MATH 2011 Mid-term Mock

1. Let S_1 be the surface of the equation $y = x^3 - 2$ and S_2 be the surface of the equation $6x^2 - y + 2z^2 = 18$. Find the intersection curve of S_1 and S_2 and represent it using a vector-valued function $\vec{r}(t)$.

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2. (a.) Write down a parametric equation of the line through $(-3, 8, 2)$ that is parallel to $5\vec{i} - 7\vec{j} + 9\vec{k}$.

(b.) Find an equation of the plane that passes through the three points $(3, 7, -4)$, $(2, -2, -6)$ and $(1, 8, -3)$.

(c.) Determine whether the line in (a) and the plane in (b) intersects if so, find the coordinate of the intersection.

3. Given z implicitly defined as a function of x and y through the equation $x^3y^2 + e^{y^z} - \sqrt{xyz} = 0$

Use the implicit differentiation to derive $\frac{\partial z}{\partial x}$.

4. Find $\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1,e)}$ of $z = x^3 + y^2$, $x = v^u$ and $y = ue^v$.

and $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,e)}$

5. (a) Let S be the surface of the equation $6z - 4x^2 - y^{10} = 0$.

Find all points on the surface S at which the tangent plane is parallel to the plane $x + 5y - 3z = 2$

(b) Determine the equation(s) of the tangent plane(s) at the point(s) in (a)

Solutions to Mock (MATH 2011) (midterm)

1. Let $x = 2t$, then $y = 8t^3 - 2$ and

$$6(2t)^2 - (8t^3 - 2) - 2z^2 = 18$$
$$24t^2 - 8t^3 + 2 + 2z^2 = 18$$
$$z^2 = 8 - 12t^2 + 4t^3$$
$$z = 2\sqrt{2 - 3t^2 + t^3}$$

$\therefore \vec{r}(t) = \left(2t,\ 8t^3 - 2,\ 2\sqrt{2 - 3t^2 + t^3}\right)$

2. a. $\vec{r}(t) = (-3, 8, 2) + t(5, -7, 9)$

$= (-3 + 5t,\ 8 - 7t,\ 2 + 9t)$

$\therefore$ The parametric equation is $\begin{cases} x = 5t - 3 \\ y = 8 - 7t \\ z = 2 + 9t \end{cases}$

b. $\big((3, 7, -4) - (2, -2, -6)\big) \times \big((3, 7, -4) - (1, 8, -3)\big) = (1, 9, 2) \times (2, -1, -1)$

$= (-7, 5, -19)$

~~Let~~ $\therefore$ The plane required is:

$$-7x + 5y - 19z = -7(1) + 5(8) - 19(-3)$$
$$\Leftrightarrow \quad 7x - 5y + 19z + 90 = 0$$

c. Put the answer in (a) into the answer in (b), we have:

$$7(5t - 3) - 5(8 - 7t) + 19(2 + 9t) + 90 = 0$$
$$35t + 35t + 171t - 21 - 40 + 38 = -90$$
$$241t = -90 + 23$$
$$t = \frac{-67}{241}$$

$\therefore x = 5\left(\frac{-67}{241}\right) - 3 = \frac{-1058}{241}$

$y = 8 - 7\left(\frac{-67}{241}\right) = \frac{2397}{241}$

$z = 2 + 9\left(\frac{-67}{241}\right) = \frac{-121}{241}$

$\therefore$ The intersection exists and its coordinate is $\left(\frac{-1058}{241}, \frac{2397}{241}, \frac{-121}{241}\right)$

3. $\frac{\partial}{\partial x}\left(x^3y^2 + e^{y^z} - \sqrt{xyz}\right) = \frac{\partial}{\partial x}(0)$

$$3x^2y^2 + e^{y^z} y^z \ln y \frac{\partial z}{\partial x} - \frac{y}{2\sqrt{xyz}}\left(z + x\frac{\partial z}{\partial x}\right) = 0$$
$$\left(e^{y^z} y^z \ln y + \frac{xy}{2\sqrt{xyz}}\right)\frac{\partial z}{\partial x} = -\left(3x^2y^2 - \frac{yz}{2\sqrt{xyz}}\right)$$
$$\frac{\partial z}{\partial x} = \frac{yz - 6x^2y^2\sqrt{xyz}}{2e^{y^z} y^z (\ln y)\sqrt{xyz} + xy}$$

4. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$= (3x^2)(v^u \ln v) + (2y)(e^v)$

$= 3v^{2u}\ln v + 2ue^{2v}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

$= (3x^2)(uv^{u-1}) + (2y)(ue^v)$

$= 3uv^{3u-1} + 2u^2e^{2v}$

Solutions (cont'd)

4. (cont'd) $\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1,e)} = 3(e)^{2(1)}\ln(e) + 2(1)e^{2(e)}$

$= 3e^2 + 2e^{2e}$

$\cancel{= 5e}$

$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,e)} = 3(1)(e)^{3(1)-1} + 2(1)^2 e^{2(e)}$

$= 3e^2 + 2e^{2e}$

5. a. Let $f(x,y,z) = 6z - 4x^2 - y^{10}$

Then, $\nabla f = (-8x, -10y^9, 6)$ and ∇f has to be in the same direction to $(1,5,-3)$, i.e. $(-8x, -10y^9, 6) = k(1,5,-3)$ where k is a constant scalar.

i.e. $(-8x, -10y^9, 6) = \frac{k}{-2}(-2, -10, 6)$

and leads to $-8x = -2$, $-10y^9 = -10$

i.e. $x = \frac{1}{4}$, $y = 1$

Find z: $6z - 4(\frac{1}{4})^2 - (1)^{10} = 0$

$6z - \frac{1}{4} - 1 = 0$

$z = \frac{5/4}{6} = \frac{5}{24}$

$\therefore$ The point is $(\frac{1}{4}, 1, \frac{5}{24})$

b. The tangent plane is: $x + 5y - 3z = (\frac{1}{4}) + 5(1) - 3(\frac{5}{24})$

$= \frac{1}{4} + 5 - \frac{5}{8}$

$= \frac{37}{8}$