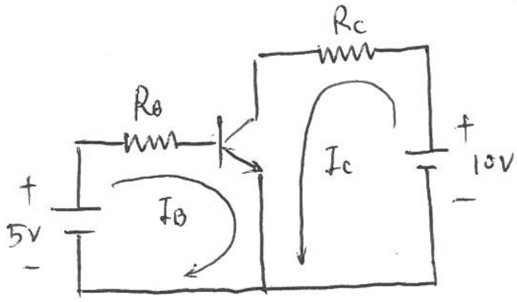
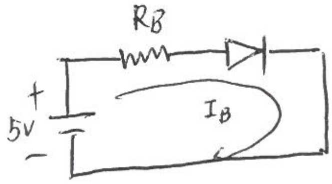
L9. Circuit Analysis

We have done some circuit analysis, such as calculating I_B in the transistor circuit.



To do that, we look at the left loop :



and get $I_B = \frac{5-0.7}{R_B}$

① Ω's law (Component I-V)
② Circuit law

What is the physics background for this ?

Are there any systematic ways for circuit analysis ?

1. Terms for circuit analysis :

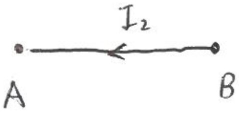
1) Node : an electrical point connecting terminals of two or more circuit elements.

2) Branch : circuit element between two nodes.

3) Loop : any circuit branch that ends at its starting node, without passing an intermediate node more than once.

4) Current :

$I_{AB} = I_1$



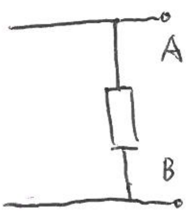
$I_{BA} = I_2$

① Label direction
② Actual direction.

$I_{AB} = -I_{BA}$

The actual current direction is labeled as the direction on which positive charges flow. So, $I_{AB} = 1A$ / $I_{AB} = -1A$ represent currents on different directions.

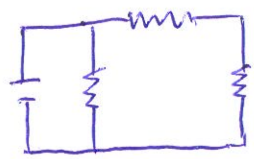
5) Voltage drop :



$V_{AB} = V_A - V_B$, $V_{BA} = -V_{BA}$

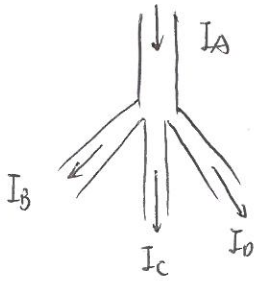
$V_A = V_A - GND$.

6) Example : loops/nodes



2. Kirchhoff's current law (KCL)

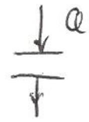
Physics basis : Conservation of charges (charges can't be created or destroyed).



KCL : The algebraic sum of all branch currents entering and leaving a node is zero at all instants of time.

Three forms :
① Entering : $I_A - I_B - I_C - I_D = 0$
② Leaving : $-I_A + I_B + I_C + I_D = 0$
③ Entering = Leaving : $I_A = I_B + I_C + I_D$

① Capacitor



3. Kirchhoff's voltage law (KVL)

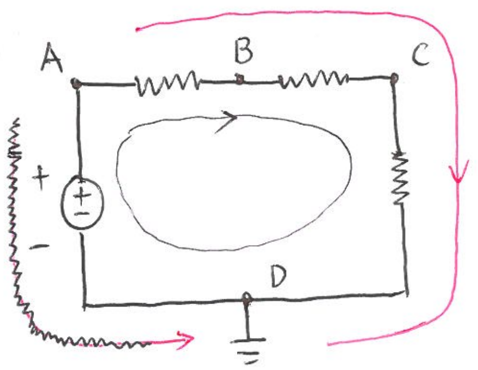
(Gravitational field ?)

Physics Basis : Conservation of energy $\Rightarrow$ Consider moving a charge around a loop.

KVL : The algebraic sum of all branch voltages around any loop is zero at all instants of time.

② Phone example

Example :



$$E_{AB} + E_{BC} + E_{CD} + E_{DA} = 0$$

$$Q(V_{AB} + V_{BC} + V_{CD} + V_{DA}) = 0 \quad Q \neq 0$$

$$\Rightarrow V_{AB} + V_{BC} + V_{CD} + V_{DA} = 0$$

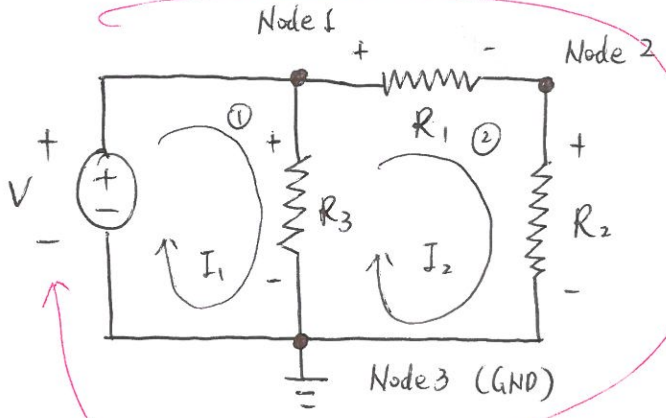
Two forms :

$$V_{AB} + V_{BC} + V_{CD} + V_{DA} = 0$$

$$V_{AB} + V_{BC} + V_{CD} = -V_{DA} = V_{AD}$$

: The algebraic sums of all branch voltages on any path between two nodes are equal.

4. KVL Example :



Given V, R_1 R_2 R_3,
5V = V, 50Ω = R_1, 75Ω = R_2, 25Ω = R_3

Solve I_1 I_2.

General procedure :

Step 1 : Define / Label loops. L_1 & L_2

Step 2 : Set up KVL equation for all loops.

$V_{13} + V_{31} = 0$

L_1 : $V_{13} - V = 0$ [Use the Loop current direction (arrow) to determine the "+" "−" sign before the voltage.

For example, ↻ hits R_3 at the "+" terminal. Then, "+" V_{13}. ↻ hits the voltage source at the "−" terminal, So, "−" V.

L_2 : $V_{12} + V_{23} - V_{13} = 0$ [$V_{12} + V_{23} + V_{31} = 0$]

Step 3 : Write voltages in terms of loop currents (I_1, I_2). Ω's law

$$\begin{cases} V_{13} - V = 0 \\ V_{12} + V_{23} - V_{13} = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} I_{13} \cdot R_3 - V = 0 \\ I_2 \cdot R_1 + I_2 R_2 - I_{13} \cdot R_3 = 0 \end{cases}$$

what is I_{13} ?

I_{13} is the "actual total" current entering R_3 from node 1 $\Rightarrow$ So, $I_{13} = I_1 - I_2$

$$\Downarrow$$

$$\begin{cases} (I_1 - I_2) R_3 - V = 0 \\ I_2 R_1 + I_2 R_2 - (I_1 - I_2) R_3 = 0 \end{cases}$$

two equations & two unknowns.

$$\Downarrow$$

$$\begin{cases} R_3 I_1 - R_3 I_2 = V \\ -R_3 I_1 + (R_1 + R_2 + R_3) I_2 = 0 \end{cases}$$

Consider $V = 5V$
$R_1 = 50\,\Omega$
$R_2 = 75\,\Omega$
$R_3 = 25\,\Omega$

$$\Downarrow$$

Step 4 : Solve the equations.

$$\begin{cases} 25 I_1 - 25 I_2 = 5 & ① \\ -25 I_1 + (50 + 75 + 25) I_2 = 0 & ② \end{cases}$$

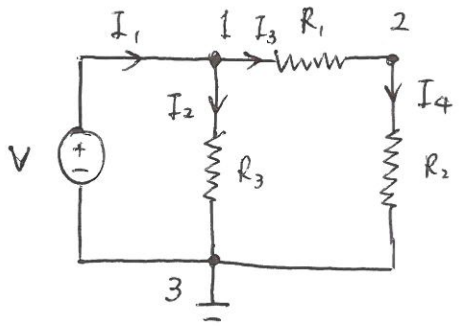
$① + ② \Rightarrow 125 I_2 = 5 \Rightarrow I_2 = 0.04 A$

$\Downarrow ①$

From ①, $25 I_1 = 5 + 25 \times 0.04 = 6 \Rightarrow I_1 = 0.24 A$.

Then, we can determine V_1 V_2 ------

5. KCL Example : [Nodal Analysis]



$V = 5\,V$

$R_1 = 50\,\Omega$

$R_2 = 75\,\Omega$

$R_3 = 25\,\Omega$

Determine all voltages & currents in the circuit.

Step 1 : For a circuit with "n" nodes ($n = 3$, here), make one node as the GND (3, here) and apply KCL to the other "n-1" nodes.

Step 2 : Set up KCL equations for nodes 1 & 2. [We need to label some currents]

N_1 : $I_1 = I_3 + I_2$

N_2 : $I_3 = I_4$

"Ω's law"

Step 3 : Express branch currents in terms of node voltages. (V_1, V_2)

N_1 : $I_1 = \frac{V_1 - V_2}{R_1} + \frac{V_1}{R_3}$ [Note that we can't do this for I_1]

N_2 : $\frac{V_1 - V_2}{R_1} = \frac{V_2}{R_2}$ [Three unknowns ?]

Step 4 : Express the node voltages in terms of known voltage ($V_1 = V$)

N_1 : $I_1 = \frac{V - V_2}{R_1} + \frac{V}{R_3}$

N_2 : $\frac{V - V_2}{R_1} = \frac{V_2}{R_2}$

two equations & two variables (I_1, V_2) $\Longrightarrow$ solve them.

Numerical Results :

$$\begin{cases} I_1 = \dfrac{V-V_2}{R_1} + \dfrac{V}{R_3} & \text{①} \\ \dfrac{V-V_2}{R_1} = \dfrac{V_2}{R_2} & \text{②} \end{cases}$$

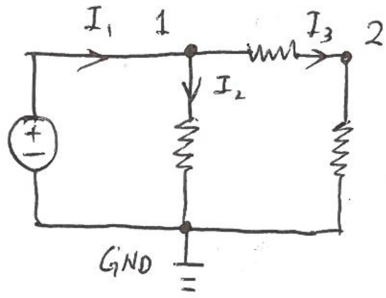
② $\dfrac{V}{R_1} = \dfrac{V_2}{R_1} + \dfrac{V_2}{R_2} \Rightarrow \dfrac{V}{R_1} = \left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) V_2 \Rightarrow \dfrac{V}{R_1} = \dfrac{R_1+R_2}{R_1 R_2} V_2$

$$\Rightarrow V_2 = \frac{R_2}{R_1+R_2} V = \frac{75}{50+75} \times 5 = 3V$$

① $I_1 = \dfrac{5-3}{50} + \dfrac{5}{25} = \dfrac{12}{50} = \dfrac{6}{25} A$.

Then, we can determine other values

6. GND : ☆



For the above analysis, we use node 3 as GND.

We get I_1 I_2 I_3

$V_1 = 5V$, $V_2 = 3V$, $V_3 = 0$

Now, what happens if we change the GND, say let $V_2 = 0$.

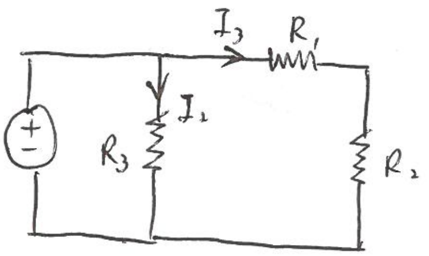
Will this change I_1 I_2 I_3 ? No

Will this change V_1 V_2 V_3 ? Yes, $\Rightarrow$ $V_1 = 2V$, $V_2 = 0$, $V_3 = -3V$

But, $V_1 - V_2$, $V_1 - V_3$ & $V_2 - V_3$ don't change.

So, GND is just a reference.

Some students have found that, in fact, we don't need KCL & KVL for analysing the above circuit.

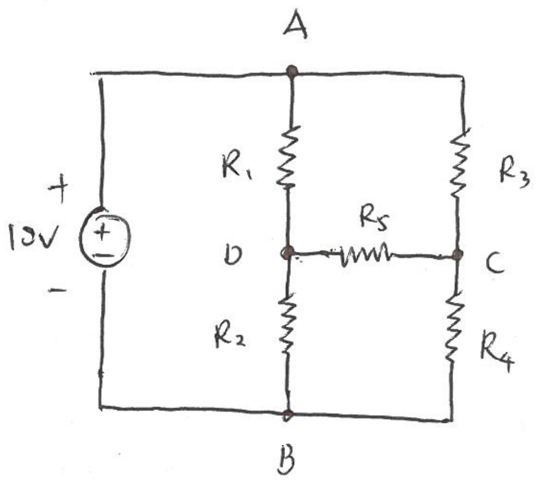


In fact, $I_1 = \frac{V}{R_3}$

$I_3 = \frac{V}{R_1+R_2}$

But, the typical circuit we meet in practice is not that simple:

7. A Complex example:



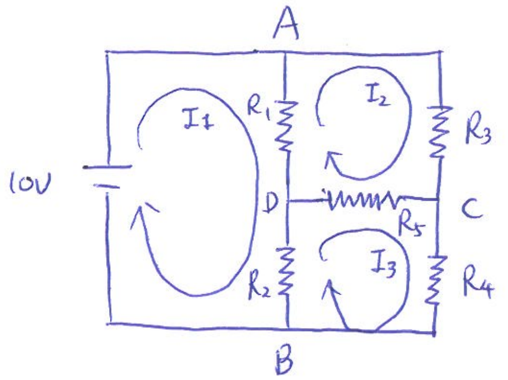
There are 4 nodes & 6 branches (containing one circuit component.)

Can we find a simple method to solve the circuit?

No.

No worries! We call KCL & KVL the "systematic" circuit analysis methods, because they can be utilized for solving "any" circuit.

Step 1 : Label Loops to include all branches.



Note that there are other ways to label loops. (Compare to your lecture notes.)

Step 2 : Write down the loop equations (KVL)

$$\begin{cases} L1: & V_{AD} + V_{DB} + V_{BA} = 0 \\ L2: & V_{AC} + V_{CD} + V_{DA} = 0 \\ L3: & V_{DC} + V_{CB} + V_{BD} = 0 \end{cases}$$

Here, $V_{BA} = -10\,V$

Step 3 : Express voltages in terms of Loop current.

V_{AD} is the voltage drop over R_1. By Ω's law, it should be $I_{AD} \times R_1$

Here, I_{AD} is the effective current going through R_1 with $I_{AD} = I_1 - I_2$

Similarly, $I_{DB} = I_1 - I_3$. Thus, we have

$$\begin{cases} L1: & \underbrace{(I_1 - I_2)R_1}_{V_{AD}} + \underbrace{(I_1 - I_3)R_2}_{V_{DB}} \underbrace{- 10}_{V_{BA}} = 0 \\ L2: & \underbrace{I_2 R_3}_{V_{AC}} + \underbrace{(I_2 - I_3)R_5}_{V_{CD}} + \underbrace{(I_2 - I_1)R_1}_{V_{DA} = -V_{AD}} = 0 \\ L3: & \underbrace{(I_3 - I_2)R_5}_{V_{DC}} + \underbrace{I_3 R_4}_{V_{CB}} + \underbrace{(I_3 - I_1)R_2}_{V_{BD}} = 0 \end{cases}$$

$\longrightarrow$ Note that here we didn't label the positive & negative terminals, but just use the fact $\underset{D \Rightarrow A}{V_{DA}} = \underset{D \Rightarrow A}{I_{DA}} \cdot R_1$

Step 4 : Reorganize the three equations with respect to I_1 I_2 I_3

$$\begin{cases} (R_1+R_2)I_1 - R_1 I_2 - R_2 I_3 = 10 & L_1 & ① \\ -R_1 I_1 + (R_1+R_2+R_5) I_2 - R_5 I_3 = 0 & L_2 & ② \\ -R_2 I_1 - R_5 I_2 + (R_2+R_4+R_5) I_3 = 0 & L_3 & ③ \end{cases}$$

3 Variables & 3 equations $\Longrightarrow$ Done!

After getting I_1, I_2, I_3 we can determine currents going through all components. For example, $I_{R_1} = I_1 - I_2$, $I_{R_2} = I_1 - I_3$ .....

※ Remarks : Look at equations ① ② ③. What can you observe.

$$① \Longrightarrow \underbrace{(R_1+R_2)I_1}_{} - \underbrace{R_1 I_2}_{} \underbrace{-R_3 I_3}_{} = 10$$

Contribution of I_1 to loop 1 voltage.

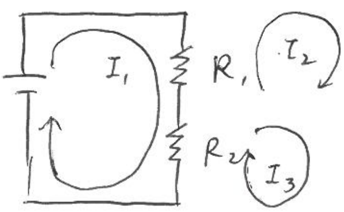
I_1 goes through two resistors.

Contribution of I_3 to loop 1 voltage

I_3 goes through R_3

(In the opposite direction, "−")

Basically, loop 1 has three components

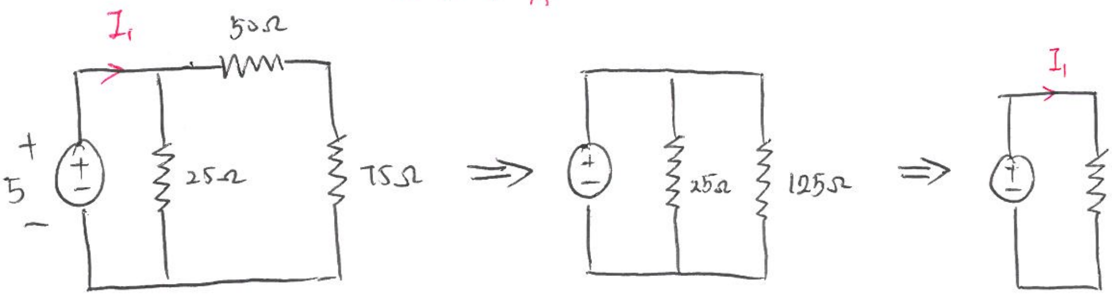


Three loop currents contribute to the loop voltages through different components and in different ways ("+" "−")

8 Circuit Simplification

For the above circuit, we can simplify the analysis without using KVL.

※ $R = \rho \cdot \frac{L}{A}$



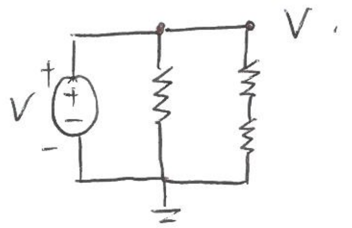
$\frac{25 \times 125}{25 + 125} \Omega$

$= \frac{25 \times 125}{150} = \frac{125}{6} \Omega$

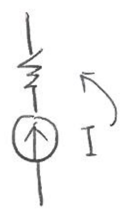
$I_1 = \frac{5}{\frac{125}{6}} = \frac{5 \times 6}{125} = \frac{6}{25} = 0.24A$

Some general rules:

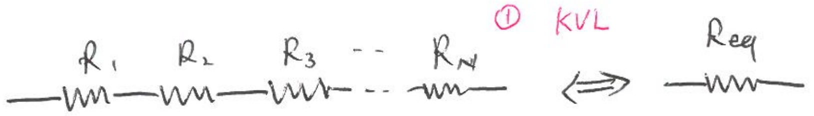
※ Branch voltage in parallel with a voltage source is known.



※ Branch current in series with a current source is known.

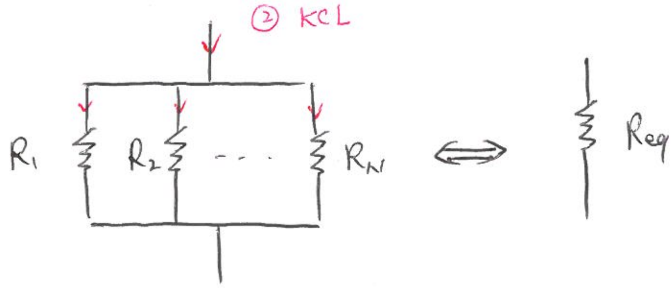


① KVL

※ 

$\Rightarrow R_{eq} = R_1 + R_2 + \cdots R_N$

② KCL

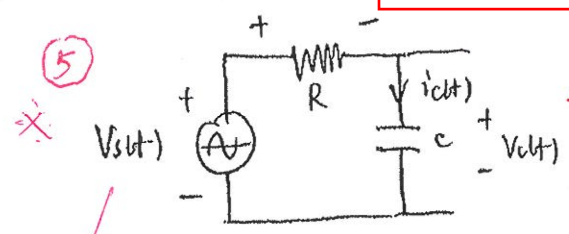
※ 

$\Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_N}$

$\Rightarrow R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_N}}$

This is just for fun. Read it if you are interested to see how we design a filter.

(5)



I-V for C:

$$i_c(t) = C\frac{dV_c(t)}{dt}$$

KVL: $V_s(t) = R\,i_c(t) + V_c(t)$

$$= RC\frac{dV_c(t)}{dt} + V_c(t)$$

$$V_s\cos(\omega t) = CR\frac{dV_c(t)}{dt} + V_c(t)$$

$$\Rightarrow \frac{dV_c(t)}{dt} + \frac{1}{RC}V_c(t) = \frac{V_s}{RC}\cos(\omega t) \qquad (1) \quad \text{Define } RC = \tau.$$

We need $V_c(t)$, whose derivative + itself is a sinusoid. ⟶ it should be a sinusoid.

$\Rightarrow$ Let's assume $V_c(t) = a\cos(\omega t) + b\sin(\omega t)$

$$\frac{dV_c(t)}{dt} = -a\omega\sin\omega t + b\omega\cos(\omega t) \xrightarrow{(1)} -a\omega\sin\omega t + b\omega\cos\omega t + \frac{a}{\tau}\cos\omega t + \frac{b}{\tau}\sin\omega t = \frac{V_s}{\tau}\cos\omega t$$

$$\Rightarrow \left(b\omega + \frac{a}{\tau}\right)\cos\omega t + \left(\frac{b}{\tau} - a\omega\right)\sin\omega t = \frac{V_s}{\tau}\cos\omega t$$

$$\Rightarrow \left.\begin{aligned} & b\omega + \frac{a}{\tau} = \frac{V_s}{\tau} \\ & \frac{b}{\tau} = a\omega \Rightarrow b = a\omega\tau \end{aligned}\right\} \Rightarrow a\tau\omega^2 + \frac{a}{\tau} = \frac{V_s}{\tau}$$

$$\Rightarrow a = \frac{V_s}{1+\omega^2\tau^2}$$

$$b = \frac{\omega\tau V_s}{1+\omega^2\tau^2}$$

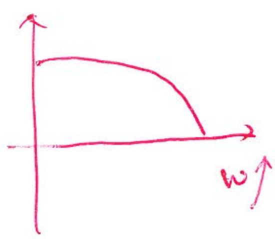
$$\Rightarrow V_c(t) = a\cos\omega t + b\sin\omega t$$

$$= \sqrt{a^2+b^2}\left[\cos\omega t\,\frac{a}{\sqrt{a^2+b^2}} + \sin\omega t\,\frac{b}{\sqrt{a^2+b^2}}\right]$$

$$= \sqrt{a^2+b^2}\cos(\omega t - \theta) \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(\omega\tau)$$

$$= \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\cos(\omega t - \theta)$$

$$= \frac{V_s}{\sqrt{1+R^2C^2\omega^2}}\cos(\omega t - \theta) \qquad \theta = \tan^{-1}(\omega RC) \qquad \omega \nearrow$$



$V_c(t) \longrightarrow i_c(t) \longrightarrow V_R(t)$

Low-pass filter