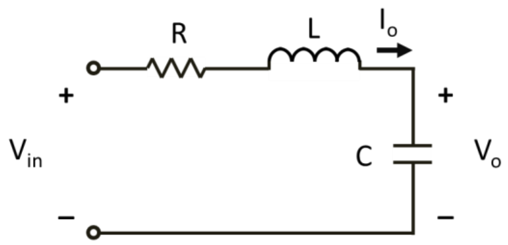
The Hong Kong University of Science and Technology
Department of Electronic and Computer Engineering

ELEC2400	ELECTRONIC CIRCUITS	FALL 2021-22

HOMEWORK 4 SOLUTION

Assume ideal op amp in all cases.

Q1. Find the transfer functions $H(s) = \frac{V_o(s)}{V_{in}(s)}$ and $G(s) = \frac{I_o(s)}{V_{in}(s)}$.



$$I_o = \frac{V_{in}}{R + sL + \frac{1}{sC}} = \frac{sC}{s^2LC + sCR + 1}V_{in}$$

Hence

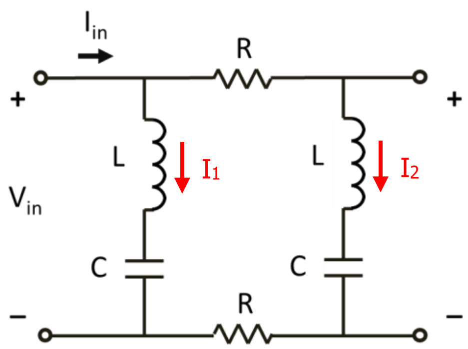
$$G(s) = \frac{I_o(s)}{V_{in}(s)} = \frac{sC}{s^2LC + sCR + 1}$$

$$V_o = \frac{I_o}{sC} = \left(\frac{sC}{s^2LC + sCR + 1}V_{in}\right)\frac{1}{sC} = \frac{1}{s^2LC + sCR + 1}V_{in}$$

Hence

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{s^2LC + sCR + 1}$$

Q2. Find the transfer functions $H(s) = \frac{V_o(s)}{V_{in}(s)}$ and $R(s) = \frac{V_o(s)}{I_{in}(s)}$.



The R, L, C and R are connected in series on the right. They form a voltage divider in which

$$V_o = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC} + R} V_{in} = \frac{s^2LC + 1}{s^2LC + 2sCR + 1} V_{in} \tag{1}$$

Hence,

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{s^2LC + 1}{s^2LC + 2sCR + 1}$$

Since

$$I_1 = \frac{V_{in}}{sL + \frac{1}{sC}} = \frac{sC}{s^2LC + 1} V_{in}$$

and

$$I_2 = \frac{V_{in}}{R + sL + \frac{1}{sC} + R} = \frac{sC}{s^2LC + 2sCR + 1} V_{in}$$

Therefore,

$$I_{in} = I_1 + I_2 = \frac{sC}{s^2LC + 1} V_{in} + \frac{sC}{s^2LC + 2sCR + 1} V_{in}$$

$$I_{in} = \frac{2sC(s^2LC + sCR + 1)}{(s^2LC + 1)(s^2LC + 2sCR + 1)} V_{in} \tag{2}$$

(1) ÷ (2),

$$R(s) = \frac{V_o(s)}{I_{in}(s)} = \left(\frac{s^2LC + 1}{s^2LC + 2sCR + 1}\right) \frac{(s^2LC + 1)(s^2LC + 2sCR + 1)}{2sC(s^2LC + sCR + 1)}$$

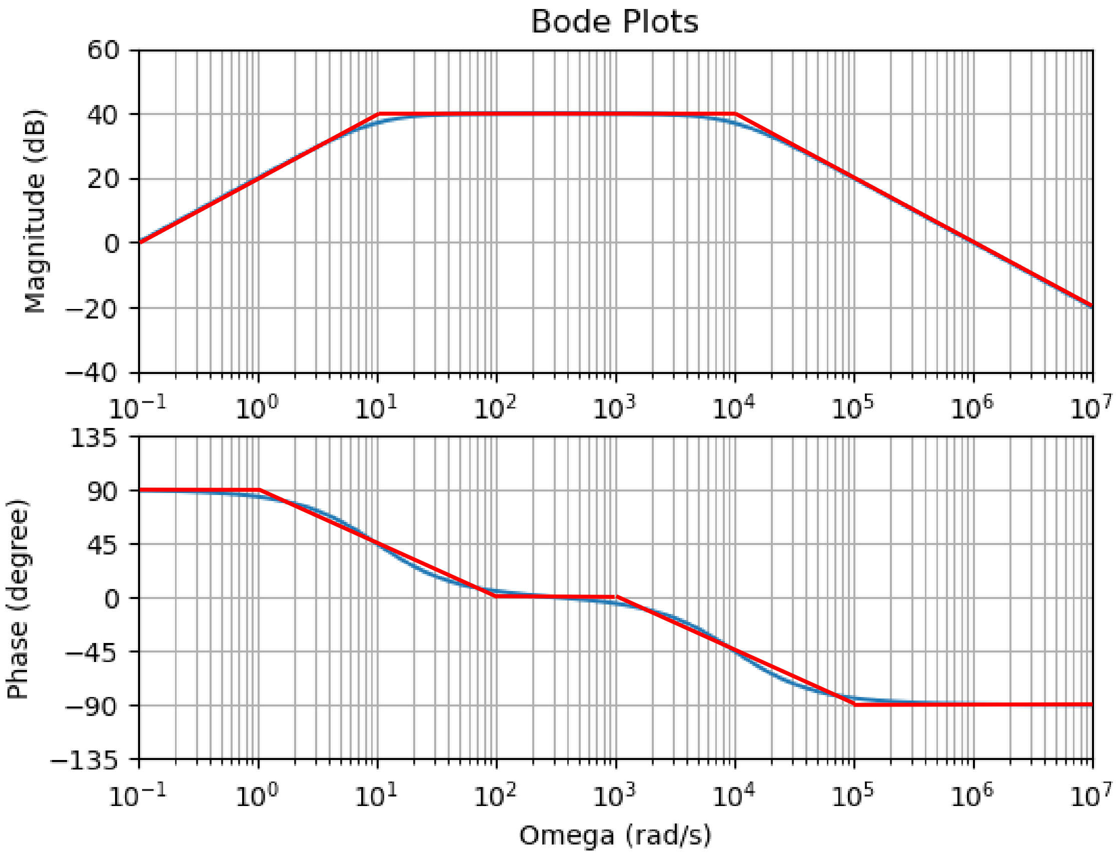
$$= \frac{(s^2LC + 1)^2}{2sC(s^2LC + sCR + 1)} \qquad \text{(4}^{\text{th}} \text{ order system)}$$

Q3. Sketch the Bode plots of

$$H(s) = \frac{10^6 s}{s^2 + 10010s + 10^5} = \frac{10^6 s}{(s + 10)(s + 10000)} = \frac{10s}{\left(1 + \frac{s}{10}\right)\left(1 + \frac{s}{10000}\right)}$$

There is a zero at $s = 0$ corresponding to $\omega = 0$ rad/s.
There are two poles at $s = -10, -10000$ corresponding to $\omega = 10, 10000$ rad/s.
At sufficiently low frequencies, $H(j\omega) \approx j10\omega$ with a 90° phase.
When $\omega = 0.1$ rad/s, $H(j\omega) = H(j0.1) \approx j$ with a magnitude $\approx 1 = 0$ dB. This is the point the magnitude plot should pass through.

The Bode plots are as shown below. Only the red sketches are required. The blue computer-generated plots are for your reference.



Q4. Sketch the Bode plots of

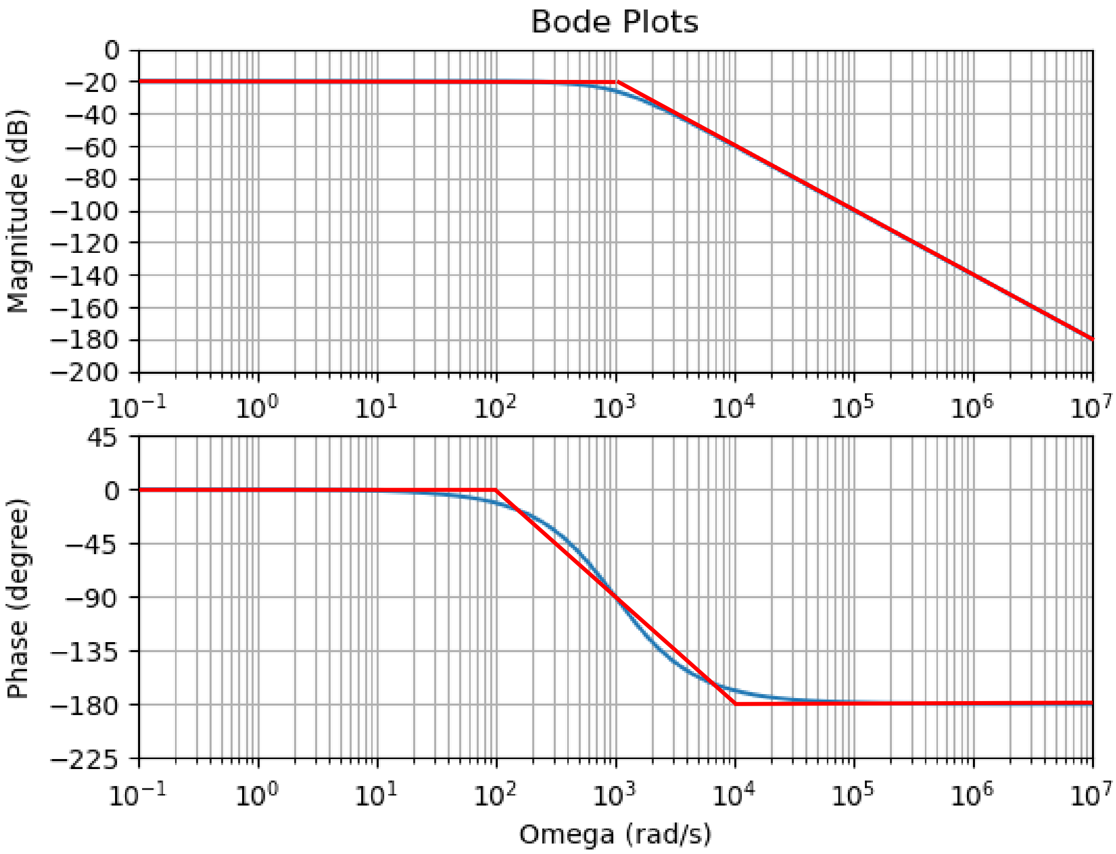
$$H(s) = \frac{10^4}{\frac{s^2}{10} + 200s + 10^5} = \frac{10^5}{s^2 + 2000s + 10^6}$$

$$= \frac{10^5}{(s+1000)(s+1000)} = \frac{0.1}{\left(1 + \frac{s}{1000}\right)\left(1 + \frac{s}{1000}\right)}$$

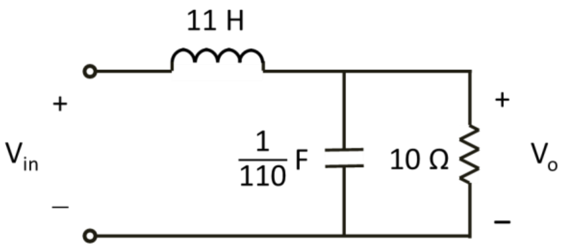
DC magnitude $= 20\log_{10} 0.1 = -20$ dB.

There are two identical poles at $s = -1000$ corresponding to $\omega = 1000$ rad/s. Their combined effects are two times that for a single pole at the same location. Hence, the magnitude roll-off is -40 dB/dec beyond 1000 rad/s and the phase shift is $-90°$/dec centered at 1000 rad/s.

The Bode plots are as shown below. Only the red sketches are required. The blue computer-generated plots are for your reference.



Q5. Find the transfer function $H(s) = \frac{V_o(s)}{V_{in}(s)}$ and sketch the Bode plots of $H(s)$.



First,

$$R||\left(\frac{1}{sC}\right) = \frac{\frac{R}{sC}}{R + \frac{1}{sC}} = \frac{R}{sCR + 1}$$

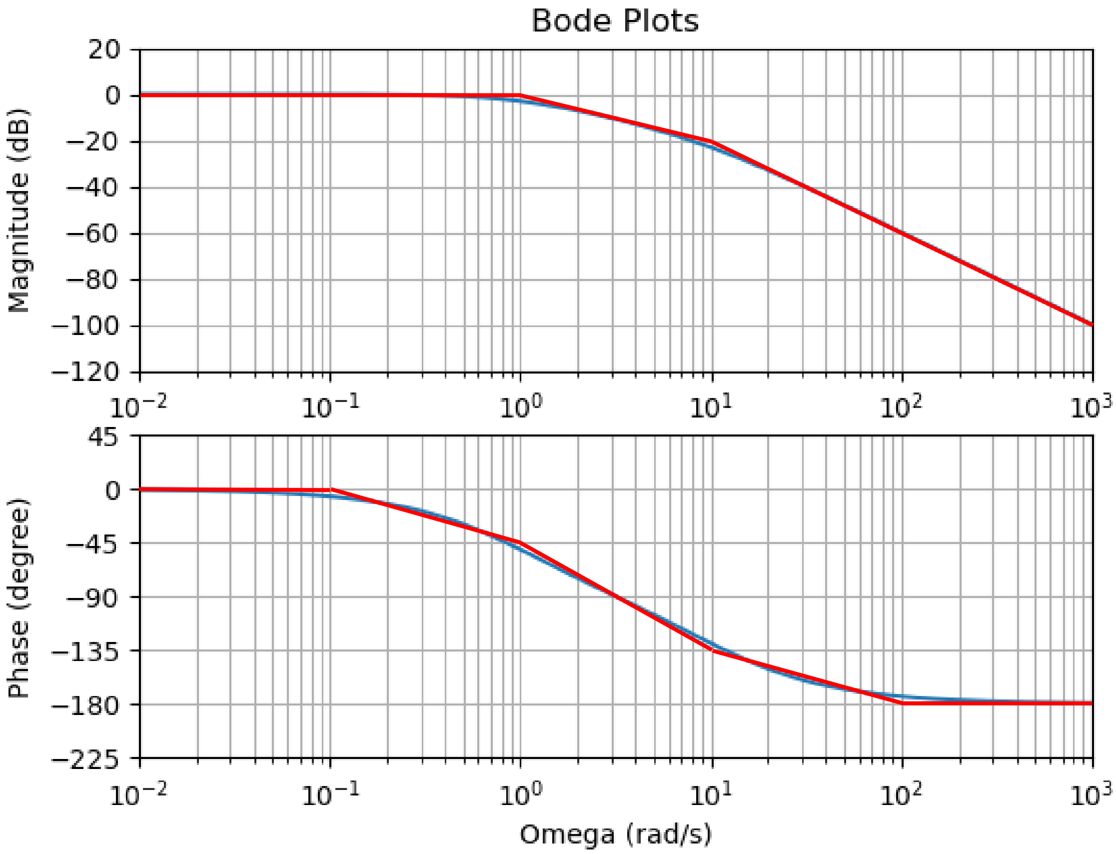
Now,

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{R||\left(\frac{1}{sC}\right)}{sL + R||\left(\frac{1}{sC}\right)} = \frac{\frac{R}{sCR + 1}}{sL + \frac{R}{sCR + 1}} = \frac{R}{s^2LCR + sL + R}$$

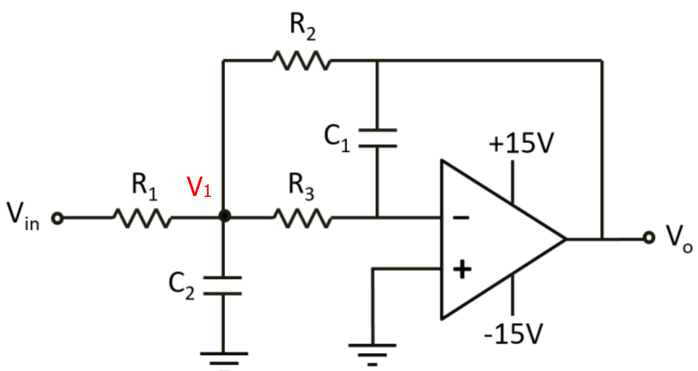
$$= \frac{10}{s^2 11\left(\frac{1}{110}\right)10 + s11 + 10} = \frac{10}{s^2 + s11 + 10} = \frac{10}{(s+1)(s+10)} = \frac{1}{(1+\frac{s}{1})(1+\frac{s}{10})}$$

DC magnitude $= 1 = 0$ dB.
There are two poles at $s = -1, -10$ corresponding to $\omega = 1, 10$ rad/s.

The Bode plots are as shown below. Only the red sketches are required. The blue computer-generated plots are for your reference.



Q6. Find the transfer function $H(s) = \frac{V_o(s)}{V_{in}(s)}$. What type of filter is this? What is the order?



Assume op amp not saturated, $V_+ = 0 = V_-$

Apply KCL at node V_-,

$$\frac{V_1 - V_-}{R_3} = \frac{V_- - V_o}{\frac{1}{sC_1}}$$

$$\frac{V_1 - 0}{R_3} = sC_1(0 - V_o)$$

$$V_1 = -sC_1R_3V_o \tag{1}$$

Apply KCL at node V_1,

$$\frac{V_{in} - V_1}{R_1} = \frac{V_1 - V_o}{R_2} + \frac{V_1}{\frac{1}{sC_2}} + \frac{V_1}{R_3}$$

$$\frac{V_{in}}{R_1} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)V_1 + sC_2V_1 - \frac{V_o}{R_2}$$

$$R_2R_3V_{in} = (R_1R_2 + R_2R_3 + R_3R_1 + sC_2R_1R_2R_3)V_1 - R_1R_3V_o$$

Substituting (1),

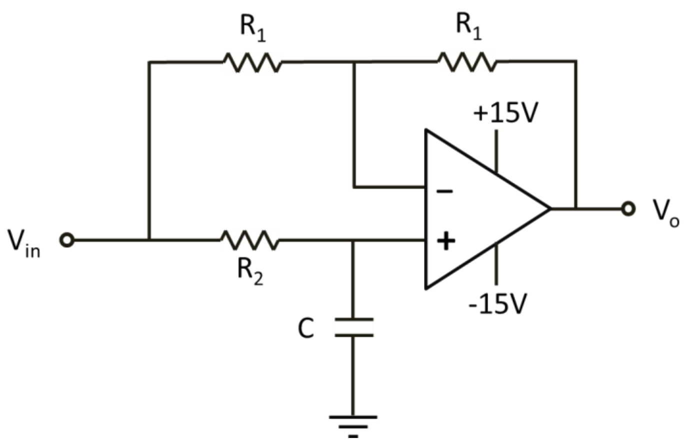
$$R_2R_3V_{in} = -(R_1R_2 + R_2R_3 + R_3R_1 + sC_2R_1R_2R_3)sC_1R_3V_o - R_1R_3V_o$$

Hence,

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{R_2R_3}{s^2C_1C_2R_1R_2{R_3}^2 + sC_1R_3(R_1R_2 + R_2R_3 + R_3R_1) + R_1R_3}$$

There are two poles and no zeros. This is a second-order low-pass filter.

Q7. Find the transfer function $H(s) = \frac{V_o(s)}{V_{in}(s)}$. Sketch the Bode plots for the case $R_1 = R_2 = 1$ kΩ and C = 1 μF. What type of filter is this? What is the order?



It can be easily seen that

$$V_- = \frac{V_{in} + V_o}{2} \tag{1}$$

R_2 and C form a voltage divider in which

$$V_+ = \frac{\frac{1}{sC}}{R_2 + \frac{1}{sC}} V_{in} = \frac{1}{sCR_2 + 1} V_{in} \tag{2}$$

Equate (1) and (2),

$$\frac{V_{in} + V_o}{2} = \frac{1}{sCR_2 + 1} V_{in}$$

$$V_o = \frac{2}{sCR_2 + 1} V_{in} - V_{in} = \frac{1 - sCR_2}{1 + sCR_2} V_{in}$$

Therefore,

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1 - sCR_2}{1 + sCR_2}$$

There is a zero at $s = \frac{1}{CR_2}$ and a pole at $s = -\frac{1}{CR_2}$, both corresponding to the same corner frequency $\omega = \frac{1}{CR_2}$, which is the frequency at which $|\pm sCR_2| = 1$, where $s = j\omega$.

Since the corner frequencies coincide, the effects of the zero and pole on the magnitude exactly cancel each other:

$$|H(j\omega)| = \frac{|1 - j\omega CR_2|}{|1 + j\omega CR_2|} = \frac{\sqrt{1 + (-\omega CR_2)^2}}{\sqrt{1 + (\omega CR_2)^2}} = 1 = \text{constant}$$

However, their effects on the phase are combined:

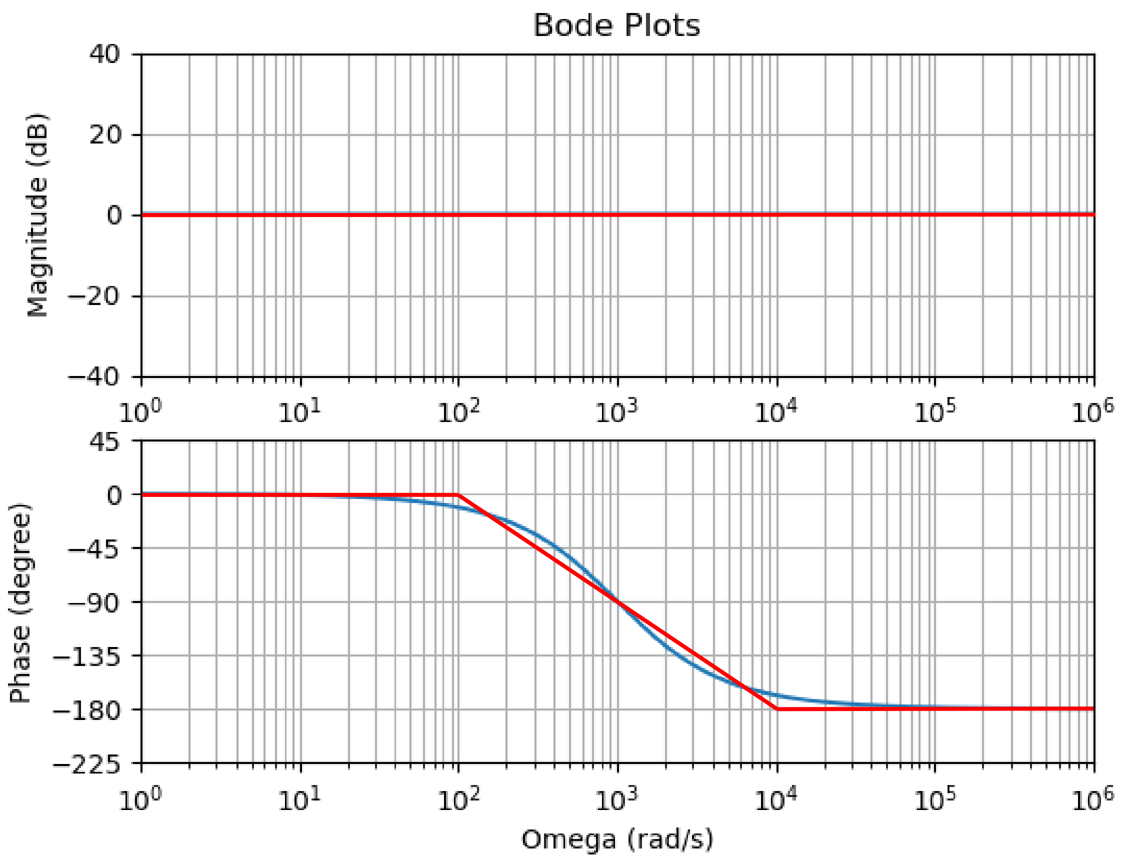
$$\angle[H(j\omega)] = \angle\left[\frac{1 - j\omega CR_2}{1 + j\omega CR_2}\right] = \angle(1 - j\omega CR_2) - \angle(1 + j\omega CR_2)$$

$$= \tan^{-1}(-\omega CR_2) - \tan^{-1}(\omega CR_2) = -2\tan^{-1}(\omega CR_2)$$

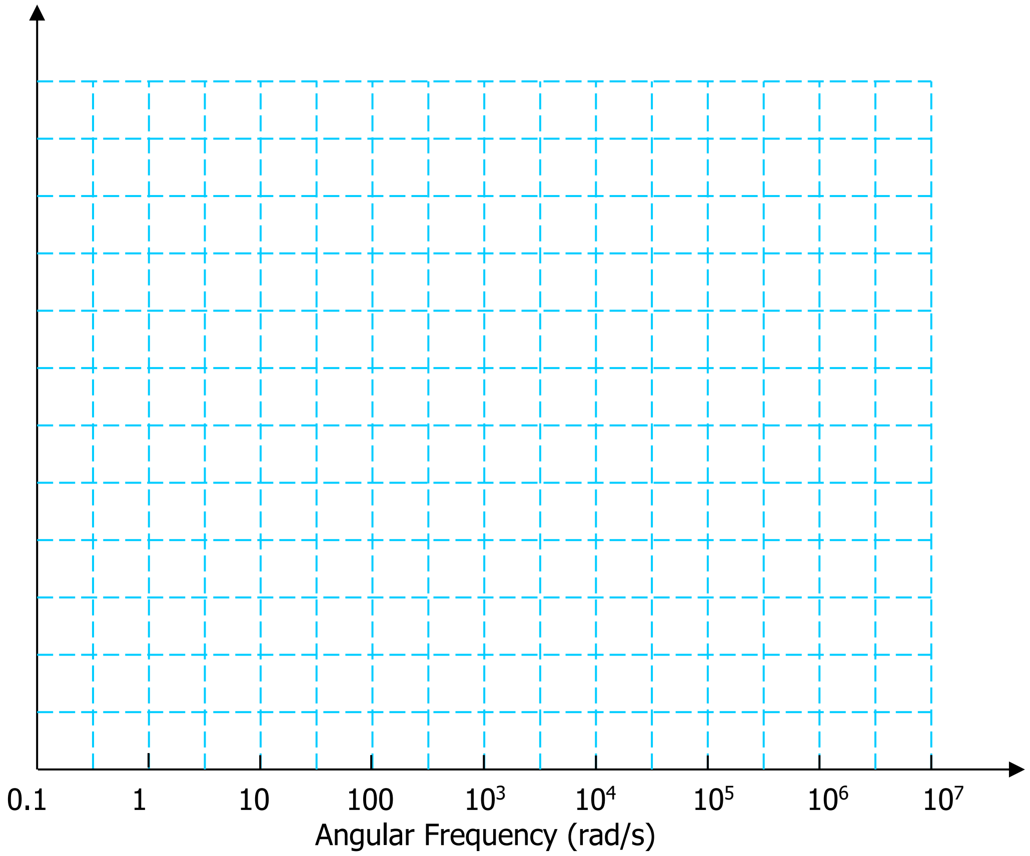
Moreover, it can be easily seen that $H(j0) = 1 = 1\angle 0°$ and $H(j\infty) = -1 = 1\angle(\pm 180°)$, but only $1\angle(-180°)$ is correct because the phase decreases with frequency.

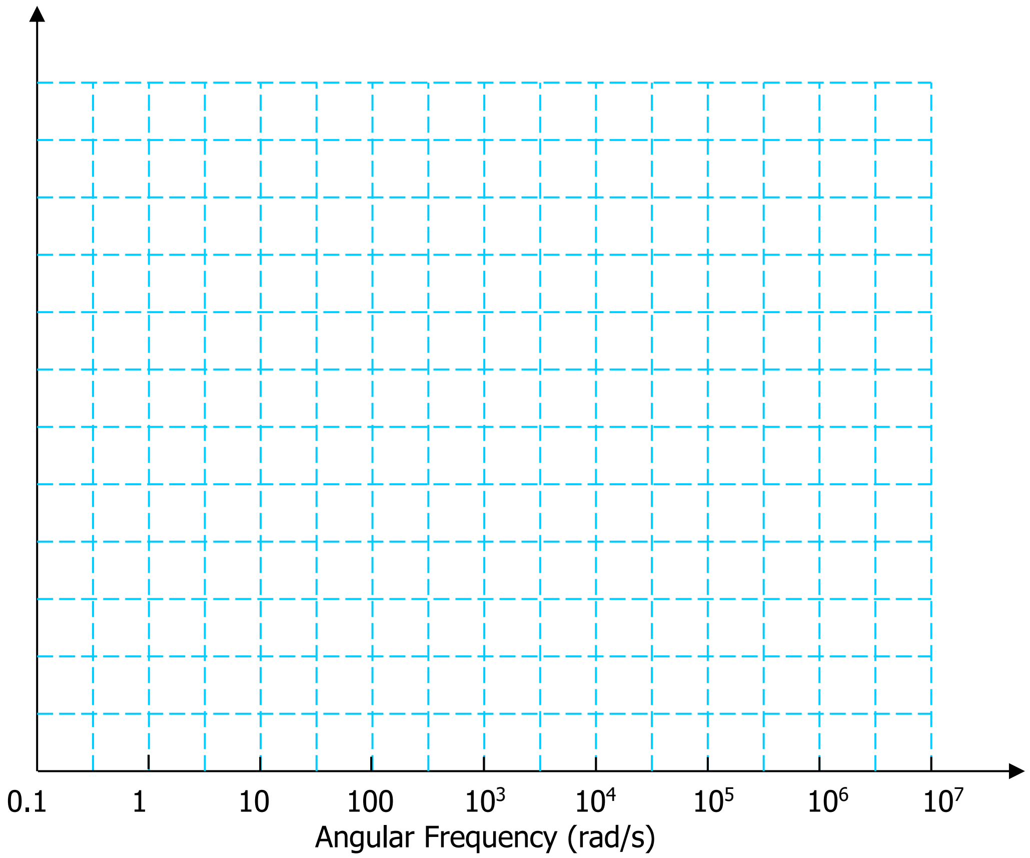
This is therefore a first-order all-pass filter.

The Bode plots for the case $R_1 = R_2 = 1$ kΩ and C = 1 μF are shown below. An ideal all-pass filter passes all frequencies with a constant gain. The only effect is the frequency-dependent phase shift it introduces.



Bode Plots
Magnitude (dB)
40
20
0
−20
−40
10^0
10^1
10^2
10^3
10^4
10^5
10^6
Phase (degree)
45
0
−45
−90
−135
−180
−225
10^0
10^1
10^2
10^3
10^4
10^5
10^6
Omega (rad/s)


0.1
1
10
100
10^3
10^4
10^5
10^6
10^7
Angular Frequency (rad/s)


0.1
1
10
100
10^3
10^4
10^5
10^6
10^7
Angular Frequency (rad/s)