COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial Problems: Logic and Proofs

Q1. (a) $r \wedge \neg q$

(b) $p \wedge q \wedge r$

(c) $r \rightarrow p$

(d) $p \wedge \neg q \wedge r$

(e) $(p \wedge q) \rightarrow r$

(f) $(p \vee q) \leftrightarrow r$

Q2. The proof using truth table is omitted here. The proof without using truth table is the following.

$$\begin{aligned}
& p \leftrightarrow q \\
\equiv & (p \rightarrow q) \wedge (q \rightarrow p) \\
\equiv & (\neg p \vee q) \wedge (\neg q \vee p) && (s \rightarrow t \equiv \neg s \vee t) \\
\equiv & ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p) && \text{(Distributive laws)} \\
\equiv & (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (\neg p \wedge p) \vee (p \wedge q) && \text{(Distributive laws)} \\
\equiv & (\neg p \wedge \neg q) \vee F \vee F \vee (p \wedge q) && \text{(Negation laws)} \\
\equiv & (\neg p \wedge \neg q) \vee (p \wedge q) && \text{(Identity laws)}
\end{aligned}$$

Q3. To prove that $\exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$ is a tautology, it suffices to show that whenever $\exists x \forall y P(x, y)$ is true, $\forall y \exists x P(x, y)$ is also true. If $\exists x \forall y P(x, y)$ is true, then there is some x, namely x_0, such that for all y, $P(x_0, y)$ is true. Therefore, for all y, x_0 makes $P(x_0, y)$ true, which implies that $\forall y \exists x P(x, y)$ is true.

Q4. Let the domain consist of all people in the world.

$$\forall x \exists y \exists z, [(y \neq z) \wedge (\forall w, (w = y) \vee (w = z) \leftrightarrow P(w, x))].$$

Q5. For all $i \in \mathbb{Z}$, i can be written as $i = 5q + r$ for some $q \in \mathbb{Z}$ and some $r \in \{0, 1, 2, 3, 4\}$. If i is not divisible by 5, r cannot be 0. There are four cases remaining.

- case 1: $i = 5q + 1$. Then we have $i^2 = 25q^2 + 10q + 1$. Then divide i^2 by 5, the remainder is 1.
- case 2: $i = 5q + 2$. Then we have $i^2 = 25q^2 + 20q + 4$. Then divide i^2 by 5, the remainder is 4.
- case 3: $i = 5q + 3$. Then we have $i^2 = 25q^2 + 30q + 9$. Then divide i^2 by 5, the remainder is 4.
- case 4: $i = 5q + 4$. Then we have $i^2 = 25q^2 + 40q + 16$. Then divide i^2 by 5, the remainder is 1.

In conclusion, if i is not divisible by 5, then dividing i^2 by 5 leaves a remainder of 1 or 4.

Q6. Suppose, for the sake of contradiction, that $\sqrt{2}+\sqrt{3}$ is rational. Then $\sqrt{2}+\sqrt{3} = \frac{p}{q}$ for some integer p and q. Now let's consider $(\sqrt{2}+\sqrt{3})^2$. We have that

$$\begin{aligned}\frac{p^2}{q^2} &= (\sqrt{2}+\sqrt{3})^2 \\ &= 5 + 2\sqrt{6},\end{aligned}$$

and consequently, we have that

$$\sqrt{6} = \frac{p^2 - 5q^2}{2q^2},$$

which implies that $\sqrt{6}$ is rational.

Next, we will prove that $\sqrt{6}$ is not rational, which leads to a contradiction. Suppose that $\sqrt{6}$ is rational. Then $\sqrt{6} = \frac{m}{n}$ for some integer m and n that are co-prime. Then we have that

$$m^2 = 6n^2.$$

Consequently, m must be a multiple of 6, i.e., $m = 6k$ for some integer k. Replacing m with $6k$, we get

$$n^2 = 6k^2.$$

So n must be a multiple of 6. Since m and n have a common factor 6, they are not co-prime. This is a contradiction, and it completes the proof.

Q7. Suppose, for the sake of contradiction, that there is some rational solution $\frac{p}{q}$ where p and q are two co-prime integers. Then we have that

$$\frac{p^3}{q^3} + \frac{p}{q} + 1 = 0,$$

or, equivalently,

$$p^3 + pq^2 + q^3 = 0. \qquad (*)$$

Next we will show that no co-prime integers p and q will make equation $(*)$ hold, which leads to a contradiction. Since p and q are co-prime, there are three cases.

(i) p is odd, and q is odd. Then we have that p^3, pq^2, and q^3 are all odd. As a result $p^3 + pq^2 + q^3$ is also odd and cannot be 0.

(ii) p is odd, and q is even. Then we have that p^3 is odd, pq^2 is even, and q^3 is even. Again, $p^3 + pq^2 + q^3$ is odd and cannot be 0.

(iii) p is even, and q is odd. Similar to that in (ii), we can show $p^3 + pq^2 + q^3$ is odd and cannot be 0.

Q8. Let p be one of these six people. One of the following two cases must hold for the remaining five people.

(i) At least three of them have shaken hands with p. Now consider these three people. If any two of them have shaken hands, then these two people, together with p, form a triangle. Otherwise, no pair within the three people have shaken hands, they form a triangle.

(ii) At least three of them haven't shaken hands with p. The proof is similar to that in (i).