L07: Cryptography

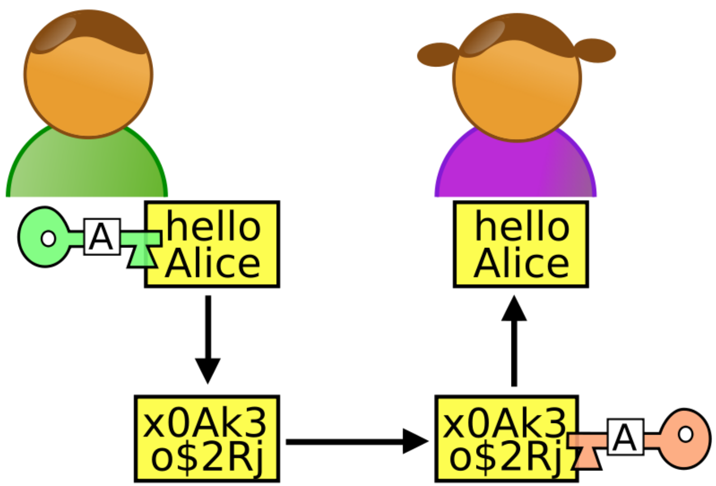
- Cryptography is the study of methods for sending and receiving secret messages through insure channels

- Outline:
  - **Private Key Cryptography**
  - Key Exchange
  - Public Key Cryptography and RSA

- Reading: Rosen 4.5, 4.6

Private Key Cryptography

- In private key cryptography, the sender (Alice) and the receiver (Bob) first agree on a common secret key in advance



Caesar Cipher (Shift Cypher)

- Encryption
  - The secret key k is a number from $\mathbf{Z}_{26}$
  - Replace each letter by an integer from $\mathbf{Z}_{26}$
  - The encryption function is $f(p) = (p + k) \bmod 26$. It replaces each integer p by $f(p)$.
  - Replace each integer by the corresponding letter
- Decryption
  - Just replace $f(p)$ with $f^{-1}(p) = (p - k) \bmod 26$ in the procedure above.

Caesar Cipher: Example

- **Example**
  Encrypt the message "MEET YOU IN THE PARK" using $k = 3$
- **Solution**
  - Replace letters by numbers:
    $12\ 4\ 4\ 19 \quad 24\ 14\ 20 \quad 8\ 13 \quad 19\ 7\ 4 \quad 15\ 0\ 17\ 10.$
  - Replace each of these numbers p by $f(p)$:
    $15\ 7\ 7\ 22 \quad 1\ 17\ 23 \quad 11\ 16 \quad 22\ 10\ 7 \quad 18\ 3\ 20\ 13.$
  - Translating the numbers back to letters
    "PHHW BRX LQ WKH SDUN."

Affine Ciphers

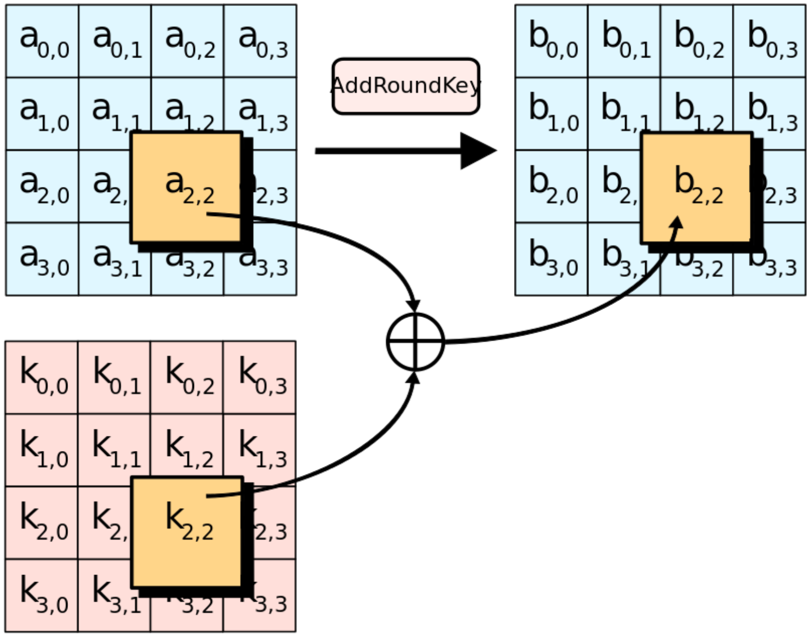
- The shift cipher is easy to break:
  - Just try all 26 possible keys!
- Affine ciphers make it (a little bit) safer by using both additions and multiplications
- Use the function $f(p) = (ap + b) \bmod 26$
  - The (a, b) pair is the secret key
  - Now there are $26^2 = 676$ possible secret keys
- However, suppose $a = 13, b = 1$
  - $f(1) = f(3) = 14$
  - If we receive a 14, which number does it decrypt to?
- How to fix?
  - Choose a such that $\gcd(a, 26) = 1$, e.g., $a = 7$
  - Then $ax + b \equiv y \pmod{p}$ has a unique solution

Block Ciphers

- Each character is a number between 0 and 255
  - A byte = 8 bits
- Partition the message into blocks of k characters
  - Treat each block as a big number of $8k$ bits
  - Use arithmetic modulo 2^{8k}
- Example
  - Choose $k = 10$
  - Encryption: $f(x) = ax + b \bmod 2^{80}$
  - Decryption: $f^{-1}(y) = a^{-1}(y - b) \bmod 2^{80}$
  - Now there are $\frac{2^{80}}{2} \times 2^{80} = 2^{159}$ different keys
- There are libraries on arbitrary-precision arithmetic

Advanced Encryption Standard (AES)

- Used in Transport Layer Security (TLS)
  - Previously known as Secure Sockets Layer (SSL)
  - Provides security for https, email, etc.
- A block cipher
  - Block size 128 bits
  - Key lengths: 128, 192, 256 bits
- Complicated operations that make it very difficult to break

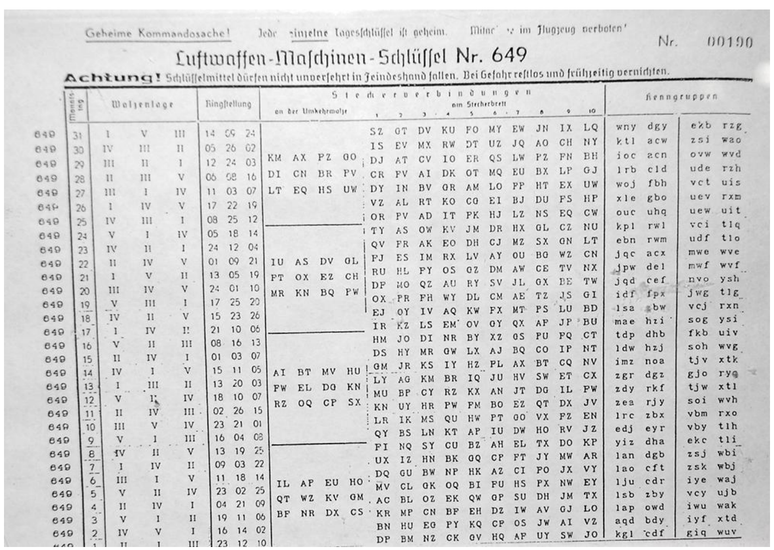


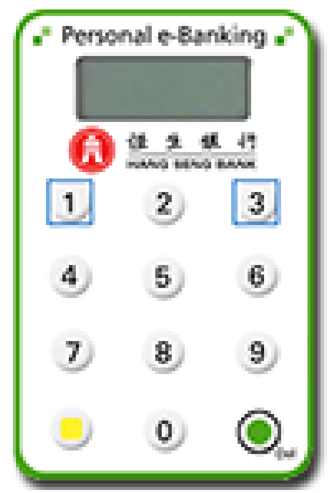
Outline

- Private Key Cryptography
- **Key Exchange**
- Public Key Cryptography and RSA

Problems with private-key cryptography

- How to send the secrete key?
  - Keys had to be transmitted in physical form in World War II
- How to distribute different keys to different customers?





New Security Device

The Key Exchange Puzzle

- Alice wants to send a valuable item to Bob, but the postman cannot be trusted
  - Alice can put an (unbreakable) lock on the box, but Bob cannot open it without the key
- Solution
  - Alice puts her lock on the box, and send its to Bob.
  - Bob, after receiving the box, puts his lock on the box as well, and returns to Alice.
  - Alice, after receiving the box, takes off her lock, and sends it back to Bob.
  - Bob takes off his lock and opens the box.

Modular Exponentiation

- How to compute
$$a^n \bmod m$$
efficiently for large n?
- Repeated squaring method
  - Compute
    $a^2 \bmod m$
    $a^{2^2} \bmod m = a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$
    $a^{2^3} \bmod m = a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$
    ...
  - Write n in binary $n = (b_k \ldots b_1 b_0)_2$
  - $a^n \equiv a^{b_0 \cdot 1} \cdot a^{b_1 \cdot 2} \cdot a^{b_2 \cdot 2^2} \cdots \pmod m$
- Example: $n = 50 = (110010)_2$
  - $a^{50} \equiv a^{2^1} a^{2^4} a^{2^5} \pmod m$

A Hard Problem: Discrete Logarithm

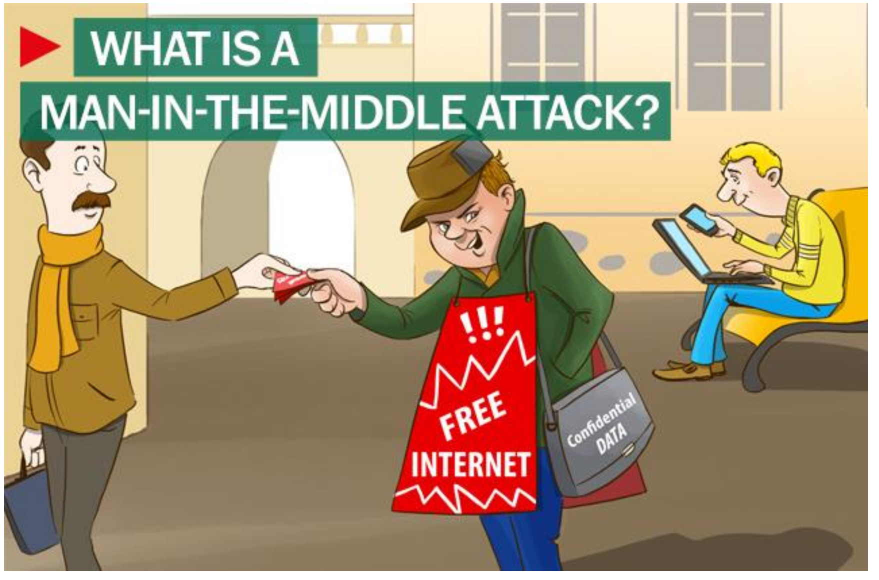
- “Locks” in cryptography correspond to problems that are believed to be computationally difficult
- Yes, if you can solve these problems, you can break current crypto systems
- Discrete logarithm is one such problem
  - Opposite of modular exponentiation
  - Given a prime p (potentially very large) and $r, a \in \mathbf{Z}_p$, find $x \in \mathbf{Z}_p$ such that
$$r^x \equiv a$$
- In 2015, it was reported that for 512-bit primes, the problem can be solved with a few thousands of CPUs in a week
  - Estimated cost to break 1024 bits: US$100 million.

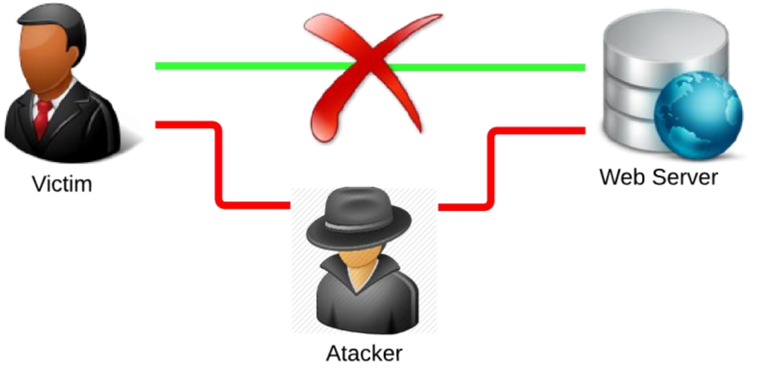
Diffe-Hellman Key Exchange

- Fix p and a
  - E.g., hardcoded in the TLS library
- The protocol
  1) Alice chooses a secret integer k_1 and sends $a^{k_1} \bmod p$ to Bob.
     Secure to eavesdropping: Even this value is known to attackers, they cannot compute k_1
  2) Bob chooses a secret integer k_2 and sends $a^{k_2} \bmod p$ to Alice.
  3) Alice computes $(a^{k_2})^{k_1} \bmod p$.
  4) Bob computes $(a^{k_1})^{k_2} \bmod p$.
- The shared key is

$$\left(a^{k_1}\right)^{k_2} \bmod p = \left(a^{k_2}\right)^{k_1} \bmod p$$

Man-in-the-middle Attack





- If attacker intercepts all traffic between two parties
- Diffe-Hellman protocol can be compromised
- Attacker
  - communicates with Alice pretending as Bob
  - communicates with Bob pretending as Alice

Outline

- Private Key Cryptography
- Key Exchange
- **Public Key Cryptography and RSA**

Public Key Cryptography

- All previous ciphers need a common private key
- Encryption and decryption are symmetric
- The key has to be
  - communicated physically in private
  - using the DH protocol (secure to eavesdropping but not man-in-the-middle)
- Public key cryptography
  - Encryption and decryption are asymmetric
  - Everyone has
    - a public key: shared with everyone else
    - a private key: kept in private

The RSA Cryptosystem

Ronald Rivest
(Born 1948)

Adi Shamir
(Born 1952)

Leonard
Adelman
(Born 1945)

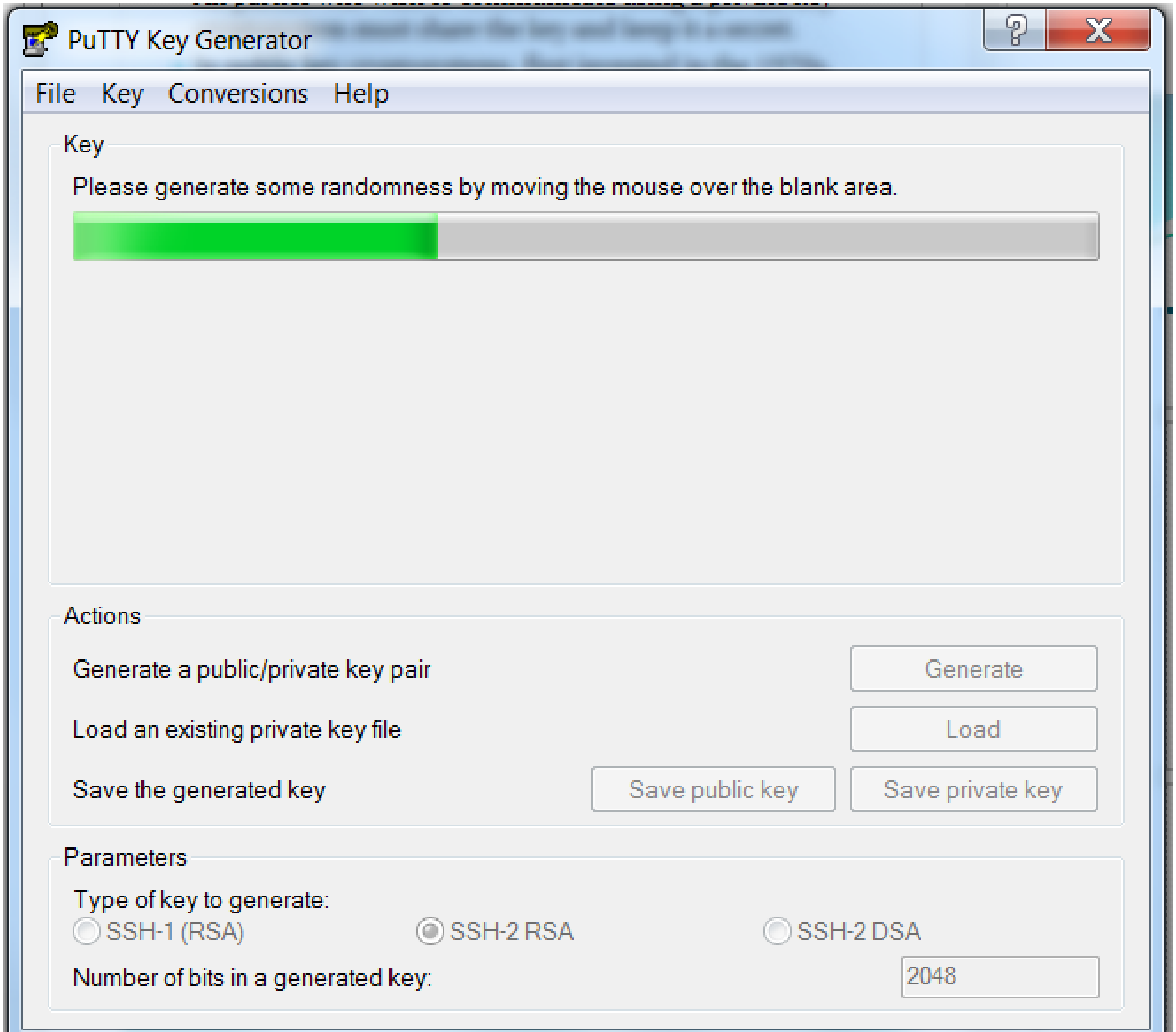
Clifford Cocks
(Born 1950)

- RSA was introduced in 1976 by RSA.
- In fact, Clifford Cocks, working secretly for the UK government, discovered it 3 years earlier.
  - Made known to public in late 1990s.

Another Difficult Problem: Factoring

- Let $n = pq$, where p and q are large primes (e.g. 1024 bits or longer)
- The factoring problem: Given n, find p and q
- On the other hand, it is known how to find large primes efficiently
- Public key in RSA: n and e, such that e is relatively prime to $(p-1)(q-1)$
- Private key in RSA: p and q
- Everyone uses a different set of keys

Key Generator



This is My Public Key

```
---- BEGIN SSH2 PUBLIC KEY ----
Comment: "rsa-key-20161118"
AAAAB3NzaC1yc2EAAAABJQAAAQEAkrwKeUwwz0jThhh2NSS8EJhEDl8VDzyCh8Rw
y2NJ6nHymOwyCWicUhjiY7wPOMljt6XFlmnAHACz0JhAg/hAHHYF8bdJJZ4slZrM
kNRQ0ZUDVDvacygKjeXDjneCvFrS+78ancE7gGGkZMaxWf4NsQVCoX3wRMuk6cHs
mrwGINYWGCHshjLAnzYwPvLegvlPszh1zhgzziMGNU08wf/q8WOrZmrtHB4epWhI
aSEjNIZmDlbkyy8SwW4y/7GjVKNLpnObUHh7qqBDnmWd5HnMWAEuHxbAhMXqIWIS
UKe8cwnFBWHpHCXMCyoCI1uJNhfjtj2hq7QKkejH/jCJ5U26pQ==
---- END SSH2 PUBLIC KEY ----
```

RSA Encryption

- Let $x < n$ be the message to be encrypted
- Alice encrypts it as

$$C = x^e \bmod n$$

  - (n, e) is Bob's public key
  - Sends C to Bob
  - C may be eavesdropped
- Security
  - Exponentiation can be computed efficiently
    - Proportional to the length of the key
  - Computing x from (n, e) is believed to be difficult
    - Similar to discrete logarithm

RSA Decryption

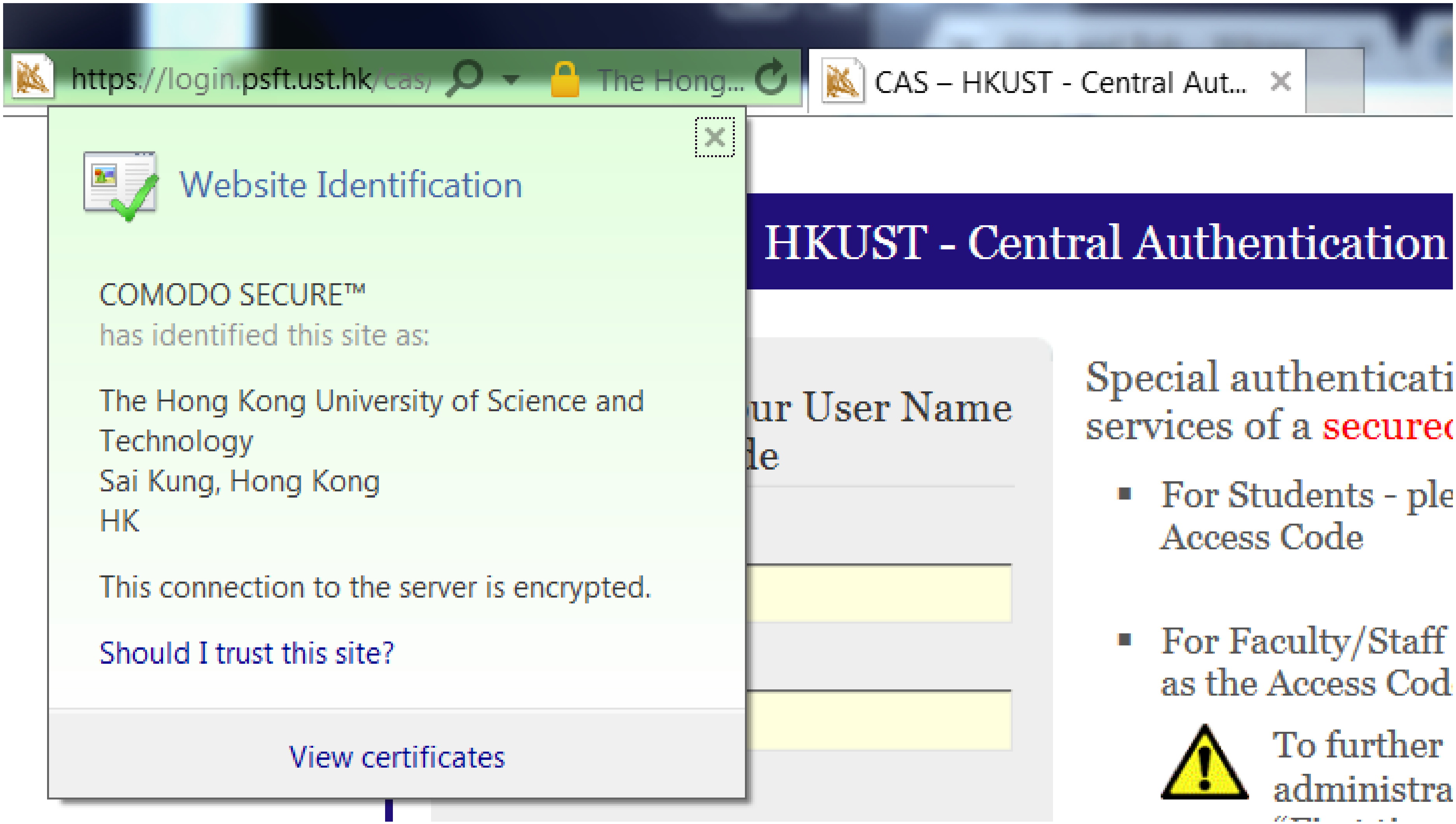
- Bob receives C
- Bob decrypts x from C using his private key (p, q)
  - Find d, the inverse of e modulo $(p-1)(q-1)$, i.e.,
    $$de \equiv 1 \pmod{(p-1)(q-1)}$$
  - Compute
    $$C^d \bmod n$$
- Will show later that $C^d \equiv (x^e)^d \equiv x^{de} \equiv x \pmod{n}$
- Security:
  - It's hard to find d without knowing (p, q)

RSA in Use

- Sending secrete messages
  - Divide message into blocks such that each block is a number $< n$
  - Alice encodes her message using Bob's public key
  - Bob decodes the message using his private key
- Digital signatures (authentication)
  - Alice encodes her message using her private key
    - Computes $C = x^d \bmod n$
  - Bob (or anyone else) decodes the message using Alice's public key
    - Computes $C^e \bmod n = x^{de} \bmod n = x$
  - He will know the message indeed came from Alice

RSA in Use

- How to prevent man-in-the-middle attacks?
- How to make sure that Alice's public key indeed belongs Alice?
- Certificate authority (CA)
  - A small number of trusted third parties: Comodo, Symantex, GoDaddy, GlobalSign, …
- How to make sure that a CA's public key indeed belongs to that CA?
- Built into Internet browsers
- How can I trust my browser and the CAs?
- Well, you have to …


https://login.psft.ust.hk/cas/
The Hong...
CAS – HKUST - Central Aut...
Website Identification
COMODO SECURE™
has identified this site as:
The Hong Kong University of Science and Technology
Sai Kung, Hong Kong
HK
This connection to the server is encrypted.
Should I trust this site?
View certificates
HKUST - Central Authentication
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RSA: Correctness

- **Proof plan**

  We want to show

$$C^d = x^{de} \equiv x \pmod{n}.$$

  Step 1: Show that

$$x^{de} \equiv x \pmod{p}$$
$$x^{de} \equiv x \pmod{q}$$

  Step 2: Show that

$$x^{de} \equiv x \pmod{pq}$$

Fermat's Little Theorem

Pierre de Fermat
(1601-1665)

- **Theorem**
  If p is prime and a is an integer not divisible by p, then
  $$a^{p-1} \equiv 1 \pmod{p}$$
- Proof omitted
- Useful in computing the remainders of large powers
- **Example:**
  Find $7^{222} \bmod 11$.
  By the theorem, we know that $7^{10} \equiv 1 \pmod{11}$, and so $(7^{10})^k \equiv 1 \pmod{11}$, for any positive integer k.
  Therefore,
  $$7^{222} = 7^{22 \cdot 10+2} = (7^{10})^{22} \cdot 7^2 \equiv 1^{22} \cdot 49 \equiv 5 \pmod{11}$$

RSA Correctness Step 1

- **Proof**

  We know d is the inverse of e modulo $(p-1)(q-1)$, so
  $$de = 1 + k(p-1)(q-1).$$
  It follows that
  $$\begin{aligned} C^d &\equiv (x^e)^d \pmod{p} \\ &\equiv x^{de} \pmod{p} \\ &\equiv x^{1+k(p-1)(q-1)} \pmod{p} \end{aligned}$$

  Case 1: $x^{k(q-1)}$ is not a multiple of p.

  Applying Fermat's Little Theorem with $a = x^{k(q-1)}$:
  $$x^{k(p-1)(q-1)} \equiv 1 \pmod{p},$$
  so
  $$x^{1+k(p-1)(q-1)} \equiv x \pmod{p},$$

RSA Correctness Step 1 (cnt'd)

- Case 2: $x^{k(q-1)}$ is a multiple of p.
  Then
  $$x^{k(p-1)(q-1)} \equiv 0 \pmod{p}$$
  $$x^{1+k(p-1)(q-1)} \equiv 0 \pmod{p}$$
  On the other hand, since $x^{k(q-1)}$ is a multiple of p and p is a prime, then x must be a multiple of p. So
  $$x \equiv 0 \pmod{p}$$
  Thus in this case, we have
  $$x^{1+k(p-1)(q-1)} \equiv x \equiv 0 \pmod{p}.$$
- The proof for $x^{de} \equiv x \pmod{q}$ is symmetric.

RSA Correctness Step 2

- A simple property of prime numbers
  - If p and q are both primes and $p \mid z, q \mid z$, then $pq \mid z$
- Proof of Step 2:
  - We already have

$$x^{de} \equiv x \pmod{p}$$
$$x^{de} \equiv x \pmod{q}$$

  - So,

$$p \mid \left(x^{de} - x\right)$$
$$q \mid (x^{de} - x)$$

  - Therefore,

$$pq \mid (x^{de} - x)$$
$$x^{de} \equiv x \pmod{pq}$$