \delta_2^{01}(\mathbf{u}, \mathbf{0}) \\
&= \sqrt{mean(\mathbf{u})^2 + std(\mathbf{u})^2}, \\
\mathsf{R}(\mathbf{u}) &= \frac{\delta_2^{01}(\mathbf{u}, \mathbf{0})}{\delta_2^{01}(\mathbf{u}, \mathbf{1}) + \delta_2^{01}(\mathbf{u}, \mathbf{0})} \\
&= \frac{\sqrt{mean(\mathbf{u})^2 + std(\mathbf{u})^2}}{\sqrt{(1 - mean(\mathbf{u}))^2 + std(\mathbf{u})^2} + \sqrt{mean(\mathbf{u})^2 + std(\mathbf{u})^2}}.
\end{aligned}
$$

The fact that all those aggregations can be expressed with $mean(\mathbf{u})$ and $std(\mathbf{u})$ means that even for $n > 2$ TOPSIS is in a sense a two-dimensional method. The same could have certainly been also stated earlier since the aggregations are originally expressed with $\delta_2^{01}(\mathbf{u}, \mathbf{1})$ and $\delta_2^{01}(\mathbf{u}, \mathbf{0})$, which equally makes the aggregations functions of two parameters. What is interesting, however, is that $mean(\mathbf{u})$ and $std(\mathbf{u})$ are much more basic in nature than $\delta_2^{01}(\mathbf{u}, \mathbf{1})$ and $\delta_2^{01}(\mathbf{u}, \mathbf{0})$, therefore a more direct dependence between the vectors and the final result of the method is observable.

Since all the discussed TOPSIS aggregations are functions of merely two parameters: $mean(\mathbf{u})$ and $std(\mathbf{u})$, it is possible to visualize them in MSD-space assigning a color from some pre-defined color map to each value of the aggregation function. The change of color swiftly reveals the preferences as expressed by aggregations.

More precisely, for any two alternatives $\mathbf{u}_k \in US$, $\mathbf{u}_l \in US$ and $\mathbf{u}_k \neq \mathbf{u}_l$, if $\mathbf{u}_k$ is to be preferred to $\mathbf{u}_l$ under aggregation $\mathsf{I}(\mathbf{u})$, then $\mathsf{I}(\mathbf{u}_k) > \mathsf{I}(\mathbf{u}_l)$ must hold. Analogously for $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ aggregations. By the respective definitions of aggregations, this preference requires that: under aggregation $\mathsf{I}(\mathbf{u})$: $\delta_2^{01}(\mathbf{u}_k, \mathbf{1}) < \delta_2^{01}(\mathbf{u}_l, \mathbf{1})$ and under aggregation $\mathsf{A}(\mathbf{u})$: $\delta_2^{01}(\mathbf{u}_k, \mathbf{0}) > \delta_2^{01}(\mathbf{u}_l, \mathbf{0})$. More interestingly, under aggregation $\mathsf{R}(\mathbf{u})$: $\frac{\delta_2^{01}(\mathbf{u}_k,\mathbf{0})}{\delta_2^{01}(\mathbf{u}_k,\mathbf{0})+\delta_2^{01}(\mathbf{u}_k,\mathbf{1})} > \frac{\delta_2^{01}(\mathbf{u}_l,\mathbf{0})}{\delta_2^{01}(\mathbf{u}_l,\mathbf{0})+\delta_2^{01}(\mathbf{u}_l,\mathbf{1})} \Leftrightarrow \frac{\delta_2^{01}(\mathbf{u}_k,\mathbf{0})}{\delta_2^{01}(\mathbf{u}_k,\mathbf{1})} > \frac{\delta_2^{01}(\mathbf{u}_l,\mathbf{0})}{\delta_2^{01}(\mathbf{u}_l,\mathbf{1})}$ for $\mathbf{u}_k \neq \mathbf{1}$ and $\mathbf{u}_l \neq \mathbf{1}$, which is simultaneously equivalent to $\frac{\delta_2^{01}(\mathbf{u}_k,\mathbf{1})}{\delta_2^{01}(\mathbf{u}_k,\mathbf{0})} < \frac{\delta_2^{01}(\mathbf{u}_l,\mathbf{1})}{\delta_2^{01}(\mathbf{u}_l,\mathbf{0})}$ for $\mathbf{u}_k \neq \mathbf{0}$ and $\mathbf{u}_l \neq \mathbf{0}$; see Appendix B for a mathematical justification of this equivalence.

Figure 7 visualizes the three aggregations in two exemplary MSD-spaces, for $n = 4$ and $n = 5$. The used color map reflects the preference in the following manner: dark blue—the least preferred, dark red—the most preferred. Clearly, all of the discussed aggregations place the least preferred alternative in the lower-left vertex of the MSD-space (dark blue), where the anti-ideal point is situated. Analogously, the most preferred point with respect to all the considered aggregations (dark-red) is situated in the lower-right vertex of the MSD-space, where the ideal point resides. Figure 7 allows to quickly notice the differences between the aggregations, depicted by different isolines in the visualizations. Thus, the isolines of aggregation $\mathsf{I}(\mathbf{u})$ are concetric around $(1, 0)$, the isolines of aggregation $\mathsf{A}(\mathbf{u})$ are concetric around $(0, 0)$, while the isolines of aggregation $\mathsf{R}(\mathbf{u})$ are arch-like with their focus in $(1, 0)$ on the right-hand side of the space ($mean(\mathbf{u}) > 0.5$) and arch-like with their focus in $(0, 0)$ on the left-hand side of the space ($mean(\mathbf{u}) < 0.5$).

The differences between the three aggregations with respect to the isolines are a visual evidence of the ordinal non-equivalence of the aggregations. Moreover, it can also be observed that the isolines are the same under a chosen aggregation regardless of the number of criteria, which means that each of the aggregations works consistently no matter how many criteria are used to describe the alternatives.

All these visual-based discussions confirm that the introduced MSD-space is a practical tool for a swift comparison of TOPSIS aggregations.

4.2. Preference-related Interplay of mean($\mathbf{u}$) *and std*($\mathbf{u}$)

Adding color within the MSD-space to represent the values of the aggregation functions allows not only to conduct visual-based analysis of different aggregations, but also reveals the preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$. This, in turn shows the effects that changes to the values of criteria can have on the resulting ratings and rankings of alternatives. The awareness of such particular trade-offs between $mean(\mathbf{u})$ and $std(\mathbf{u})$ under different aggregations is important because they explain how carefully designed changes to the criteria may positively influence the final results of TOPSIS.

Formally, in preferential contexts $mean(\mathbf{u})$ and $std(\mathbf{u})$ behave like criteria. Their interplay is summarized in Table 1, showing under which aggregations and further conditions $mean(\mathbf{u})$ and $std(\mathbf{u})$ act like type 'cost' or type 'gain' criteria.

Table 1: Preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$

aggregation	$mean(\mathbf{u})$	$std(\mathbf{u})$
$\mathsf{I}(\mathbf{u})$	gain	cost
$\mathsf{A}(\mathbf{u})$	gain	gain
$\mathsf{R}(\mathbf{u})$	gain	$mean(\mathbf{u}) < 0.5$: gain $mean(\mathbf{u}) = 0.5$: neutrality $mean(\mathbf{u}) > 0.5$: cost

In particular, under all of the considered aggregations, increasing the $mean(\mathbf{u})$ of the alternatives always results in higher values of the respective aggregations, provided the $std(\mathbf{u})$ remains unchanged. In the context of our running example, this means that an increase in a student's $mean(\mathbf{u})$ will have a positive effect on their place in the final ranking (provided $std(\mathbf{u})$ remains unchanged) no matter which aggregation was chosen. In the example, this would naturally require getting some higher grades by a student. On the other hand, $std(\mathbf{u})$ acts differently under different aggregations. For example, it is of type 'gain' under the $\mathsf{R}(\mathbf{u})$ aggregation provided $mean(\mathbf{u})$ remains unchanged and does not exceed 0.5. Thus, a student who has a constant $mean(\mathbf{u}) < 0.5$, could go higher in the final ranking if she/he made her/his grades more variant, as it would increase $std(\mathbf{u})$ and the value of the $\mathsf{R}(\mathbf{u})$ aggregation function (see the subsequent section for a more detailed discussion on when making the grades more variant may actually prove beneficial).

All of the dependencies mentioned in Table 1 are n-independent as is exemplified in Figure 7 for $n = 4$ (left) and $n = 5$ (right). The dependencies are formally justified in the Appendix C and are easily noticeable thanks to the color map used in Figure 7:

- when one moves horizontally from left to right, one increases $mean(\mathbf{u})$ only; this corresponds to increasing the value of each aggregation ($\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$) for a given $std(\mathbf{u})$,

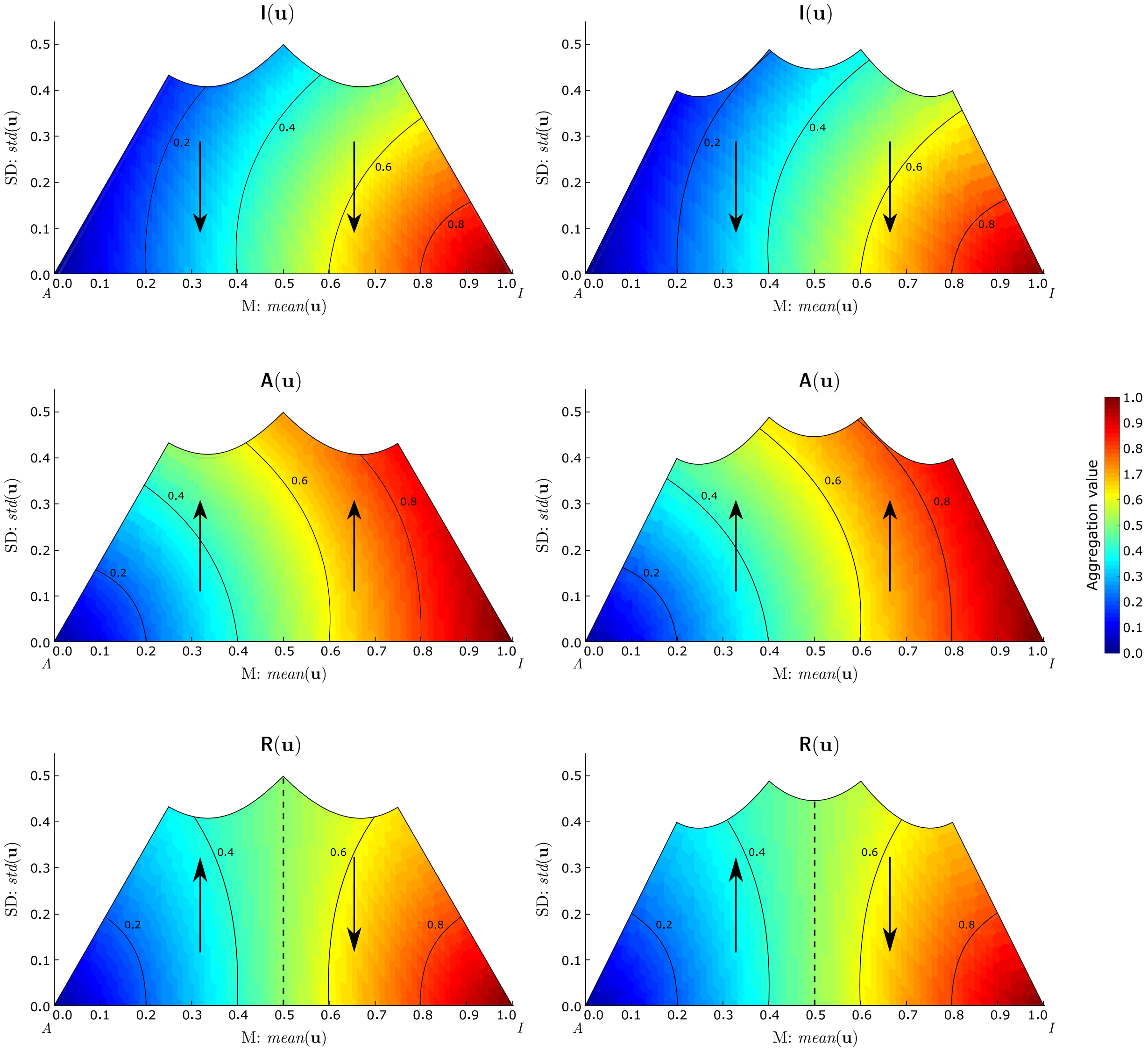

Figure 7: The visualization of preference as expressed by aggregations: $\mathsf{I}(\mathbf{u})$ (top), $\mathsf{A}(\mathbf{u})$ (middle) and $\mathsf{R}(\mathbf{u})$ (bottom) in the MSD-spaces defined for $n = 4$ (left) and $n = 5$ (right). The color map reflects the preference: dark blue – the least preferred, dark red – the most preferred (see the second bullet point for the interpretation of the arrows)

- when one moves vertically from bottom to top, one increases $std(\mathbf{u})$ only; this corresponds to decreasing $\mathsf{I}(\mathbf{u})$, increasing $\mathsf{A}(\mathbf{u})$, and to different effects on the value of $\mathsf{R}(\mathbf{u})$ depending on $mean(\mathbf{u})$ (the corresponding changes of the aggregations are indicated by arrows).

The preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$ directly influences the rankings of alternatives, which are the final result of TOPSIS. It can, thus, be regarded as an operational implication of the IA-MSD property. Examples of how slight changes to the values on the criteria of alternatives influence the alternative's position in the ranking due to the trade-offs between $mean(\mathbf{u})$ and $std(\mathbf{u})$ are discussed in the next section.

5. Case studies

In this section, we discuss two case studies that show how MSD-space can help visualize the relations between each alternative's properties and their rating. The first case study analyzes a hypothetical dataset of student grades that highlights the possible relations between alternatives. The second case study focuses on a real-world dataset of bus specifications and showcases the use of MSD-space in practical applications. Both studies will serve to discuss potential intervention actions that can be undertaken to influence the final ranking.

5.1. Student grades

To increase the rating of an alternative, the values of some of its criteria should be improved upon. Unfortunately, such a ploy may be very difficult, costly, or simply impossible. For example, in a dataset where students are described by grades obtained in different subjects, it would mean that a student has a chance to go higher in the ranking, provided they get better grades in some subjects and at least the same grades in all the others. The difficulty of such an achievement has been experienced by many, and is often simply unattainable. What remains in such cases is to improve the values of some criteria at the cost of worsening those of some others. The properties of TOPSIS described in this paper elucidate such endeavors, showing clearly that even even when $mean(\mathbf{u})$ *decreases* slightly, the simultaneous increase of $std(\mathbf{u})$ may have a positive overall effect, depending on the aggregation and value of $mean(\mathbf{u})$ (see the arrows in Figure 7). Therefore, carefully designed changes to the values of the criteria may positively influence the final results.

To illustrate how changes of $mean(\mathbf{u})$ and $std(\mathbf{u})$ affect the final rankings, a set of 19 alternatives has been prepared. Each alternative is described by three equally important criteria. Following the running example, the alternatives are students, while the criteria are subjects, with the average grades obtained by these students in those subjects serving as descriptions of the alternatives. The representations of all considered exemplary alternatives are shown in Table 4. Because the table provides matrix-like representations of alternatives, denotations throughout this section will reflect this fact, e.g. $\mathbf{U}_{i,*}$ (i-th row of a matrix) will denote the i-th alternative, $\mathbf{U}_{*,j}$ (j-th column of a matrix) will denote the j-th utility space dimension, while $\mathbf{U}_{i,j}$ will denote the value of j-th utility space coordinate of the i-th alternative.

The alternatives have been chosen to represent some characteristic points in the MSD-space (Figure 8), e.g. the worst possible alternative ($\mathbf{U}_{1,*} = \mathbf{0}$) or best possible alternative ($\mathbf{U}_{5,*} = \mathbf{1}$). Our goal is also to illustrate examples of trade-offs and compensation between the values of $mean(\mathbf{u})$ and $std(\mathbf{u})$, and to show how they influence the final rankings.

Notice that values of any aggregation directly determine the ranking of the alternatives, e.g. in the case of aggregation $\mathsf{R}(\mathbf{u})$ (see the last column of Table 4) the ranking is: $\mathbf{U}_{5,*} \succ_R \mathbf{U}_{4,*} \succ_R \mathbf{U}_{8,*} \succ_R \mathbf{U}_{16,*} \succ_R \mathbf{U}_{11,*} \succ_R \mathbf{U}_{15,*} \succ_R \mathbf{U}_{18,*} \succ_R \mathbf{U}_{3,*} \sim_R \mathbf{U}_{7,*} \sim_R \mathbf{U}_{19,*} \sim_R \mathbf{U}_{10,*} \sim_R \mathbf{U}_{14,*} \succ_R \mathbf{U}_{17,*} \succ_R \mathbf{U}_{13,*} \succ_R \mathbf{U}_{9,*} \succ_R \mathbf{U}_{12,*} \succ_R \mathbf{U}_{6,*} \succ_R \mathbf{U}_{2,*} \succ_R \mathbf{U}_{1,*}$. In the used notation, $\mathbf{U}_{i,*} \succ \mathbf{U}_{j,*}$: $\mathbf{U}_{i,*}$ is preferred over $\mathbf{U}_{j,*}$; $\mathbf{U}_{i,*} \sim \mathbf{U}_{j,*}$: $\mathbf{U}_{i,*}$ and $\mathbf{U}_{j,*}$ are indifferent; $\mathbf{U}_{i,*} \prec \mathbf{U}_{j,*}$: $\mathbf{U}_{j,*}$ is preferred over $\mathbf{U}_{i,*}$.

Let us look at the exemplary alternatives from perspectives corresponding to the three aggregations: $\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$. Aggregation $\mathsf{I}(\mathbf{u})$ ranks alternative $\mathbf{U}_{3,*}$ higher than alternative $\mathbf{U}_{2,*}$. Both alternatives have the same $std(\mathbf{u}) = 0$, but $mean(\mathbf{U}_{3,*}) = 0.5 > mean(\mathbf{U}_{2,*}) = 0.33$. The ranking of those two alternatives (i.e. $\mathbf{U}_{3,*} \succ_I \mathbf{U}_{2,*}$) is the same also under $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ (i.e. $\mathbf{U}_{3,*} \succ_A \mathbf{U}_{2,*}$ and $\mathbf{U}_{3,*} \succ_R \mathbf{U}_{2,*}$). This derives from the fact that $mean(\mathbf{u})$ is of type 'gain' under all the considered aggregations. Thus, within alternatives with the same $std(\mathbf{u})$, the increase of $mean(\mathbf{u})$ for one alternative will always increase the value of its aggregation function and (possibly) move the alternative up the resulting ranking. However, increasing the $mean(\mathbf{u})$ is often difficult or even impossible, thus other actions can be considered. To this end, let us have a closer look at alternatives $\mathbf{U}_{3,*}$, $\mathbf{U}_{7,*}$, $\mathbf{U}_{10,*}$ and $\mathbf{U}_{14,*}$, which happen to occupy the very middle part of the MSD-space (Figure 8). They are all characterized by $mean(\mathbf{u}) = 0.5$, but $std(\mathbf{U}_{3,*}) < std(\mathbf{U}_{7,*}) < std(\mathbf{U}_{10,*}) < std(\mathbf{U}_{14,*})$.

Under aggregation $\mathsf{I}(\mathbf{u})$ we obtain the following ranking: $\mathbf{U}_{3,*} \succ_I \mathbf{U}_{7,*} \succ_I \mathbf{U}_{10,*} \succ_I \mathbf{U}_{14,*}$, which places alternative $\mathbf{U}_{3,*}$ as the best and alternative $\mathbf{U}_{14,*}$ as the worst. This is due to the fact that $std(\mathbf{u})$ is of type 'cost' under $\mathsf{I}(\mathbf{u})$. Thus within students of the same $mean(\mathbf{u})$ this aggregation rates higher those with less

Table 2: Descriptions of the exemplary alternatives in terms of US, MSD-space and the three aggregations

Alternatives	Subjects			US			MSD-space		Aggregations		
	Math	Bio	Art	$\mathbf{U}_{*,1}$	$\mathbf{U}_{*,2}$	$\mathbf{U}_{*,3}$	$mean(\mathbf{u})$	$std(\mathbf{u})$	$\mathsf{I}(\mathbf{u})$	$\mathsf{A}(\mathbf{u})$	$\mathsf{R}(\mathbf{u})$
$\mathbf{U}_{1,*}$	0.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathbf{U}_{2,*}$	33.33	2.67	2.67	0.33	0.33	0.33	0.33	0.00	0.33	0.33	0.33
$\mathbf{U}_{3,*}$	50.00	3.50	3.50	0.50	0.50	0.50	0.50	0.00	0.50	0.50	0.50
$\mathbf{U}_{4,*}$	66.67	4.33	4.33	0.67	0.67	0.67	0.67	0.00	0.67	0.67	0.67
$\mathbf{U}_{5,*}$	100.00	6.00	6.00	1.00	1.00	1.00	1.00	0.00	1.00	1.00	1.00
$\mathbf{U}_{6,*}$	20.00	2.25	3.75	0.20	0.25	0.55	0.33	0.15	0.32	0.37	0.35
$\mathbf{U}_{7,*}$	67.64	2.50	3.62	0.68	0.30	0.52	0.50	0.15	0.48	0.52	0.50
$\mathbf{U}_{8,*}$	45.00	4.75	5.00	0.45	0.75	0.80	0.67	0.15	0.63	0.68	0.65
$\mathbf{U}_{9,*}$	0.00	2.25	4.75	0.00	0.25	0.75	0.33	0.31	0.26	0.46	0.38
$\mathbf{U}_{10,*}$	62.99	5.00	1.35	0.63	0.80	0.07	0.50	0.31	0.41	0.59	0.50
$\mathbf{U}_{11,*}$	25.00	4.75	6.00	0.25	0.75	1.00	0.67	0.31	0.54	0.74	0.62
$\mathbf{U}_{12,*}$	0.00	1.00	5.25	0.00	0.00	0.85	0.28	0.40	0.18	0.49	0.37
$\mathbf{U}_{13,*}$	0.00	1.46	5.54	0.00	0.09	0.91	0.33	0.41	0.22	0.53	0.40
$\mathbf{U}_{14,*}$	0.00	3.50	6.00	0.00	0.50	1.00	0.50	0.41	0.35	0.65	0.50
$\mathbf{U}_{15,*}$	9.00	5.55	6.00	0.09	0.91	1.00	0.67	0.41	0.47	0.78	0.60
$\mathbf{U}_{16,*}$	100.00	6.00	1.75	1.00	1.00	0.15	0.72	0.40	0.49	0.82	0.63
$\mathbf{U}_{17,*}$	0.00	1.00	6.00	0.00	0.00	1.00	0.33	0.47	0.18	0.58	0.41
$\mathbf{U}_{18,*}$	100.00	6.00	1.00	1.00	1.00	0.00	0.67	0.47	0.42	0.82	0.59
$\mathbf{U}_{19,*}$	37.00	3.10	4.55	0.37	0.42	0.71	0.50	0.15	0.48	0.52	0.50

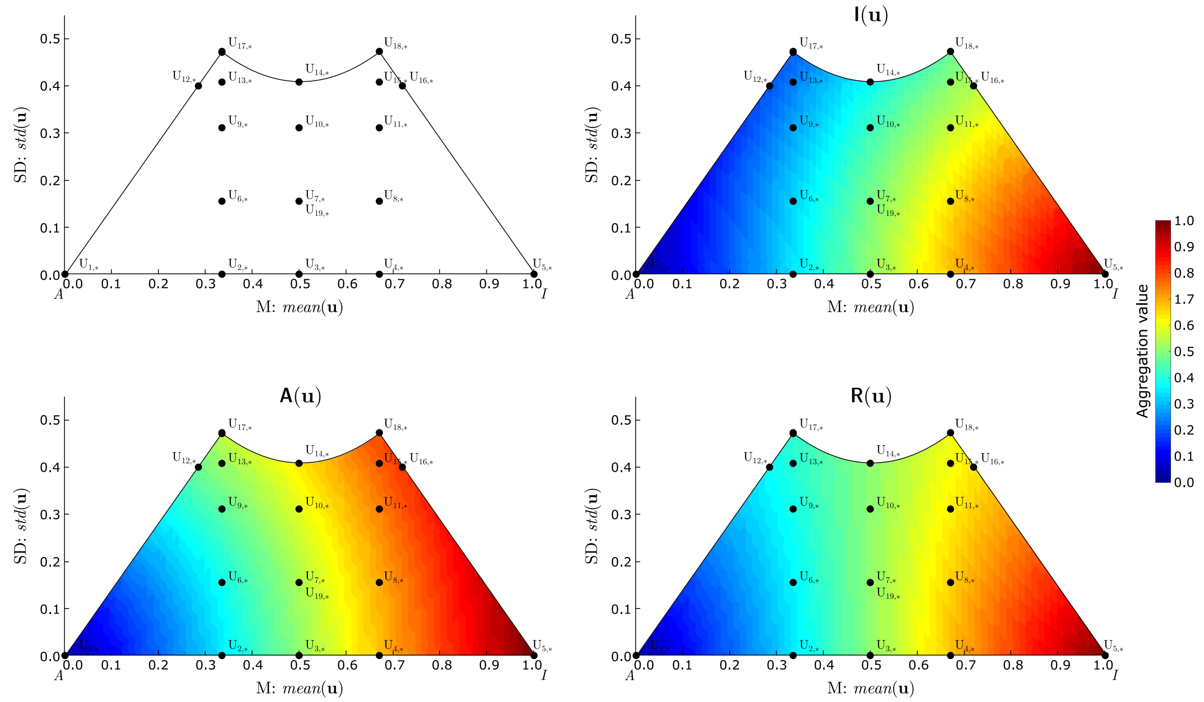

Figure 8: Visualizations of the student case study alternatives in MSD-space. MSD-space presented without any aggregations, as well as with the three considered aggregations: $\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$.

diverse grades. For example, alternative $\mathbf{U}_{3,*}$ with utilities $\mathbf{U}_{3,1} = \mathbf{U}_{3,2} = \mathbf{U}_{3,3} = 0.5$ is clearly preferred over alternative $\mathbf{U}_{14,*}$ with utilities $\mathbf{U}_{14,1} = 0$, $\mathbf{U}_{14,2} = 0.5$, $\mathbf{U}_{14,3} = 1$.

Aggregation $\mathsf{A}(\mathbf{u})$ creates the opposite ranking: $\mathbf{U}_{3,*} \prec_I \mathbf{U}_{7,*} \prec_I \mathbf{U}_{10,*} \prec_I \mathbf{U}_{14,*}$, as $std(\mathbf{u})$ is of type 'gain' under $\mathsf{A}(\mathbf{u})$, which means that within students of the same $mean(\mathbf{u})$ this aggregation rates higher those with more diverse grades. Thus alternative $\mathbf{U}_{14,*}$ is now clearly preferred over alternative $\mathbf{U}_{3,*}$.

Interestingly, for aggregation $\mathsf{R}(\mathbf{u})$, $std(\mathbf{u})$ does not influence the preference at all when $mean(\mathbf{u}) = 0.5$, and the four considered alternatives are ranked $\mathbf{U}_{3,*} \sim_R \mathbf{U}_{7,*} \sim_R \mathbf{U}_{10,*} \sim_R \mathbf{U}_{14,*}$. The differences in the rankings created by the different aggregations are clearly noticeable after the alternatives are depicted within the MSD-space sketched on top of the isolines of the aggregation function (Figure 8). In particular, the straight vertical isoline in green for $mean(\mathbf{u}) = 0.5$ in Figure 8 exposes that $std(\mathbf{u})$ is not a criterion under aggregation $\mathsf{R}(\mathbf{u})$ when $mean(\mathbf{u}) = 0.5$.

Table 3: Exemplary rankings of subsets S_1 and S_2 under different aggregations

S_1	S_2
$\mathbf{U}_{2,*} \succ_I \mathbf{U}_{6,*} \succ_I \mathbf{U}_{9,*} \succ_I \mathbf{U}_{12,*}$	$\mathbf{U}_{4,*} \succ_I \mathbf{U}_{8,*} \succ_I \mathbf{U}_{11,*} \succ_I \mathbf{U}_{16,*}$
$\mathbf{U}_{12,*} \succ_A \mathbf{U}_{9,*} \succ_A \mathbf{U}_{6,*} \succ_A \mathbf{U}_{2,*}$	$\mathbf{U}_{16,*} \succ_A \mathbf{U}_{11,*} \succ_A \mathbf{U}_{8,*} \succ_A \mathbf{U}_{4,*}$
$\mathbf{U}_{9,*} \succ_R \mathbf{U}_{12,*} \succ_R \mathbf{U}_{6,*} \succ_R \mathbf{U}_{2,*}$	$\mathbf{U}_{4,*} \succ_R \mathbf{U}_{8,*} \succ_R \mathbf{U}_{16,*} \succ_R \mathbf{U}_{11,*}$

The isolines of $\mathsf{R}(\mathbf{u})$ in MSD-space also demonstrate the similarity of $\mathsf{R}(\mathbf{u})$ to $\mathsf{I}(\mathbf{u})$ on one hand, and to $\mathsf{A}(\mathbf{u})$ on the other: its right part resembles $\mathsf{I}(\mathbf{u})$ (with $std(\mathbf{u})$ being of type 'cost' for $mean(\mathbf{u}) > 0.5$), while its left part resembles $\mathsf{A}(\mathbf{u})$ (with $std(\mathbf{u})$ being of type 'gain' for $mean(\mathbf{u}) < 0.5$). To be precise, the particular isolines of $\mathsf{I}(\mathbf{u})$ and $\mathsf{A}(\mathbf{u})$ are slightly different in shape from those of $\mathsf{R}(\mathbf{u})$, however, the common character of those isolines is clearly noticeable.

The phenomenon of opposite rankings under different aggregations is further corroborated with subsets $\{\mathbf{U}_{2,*}, \mathbf{U}_{6,*}, \mathbf{U}_{9,*}\}$ ($mean(\mathbf{u}) = 0.33$) and $\{\mathbf{U}_{4,*}, \mathbf{U}_{8,*}, \mathbf{U}_{11,*}\}$ ($mean(\mathbf{u}) = 0.67$). The first subset is ranked $\mathbf{U}_{2,*} \prec_A \mathbf{U}_{6,*} \prec_A \mathbf{U}_{9,*}$ and $\mathbf{U}_{2,*} \prec_R \mathbf{U}_{6,*} \prec_R \mathbf{U}_{9,*}$, but $\mathbf{U}_{2,*} \succ_I \mathbf{U}_{6,*} \succ_I \mathbf{U}_{9,*}$, demonstrating the resemblance of $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ for $mean(\mathbf{u}) < 0.5$. Analogous situation is observable for the second subset, which is ranked in the same way by $\mathsf{I}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ and in a different way by $\mathsf{A}(\mathbf{u})$, demonstrating the resemblance of $\mathsf{I}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ for $mean(\mathbf{u}) > 0.5$

Having considered different cases where either $mean(\mathbf{u})$ or $std(\mathbf{u})$ has a common value, we will now investigate situations when they have different values and their differences can compensate each other. Let us consider two subsets: $S_1 = \{\mathbf{U}_{2,*}, \mathbf{U}_{6,*}, \mathbf{U}_{9,*}, \mathbf{U}_{12,*}\}$ and $S_2 = \{\mathbf{U}_{4,*}, \mathbf{U}_{8,*}, \mathbf{U}_{11,*}, \mathbf{U}_{16,*}\}$. Let us contemplate alternatives from S_1. Interestingly, or even surprisingly, $\mathbf{U}_{12,*}$ is ranked higher under $\mathsf{A}(\mathbf{u})$ than $\mathbf{U}_{2,*}$, $\mathbf{U}_{6,*}$ or even $\mathbf{U}_{9,*}$ despite having lower mean. The high value of $std(\mathbf{U}_{12,*})$ compensated the lower $mean(\mathbf{U}_{12,*})$ and placed this alternative higher in the ranking. Recalling the interpretation of the example, this means that under $\mathsf{A}(\mathbf{u})$ students with very diverse grades could be preferred over students with less diverse grades even when their $mean(\mathbf{u})$ is lower. A similarly surprising ranking is produced under $\mathsf{I}(\mathbf{u})$ for alternatives from S_2, in which case $\mathbf{U}_{16,*}$ is ranked lower than $\mathbf{U}_{4,*}$, $\mathbf{U}_{8,*}$ or even than $\mathbf{U}_{11,*}$, despite fact that $\mathbf{U}_{16,*}$ has a higher mean. Here, $std(\mathbf{u})$ is of type 'cost' for $\mathsf{I}(\mathbf{u})$, and therefore the high $std(\mathbf{U}_{16,*})$ disadvantaged the superior value of $mean(\mathbf{U}_{16,*})$.

Finally, to investigate all possibilities let us discuss a situation where both $mean(\mathbf{u})$ and $std(\mathbf{u})$ are constant. Notice that this corresponds to different vectors in US, but single point in the MSD-space (which is the image of all those vectors). In fact, for $n > 2$ there exist potentially infinitely many such vectors (e.g. if $std(\mathbf{u}) > 0$ for $n = 3$ the representation of all such vectors constitutes parts of or the whole of a circle of radius $std(\mathbf{u})$, located in a plane orthogonal to vector $\overline{\mathbf{u}}$ and centered exactly in $\overline{\mathbf{u}}$). For example, for $n = 3$, vector $U_{7,*} = [0.29, 0.58, 0.63]$ is characterized by $mean(\mathbf{U_{7,*}}) = 0.5$ and $std(\mathbf{U_{7,*}}) = 0.15$, as is any vector composed of any permutation of its components (e.g. $[0.58, 0.63, 0.29]$ or $[0.58, 0.29, 0.63]$). Additionally, there exist also vectors composed of other values, e.g. $U_{19,*} = [0.37, 0.42, 0.71]$, that are characterized by the same $mean(\mathbf{u})$ and $std(\mathbf{u})$. As a result, all those vectors share the same point in the MSD-space, and are thus identically evaluated by all three considered aggregations (in particular, $\mathbf{U}_{7,*} \sim_I \mathbf{U}_{19,*}$ and $\mathbf{U}_{7,*} \sim_A \mathbf{U}_{19,*}$

and $\mathbf{U}_{7,*} \sim_R \mathbf{U}_{19,*}$). This shows the inherently two-dimensional nature of TOPSIS as visualized by the MSD-space.

5.2. Bus specifications

The real-world dataset presented in this case study is based on the one used by Greco et al. (2013) and Zielniewicz (2017), which describes the technical condition of 32 buses (Table 4). Each bus is characterized by eight numeric attributes considered during the periodical technical inspection of the vehicles. There are no missing values and the dataset is considered noise-free as it was carefully prepared by domain experts. There are four criteria of type 'gain' and four criteria of type 'cost':

- Speed [gain]—maximum speed [km/h],
- Pressure [gain]—compression pressure [Mpa],
- Blacking [cost]—blacking components in exhaust gas [%],
- Torque [gain]—torque [Nm],
- Summer [cost]—summer fuel consumption [l/100 km],
- Winter [cost]—winter fuel consumption [l/100 km],
- Oil [cost]—oil consumption [l/100 km],
- HP [gain]—maximum horsepower of the engine [hp].

Table 4 contains firstly the specification of buses, i.e., their representation in criteria space, then their $mean(\mathbf{u})$ and $std(\mathbf{u})$ coordinates in MSD-space[3], and finally the values of the three considered aggregation functions that determine the position of the alternative in the final rankings. The utility space coordinates were not shown to keep the table compact.

Figure 9 visualizes the 32 alternatives in MSD-space: firstly without any aggregation, then with the three analyzed aggregations ($\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$) imposed on the space through color. The very shape of the MSD-space immediately depicts that the dataset is 8-dimensional, as there are nine vertices (see Appendix A for the formal definitions of the vertices and diagonals in MSD-space). Naturally, visualization of such dataset in the criteria space or utility space would not be possible. As seen in Figure 9, MSD-space is not equally populated with representations of the buses. Uneven density of alternatives is the phenomenon of real-world datasets that limits the generality of analyses conducted on such particular datasets, making them dataset-dependent.

One of the operational implications of MSD-space visualizations is that they help to understand why certain alternatives are higher or lower in rankings, and thus increase the explainability of TOPSIS aggregations. To illustrate this, consider buses $\mathbf{b}_{03}$ and $\mathbf{b}_{08}$ from Table 4. By looking solely at the specification of the buses it could be difficult to make out their positions in the rankings. But expressing the specification with $mean(\mathbf{u})$ and $std(\mathbf{u})$ of an alternative puts us in a much more intuitive and comprehensible 2D space as presented in Figure 9. Clearly, $\mathbf{b}_{03}$ and $\mathbf{b}_{08}$ are described by very similar values of $mean(\mathbf{u})$ ($mean(\mathbf{b}_{03}) = 0.50$ and $mean(\mathbf{b}_{08}) = 0.49$) and differ mostly on $std(\mathbf{u})$ (0.22 and 0.36, respectively). The preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$ shows the trade-offs made by different aggregation functions (recall Table 1) and thus explains the ranking position of alternatives. Among $\mathbf{b}_{03}$ and $\mathbf{b}_{08}$, it is the $\mathbf{b}_{03}$ bus that has the highest $mean(\mathbf{u})$ and smallest $std(\mathbf{u})$, which puts it higher in the ranking under $\mathsf{I}(\mathbf{u})$ aggregation ($\mathbf{b}_{08} \prec_I \mathbf{b}_{03}$). On the other hand, for $\mathsf{A}(\mathbf{u})$, $std(\mathbf{u})$ is of type 'gain'. As a result, the slight difference in $mean(\mathbf{u})$ of $\mathbf{b}_{03}$ and $\mathbf{b}_{08}$ is compensated by high $std(\mathbf{u})$ of $\mathbf{b}_{08}$, and thus $\mathbf{b}_{03} \prec_A \mathbf{b}_{08}$. Furthermore, because under $\mathsf{R}(\mathbf{u})$ aggregation for $mean(\mathbf{u}) \approx 0.5$, $std(\mathbf{u})$ has hardly any influence on the ranking, $\mathbf{b}_{03} \sim_R \mathbf{b}_{08}$.

[3]While computing means and standard deviations of numeric values coming from very different ranges is formally always feasible and does not invalidate the correctness of depicting the objects within the MS-space, it might not have clear interpretations and should be used with care

Table 4: Description of alternatives in the bus specification dataset.

	Specifications								MSD-space		Aggregations		
Bus	Speed	Pressure	Blacking	Torque	Summer	Winter	Oil	HP	$mean(\mathbf{u})$	$std(\mathbf{u})$	$\mathsf{I}(\mathbf{u})$	$\mathsf{A}(\mathbf{u})$	$\mathsf{R}(\mathbf{u})$
$\mathbf{b}_{01}$	90	2	49	477	21	25	1	138	0.85	0.11	0.81	0.85	0.82
$\mathbf{b}_{02}$	85	2	52	460	21	25	1	130	0.78	0.11	0.75	0.79	0.76
$\mathbf{b}_{03}$	72	2	73	425	23	27	2	112	0.50	0.22	0.45	0.55	0.50
$\mathbf{b}_{04}$	88	2	50	480	21	24	1	140	0.86	0.10	0.82	0.86	0.83
$\mathbf{b}_{05}$	60	1	95	400	23	24	4	96	0.18	0.33	0.12	0.38	0.30
$\mathbf{b}_{06}$	78	2	63	448	21	26	1	120	0.67	0.18	0.63	0.70	0.65
$\mathbf{b}_{07}$	90	2	26	482	22	24	0	148	0.95	0.09	0.89	0.95	0.90
$\mathbf{b}_{08}$	65	2	67	402	22	23	2	103	0.49	0.36	0.38	0.61	0.50
$\mathbf{b}_{09}$	90	2	51	468	22	26	1	138	0.80	0.13	0.76	0.81	0.77
$\mathbf{b}_{10}$	76	2	65	428	27	33	2	116	0.40	0.30	0.33	0.50	0.42
$\mathbf{b}_{11}$	85	2	50	454	21	26	1	129	0.76	0.12	0.73	0.77	0.74
$\mathbf{b}_{12}$	85	2	58	450	22	25	1	126	0.72	0.15	0.69	0.74	0.70
$\mathbf{b}_{13}$	88	2	48	458	22	25	1	130	0.78	0.12	0.75	0.79	0.76
$\mathbf{b}_{14}$	75	2	64	432	22	25	1	114	0.62	0.22	0.56	0.65	0.60
$\mathbf{b}_{15}$	68	2	70	400	22	26	2	100	0.45	0.32	0.37	0.55	0.47
$\mathbf{b}_{16}$	88	2	44	478	21	25	0	138	0.88	0.09	0.85	0.89	0.86
$\mathbf{b}_{17}$	85	2	55	445	23	26	1	120	0.68	0.17	0.64	0.70	0.66
$\mathbf{b}_{18}$	90	2	40	480	22	25	0	139	0.88	0.11	0.84	0.89	0.85
$\mathbf{b}_{19}$	72	2	64	428	21	25	2	111	0.58	0.25	0.51	0.63	0.56
$\mathbf{b}_{20}$	75	2	60	440	22	26	1	120	0.64	0.18	0.60	0.66	0.62
$\mathbf{b}_{21}$	85	2	61	458	21	25	1	126	0.75	0.15	0.71	0.76	0.72
$\mathbf{b}_{22}$	68	2	88	422	22	25	3	108	0.45	0.31	0.37	0.55	0.47
$\mathbf{b}_{23}$	82	2	65	430	23	25	2	115	0.59	0.22	0.54	0.63	0.58
$\mathbf{b}_{24}$	90	2	38	482	20	24	0	146	0.96	0.06	0.93	0.96	0.93
$\mathbf{b}_{25}$	90	2	45	479	21	25	1	145	0.87	0.10	0.84	0.88	0.84
$\mathbf{b}_{26}$	90	2	34	486	21	25	0	148	0.94	0.08	0.90	0.95	0.91
$\mathbf{b}_{27}$	86	2	60	444	22	25	1	122	0.71	0.17	0.66	0.73	0.68
$\mathbf{b}_{28}$	88	2	50	475	22	25	1	142	0.83	0.11	0.79	0.83	0.80
$\mathbf{b}_{29}$	85	2	63	440	21	26	2	120	0.66	0.20	0.60	0.69	0.64
$\mathbf{b}_{30}$	72	2	85	420	22	25	3	110	0.48	0.30	0.40	0.56	0.48
$\mathbf{b}_{31}$	65	2	94	400	24	27	4	98	0.28	0.34	0.20	0.44	0.36
$\mathbf{b}_{32}$	87	2	60	460	22	25	1	131	0.76	0.14	0.72	0.77	0.73

The preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$ can serve as a guideline for decision makers as to which aggregation could suit them more. For example, the $\mathsf{A}(\mathbf{u})$ aggregation could be recommended for decision makers which favor diversified descriptions of alternatives, as high standard deviation increases $\mathsf{A}(\mathbf{u})$ aggregation when the $mean(\mathbf{u})$ is fixed. On the other hand, decision makers that are 'diversification-averse' could prefer the $\mathsf{I}(\mathbf{u})$ aggregation. However, it should be clearly stated that the final decision as to which aggregation should be used has to always be made with respect to the task at hand.

Among operational implications of MSD-space visualizations one should also mention the ability to formulate intervention actions. MSD-space colored with respect to a certain aggregation shows what actions can be taken to make an alternative go higher in the ranking under particular aggregations. To illustrate this, consider alternatives $\mathbf{b}_{20}$ and $\mathbf{b}_{29}$. Under the $\mathsf{R}(\mathbf{u})$ aggregation, bus $\mathbf{b}_{29}$ is preferred over $\mathbf{b}_{20}$ ($\mathbf{b}_{20} \prec_R \mathbf{b}_{29}$). Of course the natural way of putting $\mathbf{b}_{20}$ higher in the ranking would be to increase its value on one of the 'gain' criteria or decrease on one of the 'cost' criteria without changing the other criteria. This could be, however, hard or even impossible. One could then analyze the shape of the isolines in the $\mathsf{R}(\mathbf{u})$ aggregation (Figure 9) and conclude that moving the alternative $\mathbf{b}_{20}$ down the SD-axis (i.e. decreasing $std(\mathbf{b}_{20})$ without changing $mean(\mathbf{b}_{20})$) would be beneficial for $\mathbf{b}_{20}$'s position on the ranking. As a result, it would be enough to make the specification of the alternative less diversified (without changing its mean), to get it higher in the ranking based on the $\mathsf{R}(\mathbf{u})$ aggregation. Analogous intervention actions could be formulated for other considered aggregations.

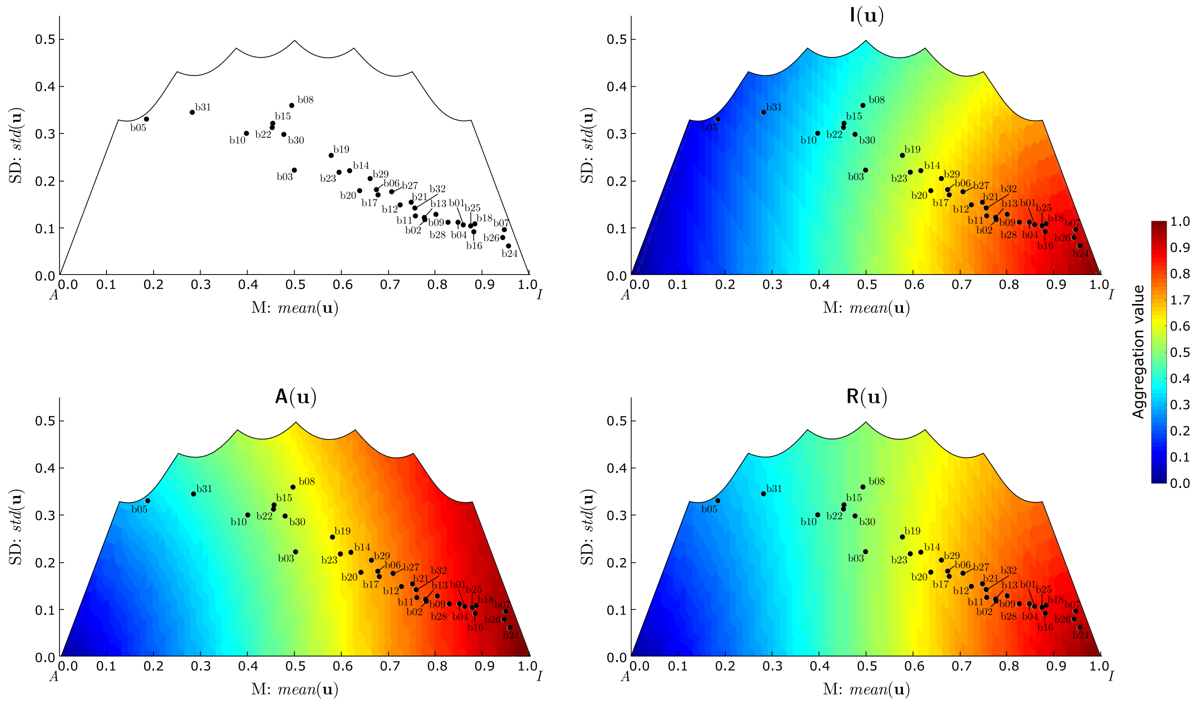

Figure 9: Visualizations of bus specification dataset in MSD-space. MSD-space presented without any aggregations, as well as with the three considered aggregations: $\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$.

6. Conclusions and Future Works

The popularity of TOPSIS motivated us to investigate its inner workings, as thorough understanding of the algebraic aspects of this approach leads to better explainability and interpretability of the rankings created by the method. Moreover, such knowledge reveals how carefully designed changes to the criteria may influence the final results and thus helps constitute potential improvement actions.

The paper's main results demonstrate that the alternative's preferences calculated in TOPSIS as their distances to ideal and anti-ideal alternatives can be actually expressed with two fundamental features of the alternatives: the mean value of their utilities ($mean(\mathbf{u})$) and the standard deviation of their utilities ($std(\mathbf{u})$). These two features, representing a measure of tendency and a measure of variation (or the proverbial variety), are easily interpretable and directly influence the final ratings and rankings of the alternatives. Thus, they naturally create a two-dimensional MSD-space of alternatives.

The IA-MSD property put forward in this paper formalizes the dependencies between the distances to ideal and anti-ideal alternatives on one hand and $mean(\mathbf{u})$ and $std(\mathbf{u})$ on the other. As a result, the aggregations of the distances applied in TOPSIS ($\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$) can also be expressed in terms of $mean(\mathbf{u})$ and $std(\mathbf{u})$, further revealing the 'gain'/'cost' type of these fundamental features. Formally, $mean(\mathbf{u})$ under $\mathsf{I}(\mathbf{u})$, $\mathsf{A}(\mathbf{u})$ and $\mathsf{R}(\mathbf{u})$ is of type 'gain', which means that for a fixed $std(\mathbf{u})$ any increase of alternative's mean will increase the alternative's ratings and possibly its ranking. On the other hand, the type of $std(\mathbf{u})$ depends on the chosen aggregation, being 'cost' under $\mathsf{I}(\mathbf{u})$, 'gain' under $\mathsf{A}(\mathbf{u})$ and conditionally 'gain'/'cost' under $\mathsf{R}(\mathbf{u})$.

Identification of 'gain'/'cost' type of the fundamental features under particular aggregations leads to revealing the preference-related interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$. The trade-offs and compensation between $mean(\mathbf{u})$ and $std(\mathbf{u})$ directly influence the rankings produced by TOPSIS. Examples of how changing the $mean(\mathbf{u})$ and $std(\mathbf{u})$ affects the final ratings and rankings of alternatives are discussed in the paper showing applicability of the results.

As far as visualization is concerned, the MSD-space serves this purpose well, since it can always be successfully depicted in a plane, as opposed to CS and US, which are n-dimensional. The depiction of the MSD-space becomes particularly useful when shown together with the isolines of the aggregation function. It can then serve as a tool for swift visual analysis and comparison of aggregation procedures as well as for visualizing how the interplay of $mean(\mathbf{u})$ and $std(\mathbf{u})$ affects the created rankings.

Lines of further investigation include analyses of modifications of the presented TOPSIS aggregations. In particular, it is very common to normalize or weigh criteria for a given ranking application. Therefore, it would be interesting to see how such criteria weighting affects the aggregation process and its visualization in MSD-space. This, in fact, shall be the subject of our follow-up paper. Moreover, not all coordinates in MSD-space are equally populated with alternatives, with some values of the mean and standard deviation being more probable than others. Therefore, it would be interesting to provide insight into the density of alternatives in MSD-space, which could help identify the actual areas where most alternatives are expected.

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