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SECTION I: Multiple Choice [10 Marks]

Notes:

- **For each multiple-choice question below, select only ONE answer.**
- **Write your answer in your ANSWER BOOKLET (not this question sheet)**

[1 Mark Per Question]

1. For a discrete random variable, X, which of the following statements is not true?
   a) We can evaluate all moments of X solely based on the probability mass function
   b) The standard deviation of X has the same units as the mean
   c) All moments of X are non-negative
   d) The conditional mean and variance of X (conditioned on some event A) are examples of moments.
   e) Trellis diagrams, which represent the decoder of a Trellis Coded Modulation communication system, implement successive products of conditional probabilities.

2. Which of the following statements about conditional probability is not true?
   a) Conditioning always increases the probability of an event
   b) Conditioning may increase or decrease the probability of an event
   c) Conditioning changes the effective sample space of the experiment
   d) The Total Theorem of Probability allows one to compute unconditional probabilities by combining conditional probabilities

3. Assume that we select balls from an urn, at random. From the following selection methods, which method does not lead to equally-probable events?
   a) With replacement with ordering
   b) Without replacement with ordering
   c) With replacement without ordering
   d) Without replacement without ordering

4. If two events have non-zero probabilities and are mutually exclusive, then they:
   a) Must be independent
   b) Cannot be independent
   c) Can be independent or not independent
   d) Always form a partition
   e) Can never form a partition

5. In a sequential experiment, if the outcome of the nth sub-experiment s_n depends only on that of the (n-1)th sub-experiment s_{n-1}, which of the following statements is not true?
   a) It is a dependent sequential experiment
   b) $P[\{s_0\}\cap\{s_1\}\cap\{s_2\}] = P[\{s_2\}\,|\,\{s_1\}]\times P[\{s_1\}\,|\,\{s_0\}]\times P[\{s_0\}]$
   c) s_n and s_{n-2} are independent
   d) For a given s_{n-1}, s_n and s_{n-2} are independent
   e) None of the above (i.e., they are all true)

6. Which of the following statements about equivalent events is not true?

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a) If two outcomes map to the same number, they belong to the same equivalent event
b) If two events are equivalent, they have the same probability
c) A random variable defined on a discrete sample space must be a discrete random variable. Vice versa.
d) The equivalent events that correspond to all possible values of a discrete random variable form a partition of the underlying sample space
e) None of the above (i.e., they are all true)

7. If the CDF of a random variable, X, has discrete jumps, then:
a) X must be a discrete random variable
b) X can be a continuous random variable.
c) X can be either discrete or continuous
d) X can be both discrete and continuous
e) The CDF of X must be constant between jumps

8. Which of the following statements about a random variable X is not true?
a) X is a random function
b) X is a function which assigns numbers to the outcomes of a random experiment
c) X can take positive and negative values
d) X can take the value "zero"
e) None of the above (i.e., they are all true)

9. Consider the following communication system: A transmitter sends a signal S to a receiver. The received signal is $R=S+N$, where N denotes independent random noise obeying a Gaussian distribution. Based on R, the receiver successfully decodes S with probability p; otherwise it makes a mistake and the transmitter tries again. Let n be the number of times that S must be transmitted before it is received successfully. Which of the following is true?
a) n is a binomial random variable
b) n is a Bernoulli random variable
c) n is a Poisson random variable
d) n is a geometric random variable
e) n is a Gaussian random variable

10. Which of the following functions does not directly represent probabilities?
a) Probability mass function
b) Cumulative distribution function
c) Conditional cumulative distribution function
d) Probability density function

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SECTION II: Problems [40 Marks]

Notes:

- **Please attempt all problems, clearly showing your working.**
- **Write your solution in your ANSWER BOOKLET (not this question sheet)**

1. [10 Marks]

Consider the following communication system: A source sends a sequence of binary digits. For each transmission, a "0" is transmitted with probability 0.25; otherwise a "1" is transmitted. Each binary digit passes through a communication channel which causes an error with probability 0.1. [I.e., an error occurs if a “1” is changed to a “0”, and vice-versa.]

a) What is the probability that a "1" is observed at the output of the channel?

b) If a "1" is observed at the output of the channel, what is the probability that a "1" was actually transmitted by the source?

c) Let A be the event that a "1" is transmitted by the source and is also observed at the output of the channel. What is the probability that A occurs for the first time when the 3rd binary digit is sent?

Solution:

(a) Let X be the information produced at the source, and Y be the information observed at the output. Then

$$\begin{aligned} P(Y=1) &= P(Y=1|X=0)P(X=0)+P(Y=1|X=1)P(X=1) \\ &= 0.1\times 0.25+(1-0.1)\times 0.75 \\ &= 0.7 \end{aligned}$$

(b)

$$\begin{aligned} P(X=1|Y=1) &= \frac{P(X=1,Y=1)}{P(Y=1)} \\ &= \frac{P(Y=1\mid X=1)P(X=1)}{P(Y=1)} \\ &= \frac{(1-0.1)\times 0.75}{0.7} \\ &\approx 0.964 \end{aligned}$$

(c)

$$P(A)=0.75\times 0.9=0.675$$

$$\therefore (1-P(A))\times(1-P(A))\times P(A)=0.325\times 0.325\times 0.675\approx 0.071$$

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2. [10 Marks]

Consider a manufacturing plant producing electronic devices. The devices are made cheaply; therefore each device on the production line fails the quality inspection with probability 0.1. Assume that 20 devices are selected from this production line at random.

a) Compute the mean number of selected products which fail the inspection. Also compute the variance.

b) Consider the relation: $\varepsilon = 2\xi + 3$, where ξ is the number of selected products failing inspection. Find the expected value and the variance of ε .

c) Suppose that out of the selected 20 devices, we *know* that at least 2 devices fail inspection. Find the probability that more than 3 devices fail.

Solution:

(1)

ξ is a binomial R.V.

$n = 20, p = 1 - 0.9 = 0.1$

$E[\xi] = n \times p = 20 \times 0.1 = 2$

$D[\xi] = n \times p \times (1 - p) = 20 \times 0.1 \times (1 - 0.1) = 1.8$

(2)

$E[\varepsilon] = E[2\xi + 3] = 2 \times D[\xi] + 3 = 2 \times 2 + 3 = 7$

$D[\varepsilon] = D[2\xi + 3] = 2^2 \times D[\xi] = 4 \times 1.8 = 7.2$

(3)

$$P\{\xi > 3|\xi \geq 2\} = \frac{P\{\xi>3 \cap \xi \geq 2\}}{P\{\xi \geq 2\}} = \frac{P\{\xi>3\}}{P\{\xi \geq 2\}} = \frac{P\{\xi \geq 2\} - P\{\xi=2\} - P\{\xi=3\}}{P\{\xi \geq 2\}} = 1 - \frac{P\{\xi=2\}+P\{\xi=3\}}{1-P\{\xi=0\}-P\{\xi=1\}}$$

$$= 1 - \frac{\binom{20}{3}\times 0.1^3 \times 0.9^{17} + \binom{20}{2}\times 0.1^2 \times 0.9^{18}}{1-0.9^{20} - \binom{20}{1}\times 0.1 \times 0.9^{19}} = 0.2186$$

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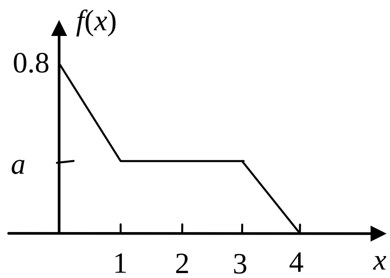
3. [10 Marks]

As part of his/her Final Year Project (FYP) on digital image processing, a student needs to select from one of two possible digital cameras: Camera A or Camera B. Both cameras have almost the same technical specifications. Thus, the main objective is to purchase the camera with the longest lifetime.

Camera A would be purchased new. The lifetime of Camera A is distributed according to the probability density function *f(x)*, as shown below. Here, x is in units of “years”.



$$f(x) = \begin{cases} (a-0.8)\text{x}+0.8, & 0 \le \text{x}<1 \\ a, & 1 \le x < 3 \\ -ax+4a, & 3 \le x \le 4 \end{cases}$$

The lifetime of Camera B, if purchased new, is exponentially distributed with probability density function:

$$\text{g}(x) = 0.5e^{-0.5x}, \quad x \ge 0$$

where, once again, x is in units of “years”. However, the student will purchase a second-hand model of Camera B, which has been used for 2 years already (for other FYPs!).

a) Compute the value of a in the expression for the probability density function of the lifetime of Camera A.

b) Compute the expected value of the lifetime of Camera A.

c) Compute the variance of the lifetime of Camera A.

d) Compute the expected value and variance of the lifetime of Camera B.

e) Which digital camera should the student choose? Justify your choice.

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Solution:

a) $a = 0.2$, because $\int_0^4 f(x)dx = 1$.

b) $E_A(x) = \int_0^4 xf(x)dx = \int_0^1(-0.6x^2 + 0.8x)\,dx + \int_1^3 0.2xdx + \int_3^4 -0.2x^2 + 0.8xdx = 0.2 + 0.8 + \frac{1}{3} = \frac{4}{3}$

c) $E_A(x^2) = \int_0^4 x^2 f(x)dx = \int_0^1(-0.6x^3 + 0.8x^2)dx + \int_1^3 0.2x^2dx + \int_3^4 -0.2x^3 + 0.8x^2dx = \frac{7}{60} + \frac{26}{15} + \frac{67}{60} = \frac{178}{60} = \frac{89}{30}$

$$D_A(x) = E_A(x^2) - \left(E_A(x)\right)^2 = \frac{107}{90}$$

d) The lifetime of camera B is exponentially distributed, according to the memoryless property of exponential distribution, the expectation and the variance does not change. Therefore, $E_B(x) = \frac{1}{\lambda} = 2$, $D_B(x) = \frac{1}{\lambda^2} = 4$.

e) Since $E_B(x) > E_A(x)$, so the camera B is preferable although it is a second-hand camera.
(The answer camera A is also acceptable since $D_B(x) > D_A(x)$.)

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4. [10 Marks]

A student must complete a sequence of assignments as part of his/her probability course. For the first assignment, the student has a 50% chance of getting a distinction. For all subsequent assignments, if a student gets a distinction on their previous assignment, they will adopt the same study pattern for the next assignment, and therefore again have a 50% chance of getting a distinction. Alternatively, if the student does not get a distinction for their previous assignment, then they will try even harder for the next assignment, and therefore the chance of getting a distinction increases to 75%.

a) What is the probability that the student gets a distinction for the first two assignments?

b) What is the probability that for the first three assignments, the student gets exactly two distinctions?

c) Given that the third assignment received a distinction, what is the probability that the second assignment also received a distinction?

d) Compute the probability that the *k*-th assignment receives a distinction.

(Hint: The following may be useful: $1+\alpha+\alpha^2+\cdots+\alpha^m=\frac{1-\alpha^{m+1}}{1-\alpha}, \quad \alpha\neq 1$)

Solution:

(1)

$$P(X_1=X_2=S)=P(X_1=S)P(X_2=S|X_1=S)=\frac{1}{2}*\frac{1}{2}=\frac{1}{4}=0.25$$

(2)

$$P_1(X_1=S,X_2=S,X_3=F)=P(X_1=S)P(X_2=S|X_1=S)P(X_3=F|X_2=S)=\frac{1}{8}$$

Also, we have that: $P_2(X_1=S,X_2=F,X_3=S)=\frac{3}{16}, \quad P_3(X_1=F,X_2=S,X_3=S)=\frac{3}{16}$

Thus, $P=P_1+P_2+P_3=\frac{1}{2}=0.5$

(3)

$$P(X_1=S)=\frac{1}{2}, P(X_2=S)=\frac{1}{2}*\frac{1}{2}+\frac{1}{2}*\frac{3}{4}=\frac{5}{8}, P(X_3=S)=\frac{5}{8}*\frac{1}{2}+\frac{3}{8}*\frac{3}{4}=\frac{19}{32}$$

$$P(X_2=X_3=S)=P(X_2=S)P(X_3=S|X_2=S)=\frac{5}{8}*\frac{1}{2}=\frac{10}{32}$$

Thus:

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$$P(X_2 = S|X_3 = S) = \frac{10}{19} = 0.526$$

(4)

Let $p_k = P(X_k = S)$, since we have

$P(X_k = S) = P(X_{k-1} = S)P(X_k = S|X_{k-1} = S) + P(X_{k-1} = F)P(X_k = S|X_{k-1} = F)$,

and

$P(X_{k-1} = F) = 1 - P(X_{k-1} = S)$,

we obtain,

$$p_k = p_{k-1} * \frac{1}{2} + (1 - p_{k-1}) * \frac{3}{4} = \frac{3}{4} + \left(-\frac{1}{4}\right)p_{k-1} = \frac{3}{4} + \left(-\frac{1}{4}\right)\frac{3}{4} + \left(-\frac{1}{4}\right)^2 p_{k-2} = \cdots$$

By $P(X_1 = S) = 1/2$, i.e., $p_1 = 1/2$, we get:

$$p_k = \frac{3}{4} + \left(-\frac{1}{4}\right)\frac{3}{4} + \left(-\frac{1}{4}\right)^2\frac{3}{4} + \cdots + \left(-\frac{1}{4}\right)^{k-2}\frac{3}{4} + \left(-\frac{1}{4}\right)^{k-1} p_1$$

From the above formula, we can calculate that:

$$p_k = \frac{3}{5} - \frac{1}{10}\left(-\frac{1}{4}\right)^{k-1}$$