The Hong Kong University of Science and Technology

Department of Electronic and Computer Engineering

ELEC2600 Fall 2019 Online Exam Group 1

Dec 17, 2019, 08:30 – 10:30

Please submit the soft copy of your exam solutions to Canvas

Instructions:

- This is a **2-hour test**.
- Please show all of your work, as marks will be given for the key steps, not just for the correct answer.
- If your results involve binomial coefficients, factorials, exponential functions, or ratios, you do not need to compute the value. For example, an answer like $\frac{2\times 3^4}{4!}e^{-4}$ doesn't need to be simplified further or evaluated explicitly.

1. (19 pts) X and Y are discrete random variables with joint PMF (Probability Mass Function) shown below:

$p_{X,Y}(j,k)$		j		
		0	1	2
k	0	$\frac{1}{6}$	0	$\frac{1}{6}$
	1	0	c	0
	2	$\frac{1}{6}$	0	$\frac{1}{6}$

(a) Find the value of c. (2 pts)
(b) Find the value of the joint CDF (Cumulative Density Function) for X and Y at the point (1, 2), $F_{X,Y}(1,2)$. (1 pt)
(c) Find the marginal PMFs of X and Y, respectively. (4 pts)
(d) Are X and Y independent? Justify your answer. (2 pts)
(e) Are X and Y correlated? Justify your answer. (5 pts)
(f) Find the correlation coefficient between X and Y. (1 pt)
(g) Find the conditional pmf of Y given $X < 1$. (3 pts)
(h) Find the conditional expected value of Y given $X < 1$. (1 pt)

Solution

(a) $\sum_{j=0}^{2}\sum_{k=0}^{2} p_{X,Y}(j,k) = \frac{1}{6} + 0 + \frac{1}{6} + 0 + c + 0 + \frac{1}{6} + 0 + \frac{1}{6} = 1$ (1 pt)

$c = \frac{1}{3}$ (1 pt)

(b)

$$F_{X,Y}(1,2) = p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(0,2) + p_{X,Y}(1,0) + p_{X,Y}(1,1) + p_{X,Y}(1,2)$$

$$= \frac{1}{6} + 0 + \frac{1}{6} + 0 + \frac{1}{3} + 0$$
$$= \frac{2}{3} \quad \text{(1 pt)}$$

(c)

$$p_X\,(0) = p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(0,2)$$
$$= \frac{1}{6} + 0 + \frac{1}{6}$$
$$= \frac{1}{3}$$
$$p_X\,(1) = p_{X,Y}(1,0) + p_{X,Y}(1,1) + p_{X,Y}(1,2)$$
$$= 0 + \frac{1}{3} + 0$$
$$= \frac{1}{3}$$
$$p_X\,(2) = p_{X,Y}(2,0) + p_{X,Y}(2,1) + p_{X,Y}(2,2)$$
$$= \frac{1}{6} + 0 + \frac{1}{6}$$
$$= \frac{1}{3}$$
$$p_X\,(x) = \begin{cases} \frac{1}{3}, & x = 0,1,2 \\ 0, & therwise \end{cases} \quad \text{(2 pts)}$$
$$p_Y\,(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0) + p_{X,Y}(2,0)$$
$$= \frac{1}{6} + 0 + \frac{1}{6}$$
$$= \frac{1}{3}$$
$$p_Y\,(1) = p_{X,Y}(0,1) + p_{X,Y}(1,1) + p_{X,Y}(2,1)$$
$$= 0 + \frac{1}{3} + 0$$
$$= \frac{1}{3}$$
$$p_Y\,(2) = p_{X,Y}(0,2) + p_{X,Y}(1,2) + p_{X,Y}(2,2)$$
$$= \frac{1}{6} + 0 + \frac{1}{6}$$
$$= \frac{1}{3}$$
$$p_Y\,(y) = \begin{cases} \frac{1}{3}, & y = 0,1,2 \\ 0, & therwise \end{cases} \quad \text{(2 pts)}$$

(d) No. (1 pt)

$$p_X\,(0) p_Y\,(0) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \neq p_{X,Y}(0,0) = \frac{1}{6} \quad \text{(1 pt)}$$

(e) No. (1 pt)

$$
\begin{aligned}
E[XY] &= \sum_{j=0}^{2}\sum_{k=0}^{2} jk p_{X,Y}(j,k) \\
&= 0\times 0\times \frac{1}{6} + 0\times 1\times 0 + 0\times 2\times \frac{1}{6} + 1\times 0\times 0 + 1\times 1\times \frac{1}{3} + 1\times 2\times 0 + 2\times 0 \\
&\qquad \times \frac{1}{6} + 2\times 1\times 0 + 2\times 2\times \frac{1}{6} \\
&= 1 \quad \text{(1 pt)}
\end{aligned}
$$

$$E[X] = \sum\nolimits_{x=0}^{2} x\, p_X(x) = 0\times\frac{1}{3} + 1\times\frac{1}{3} + 2\times\frac{1}{3} = 1 \quad \text{(1 pt)}$$

$$E[Y] = \sum\nolimits_{y=0}^{2} y\, p_Y(y) = 0\times\frac{1}{3} + 1\times\frac{1}{3} + 2\times\frac{1}{3} = 1 \quad \text{(1 pt)}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 1 - 1\times 1 = 0 \quad \text{(1 pt)}$$

(f) X and Y are uncorrelated, $\rho_{X,Y} = 0$. (1 pt)

(g) $P[X < 1] = P[X = 0] = \frac{1}{3}$ (1 pt)

$$
\begin{aligned}
p_{Y|X<1}(y|X<1) &= \frac{P[\{Y=y\}\cap\{X<1\}]}{P[X<1]} \\
&= \begin{cases} \dfrac{\frac{1}{6}}{\frac{1}{3}} = \dfrac{1}{2}, & y = 0 \\ \dfrac{0}{\frac{1}{3}} = 0, & y = 1 \\ \dfrac{\frac{1}{6}}{\frac{1}{3}} = \dfrac{1}{2}, & y = 2 \\ 0, & otherwise \end{cases} \\
&= \begin{cases} \dfrac{1}{2}, & y = 0,2 \\ 0, & otherwise \end{cases} \quad \text{(2 pts)}
\end{aligned}
$$

(h) $E[Y|X<1] = \sum_{y=0}^{2} y p_{Y|X<1}(y|X<1) = 0\times\frac{1}{2} + 2\times\frac{1}{2} = 1$ (1 pt)

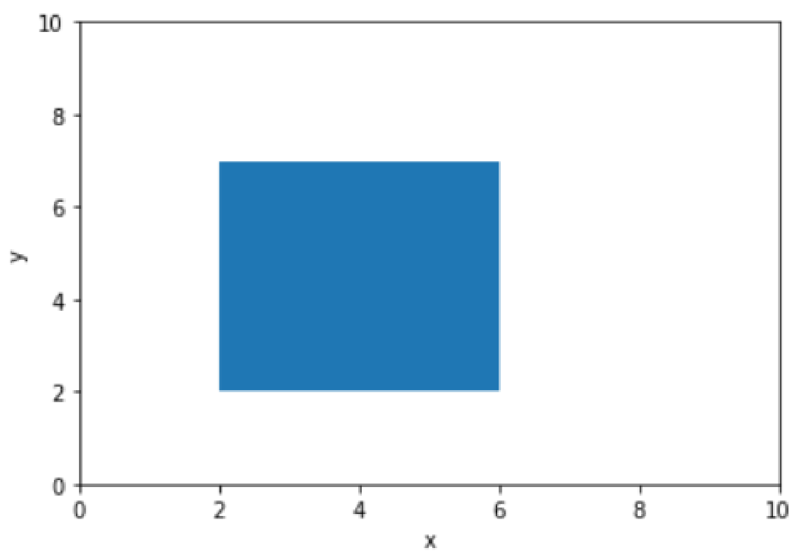
2. (32 pts) Suppose the lifetimes of a new iPhone and a new iPad, X and Y, are continuous and uniformly distributed in [2, 6] and [2, 7], respectively. Assume that the lifetimes of a new iPhone and a new iPad are independent.
   (a) Plot the region where the joint PDF (Probability Density Function) of X and Y are non-zero on the X, Y plane. (2 pts)
   (b) Find $P[X \le 4 \cap 5 < Y < 6]$. (4 pts)
   (c) Find the probability that the sum of the lifetimes of a new iPhone and a new iPad is greater than 8. (4 pts)
   (d) Let Z be the sum of the lifetimes of a new iPhone and a new iPad. Find the CDF and PDF of Z. (22 pts)

Solution

(a) $f_X(x) = \begin{cases} \frac{1}{4}, & 2 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$, $f_Y(y) = \begin{cases} \frac{1}{5}, & 2 \le y \le 7 \\ 0, & \text{otherwise} \end{cases}$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
$$= \begin{cases} \frac{1}{20}, & 2 \le x \le 6, \quad 2 \le y \le 7 \\ 0, & otherwise \end{cases}$$

The region where the joint PDF (Probability Density Function) of X and Y are non-zero is the shaded region below. (2 pt)



(b)

$$P[X \le 4 \cap 5 < Y < 6] = \int_2^4 \int_5^6 \frac{1}{20} dy dx \quad \text{(3 pts)}$$
$$= \frac{1}{10} \quad \text{(1 pt)}$$

(c)

$$P[X + Y > 8] = \int_2^6 \int_{8-x}^7 \frac{1}{20} dy dx \quad \text{(3 pts)}$$

$$= \frac{3}{5} \quad \text{(1 pt)}$$

(d) $F_Z(z) = P[X + Y \leq z] = P[Y \leq z - X]$

When $z < 4$, $F_Z(z) = 0$. (1 pt)

When $4 \leq z < 8$

$$F_Z(z) = \int_2^{z-2} \int_2^{z-x} \frac{1}{20}\, dy dx \quad \text{(3 pts)}$$

$$= \frac{1}{40} z^2 - \frac{1}{5} z + \frac{2}{5} \quad \text{(1 pt)}$$

When $8 \leq z < 9$

$$F_Z(z) = \int_2^{6} \int_2^{z-x} \frac{1}{20}\, dy dx \quad \text{(3 pts)}$$

$$= \frac{1}{5} z - \frac{6}{5} \quad \text{(1 pt)}$$

When $9 \leq z < 13$

$$F_Z(z) = \int_2^{z-7} \int_2^{7} \frac{1}{20}\, dy dx + \int_{z-7}^{6} \int_2^{z-x} \frac{1}{20}\, dy dx \quad \text{(3 pts)}$$

$$= -\frac{1}{40} z^2 + \frac{13}{20} z - \frac{129}{40} \quad \text{(1 pt)}$$

Alternatively,

$$F_Z(z) = 1 - \int_{z-7}^{6} \int_{z-x}^{7} \frac{1}{20}\, dy dx \text{ (3 pts)}$$

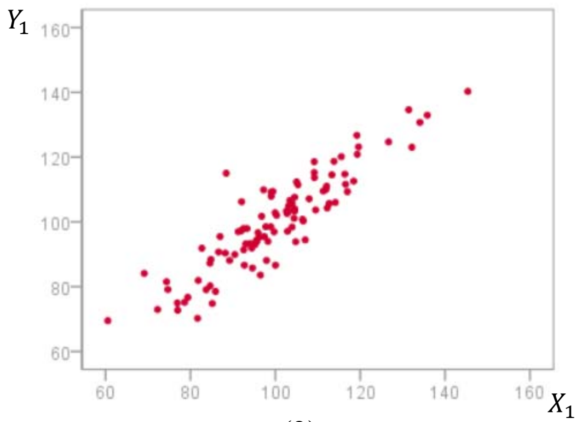
$$= -\frac{1}{40} z^2 + \frac{13}{20} z - \frac{129}{40} \quad \text{(1 pt)}$$

When $z \geq 13$, $F_Z(z) = 1$. (1 pt)

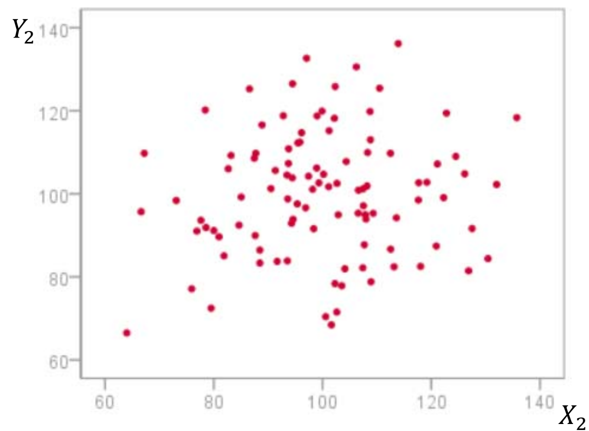
$$F_Z(z) = \begin{cases} 0, & z < 4 \\ \frac{1}{40} z^2 - \frac{1}{5} z + \frac{2}{5}, & 4 \leq z < 8 \\ \frac{1}{5} z - \frac{6}{5}, & 8 \leq z < 9 \\ -\frac{1}{40} z^2 + \frac{13}{20} z - \frac{129}{40}, & 9 \leq z < 13 \\ 1, & z \geq 13 \end{cases}$$

$$f_Z(z) = \frac{dF_Z(z)}{dx} = \begin{cases} \frac{1}{20} z - \frac{1}{5}, & 4 \leq z < 8 \\ \frac{1}{5}, & 8 \leq z < 9 \\ -\frac{1}{20} z + \frac{13}{20}, & 9 \leq z < 13 \\ 0, & otherwise \end{cases} \quad \text{(8 pts)}$$

3. (9 pts) The scatter plots of two pairs of random variables (X_k, Y_k), for $k = 1, 2$ are shown below.



(2) ρ_1



(1) ρ_2

Let ρ_k be the correlation coefficient between X_k and Y_k for $k = 1, 2$.

(a) Is ρ_1 positive? Justify your answer. (3 pts)
(b) Is $\rho_1 < \rho_2$? Justify your answer. (3 pts)
(c) Which value, $|\rho_1|$ or $|\rho_2|$ is closer to 1? Justify your answer. (3 pts)

Solution

(a) Yes. (1 pt)
The scatter plot of (X_1, Y_1) shows that if X_1 is greater than its mean, Y_1 is also greater than its mean. Hence, the covariance between X_1 and Y_1 is positive and ρ_1 is positive. (2 pts)

(b) No. (1 pt)
The scatter plot of (X_2, Y_2) is more spread out than the scatter plot of (X_1, Y_1). Hence, the covariance of X_1 and Y_1 is higher than the covariance of X_2 and Y_2, that is $\rho_1 > \rho_2$. (2 pts)

(c) Compare to $|\rho_2|$, $|\rho_1|$ is closer to 1. (1 pt)
The points in the scatter plot of (X_1, Y_1) are closer to each other in a linear shape; while the points in the scatter plot of (X_2, Y_2) are far away from each other in a round shape. Hence, $|\rho_1|$ is closer to 1. (2 pts)

4. (15 pts) Suppose an Urn contains enough coupons. Assume that the value of each coupon is uniformly distributed in $\{10, 20, 30, 40, 50\}$. Further assume that the value of each coupon is independent of each other.
   (a) Suppose you pick two coupons at random out of the urn. Find the PMF of the sum of the values of the two coupons. (5 pts)
   (b) Let X and Y denote the values of the two coupons, respectively. Let Z denote the sum of the values for the two coupons. Determine the PMF of Z <u>analytically</u> based on the PMFs of X and Y. (3 pts)
   (c) Suppose you pick two coupons at random out of the urn and check the sum of the values of the two coupons. Let N be the number of coupons you picked such that the sum of the values of the two coupons is 80 or more. Find the expected value of N. (3 pts)
   (d) Now, you pick one coupon at random, and check whether the value of the coupon is greater than or equal to 40. If it is, keep the coupon, and pick again to find another coupon with value of 40 or more. Let M be the number of coupons you picked such that the sum of the values of the two coupons is 80 or more. Find the expected value of $\boldsymbol{M}$. (2 pts)
   (e) Compare the expected values computed in (c) and (d). Which picking method is better? Justify your answer. (2 pts)

Solution

(a) Let Z be the sum of the values of the two coupons, Let X and Y be the values of the two picked coupons, respectively

$$p_Z(20) = P[X = 10 \cap Y = 10] = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$$

$$\begin{aligned} p_Z(30) &= P[X = 10 \cap Y = 20] + P[X = 20 \cap Y = 10] \\ &= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\ &= \frac{2}{25} \end{aligned}$$

$$\begin{aligned} p_Z(40) &= P[X = 10 \cap Y = 30] + P[X = 30 \cap Y = 10] + P[X = 20 \cap Y = 20] \\ &= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\ &= \frac{3}{25} \end{aligned}$$

$$\begin{aligned} p_Z(50) &= P[X = 10 \cap Y = 40] + P[X = 40 \cap Y = 10] + P[X = 20 \cap Y = 30] \\ &\quad + P[X = 30 \cap Y = 20] \\ &= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\ &= \frac{4}{25} \end{aligned}$$

$$\begin{aligned} p_Z(60) &= P[X = 10 \cap Y = 50] + P[X = 50 \cap Y = 10] + P[X = 20 \cap Y = 40] \\ &\quad + P[X = 40 \cap Y = 20] + P[X = 30 \cap Y = 30] \\ &= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\ &= \frac{1}{5} \end{aligned}$$

$$
\begin{aligned}
p_Z(70) &= P[X = 20 \cap Y = 50] + P[X = 50 \cap Y = 20] + P[X = 30 \cap Y = 40] \\
&\quad + P[X = 40 \cap Y = 30] \\
&= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\
&= \frac{4}{25}
\end{aligned}
$$

$$
\begin{aligned}
p_Z(80) &= P[X = 30 \cap Y = 50] + P[X = 50 \cap Y = 30] + P[X = 40 \cap Y = 40] \\
&= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\
&= \frac{3}{25}
\end{aligned}
$$

$$
\begin{aligned}
p_Z(90) &= P[X = 40 \cap Y = 50] + P[X = 50 \cap Y = 40] \\
&= \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \\
&= \frac{2}{25}
\end{aligned}
$$

$$
p_Z(100) = P[X = 50 \cap Y = 50] = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}
$$

$$
p_Z(z) = \begin{cases} \frac{1}{25}, & z = 20, 100 \\ \frac{2}{25}, & z = 30, 90 \\ \frac{3}{25}, & z = 40, 80 \\ \frac{4}{25}, & z = 50, 70 \\ \frac{1}{5}, & z = 60 \\ 0, & otherwise \end{cases} \quad \text{(5 pts)}
$$

(b) Suppose $X = x$, then $Z = z$ if only if $Y = z - x$. The event $Z = z$ is the union of the pairwise disjoint events $X = x$ and $Y = z - x$. (1 pt)
$p_Z(z) = P[Z = z] = \sum_{x=10}^{50} p_X(x) p_Y(z - x)$, for $z = 20, 30, 40, \dots, 100$ (2 pts)

(c) $P[Z \geq 80] = \sum_{z=80}^{100} p_Z(z) = \frac{3}{25} + \frac{2}{25} + \frac{1}{25} = \frac{6}{25}$ (2 pts)
$N/2$ is a geometric random variable.
$E[N] = 2 \times \frac{1}{\frac{6}{25}} = \frac{25}{3}$ (1 pt)

(d) Let M_1 be the number of coupons picked until you pick the first coupon with value greater than or equals to 40. Let M_2 be the number of coupons until you pick the second coupon with value greater than or equals to 40.
M_1 and M_2 are geometric random variables with parameter $p_1 = \frac{2}{5}$ and $p_2 = \frac{2}{5}$. (1 pt)

$$
\begin{aligned}
E[M] &= E[M_1 + M_2] \\
&= E[M_1] + E[M_2] \\
&= \frac{5}{2} + \frac{5}{2} \\
&= 5 \quad \text{(1 pt)}
\end{aligned}
$$

(e) The method in (d) is better. (1 pt)
The expected value of the number of coupons to pick in (d) is less than the expected value of the number of coupons to pick in (c). (1 pt)

5. (11 pts) Consider a communications system where a constant level signal s is transmitted to the receiver. The received signal is given by $Y(t) = s + N(t)$, where $N(t)$ denotes the additive noise. We sample the received signal at n time instances and obtain n samples, $Y(i) = s + N(i)$, for $i = 1, \dots, n$. Assume the sampled noise $N(i)$, $i = 1, \dots, n$, are independent and identically distributed with mean value 0 and variance 1. We use the sample mean $M_n = \frac{1}{n}\sum_{i=1}^{n} Y(i)$ to estimate s.
   (a) Use the Chebyshev inequality to determine how many samples are required such that the sample mean is within 0.1 of the value of s with probability greater than or equal to 99%. (4 pts)
   (b) Use the central limit theorem to approximate the number of samples are required such that the sample mean is within 0.1 of the value of s with probability greater than or equal to 99%. (4 pts)
   (c) Explain the difference in the estimated sample sizes in parts (a) and (b). (1 pt)
   (d) Show how $|M_n - s|$ will change as n increases. (2 pts)

Solution

(a) $E[M_n] = s, Var[M_n] = \frac{\sigma^2}{n} = \frac{1}{n}$ (1 pt)
From the Chebyshev inequality:

$$\begin{aligned} P[|M_n - s| \le \varepsilon] &\ge 1 - \frac{\sigma^2}{n \times \varepsilon^2} \text{ (1 pt)} \\ &= 1 - \frac{1}{n \times 0.1^2} \\ &= 0.99 \text{(1 pt)} \\ n &= 10000 \text{ (1 pt)} \end{aligned}$$

(b) For large enough sample size n, distribution of the sample mean M_n approaches a Gaussian distribution with mean s and variance $\frac{1}{n}$. Thus

$$\begin{aligned} P[|M_n - s| \le \varepsilon] &= 1 - 2\text{Q}\left(\frac{\varepsilon}{\sqrt{\frac{\sigma^2}{n}}}\right) \text{ (1 pt)} \\ &= 1 - 2\text{Q}\left(0.1\sqrt{n}\right) \\ &\ge 0.99 \text{ (1 pt)} \\ \text{Q}\left(0.1\sqrt{n}\right) &\le 0.005 \text{ (1 pt)} \\ n &\ge 666 \text{ (1 pt)} \end{aligned}$$

(c) The result in (a) guarantees that we estimate is within 0.1 of sample mean with at least 99% probability, but results in a large number of measurements. Due to the approximation, the result in (b) is not an absolute guarantee, but results in a small number of measurements. (1 pt)

(d) $|M_n - s| = \left|\frac{1}{n}\sum_{i=1}^{n} Y(i) - s\right| = \left|\frac{1}{n}\sum_{i=1}^{n}(N(i) + s) - s\right| = \left|\frac{1}{n}\sum_{i=1}^{n} N(i)\right|$

$$E[(M_n - s)^2] = \frac{1}{n^2}\sum_{i=1}^{n} E\left[\left(N(i)\right)^2\right] = \frac{1}{n}Var[N(i)] = \frac{1}{n}$$

As we increase n, the estimation should get "better". That is $|M_n - s|$ decreases. (2 pts)

6. Part I: (9 pts) Suppose that S_n is a binomial counting process with parameter $p = 0.1$
   (a) Find $E[S_2]$, $Var[S_5]$and $C_s(2,5) = Cov(S_2, S_5)$. (3 pts)
   (b) Find $P[S_2 \leq 2]$. (2 pts)
   (c) Find $P[S_5 = 5|S_2 = 2]$. (2 pts)
   (d) Find $P[S_5 = 5 \cap S_2 = 2]$. (2 pts)

   Part II: (5 pts) We flip a fair coin. A random process $X(t)$ is defined as: $X(t) = t + 1$ if the head faces up, and $X(t) = 2t - 3$ if the tail faces up.
   (a) Find $E[X(t)]$. (1 pt)
   (b) Find $F_{X(t)}(x)$ for $t = 0$. (4 pts)

Solution Part I

(a) $E[S_2] = 2p = 0.2$ (1 pt)
Define $\sigma^2 = p(1-p) = 0.09$ to be the variance of one step/Bernoulli trial
$Var[S_5] = 5\sigma^2 = 0.45$ (1 pt)
$C_s(2,5) = \sigma^2 \min(2,5) = 2\sigma^2 = 0.18$ (1 pt)

(b) $P[S_2 \leq 2] = \sum_{k=0}^{2} \binom{2}{k} 0.1^k (1-0.1)^{2-k} = 1$ (2 pts)

(c) Solution 1:

$$P[S_5 = 5|S_2 = 2] = \frac{P[S_5 = 5 \cap S_2 = 2]}{P[S_2 = 2]}$$
$$= \frac{\binom{3}{3} 0.1^3 (1-0.1)^{3-3} \binom{2}{2} 0.1^2 (1-0.1)^{2-2}}{\binom{2}{2} 0.1^2 (1-0.1)^{2-2}} \quad \text{(2 pts)}$$
$$= 0.1^3$$

Solution 2: Since the binomial counting process is an independent and stationary increment process,

$$P[S_5 = 5|S_2 = 2] = P[S_{5-2} = 5-2] = P[S_3 = 3] = 0.1^3$$

(d)

$$P[S_5 = 5 \cap S_2 = 2] = P[S_5 = 5|S_2 = 2]P[S_2 = 2]$$
$$= P[S_3 = 3]P[S_2 = 2]$$
$$= \binom{3}{3} 0.1^3 (1-0.1)^{3-3} \binom{2}{2} 0.1^2 (1-0.1)^{2-2} \quad \text{(2 pts)}$$
$$= 0.1^5$$

Solution Part II

(a) $E[X(t)] = 0.5 \times (t+1) + 0.5 \times (2t-3) = 1.5t - 1$ (1 pt)

(b) $X(0) = \begin{cases} 1, & \text{if head faces up} \\ -3, & \text{if tail faces up} \end{cases}$ (1 pt)

$$F_{X(0)}(x) = P[X(0) \leq x]$$
$$= \begin{cases} 0, & x < -3 \\ 0.5, & -3 \leq x < 1 \\ 1, & x \leq 1 \end{cases} \quad \text{(3 pts)}$$

Q-function Table

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-9}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						