COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial Problems: Combinatorics

Q1. (a) Let x_i be the number of scoops of flavor i. We have that the number of different dishes is equal to the number of nonnegative solutions to $\sum_{i=1}^{28} x_i = 3$. As in the tutorial, this is $\binom{30}{3} = 4060$.

(b) By the product rule, it is $28 \times 8 \times 12 = 2688$.

(c) By the product rule, it is $\binom{30}{3} \times \binom{8}{2} \times \binom{12}{3}$.

Q2. There are $\binom{10}{5} = 252$ different five-element subsets of these 10 integers. For any five distinct positive integers not exceeding 50, their sum is at least 1 and is at most 250, so there are at most 250 different sums. By the pigeonhole principle, at least two different five-elements have the same sum.

Q3. Let n be the number of people in the party. Let k be the number of people who don't know any people in the party. If $k \geq 2$, we are done. Without loss of generality, we assume that $k \leq 1$. Consider the remaining $n-k$ people, each of them know at least 1 other people and at most $n-k-1$ people in the party (This also indicates that $n-k \geq 2$). By the pigeonhole principle, at least 2 of these $n-k$ people know the same number of other people.

A proof in graph language We denote each people in the party by a vertex. If two people know each other, then we connect them by an edge. Now consider the connected components of this graph (you may refer to wikipedia if you don't know connected components). If there are two connected components of size 1, then we are done. Otherwise, there must be a connected component of size k where $k \geq 2$. In this connected component, each people know at least 1 and at most $k-1$ other people. By the pigeonhole principle, there must be two people know the same number of other people.

Q4. We partition all integers not exceeding $2n$ into n groups $\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$. It is easy to see that the two integers in each group are relatively prime. Since we select $n+1$ distinct integers, by the pigeonhole principle, at least 2 of them must be in the same group and are relatively prime.

Q5. Out of n people, we want to select a committee with a leader. Now let's count the number of ways to do this. We count in two ways.

(i) We first select a leader, and then select the rest of the committee. There n ways to select a leader from n people. For the remaining $n-1$ people, they are either in or not in the committee, so there are 2^{n-1} to select the rest of the committee. Applying the product rule, there are $n2^{n-1}$ ways to select a committee with a leader from n people.

(ii) We first select all the committee members, and then select a leader from these members. Consider a committee of size k. There are $C(n,k)$ ways to select all the committee members and there are k ways to select a leader, so there are $kC(n,k)$ ways to select a committee of size k. Since k can be from 1 to n, the total number of ways to select such a committee and a leader is $\sum_{k=1}^{n} kC(n,k)$.

Since in (i) and (ii), we count the same things, we have that $n2^{n-1} = \sum_{k=1}^{n} kC(n,k)$.

Q6. Let A be the set of all nonnegative integral solutions to inequality

$$x_1 + x_2 + x_3 \leq 10.$$

Let B be the set of all nonnegative integral solutions to the equality

$$y_1 + y_2 + y_3 + y_4 = 10.$$

We claim that $|A| = |B|$. To see this, consider the following function $f : A \to B$.

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3, 10 - x_1 - x_2 - x_3)$$

One can verify that f is a bijection (we leave this as an exercise), so by the bijection principle, $|A| = |B|$. As in the tutorial, we know that $|B| = \binom{13}{3}$, so $|A| = \binom{13}{3}$.

Q7. (a) We first choose 10 out of 30 issues and put them into the first box. Then we choose 10 out of the remaining 20 issues and put them into the second box. At last, we put all the rest into the third box. By the product rule, the total number of ways to do this is $\binom{30}{10}\binom{20}{10}\binom{10}{10}$.

(b) If the boxes are identical, then each arrangement in (b) is counted 3! in (a). Hence, the number of arrangements of (b) is $\binom{30}{10}\binom{20}{10}\binom{10}{10}/3!$.

Q8. We do this task in two phases.

(i) We first seat 8 girls in a row. There are 8! ways to carry out the first phase.

(ii) Then we add boys. Since boys cannot sit together, between any two girls (we call this a slot), we can insert at most one boy. There are 9 slots, including the one before the first girl and the one after the last girl. Out of these 9 slots, we need to choose 6 to seat boys, and these boys are distinguishable, so there are $\binom{9}{6}6!$ to carry out the second phase.

Putting this together, there are $8!\binom{9}{6}6!$ ways to carry out our task.

Q9. Let $U = \{i \in Z : 1 \leq i \leq pq\}$. Let $A = \{i \in U : i \text{ is relatively prime to } pq\}$. Let $P = \{i \in U : p \text{ divides } i\}$. Let $Q = \{i \in U : q \text{ divides } i\}$. Since p and q are primes, we have that $\bar{A} = P \cup Q$. By the principle of inclusion and exclusion, we have

$$\begin{aligned} |A| &= |U| - |P \cup Q| \\ &= |U| - |P| - |Q| + |P \cap Q| \\ &= pq - q - p + 1 \\ &= (p-1)(q-1). \end{aligned}$$