COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial 10

Equations

Expectation. $E(X) = \sum_{x \in S} p(s)X(s)$ and for discrete probability,

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

Linearity of Expectations.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$E(aX + b) = aE(X) + b$$

Independence. The random variables X and Y on a sample space S are independent if

$$p(X = r_1 and Y = r_2) = p(X = r_1) \cdot (p(Y = r_2)),$$

and so

$$E(XY) = E(X)E(Y)$$

Variance.

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s) = E\left((X - E(X))^2\right) = E(X^2) - E(X)^2$$

Corollary

$$V(aX + b) = a^2 V(X)$$

Independence. For n independent variables $X_1, X_2, \cdots, X_n$,

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$$

EP4-9. Let X and Y be two independent random variables. Express the variance of $X - Y$, $V(X - Y)$, in terms of $V(X)$ and $V(Y)$.

Solution

$$V(X-Y) = V(X+(-Y)) = V(X)+V(-Y) = V(X)+(-1)^2V(Y) = V(X)+V(Y)$$

EP4-10. Each pixel in a 32×8 vertical display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows.

(a) What is the expected value of X?

(b) What is the variance of X?

Solution

(a)Let $S_{ij}, i \in [1,8], j \in [1,32]$, denote the i^{th} pixel of j^{th} horizontal line is on ($S_{ij} = 1$) or off ($S_{ij} = 0$). We define the following indicator random variables:

$$X_j = \begin{cases} 1 & j^{th}\text{horizontal line is shown} \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } j \in [1,32]$$

We are interested in the number of horizontal lines that the display shows, where $X = \sum X_j$. S_{ij} are independent and X_j are independent. We have

$$\begin{aligned} E(X) = E(\sum_{j=1}^{32} X_j) &= \sum_{j=1}^{32} E(X_j) \\ &= \sum_{j=1}^{32} E(\prod_{i=1}^{8} S_{ij}) \\ &= \sum_{j=1}^{32} \prod_{i=1}^{8} E(S_{ij}) \\ &= \sum_{j=1}^{32} \prod_{i=1}^{8} \frac{1}{2} \\ &= \sum_{j=1}^{32} \frac{1}{2^8} \\ &= \frac{32}{2^8} \\ &= \frac{1}{8} \end{aligned}$$

(b)

$$
\begin{aligned}
V(X) &= E(X^2) - E(X)^2 \\
&= E((\sum_{j=1}^{32} X_j)^2) - \frac{1}{2^6} \\
&= \sum_{j=1}^{32} E(X_j^2) + 2\sum_{j_1=1}^{32}\sum_{j_2=j_1+1}^{32} E(X_{j_1}X_{j_2}) - \frac{1}{2^6} \\
&= \sum_{j=1}^{32} E((\prod_{i=1}^{8} S_{ij})^2) + 2\sum_{j_1=1}^{32} E(X_{j_1}) \sum_{j_2=j_1+1}^{32} E(X_{j_2}) - \frac{1}{2^6} \\
&= \sum_{j=1}^{32} E((\prod_{i=1}^{8} S_{ij})^2) + 2\binom{32}{2}\frac{1}{2^8}\frac{1}{2^8} - \frac{1}{2^6} \\
&= \sum_{j=1}^{32} E(\prod_{i=1}^{8} S_{ij}^2) + \frac{31}{2^{11}} - \frac{1}{2^6} \\
&= \sum_{j=1}^{32}\prod_{i=1}^{8} E(S_{ij}^2) + \frac{31}{2^{11}} - \frac{1}{2^6} \\
&= \sum_{j=1}^{32}\prod_{i=1}^{8} \frac{1}{2} + \frac{31}{2^{11}} - \frac{1}{2^6} \\
&= \frac{32}{2^8} + \frac{31}{2^{11}} - \frac{1}{2^6} \\
&= \frac{255}{2048}
\end{aligned}
$$

EP4-11. A biased coin is tossed n times, and a head shows up with probability p on each toss. A run is a maximal sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$.

Solution

Let $S_i, i \in \{1, \cdots, n\}$, denote the i^{th} element of the sequence of coin tosses. We define the following indicator random variables:

$$X_1 = 1$$

$$X_i = \begin{cases} 1 & S_i \neq S_{i-1} \\ 0 & \text{otherwise} \end{cases} \text{for each } i \in \{2, \cdots, n\}$$

X_i indicates the event that a new run begins at position i in the sequence. The random variable X that we are interested in, i.e., the number of runs in the sequence of random coin tosses, can be computed as $X = \sum_{i=1}^{n} X_i$. We first compute the following probabilities:

$$
\begin{aligned}
P(X_1 = 1) &= 1 \\
P(X_i = 1) &= P(S_i \neq S_{i-1}) \\
&= P(S_i = T \wedge S_{i-1} = H) \vee P(S_i = H \wedge S_{i-1} = T) \\
&= (1-p)p + p(1-p) \\
&= 2p(1-p)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E(X) = \sum_{i=1}^{n} E(X_i) &= \sum_{i=1}^{n} P(X_i = 1) \\
&= 1 + \sum_{i=2}^{n} 2p(1-p) \\
&= 1 + 2(n-1)p(1-p)
\end{aligned}
$$

EP4-12. Each of 1000 voters votes independently for a candidate A with probability 1/2.

(a) What is the probability that A gets exactly 500 votes?

(b) What is the probability that A gets at least 500 votes?

Solution

(a) $p(X = 500) = \frac{\binom{1000}{500}}{2^{1000}}$

(b) $p(X = n) = \frac{\binom{1000}{n}}{2^{1000}} = \frac{\binom{1000}{1000-n}}{2^{1000}} = p(X = 1000 - n)$

$$p(X \geq 500) = \sum_{x=500}^{1000} p(X = x),$$

and we know

$$\sum_{x=500}^{1000} p(X = x) = \sum_{x=0}^{500} p(X = x), \sum_{x=0}^{1000} p(X = x) = 1$$

$$\sum_{x=0}^{1000} p(X = x) = \sum_{x=500}^{1000} p(X = x) + \sum_{x=0}^{500} p(X = x) - p(X = 500)$$

$$\Leftrightarrow 1 = 2 \sum_{x=500}^{1000} p(X = x) - p(X = 500)$$

$$p(X \geq 500) = \sum_{x=500}^{1000} p(X = x) = \frac{1 + p(X = 500)}{2} = \frac{1 + \frac{\binom{1000}{500}}{2^{1000}}}{2}$$

EP1-22. Prove by induction on $n \geq 0$ that

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Solution

Basis: $P(0)$ is true as $(a+b)^0 = 1$ and $\binom{0}{0}a^0b^0 = 1$.

Inductive step : $P(k) \to P(k+1)$ very positive integer k

Induction hypothesis $P(k)$:

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}.$$

So, assuming $P(k)$, we have

$$\begin{aligned}
(a+b)^{k+1} &= a(a+b)^k + b(a+b)^k \\
&= \sum_{i=0}^{k} \binom{k}{i} a^{i+1} b^{k-i} + \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1} \\
&= \sum_{j=1}^{k+1} \binom{k}{j-1} a^j b^{k-j+1} (\text{replace } j = i+1) + \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1} \\
&= \binom{k}{k} a^{k+1} b^0 + \sum_{j=1}^{k} \binom{k}{j-1} a^j b^{k-j+1} \quad + \quad \binom{k}{0} a^0 b^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^i b^{k-i+1} \\
&= \binom{k+1}{k+1} a^{k+1} + \binom{k+1}{0} b^{k+1} + \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1} \right) a^i b^{(k+1)-i} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{(k+1)-i}
\end{aligned}$$

$P(k+1)$ is true.