单元6.4 无向图的连通度

第二编 图论　　第七章 图

7.4 无向图的连通度

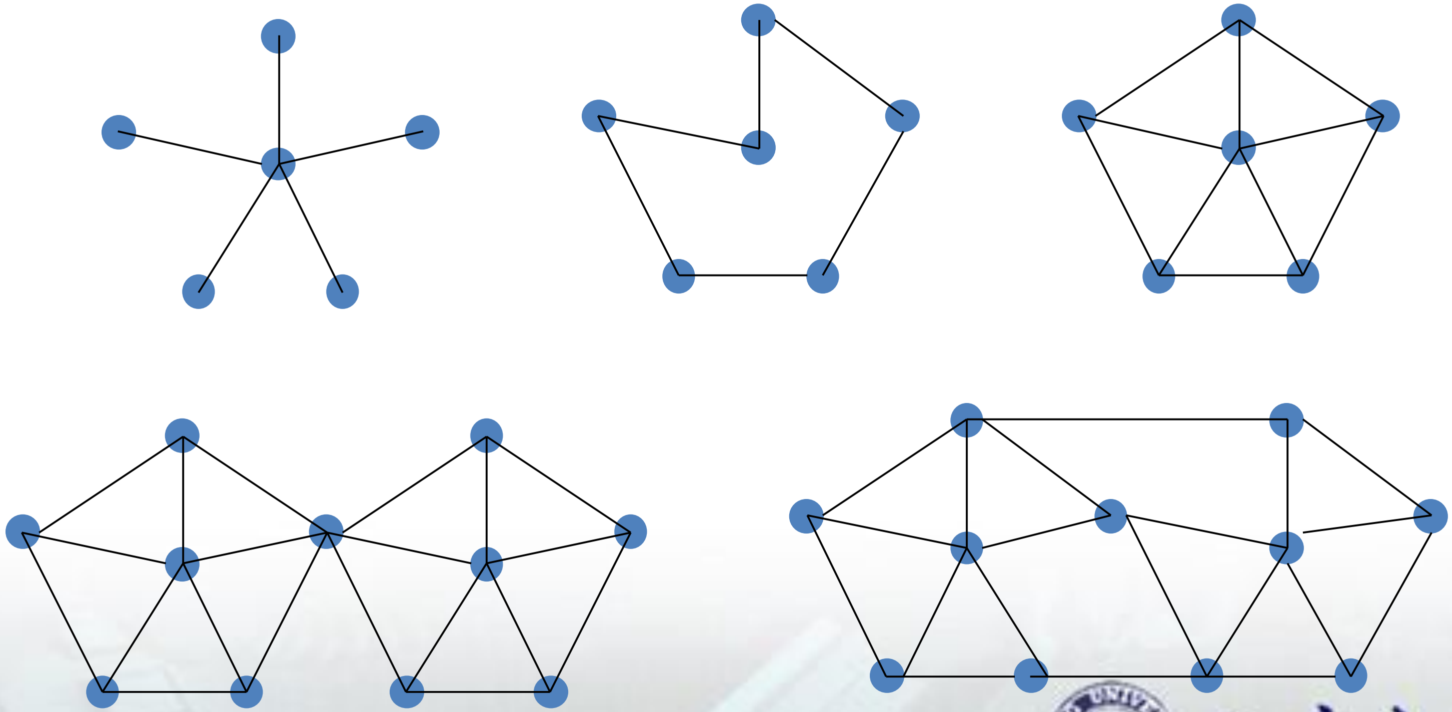
北京大学

内容提要

- 点割集、点连通度
- 边割集、边连通度

如何定量比较连通性?

- 如何定义一个图比另一个图的连通性更好?

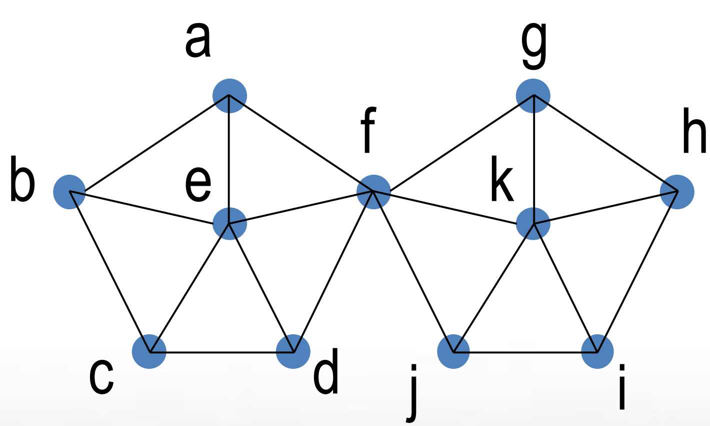


点连通度、边连通度

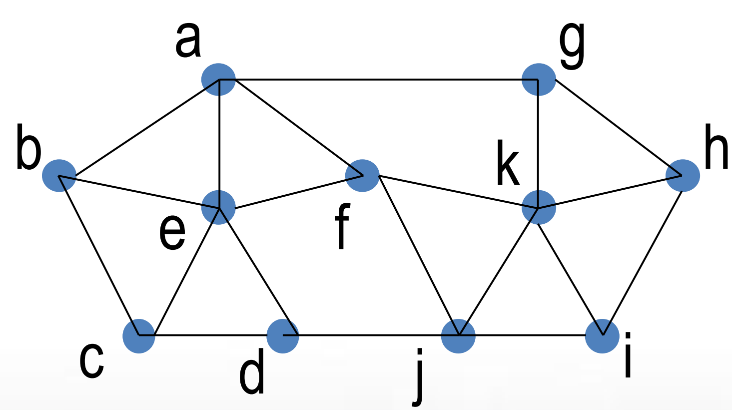
- 为了破坏连通性,至少需要删除多少个顶点?
- 为了破坏连通性,至少需要删除多少条边?
- “破坏”连通性:
  - $p(G-V') > p(G)$
  - $p(G-E') > p(G)$
  - 连通分支数增加

点割集

- 点割集： $G=\langle V,E\rangle$, $\varnothing\neq V'\subset V$, (1) $p(G-V')>p(G)$;
  (2) $\forall V''\subset V'$, $p(G-V'')=p(G)$ (极小性条件)
- 例 G_1: {f},{a,e,c},{g,k,j}, {b,e,f,k,h}不是
  G_2: {f}不是,{a,e,c},{g,k,j},{b,e,f,k,h}



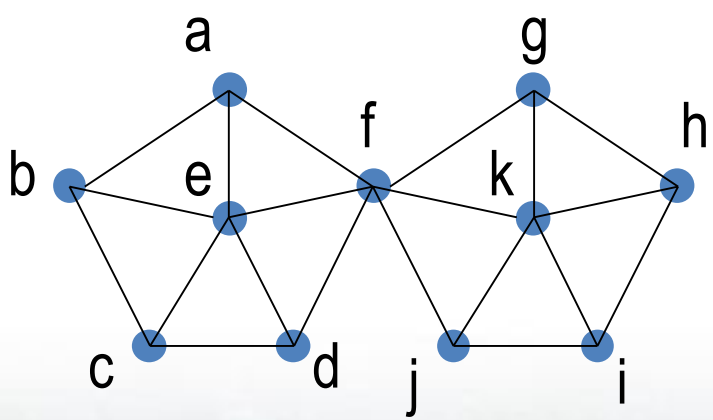
G_1



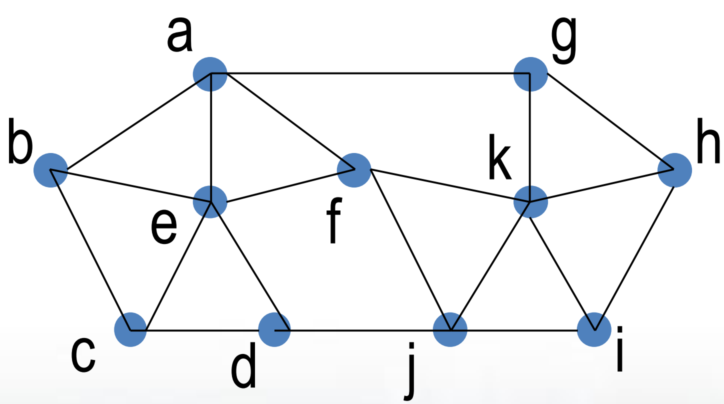
G_2

割点

- v是割点 ⇔ {v}是割集
- 例: G_1中f是割点, G_2中无割点



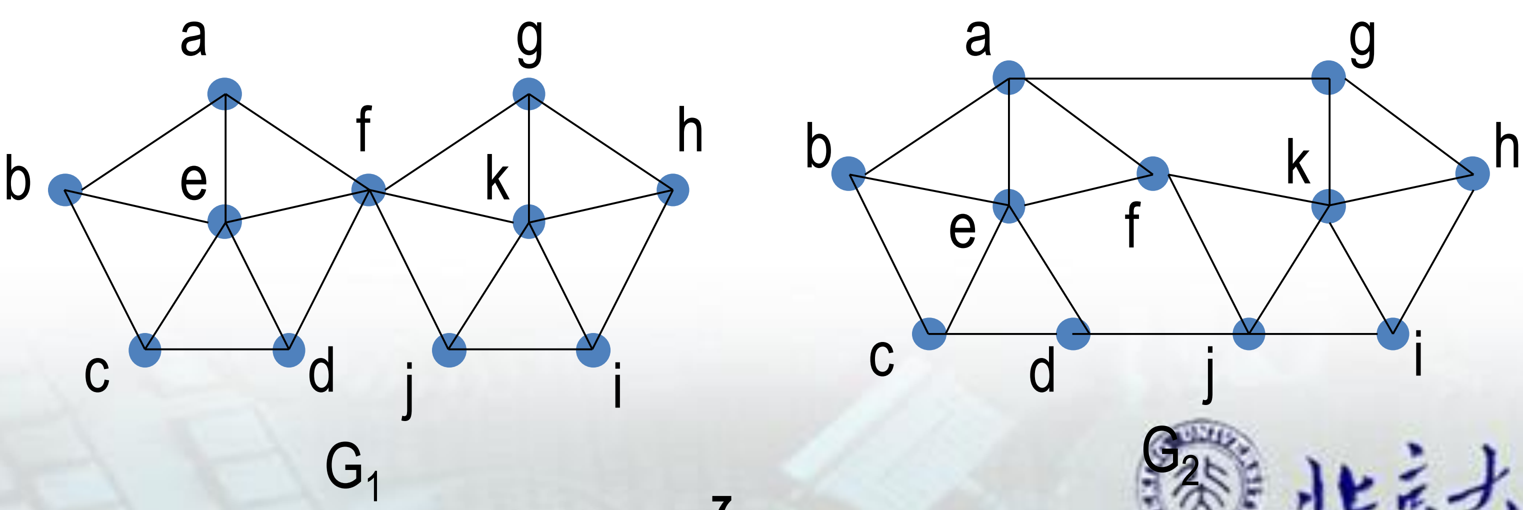
G_1



G_2

边割集

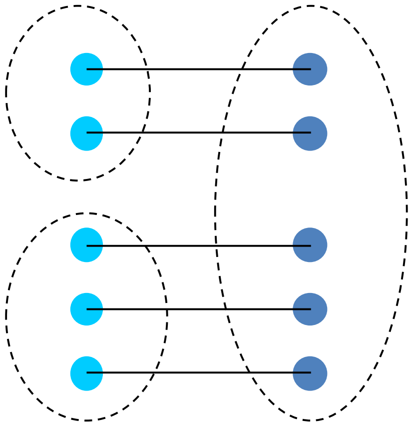
- 边割集：G=<V,E>, $\varnothing \neq E' \subset E$, (1) $p(G-E')>p(G)$;
  (2) $\forall E'' \subset E'$, $p(G-E'')=p(G)$ (极小性条件)
- 例：G_1: {(a,f),(e,f),(d,f)}, {(f,g),(f,k),(j,k),(j,i)},
  {(c,d)}不是， {(a,f),(e,f),(d,f),(f,g),(f,k),(f,j)}不是
  G_2: {(b,a),(b,e),(b,c)}



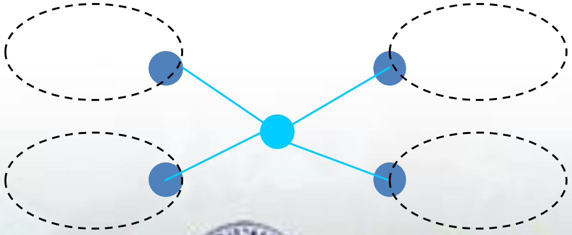
引理1

- 设E’是边割集,则p(G-E’)=p(G)+1.
- 证: 如果p(G-E’)>p(G)+1,

则E’不是边割集, 因为不满足定义中的极小性. #

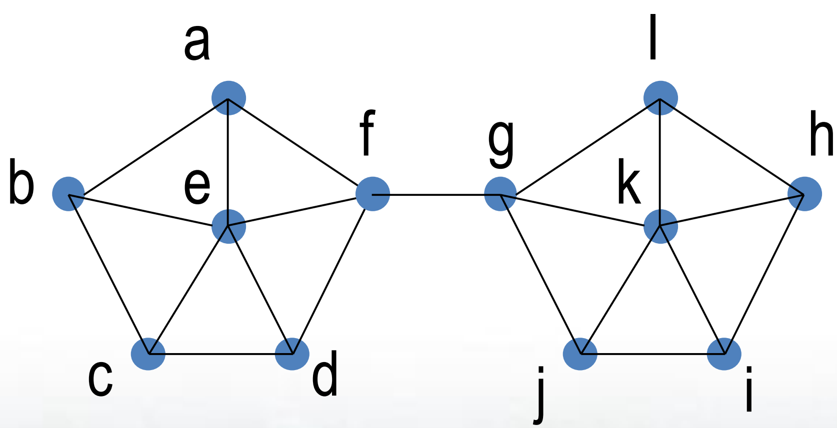


- 注: 点割集无此性质

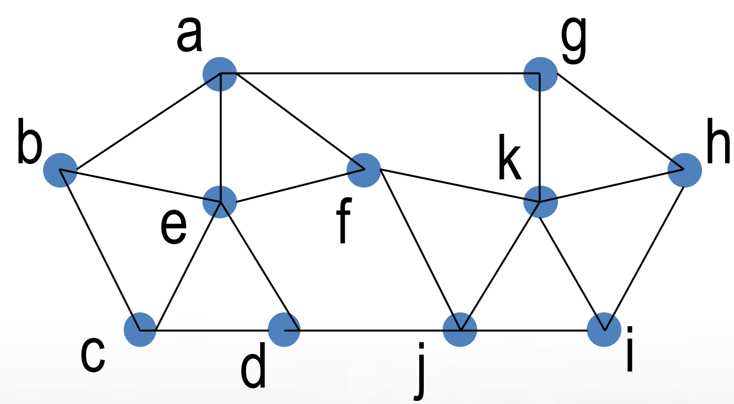


割边 (桥)

- (u,v)是割边 ⇔ {(u,v)}是边割集
- 例: G_1中(f,g)是桥, G_2中无桥



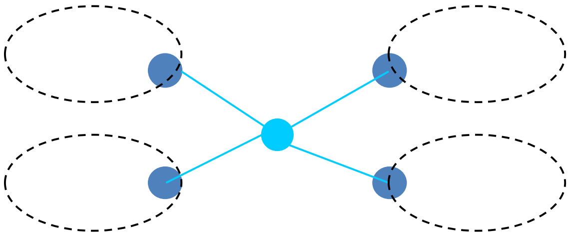
G_1



G_2

扇形割集

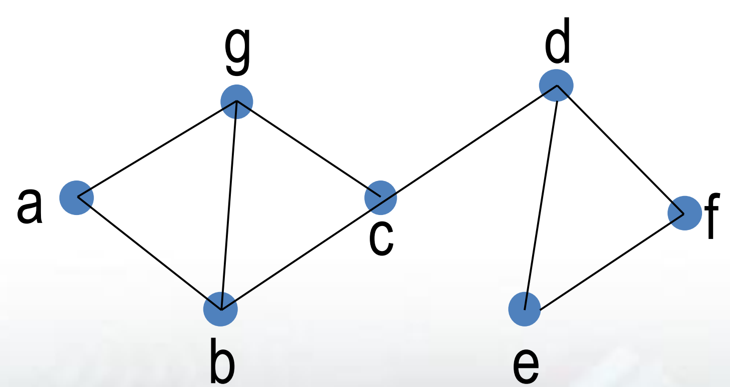
- $I_G(v)$不一定是边割集(不一定极小)
- $I_G(v)$是边割集$\Leftrightarrow$ v不是割点



- 扇形割集: 边割集$E' \subseteq I_G(v)$

扇形割集举例

- {(a,g),(a,b)},{(g,a),(g,b),(g,c)},
- {(c,d)}, {(d,e),(d,f)}
- {(a,b),(g,b),(g,c)}不是



点连通度

- G是无向连通非完全图,

  $\kappa(G) = \min\{\ |V'|\ |\ V'$是G的点割集$\}$

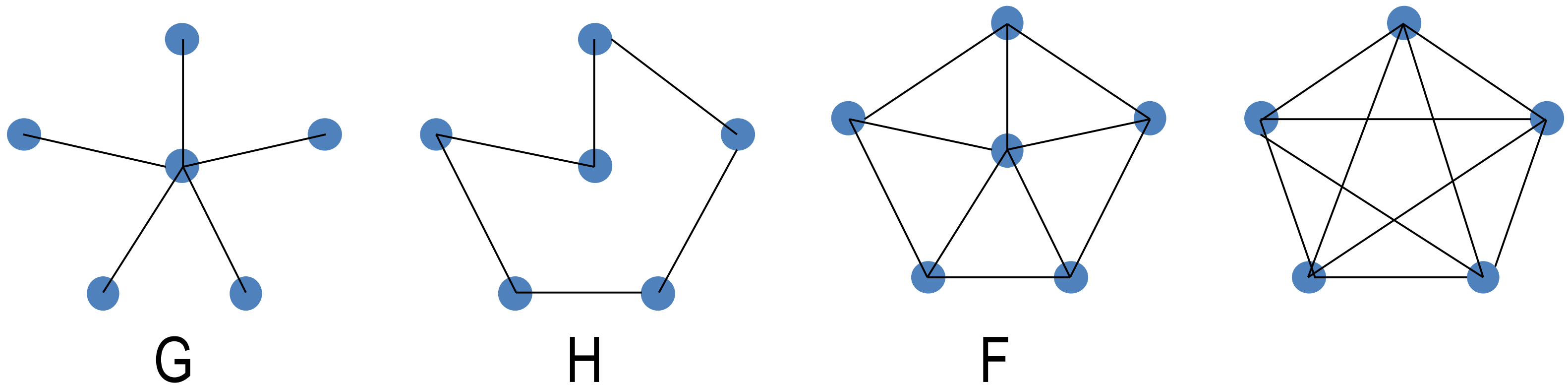
- 规定: $\kappa(K_n) = n-1$

  G非连通: $\kappa(G)=0$

(平凡图N_1连通, 但$\kappa(N_1) = \kappa(K_1) = 0$)

点连通度举例

- $\kappa(G)=1$, $\kappa(H)=2$, $\kappa(F)=3$, $\kappa(K_5)=4$



边连通度

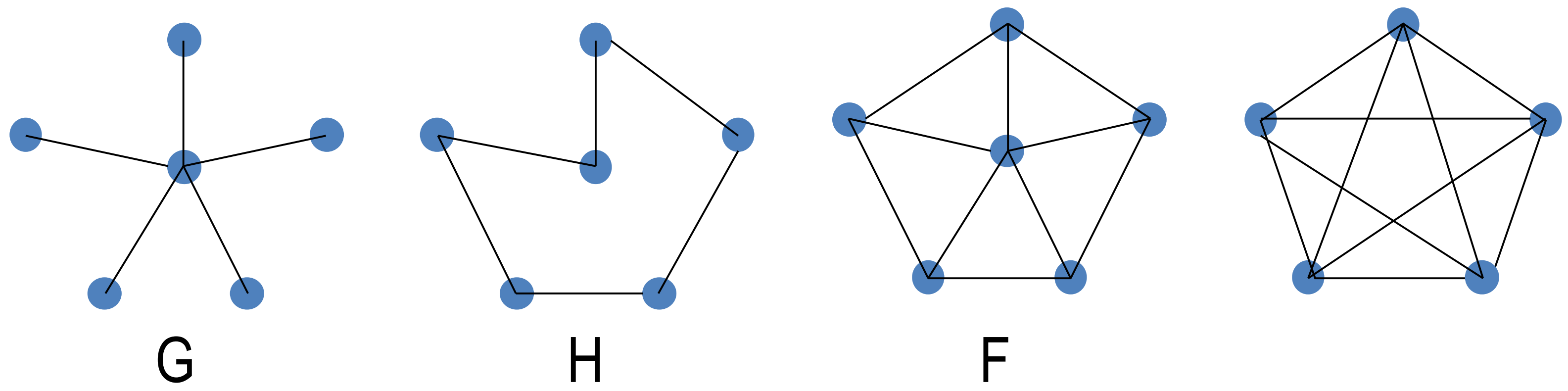
- G是无向连通图,

$$\lambda(G) = \min\{\ |E'|\ \mid E'\text{是}G\text{的边割集}\ \}$$

- 规定: G非连通: $\lambda(G)=0$

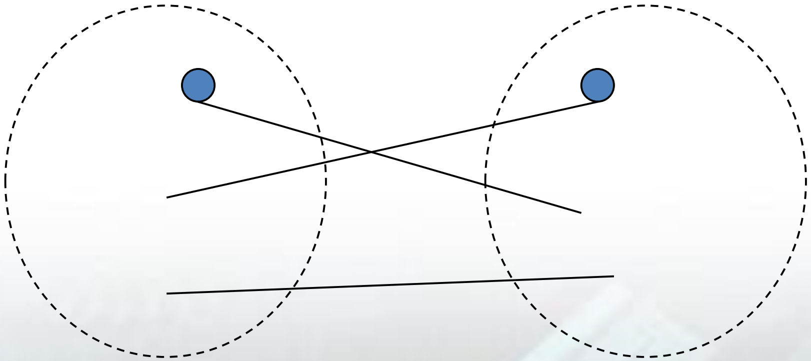
边连通度举例

- $\lambda(G)=1$, $\lambda(H)=2$, $\lambda(F)=3$, $\lambda(K_5)=4$



引理2

- 设E’是非完全图G的最小边割集,
  G-E’的两个(引理1)连通分支是G_1,G_2,
  则存在$u \in V(G_1)$, $v \in V(G_2)$, 使得$(u,v) \notin E(G)$.



引理2证明

- 证: (反证) 否则

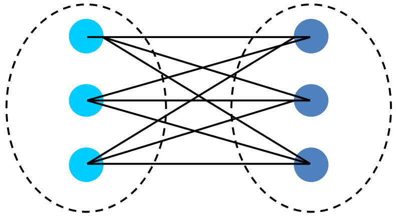
  $\lambda(G) = |E'| = |V(G_1)|\times|V(G_2)|$

  $\geq |V(G_1)|+|V(G_2)|-1=n-1$,

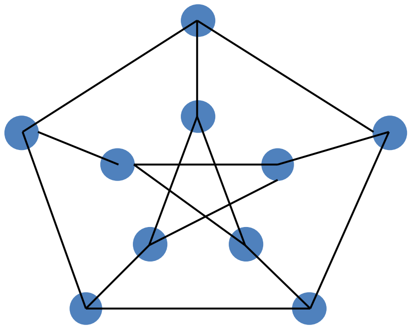
  与G非完全图相矛盾! #

- $a\geq 1 \wedge b\geq 1 \Rightarrow (a-1)(b-1) \geq 0$

  $\Leftrightarrow ab-a-b+1 \geq 0 \Leftrightarrow ab \geq a+b-1$.



k-(边)连通图



- k-连通图：$\kappa(G) \geq k$
- K-边连通图：$\lambda(G) \geq k$
- 例：彼得森图：$\kappa=3, \lambda=3$

  它是1-连通图, 2-连通图,3-连通图,

  但不是4-连通图

  它是1-边连通图, 2-边连通图, 3-边连通图,

  但不是4-边连通图

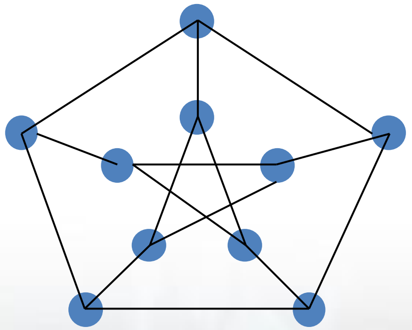
定理

- 定理：对3-正则图G,

$$\kappa(G) = \lambda(G).$$

- 证明：(作业). #

- 彼得森图：$\kappa=3, \lambda=3$



Whitney定理

- 定理7.10(Whitney不等式): 任意G,

$$\kappa(G) \le \lambda(G) \le \delta(G).$$

- 推论: k-连通图一定是k-边连通图. #

Whitney定理证明

- 目标: $\kappa \leq \lambda \leq \delta$.
- 证明: 不妨设G是 3阶以上 连通 非完全 简单图. (否则可直接验证结论成立).

Whitney定理证明

- 第一部分:$\lambda \le \delta$

- 证明: 设 $d_G(v) = \delta$.

  $I_G(v) = \{ (u,v) \mid (u,v) \in E(G) \}$

  则必有扇形边割集 $S \subseteq I_G(v)$,

  所以, $\lambda \le |S| \le |I_G(v)| = \delta$.

Whitney定理证明

- 第二部分:$\kappa \le \lambda$

- 证明: 设边割集E’满足|E’|=λ.
  根据引理1和引理2,
  设G-E’的两个连通分支是G_1和G_2,
设$u \in V(G_1)$,$v \in V(G_2)$,使得$(u,v) \notin E(G)$.

Whitney定理证明

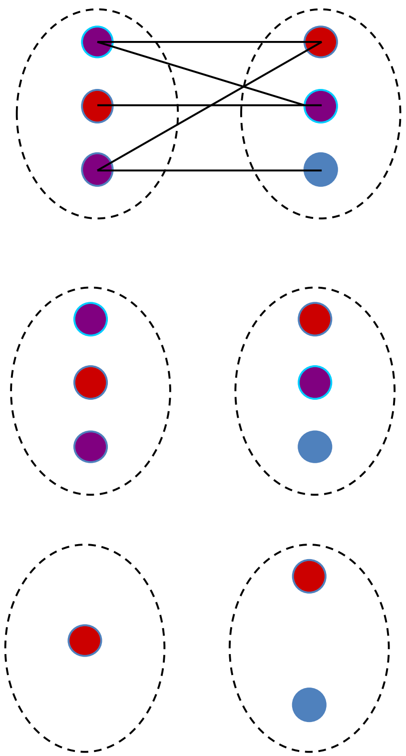
- 如下构造V”：对任何$e \in E'$，

选择e的异于u,v的一个端点放入V”.

则 $u,v \in G-V'' \subseteq G-E' = G_1 \cup G_2$,

所以 V”中含有点割集V’.

故 $\kappa \le |V'| \le |V''| \le |E'| = \lambda$. #



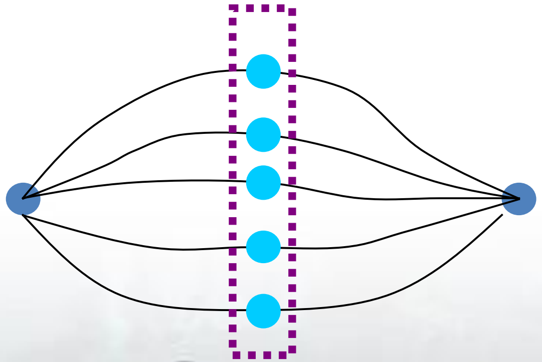
x-y割

- 如果 x,y是G中不相邻顶点,

  $S \subseteq V(G) - \{x,y\}$,

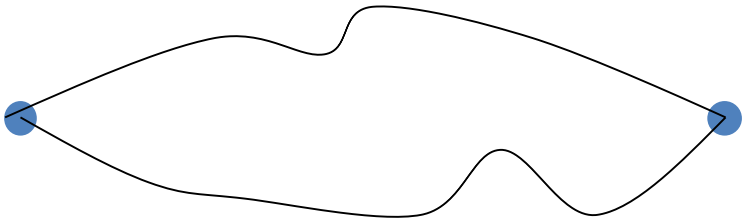
  在G-S中x与y不连通,

则 S称为G中的x-y割

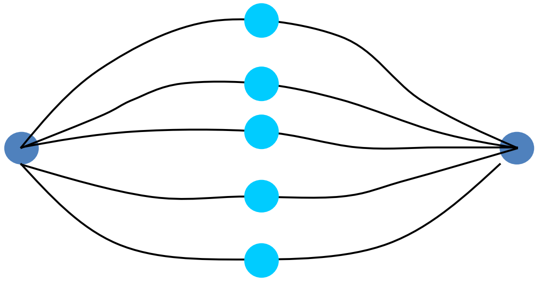


独立路径

- 两条除起点和终点外无其他公共顶点的路径



Menger定理



- 定理(Menger,1927): 在图G中,
  最小的x-y割包含的顶点数
  = 最大的x-y独立路径的条数. #

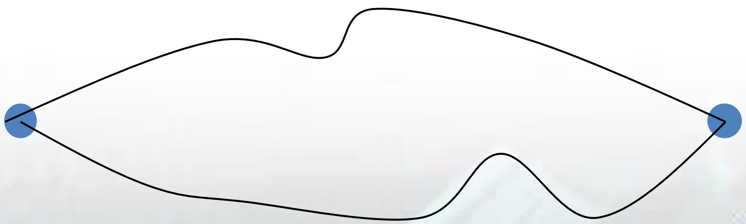
- 最小-最大(min-max)定理

2-连通的充分必要条件

- 定理7.15: 3阶以上无向简单连通图G是2-连通图

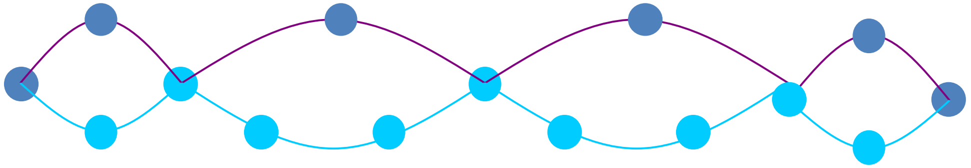
$\Leftrightarrow$ G中任两顶点共圈

$\Leftrightarrow$ G中任两顶点之间有2条以上独立路径. #



边不交路径

- 两条无公共边(但可能有公共顶点)的路径



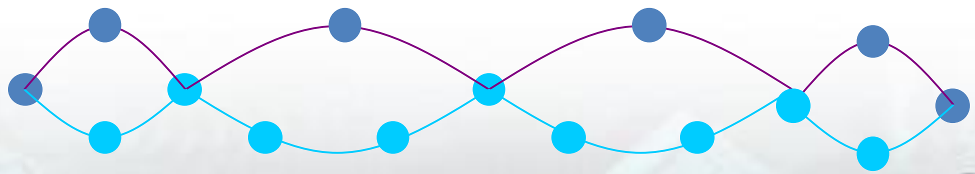
2-边连通的充分必要条件

- **定理7.16**:

  3阶以上无向图G是2-边连通图

  ⇔ G中任2顶点共简单回路

  ⇔ G中任2顶点间有2条以上边不交路径. #



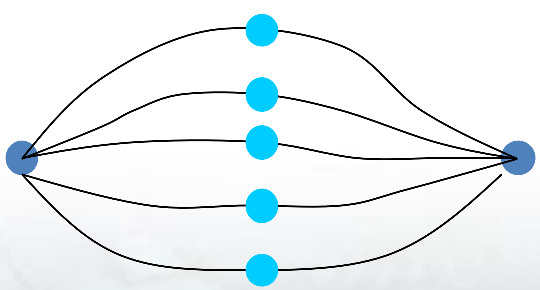
k-(边)连通的充分必要条件

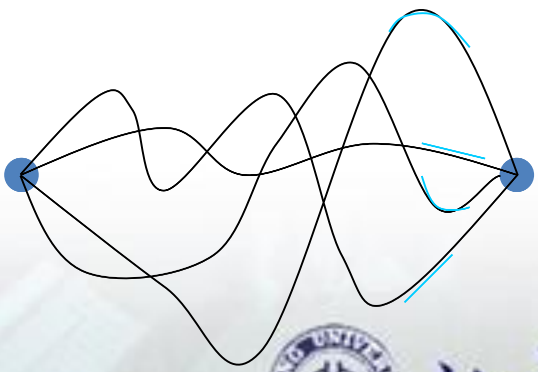
- 定理：3阶以上无向图G是k-连通图

⇔ G中任2顶点间有k条以上独立路径. #

- 定理：3阶以上无向图G是k-边连通图

⇔ G中任2顶点间有k条以上边不交路径. #





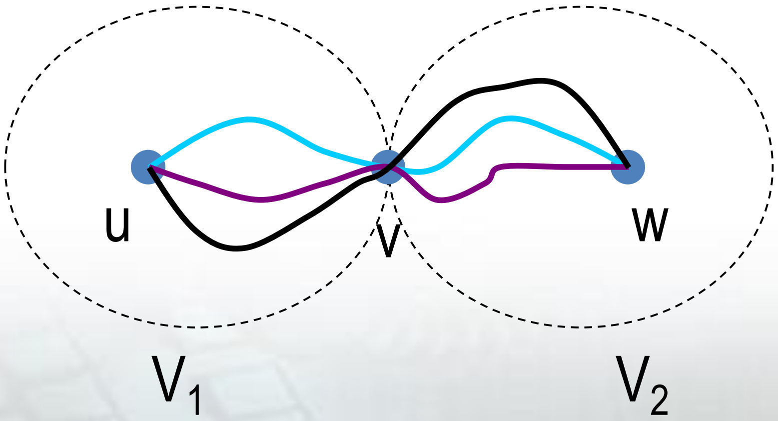
割点的充分必要条件

- **定理7.17:**

  无向连通图G中顶点v是割点

  ⇔ 可把V(G)-{v}划分成V_1与V_2, 使得从V_1中任意顶点u到V_2中任意顶点w的路径都要经过v.

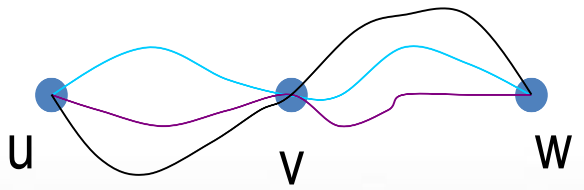
  #



割点的充分必要条件

- 推论: 无向连通图G中顶点v是割点

⇔ 存在与v不同的顶点u和w,使得从顶点u到w的路径都要经过v. #

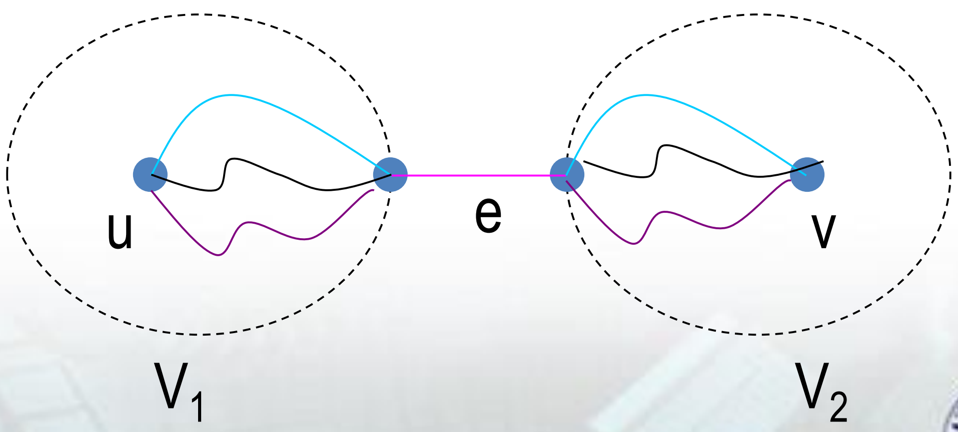


桥的充分必要条件

- 定理7.18-19: 无向连通图G中边e是桥

⇔ G的任何圈都不经过e

⇔ 可把V(G)划分成V_1与V_2, 使得从V_1中 任意顶点u到V_2中任意顶点v的路径都要经过e. #

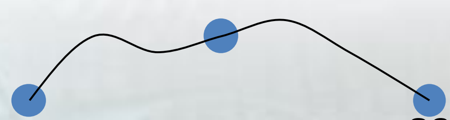


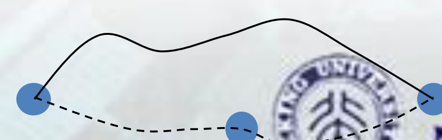
块(block)

- 块: 极大无割点连通子图

块的充分必要条件

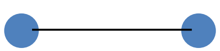
- 定理7.20: G是3阶以上无向简单连通图. 则G是块 ⇔ G中任意2顶点共圈 ⇔ G中任意1顶点与任意1边共圈 ⇔ G中任意2边共圈 ⇔ G中任意2顶点与任意1边,有路径连接这2顶点并经过这1边 ⇔ G中任意3顶点,有路径连接其中2顶点并经过第3点⇔G中任意3顶点,有路径连接其中2顶点并不经过第3点. #





几个概念的比较

- 块: 极大无割点连通子图
- 2-连通图: $\kappa\geq2$, 即连通无割点图
- 2-边连通图: $\lambda\geq2$, 即连通无桥图
- 2-连通 $\subset$ 2-边连通 (可能 $\kappa<\lambda$)
- 2-连通 $\subset$ 块 (K_2是块,不是2-连通)
- 块 $\neq$ 2-边连通(K_2是块,不是2-边连通; 8字形图是2-边连通,不是块)



定理7.14

- n阶简单连通图的κ,λ,δ之间关系有且仅有3种可能:

  (1) $\kappa = \lambda = \delta = n-1$

  (2) $1 \le 2\delta-n+2 \le \kappa \le \lambda = \delta \le n-2$

  (3) $0 \le \kappa \le \lambda \le \delta < \lfloor n/2 \rfloor$

- 注: $1 \le 2\delta-n+2 \Leftrightarrow (n-1)/2 \le \delta$

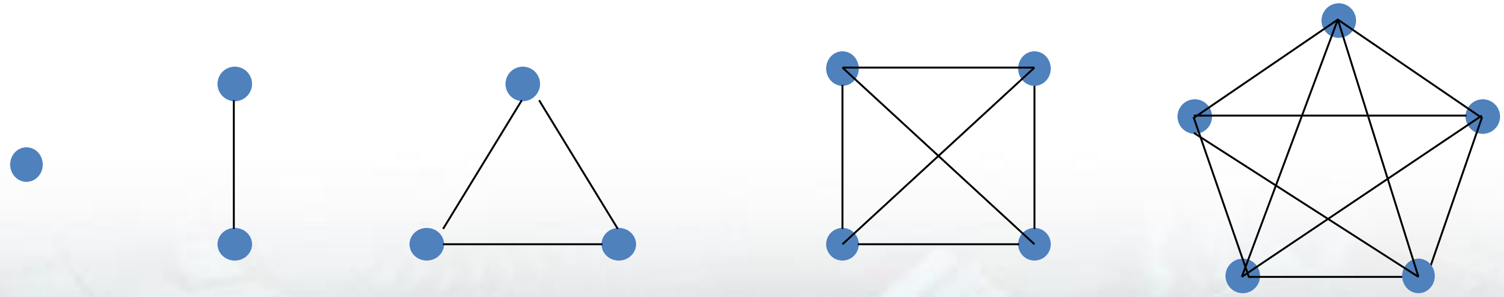
  $\Leftrightarrow \lfloor n/2 \rfloor \le \delta$

定理7.14证明(有)(1)

- 目标: (有): (1) $\kappa = \lambda = \delta = n\text{-}1$.
- 构造:　令 $G = K_n$即可.

- 注意: 非连通图 $\Rightarrow \kappa=\lambda=0$

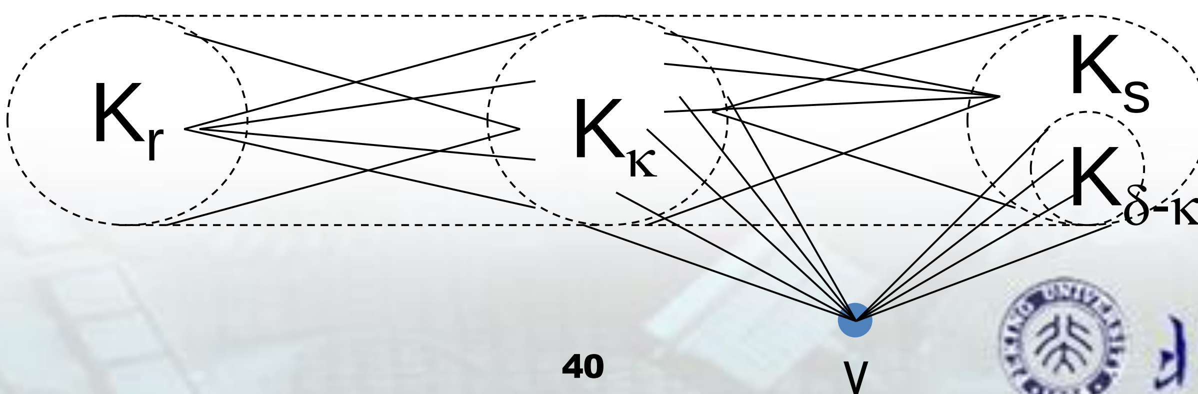
  但是K_1连通, $\kappa(K_1)=\lambda(K_1)=\delta(K_1)=0$



定理7.14证明(有)(2)

- 目标: $1 \le 2\delta - n + 2 \le \kappa \le \lambda = \delta \le n-2$
- 构造: 令 $r = \lfloor (n-\kappa)/2 \rfloor$, $s = \lfloor (n-\kappa-1)/2 \rfloor$,

$r+s = n-\kappa-1$. 令 $G' = K_\kappa + (K_r \cup K_s)$. 给 G' 增加顶点 v, 使得 v 与 K_κ 中所有顶点相邻, 与 K_s 中 $\delta-\kappa$ 个顶点相邻, 就得到 G.



定理7.14证明(有)(2)

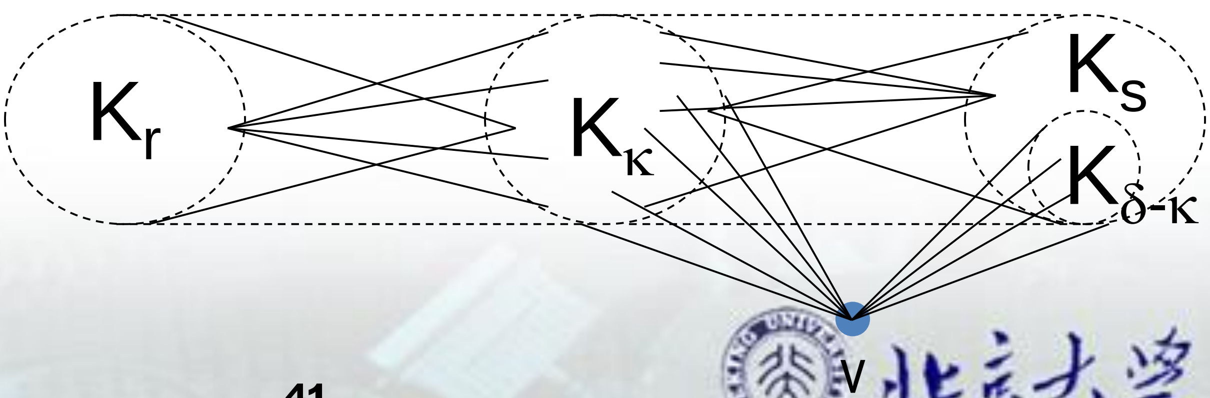
- 分析: $\delta(G)=\delta$:

K_κ: $d(u) = \kappa-1+r+s+1 = n-1 \geq \delta$,

K_r: $d(u) = r-1+\kappa \geq \delta$,

K_s: $d(u) = s-1+\kappa \geq \delta$,

v: $d(v) = \delta$.

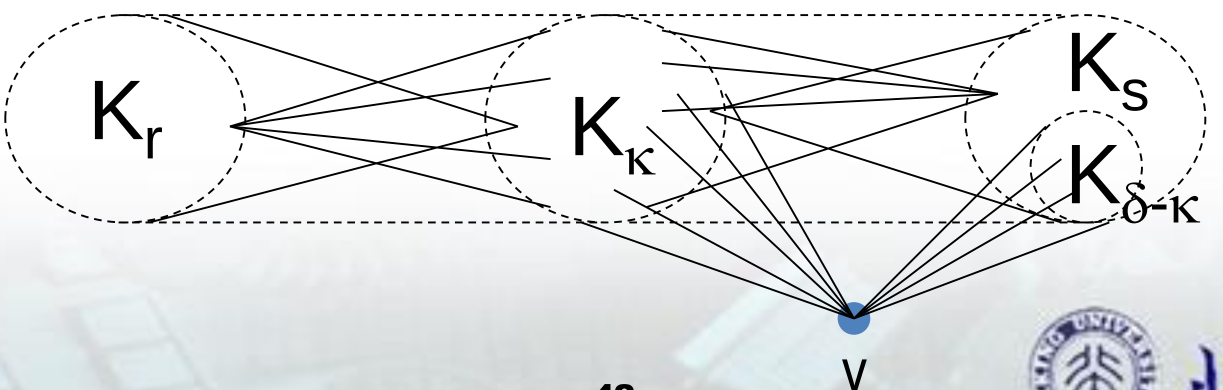


定理7.14证明(有)(2)

- 分析:

$\kappa(G)=\kappa$: 删除K_κ.

$\lambda(G)=\lambda=\delta$: 删除$I_G(v)$.



定理7.14证明(有)(3)

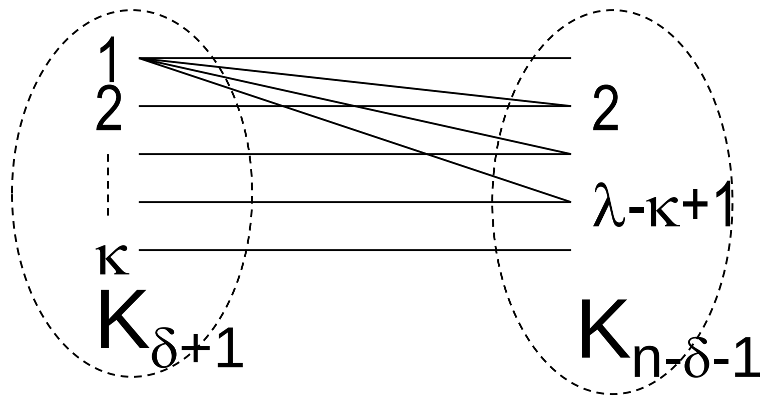
- 目标: $0 \le \kappa \le \lambda \le \delta < \lfloor n/2 \rfloor$
- 构造:令 $G' = K_{\delta+1} \cup K_{n-\delta-1}$, 设

$V(K_{\delta+1}) = \{u_1, u_2, \ldots, u_{\delta+1}\}$,

$V(K_{n-\delta-1}) = \{v_1, v_2, \ldots, v_{n-\delta-1}\}$,

给 G' 增加边 (u_i, v_i), $i = 1, 2, \ldots, \kappa$,

以及 (u_1, v_i), $i = 2, 3, \ldots, \lambda-\kappa+1$, 就得到 G.



定理7.14证明(有)(3)

- 分析: $\delta(G)=\delta$:

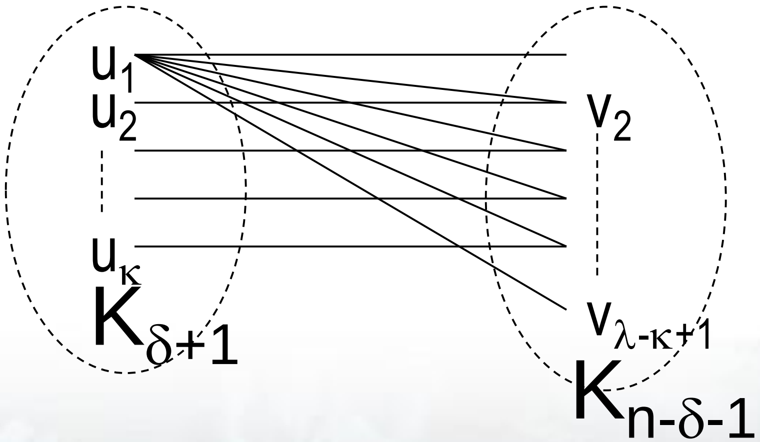
  $K_{\delta+1}$: $d(u)\geq\delta$, $K_{n-\delta-1}$: $d(u)\geq n-\delta-2\geq\delta$.

$\kappa(G)=\kappa$: 删除$\{ u_i \mid i=1,2,\ldots,\kappa \}$,

$\lambda(G)=\lambda$: 删除

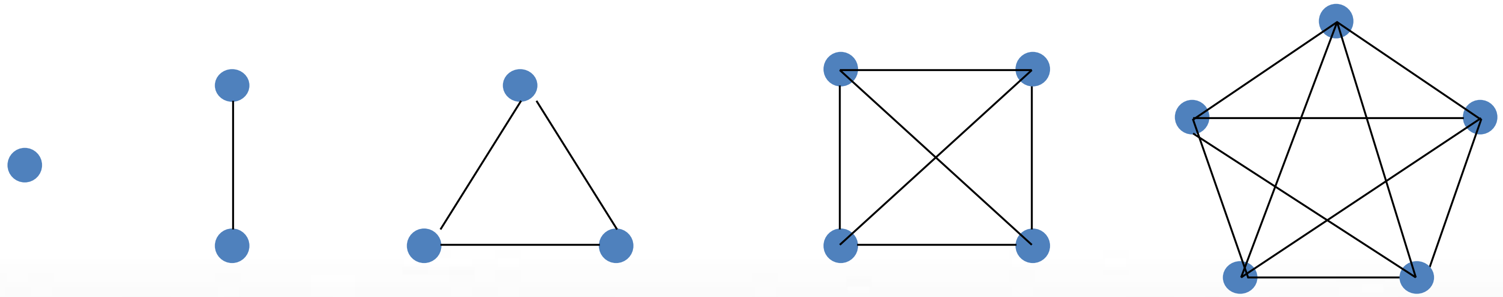
$\{(u_i,v_i) \mid i=1,2,\ldots,\kappa\} \cup$

$\{(u_1,v_i) \mid i=2,3,\ldots,\lambda-\kappa+1\}$



定理7.14(仅有)(1)

- 如果 G是完全图, 则 $G=K_n$,
  所以 $\kappa = \lambda = \delta = n-1$.



定理7.14(仅有)(2)(3)

- (2) $1 \le 2\delta - n + 2 \le \kappa \le \lambda = \delta \le n-2$

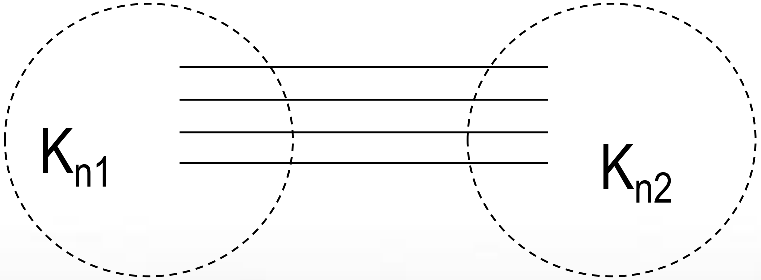
  $\delta \ge \lfloor n/2 \rfloor$时, 定理7.12, 7.13.

- (3) $0 \le \kappa \le \lambda \le \delta < \lfloor n/2 \rfloor$

  $\delta < \lfloor n/2 \rfloor$时, Whitney定理. #

定理7.11

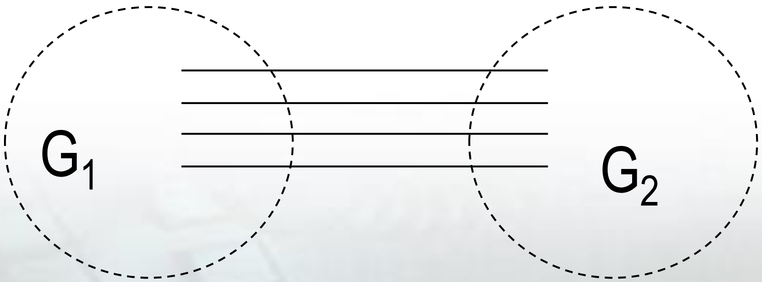
- G是n阶简单无向连通图,$\lambda(G)<\delta(G)$,则存在G*以G为生成子图,G*由完全图K_{n1}和K_{n2},以及它们之间的$\lambda(G)$条边组成,$\lambda(G)+2\leq n_1\leq\lfloor n/2\rfloor$.

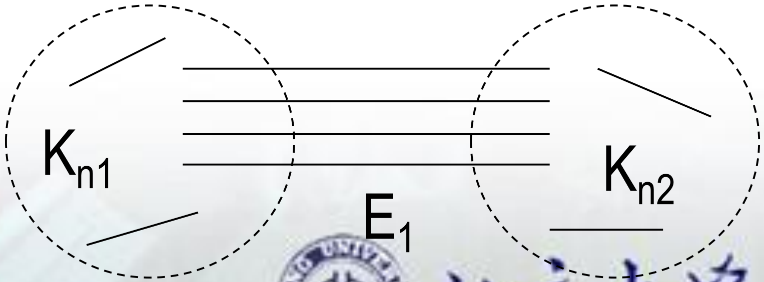


定理7.11证明

- 证: 设E_1是G的最小边割,$|E_1|=\lambda(G)$.

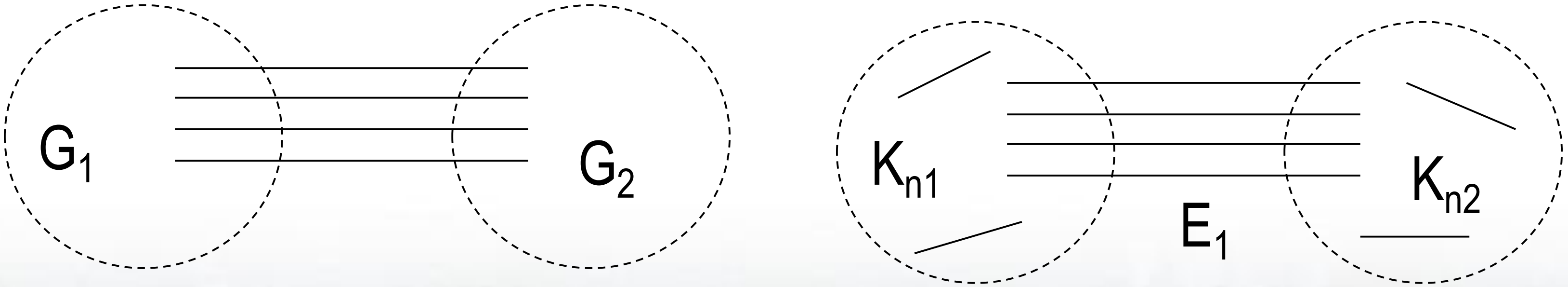
  设$G-E_1$的2个连通分支是G_1与G_2, $|V(G_1)|=n_1$, $|V(G_2)|=n_2$, 不妨设$n_1 \le n_2$, 显然$n_1+n_2=n$, $n_1 \le \lfloor n/2 \rfloor$.





定理7.11证明

- 给G_1加新边使它成为K_{n1}，
  给G_2加新边使它成为K_{n2}，
  令 $G^* = K_{n1} \cup E_1 \cup K_{n2}$.



定理7.11证明

- $\lambda(G)<\delta(G)\leq\delta(G^*)\leq n_1-1+\lfloor\lambda(G)/n_1\rfloor$

$\Rightarrow \lambda(G)<n_1-1+\lambda(G)/n_1 \Leftrightarrow (n_1-1)(n_1-\lambda(G))>0$

$\Rightarrow \lambda(G)<n_1 \Rightarrow \lambda(G) \leq n_1-1.$

$\lambda(G)=n_1-1 \Rightarrow \lambda(G)=n_1-1+\lfloor\lambda(G)/n_1\rfloor$

$\Rightarrow \lambda(G)<\delta(G)\leq\delta(G^*)\leq\lambda(G)$ (矛盾!)

$\lambda(G)<n_1-1 \Rightarrow \lambda(G) \leq n_1-2 \Rightarrow \lambda(G)+2\leq n_1$. #

推论

- (1) $\delta(G)\le\delta(G^*)\le n_1-1\le\lfloor n/2\rfloor-1$

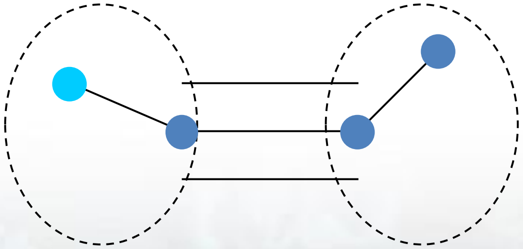
  (2) G*中有不相邻顶点u,v,使得

  $$d_{G^*}(u)+d_{G^*}(v)\le n-2$$

  (3) $d(G)\ge d(G^*)\ge 3$
- 证明: (2)$u\in G_1, v\in G_2$,在G中不相邻,则在G*中仍然不相邻.

  (3) $d(G)=\max d(u,v)$

  $\lambda(G)\le n_1-2$ #



定理7.12

- G是6阶以上连通简单无向图.

(1) $\delta(G) \geq \lfloor n/2 \rfloor \Rightarrow \lambda(G)=\delta(G)$

(2) 任意一对不相邻顶点u,v都有

$$d(u)+d(v) \geq n-1,$$

$\Rightarrow \lambda(G)=\delta(G)$.

(3) $d(G) \leq 2 \Rightarrow \lambda(G)=\delta(G)$.   #

定理7.13

- 定理7.13 G是n阶简单连通无向非完全图,则 $2\delta(G)-n+2 \le \kappa(G)$.
- 证: 设V_1是G的点割集, $|V_1|=\kappa(G)$, 设$G-V_1$的连通分支是$G_1,G_2,\ldots,G_s(s\ge2)$, 设$|V(G_1)|=n_1$, $|V(G_2)|+\ldots+|V(G_s)|=n_2$, 则$n_1+ n_2+\kappa(G)=n$.

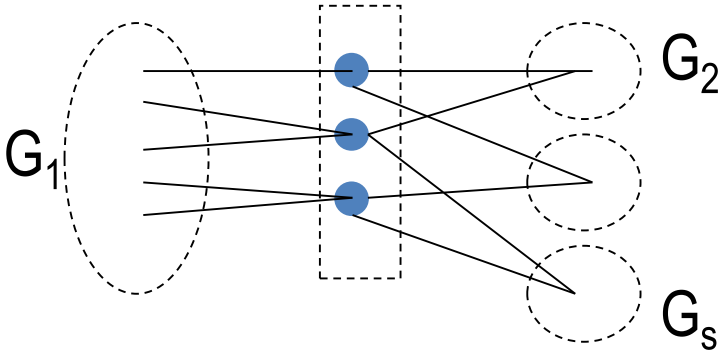
$\delta(G)\le n_1-1+\kappa(G)=n_1+\kappa(G)-1$,

并且 $\delta(G)\le n_2+\kappa(G)-1$

$\Rightarrow 2\delta(G) \le n_1+\kappa(G)+n_2+\kappa(G)-2$

$= n+\kappa(G)-2$

$\Rightarrow \kappa(G) \ge 2\delta(G)-n+2.$ #



小结

- 点割集, 边割集, 点(边)连通度κ(λ);
- κ, λ, δ之间关系, Whitney定理 等
- k-(边)连通图, Menger定理 等
- 2-(边)连通,割点,桥,块的充要条件