COMP 2711H Discrete Mathematical Tools for Computer Science
Tutorial Problems: Discrete Probability

Problem 1. Timothy can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will late. The probability that he is late when he rides his bicycle is only 5%. Timothy arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

(a) Suppose the boss assumes that there is a $1/3$ chance that Timothy takes each of the three ways he can get to work. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem under this assumption?

(b) Suppose the boss knows that Timothy drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem using this information?

Problem 2. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

(a) What is the expected value of X_n?

(b) What is the variance of X_n?

Problem 3. Consider the following game. A person flips a coin repeatedly until a head comes up. This person receives a payment of 2^n dollars if the first head comes up at the n-th flip.

(a) Let X be a random variable equal to the amount of money the person wins. Show that the expected value of X does not exist (that is, it is infinite). Show that a *rational gambler*, that is, someone willing to pay to play the game as long as the price to play is not more than the expected payoff, should be willing to wager any amount of money to play this game. (This is knows as the *St. Petersburg paradox*. Why do you suppose it is called a paradox?)

(b) Suppose that the person receives 2^n dollars if the first head comes up on the n-th flip where $n < 10$ and $2^{10} = 1024$ dollars if the first head comes up on or after the tenth flip. What is the expected value of the amount of money the person wins? How much should a rational gambler be willing to pay to play this game?

Problem 4. Let X be a nonnegative random variable with mean μ. Show that for any positive real a,

$$p(X \geq a) \leq \mu/a.$$

This inequality is called *Markov's inequality*.

Problem 5. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $10\sqrt{n}$.

Problem 6. You are given a biased coin (that is, the probability of getting "head" is not equal to $1/2$). Describe how you can use this biased coin to simulate a fair coin. In order to simulate one toss of a fair coin, you are allowed to toss the biased coin more than once. Further, you may assume that the probability of getting head for the biased coin remains the same for each coin toss.