HKUST

MATH2011 Introduction to Multivariable Calculus

Midterm Examination

Name:________________

18:00-19:00; Mar. 29, 2014

Student I.D.:________________

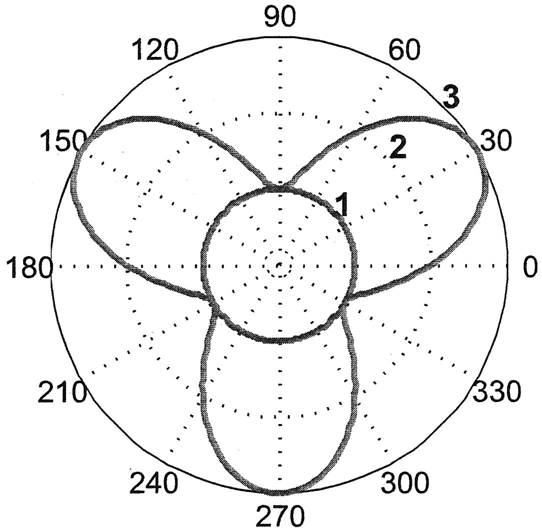
Lecture Section:________________

Directions:

- DO NOT open the exam until instructed to do so.
- Please have your student ID ready for checking.
- You may not use a calculator during the exam.
- You may write on both sides of the examination papers.
- You must show the steps in order to receive full credits.

Question No.	Points	Out of
1		25
2		30
3		20
4		25
Total		100

1. (25pts) Find the area of the region between the curve $r = 2 + \sin 3\theta$ and the unit circle centered at the origin.



$$\text{Area of the region} = \int_0^{2\pi} \frac{1}{2}\left[(2+\sin 3\theta)^2 - 1^2\right] d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} + 2\sin 3\theta + \frac{1}{2}\sin^2 3\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} + 2\sin 3\theta + \frac{1}{4}(1-\cos 6\theta)\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{7}{4} + 2\sin 3\theta - \frac{1}{4}\cos 6\theta\right) d\theta$$

$$= \left(\frac{7}{4}\theta - \frac{2}{3}\cos 3\theta - \frac{1}{24}\sin 6\theta\right)\Big|_0^{2\pi} = \frac{7\pi}{2}$$

2. (30pts) A hunter stands 10 m horizontally away from a bird and 5 m vertically below the bird on a tree. Assuming the arrow is shot at an angle 45^o relative to the horizontal direction.
a.(20pts) In order to shoot the bird, what is the initial speed $|\vec{v}_0|$ of the arrow?
b.(10pts) Calculate the arc-length of the trajectory travelled by the arrow from the hunter to the bird with an initial speed obtained in question **a.**

Hint: 1). The gravitational acceleration is $g = 10$ m/s^2.
2). $\int \sqrt{x^2+1}dx = \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + c$, where c is a constant

(a) $\vec{a} = \langle 0, -10 \rangle$ acceleration

$\vec{V}_0 = \langle V_0 \cos 45°, V_0 \sin 45° \rangle = \langle \frac{V_0}{\sqrt{2}}, \frac{V_0}{\sqrt{2}} \rangle$ initial velocity

$$\vec{V}(t) = \vec{V}_0 + \int_0^t \vec{a}\, d\tau = \langle \frac{V_0}{\sqrt{2}}, \frac{V_0}{\sqrt{2}} - 10t \rangle$$

This is velocity as a function of t.

$$\vec{r}(t) = \langle 0, 0 \rangle + \int_0^t \vec{V}(\tau)\, d\tau$$

$$= \langle \frac{V_0}{\sqrt{2}} t, \frac{V_0}{\sqrt{2}} t - 5t^2 \rangle \quad \text{position.}$$

The hunter stands at $(0,0)$

The bird is at $(10, 5)$

$$\begin{cases} \frac{V_0}{\sqrt{2}} t = 10 \\ \frac{V_0}{\sqrt{2}} t - 5t^2 = 5 \end{cases} \qquad \begin{aligned} t &= 1 \\ V_0 &= 10\sqrt{2} \ (m/s) \end{aligned}$$

(b) Arc length

$$L = \int_0^{t=1} |\vec{V}(\tau)|\, d\tau$$

$$= \int_0^1 \sqrt{10^2 + (10 - 10\tau)^2}\, d\tau$$

$$= 10 \int_0^1 \sqrt{1 + (1-\tau)^2}\, d\tau$$

$x = 1 - \tau$

$$= 10 \int_1^0 \sqrt{1 + x^2}\, (-dx)$$

$$= 10 \int_0^1 \sqrt{1 + x^2}\, dx$$

$$= 10 \left(\frac{x}{2}\sqrt{x^2+1} + \frac{1}{2} \ln(x + \sqrt{x^2+1}) \right) \Big|_0^1$$

$$= 10 \left(\frac{\sqrt{2}}{2} + \frac{1}{2} \ln(1 + \sqrt{2}) \right)$$

$$= 5\sqrt{2} + 5 \ln(1 + \sqrt{2}) \quad (m)$$

3. (20pts) The intersection line of two orthogonal planes (α and β) is the y-axis. The point $P_0(1,1,1)$ is on plane α. Find the equation of plane β.

Select two points on y axis

$P_1\ (0,0,0)$, $P_2\ (0,1,0)$

Plane α contains P_0, P_1, P_2

$\overrightarrow{P_1P_0} = \langle 1,1,1\rangle$, $\overrightarrow{P_1P_2} = \langle 0,1,0\rangle$

Their cross product:

$$\vec{N}_\alpha = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i} - \vec{k}$$

$$= \langle 1, 0, -1\rangle$$

Normal vector of plane α.

$\vec{N}_\beta \perp \vec{N}_\alpha$ and $\vec{N}_\beta \perp y$ axis ($\vec{j}$)

$$\vec{N}_\beta = \vec{N}_\alpha \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} + \vec{k}$$

$$= \langle 1, 0, 1\rangle$$

Equation of plane β: $\vec{N}_\beta \cdot \langle x, y, z\rangle = 0$

$$x + z = 0$$

Note that $(0,0,0)$ is in β.

4. (25pts) Find the domain and range of the function $f(x, y) = \sqrt{x^2 + y^2 - 4}$. Then sketch three level curves of the given function on xy-plane.

$x^2 + y^2 \geq 4$ is the domain

$f(x,y) \geq 0$: range is $[0, +\infty)$

Level curves $z_0 = \sqrt{x^2 + y^2 - 4}$

$x^2 + y^2 = 4 + z_0^2 \geq 2^2$

Circles centered at $(0,0)$ with radii equal or larger than 2.

Note: Circles must be labeled with radii and levels