COMP 2711H Discrete Mathematical Tools for Computer Science
2021 Fall Semester
Homework 5: Induction and Recursion
Due: Not for submission

Problem 1. Use mathematical induction to show that if a, b, and c are the lengths of the sides of a right triangle, where c is the length of the hypotenuse, then $a^n + b^n < c^n$ for all integers n with $n \geq 3$.

Problem 2. Suppose that among a group of cars on a circular track there is enough fuel for one car to complete a lap. Use mathematical induction to show that there is a car in the group that can complete a lap by obtaining gas from other cars as it travels along the track.

Problem 3. Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample.

(a) $\log_2 f(n)$ is $O(\log_2 g(n))$.

(b) $2^{f(n)}$ is $O(2^{g(n)})$.

(c) $(f(n))^2$ is $O((g(n))^2)$.

Problem 4. Let H_n denote the n-th harmonic number. Use mathematical induction to prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Problem 5. Use mathematical induction to prove that a two-dimensional $2^n \times 2^n$ checkerboard with one 1×1 square missing can be completely covered by 2×2 squares with one 1×1 square missing.

Problem 6. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \to P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

Problem 7. The *reversal* of a string is the string consisting of the symbols of the string in reverse order. The reversal of the string w is denoted by w^R.

(a) Give a recursive definition of the reversal of a string. (*Hint:* First define the reversal of the empty string. Then write a string w of length $n+1$ as xy, where x is a string of length n, and express the reversal of w appropriately.)

(b) Use induction to prove that $(w_1 w_2)^R = w_2^R w_1^R$.

Problem 8. Use induction to show that $\ell(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal nodes of T.

The set of leaves and the set of internal nodes of a full binary tree can be defined recursively as follows.

Basis step: The root r is a leaf of the full binary tree with exactly one node r. This tree has no internal nodes.

Recursive step: The set of leaves of the tree is the union of the sets of leaves of T_1 and of T_2. The internal nodes of T are the root r of T and the union of the set of internal nodes of T_1 and the set of internal nodes of T_2.

Problem 9. A *ternary* string is a string that contains only 0's, 1's, and 2's.

(a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0's. How many ternary strings of length six contain two consecutive 0's?

(b) Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same. How many ternary strings of length six do not contain consecutive symbols that are the same?

Problem 10. In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we canot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk. Write a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restricton, and solve it.