Tutorial 10

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Equations

Expectation. $E(X) = \sum_{x \in S} p(s)X(s)$ and for discrete probability,

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

Linearity of Expectations.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$E(aX + b) = aE(X) + b$$

Independence. The random variables X and Y on a sample space S are independent if

$$p(X = r_1 and Y = r_2) = p(X = r_1) \cdot (p(Y = r_2)),$$

and so

$$E(XY) = E(X)E(Y)$$

Variance.

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s) = E\left((X - E(X))^2\right) = E(X^2) - E(X)^2$$

Corollary

$$V(aX + b) = a^2 V(X)$$

Independence. For n independent variables $X_1, X_2, \cdots, X_n$,

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$$

EP4-10. Each pixel in a 32×8 vertical display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows.

(a) What is the expected value of X?

(b) What is the variance of X?

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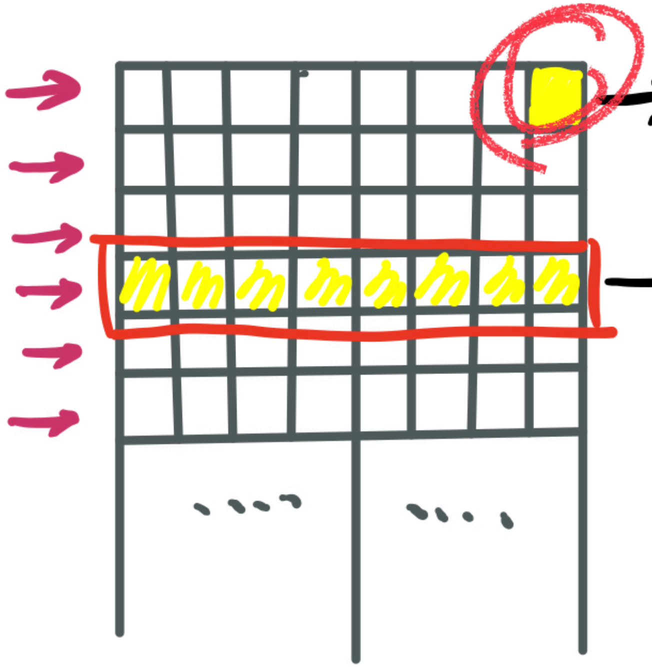

$\frac{1}{2}$ probability.

horizontal line

$X = \sum$ (horizontal lines)

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Let's define a random variable for each row.

$$Y_j = \begin{cases} 1, & \text{horizontal line is shown in } j \text{ th row.} \\ 0, & \text{no horizontal line } \cdots\cdots \end{cases}$$

where $j = [1, 32]$

Our goal $= E(\sum Y_j)$

$= \sum E(Y_j)$

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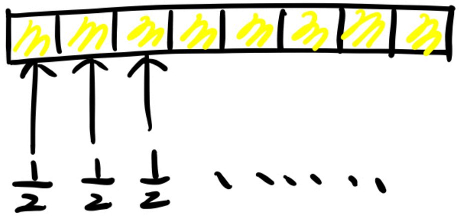
What about $E(Y_j)$



$\Leftarrow$ when $Y_j = 1$

otherwise $Y_j = 0$

$$E(Y_j) = 1 \times (\underbrace{\tfrac{1}{2} \times \tfrac{1}{2} \times \cdots \times \tfrac{1}{2}}_{8}) + 0 \times (1 - \tfrac{1}{2} \times \cdots)$$

$$= \prod_{i=1}^{8} \left(\tfrac{1}{2}\right) = \frac{1}{2^8}$$

$$\therefore \sum E(Y_j) = 32 \times \frac{1}{2^8} = 2^5 \times \frac{1}{2^8} = \frac{1}{8}$$

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$$V(X) = \underline{E(X^2)} - \underline{E(X)}^2$$

?

$$\left(\frac{1}{2^3}\right)^2 = \frac{1}{2^6}$$

$$E(X^2) = E\left(\left(\sum Y_j\right)^2\right)$$

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$$E(X^2) = E((\sum Y_j)^2)$$

$$E\left(\begin{array}{l} (Y_1 + Y_2 + Y_3 + \cdots + Y_{32}) \\ \times (Y_1 + Y_2 + Y_3 + \cdots + Y_{32}) \end{array} \right)$$

$$= E\left(\begin{array}{l} Y_1 Y_1 + Y_1 Y_2 + \cdots + Y_1 Y_{32} \\ + Y_2 Y_1 + Y_2 Y_2 + \cdots + Y_2 Y_{32} \\ + \cdots \\ + Y_{32} Y_1 + Y_{32} Y_2 + \cdots + Y_{32} Y_{32} \end{array} \right)$$

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$$= E\left(\begin{array}{l} Y_1 Y_1 + Y_1 Y_2 + \cdots + Y_1 Y_{32} \\ + Y_2 Y_1 + Y_2 Y_2 + \cdots + Y_2 Y_{32} \\ + \cdots \\ + Y_{32} Y_1 + Y_{32} Y_2 + \cdots + Y_{32} Y_{32} \end{array} \right)$$

$$= E\left(\sum (Y_j)^2\right) + 31 \times \overset{2^5}{32} \times \frac{1}{2^8} \times \frac{1}{2^8}$$

$$\frac{31}{2^{11}}$$

$$\sum E(Y_j)^2$$

$$E[(Y_j)^2] = 0 \times \ldots + 1 \times P(\underbrace{Y_j}_{0,\ 1} = 1) = \frac{1}{2^8}$$

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(a) What is the expected value of X?

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$$\therefore V(x) = E(x^2) - E(x)^2$$

$$= \frac{32}{2^8} + \frac{31}{2^{11}} - \frac{1}{2^6}$$

$$= \frac{2^5 \times 2^3 + 31 - 2^5}{2^{11}}$$

$$= \frac{7 \times 32 + 31}{2024} = \frac{255}{2024}$$

EP4-11. A biased coin is tossed n times, and a head shows up with probability p on each toss. A run is a maximal sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$.

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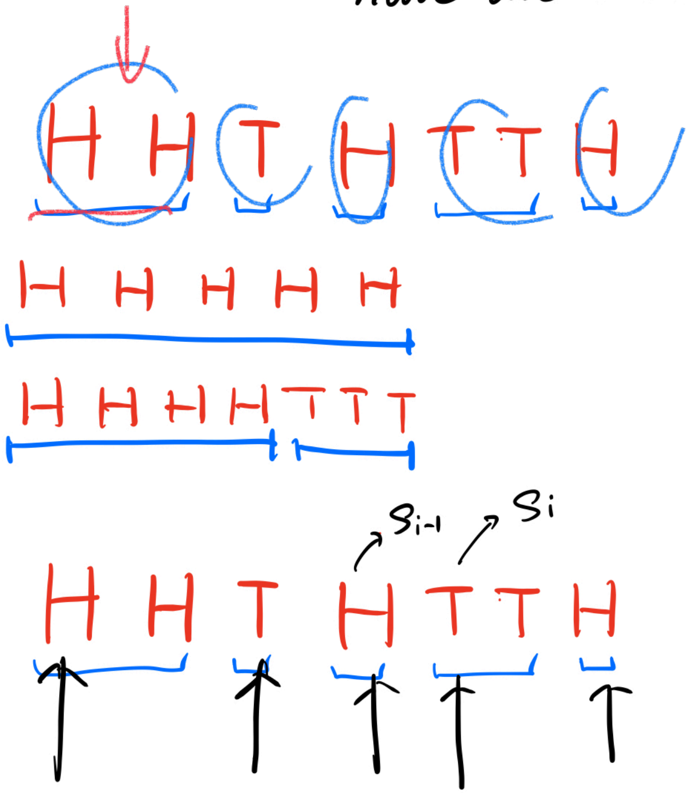
A "run": maximal sequence
have the same outcome



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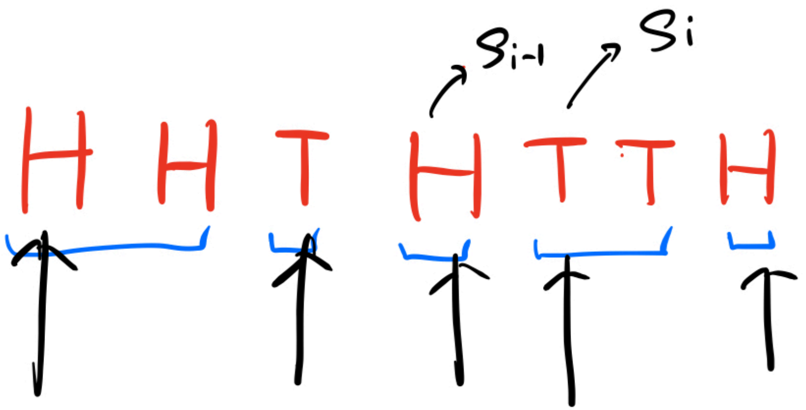

considering the start of the run:

$$X_i = \begin{cases} 1, & S_i \neq S_{i-1} \\ 0, & \text{otherwise.} \end{cases}, \quad i \geq 2.$$

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S_{i-1} S_i

H H T H T T H

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The expected number of runs

$$X = \sum X_i$$

$$\begin{aligned} &P(X_i = 1) \\ &= P(S_i \neq S_{i-1}) \\ &= P(S_i = H \wedge S_{i-1} = T) \\ &\quad \vee P(S_i = T \wedge S_{i-1} = H) \\ &= p(1-p) + (1-p)p \\ &= 2p(1-p) \end{aligned}$$

EP4-11. A biased coin is tossed n times, and a head shows up with probability p on each toss. A run is a maximal sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$.

$$\therefore E(X) = E\left(\sum X_i\right)$$

$$= E(X_1) + E\left(\sum_{j=2}^{n} X_j\right)$$

$$= 1 + (n-1)(2p(1-p))$$

$$= 1 + 2(n-1)p(1-p)$$

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H H T H T T H

my misunderstanding: a minimal sequence where the start and the end is the same.

EP4-12. Each of 1000 voters votes independently for a candidate A with probability 1/2.

(a) What is the probability that A gets exactly 500 votes?

(b) What is the probability that A gets at least 500 votes?

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(a) $$p(X = 500) = \frac{\text{Count (exactly 500)}}{\text{Count (all cases)}}$$

$$= \frac{\binom{1000}{500}}{2^{1000}}$$

another way of computing

$$p(X=500) = \binom{1000}{500}\left[\left(\frac{1}{2}\right)^{500}\left(\frac{1}{2}\right)^{500}\right]$$

EP4-12. Each of 1000 voters votes independently for a candidate A with probability 1/2.

(a) What is the probability that A gets exactly 500 votes?

(b) What is the probability that A gets at least 500 votes?

$$P(x \geq 500)$$

$$= \sum_{k=500}^{1000} P(x = k)$$

$$= \sum_{k=500}^{1000} \frac{\binom{1000}{k}}{2^{1000}} = \frac{\binom{1000}{500} + \overset{\binom{1000}{499}}{\overset{\shortparallel}{\binom{1000}{501}}} + \cdots \overset{\binom{1000}{0}}{\overset{\shortparallel}{\binom{1000}{1000}}}}{2^{1000}}$$

EP4-12. Each of 1000 voters votes independently for a candidate A with probability 1/2.

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Another way of counting.

$$p(X=n) = \frac{\binom{1000}{n}}{2^{1000}} = \frac{\binom{1000}{1000-n}}{2^{100}} = p(X=1000-n)$$

$$\sum_{k=500}^{1000} p(x=k) = \sum_{k=0}^{500} p(x=k)$$

$$\sum_{k=0}^{1000} p(x=k) = 1$$

$$= \sum_{k=0}^{500} p(x=k) + \sum_{k=500}^{1000} p(x=k)$$

$$- p(x=500)$$

$$\sum_{k=500}^{1000} p(x=k) = \frac{1+p(x=500)}{2}$$

Ep 1-22

QB1

prove that

$$n \geq 0, \ (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Ep 1-22

prove that

$n \geq 0, \ (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

P(n)

Basis: P(0) is true.

$(a+b)^0 = 1$

$\binom{0}{0} a^0 b^0 = 1$

Inductive step: $P(k) \rightarrow P(k+1)$

$P(k) = (a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$

$P(k+1)$: $(a+b)^{k+1} \longrightarrow \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{k+1-i}$

$$p(k+1):\ (a+b)^{k+1} \longrightarrow \sum_{i=0}^{k+1}\binom{k+1}{i}a^{i}b^{k+1-i}$$

$$= (a+b)^{k}(a+b)$$

$$= \left(\sum_{i=0}^{k}\binom{k}{i}a^{i}b^{k-i}\right)(a+b)$$

$$= \sum_{i=0}^{k}\binom{k}{i}a^{i+1}b^{k-i} + \sum_{i=0}^{k}\binom{k}{i}a^{i}b^{k-i+1}$$

$$= \sum_{j=1}^{k+1}\binom{k}{j-1}a^{j}b^{k-(j-1)} + \sum_{i=0}^{k}\binom{k}{i}a^{i}b^{k-i+1}$$

$$= \sum_{j=1}^{k+1} \binom{k}{j-1} a^j b^{k-(j-1)} + \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1}$$

$$= \binom{k}{k} a^{k+1} \underbrace{b^{k-(k+1-1)}}_{=\,1} + \boxed{\begin{aligned} &\sum_{j=1}^{k} \binom{k}{j-1} a^j b^{k-j+1} \\ &\sum_{i=1}^{k} \binom{k}{i} a^i b^{k-i+1} \end{aligned}}$$

$$+ \binom{k}{0} a^0 b^{k+1} +$$

$$= \binom{k+1}{k+1} a^{k+1} + \binom{k+1}{0} b^{k+1}$$

$$+ \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1} \right) a^i b^{(k+1)-i}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{(k+1)-i}$$