Department of Electronic and Computer Engineering

ELEC 2600: Probability and Random Processes in Engineering

2020 Spring Semester

Final Exam Session 2

May 24, 2020, 19:30 – 21:30

Name: ______________________________

Student ID: ______________________________

Instructions:

1. This is a **2-hour open-book test**.
2. Calculator is allowed.
3. There are **5 computational problems**.
4. Try to attempt all problems.
5. Answer the problems of **Section II** on paper.
6. Please show all of your work, as marks will be given for the key steps, not just for the correct answer.
7. If your results involve binomial coefficients, factorials, exponential functions, or ratios, you do not need to compute the value. For example, an answer like $\frac{2\times 3^4}{4!}e^{-4}$ doesn't need to be simplified further.
8. You do not need to compute any matrix inverse. It is sufficient to give the numerical values of the matrix that needs to be inverted. No matrix determinants are required to be computed unless explicitly asked for.
9. The distribution of marks is shown in the table below:

Question/Problem	**Mark**
1	12
2	12
3	10
4	14
5	10
Total	**58**

1. (12 pts) The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2xy, & 0 < x < 1, 0 < y < 2x \\ 0, & otherwise \end{cases}$$

   (a) Find $P[X < Y]$. (3 pts)
   (b) Find the marginal pdf of X, $f_X(x)$. (3 pts)
   (c) Find the conditional pdf of Y given X, $f_{Y|X}(y|x)$, for $0 < x < 1$. (3 pts)
   (d) Find the conditional expected value of Y given $X = x$, $E[Y|X = x]$, for $0 < x < 1$. (3 pts)

2. (12 pts) Let X be the lifetime (measured in years) of a component in a system, which is continuous and uniformly distributed in the interval $[1, 2]$. The system also has a backup component that can take over when the original component fails so that the system can provide continuous service. Let Y be the lifetime of the backup component, which is exponentially distributed with mean 1 year. Assume that the lifetimes of the original and backup components are independent. Let $Z = X + Y$ be the total lifetime of the system.
   (a) Find the mean of Z. (2 pts)
   (b) Find the PDF of Z. Note that you will need different expressions for different ranges of Z. For partial credit, give the ranges of Z. For full credit, find the correct expressions in each of these ranges. (7 pts)
   (c) Find the probability that the backup component lasts for between 2 and 3 years, i.e. $P[2 \leq Y \leq 3]$. (3 pts)

3. (10 pts) Consider a discrete-time communications receiver that receives a signal Y_i consisting of a constant signal, 3, corrupted by additive noise N_i, i.e.,

$$Y_i = 3 + N_i \text{ for } i = 1, \dots, n.$$

   Assume that the noise is an independent and identically distributed (i.i.d.) random process, where each N_i is a continuous random variable that is uniformly distributed on $[-1, 1]$. Let $M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ be the sample mean.
   (a) Find $E[Y_2]$. (2 pts)
   (b) Find $\mathrm{Var}[Y_2]$. (2 pts)
   (c) Find $E[Y_2 Y_3]$. (2 pts)
   (d) Find $E[M_8]$. (1 pt)
   (e) Find $\mathrm{Var}[M_8]$. (1 pt)
   (f) According to the central limit theorem, the distribution of M_{100} is approximately Gaussian. Estimate $P[|M_{100} - 3| > 0.1]$ by assuming that M_{100} is Gaussian distributed. Express your answer in terms of the Q function, $Q(x) = P[X > x]$, where X is a Gaussian with zero mean and variance 1. (2 pts)

4. (14 pts) A Gaussian Random Process is one where all finite order distributions are jointly Gaussian distributed. Suppose that $X(t)$ is a continuous-time Gaussian random process with mean and covariance functions given by

$$m_X(t) = E[X(t)] = \cos\left(\frac{\pi t}{2}\right) \text{ and } C_X(t_1, t_2) = 4e^{-0.5|t_1 - t_2|}.$$

   (a) Write the variance function for $X(t)$, $\text{Var}[X(t)]$, for all $-\infty < t < \infty$. (1 pt)
   (b) Give the first order PDF for $X(1)$, $f_{X(1)}(x)$, for $-\infty < x < \infty$. (3 pts)
   (c) Define a random vector $\vec{Y} = [X(1)\ X(4)]^T$. Give the second order PDF for $\vec{Y}$, $f_{\vec{Y}}(\vec{y}) = f_{X(1)X(4)}(x_1, x_2)$ where $\vec{y} = [x_1, x_2]^T$ for $-\infty < x_1, x_2 < \infty$.
   Hint: You can answer this by giving the general form of the multivariate Gaussian and specifying the mean vector and covariance matrix. (4 pts)
   (d) Find the correlation coefficient between $X(1)$ and $X(4)$. (2 pts)
   (e) Define a random variable $Z = 2X(1) + 3X(2)$. Give the probability density function of Z, $f_Z(z)$ for $-\infty < z < \infty$. (4 pts)

5. (10 pts) Suppose that the total number of requests to a web server received between time 0 and time t, $N(t)$, is given by a Poisson random process with rate $\lambda = 6$ requests per minute. Assume that time t is measured in minutes.
   (a) Give the value of $E[N(2.5)]$. (1 pt)
   (b) Give the value of $\text{Var}[N(2.5)]$. (1 pt)
   (c) Give the value of $C_N(2.5, 1.5) = \text{Cov}(N(2.5), N(1.5))$. (1 pt)
   (d) Find the probability that exactly 10 requests are received in the first minute. (1 pt)
   (e) Find $P[N(1) = 10 \cap N(2) = 12]$. (2 pts)
   (f) Find the probability that the first request occurs before $t = 2$ seconds. Note that more than one request can occur before 2 seconds. (2 pts)
   (g) Find the probability that the first request occurs between $t = 1$ and $t = 2$ seconds. (2 pts)