COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial Problems: Discrete Probability

Q1. We define 4 events BUS, CAR, $BIKE$, and $LATE$.

(a) We are given that $p(BUS) = p(CAR) = p(BIKE) = 1/3$. We are also given that $p(LATE|BUS) = 0.2$, $p(LATE|CAR) = 0.5$, and $p(LATE|BIKE) = 0.05$. Thus

$$\begin{aligned} p(LATE) &= p(LATE|BUS) \cdot p(BUS) + p(LATE|CAR) \cdot p(CAR) + \\ &\quad\ p(LATE|BIKE) \cdot p(BIKE) \\ &= 0.2/3 + 0.5/3 + 0.05/3 = 0.25. \end{aligned}$$

and

$$p(CAR|LATE) = \frac{p(CAR \bigcap LATE)}{p(LATE)} = \frac{p(LATE|CAR) \cdot p(CAR)}{p(LATE)} = \frac{(0.5)(1/3)}{0.25} = 2/3.$$

(b) We are given that $p(BUS) = 0.1$, $p(CAR) = 0.3$, and $p(BIKE) = 0.6$. We are also given that $p(LATE|BUS) = 0.2$, $p(LATE|CAR) = 0.5$, and $p(LATE|BIKE) = 0.05$. Calculations as in part (a) show that

$$p(LATE) = 0.2 \cdot 0.1 + 0.5 \cdot 0.3 + 0.05 \cdot 0.6 = 0.2.$$

and

$$p(CAR|LATE) = \frac{p(CAR \bigcap LATE)}{p(LATE)} = \frac{p(LATE|CAR) \cdot p(CAR)}{p(LATE)} = \frac{0.5 \cdot 0.3}{0.2} = 3/4.$$

Q2. Let Y_i be a random variable associated with the ith flip such that $Y_i = 1$ if the ith flip is a tail and $Y_i = -1$ if the ith flip is a head. It is easy to see that $E[Y_i] = 0$ and that $Var[Y_i] = 1$.

We also observe that $X_n = \sum_{i=1}^{n} Y_i$. By the linearity of expectation, we have that

$$E[X_n] = \sum_{i=1}^{n} E[Y_i] = 0.$$

Since all Y_i's are independent, we have that

$$Var[X_n] = \sum_{i=1}^{n} Var[Y_i] = n.$$

Q3. (a) $E[X] = \sum_{i=1}^{\infty} Pr(X = 2^i)2^i = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \infty$.

(b) Let Y be the number of flips we need to obtain the first head. $Pr(Y \geq 10) = \sum_{i=10}^{\infty} \frac{1}{2^i} = \frac{1}{2^9}$. Let Z be the amount of money the player wins. $Pr(Z = 1024) = Pr(Y \geq 10) = \frac{1}{2^9}$. $Pr(Z = 2^i) = Pr(Y = i) = \frac{1}{2^i}$ for $i = 1, \ldots, 9$. By the definition, it is easy to obtain that $E[Z] = 11$.

Q4. Let Y be a random variable such that $Y = 1$ if $X \geq a$; $Y = 0$ if $X < a$. It is easy to see that $Y \leq \frac{X}{a}$, so we have

$$p(X \geq a) = p(Y = 1) = E[Y] \leq \frac{E[X]}{a} = \frac{\mu}{a}.$$

Q5. Let X be the number of tails we obtain after n trials. One can show that $E[X] = \frac{n}{2}$, $Var[X] = \frac{n}{4}$, and $\sigma(x) = \sqrt{Var[X]} = \sqrt{n}/2$. Note that $10\sqrt{n} = 20\sigma$, by the Chebyshev's inequality,

$$p(|X - n/2| \geq 10\sqrt{n}) = p(|X - n/2| \geq 20\sigma) \leq \frac{1}{400}.$$

Q6. Let's say that in each flip, we get a head with probability p and a tail with probability $1 - p$. To simulate a fair coin, we need to construct an event which occurs with probability $1/2$. Now consider the following experiment. In each trial, we flip the coin twice. We repeat the trial over and over until

(i) we get a head in the first flip and a tail in the second flip, then we output 1; or

(ii) we get a tail in the first flip and a head in the second flip, we output 0.

Next we will show that $p(\text{output} = 1) = 1/2$. In each trial, 0 and 1 are equally likely to be output (with probability $p(1 - p)$). Thus, they are equally likely to be the output of the whole experiment.