# 单元6.1 图的基本概念

第二编 图论　第七章 图

7.1 图的基本概念

# 内容提要

- 图、无向图、有向图、简单图
- 相邻、关联
- 度、度序列、握手定理
- 图同构
- 图族
- 图运算

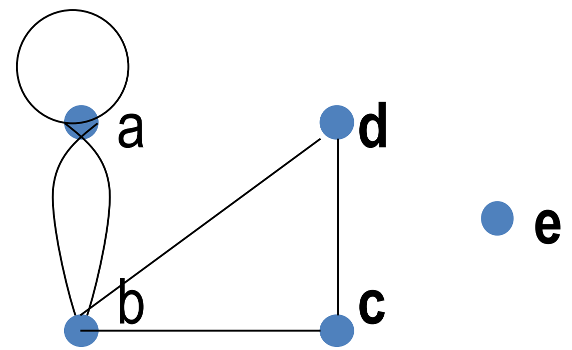
# 无序积

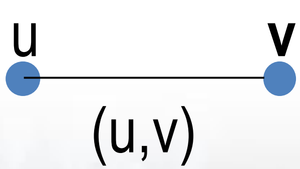
$$A\&B = \{ \{a,b\} \mid x\in A \land y\in B \}$$

- 记{a,b}=(a,b)
- 允许 a=b
- (a,b)=(b,a)

# 无向图

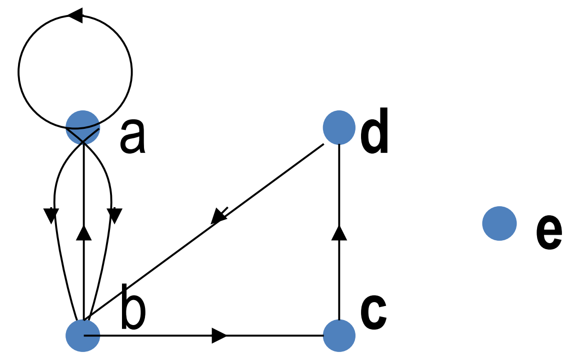
- 图 G=<V,E>
  - V≠∅　顶点集 V(G)
  - E⊆V&V　边集(多重集) E(G)
- 例： G=<V,E>
  - V={a,b,c,d,e}
  - E={(a,a),(a,b),(a,b),(b,c),(c,d),(b,d)}
- 顶点、边



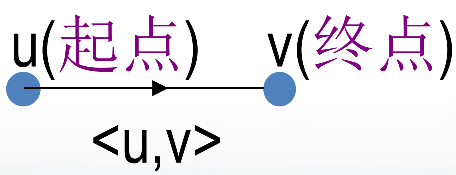



# 有向图

- 有向图 D=<V,E>
  - V≠∅, 顶点集 V(D)
  - E⊆V×V, 边集(多重集) E(D)
- 例： D=<V,E>
  - V={a,b,c,d,e}
  - E={ <a,a>,<a,b>,<a,b>, <b,a>,<b,c>,<c,d>,<d,b> }



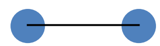

- 有向边



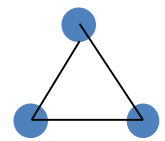
# n阶图、有限图、零图、平凡图、空图

- n阶图： $|V(G)|=n$

1阶图



2阶图

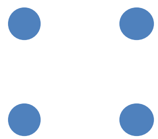


3阶图

4阶图

- 有限图： $|V(G)|<\infty$
- 零图$N_n$： $E=\varnothing$
- 平凡图： 1阶零图$N_1$
- 空图： $V=E=\varnothing$

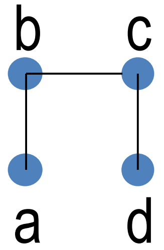
$N_1$



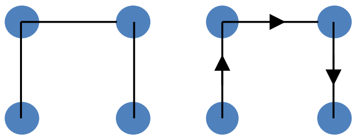
$N_4$

# 标定图、非标定图、底图

- 标定图：顶点或边带标记的图



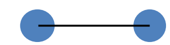

- 非标定图：顶点和边不带标记的图



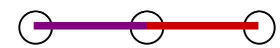
- 底图(基图)：有向图去掉边的方向后得到的无向图

# 相邻、邻接、关联

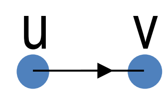
- 有边相连的两个顶点是相邻的



- 有公共顶点的两条边是相邻的

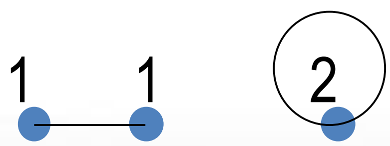


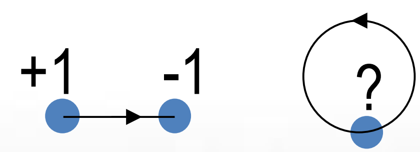
- u邻接到v, v邻接于u




- 一条边的端点与这条边是关联的

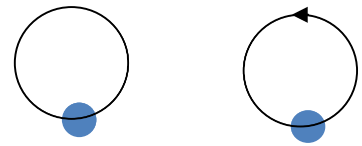
- 关联次数



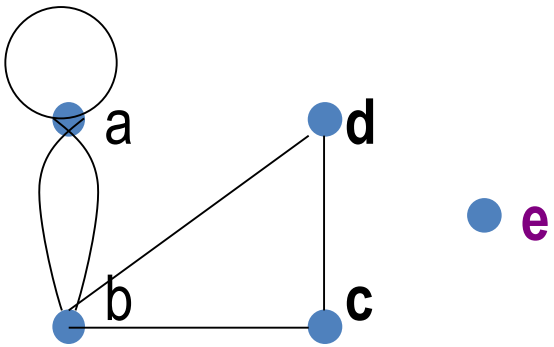



# 环、孤立点、平行边

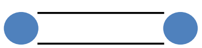
- 环：只与一个顶点关联的边



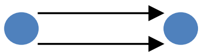
- 孤立点：不与任何边关联的顶点

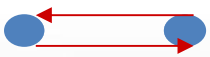



- 平行边
  - 端点相同的两条无向边是平行边



  - 起点与终点相同的两条有向边是平行边

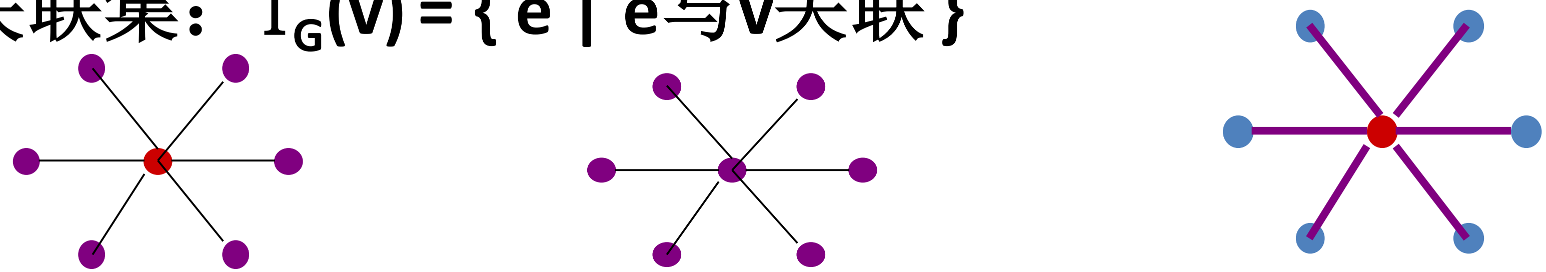




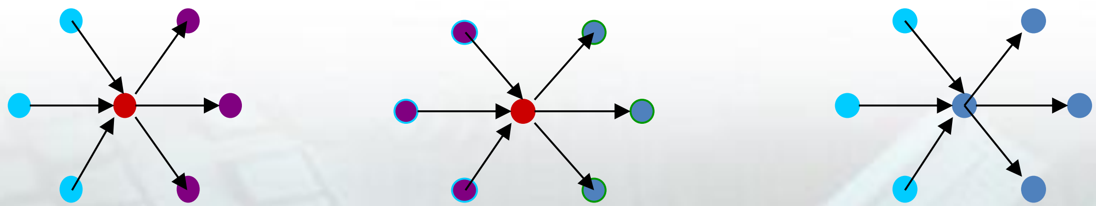
非平行边

# 邻域、闭邻域、关联集

- 邻域：$N_G(v)=\{\, u\in V(G) \mid (u,v)\in E(G)\wedge u\neq v \,\}$
- 闭邻域：$\overline{N_G(v)} = N_G(v)\cup\{v\}$
- 关联集：$I_G(v) = \{\, e \mid e$与$v$关联$\,\}$

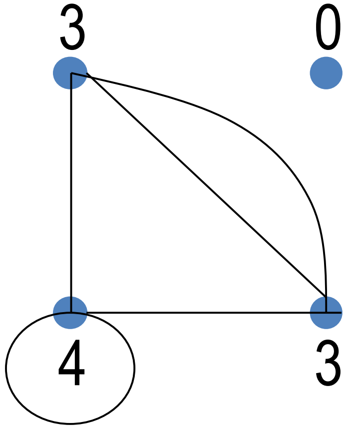


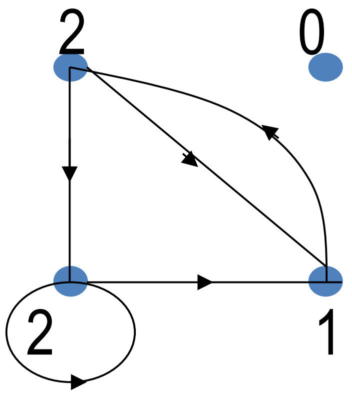
- 后继：$\Gamma_D^+(v)=\{u\in V(D) \mid \langle v,u\rangle\in E(D)\wedge u\neq v\}$
- 前驱：$\Gamma_D^-(v)=\{u\in V(D) \mid \langle u,v\rangle\in E(D)\wedge u\neq v\}$
- (闭)邻域：$N_D(v)=\Gamma_D^+(v)\cup\Gamma_D^-(v)$　　$\overline{N_D(v)} = N_D(v)\cup\{v\}$

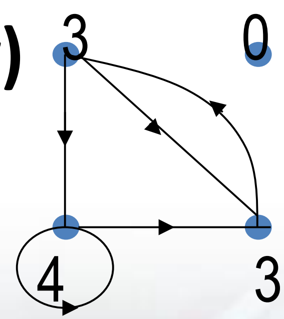


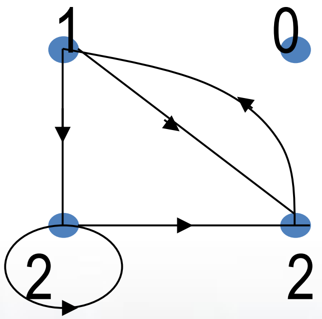
# 顶点的度

- 度：$d_G(v)$ = 与v关联的边的次数之和
- 出度：$d_D^+(v)$ =与v关联的出边的次数之和
- 入度：$d_D^-(v)$ =与v关联的入边的次数之和
- 度：$d_D(v) = d_D^+(v) + d_D^-(v)$



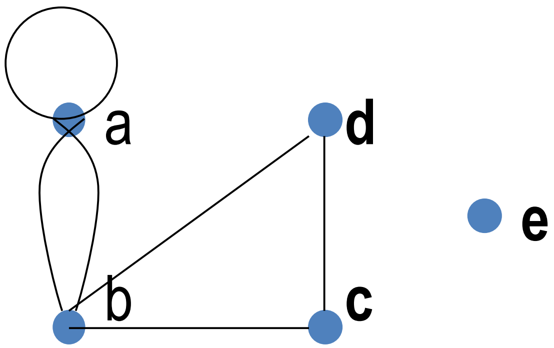



# 最大度、最小度

- 最大度 $\Delta(G) = \max\{ d_G(v) \mid v \in V(G) \}$
- 最小度 $\delta(G) = \min\{ d_G(v) \mid v \in V(G) \}$
- 最大出度 $\Delta^+(D) = \max\{ d_D^+(v) \mid v \in V(D)$
- 最小出度 $\delta^+(D) = \min\{ d_D^+(v) \mid v \in V(D) \}$
- 最大入度 $\Delta^-(D) = \max\{ d_D^-(v) \mid v \in V(D)$
- 最小入度 $\delta^-(D) = \min\{ d_D^-(v) \mid v \in V(D) \}$

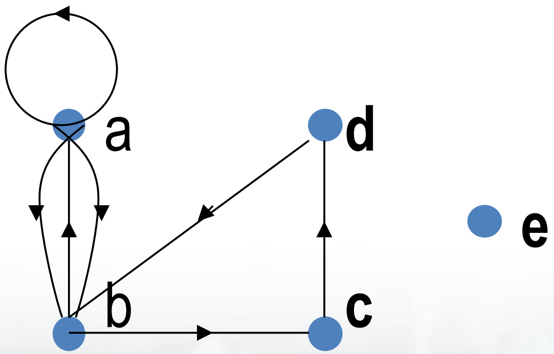
# 举例

- $\Delta=4, \delta=0$




- $\Delta^{+}=3, \delta^{+}=0$

  $\Delta^{-}=3, \delta^{-}=0$

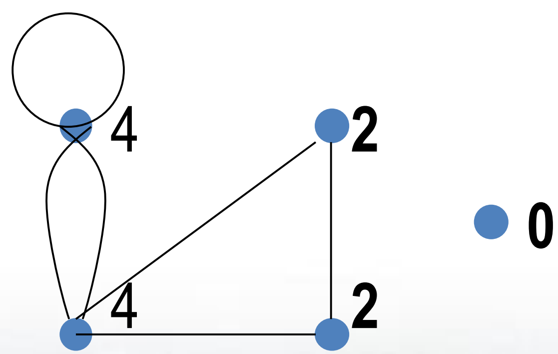


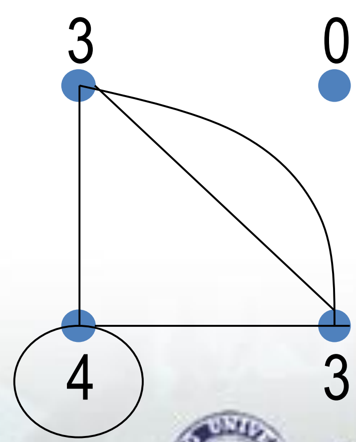
# 图论基本定理(握手定理)

- 设G=<V,E>是无向图,

  $V=\{v_1,v_2,\ldots,v_n\}$, $|E|=m$, 则

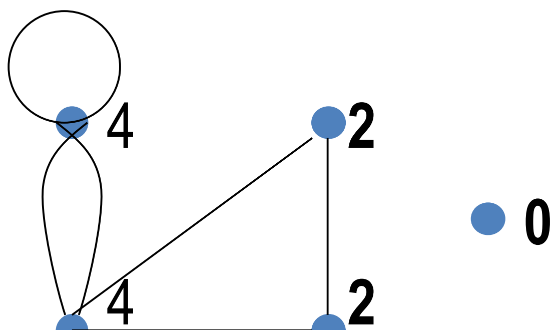
  $$d(v_1)+d(v_2)+\ldots+d(v_n)=2m. \quad \#$$

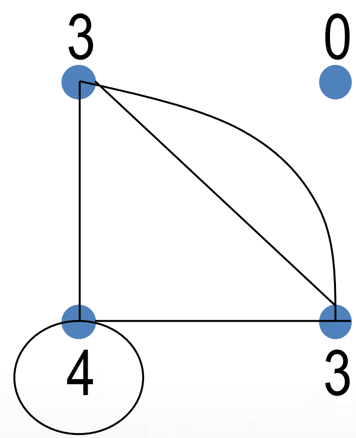


# 握手定理推论

任何图中奇度顶点的个数是偶数. #



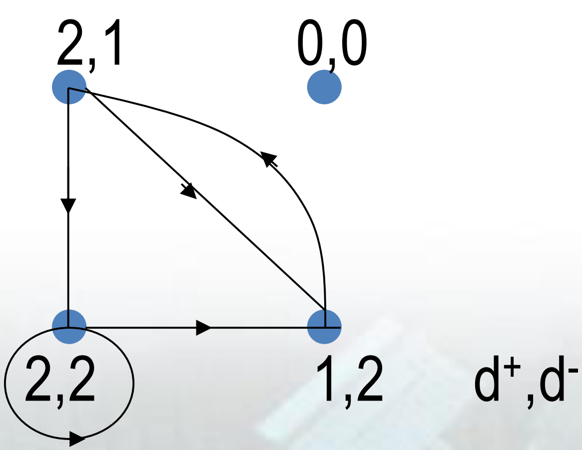



# 图论基本定理(握手定理)

- 设D=<V,E>是有向图, $V=\{v_1,v_2,\ldots,v_n\}$, $|E|=m$, 则

$$d^+(v_1)+d^+(v_2)+\ldots+d^+(v_n)$$

$$= d^-(v_1)+d^-(v_2)+\ldots+d^-(v_n) = m. \quad \#$$

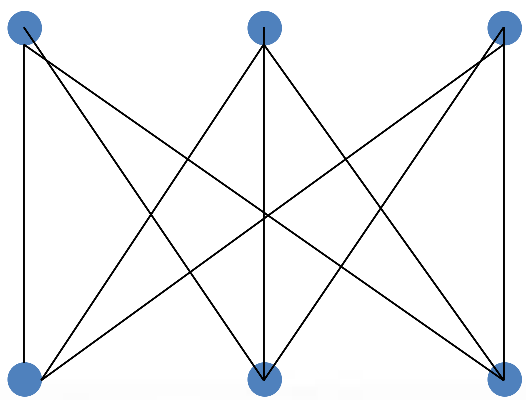


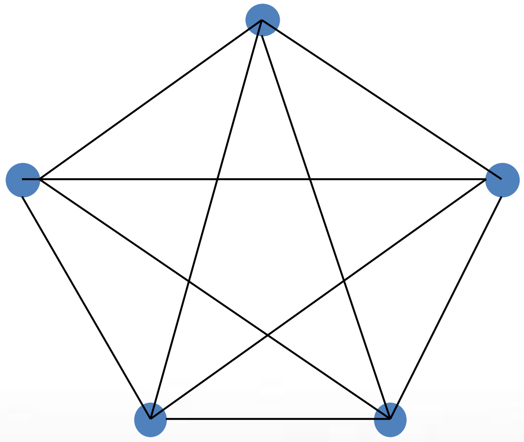
# 简单图、k-正则图

- 简单图：既无环也无平行边的图

$$\Rightarrow 0 \le \Delta(G) \le n\text{-}1$$

- k-正则图：所有顶点的度都是k





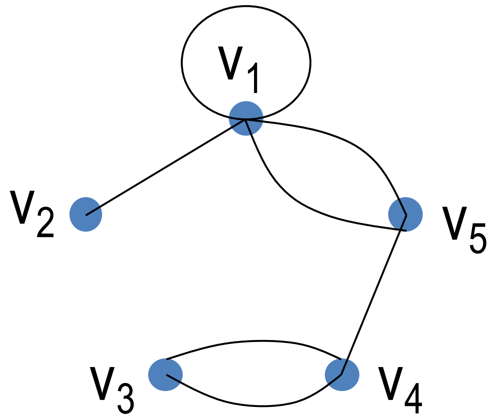
# 度数列

- 设$G=<V,E>,V=\{v_1,v_2,\ldots,v_n\}$, 称

$$d = ( d(v_1),d(v_2),\ldots, d(v_n) )$$

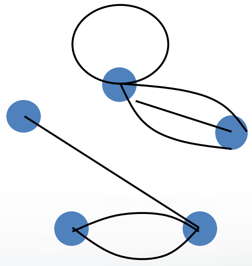
为G的度数列

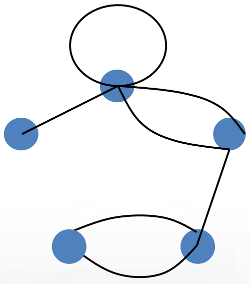
- 例： $d = ( 5, 1, 2, 3, 3 )$



# 可图化

- 设非负整数列d=( $d_1$,$d_2$, ..., $d_n$ ), 若存在图G, 使得G的度数列是d, 则称d为可图化的
- 可图化举例：度数列 d = ( 5, 3, 3, 2, 1 )





# 可图化充要条件

非负整数列$d=(d_1,d_2,\ldots,d_n)$可图化$\Leftrightarrow$

$d_1+d_2+\ldots+d_n \equiv 0 \pmod 2$.

- 证: $(\Rightarrow)$ 握手定理

$(\Leftarrow)$ 奇数度点两两之间连一边,

剩余度用环来实现. #

# 可简单图化

- 设非负整数列$d=(d_1, d_2, \ldots, d_n)$, 若存在简单图G, 使得G的度数列是d, 则称d为可简单图化的

- 例：$d = (5, 3, 3, 2, 1)$ 不可简单图化
  - $\Delta=n \quad (\Delta\le n-1)$

- 例：$d = (4, 4, 3, 2, 1)$ 不可简单图化
  - $(n-1, n-1, \ldots, 1)$

# 可简单图化充要条件(Havel定理)

- 设非负整数列$d=(d_1,d_2,\ldots,d_n)$满足:

$$d_1+d_2+\ldots+d_n \equiv 0 \pmod 2),$$

$$n-1 \geq d_1 \geq d_2 \geq \ldots \geq d_n \geq 0,$$

则d可简单图化$\Leftrightarrow$

$$d'=(d_2-1,d_3-1,\ldots,d_{d1+1}-1,d_{d1+2},\ldots,d_n)$$

可简单图化

- 例: d=(4,4,3,3,2,2), d'=(3,2,2,1,2)

# Havel定理举例

- 判断下列非负整数列是否可简单图化. (1) (5,5,4,4,2,2)　　(2)(4,4,3,3,2,2)
- 解: (1) (5,5,4,4,2,2), (4,3,3,1,1),

　　(2,2,0,0), (1,-1,0), 不可简单图化.

(2) (4,4,3,3,2,2), (3,2,2,1,2), (3,2,2,2,1),

(1,1,1,1), (0,1,1), (1,1), 可简单图化. #

# 可简单图化充要条件

- 定理7.4(P.Erdös,T.Gallai,1960):

  设非负整数列$d=(d_1,d_2,\dots,d_n)$满足:

$$n-1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0,$$

  则d可简单图化$\Leftrightarrow$

$$d_1+d_2+\dots+d_n=0 \pmod 2$$

  并且对$r=1,2,\dots,n-1$有

$$d_1+d_2+\dots+d_r \leq r(r-1)+\min\{r,d_{r+1}\}+\min\{r,d_{r+2}\}+\dots+\min\{r,d_n\}.$$

  #

# 定理7.4等价形式

- 定理7.4' (P.Erdös,T.Gallai,1960):非负整数列 $d=(d_1,d_2,\ldots,d_n)$可简单图化 $\Leftrightarrow$ $d_1+d_2+\ldots+d_n=0 \pmod 2$

 并且对$r=1,2,\ldots,n$有

 $d_1+d_2+\ldots+d_r \le r(r-1)+\min\{r,d_{r+1}\}+\min\{r,d_{r+2}\}+\ldots+\min\{r,d_n\}$. #

- 说明: $n-1\ge d_1\ge d_2\ge\ldots\ge d_n\ge 0 \Rightarrow$

$$d_1+d_2+\ldots+d_n \le n(n-1).$$

# 定理7.4举例

- 下列非负整数列是否可简单图化?

(1) (5,4,3,2,2,1)

(2) (5,4,4,3,2)

(3) (3,3,3,1)

(4) (6,6,5,4,3,3,1)

(5) (5,5,3,3,2,2,2)

(6) $(d_1,d_2,\ldots,d_n)$, $d_1>d_2>\ldots>d_n\geq 1$

# 定理7.4举例

- (1) $5+4+3+2+2+1=17\neq0 \pmod 2$.

不可(简单)图化.

- (2) $5+4+4+3+2=18=0 \pmod 2$.

但是$d_1=5>n-1=4$,不满足$n-1\geq d_1$, 不可简单图化.

( 或者: 但是$r=1$时, $d_1=5>1(1-1)+\min\{1,4\}+\min\{1,4\}+\min\{1,3\}+\min\{1,2\}=4$, 不可简单图化.)

- (3) $3+3+3+1=10=0 \pmod 2$. $d_1=3=n-1$,满足$n-1\geq d_1$,

但是$r=2$时, $d_1+d_2=6 > 2(2-1) + \min\{2,3\} + \min\{2,1\} = 5$,

不可简单图化.

# 定理4举例(4)

- (4) 6+6+5+4+3+3+1=28=0(mod 2).

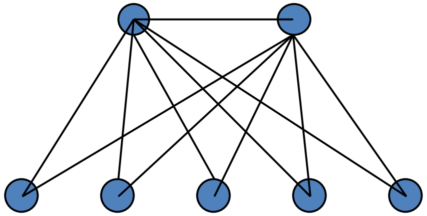
$d_1$=6=n-1,满足n-1≥$d_1$. r=1,2时,

$d_1$=6≤1(1-1)+min{1,6}+min{1,5}+ …=6,

$d_1+d_2$=12>2(2-1)+min{2,5}+…=11,

不可简单图化.

- 或:6,6,*,*,*,*,1不可简单图化



# 定理4举例(5)

- (5) $5+5+3+3+2+2+2=22=0(\mathrm{mod}\ 2)$.

$d_1=5<n-1$,满足$n-1\geq d_1$. $r=1,2,\ldots,7$时,

$d_1=5<1(1-1)+\min\{1,5\}+\min\{1,5\}+\ \ldots=6$,

$d_1+d_2=10<2(2-1)+\min\{2,3\}+\ldots=12$,

$d_1+d_2+d_3=13<3(3-1)+\min\{3,3\}+\ldots=15$,

$d_1+d_2+d_3+d_4=16<4(4-1)+\min\{4,2\}+\ldots=18$,

# 定理4举例(5)

$d_1+d_2+d_3+d_4=16<4(4-1)+\min\{4,2\}+\ldots=18$,

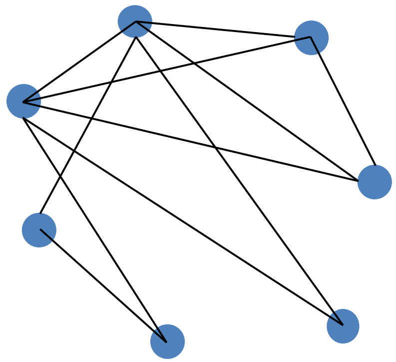
$d_1+d_2+\ldots+d_5=18<5(5-1)+\min\{5,2\}+\ldots=24$,

$d_1+d_2+\ldots+d_6=20<6(6-1)+\min\{6,2\}=32$,

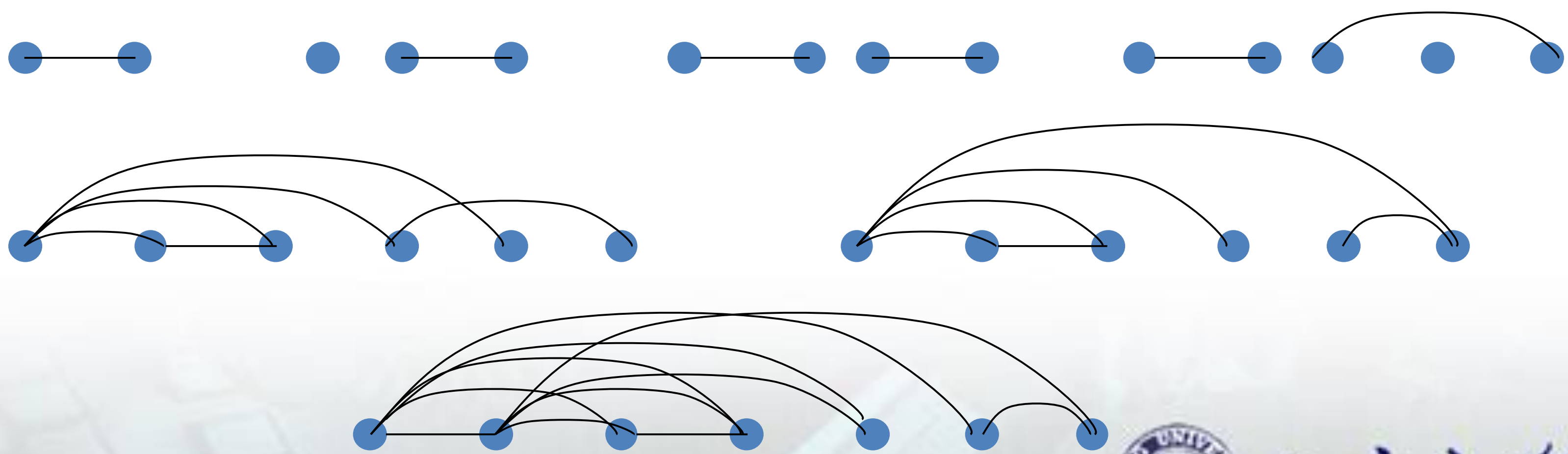
$d_1+d_2+\ldots+d_7=22<7(7-1)=42$,

可简单图化.

# 定理4举例(5)



**(5,5,3,3,2,2,2), (4,2,2,1,1,2), (4,2,2,2,1,1), (1,1,1,0,1),(1,1,1,1), (0,1,1),(1,1)**



# 定理4举例(6)

- (6) $d_1>d_2>\ldots>d_n\geq 1$,

  $d_{n-1}\geq 2$, $d_{n-2}\geq 3$,…, $d_1\geq n$,

  不满足$n-1\geq d_1$,

  不可简单图化.　　　#

# Paul Erdös(1913-1996)

- 保罗•爱尔特希, 匈牙利人
- 廿世纪数学界的传奇人物
- “Another roof, another proof.”

# Paul Erdös

*"My mother said, 'Even you, Paul, can be in only one place at one time.'*

*Maybe soon I will be relieved of this disadvantage.*

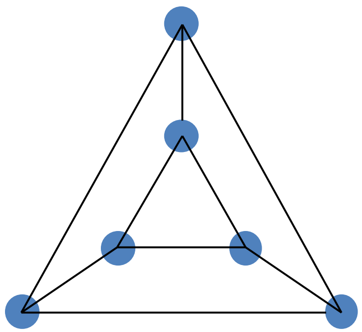
*Maybe, once I've left, I'll be able to be in many places at the same time.*

*Maybe then I'll be able to collaborate with Archimedes and Euclid."*

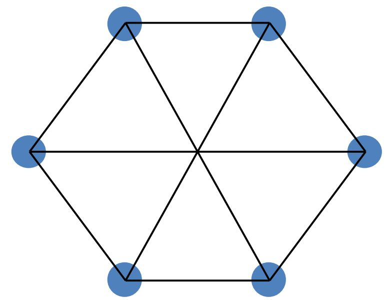
# 图同构

- 无向图 $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$，若存在双射 $f:V_1 \to V_2$ 满足 $\forall u,v \in V_1$，$(u,v) \in E_1 \leftrightarrow (f(u),f(v)) \in E_2$，则称 $G_1$ 与 $G_2$ 同构, 记作 $G_1 \cong G_2$
- 同构的图,其图论性质完全一样
- NAUTY算法
- 有向图 $D_1=<V_1,E_1>$, $D_2=<V_2,E_2>$，若存在双射 $f:V_1 \to V_2$，满足 $\forall u,v \in V_1$，$<u,v> \in E_1 \leftrightarrow <f(u),f(v)> \in E_2$，则称 $D_1$ 与 $D_2$ 同构, 记作 $D_1 \cong D_2$

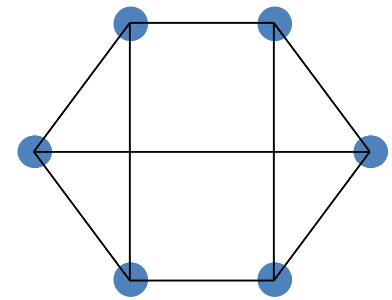
# 图同构(举例)



$G_1$



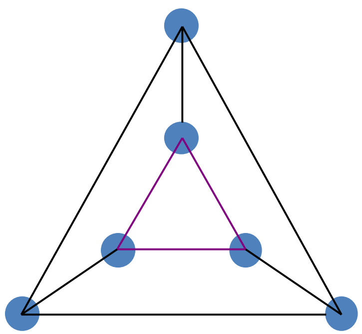
$G_2$



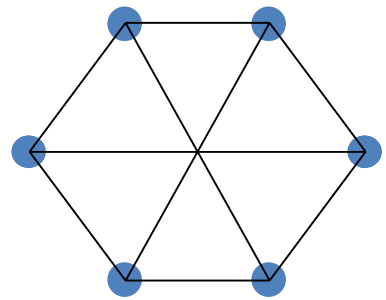
$G_3$

$G_1 \cong G_3$， $G_1 \ncong G_2$

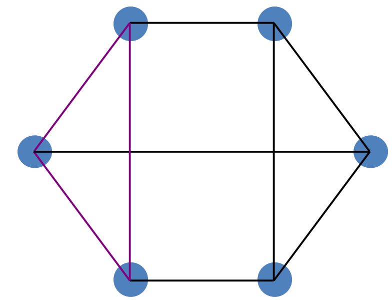
# 图同构(举例)



$G_1$

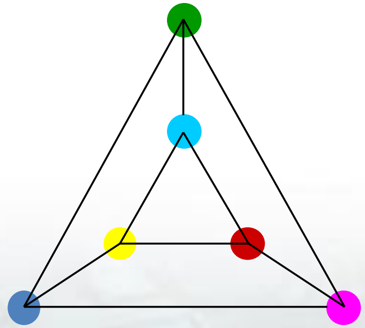


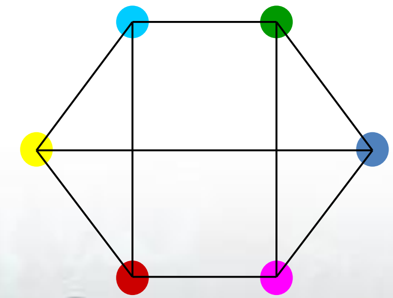
$G_2$



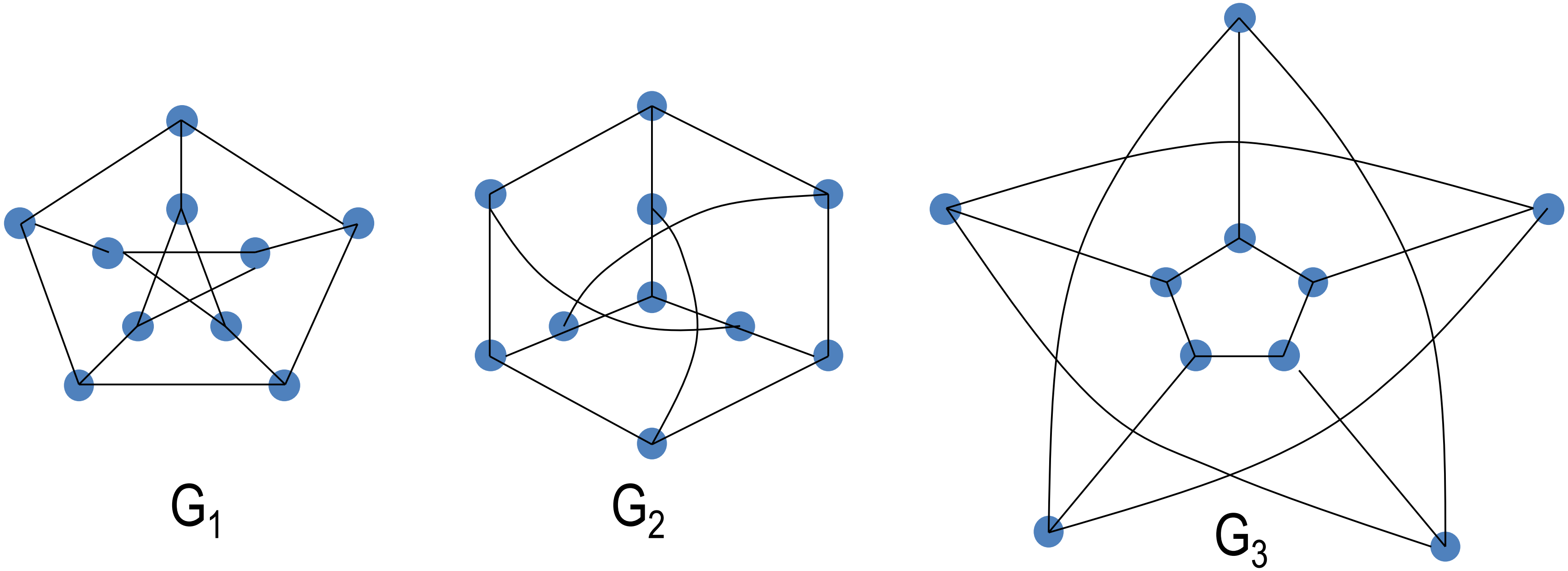
$G_3$

$G_1 \cong G_3$，　$G_1 \ncong G_2$



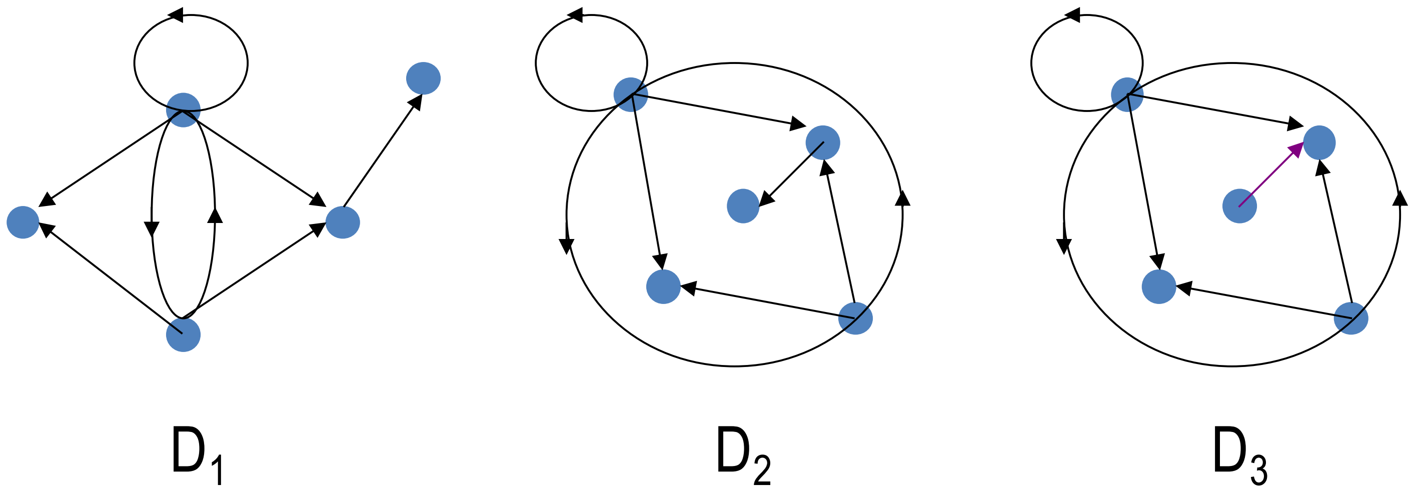


# 图同构(举例)




$G_1 \cong G_2 \cong G_3$

# 图同构(举例)



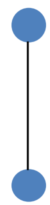

$D_1 \cong D_2$， $D_2 \ncong D_3$

# 图族(graph class)

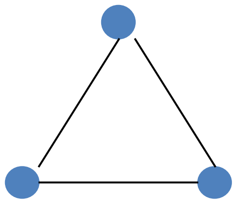
- 完全图,有向完全图,竞赛图
- 柏拉图图,彼德森图,库拉图斯基图
- r部图,二部图(偶图),完全r部图
- 路径,圈,轮,超立方体

# 完全图

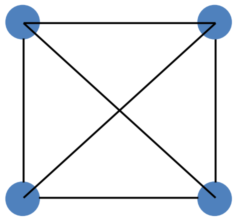
$K_1$



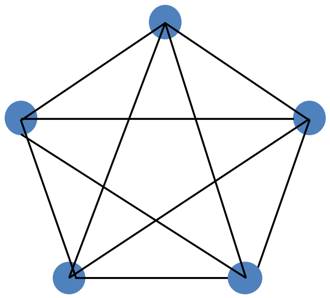
$K_2$



$K_3$

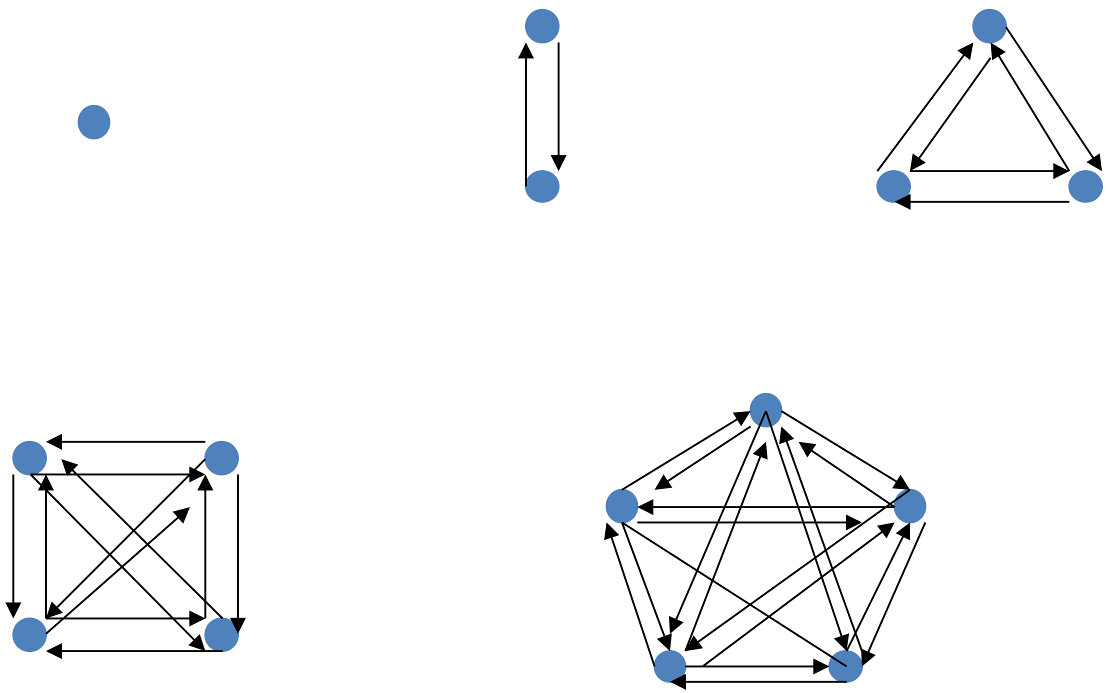


$K_4$

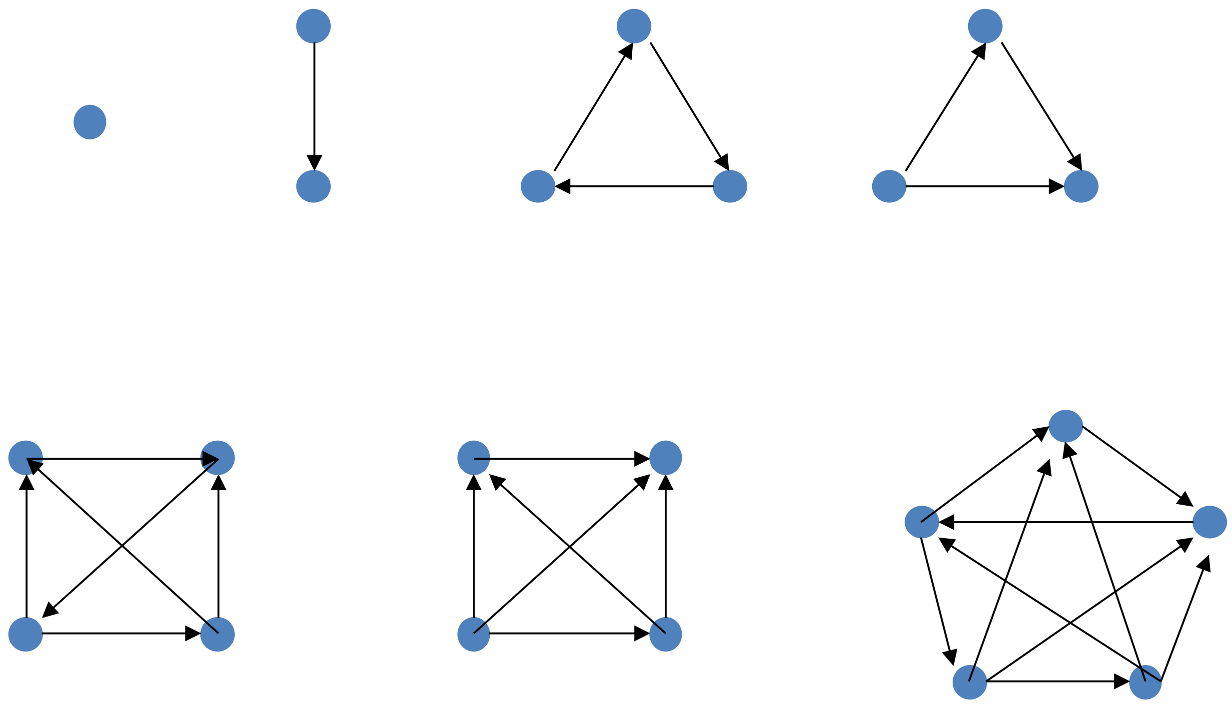


$K_5$

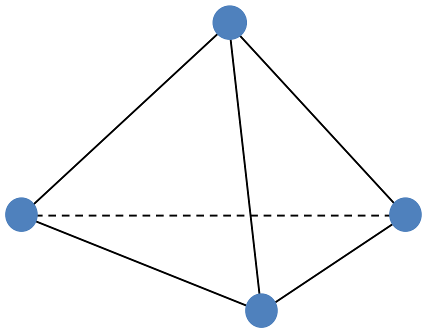
# 有向完全图



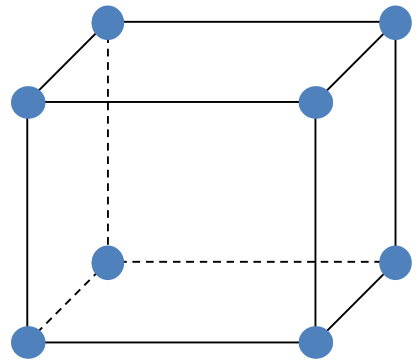
# 竞赛图



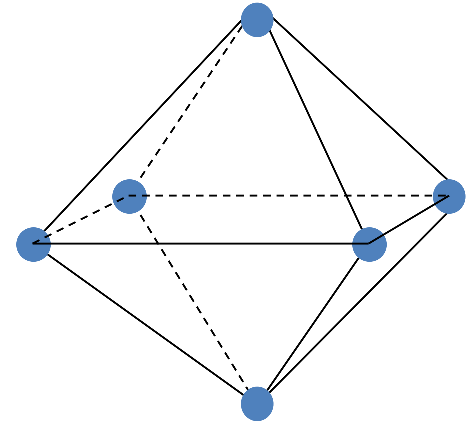
# 柏拉图图



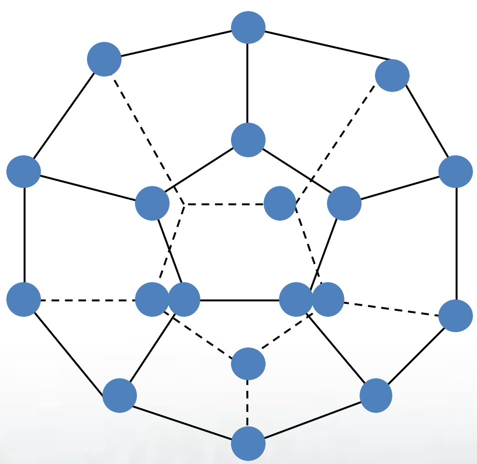
正四面体图



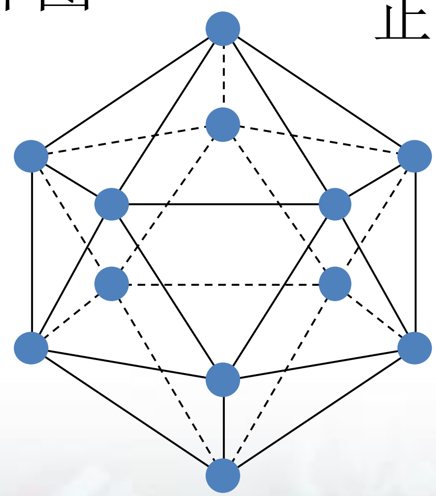
正六面体图



正八面体图

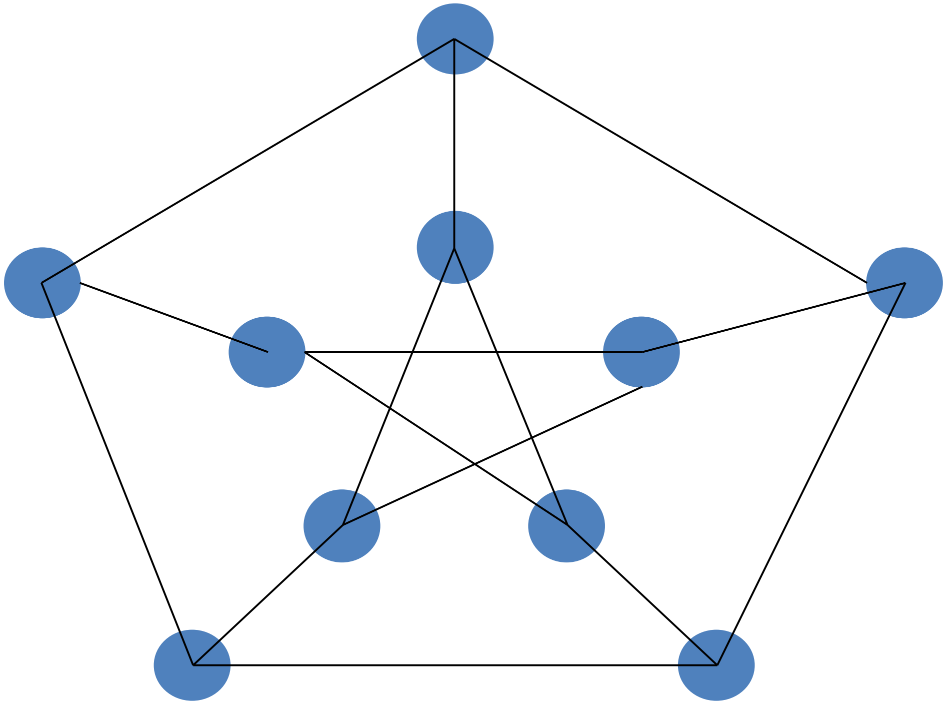


正十二面体图



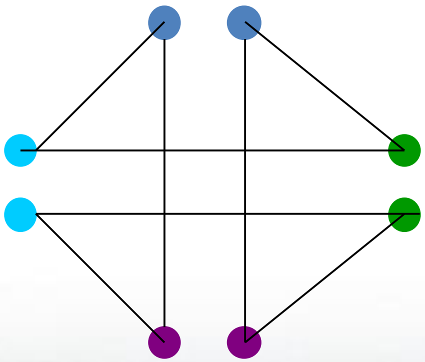
正二十面体图

# 彼德森图



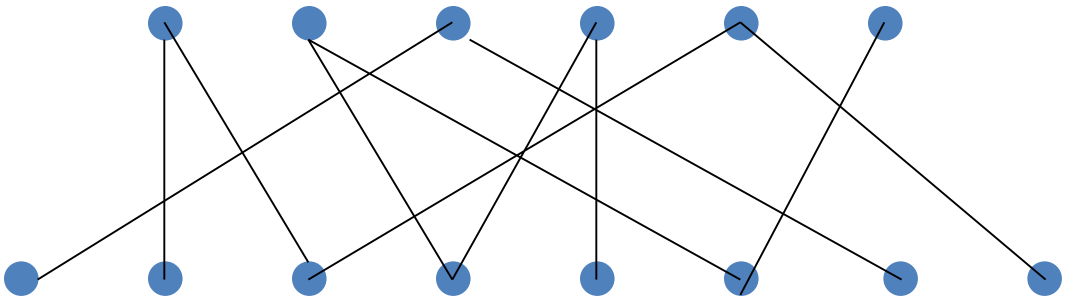
# r部图

- **r部图**: $G=\langle V,E\rangle, V=V_1\cup V_2\cup\ldots\cup V_r$, $V_i\cap V_j=\varnothing\ (i\neq j)$, $E\subseteq \bigcup_{i\neq j}(V_i\& V_j)$,
- 也记作 $G=\langle V_1,V_2,\ldots,V_r\,;\,E\rangle$.

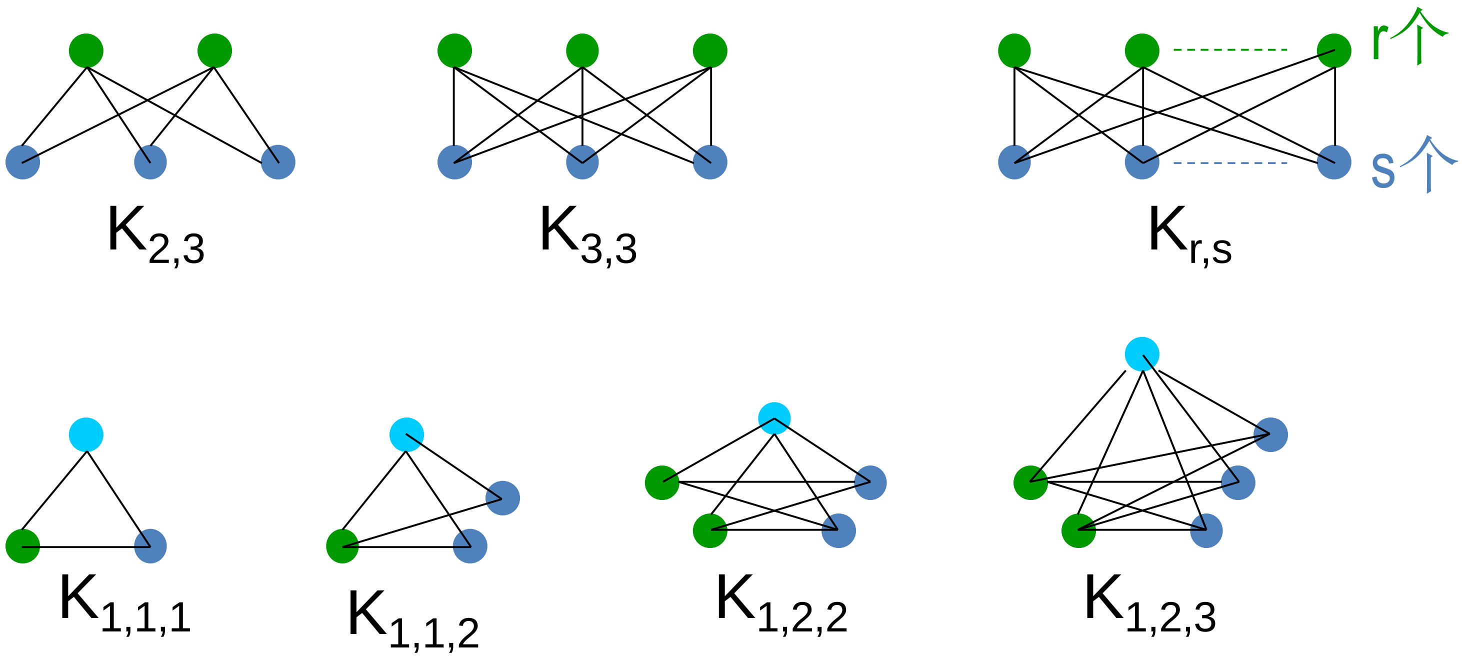


# 二部图(偶图)

- 二部图: $G=<V_1,V_2; E>$, 也称为偶图

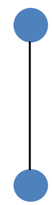


# 完全r部图

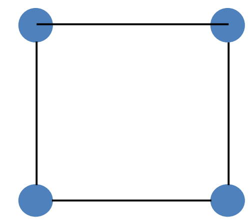


# 超立方体

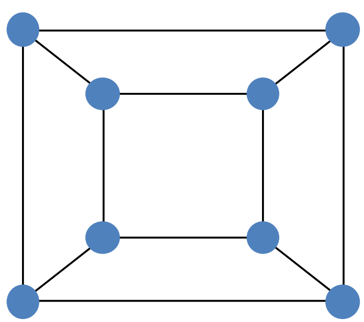
$Q_0$



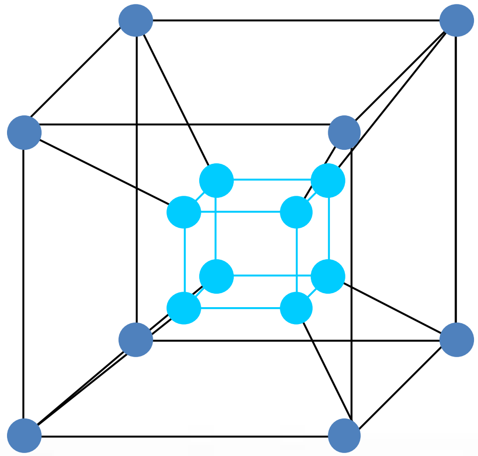
$Q_1$



$Q_2$



$Q_3$



$Q_4$

# 子图,生成子图

- 子图(subgraph): $G=<V,E>$, $G'=<V',E'>$,

  $$G'\subseteq G \Leftrightarrow V'\subseteq V \wedge E'\subseteq E$$

- 真子图(proper subgraph):

  $$G'\subset G \Leftrightarrow G'\subseteq G \wedge (V'\subset V \vee E'\subset E)$$

- 生成子图(spanning subgraph):

  G'是G的生成子图 $\Leftrightarrow G'\subseteq G \wedge V'=V$

# 导出子图

- 导出子图: $G=<V,E>$,
- 若$V_1 \subset V$, $E_1 = E \cap V_1 \& V_1$,则称

$$G[V_1] = <V_1,E_1>$$

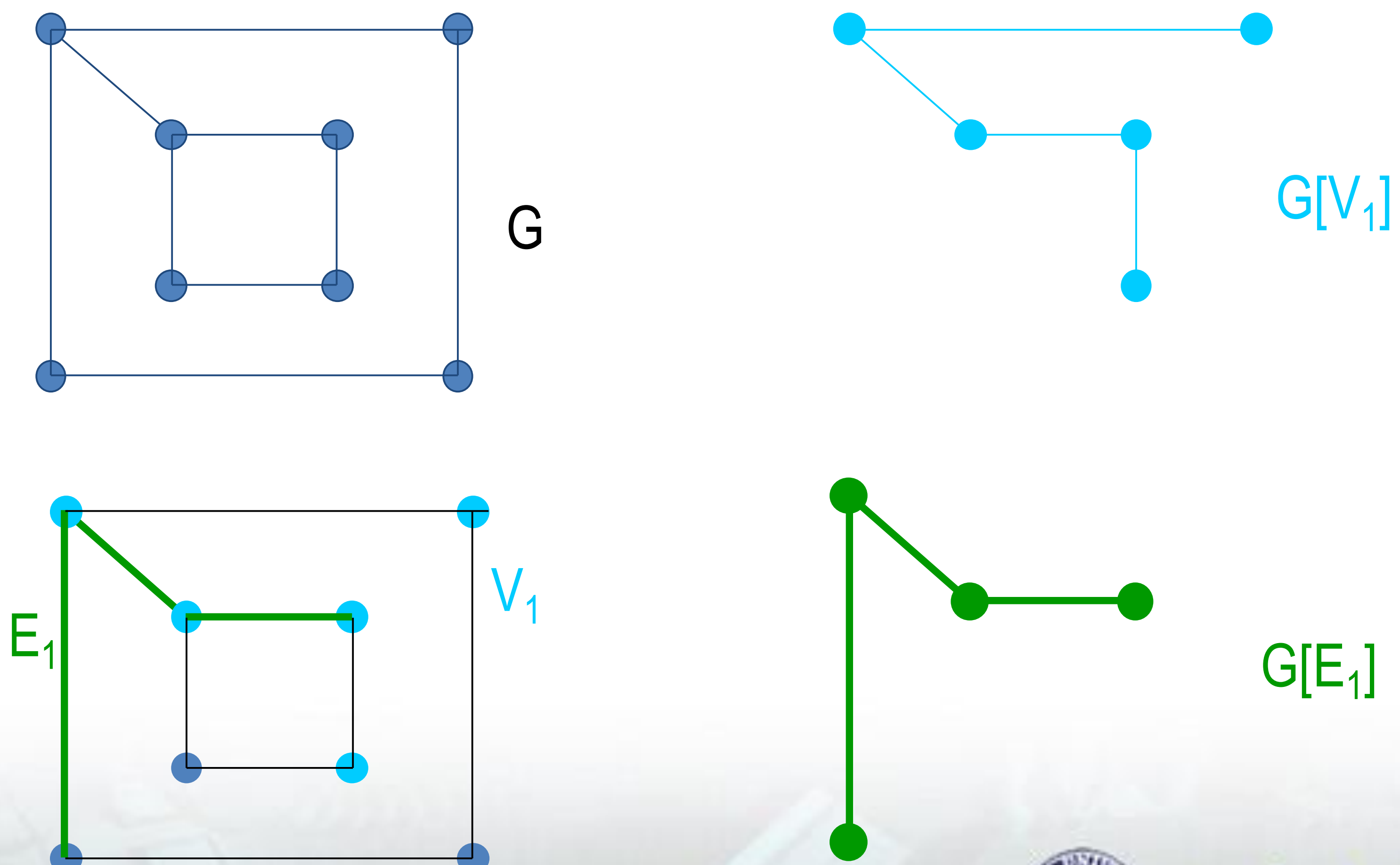
为由$V_1$导出的子图

- 若$\varnothing \neq E_1 \subset E$, $V_1=\{v \mid v$与$E_1$中的边关联$\}$,

则称 $G[E_1] = <V_1,E_1>$

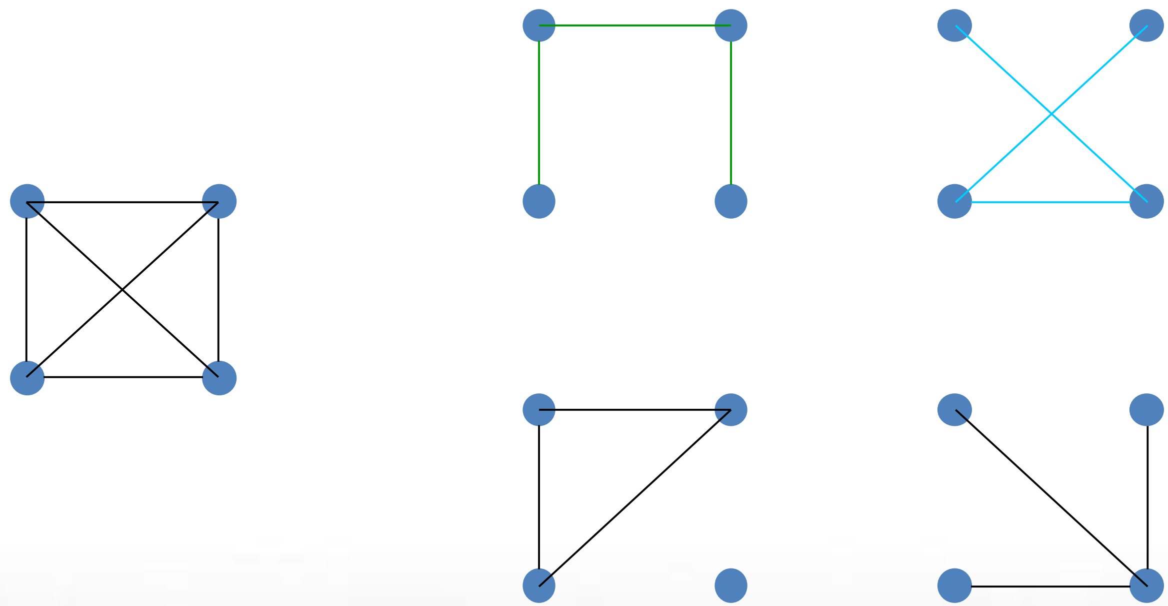
为由$E_1$导出的子图

# 导出子图(举例)



# 补图

- 补图: $G=\langle V,E\rangle$, $\overline{G}=\langle V,E(K_n)-E\rangle$
- 自补图(self-complement graph): $G\cong\overline{G}$



# 例7.5

- (1) 画出5阶4条边的所有非同构的无向简单图； (2) 画出4阶2条边的所有非同构的有向简单图.
- (1) $\Sigma d(v)=2m=8$, $\Delta\le n-1=4$,

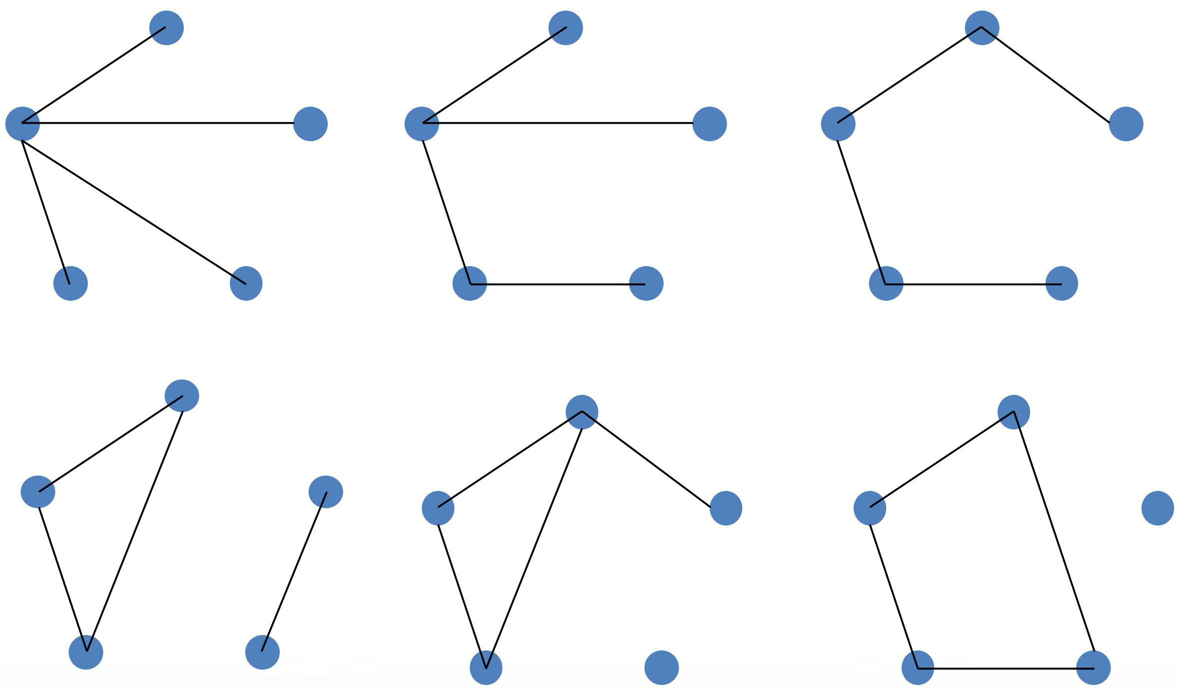
  (4,1,1,1,1),(3,2,1,1,1),(2,2,2,1,1),

  (3,2,2,1,0),(2,2,2,2,0)
- (2) $\Sigma d^+(v)=\Sigma d^-(v)=m=2$, $\Sigma d(v)=2m=4$,

  (2,1,1,0),(1,1,1,1),(2,2,0,0)

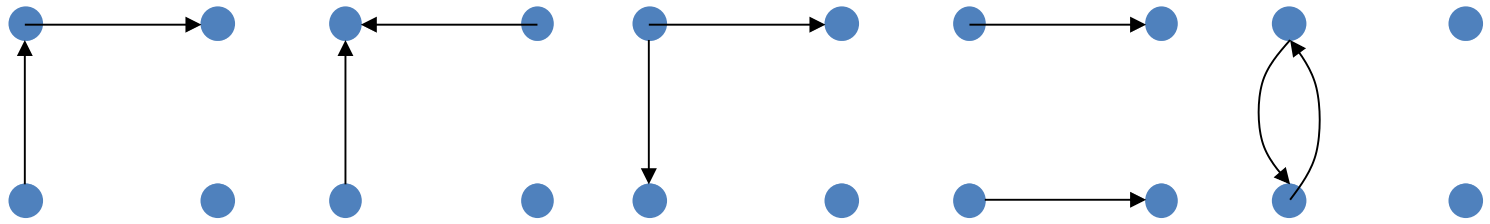
# 例7.5(1)

- 5阶4条边的所有非同构无向简单图



# 例7.5(2)

- 4阶2条边的所有非同构有向简单图.



# 图的运算

- 删除,收缩,加新边,不交
- 并图,交图,差图,环和
- 联图,积图

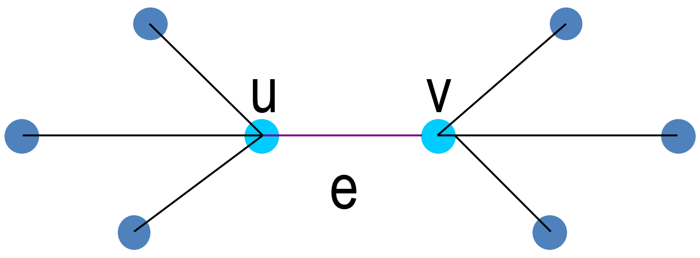
# 删除(delete)

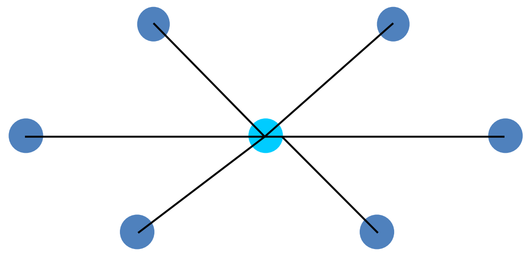
- G-e = < V, E-{e} >，删除边e
- G-E’ = < V, E-E’ >，删除边集E’
- G-v = < V-{v}, E-$I_G$(v) >，删除顶点v以及v所关联的所有边
- G-V’ = < V-V’, E-$I_G$(V’) >,删除顶点集V’以及V’所关联的所有边

# 收缩、加新边

- G\e: e=(u,v), 删除e,合并u与v
- G∪(u,v) = <V,E∪{(u,v)}>
  - 在u与v之间加新边

也写成G+(u,v)



# 不交

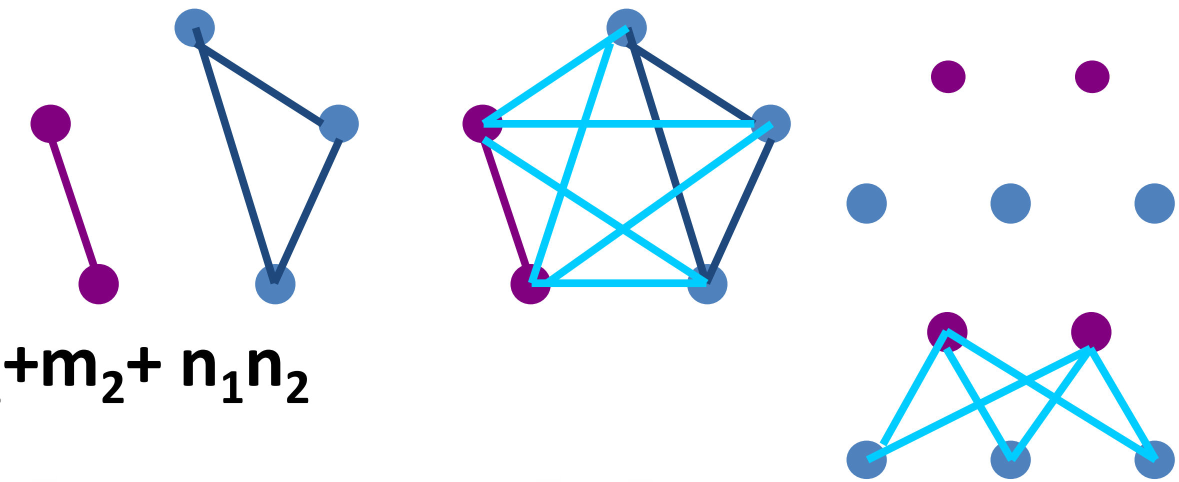
- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$,
- $G_1$与$G_2$不交 $\Leftrightarrow V_1 \cap V_2=\varnothing$
- $G_1$与$G_2$边不交(边不重) $\Leftrightarrow E_1 \cap E_2=\varnothing$

# 并图,交图,差图,环和

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$, 都无孤立点
- 并图: $G_1 \cup G_2 =<V(E_1 \cup E_2), E_1 \cup E_2>$
- 交图: $G_1 \cap G_2 =<V(E_1 \cap E_2), E_1 \cap E_2>$
- 差图: $G_1 - G_2 =<V(E_1 - E_2), E_1 - E_2>$
- 环和: $G_1 \oplus G_2 =<V(E_1 \oplus E_2), E_1 \oplus E_2>$

# 联图

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$, 不交无向图
- $G_1+G_2=<V_1\cup V_2,E_1\cup E_2\cup(V_1\&V_2)>$
- $K_r+K_s=K_{r+s}$
- $N_r+N_s=K_{r,s}$
- $n=n_1+n_2$, $m=m_1+m_2+n_1n_2$



# 积图

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$, 无向简单图
- $G_1\times G_2=<V_1\times V_2, E>$, 其中

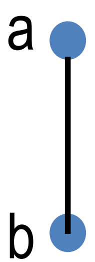
$E = \{ (<u_i,u_j>,<u_k,u_s>) \mid$

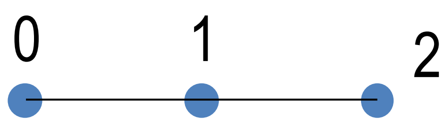
$(<u_i,u_j>,<u_k,u_s>\in V_1\times V_2) \wedge$

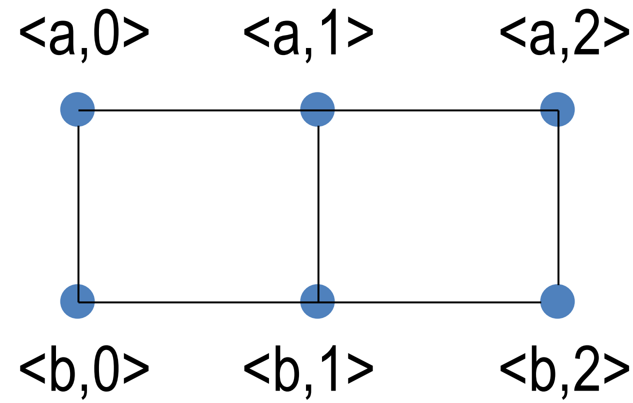
$((u_i=u_k \wedge u_j$与$u_s$相邻$) \vee (u_j=u_s \wedge u_i$与$u_k$相邻$)) \}$

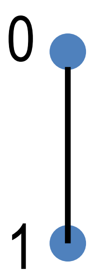
- $n=n_1 n_2$, $m=n_1 m_2 + n_2 m_1$

# 积图(举例)

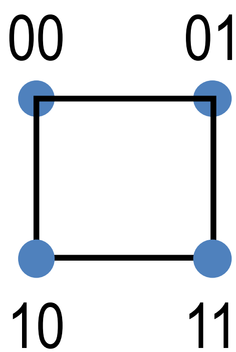


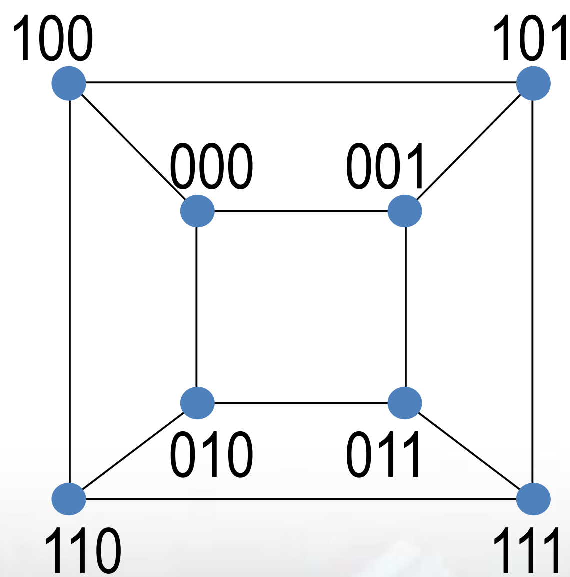




$Q_1=K_2$




$Q_2=K_2\times Q_1$




$Q_k=K_2\times Q_{k-1}$

# 小结

- 无向图,有向图,简单图,相邻,关联
- 度, 握手定理
- 图同构
- 图族(完全图, 零图, … )
- 图运算(并图, 联图, 积图, …)