COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial 11

EP5-5. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n-1) + 2 & n > 1 \end{cases}$$

Solution

For $n > 1$,

$$T(n) + 1 = 3(T(n-1) + 1)$$
$$T(n) + 1 = 3^{(n-1)}(T(1) + 1)$$
$$T(n) = 2 \cdot 3^{(n-1)} - 1$$

EP5-6. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 5 & n = 1 \\ 2T(n-1) + 3n + 1 & n > 1 \end{cases}$$

Solution

For $n > 1$,

$$T(n) + 3n + 7 = 2(T(n-1) + 3(n-1) + 7)$$
$$T(n) + 3n + 7 = 2^{(n-1)}(T(1) + 3 + 7)$$
$$T(n) = 15 \cdot 2^{(n-1)} - 3n - 7$$

EP5-7. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ nT(n-1) + n & n > 1 \end{cases}$$

Solution

For $n > 1$,

$$\frac{T(n)}{n!} = \frac{T(n-1)}{(n-1)!} + \frac{1}{(n-1)!}$$
$$\frac{T(n)}{n!} = \sum_{i=1}^{n-1} \frac{1}{i!} + \frac{T(1)}{1!}$$
$$T(n) = n!(\sum_{i=1}^{n-1} \frac{1}{i!} + 1)$$

EP5-8. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2^n - T(n-1) & n > 1 \end{cases}$$

Solution

For $n > 1$,

$$T(n) - \frac{2}{3}2^n = -(T(n-1) - \frac{2}{3}2^{n-1})$$

$$T(n) - \frac{2}{3}2^n = (-1)^{(n-1)}(T(1) - \frac{2}{3}2^1)$$

$$T(n) = (-1)^{(n-1)}(-\frac{1}{3}) + \frac{2}{3}2^n$$

EP5-9. Find a closed-form solution for the following recurrence where you are allowed to consider $H_n = \sum_{i=1}^{n} \frac{1}{i}$ as a known function

$$T(n) = \begin{cases} 1 & n = 2 \\ (n-1)T(n-1) + (n-2)! & n > 2 \end{cases}$$

Solution

For $n > 2$,

$$\frac{T(n)}{(n-1)!} = \frac{T(n-1)}{(n-2)!} + \frac{1}{n-1}$$

$$\frac{T(n)}{(n-1)!} = \frac{T(2)}{1!} + \sum_{i=2}^{n-1} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} = H_{n-1}$$

$$T(n) = (n-1)!H_{n-1}$$

EP5-11. Find a closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 3 & n = 1 \\ 6T(n/6) + 3n - 1 & n > 1 \end{cases}$$

where n is a power of 6.

Solution

For $n > 1$, let $k = \log_6 n$, then $n = 6^k$ and

$$T(6^k) = 6T(6^{k-1}) + 3 \cdot 6^k - 1$$

$$\frac{T(6^k)}{6^k} = \frac{T(6^{k-1})}{6^{k-1}} + 3 - 6^{-k}$$

$$\frac{T(6^k)}{6^k} = \frac{T(1)}{1} + 3k - \sum_{i=1}^{k} 6^{-i} = 3k + 3 - \frac{1 - 6^{-k}}{5}$$

$$T(n) = T(6^k) = 3(k+1) \cdot 6^k - \frac{6^k - 1}{5} = 3(log_6 n + 1)n - \frac{n-1}{5} = 3nlog_6 n + \frac{14n}{5} + \frac{1}{5}$$