ELEC 2600 Probability and Random Processes in Engineering

2012 Spring Semester - Midterm Exam Paper with Marking Scheme

March 16, 2012

SECTION I: Multiple Choice [20 Marks]

1. [4'] Which of the following statements is wrong? (B)

   A. The sub-experiments in a sequential experiment can be either dependent or independent;

   B. A uniform random variable has flat PDF, so its variance will be 0;

   C. The Binomial distribution describes the number of successes in several independent trails;

   D. Poisson distribution is an approximation to the Binomial distribution with large n and small p.

2. [4'] We perform n independent sequential experiments. In each experiment, there're three outcomes A, B, and C. Also, $P[A] = p$, $P[B] = q$, and $P[C] = 1 - p - q$. What's the probability that, among the n experiments, A occurs a times and B occurs b times ($a \leq n$, $b \leq n$)? (D)

   A. $\frac{n!}{a!b!} p^a q^b$;

   B. $\frac{n!}{a!(n-a)!} p^a \times \frac{n!}{b!(n-b)!} q^b$;

   C. $\frac{(n!)^2}{a!b!(n-a)!(n-b)!} p^a q^b (1-p-q)^{n-a-b}$;

   D. $\frac{n!}{(n-a-b)!a!b!} p^a q^b (1-p-q)^{n-a-b}$.

3. [4'] We want to measure the distance from the "redbird" to the seafront cafeteria. For each measurement, the error X follows Gaussian distribution with mean $\mu = 0$ and variance $\sigma^2 = 9$. What is the probability that $|X| > 10$? (A)

   (Hint: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} e^{-\frac{t^2}{2}} dt$)

   A. $2\Phi(-\frac{10}{3})$;

   B. $2Q(\frac{10}{9})$;

   C. $1 - 2Q(\frac{10}{3})$;

   D. $1 - 2\Phi(-\frac{10}{9})$.

4. [4'] Suppose we roll a fair dice (each digit occurs with equal probability) and we want to get a six. What's the probability that it will take 4 more trials until we finally get a six for the first time, given that we have already thrown it for 2 times and get two non-six digits? (C)

   A. $(\frac{1}{6})^3 \cdot \frac{5}{6}$;

   B. $(\frac{5}{6})^5 \cdot \frac{1}{6}$;

   C. $(\frac{5}{6})^3 \cdot \frac{1}{6}$;

   D. $(\frac{1}{6})^5 \cdot \frac{5}{6}$.

5. [4'] In an experiment, one of the three events A_1, A_2 and A_3 occurs. Also $P[A_1] = 0.4$, $P[A_2] = 0.1$, and $P[A_3] = 0.5$. Random variable X follows exponential distribution given A_i ($i = 1,2,3$) happens. Besides, $\mathrm{VAR}[X|A_1] = 3.24$, $\mathrm{VAR}[X|A_2] = 5.76$, and $\mathrm{VAR}[X|A_3] = 0.36$. What is $E[X]$? (B)

   A. 2.052;
   B. 1.26;
   C. 2.1;
   D. 1.10.

SECTION II: Problems [80 Marks]

1. [20 Marks]

$P[A] = 0.7$, $P[B] = 0.6$, find the following probability.

(a) If $P[A \cup B] = 0.9$.

(i) [4'] Compute $P[A \cap B]$;

(ii) [4'] Compute $P[A^c \cap B]$;

(iii) [4'] Compute $P[A \cap B^c]$;

(iv) [4'] Compute $P[A^c \cap B^c]$;

(b) [4'] If $P[A|B] = 0.56$, compute $P[B|A]$.

Solution:

(a) Because $P[A] = 0.7$, $P[B] = 0.6$ and $P[A \cup B] = 0.9$,

(i) $P[A \cap B] = P[A] + P[B] - P[A \cup B]$[3′] $= 0.7 + 0.6 - 0.9 = 0.4$[1′];

(ii) $P[A^c \cap B] = P[B] - P[A \cap B]$[3′] $= 0.6 - 0.4 = 0.2$[1′];

(iii) $P[A \cap B^c] = P[A] - P[A \cap B]$[3′] $= 0.7 - 0.4 = 0.3$[1′];

(iv) $P[A^c \cap B^c] = P[(A \cup B)^c]$[2′] $= 1 - P[A \cup B]$[1′] $= 1 - 0.9 = 0.1$[1′];

(b) Since $P[A|B] = \frac{P[A \cap B]}{P[B]}$[1′], then $P[B|A] = \frac{P[A \cap B]}{P[A]}$[1′] $= \frac{P[A|B] \times P[B]}{P[A]}$[1′] $= \frac{0.56 \times 0.6}{0.7} =$ 0.48[1′].

2. [20 Marks]

Suppose that X, the number of items in an order to an online retailer, is a discrete random variable whose probability mass function is tabulated as follows. For the values that not listed, the corresponding probabilities are zero.

x	1	2	3	4	5
$P_X(x)$	0.3	0.25	0.2	0.15	0.1

Suppose that each item weighs 2kg, and the shipping cost, Y, depends upon the total weight of the order. In particular, if the weight is less than or equal to 5kg, the cost is \$50. For each kilogram over 5kg, an extra cost \$10 is charged (i.e. if the order is 6kg, the total cost is \$60).

(a) Find out the statistics of X:

(i) [3'] Find out $E[X]$;

(ii) [3'] Find out $E[X^2]$;

(b) [3'] Tabulate the relationship between X and Y for $X \in \{1, 2, 3, 4, 5\}$;

(c) [4'] Tabulate the relationship between Y and $P_Y(y)$, ignore the values with zero probabilities;

(d) Find out the statistics of the shipping cost, Y:

(i) [3'] Find out $E[Y]$;

(ii) [4'] Find out $\mathrm{VAR}[Y^2]$.

Solution:

According to the problem,

(a) For X, we have

(i) $E[X] = \sum_{x=1}^{5} xP_X(x) = 1 \times 0.3 + 2 \times 0.25 + 3 \times 0.2 + 4 \times 0.15 + 5 \times 0.1$[2.5′] $=$ 2.5[0.5′];

(ii) $E[X^2] = \sum_{x=1}^{5} x^2 P_X(x) =$ $1 \times 0.3 + 4 \times 0.25 + 9 \times 0.2 + 16 \times 0.15 + 25 \times 0.1$[2.5′] = 8[0.5′];

(b) We have $Y = \begin{cases} 20X, & X > 2 \\ 50, & X \le 2 \end{cases}$, so the relationship between X and Y can be tabulated as belows,

X	1	2	3	4	5
Y	50[0.5']	50[0.5']	60[1']	80[0.5']	100[0.5']

(c) The relationship between Y and $P_Y(y)$ can be listed as belows,

y	50	60	80	100
$P_Y(y)$	0.55[1']	0.2[1']	0.15[1']	0.1[1']

(d) For Y, we have

(i) $E[Y] = \sum_{y=1}^{4} yP_Y(y)$ [2′] $= 50 \times 0.55 + 60 \times 0.2 + 80 \times 0.15 + 100 \times 0.1 =$ 61.5[1′];

(ii) $E[Y^2] = \sum_{y=1}^{4} y^2 P_Y(y)$ 0.5′] $= 50^2 \times 0.55 + 60^2 \times 0.2 + 80^2 \times 0.15 + 100^2 \times 0.1 =$ 4055[0.5′],

$E[Y^4] = \sum_{y=1}^{4} y^4 P_Y(y)$[0.5′] $= 50^4 \times 0.55 + 60^4 \times 0.2 + 80^4 \times 0.15 + 100^4 \times 0.1 =$ 2.21735×10^7[0.5′], therefore

$\mathrm{VAR}[Y^2] = E[Y^4] - (E[Y^2])^2$[1′] $= 2.21735 \times 10^7 - 4055^2 = 5730475$[1′].

3. [20 Marks]

A continuous random variable X has the following cumulative distribution function, which is also continuous:

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ Ax^2, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) [4'] Find out the coefficient A;

(b) [4'] Find the probability density function of X;

(c) [4'] Find the probability that $X \in (0.1, 0.4)$;

(d) [4'] Given we know that $X < 0.5$, find the probability that $X \in (0.1, 0.4)$;

(e) [4'] Find the expected value and the variance of X.

Solution:

(a) Since $F_X(x)$ is continuous, $F_X(1) = F_X(1+\epsilon) = 1$, where ϵ is an arbitrarily small positive quantity, therefore $F_X(1) = A = 1$[4'];

(b) By differentiating $F_X(x)$, we can obtain the following probability density function,

$$f_X(x) = \begin{cases} 2x[1'], & 0 < x \leq 1[1'] \\ 0\ [1'], & \text{otherwise}[1'] \end{cases}$$

(c) $P[0.1 < X < 0.4][1'] = F_X(0.4) - F_X(0.1)[1'] = 0.16A - 0.01A[1'] = 0.15A = 0.15[1']$;

(d) Since $P[X < 0.5] = F_X(0.5) = 0.25$, therefore

$$P[0.1 < X < 0.4 | X < 0.5][1'] = \frac{P[0.1 < X < 0.4][1']}{P[X < 0.5][1']} = \frac{0.15}{0.25} = 0.6[1']$$

(e) $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx[1'] = \int_0^1 2x^2\, dx = \frac{2}{3}[0.5']$, $E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx[0.5'] = \int_0^1 2x^3\, dx = \frac{1}{2}[0.5']$,

so $\mathrm{VAR}[X] = E[X^2] - (E[X])^2[1'] = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}[0.5']$.

4. [20 Marks]

There're 5 different unfair coins and they all look the same. If we toss these 5 coins, the probabilities of getting a head are $P_1[\text{head}] = 0$, $P_2[\text{head}] = 0.25$, $P_3[\text{head}] = 0.5$, $P_4[\text{head}] = 0.75$ and $P_5[\text{head}] = 1$, respectively. Now we choose a coin randomly (equally probable). Find out the probabilities of the following events:

(a) [5'] We toss the coin and get a head;

(b) [5'] We toss the coin twice and get two heads;

(c) [5'] The coin we toss is the 4th coin given we get a head;

(d) [5'] We toss the coin twice, and get a head at the second toss given we already get a head at the first toss.

Solution:

(a) According to the total probability theorem,

$$\begin{aligned} P[\text{head}] = {} & P[\text{head}|1^{\text{st}}\text{ coin}]P[1^{\text{st}}\text{ coin}] + P[\text{head}|2^{\text{nd}}\text{ coin}]P[2^{\text{nd}}\text{ coin}] \\ & + P[\text{head}|3^{\text{rd}}\text{ coin}]P[3^{\text{rd}}\text{ coin}] + P[\text{head}|4^{\text{th}}\text{ coin}]P[4^{\text{th}}\text{ coin}] \\ & + P[\text{head}|5^{\text{th}}\text{ coin}]P[4^{\text{th}}\text{ coin}][2'] \\ & = 0 \times 0.2 + 0.25 \times 0.2 + 0.5 \times 0.2 + 0.75 \times 0.2 + 1 \times 0.2[2'] = 0.5[1'] \end{aligned}$$

(b) According to the total probability theorem,

$$\begin{aligned}P[2\text{ heads}] = {} & P[2\text{ heads}|1^{\text{st}}\text{ coin}]P[1^{\text{st}}\text{ coin}] + P[2\text{ heads}|2^{\text{nd}}\text{ coin}]P[2^{\text{nd}}\text{ coin}] \\ & + P[2\text{ heads}|3^{\text{rd}}\text{ coin}]P[3^{\text{rd}}\text{ coin}] + P[2\text{ heads}|4^{\text{th}}\text{ coin}]P[4^{\text{th}}\text{ coin}] \\ & + P[2\text{ heads}|5^{\text{th}}\text{ coin}]P[4^{\text{th}}\text{ coin}][2'] \\ & = 0 \times 0.2 + 0.25^2 \times 0.2 + 0.5^2 \times 0.2 + 0.75^2 \times 0.2 + 1^2 \times 0.2[2'] \\ & = 0.375[1']\end{aligned}$$

(c) Given we got a head, according to the Bayes' rule,

$$P[4^{\text{th}}\text{coin}|\text{head}][1'] = \frac{P[\text{head}|4^{\text{th}}\text{ coin}]P[4^{\text{th}}\text{ coin}]}{P[\text{head}]}[2'] = \frac{0.15}{0.5}[1.5'] = 0.3[0.5'];$$

(d) Given we got a head at the 1st toss, according to the Bayes' rule,

$P[\text{get a head at the }2^{\text{nd}}\text{ toss}|\text{get a head at the }1^{\text{st}}\text{ toss}][1'] = \frac{P[\text{get 2 heads}]}{P[\text{get a head at the }1^{\text{st}}\text{ toss}]}[2'] = \frac{0.375}{0.5}[1'] = 0.75[1']$. As can be seen, the confidence of getting a head at the second toss is higher if we get a head at the first toss.