Supplementary Exercises on Probability

Note: These exercises are meant to help you revise the material that you have learnt in class when preparing for the exam. Please note that when solutions are given below for the problems, they are at most sketch solutions and do not provide derivations of the answers. For assignments and exams you are expected to provide full derivations. See the posted solutions to assignments for examples of this.

1. You are given a deck of 52 cards which have printed on them a pair of symbols: an integer between 1 and 13, followed by one of the letters S, H, D, or C. There are $13 \times 4 = 52$ such possible combinations, and you may assume that you have one of each type, handed to you face down. Suppose the cards are randomly distributed and you turn them over one at a time. What is the expected number of cards that you need to turn over before finding the card [11,H]?

2. Suppose we have one loaded die and one fair die, where the probabilities for the loaded die are

$$P(1) = P(2) = P(5) = P(6) = 1/6, \quad P(3) = 1/7, \quad P(4) = 4/21.$$

   (a) What is the probability that a toss of these two dice produces a sum of either seven or eleven?

   (b) What is the expected value of the sum of the values of the dice?

   (c) What is the expected value of the product of the values of the dice?

3. Consider the following experiments. For each of them find the expected value and the variance.

   (a) Roll a 6-sided die and a 4-sided die. Add the two values together. Both dice are fair.

   (b) Roll a 2-sided die and a 3-sided die. Multiply the two values together. Both dice are fair.

   (c) Roll a 3-sided die and another 3-sided die. Add the two values together. The first die has probability p_1 of being 1, p_2 of being 2, and p_3 of being 3. The second die is fair.
   (Express the expected value and variance in terms of p_1, p_2, and p_3.)

4. You play a game of throwing two six-sided dice (with integer values $1, 2, \ldots, 6$). You lose a dollar if the two dice have different values. You win d dollars if the two dice land on the same value d (i.e., you throw double d's).

(a) What is the expected value of playing this game once?

(b) What is the variance of playing this game once?

(c) What is the standard deviation of playing this game once?

(d) What is the expected value of playing this game n times?

(e) What is the variance of playing this game n times?

(f) What is the standard deviation of playing this game n times?

5. Consider an s-sided die, where the sides are marked with the numbers $1, 2, \ldots, s$, and each side has equal probability ($1/s$) of being rolled.

   (a) What is the expected value of one roll of this die?

   (b) What is the expected value of the sum of n (independent) rolls of this die?

   (c) What is the variance of one roll of this die?

   (d) What is the variance of the sum of n (independent) rolls of this die?

6. A coin is tossed n times, each time with an independent probability p of coming up heads and $1 - p$ of coming up tails. Let H be the number of heads occurring.

   (a) What is the expected value of H?

   (b) What is the variance of H?

7. We assume that each child born has probability $1/2$ of being a boy or a girl. We also assume that children's sexes are independent (even within a family).

   (a) Mr Jones has two children. Knowing that the older one is a boy, what is the probability that they are both boys?

   (b) Mr Smith has two children. Knowing that at least one is a boy, what is the probability that they are both boys?

   (c) Mr Johnson has three children. Knowing that at least one is a boy, what is the probability that at least one is a girl?

8. Box A contains 2 black and 5 white marbles, and box B contains 1 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

   (a) What is the probability that the marble is black?

   (b) Given that the marble is white, what is the probability that it came from box A?

9. Let X and Y be two independent random variables. Express the variance of $X - Y$, $V(X - Y)$, in terms of $V(X)$ and $V(Y)$.

10. Each pixel in a 32×8 vertical display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows.

    (a) What is the expected value of X?

    (b) What is the variance of X?

11. A biased coin is tossed n times, and a head shows up with probability p on each toss. A *run* is a maximal sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n - 1)p(1 - p)$.

12. Each of 1000 voters votes independently for a candidate A with probability 1/2.

    (a) What is the probability that A gets exactly 500 votes?

    (b) What is the probability that A gets at least 500 votes?

Solutions/Hints

1. $(\sum_{i=1}^{52} i)/52 = 53/2$

2. (a) $1/6 + 1/18 = 2/9$

   (b) $74/21 + 7/2 = 295/42$

   (c) $74/21 \times 7/2 = 37/3$

3. (a) $E(X) = 7/2 + 5/2 = 6$, $V(X) = 35/12 + 5/4 = 25/6$

   (b) $E(X) = 3/2 \times 2 = 3$, $V(X) = 8/3$

   (c)
$$\begin{aligned} E(X) &= p_1 + 2p_2 + 3p_3 + 2 \\ V(X) &= p_1(1 - p_1 - 2p_2 - 3p_3)^2 + \\ &\quad p_2(2 - p_1 - 2p_2 - 3p_3)^2 + \\ &\quad p_3(3 - p_1 - 2p_2 - 3p_3)^2 + 2/3 \end{aligned}$$

4. (a) $-1/4$

   (b) $475/144$

   (c) $5\sqrt{19}/12$

   (d) $-n/4$

   (e) $475n/144$

   (f) $5\sqrt{19n}/12$

5. (a) $(s + 1)/2$

   (b) $(s + 1)n/2$

   (c) $(s + 1)(s - 1)/12$

   (d) $(s + 1)(s - 1)n/12$

6. (a) np

   (b) $np(1 - p)$

7. (a) $1/2$

   (b) $1/3$

   (c) $6/7$

8. (a) $17/70$

(b) Let $P(\text{A} \mid \text{white})$ denote the desired probability.

$$\begin{aligned} P(\text{A} \mid \text{white}) &= \frac{P(\text{white} \,\&\, \text{A})}{P(\text{white})} \\ &= \frac{P(\text{white} \mid \text{A})P(\text{A})}{P(\text{white})} \\ &= \frac{P(\text{white} \mid \text{A})P(\text{A})}{P(\text{white} \mid \text{A})P(\text{A}) + P(\text{white} \mid \text{B})P(\text{B})} \\ &= \frac{5/7}{5/7 + 4/5} = \frac{25}{53} \end{aligned}$$

9. $V(X - Y) = V(X) + V(Y)$

10. (a) $1/8$

    (b) $255/2048$

11. Let S_i, $i \in \{1, \ldots, n\}$, denote the ith element of the sequence of coin tosses. We define the following indicator random variables:

$$\begin{aligned} X_1 &= 1 \\ X_i &= \begin{cases} 1 & S_i \neq S_{i-1} \\ 0 & \text{otherwise} \end{cases}, \text{ for each } i \in \{2, \ldots, n\}. \end{aligned}$$

X_i indicates the event that a new run begins at position i in the sequence. The random variable X that we are interested in, i.e., the number of runs in the sequence of random coin tosses, can be computed as $X = \sum_{i=1}^{n} X_i$. We first compute the following probabilities:

$$\begin{aligned} P(X_1 = 1) &= 1 \\ P(X_i = 1) &= P(S_i \neq S_{i-1}) \\ &= P((S_{i-1} = H \wedge S_i = T) \vee (S_{i-1} = T \wedge S_i = H)) \\ &= p(1-p) + (1-p)p \\ &= 2p(1-p). \end{aligned}$$

Hence,

$$\begin{aligned} E(X) = \sum_{i=1}^{n} E(X_i) &= \sum_{i=1}^{n} P(X_i = 1) \\ &= 1 + \sum_{i=2}^{n} [2p(1-p)] \\ &= 1 + 2(n-1)p(1-p). \end{aligned}$$

12. (a) $p = \binom{1000}{500}(1/2)^{500}(1/2)^{1000-500} = \binom{1000}{500}(1/2)^{1000}$

    (b) $(1+p)/2$