COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial 9

QB4-1. Timothy can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will late. The probability that he is late when he rides his bicycle is only 5%. Timothy arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

(a) Suppose the boss assumes that there is a 1/3 chance that Timothy takes each of the three ways he can get to work. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem under this assumption?

(b) Suppose the boss knows that Timothy drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem using this information?

Solution

Define events R (riding bicycle), C (driving car), B (bus), L (arriving late). What we can get directly:

$$P(L|C) = 0.5, P(L|B) = 0.2, P(L|R) = 0.05$$

And then we can derive:

$$P(\bar{L}|C) = 0.5, P(\bar{L}|B) = 0.8, P(\bar{L}|R) = 0.95$$

(a) Now we have more information:

$$P(C) = P(B) = P(R) = 1/3$$

The target is $P(C|B)$. We have:

$$\begin{aligned}
P(C|L) &= \frac{P(C \cap L)}{P(L)} \\
&= \frac{P(C)P(L|C)}{P(L|C)P(C) + P(L|R)P(R) + P(L|B)P(B)} \\
&= \frac{1/3 \cdot 0.5}{1/3 \cdot 0.5 + 1/3 \cdot 0.05 + 1/3 \cdot 0.2} \\
&= 2/3
\end{aligned}$$

(b)

$$P(C) = 0.3, P(B) = 0.1, P(R) = 0.6$$

The target is $P(C|B)$. We have:

$$
\begin{aligned}
P(C|L) &= \frac{P(C \cap L)}{P(L)} \\
&= \frac{P(C)P(L|C)}{P(L|C)P(C) + P(L|R)P(R) + P(L|B)P(B)} \\
&= \frac{0.3 \cdot 0.5}{0.3 \cdot 0.5 + 0.6 \cdot 0.05 + 0.1 \cdot 0.2} \\
&= 0.75
\end{aligned}
$$

EP4-7. We assume that each child born has probability 1/2 of being a boy or a girl. We also assume that children's sexes are independent (even within a family).

(a) Mr Jones has two children. Knowing that the older one is a boy, what is the probability that they are both boys?

(b) Mr Smith has two children. Knowing that at least one is a boy, what is the probability that they are both boys?

(c) Mr Johnson has three children. Knowing that at least one is a boy, what is the probability that at least one is a girl?

Solution (a) Define events OB (older one is a boy), BB (both boys). Our target is $P(BB|OB)$. We have:

$$
P(BB|OB) = \frac{P(BB \cap OB)}{P(OB)} = \frac{P(OB|BB)P(BB)}{P(OB)} = \frac{1 \cdot 0.25}{0.5} = 0.5
$$

(b) Define events A (at least one is a boy), BB (both boys). Our target is $P(BB|A)$. We have:

$$
P(BB|A) = \frac{P(BB \cap A)}{P(A)} = \frac{P(A|BB)P(BB)}{1 - P(\bar{A})} = \frac{1 \cdot 0.25}{0.75} = 1/3
$$

(c) Define events B (at least one is a boy), G (at least one is a girl). Our target is $P(G|B)$. We have:

$$
\begin{aligned}
P(G|B) &= \frac{P(G \cap B)}{P(B)} \text{[G and B are symmetric, transferring to P(B—G)P(G) is useless]} \\
&= \frac{P(B) + P(G) - P(G \cup B)}{1 - P(\bar{B})} \\
&= \frac{(1 - P(\bar{B})) + (1 - P(\bar{G})) - (1)}{1 - P(\bar{B})} \\
&= \frac{(1 - 1/8) + (1 - 1/8) - 1}{1 - 1/8} = 6/7
\end{aligned}
$$

EP4-8. Box A contains 2 black and 5 white marbles, and box B contains 1 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

(a) What is the probability that the marble is black?

(b) Given that the marble is white, what is the probability that it came from box A?

Solution Define events K (black marbles), W (white marbles), A (select box A), B (select box B). What we can get directly:

$$P(K|A) = 2/7, P(W|A) = 5/7, P(K|B) = 1/5, P(W|B) = 4/5,$$

and:

$$P(A) = P(B) = 1/2$$

(a) Our target is $P(K)$, we have:

$$P(K) = P(K \cup A) + P(K \cup B) = P(K|A)P(A) + P(K|B)P(B) = 2/7 \cdot 1/2 + 1/5 \cdot 1/2 = 17/70$$

(b) Our target is $P(A|W)$, we have:

$$P(A|W) = \frac{P(A \cap W)}{P(W)} = \frac{P(W|A)P(A)}{1 - P(K)} = \frac{5/7 \cdot 0.5}{53/70} = 25/53$$