COMP 2711H Discrete Mathematical Tools for Computer Science
Tutorial Problems: Induction and Recursion

Problem 1. Use mathematical induction to prove that 43 divides $6^{n+1}+7^{2n-1}$ for every positive integer n.

Problem 2. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 5-cent stamps. Prove that $P(n)$ is true for $n \geq 8$. You should give two different proofs, using weak and strong induction, respectively.

Problem 3. Use mathematical induction to show that if you draw lines in the plane you only need two colors to color the regions formed so that no two regions that have an edge in common have a common color.

Problem 4. Arrange the following running times in order of increasing asymptotic complexity. Just give the answer; no explanation is needed.

$$n^3, \ \sqrt{2n}, \ n+10, \ \log(n^4), \ 20^n, \ 2^n, \ n^2 \log n$$

Note that you must write function $f(n)$ before function $g(n)$ if $f(n) = O(g(n))$.

Problem 5. Use mathematical induction to prove that n lines separate the plane into $(n^2+n+2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

Problem 6. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.

Problem 7. Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$.

Recursive step: If $(a,b) \in S$, then $(a,b+1) \in S$, $(a+1,b+1) \in S$, and $(a+2,b+1) \in S$.

(a) List the elements of S produced by the first four applications of the recursive definition.

(b) Use induction to prove that $a \leq 2b$ whenever $(a,b) \in S$.

Problem 8. Recursively define the set of bit strings that have more zeros than ones.

Problem 9. In this problem, you need to determine the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1, a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$.

(a) Find the answer to this problem by first writing a recurrence relation for the number of such sequences, and then solving it (using, say, the iterative approach).

(b) Find another way of solving this problem, which does not involve writing a recurrence relation.

Problem 10. Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes.

Problem 11. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmission, the other two signals require 2 microseconds each for transmission, and a signal in a message is followed immediately by the next signal? You need to give a closed-form solution.