COMP 2711H Discrete Mathematical Tools for Computer Science
Solutions to Tutorial 2

QB1-4. Use existential and universal quantifiers to express the statement "Everyone has exactly two biological parents" using the propositional function $P(x, y)$ which represents "x is the biological parent of y."

Solution: Let the domain consist of all people in the world.

$$\forall x \exists y \exists z, [(y \neq z) \wedge (\forall w, (w = y) \vee (w = z) \leftrightarrow P(w, x))]$$

QB1-5. Prove that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5. (*Hint:* Use a proof by cases, where the cases correspond to the possible remainders for the integer when it is divided by 5.)

Solution: For all $i \in \mathbb{Z}$, i can be written as $i = 5q + r$ for some $q \in \mathbb{Z}$ and some $r \in \{0, 1, 2, 3, 4\}$. If i is not divisible by 5, r cannot be 0. There are four cases remaining.

- case 1: $i = 5q + 1$. Then we have $i^2 = 25q^2 + 10q + 1$. Then divide i^2 by 5, the remainder is 1.
- case 2: $i = 5q + 2$. Then we have $i^2 = 25q^2 + 20q + 4$. Then divide i^2 by 5, the remainder is 4.
- case 3: $i = 5q + 3$. Then we have $i^2 = 25q^2 + 30q + 9$. Then divide i^2 by 5, the remainder is 4.
- case 4: $i = 5q + 4$. Then we have $i^2 = 25q^2 + 40q + 16$. Then divide i^2 by 5, the remainder is 1.

In conclusion, if i is not divisible by 5, then dividing i^2 by 5 leaves a remainder of 1 or 4.

QB1-7. Prove that there is no rational number r for which $r^3 + r + 1 = 0$.

Solution: Suppose, for the sake of contradiction, that there is some rational solution $\frac{p}{q}$ where p and q are two co-prime integers. Then we have that

$$\frac{p^3}{q^3} + \frac{p}{q} + 1 = 0,$$

or, equivalently,

$$p^3 + pq^2 + q^3 = 0. \qquad (*)$$

Next we will show that no co-prime integers p and q will make equation $(*)$ hold, which leads to a contradiction. Since p and q are co-prime, there are three cases.

(1) p is odd, and q is odd. Then we have that p^3, pq^2, and q^3 are all odd. As a result $p^3 + pq^2 + q^3$ is also odd and cannot be 0.

(2) p is odd, and q is even. Then we have that p^3 is odd, pq^2 is even, and q^3 is even. Again, $p^3 + pq^2 + q^3$ is odd and cannot be 0.

(3) p is even, and q is odd. Similar to that in (ii), we can show $p^3 + pq^2 + q^3$ is odd and cannot be 0.

QB1-8. A *triangle* is a set of three people such that either every pair has shaken hands or no pair has shaken hands. Prove that among every six people there is a triangle.

Solution: Let p be one of these six people. One of the following two cases must hold for the remaining five people.

(1) At least three of them have shaken hands with p. Now consider these three people. If any two of them have shaken hands, then these two people, together with p, form a triangle. Otherwise, no pair within the three people have shaken hands, they form a triangle.

(2) At least three of them haven't shaken hands with p. The proof is similar to that in (i).

EP1-11. Let the universe be the set of all positive integers. Are the following quantified statements true?

(a) $\exists x(\forall y(x^2 < y + 1))$

(b) $\forall x(\exists y(x^2 + y^2 < 12))$

(c) $\forall x(\forall y(x^2 + y^2 > 0))$

Solution:

(a) T. (x = 1)

(b) F. (x = 4)

(c) T.

EP1-14. Prove that $p \to \neg q$, $r \to q$ and r imply $\neg p$.

Solution: The question wants us to prove $((p \to \neg q) \wedge (r \to q) \wedge r) \to \neg p$.

$p \to \neg q \equiv q \to \neg p$ because of contrapositive equivalence.

By hypothetical syllogism,

$$(r \wedge (r \to q)) \to q$$

$$(q \wedge (q \to \neg p)) \to \neg p$$

So we get $\neg p$ finally.