Supplementary Exercises on Counting

Note: These exercises are meant to help you revise the material that you have learnt in class when preparing for the exam. Please note that when solutions are given below for the problems, they are at most sketch solutions and do not provide derivations of the answers. For assignments and exams you are expected to provide full derivations. See the posted solutions to assignments for examples of this.

1. In how many different ways can five persons be seated on a bench?
2. How many three-digit odd numbers can be formed with the digits $1, 2, \ldots, 9$ if no digit is repeated in any number?
3. In how many ways can three boys and three girls be seated in a row if boys and girls alternate?
4. In how many ways can two letters be mailed if five letter boxes are available?
5. In how many ways can 10 boys take positions in a straight line if two particular boys must not stand side by side?
6. In how many ways can the offices of chairman, vice-chairman, secretary, and treasurer be filled from a committee of seven?
7. How many three-digit numbers greater than 300 can be formed with the digits $1, 2, \ldots, 6$ if no digit is repeated in any number?
8. A bag contains nine balls numbered $1, 2, \ldots, 9$. In how many ways can two balls be drawn so that (a) both are odd? (b) their sum is odd?
9. Two dice can be tossed in 36 ways. In how many of these is the sum equal to (a) 4; (b) 7; (c) 11?
10. Four delegates are to be chosen from eight members of a club. (a) How many choices are possible? (b) How many contain member A? (c) How many contain A or B but not both?
11. How many committees of two or more can be selected from 10 people?
12. A lady gives a dinner party for six guests. (a) In how many ways may they be selected from among 10 friends? (b) In how many ways if two of the friends will not attend the party together?

13. A committee of five is to be selected from 12 seniors and eight juniors. In how many ways can this be done if the committee is to contain at least three seniors and one junior?

14. In how many ways may 12 persons be divided into three groups of four persons each?

15. From an ordinary deck of playing cards, in how many different ways can five cards be dealt consisting of three kings and a pair?

16. A license plate consists of either: (a) three letters followed by three digits (standard plate), or (b) five letters (vanity plate). Let L be the set of all possible license plates. Compute $|L|$, the number of different license plates, using the sum and product principles.

17. In how many different ways can the letters in the name of the popular 1980's band *BANANARAMA* be arranged?

18. How many different paths are there from point $(0, 0, 0)$ to point $(12, 24, 36)$ if every step increments one coordinate and leaves the other two unchanged?

19. In how many different ways can $2n$ students be paired up?

20. Describe a bijection between the set of 30-bit sequences with 10 zeros and 20 ones and the set of paths from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate).

21. Calculate the coefficient of
(a) $x^{17}y^3$ in the expansion of $(x + y)^{20}$
(b) $x^{17}y^3$ in the expansion of $(x - y)^{20}$.

22. Calculate each of the following:
(a) the fifteenth term of $(1 + \frac{1}{x})^{18}$
(b) the fourth term of $(x - \frac{1}{x})^{10}$.

23. Consider that $(x + b)^5 = (x + b)(x + b)(x + b)(x + b)(x + b)$.
(a) Upon multiplying out, and before collecting like terms, how many terms will be produced?
(b) How many times will the term x^3b^2 be produced? In other words, upon adding those like terms, what number will be the coefficient of x^3b^2?

24. In what binomial does the term a^8b^4 occur?

25. Expand $(a - b)^5$.

26. Prove the identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

for $0 \leq k < n$.

27. Prove the identity

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

for $0 \leq k < n$.

28. How many ways are there to choose five from 10 persons as committee members and make one of them the president?

29. How many ways are there to choose five from 10 persons to do different jobs?

30. An n-input, m-output boolean function is a function from $\{\texttt{TRUE}, \texttt{FALSE}\}^n$ to $\{\texttt{TRUE}, \texttt{FALSE}\}^m$.
(a) How many n-input, 1-output boolean functions are there?
(b) How many n-input, m-output boolean functions are there?

31. For any $n \geq 0$ and $0 \leq k \leq n$, what value of k achieves the maximum value of $\binom{n}{k}$?

32. Consider identical k-substrings at different positions as different.
(a) How many k-substrings does an n-string have?
(b) How many substrings does an n-string have in total?

33. In how many ways can n professors sit around a circular conference table? Consider two seatings to be the same if one can be rotated to form the other.

34. How many ways are there to choose from the set $\{1, 2, \ldots, 100\}$ three distinct numbers so that their sum is even?

35. Prove the identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$$

36. Use the binomial theorem to prove that $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$.

37. Use the binomial theorem to prove that $2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$.

38. What is the coefficient of $x^5 y^{13}$ in the expansion of $(3x - 2y)^{18}$?

39. Compare $\binom{7}{2}\binom{5}{2}$ and $\frac{7!}{2!\,2!\,3!}$.

40. Compare $\binom{12}{3}\binom{9}{4}$ and $\frac{12!}{3!\,4!\,5!}$.

41. Compare $\binom{n}{k}\binom{n-k}{r}$ and $\frac{n!}{k!\,r!\,(n-k-r)!}$.

42. How many different signals can be created by lining up nine flags in a vertical column if three of them are white, two are red and four are blue?

43. In how many ways can $2n$ elements be partitioned into two sets with n elements each?

44. Twelve contestants are divided into four teams of three contestants each, in which Ann and Bob want to be in the same team. How many ways are there to choose the teams so that Ann and Bob are together?

45. How many four-digit numbers can be formed using only the digits 3,4,5,6,7?

46. How many of the numbers in Problem 45 have some digit(s) repeated?

47. How many of the numbers in Problem 45 are even?

48. There are 20 books arranged in a row on a shelf. How many ways are there to select six books so that no two adjacent books are selected?

49. Mr. and Mrs. Grumperson have collected 13 identical pieces of coal as Christmas presents for their beloved children, Lucy and Spud. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the two children and the set of 14-bit sequences with exactly 1 one.

50. On reflection, Mr. and Mrs. Grumperson decide that each of their two children should receive at least two pieces of coal for Christmas. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the two Grumperson children given this constraint and the set of 9-bit sequences with exactly 1 one.

51. Describe a bijection between the set of 110-bit sequences with exactly 10 ones and the set of solutions over the natural numbers to the equation $x_1 + x_2 + \cdots + x_{10} \leq 100$.

52. Describe a bijection between the set of solutions to the inequality in the preceding problem and the set of sequences $(y_1, y_2, \ldots, y_{10})$ such that $0 \leq y_1 \leq y_2 \leq \cdots \leq y_{10} \leq 100$.

53. How many ways are there to distribute six identical (non-distinguishable) coins to three individuals who are distinguishable?

54. How many ways are there to put 14 identical objects in three different boxes with at least eight objects in one box?

Solutions/Hints

1. $5! = 120$

2. $8 \cdot 7 \cdot 5 = 280$

3. $3! \cdot 3! \cdot 2 = 72$

4. $5 \cdot 5 = 25$

5. $8 \cdot 9! = 10! - 2 \cdot 9!$

6. $7^{\underline{4}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

7. $4 \cdot 5 \cdot 4 = 80$

8. (a) $\binom{5}{2}$; (b) $\binom{5}{1} \cdot \binom{4}{1} = 20$

9. (a) 3; (b) 6; (c) 2

10. (a) $\binom{8}{4}$; (b) $\binom{7}{3}$; (c) $2 \cdot \binom{6}{3}$

11. $2^{10} - \binom{10}{0} - \binom{10}{1} = 2^{10} - 11$

12. (a) $\binom{10}{6} = 210$; (b) $210 - \binom{8}{4} = 140$

13. The committee may consist of three seniors and two juniors or of four seniors and one junior. A committee of three seniors and two juniors can be selected in $\binom{12}{3} \cdot \binom{8}{2} = 6160$ ways, and a committee of four seniors and one junior can be selected in $\binom{12}{4} \cdot \binom{8}{1} = 3960$ ways. In all, a committee may be selected in $6160 + 3960 = 10120$ ways.

14. One group of four can be selected in $\binom{12}{4}$ ways, then another in $\binom{8}{4}$ ways, and the third in one way. Since the order in which the groups are formed is now immaterial, the division may be made in $\frac{\binom{12}{4} \cdot \binom{8}{4} \cdot 1}{3!} = 5775$ ways.

15. Three kings can be selected from the four kings in $\binom{4}{3}$ ways, another kind can be selected in 12 ways, and two cards of this kind can be selected in $\binom{4}{2}$ ways. Thus, three kings and another pair can be dealt in $\binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 288$ ways.

16. $26^3 \cdot 10^3 + 26^5$

17. $\frac{10!}{5!\,2!}$

18. $\frac{72!}{12!\,24!\,36!}$

19. $\frac{(2n)!}{2^n n!}$

20. Map the 30-bit sequence $b_1 b_2 \cdots b_{30}$ to a path where the i-th step is right if $b_i = 0$ and up if $b_i = 1$.

21. (a) 1140; (b) -1140

22. (a) $3060x^{-14}$; (b) $-120x^4$

23. (a) 32; (b) 10

24. $(a+b)^{12}$

25. $(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

26. Hint: note that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ can be written as $\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}$.

27. Hint: note that $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ can be written as $\binom{n}{k}\binom{n-k}{1} = \binom{n}{1}\binom{n-1}{k}$.

28. $\binom{10}{5} \cdot \binom{5}{1}$

29. $10^{\underline{5}}$

30. (a) 2^{2^n}; (b) $(2^m)^{2^n}$

31. $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$

32. (a) $n-k+1$; (b) $\sum_{k=1}^{n}(n-k+1)$

33. $(n-1)!$

34. $\binom{50}{1} \cdot \binom{50}{2} + \binom{50}{3}$ or $\binom{100}{3}/2$

35. Hint: pair up $k\binom{n}{k}$ and $(n-k)\binom{n}{n-k}$

36. Hint: $(1+2)^n$

37. Hint: $(3-1)^n$

38. $3^5 \cdot (-2)^{13} \cdot \binom{18}{5}$

39. equal

40. equal

41. equal

42. $\binom{9}{3} \cdot \binom{6}{2} \cdot \binom{4}{4} = 1260$

43. There are $\frac{1}{2}\binom{2n}{n}$ unordered partitions and $\binom{2n}{n}$ ordered partitions.

44. $10 \cdot \frac{\binom{9}{3}\binom{6}{3}}{3!} = 2800$

45. $5^4 = 625$

46. $5^4 - 5^{\underline{4}} = 505$

47. $5^3 \cdot 2 = 250$

48. The answer is $\binom{15}{6}$. Each of the first five chosen books is followed by a non-chosen one. The chosen one and the non-chosen one are bound together. So the problem is equivalent to selecting six books from 15 ones.

49. Map a distribution in which Lucy gets l pieces and Spud gets s pieces to a 14-bit sequence with l zeros, a one, and then s zeros.

50. Map a distribution in which Lucy gets $l \geq 2$ pieces and Spud gets $s \geq 2$ pieces to a 9-bit sequence with exactly $l - 2$ zeros, a one, and then $s - 2$ zeros.

51. Let x_1 be the number of zeros before the first 1, x_2 be the number of zeros between the first and second 1, etc. Note that zeros after the tenth 1 do not contribute to the value of any of the variables $x_1, x_2, \ldots, x_{10}$; this allows us to count solutions to the inequality (≤ 100) rather than the equality ($= 100$).

52. Let $y_i = x_1 + \cdots + x_i$ for each i from 1 to 10.

53. $\binom{8}{2} = 28$ (the number of ways to place two barriers among $6 + 2$ objects)

54. $3\binom{8}{2}$ (put eight in one of the three boxes, then distribute the remaining six in three boxes)