COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial Problems: Induction and Recursion

Q1. Base case: when $n = 1$, it is easy to see that $6^{1+1} + 7^{2-1} = 43$ is divided by 43.
Inductive step: suppose that 43 divides $6^{k+1} + 7^{2k-1}$ for some positive integer k. We shall prove that 43 divides $6^{k+2} + 7^{2k+1}$.

$$\begin{aligned} 6^{k+2} + 7^{2k+1} &= 6 \cdot 6^{k+1} + 49 \cdot 7^{2k-1} \\ &= 6 \cdot (6^{k+1} + 7^{2k-1}) + 43 \cdot 7^{2k-1} \end{aligned}$$

By the inductive hypothesis, 43 divides $6 \cdot (6^{k+1} + 7^{2k-1})$, and obviously, 43 divides $43 \cdot 7^{2k-1}$, so 43 divides $6^{k+2} + 7^{2k+1}$. This completes the proof.

Q2. (a) *Weak induction* $P(8)$ is true since we can use one 3-cent stamp and one 5-cent stamp to form a postage of 8 cents. Now suppose $P(k)$ is true for some integer $k \geq 8$. We shall show that $P(k+1)$ is true. Since $P(k)$ is true, consider a postage of k cents formed by 3-cent and 5-cent stamp. It is the case either

(i) the postage of k cents uses a 5-cent stamp. By replacing it with two 3-cent stamps, we get a postage of $k+1$ cents. Or

(ii) the postage of k cents uses only 3-cent stamps. Since $k \geq 8$, there are at least three 3-cent stamps. By replacing them with two 5-cent stamps, we get a postage of $k+1$ cents.

So, $P(k+1)$ is true. This completes the proof.

(b) *Strong induction* One can easily verify that $P(8)$, $P(9)$, $P(10)$ are true. Now suppose that $P(i)$ is true for all $i \in \{8, \ldots, k\}$ for some integer $k \geq 10$. We shall prove that $P(k+1)$ is true. Since $k \geq 10$, we have that $8 \leq k-2 \leq k$. By the inductive hypothesis, we can form a postage of $k-2$ cents. Then by adding one more 3-cent stamp, we form a postage of $k+1$ cents. So $P(k+1)$ is true. This completes the proof.

Q3. We prove it by induction on the number of lines.
Base case: when there is only one line, obviously we only need two colors.
Inductive step: Suppose that when there are k lines, we only need two colors. We claim that when there are $k+1$ lines, we can use two colors to color the regions as follows.

(i) We first remove an arbitrary line l. By the inductive hypothesis, we can use two colors to color the regions formed by the remaining k lines.

(ii) Then we add l back, and reverse the colors of the regions on one side of l.

This completes the proof.

Q4. $\log(n^4)$, $\sqrt{2n}$, $n+10$, $n^2 \log n$, n^3, 2^n, 20^n

Q5. It is clear the statement is true when $n = 1$. Now suppose that statement is true when $n = k$ for some positive integer k. We prove that the statement is also true when $n = k+1$. By inductive hypothesis, we have k lines separating the plane into $(k^2+k+2)/2$ regions. Then we add one more line. This line intersects with all other

k lines at different points, so it is divided into $k+1$ parts. Each part separate the region to which it belongs into two. Hence we get $k+1$ more regions, and the total number of region is $(k^2+k+2)/2+k+1 = ((k+1)^2+k+3)/2$. This completes the proof.

Q6. (a) *A proof by strong induction*

Base case: When we split a pile of one stone, the statement is obviously true.
Inductive step: Now suppose that when we split a pile of i stones, the sum of products is $i(i-1)/2$ for all positive integer $i \le k$. We shall prove that the sum of products is $(k+1)k/2$ when we split a pile of $k+1$ stones. Consider how you split a $(k+1)$-sized pile.

(1) First you split it into two piles: pile P_1 of size s_1 and P_2 of size s_2. And $s_1+s_2 = k+1$.

(2) You split P_1 and P_2 again and again into s_1+s_2 piles of one stone.

The product you obtain at step(1) is s_1s_2. By inductive hypothesis, the sum of products you obtain in step(2) is $s_1(s_1-1)/2+s_2(s_2-1)/2$. The total sum of the products is

$$\begin{aligned} s_1s_2+s_1(s_1-1)/2+s_2(s_2-1)/2 &= (2s_1s_2+s_1^2+s_2^2-s_1-s_2)/2 \\ &= [(s_1+s_2)^2-(s_1+s_2)]/2 \\ &= (k+1)k/2 \end{aligned}$$

This completes the proof

(b) *A proof without induction*

We let two stone shake hands whenever they are separated into different piles. We claim that the number of handshakes that occur in the whole procedure is equal the sum of products. To see this, consider a moment when a big pile is split into two small piles of r and s stones respectively. The number of handshaking occurs in this splitting is rs, which is equal to the "product". At the end, each stone is separated with all other stones, so it has shaken hands with all the $n-1$ other stones. The total number of handshakes is $(n-1)n/2$.

Q7. (a) We list the elements step by step.

(0) basis step: $(0,0)$

(0) first step: $(0,1),(1,1),(2,1)$

(0) second step: $(0,2),(1,2),(2,2),(3,2),(4,2)$

(0) third step: $(0,3),\ldots,(6,3)$

(0) forth step: $(0,4),\ldots,(8,4)$

(b) We use **structural induction**. First it is easy to see that $(0,0)$ has the property. Suppose that $(a,b) \in S$ which has the property that $a \le 2b$. Now we prove that the three new element $(a,b+1),(a+1,b+1),(a+2,b+1)$, which are generated from it, also have this property. To see this, it suffices to observe the following.

$$a \le 2b \Rightarrow a \le 2b+2, a+1 \le 2b+2, a+2 \le 2b+2$$

This completes the proof.

Q8. Let S be the set of bit strings that have more zeros than ones.

(i) $0 \in S$

(ii) if $a, b \in S$, then $ab, ab1, 1ab, a1b \in S$.

Q9. (a) We define $T(n)$ to be the number of such sequences whose last term is n. Now consider the second-last term of the sequence (i.e. term just before n). Since this term can be any number in $\{1, \ldots, (n-1)\}$, we have that

$$T(n) = \sum_{i=1}^{n-1} T(i).$$

The above recurrence follows from the fact that $T(i)$ represents the number of strictly increasing sequences that begin with 1 and end with i. Note that $T(n-1) = \sum_{i=1}^{n-2} T(i)$ for $n \geq 3$. By plugging this in, we obtain that for $n \geq 3$,

$$T(n) = \sum_{i=1}^{n-1} T(i) = T(n-1) + \sum_{i=1}^{n-2} T(i) = 2T(n-1).$$

Using the facts that $T(1) = 1, T(2) = 1$, we can derive that $T(n) = 2^{n-2}$ for $n \geq 2$. (Steps omitted.)

(b) We observe that the sequence is uniquely determined by the set of terms between the first term and the last term, and that those terms can be any subset of $\{2, 3, \ldots, n-1\}$. Since $\{2, 3, \ldots, n-1\}$ has 2^{n-2} subsets, there are 2^{n-2} possible sequences.

Q10. Let $T(n)$ be the number of ways we can cover $2 \times n$ checkerboard. It is easy to see $T(1) = 1$ and $T(2) = 2$. For $n \geq 3$, consider how we cover the last one or two column of the check board. There are only two cases.

(i) We cover the last column with a vertical dominoe.

(ii) We cover the last two columns with two horizontal dominoes.

In the case (i), there are $T(n-1)$ ways to cover the remaining $2 \times (n-1)$ checkerboard. In case (ii), there are $T(n-2)$ ways to cover the remaining $2 \times (n-2)$ checkerboard. In total there are $T(n) = T(n-1) + T(n-2)$ ways. $T(17) = 2584$.

Q11. Let $T(n)$ be the number of different messages can be transmitted in n micro seconds. Now consider the last signal we send, it can be any of the three signals.

(i) if it is the signal A requiring 1 microsecond, there are $n-1$ microseconds left, so we have $T(n-1)$ such messages.

(ii) if it is the signal B requiring 2 microsecond, there are $n-2$ microseconds left, so we have $T(n-2)$ such messages.

(iii) if it is the signal C requiring 2 microsecond, there are $n-2$ microseconds left, so we have $T(n-2)$ such messages.

So we have that $T(n) = T(n-1) + 2T(n-2)$.

To solve this recurrence, we use the characteristic equation. The roots of the characteristic equation $x^2 = x + 2$ are $x = 2$ and $x = -1$, so the solution is of the form $a2^n + b(-1)^n$ for some real number a and b. Now we consider the base case $T(1) = 1$ and $T(2) = 3$. Since $a2^1 + b(-1)^1 = 1$ and $a2^2 + b(-1)^2 = 3$, we have that $a = \frac{2}{3}$ and $b = \frac{1}{3}$. Hence the solution is $T(n) = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$.