# The Hong Kong University of Science and Technology

## Department of Electronic and Computer Engineering

## ELEC 210: Probability and Random Processes in Electrical Engineering

## 2010 Fall Semester

## Final Examination

## December 14, 2010

**Name:** ________________________________

**Student ID:** ________________________________

# Instructions:

1. This is a 3-hour test.
2. You may use a non-programmable calculator.
3. There are **2 sections**:
   a. Section 1: 10 Multiple Choice Questions
   b. Section II: 6 Problems
4. Try to attempt all questions.
5. Answer each question in your **answer booklet**.
6. The distribution of marks is shown in the table below:

| **Question** | **Mark** |
|---|---|
| Multi. Choice | 10 |
| 1 | 6 |
| 2 | 7 |
| 3 | 6 |
| 4 | 7 |
| 5 | 7 |
| 6 | 7 |
| **Total** | **50** |

# SECTION I: Multiple Choice [10 Marks]

**Notes:**

- **For each multiple-choice question below, select only ONE answer.**
- **Write your answer in your ANSWER BOOKLET (not this question sheet)**

**[1 Mark Per Question]**

1. Which of the following statements about two random variables, $X$ and $Y$, is not true:
   a) The correlation of $X$ and $Y$ is symmetrical
   b) The covariance of $X$ and $Y$ is symmetrical
   c) The correlation coefficient of $X$ and $Y$ is symmetrical
   d) The covariance and correlation have the same units
   e) The covariance and correlation coefficient have the same units

2. Which of the following statements is true:
   a) Correlation of $X$ and $Y$ gives a meaningful measure of *how much* $X$ and $Y$ vary together
   b) Scaling $X$ or $Y$ by a constant does not change their covariance
   c) Scaling $X$ or $Y$ by a constant does not change their correlation
   d) Scaling $X$ or $Y$ by a constant does not change their correlation coefficient
   e) None of the above

3. Consider the following communication system: A transmitter sends a binary value $B$ to a receiver. The received signal is $R=B+N$, where $N$ denotes independent random noise obeying a Gaussian distribution. Based on $R$, the receiver successfully decodes $B$ with probability $p$; otherwise it makes a mistake and the transmitter tries again. Let $n$ be the number of times that $B$ must be transmitted before it is received successfully. Which of the following is not true:
   a) Conditioned on $B$, the received signal $R$ is a Gaussian random variable
   b) Conditioned on $B$, the received signal $R$ has zero-mean
   c) Conditioned on $N$, the received signal $R$ is a Bernoulli random variable
   d) The received signal $R$ is a continuous random variable
   e) $n$ is a geometric random variable

4. Assume that $Y_1$, $Y_2$, $Y_3$ are IID jointly Gaussian random variables. Identify which of the following is not true:
   a) $Y_1$ and $Y_3$ are jointly Gaussian distributed
   b) Conditioned on $Y_2$, $Y_1$ and $Y_3$ are jointly Gaussian distributed
   c) For constants $a$, $b$, and $c$, the variable $aY_1+bY_2+c$ is Gaussian distributed
   d) $Y_1/Y_3$ is Gaussian distributed
   e) $Y_1$ and $Y_3$ are uncorrelated

5. Assume that we repeatedly roll a fair die. Let $X_n$ denote the number of the die on the $n^{th}$ roll. Also, let $S_n = X_1 + \ldots + X_n$. Which of the following is not true:

a) As we roll more and more die, the distribution of $S_n$ approaches a Gaussian
b) As we roll more and more die, the average value of $S_n$ approaches 3.5*$n$
c) As we roll more and more die, the variance of $S_n$ approaches $n$*VAR[$X_n$], with VAR[$X_n$] denoting the variance of $X_n$
d) $S_n$ is a binomial random variable
e) None of the above (i.e., they are all true)

6. Which of the following statements about joint distribution of random variables X ad Y is not true:
   a) The marginal densities can always be recovered from the joint density.
   b) A product form event defined on X and Y has a rectangular shape in the X-Y plane.
   c) The joint CDF is non-decreasing.
   d) The joint CDF is the product of the two marginal CDFs.
   e) None of the above (i.e., they are all true)

7. Suppose $X_1$ and $X_2$ are two uncorrelated random variables. Which of the following statements is not true:
   a) Mean of the sum equals the sum of the mean: $E[X_1 + X_2] = E[X_1]+E[X_2]$.
   b) Mean of the product equals the product of the mean: $E[X_1X_2] = E[X_1]E[X_2]$.
   c) Variance of the sum equals the sum of the variance: $Var[X_1 + X_2] = Var[X_1]+Var[X_2]$.
   d) The covariance between $X_1$ and $X_2$ is zero: $Cov[X_1 X_2]=0$.
   e) None of the above (i.e., they are all true)

8. Which of the following statements about random process is not true:
   a) Random process assigns a time function to every outcome of an experiment.
   b) Events of interest for a random process concern the value of the random process at specific instants.
   c) A random process is uniquely specified by the collection of all n-th order distribution.
   d) The underlying experiment for a Bernoulli random process is a Bernoulli experiment.
   e) None of the above (i.e., they are all true)

9. Which of the following statements about random process is not true:
   a) The mean and variance of a random process can not be constant.
   b) The mean and variance of a random process can be a function of time.
   c) The mean of an i.i.d. process is a constant.
   d) The covariance function of an i.i.d. process is a delta function.
   e) None of the above (i.e., they are all true)

10. Which of the following statements about discrete time random process is not true:
    a) A process is said to be an independent and stationary increment (i.s.i.) process if its increments are both independent and stationary.
    b) Sum Processes are ISI.
    c) A sum process $S_n$ is obtained by taking the sum of all past values of a random process $X_n$.
    d) ISI Processes are Sum Processes.
    e) None of the above (i.e., they are all true)

# SECTION II: Problems [40 Marks]

**Notes:**

- **Please attempt all problems, clearly showing your working.**
- **Write your solution in your ANSWER BOOKLET (not this question sheet)**

**Question 1 (6 marks):**

An store sells a batch of 60 iPods, 10 of which are defective. Suppose that you purchase 2 iPods from this store for your FYP project. What is the probability that:

(a) At least 1 of the items that you buy are defective. (**2 marks**)

(b) None of the items that you buy are defective. (**1 marks**)

(c) Suppose that if any of the iPods that you purchased were defective, then you returned them to the store to swap for new iPods and you can only go back to the store for one time.
What is the probability that both the 2 iPods are non-defective? (**3 marks**)

Solution:

(a) Let $N$ denote the number of iPods that you buy are defective.

$$P[N \geq 2] = \sum_{n=1}^{2} P[N = n]$$

$$= \left[\binom{10}{1}\binom{50}{1} + \binom{10}{2}\right] / \binom{60}{2}$$

$$= 0.3079$$

(b) $P[N = 0] = \binom{50}{2} / \binom{60}{2}$

$$= 0.6921$$

(c) The event that both the 2 iPods are non-defective is equivalent to the union of the event that both the 2 iPods are non-defective and the event that all the defective iPods are swapped by non-defective iPods.

$$P[N = 0] = \sum_{n=0}^{2} \{P[N = n] \times P[\text{"the swapped iPods are non} - \text{defective"}]\}$$

$$= \frac{\binom{50}{2}}{\binom{60}{2}} + \frac{\binom{10}{1}\binom{50}{1}}{\binom{60}{2}} \frac{\binom{49}{1}}{\binom{58}{1}} + \frac{\binom{10}{2}}{\binom{60}{2}} \frac{\binom{50}{2}}{\binom{58}{2}}$$

the value here is 50

$$= 0.6921 + 0.2387 + 0.0188$$

$$= 0.9496$$

the value here is 0.9496

**Question 2** (**7 marks**):

The classrooms at HKUST use the same quantity of projectors manufactured by three companies A, B, and C. The defective probabilities of the projectors from three manufacturers are 0.001, 0.005, and 0.01, respectively. Assume that these projectors are randomly installed in different classrooms. One day, the projector in the classroom for ELEC210 does not work.

a) Find the probability that the manufacturer for the defective projector was A. (**4 marks**)

b) Find the probability that the manufacturer for the defective projector was C. (**3 marks**)

**Solution:**

(a) Let $X$ represent the company whose projector is used in ELEC210 classroom, $Y$ represent the quality of the projector used in this classroom, i.e.,

| $X$=1 | $X$=2 | $X$=3 |
|---|---|---|
| From company A | From company B | From company C |

| $Y$=0 | $Y$=1 |
|---|---|
| The projector is defective. | The projector is good. |

$$P(X=1|Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}$$

$$=\frac{P(Y=0|X=1)P(X=1)}{\sum_k P(Y=0|X=k)P(X=k)}$$

$$=\frac{0.001\times\frac{1}{3}}{(0.001+0.005+0.01)\times\frac{1}{3}}$$

$$=\frac{1}{16}$$

(b) $$P(X=3|Y=0)=\frac{P(X=3,Y=0)}{P(Y=0)}$$

$$=\frac{P(Y=0|X=3)P(X=3)}{\sum_k P(Y=0|X=k)P(X=k)}$$

$$=\frac{0.01\times\frac{1}{3}}{(0.001+0.005+0.01)\times\frac{1}{3}}$$

$$=\frac{5}{8}$$

**Question 3 (6 marks):**

As a network engineer, you are asked to connect computers to the Internet, where each computer needs a Category-5 (CAT-5) cable of length 2m. Every time that you perform this task, your supplier provides you with a length of CAT-5 cable which is 10m on average and follows an exponential distribution. You cut the cable into 2m strips (assume that this is exact, and there is no loss in cutting the cable), but any "left-over" cable which is below 2m is discarded. You may assume that there are unlimited computers requiring connection to the Internet.

a) Find the probability mass function for the number of computers that you can connect to the Internet each time you perform the task. (**4 marks**)

b) What's the distribution of the random variable $Y$? (**2 marks**)

**Solution:**

(a) $\because \frac{1}{\lambda} = 10, \therefore \lambda = 0.1$

Let $X$ denote the total length of the CAT-5 cable, $Y$ be the number of computers that you can connect to the internet, then

$$Y = \left\lfloor \frac{X}{2} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is the floor function. Then

$$\begin{aligned} P(Y = k) &= P\left(\left\lfloor \frac{X}{2} \right\rfloor = k\right) \\ &= P\big(2k \leq X < 2(k+1)\big) \\ &= F_X\big(2(k+1)\big) - F_X(2k) \\ &= \left(1 - e^{-0.1(2(k+1))}\right) - \left(1 - e^{-0.1\times 2k}\right) \\ &= e^{-0.2k}\left(1 - e^{-0.2}\right) \end{aligned}$$

(b) By denote $p = 1 - e^{-0.2}$

$$P(Y = k) = (1-p)^k p$$

$Y$ is a Geometric random variable.

**Question 4** (**7 marks**)**:**

Two students, Tom and Jerry, bought their iPhone at the same time. Suppose that it is known that the lifetime of any given iPhone is 4 years on average.

a) Suggest a suitable probability distribution for modeling the lifetime of each iPhone device. (**2 marks**)

b) Compute the probability that both iPhones fail within a specified time period *T*. (**2 marks**)

c) Compute the probability that Tom's iPhone operates for at least 1 year longer than Jerry's. (**3 marks**)

**Solution:**

(a) Exponential distribution.

(b) $\because \frac{1}{\lambda} = 4, \therefore \lambda = 0.25$

Let X be the lifetime of Tom's iPhone, Y be the lifetime of Jerry's iPhone, then

$$\begin{aligned} P(max(X,Y) \leq T) &= P(X \leq T, Y \leq T) \\ &= P(X \leq T)P(Y \leq T) \\ &= (1 - e^{-0.25T})^2 \\ &= 1 - 2e^{-0.25T} + e^{-0.5T} \end{aligned}$$

*(c)* $$\begin{aligned} P(X - Y \geq 1) &= P(X \geq Y + 1) \\ &= \int_0^\infty P(X \geq y + 1) f_Y(y) dy \\ &= \int_0^\infty \left\{ \int_{y+1}^\infty 0.25 e^{-0.25x} dx \right\} 0.25 e^{-0.25y} dy \\ &= \frac{1}{2} e^{-0.25} \int_0^\infty 0.5 e^{-0.5y} dy \\ &= \frac{1}{2} e^{-0.25} \end{aligned}$$

**Question 5 (7 marks):**

Suppose that we have an assembly line which produces iPads. We wish to measure the *Defective Probability*, which is the probability that any randomly-selected iPad on the assembly line is not working properly. To estimate this probability, we will take a random selection of devices from the assembly line and test whether each one is defective or not. As an engineer, your design task is to determine how many iPads should be tested to ensure that there is at least a 95% chance that your estimate of the Defective Probability is within at least 0.05 of the true probability.

(a) Solve this design problem using Chebyshev's Inequality. (**3 marks**)

(b) Solve this design problem using the Central Limit Theorem. (**3 marks**)

(c) Comment on the differences between the results in (a) and (b), and comment on which result would be the most meaningful in practice. (**1 marks**)

**Solution:**

(a) Let $N$ denote the number of iPads should be tested and $p$ denote the Defective Probability.

Let $X_n$ denote the test result of n$^{th}$ selected iPad, thus $X_n$ are independent Bernoulli random variables with parameter $p$.

Each $X_n$ assumes values 1 and 0 with probability $p$ and $1-p$, respectively.

$E[X_n] = p$

$E[X_n^2] = p$

$\text{Var}[X_n] = p(1-p) \leq 1/4$

Let $\hat{p} = (1/N)\sum_{n=1}^{N} X_n$ as the estimate of $p$.

$E[\hat{p}] = p$

$\text{Var}[\hat{p}] = \text{Var}[X_n]/N \leq 1/(4N)$

From the proof of the Weak Law

$$P[|\hat{p}-p| < 0.05] \geq 1 - (\text{Var}\,[\hat{p}]/0.05^2)$$

$$\geq 1 - 1/(4 \times N \times 0.05^2)$$

$$\geq 0.95$$

$$N \geq 1/(4 \times 0.05^3)$$

$$= 2000$$

(b) With the same assumption to (a), by using CLT

$$P[|\hat{p}-p| < 0.05] \approx 1 - 2Q\left(0.05/\sqrt{\text{Var}[\hat{p}]}\right)$$

$$\geq 1 - 2Q\left(0.05/\sqrt{(1/4N)}\right)$$

$$= 1 - 2Q(0.1\sqrt{N})$$

Let $1 - 2Q(0.1\sqrt{N}) = 0.95$, then

$Q(0.1\sqrt{N}) = 0.025$

$0.1\sqrt{N} = 1.95$

$N = 380.25$

Thus at least 381 iPads should be tested.

(c) The result in (a) guarantees that we estimate $p$ to within 0.05 with at least 95% probability, but results in a large number of samples.

Due to the approximation, the result in (b) is not an absolute guarantee, but results in a much more manageable number of samples.

In practice, the result in (b) would be the most meaningful one.

**Question 6** (**7 marks**):

Two buses serve the transportation from HKUST to Choi Hung: mini bus 11 and public bus 91. According to the statistics from the history data, it is known that the waiting time for bus 11 is exponentially distributed with mean 10 minutes and its transportation time is also exponentially distributed with mean 15 minutes. For bus 91, the waiting time is uniformly distributed from 6 minutes to 10 minutes, and its transportation time is also uniformly distributed from 20 minutes to 25 minutes. (Assume the transportation time is independent from the waiting time for both buses.)

a) On average, if you take bus 91, what is the time you need to spend from your arrival to HKUST station to your arrival to Choi Hung station? (**1 marks**)

b) In terms of average consuming time (waiting time plus transportation time), which bus do you prefer? (**2 marks**)

c) Find the probability density function of your consuming time if you decide to take bus 11. (**4 marks**)

**Solution:**

a) Average time = 8 + 22.5 = 30.5 minutes.

b) Bus 11, since its average time is 10 + 15 = 25 minutes.

c) Let X be your waiting time for bus 11 and Y be the time you spend on 11, then the total time is given by: $Z = X + Y$, since both X and Y are exponentially distributed, for each $z > 0$, we have:

$$F_Z(z) = P[Z \leq z]$$
$$= P[X + Y \leq z]$$
$$= \int_{x=0}^{z} \int_{y=0}^{z-x} f_{XY}(x,y)dydx$$

Since $f_{XY(x,y)} = f_X(x)f_Y(y)$, we can evaluate the double integration which gives us:

$$F_z(z) = 1 - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} + \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 z}$$

And thus:

$$f_Z(z) = \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 z} - e^{-\lambda_1 z}\right), & for\ z \geq 0 \\ 0, & for\ other\ z \end{cases}$$

Here $\lambda_1 = \frac{1}{10}, \lambda_2 = \frac{1}{15}$.