COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial 8

From Wiki

Combinatorial proof often refers to 2 types of proofs:

Double counting. Counting the number of elements in two different ways to obtain the different expressions in the identity. Since those expressions count the same objects, they must be equal to each other.

Bijective proof. Two sets are shown to have the same number of members by exhibiting a bijection, i.e. a one-to-one correspondence, between them.

EP3-26. Prove the identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

for $1 \leq k < n$. (Use both combinatorial and algebraic proofs)

Solution

Algebraic proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Combinatorial proof:

Rewrite the equation as:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \iff k\binom{n}{k} = n\binom{n-1}{k-1} \iff \binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}$$

Think of such a task, choose 1 leader and $k-1$ workers from n persons. The left hand side selects k persons from n persons first and then selects 1 leader from the k persons; the right hand side selects 1 leader from the n persons first and then selects $k-1$ workers from $n-1$ rest persons. The result should be equivalent.

EP3-27. Prove the identity

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

for $0 \leq k < n$. (Use both combinatorial and algebraic proofs)

Solution

Algebraic proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\frac{n}{n-k}\binom{n-1}{k} = \frac{n}{n-k}\frac{(n-1)!}{k!(n-k-1)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Combinatorial proof:

Rewrite the equation as:

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} \iff (n-k)\binom{n}{k} = n\binom{n-1}{k} \iff \binom{n}{k}\binom{n-k}{1} = \binom{n}{1}\binom{n-1}{k}$$

Think of such a task, choose 1 leader and k workers from n persons. The left hand side selects k workers from n persons first and then selects 1 leader from the rest $n-k$ persons; the right hand side selects 1 leader from the n persons first and then selects k workers from $n-1$ rest persons. The result should be equivalent.

EP3-35. Prove the identity (use both combinatorial and algebraic proofs)

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Solution

Algebraic proof:

$$2\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n}(k\binom{n}{k} + (n-k)\binom{n}{n-k}) = \sum_{k=1}^{n}(k\binom{n}{k} + (n-k)\binom{n}{k})$$

$$= \sum_{k=1}^{n} n\binom{n}{k} = n2^n = 2n2^{n-1} \iff \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

Combinatorial proof:

Think of such a task, choose 1 leader and any number of workers from n persons. The left hand side sum up the cases for each k as the total number of person selected and selects k persons (worker and leader) from n persons $\binom{n}{k}$ first and then selects 1 leader from the k persons; the right hand side selects 1 leader from the n persons first and then for the rest $n-1$ persons, each can be selected as worker or not, so 2^{n-1} chooses. The result should be equivalent.

EP3-37. Use the binomial theorem to prove that $2^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}$

Solution

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k} = (3-1)^n = 2^n$$

QB5-4. Arrange the following running times in order of increasing asymptotic complexity. Just give the answer; no explanation is needed.

$$n^3, \sqrt{2n}, n+10, \log(n^4), 20^n, 2^n, n^2\log n$$

Note that you must write function $f(n)$ before function $g(n)$ if $f(n) = O(g(n))$.

Solution

$$\log(n^4), \sqrt{2n}, n+10, n^2\log n, n^3, 2^n, 20^n$$