Supplementary Exercises on Recurrences

Note: These exercises are meant to help you revise the material that you have learnt in class when preparing for the exam. Please note that when solutions are given below for the problems, they are at most sketch solutions and do not provide derivations of the answers. For assignments and exams you are expected to provide full derivations. See the posted solutions to assignments for examples of this.

1. Suppose that the Towers of Hanoi problem has four poles in a row instead of three. A disk can be transferred from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Consider the algorithm that works by (i) first moving the top $n-2$ disks to another pole (not the destination one), (ii) moving the two largest disks to the destination pole and then (iii) movnog the remaining $n-2$ disks to the destination pole. Let $T(n)$ be the number of moves needed by this algorithm to transfer the entire tower of n disks from the leftmost pole to the rightmost pole.
   (a) Find $T(1)$, $T(2)$ and $T(3)$.
   (b) Find $T(4)$.
   (c) Find a recurrence relation expressing $T(n)$ in terms of $T(n-2)$, for all integers $n \geq 3$.
2. In a Double Towers of Hanoi problem there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one-by-one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let $T(n)$ be the number of moves needed to transfer a tower of $2n$ disks from one pole to another.
   (a) Find $T(1)$ and $T(2)$.
   (b) Find $T(3)$.
   (c) Find a recurrence relation expressing $T(n)$ in terms of $T(n-1)$, for all integers $n \geq 2$.
3. A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions. Let $T(n)$ be the maximum number of regions into which n lines divide a plane, where n is a positive integer. Derive a recurrence relation for $T(n)$ in terms of $T(n-1)$, for all integers $n \geq 2$.

4. A person saving for retirement makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 6% per year compounded monthly, and each month she adds an additional \$100 to the account.

   (a) For each nonnegative integer n, let $A(n)$ be the amount in the account at the end of n months. Find a recurrence relation relating $A(n)$ to $A(n-1)$.

   (b) Find a closed-form solution for the recurrence.

   (c) Use mathematical induction to prove the correctness of the solution for $A(n)$.

   (d) How much will the account be worth at the end of 20 years? At the end of 40 years?

5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

6. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 5 & n = 1 \\ 2T(n-1) + 3n + 1 & n > 1 \end{cases}$$

7. Find a solution in summation form for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ nT(n-1) + n & n > 1 \end{cases}$$

8. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 0 \\ 2^n - T(n-1) & n > 0 \end{cases}$$

9. Find a closed-form solution for the following recurrence where you are allowed to consider $H_n = \sum_{i=1}^{n} \frac{1}{i}$ as a known function

$$T(n) = \begin{cases} 1 & n = 2 \\ (n-1)T(n-1) + (n-2)! & n > 2 \end{cases}$$

10. A chain letter works as follows:
One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, each of whom sends a copy to five friends, and so forth. Let $T(n)$ denote the number of persons who will have received copies of the letter after the nth repetition of this process, assuming that no person receives more than one copy. Let $T(0) = 1$.

(a) Describe $T(n)$ as a recurrence relation.

(b) Find a closed-form solution for the recurrence.

(c) How many people will have received copies of the letter after the 10th repetition of this process?

11. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 3 & n = 1 \\ 6T(n/6) + 3n - 1 & n > 1 \end{cases}$$

where n is a power of 6.

12. Consider the following recurrence where n is a power of 3:

$$T(n) = \begin{cases} 2 & n = 1 \\ 4T(n/3) + 3n - 5 & n > 1 \end{cases}$$

(a) Find a closed-form solution for the recurrence.

(b) Find a real number r such that $T(n) = \Theta(n^r)$.

(c) If $T(1)$ is changed to 4, how does the value of r in (b) change?

13. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 4 & n = 1 \\ 3T(n/2) + n^2 - 2n + 1 & n > 1 \end{cases}$$

where n is a power of 2.

14. Consider the following recurrence inequality:

$$T(n) \leq 3T(n-1) + 10, \quad T(0) = 0.$$

Find a constant c such that $T(n) \leq c^n$.

15. Consider the following recurrence inequality:

$$T(n) \leq 3T\left(\frac{n}{4}\right) + cn.$$

Find a constant k such that $T(n) \leq kn$. You may assume that n is a power of 4.

16. Consider the following recurrence inequality:

$$T(n) \leq 3T\left(\frac{n}{7}\right) + cn.$$

Find a constant k such that $T(n) \leq kn$. You may assume that n is a power of 7.

17. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

    - Rabbit pairs are not fertile during their first two months of life, but thereafter give birth to one new male/female pair at the end of every month.
    - No deaths occur during the year.

    (a) Let $T(n)$ denote the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $T(0) = 1$. Find a recurrence relation relating $T(n)$ to $T(n-3)$ and $T(n-1)$.

    (b) Compute $T(n)$ for $n = 0, \ldots, 6$.

    (c) How many rabbits will there be at the end of the year?

18. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

    - Rabbit pairs are not fertile during their first two months of life, but thereafter give birth to two new male/female pairs at the end of every month.
    - No deaths occur during the year.

    (a) Let $T(n)$ denote the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $T(0) = 1$. Find a recurrence relation relating $T(n)$ to $T(n-3)$ and $T(n-1)$.

    (b) Compute $T(n)$ for $n = 0, \ldots, 6$.

    (c) How many rabbits will there be at the end of the year?

Solutions/Hints

1. (a) $T(1) = 1$
   $T(2) = 1 + 1 + 1 = 3$
   $T(3) = T(1) + (1 + 1 + 1) + T(1) = 5$

   (b) $T(4) = T(2) + (1 + 1 + 1) + T(2) = 9$

   (c) $T(n) = T(n - 2) + (1 + 1 + 1) + T(n - 2) = 2T(n - 2) + 3$

2. (a) $T(1) = 2$
   $T(2) = 6$

   (b) $T(3) = 14$

   (c) $T(n) = 2T(n - 1) + 2$

3. $T(n) = T(n - 1) + n$

4. (a) $A(0) = 1000$
   $A(n) = 1.005 \cdot A(n - 1) + 100$ for $n > 0$

   (b) $A(n) = 1000 \cdot 1.005^n + 100 \sum_{i=0}^{n-1} 1.005^i = 21000 \cdot 1.005^n - 20000$

   (c) Apply the weak principle of mathematical induction

   (d) $A(240) = 49{,}514.29$
   $A(480) = 210{,}106.53$

5. $T(n) = 2 \cdot 3^{n-1} - 1$

6. $T(n) = 5 \cdot 2^{n-1} + \sum_{i=2}^{n} 2^{n-i}(3i + 1) = (3n + 6)2^{n-1} - 3n - 1 - 3\sum_{i=1}^{n-2} i \cdot 2^i = 15 \cdot 2^{n-1} - 3n - 7$

7. $T(n) = \sum_{i=0}^{n-1} n!/i! = \sum_{i=0}^{n-1} n^{\underline{i}}$

8. $T(n) = (1/3)[2^{n+1} - (-1)^{n+1}]$

9. $T(n) = (n - 1)! \sum_{i=1}^{n-1} \frac{1}{i} = (n - 1)!\, H_{n-1}$

10. (a) $T(0) = 1$ and $T(n) = T(n - 1) + 5^n$ for $n > 0$

    (b) $T(n) = \sum_{i=0}^{n} 5^i = (5^{n+1} - 1)/4$

    (c) 12,207,031

11. $T(n) = 3n(1 + \log_6 n) - \sum_{i=0}^{\log_6 n-1} 6^i = 3n \log_6 n + (14/5)n + 1/5$

12. (a) $T(n) = 2 \cdot 4^{\log_3 n} + \sum_{i=0}^{\log_3 n-1} 4^i(3n/3^i - 5) = (28/3) \cdot n^{\log_3 4} - 9n + 5/3$

    (b) $r = \log_3 4$

    (c) r remains the same

13. $T(n) = 4 \cdot 3^{\log_2 n} + \sum_{i=0}^{\log_2 n-1} 3^i(n^2/4^i - 2n/2^i + 1) = 4n^2 - (7/2) \cdot n^{\log_2 3} + 4n - 1/2$

14. any $c \geq 13$

15. any $k \geq \max(T(1), 4c)$

16. any $k \geq \max(T(1), 7c/4)$

17. (a) $T(n) = T(n-1) + T(n-3)$
    (b) $T(0) = 1$, $T(1) = 1$, $T(2) = 1$, $T(3) = 2$, $T(4) = 3$, $T(5) = 4$, $T(6) = 6$
    (c) 60 pairs or 120 rabbits

18. (a) $T(n) = T(n-1) + 2T(n-3)$
    (b) $T(0) = 1$, $T(1) = 1$, $T(2) = 1$, $T(3) = 3$, $T(4) = 5$, $T(5) = 7$, $T(6) = 13$
    (c) 309 pairs or 618 rabbits