MATH 2011, L1A&B, Spring 2017-18

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1. Compute the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$, in polar coordinate. $(\sin(\pi/6) = 1/2)$.

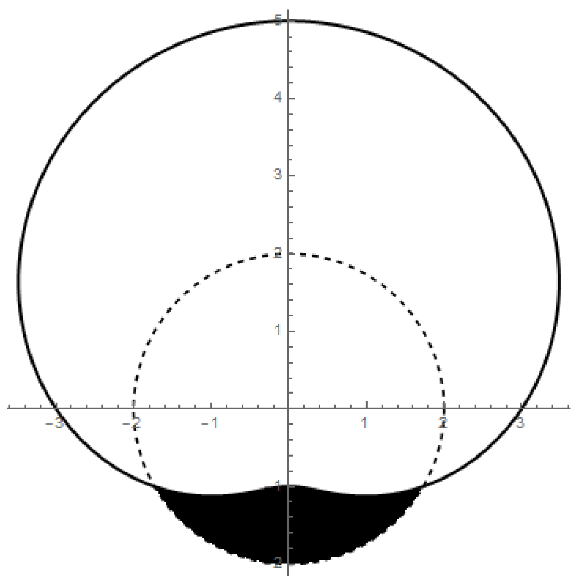

Figure 1: Thick line is $r = 3 + 2\sin\theta$, dash line is $r = 2$. Compute the area of shadowed region.

Solution 1. *First, to find intersections, solve*

$$\begin{cases} r = 3 + 2\sin\theta, \\ r = 2. \end{cases}$$

Equivalently,

$$3 + 2\sin\theta = 2, \ \textit{or,} \ \sin\theta = -\frac{1}{2}$$

we get,

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

The area is then given by,

$$\frac{1}{2}\int_{11\pi/6}^{7\pi/6} [2^2 - (3 + 2\sin\theta)^2] d\theta = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}.$$

2. (1) Write down the formula of orthogonal projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ through inner product.

   (2)Denote the orthogonal projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ by $Proj_{\boldsymbol{b}}(\boldsymbol{a})$. Prove that $\boldsymbol{b} - Proj_{\boldsymbol{b}}(\boldsymbol{a})$ is perpendicular to $\boldsymbol{a}$.

   Solution 2. *(1)*

$$Proj_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$$

   (2)

$$(\boldsymbol{b} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a})\cdot\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b} = 0.$$

3. Find the constant c defined as follows:
   (1)$c = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{u}}$
   (2)$\boldsymbol{u}\times\boldsymbol{v} = c\boldsymbol{v}\times\boldsymbol{u}$
   (3)$c = (a\boldsymbol{u} + b\boldsymbol{v})\cdot(\boldsymbol{u}\times\boldsymbol{v})$, where a, b are two arbitrary constants.

   Solution 3. *(1) $c = 1$, since inner product is symmetric.*
   (2) $c = -1$, since cross product is antisymmetric.
   (3) $c = 0$, since inner product is bilinear and the cross product of given two vectors is a vector that perpendicular to two given vectors.

4. Find the vector-valued function for the line tangent to the curve

$$\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \sqrt{3}\sin t\boldsymbol{j} + 2\cos t\boldsymbol{k},$$

   at $t = \frac{\pi}{4}$.

   Solution 4. *Tangent vector: $\boldsymbol{r}'(t) =< \cos(t), \sqrt{3}\cos(t), -2\sin(t) >$, $\boldsymbol{r}'(\frac{\pi}{4}) =< \sqrt{2}/2, \sqrt{6}/2, -\sqrt{2} >$, $\boldsymbol{r}(\pi/4) =< \sqrt{2}/2, \sqrt{6}/2, \sqrt{2} >$, so the equation of tangent line is*

$$\boldsymbol{r}(\frac{\pi}{4}) + \boldsymbol{r}'(\frac{\pi}{4})t =< \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t, \frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}t, \sqrt{2} - \sqrt{2}t >$$

5. Compute the length of the curve $\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \sqrt{3}\sin t\boldsymbol{j} + 2\cos t\boldsymbol{k}$ for the segment of $0 \leq t \leq \pi/2$.

   Solution 5.

$$|\boldsymbol{r}'(t)| = \sqrt{4} = 2,$$

$$Length = \int_0^{\pi/2} 2\,dt = \pi.$$

6. Given four points $A(2,1,0), B(1,1,1), C(3,0,1), D(1,0,2)$.
   (1) Find the equations of the plane ABC and plane BCD respectively.
   (2) Find the vector-valued function for the intersection line of the above two planes.

   Solution 6. *(1) Plane ABC, normal vector, $\boldsymbol{n}_1 = \vec{AB}\times\vec{AC} =< 1, 2, 1 >$.*

$$< 1, 2, 1 > \cdot(< x, y, z > - < 2, 1, 0 >) = 0,$$

$$x + 2y + z = 4,$$

   Plane BCD, normal vector, $\boldsymbol{n}_2 = \vec{BC}\times\vec{BD} =< -1, -2, -2 >$.

$$< -1, -2, -2 > \cdot(< x, y, z > - < 1, 1, 1 >) = 0,$$

$$x + 2y + 2z = 5,$$

(2) Note that the intersection line is BC, direction vector : $< 2, -1, 0 >$, equation of line,

$$< 1, 1, 1 > + < 2, -1, 0 > t =< 1 + 2t, 1 - t, 1 >$$

Also, you may as usual find direction vector by doing cross product of normal vectors.