Tutorial 11: Recursion

COMP 2711H

Chengbo Zheng, 11. 25

1. Suppose that the Towers of Hanoi problem has four poles in a row instead of three. A disk can be transferred from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Consider the algorithm that works by (i) first moving the top $n-2$ disks to another pole (not the destination one), (ii) moving the two largest disks to the destination pole and then (iii) movnog the remaining $n-2$ disks to the destination pole. Let $T(n)$ be the number of moves needed by this algorithm to transfer the entire tower of n disks from the leftmost pole to the rightmost pole.

   (a) Find $T(1)$, $T(2)$ and $T(3)$.

   (b) Find $T(4)$.

   (c) Find a recurrence relation expressing $T(n)$ in terms of $T(n-2)$, for all integers $n \geq 3$.

2. In a Double Towers of Hanoi problem there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one-by-one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let $T(n)$ be the number of moves needed to transfer a tower of $2n$ disks from one pole to another.

   (a) Find $T(1)$ and $T(2)$.

   (b) Find $T(3)$.

   (c) Find a recurrence relation expressing $T(n)$ in terms of $T(n-1)$, for all integers $n \geq 2$.

The Towers of Hanoi Problem has not been covered in the lecture yet.

The solution will be released in this week.

Check it and email me if you have any questions.

EP5-5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

Solving First-Order Linear Recurrence

1 * 3 ^ n + 2 * (3^n - 1) / (3 - 1)

- Solve the following recurrence:
  $f(0) = a$,
  $f(n+1) = bf(n) + c$, where a, b, c are constants.
- Note: the recurrence is more commonly written as
  $f(n) = bf(n-1) + c$ for $n > 1$
- **Solution**

$$\begin{aligned} f(n) &= bf(n-1) + c \\ &= b(b(f(n-2) + c) + c \\ &= b^2 f(n-2) + bc + c \\ &= b^2(bf(n-3) + c) + bc + c \\ &= b^3 f(n-3) + b^2 c + bc + c \\ &\ldots \\ &= b^n f(0) + b^{n-1}c + b^{n-2}c + \cdots + c \\ &= ab^n + c(b^{n-1} + b^{n-2} + \cdots + 1) = ab^n + c \cdot \frac{b^n - 1}{b-1} \end{aligned}$$

EP5-5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & \boxed{n = 1} \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

Solving First-Order Linear Recurrence

1 * 3 ^ n + 2 * (3^n - 1) / (3 - 1)

- Solve the following recurrence:
  $f(0) = a,$
  $f(n+1) = bf(n) + c$, where a, b, c are constants.
- Note: the recurrence is more commonly written as
  $f(n) = bf(n-1) + c$ for $n > 1$
- **Solution**

$$\begin{aligned} f(n) &= bf(n-1) + c \\ &= b(b(f(n-2) + c) + c \\ &= b^2 f(n-2) + bc + c \\ &= b^2(bf(n-3) + c) + bc + c \\ &= b^3 f(n-3) + b^2 c + bc + c \\ &\dots \\ &= b^n f(0) + b^{n-1}c + b^{n-2}c + \cdots + c \\ &= ab^n + c(b^{n-1} + b^{n-2} + \cdots + 1) = ab^n + c \cdot \frac{b^n - 1}{b - 1} \end{aligned}$$

EP5-5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

$$T(n) = 3T(n-1) + 2$$
$$= 3(3T(n-2)) + 2\times 3 + 2$$
$$= 3(3(3T(n-3))) + 2\times 3\times 3 + 2\times 3 + 2$$
$$\cdots\cdots$$
$$= 3^{n-1}T(1) + 2\times 3^{n-2} + 2\times 3^{n-3} + \cdots + 2$$
$$= (3^{n-1} + 3^{n-2} + \cdots + 3^1 + 3^0) + (3^{n-2} + 3^{n-3} + \cdots + 3^0)$$
$$= \frac{3^n - 1}{2} + \frac{3^{n-1} - 1}{2}$$
$$= \frac{3\cdot 3^{n-1} - 1 + 3^{n-1} - 1}{2}$$
$$= 2\cdot 3^{n-1} - 1$$

Solving First-Order Linear Recurrence

- Solve the following recurrence:
  $f(0) = a$,
  $f(n+1) = bf(n) + c$, where a, b, c are constants.
- Note: the recurrence is more commonly written as
  $f(n) = bf(n-1) + c$ for $n > 1$
- **Solution**

$$\begin{aligned} f(n) &= bf(n-1) + c \\ &= b(b(f(n-2) + c) + c \\ &= b^2 f(n-2) + bc + c \\ &= b^2(bf(n-3) + c) + bc + c \\ &= b^3 f(n-3) + b^2 c + bc + c \\ &\ldots \\ &= b^n f(0) + b^{n-1}c + b^{n-2}c + \cdots + c \\ &= ab^n + c(b^{n-1} + b^{n-2} + \cdots + 1) = ab^n + c\cdot\frac{b^n - 1}{b-1} \end{aligned}$$

EP5-5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

For $n > 1$,

$$T(n) = 3T(n-1) + 2$$

$$\Rightarrow T(n) + 1 = 3(T(n-1) + 1)$$

$$\Rightarrow T(n) + 1 = 3^{(n-1)}(T(1) + 1)$$

$$= 2 \cdot 3^{(n-1)}$$

EP5-6. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 5 & n = 1 \\ 2T(n-1) + 3n + 1 & n > 1 \end{cases}$$

$$T(n) = 2T(n-1) + 3n + 1$$

$$(T(n) + \underline{\quad}) = 2(T(n-1) + \underline{\quad})$$

EP5-6. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 5 & n = 1 \\ 2T(n-1) + 3n + 1 & n > 1 \end{cases}$$

$$T(n) = 2T(n-1) + 3n + 1$$

$$\boxed{\begin{aligned} &T(n) + 3n + a \\ &= 2(T(n-1) + 3(n-1) + a) \end{aligned}}$$

$$2a - 6 - a = 1$$

$$\Rightarrow a = 7$$

$$T(n) + 3n + 7$$

$$= 2(T(n-1) + 3(n-1) + 7)$$

$$\Rightarrow T(n) + 3n + 7$$

$$= 2^{n-1}(T(1) + 3 + 7)$$

$$\therefore T(n)$$

$$= 15 \cdot 2^{(n-1)} - 3n - 7$$

EP5-7. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ nT(n-1) + n & n > 1 \end{cases}$$

Solving First-Order Linear Recurrence

- Solve the following recurrence:
  $f(0) = a$,
  $f(n+1) = bf(n) + c$, where a, b, c are constants.
- Note: the recurrence is more commonly written as
  $f(n) = bf(n-1) + c$ for $n > 1$
- **Solution**

$$\begin{aligned} f(n) &= bf(n-1) + c \\ &= b(b(f(n-2) + c) + c \\ &= b^2 f(n-2) + bc + c \\ &= b^2(bf(n-3) + c) + bc + c \\ &= b^3 f(n-3) + b^2 c + bc + c \\ &\dots \\ &= b^n f(0) + b^{n-1}c + b^{n-2}c + \cdots + c \\ &= ab^n + c(b^{n-1} + b^{n-2} + \cdots + 1) = ab^n + c \cdot \frac{b^n - 1}{b - 1} \end{aligned}$$

$T(n) = n\,T(n-1) + n$

$\boxed{} = \text{constant} \times \boxed{}$?

Something about n

$n-1$

EP5-7. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ nT(n-1) + n & n > 1 \end{cases}$$

$$\frac{T(n)}{n!} = \frac{T(n-1)}{(n-1)!} + \frac{1}{(n-1)!}$$

$$R(n) = R(n-1) + \frac{1}{(n-1)!}$$

$$= R(n-2) + \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

$$= \cdots\cdots$$

$$= R(1) + \sum_{i=1}^{n-1} \frac{1}{i!}$$

$$= \frac{T(1)}{1!} + \sum \frac{1}{i}$$

$$\Rightarrow T(n) = n!\left(\sum_{i=1}^{n-1} \frac{1}{i!} + 1\right)$$

EP5-8. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2^n - T(n-1) & n > 1 \end{cases}$$

$$T(n) = 2^n - T(n-1)$$

$$\left(T(n) + \boxed{\quad}\right) = -\left(T(n-1) + \boxed{\quad}\right)$$

EP5-8. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2^n - T(n-1) & n > 1 \end{cases}$$

$$T(n) = 2^n - T(n-1)$$

$$\left(T(n) + \underset{\underset{a2^n}{\Downarrow}}{\boxed{\quad}}\right) = -\left(T(n-1) + \underset{\underset{a2^{n-1}}{\downarrow}}{\boxed{\quad}}\right)$$

$$-a2^{n-1} - a2^n = 2^n$$

$$-3a \times 2^{n-1} = 2 \times 2^{n-1}$$

$$\Rightarrow \quad a = -\frac{2}{3}$$

$$T(n) - \frac{2}{3}2^n = -\left(T(n-1) - \frac{2}{3}2^{n-1}\right)$$

$$= (-1)^{(n-1)}\left(T(1) - \frac{2}{3}2^1\right)$$

$$= (-1)^{(n-1)}\left(-\frac{1}{3}\right)$$

$$\therefore T(n) = (-1)^{(n-1)}\left(-\frac{1}{3}\right) + \frac{2}{3}2^n$$

EP5-9. Find a closed-form solution for the following recurrence where you are allowed to consider $H_n = \sum_{i=1}^{n} \frac{1}{i}$ as a known function

$$T(n) = \begin{cases} 1 & n = 2 \\ (n-1)T(n-1) + (n-2)! & n > 2 \end{cases}$$

$$T(n) = (n-1)T(n-1) + (n-2)!$$

$$\Rightarrow \frac{T(n)}{n!} = \frac{(n-1)T(n-1)}{n!} + \frac{1}{n(n-1)}$$

?

$$\Rightarrow \frac{T(n)}{(n-1)!} = \frac{T(n-1)}{(n-2)!} + \frac{1}{n-1}$$

$$R(n) = R(n-1) + \frac{1}{n-1}$$

$$= R(2) + \quad ?$$

EP5-9. Find a closed-form solution for the following recurrence where you are allowed to consider $H_n = \sum_{i=1}^{n} \frac{1}{i}$ as a known function

$$T(n) = \begin{cases} 1 & n = 2 \\ (n-1)T(n-1) + (n-2)! & n > 2 \end{cases}$$

$$\Rightarrow \frac{T(n)}{(n-1)!} = \frac{T(n-1)}{(n-2)!} + \frac{1}{n-1}$$

$$R(n) = R(n-1) + \frac{1}{n-1}$$

$$= R(2) + \sum_{i=3}^{n} \frac{1}{i-1}$$

$$\boxed{1 + \sum_{i=2}^{n-1} \frac{1}{i}}$$

$$= \sum_{i=1}^{n-1} \frac{1}{i} = H_{n-1}$$

$$\therefore T(n) = (n-1)!\, H_{n-1}$$

EP5-11. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 3 & n = 1 \\ 6T(n/6) + 3n - 1 & n > 1 \end{cases}$$

where n is a power of 6.

$$T(n) = 6T(n/6) + 3n - 1$$

we only consider when $n = 6^0, 6^1, 6^2, \dots$

$$\therefore \begin{cases} n = 6^k \quad (k = 0, 1, 2, 3, \dots) \\ n/6 = 6^{k-1} \end{cases}$$

$$T(6^k) = 6T(6^{k-1}) + 3 \cdot 6^k - 1$$

$$\boxed{T(6^k) + a(k) = 6\left(T(6^{k-1}) + a(k-1)\right)}$$

EP5-11. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 3 & n = 1 \\ 6T(n/6) + 3n - 1 & n > 1 \end{cases}$$

where n is a power of 6.

$$T(n) = 6T(n/6) + 3n - 1$$

we only consider when $n = 6^0, 6^1, 6^2, \cdots$

$$\therefore \begin{cases} n = 6^k \quad (k = 0, 1, 2, 3, \cdots) \\ n/6 = 6^{k-1} \end{cases}$$

$$T(6^k) = 6T(6^{k-1}) + 3 \cdot 6^k - 1$$

$$\frac{T(6^k)}{6^k} = \frac{T(6^{k-1})}{6^{k-1}} + 3 - \frac{1}{6^k}$$

$$\Rightarrow \frac{T(6^k)}{6^k} = \frac{T(1)}{6^0} + 3k - \sum_{i=1}^{k} 6^{-i}$$

$$= 3 + 3k - \frac{1 - 6^{-k}}{5}$$

EP5-11. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 3 & n = 1 \\ 6T(n/6) + 3n - 1 & n > 1 \end{cases}$$

where n is a power of 6.

$T(n) = 6T(n/6) + 3n - 1$

we only consider when $n = 6^0, 6^1, 6^2, \cdots$

$\therefore \begin{cases} n = 6^k \quad (k = 0, 1, 2, 3, \cdots) \\ n/6 = 6^{k-1} \end{cases}$

$T(6^k) = 6T(6^{k-1}) + 3 \cdot 6^k - 1$

$\frac{T(6^k)}{6^k} = \frac{T(6^{k-1})}{6^{k-1}} + 3 - \frac{1}{6^k}$

$\Rightarrow \frac{T(6^k)}{6^k} = \frac{T(1)}{6^0} + 3k - \sum_{i=1}^{k} 6^{-i}$

$= 3 + 3k - \frac{1 - 6^{-k}}{5}$

$T(n) = 3n(1 + k) - \frac{6^k - 1}{5}$

$6^k \rightarrow n$

$k = \log_6 n$

$= 3n(1 + \log_6 n) - \frac{n-1}{5}$

$= 3n \log_6 n + \frac{14n}{5} + \frac{1}{5}$