一阶谓词演算自然推演系统$N_{\mathfrak{L}}$

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复习

$\mathfrak{L}$生成的一阶语言:

- 符号库: $\left\{\begin{array}{l}\text{个体变元}\\\text{个体常元}\\\text{谓词}\\\text{函数}\\\text{量词}\\\text{联结词}\\\text{辅助符号}\end{array}\right.$
- 公式: $\left\{\begin{array}{l}\text{项}\\\text{公式}\\\text{自由与约束}\end{array}\right.$

推演系统$\mathrm{N}_{\mathfrak{L}}$的构成

给定非逻辑符号集$\mathfrak{L}$, $\mathrm{N}_{\mathfrak{L}}$的构成如下:

- 形式语言:
  - $\mathfrak{L}$生成的一阶语言
- 形式推理:
  - 形式公理: $\varnothing$
  - 形式规则: 15条

    (1)–(10) 如N

增加前提律

$$\frac{\text{若}\Gamma \vdash \alpha}{\text{则}\Gamma,\ \beta \vdash \alpha} \qquad (+)$$

思考题:

为什么把(+)作为规则，而不是象在命题演算那样作为原定理?

$\forall$消去律

$$\frac{\text{若}\Gamma \vdash \forall x\alpha,\quad \text{且}t\text{对}x\text{在}\alpha\text{中自由,}}{\text{则}\Gamma \vdash \alpha(x/t)} \qquad (\forall -)$$

直观含义:

若Γ能保证对任意的x, $\alpha(x)$都成立,

则Γ也能保证当α中的x''取值''为t的时候也成立。

注意:条件''t对x在α中自由''不可少。

$\forall$引入律

$$\frac{\text{若}\Gamma \vdash \alpha,\quad \text{且}x\text{不在}\Gamma\text{的任何公式中自由出现}}{\text{则}\Gamma \vdash \forall x\alpha} \qquad (\forall+)$$

直观含义:

若$\Gamma(x)$能保证$\alpha(x)$成立，但Γ没有对x作任何限制，则Γ也能保证对任意的x，$\alpha(x)$都成立。

注意:条件"x不在Γ的任何公式中自由出现"不可少.

∃消去律

$$\frac{\text{若}\Gamma, \alpha \vdash \beta, \quad \text{且}x\text{不在}\Gamma \cup \{\beta\}\text{的任何公式中自由出现}}{\text{则}\Gamma,\ \exists x\alpha \vdash \beta} \quad (\exists-)$$

直观含义:

若Γ和$\alpha(x)$一起才能保证$\beta(x)$成立,

但Γ和β的性质与x无关

则Γ和$\exists x\alpha(x)$也能保证$\beta(x)$也成立。

思考题: 条件"α成立"和"$\exists x\alpha$成立"哪个更强?

注意:条件"x不在$\Gamma \cup \{\beta\}$的任何公式中自由出现"不可少.

∃引入律

$$\frac{\text{若}\Gamma \vdash \alpha(x/t),\quad \text{且}t\text{对}x\text{在}\alpha\text{中自由},}{\text{则}\Gamma \vdash \exists x\alpha} \qquad (\exists +)$$

直观含义：

若Γ能保证当α中的x"取值"为t的时候成立，

则Γ一定能保证有一个x使α成立.

(t是使得$\exists x\alpha$成立的"证据".)

注意:条件"t对x在α中自由"不可少。

两个特例

注意到：$\alpha(x/x) = \alpha$，且x对x在α中自由.

$$\frac{\text{若}\Gamma \vdash \forall x\alpha,}{\text{则}\Gamma \vdash \alpha} \qquad ((\forall-)\text{中取}t\text{为}x)$$

$$\frac{\text{若}\Gamma \vdash \alpha,}{\text{则}\Gamma \vdash (\exists x)\alpha} \qquad ((\exists+)\text{中取}t\text{为}x)$$

$\mathrm{N}_{\mathfrak{L}}$的形式证明序列

若有限序列

$$\Gamma_1 \vdash \alpha_1,\ \Gamma_2 \vdash \alpha_2,\ \cdots,\ \Gamma_n \vdash \alpha_n$$

满足:

(1) 每个$\Gamma_i (i : 1 \leq i \leq n)$都是$\mathrm{N}_{\mathfrak{L}}$的有限公式集。

(2) 每个$\Gamma_i \vdash \alpha_i$ $(1 \leq i \leq n)$ 都是对此序列中它之前的若干个$\Gamma_j \vdash \alpha_j$ $(1 \leq j < i)$ 应用$\mathrm{N}_{\mathfrak{L}}$的形式规则得到的.

则称此序列为$\mathrm{N}_{\mathfrak{L}}$中的一个(形式)证明序列.

此时也称 α_n可由Γ_n在$\mathrm{N}_{\mathfrak{L}}$中形式推出,

记为$\Gamma_n \vdash_{\mathrm{N}_{\mathfrak{L}}} \alpha_n$, 或$\Gamma_n \vdash \alpha_n$.

注：和**N**中证明序列的定义几乎一样。

N的定理也是$\mathrm{N}_{\mathfrak{L}}$的的定理

定理1

设Σ与α分别为**N**中有限公式集与公式，在$\Sigma \cup \{\alpha\}$的公式中出现的命题变元符号都在p_0，p_1，$\cdots$，p_n中，α_0，α_1，$\cdots$，α_n为$\mathfrak{L}$的公式，以α_0，α_1，$\cdots$，α_n同时分别替换$\Sigma \cup \{\alpha\}$的公式中的p_0，p_1，$\cdots$，p_n，得到$\mathfrak{L}$中的公式集Σ'与公式α'.

若$\Sigma \vdash_{\mathrm{N}} \alpha$，则$\Sigma' \vdash_{\mathrm{N}_{\mathfrak{L}}} \alpha'$.

定理1的证明

证：因$\Sigma \vdash_{\mathbf{N}} \alpha$，故存在**N**中证明序列：

$$\Sigma_1 \vdash \beta_1,\ \Sigma_2 \vdash \beta_2,\ \cdots,\ \Sigma_k \vdash \beta_k \quad (= \Sigma \vdash \alpha)$$

设$\Sigma_1 \cup \cdots \cup \Sigma_k \cup \{\beta_1, \beta_2, \cdots, \beta_k\}$的公式中出现的命题变元符号都在$p_0, \cdots, p_n, p_{n+1}, \cdots, p_{n+m}$中.任选定$\mathfrak{L}$中的m个公式$\alpha_{n+1},\ \cdots,\ \alpha_{n+m}$，将

$$\Sigma_1, \Sigma_2, \cdots, \Sigma_k,\ \beta_1, \beta_2, \cdots, \beta_k$$

的公式中出现的 $p_0, \cdots, p_n, p_{n+1}, \cdots, p_{n+m}$同时分别替换为 $\alpha_0, \cdots, \alpha_n, \alpha_{n+1},\ \cdots,\ \alpha_{n+m}$ 得到$\Sigma_1', \Sigma_2', \cdots, \Sigma_k',\ \beta_1', \beta_2', \cdots, \beta_k'$.

定理1的证明(续)

则

$$\Sigma'_1 \vdash \beta'_1,\ \Sigma'_2 \vdash \beta'_2,\ \cdots,\ \Sigma'_k \vdash \beta'_K$$

为$\mathrm{N}_{\mathfrak{L}}$中的一个证明.
(因为**N**的形式规则在写法上同$\mathrm{N}_{\mathfrak{L}}$的形式规则相同.
也可归纳法进行严格证明.)

又$\Sigma'_k = \Sigma'$, $\beta'_k = \alpha'$, 故$\Sigma' \vdash_{\mathrm{N}_{\mathfrak{L}}} \alpha'$.

一个推论

定理2 对于$\mathfrak{L}$的有限公式集Γ与公式$\alpha_1, \alpha_2, \cdots, \alpha_n$.

1. 若$\Gamma \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha_1, \alpha_2, \cdots, \alpha_n$,
   且$\alpha_1, \alpha_2, \cdots, \alpha_n \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha$,
   则$\Gamma \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha$.

2. 若$\alpha_1 \to \alpha_2 \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha_3 \to \alpha_4$,
   且$\alpha_1 \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha_2$,
   则$\alpha_3 \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha_4$.

其证明和命题情形一样。

例6

在$\mathrm{N}_{\mathfrak{L}}$中写出下列公式的证明序列:

1. $\forall x(\alpha \to \beta) \vdash\dashv \alpha \to \forall x\beta$, 若$x$不在$\alpha$中自由出现.
2. $\forall x(\alpha \to \beta) \vdash\dashv \exists x\alpha \to \beta$, 若$x$不在$\beta$中自由出现.
3. $\forall x(\alpha \to \beta) \vdash \exists x\alpha \to \exists x\beta$,
4. $\forall x(\alpha \to \beta) \vdash \forall x\alpha \to \forall x\beta$,

例6(1)($\vdash$)的证明

1. $\forall x(\alpha \to \beta) \vdash\dashv \alpha \to \forall x\beta$, 若$x$不在$\alpha$中自由出现.

证: ($\vdash$)

(1)	$\forall x(\alpha \to \beta),\ \alpha \vdash \forall x(\alpha \to \beta)$	$(\in)$
(2)	$\forall x(\alpha \to \beta),\ \alpha \vdash \alpha \to \beta$	$(\forall -)$
(3)	$\forall x(\alpha \to \beta),\ \alpha \vdash \alpha$	$(\in)$
(4)	$\forall x(\alpha \to \beta),\ \alpha \vdash \beta$	$(\to -)$
(5)	$\forall x(\alpha \to \beta),\ \alpha \vdash \forall x\beta$	
	(x不在前提中自由出现)	$(\forall +)$
(6)	$\forall x(\alpha \to \beta) \vdash \alpha \to \forall x\beta$	$(\to +)$

例6(1)($\dashv$)的证明

1. $\forall x(\alpha \to \beta) \vdash\dashv \alpha \to \forall x\beta$, 若$x$不在$\alpha$中自由出现.

证: ($\dashv$)

(1)	$\alpha \to \forall x\beta,\ \alpha \vdash \alpha \to \forall x\beta$	$(\in)$
(2)	$\alpha \to \forall x\beta,\ \alpha \vdash \alpha$	$(\in)$
(3)	$\alpha \to \forall x\beta,\ \alpha \vdash \forall x\beta$	$(\to -)$
(4)	$\alpha \to \forall x\beta,\ \alpha \vdash \beta$	$(\forall -)$
(5)	$\alpha \to \forall x\beta \vdash \alpha \to \beta$	$(\to +)$
(6)	$\alpha \to \forall x\beta \vdash \forall x(\alpha \to \beta)$	
	(x不在前提中自由出现)	$(\forall +)$

例6(2)($\vdash$)的证明

2. $\forall x(\alpha\to\beta) \vdash\dashv \exists x\alpha\to\beta$, 若$x$不在$\beta$中自由出现.

证: ($\dashv$)

$$(1)\ \forall x(\alpha\to\beta),\ \alpha\vdash\forall x(\alpha\to\beta) \qquad (\in)$$

$$(2)\ \forall x(\alpha\to\beta),\ \alpha\vdash\alpha\to\beta \qquad (\forall-)$$

$$(3)\ \forall x(\alpha\to\beta),\ \alpha\vdash\alpha \qquad (\in)$$

$$(4)\ \forall x(\alpha\to\beta),\ \alpha\vdash\beta \qquad (\to -)$$

$$(5)\ \forall x(\alpha\to\beta),\ \exists x\alpha\vdash\beta$$

(x不在$\forall x(\alpha\to\beta)$及$\beta$中自由出现) $(\exists-)$

$$(6)\ \forall x(\alpha\to\beta)\vdash\exists x\alpha\to\beta \qquad (\to +)$$

例6(1)($\dashv$)的证明

2. $\forall x(\alpha \to \beta) \vdash\dashv \exists x\alpha \to \beta$, 若$x$不在$\beta$中自由出现.

证: ($\dashv$)

(1)	$\exists x\alpha \to \beta,\ \alpha \vdash \alpha$	$(\in)$
(2)	$\exists x\alpha \to \beta,\ \alpha \vdash \exists x\alpha$	$(\exists +)$
(3)	$\exists x\alpha \to \beta,\ \alpha \vdash \exists x\alpha \to \beta$	$(\in)$
(4)	$\exists x\alpha \to \beta,\ \alpha \vdash \beta$	$(\to -)$
(5)	$\exists x\alpha \to \beta \vdash \alpha \to \beta$	$(\to +)$
(6)	$\exists x\alpha \to \beta \vdash \forall x(\alpha \to \beta)$	
	(x不在前提中自由出现)	$(\forall +)$

例6(3)的证明

3. $\forall x(\alpha \to \beta) \vdash \exists x\alpha \to \exists x\beta$

证:

证:	(1)	$\forall x(\alpha \to \beta),\ \alpha \vdash \forall x(\alpha \to \beta)$	$(\in)$
	(2)	$\forall x(\alpha \to \beta),\ \alpha \vdash \alpha \to \beta$	$(\forall -)$
	(3)	$\forall x(\alpha \to \beta),\ \alpha \vdash \alpha$	$(\in)$
	(4)	$\forall x(\alpha \to \beta),\ \alpha \vdash \beta$	$(\to -)$
	(5)	$\forall x(\alpha \to \beta),\ \alpha \vdash \exists x\beta$	$(\exists +)$
	(6)	$\forall x(\alpha \to \beta),\ \exists x\alpha \vdash \exists x\beta$	$(\exists -)$
	(7)	$\forall x(\alpha \to \beta) \vdash \exists x\alpha \to \exists x\beta$	$(\to +)$

例6(4)的证明

4. $\forall x(\alpha \to \beta) \vdash \forall x\alpha \to \forall x\beta$

证: (1) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \forall x(\alpha \to \beta)$ $(\in)$

(2) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \alpha \to \beta$ $(\forall -)$

(3) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \forall\alpha$ $(\in)$

(4) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \alpha$ $(\forall -)$

(5) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \beta$ $(\to -)$

(6) $\forall x(\alpha \to \beta),\ \forall x\alpha \vdash \forall x\beta$ $(\forall +)$

(7) $\forall x(\alpha \to \beta) \vdash \forall x\alpha \to \forall x\beta$ $(\to +)$

作业

p.559(p.184)

14. (2),(4),(5)

谢　谢

复习

$N_{\mathfrak{L}}$与N的比较:

- 符号库: $N_{\mathfrak{L}}$比N多 $\begin{cases} \text{非逻辑符号} \\ \text{个体变元} + \text{量词} \end{cases}$
- 公式: $N_{\mathfrak{L}}$比N多 $\begin{cases} \text{项(不是公式)} \\ \text{原子公式(相当于N的命题变元)} \\ \text{量词公式(自由与约束)} \end{cases}$
- 推理规则: $N_{\mathfrak{L}}$比N多 $(\forall+)$, $(\forall-)$, $(\exists+)$, $(\exists-)$
- 证明序列: 字面上两者的定义几乎一样

  N的证明序列"也是"$N_{\mathfrak{L}}$的证明序列 (关于联结词的推理是一样的.)

  $N_{\mathfrak{L}}$侧重于关于量词的推理.

思考题: N的元定理都在$N_{\mathfrak{L}}$中成立吗?

例7

若y对x在α中自由，且y不在α中自由出现，则

1.$\exists x\alpha \vdash\dashv \exists y\alpha(x/y)$.　　　2.$\forall x\alpha \vdash\dashv \forall y\alpha(x/y)$.

分析:

y对x在α中自由，且y不在α中自由出现，

则$\alpha(x/y)(y/x) = \alpha$.

$$
\begin{array}{lccccccccc}
 & & \text{自由} & & \text{约束} & & \text{约束} & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\alpha: & (\cdots & x & \cdots & x & \cdots & y & \cdots) \\
\alpha(x/y): & (\cdots & y & \cdots & x & \cdots & y & \cdots) \\
\alpha(x/y)(y/x): & (\cdots & x & \cdots & x & \cdots & y & \cdots)
\end{array}
$$

例7的证明

若y对x在α中自由，且y不在α中自由出现，则

1.$\exists x\alpha \vdash\dashv \exists y\alpha(x/y)$. 2.$\forall x\alpha \vdash\dashv \forall y\alpha(x/y)$.

证:只证1.

($\vdash$)

(1) $\alpha \vdash \alpha$ ($\in$)

(2) $\alpha \vdash \alpha(x/y)(y/x)$

(3) $\alpha \vdash \exists y\alpha(x/y)$

(x对y在$\alpha(x/y)$中自由) ($\exists+$)

(4) $\exists x\alpha \vdash \exists y\alpha(x/y)$

(x在$\exists y\alpha(x/y)$中无自由出现) ($\exists-$)

例7的证明(续)

若y对x在α中自由，且y不在α中自由出现，则

1.$\exists x\alpha \vdash\dashv \exists y\alpha(x/y)$.　　　　2.$\forall x\alpha \vdash\dashv \forall y\alpha(x/y)$.

证：只证1.

($\dashv$)

(1)　$\exists y\alpha(x/y) \vdash \exists x\big(\alpha(x/y)(y/x)\big)$　($\vdash$)

(2)　$\exists y\alpha(x/y) \vdash \exists x\alpha$

思考题：如果该例中的任何一个条件不成立，结论会如何？

例8

证明: $\forall xy\alpha \vdash \forall yx\alpha$

证:

(1)	$\forall xy\alpha \vdash \forall xy\alpha$	$(\in)$
(2)	$\forall xy\alpha \vdash \forall y\alpha$	$(\forall -)$
(3)	$\forall y\alpha \vdash \alpha$	$(\forall -)$
(4)	$\forall xy\alpha \vdash \alpha$	(Tr)
(5)	$\forall xy\alpha \vdash \forall x\alpha$	$(\forall +)$
(6)	$\forall xy\alpha \vdash \forall y\forall x\alpha$	$(\forall +)$

例9

证明: 1. $\forall x\alpha \vdash\dashv \neg\exists x\neg\alpha$ 2. $\exists x\alpha \vdash\dashv \neg\forall x\neg\alpha$

证: 只证1.

$(\dashv)$

(1) $\neg\alpha \vdash \neg\alpha$ $(\in)$

(2) $\neg\alpha \vdash \exists x\neg\alpha$ $(\exists+)$

(3) $\neg\alpha \to \exists x\neg\alpha \vdash \neg\exists x\neg\alpha \to \alpha$ (定理1)

(4) $\neg\exists x\neg\alpha \vdash \alpha$ (定理2(2))

(5) $\neg\exists x\neg\alpha \vdash \forall x\alpha$ $(\forall+)$

例9(续)

证明：1. $\forall x\alpha \vdash\dashv \neg\exists x\neg\alpha$　　　　2. $\exists x\alpha \vdash\dashv \neg\forall x\neg\alpha$

证：再证1($\vdash$).

($\vdash$)

(1) $\forall x\alpha \vdash \forall x\alpha$　　($\in$)

(2) $\forall x\alpha \vdash \alpha$　　($\forall -$)

(3) $\neg\alpha \vdash \neg\forall x\alpha$　　(定理2(2))

(4) $\exists x\neg\alpha \vdash \neg\forall x\alpha$　　($\exists -$)

(5) $\forall x\alpha \vdash \neg\exists x\neg\alpha$　　(定理2(2))

例10

证明:

1. $\exists x(\alpha \to \beta) \vdash\dashv \alpha \to \exists x\beta$ 若x不在α中自由出现.
2. $\exists x(\alpha \to \beta) \vdash\dashv \forall x\alpha \to \beta$ 若x不在β中自由出现.

例10(1)($\vdash$)的证明

1.$\exists x(\alpha \to \beta) \vdash\dashv \alpha \to \exists x\beta$　　若x不在α中自由出现.

证：($\vdash$)

(1)　$\alpha \to \beta,\ \alpha \vdash \alpha$　　($\in$)

(2)　$\alpha \to \beta,\ \alpha \vdash \alpha \to \beta$　　($\in$)

(3)　$\alpha \to \beta,\ \alpha \vdash \beta$　　($\to -$)

(4)　$\alpha \to \beta,\ \alpha \vdash \exists x\beta$　　($\exists +$)

(5)　$\alpha \to \beta \vdash \alpha \to \exists x\beta$　　($\to +$)

(6)　$\exists x(\alpha \to \beta) \vdash \alpha \to \exists x\beta$

(x不在$\alpha \to \exists x\beta$中自由出现)　　($\exists -$)

例10(1)($\dashv$)的证明

1.$\exists x(\alpha\rightarrow\beta) \vdash\dashv \alpha\rightarrow\exists x\beta$ 若x不在α中自由出现.

证: ($\dashv$)

(1)	$\neg\forall x\neg(\alpha\rightarrow\beta)\vdash\exists x(\alpha\rightarrow\beta)$	(例9)
(2)	$\neg\exists x(\alpha\rightarrow\beta)\vdash\forall x\neg(\alpha\rightarrow\beta)$	(定理2)
(3)	$\neg\exists x(\alpha\rightarrow\beta)\vdash\neg(\alpha\rightarrow\beta)$	($\forall-$)(2)
(4)	$\beta\vdash\alpha\rightarrow\beta$	(定理1)
(5)	$\neg(\alpha\rightarrow\beta)\vdash\neg\beta$	(定理2)
(6)	$\neg\alpha\vdash\alpha\rightarrow\beta$	(定理1)
(7)	$\neg(\alpha\rightarrow\beta)\vdash\alpha$	(定理2)
(8)	$\neg\exists x(\alpha\rightarrow\beta)\vdash\neg\beta,\ \alpha$	(Tr)(3,5,7)
(9)	$\neg\exists x(\alpha\rightarrow\beta)\vdash\forall x\neg\beta$	($\forall+$)(8)

例10(1)($\dashv$)的证明(续1)

1.$\exists x(\alpha \to \beta) \vdash\dashv \alpha \to \exists x\beta$　若x不在α中自由出现.

续证:

(10) $\alpha \to \exists x\beta, \neg\exists x(\alpha \to \beta) \vdash \alpha$ (+)(8)

(11) $\alpha \to \exists x\beta, \neg\exists x(\alpha \to \beta) \vdash \alpha \to \exists x\beta$ ($\in$)

(12) $\alpha \to \exists x\beta, \neg\exists x(\alpha \to \beta) \vdash \exists x\beta$ ($\to -$)(10,11)

(13) $\alpha \to \exists x\beta, \neg\exists x(\alpha \to \beta) \vdash \forall x\neg\beta$ (+)(9)

(14) $\exists x\beta \vdash \neg\forall x\neg\beta$ (例9)

(15) $\forall x\neg\beta \vdash \neg\exists x\beta$ (定理2)

(16) $\alpha \to \exists x\beta, \neg\exists x(\alpha \to \beta) \vdash \neg\exists x\beta$ (Tr)(13,15)

(17) $\alpha \to \exists x\beta \vdash \exists x(\alpha \to \beta)$ ($\neg -$)(12,16)

例10(1)($\dashv$)的证明(续2)

1.$\exists x(\alpha \to \beta) \vdash\dashv \alpha \to \exists x\beta$　若x不在α中自由出现.

另续证:

(10) $\exists x\beta \vdash \neg\forall x \neg\beta$ (例9)

(11) $\forall x \neg\beta \vdash \neg\exists x\beta$ (定理2)

(12) $\neg\exists x(\alpha \to \beta) \vdash \neg\exists x\beta$ $(Tr)(9,11)$

(13) $\neg\exists x(\alpha \to \beta) \vdash \alpha \wedge \neg\exists x\beta$ $(\wedge+)(8,12)$

(14) $\alpha \wedge \neg\exists x\beta \vdash \neg(\alpha \to \exists x\beta)$ (定理1)

(15) $\neg\exists x(\alpha \to \beta) \vdash \neg(\alpha \to \exists x\beta)$ $(Tr)(13,14)$

(16) $\alpha \to \exists x\beta \vdash \exists x(\alpha \to \beta)$ (定理2)

注：该方向没用到条件"x不在α中自由出现"

例10(2)($\vdash$)的证明

2. $\exists x(\alpha\to\beta) \vdash\dashv \forall x\alpha\to\beta$ 若x不在β中自由出现.

证: ($\vdash$)

(1) $\alpha\to\beta,\ \forall x\alpha \vdash \forall x\alpha$ $(\in)$

(2) $\alpha\to\beta,\ \forall x\alpha \vdash \alpha$ $(\forall -)$

(3) $\alpha\to\beta,\ \forall x\alpha \vdash \alpha\to\beta$ $(\in)$

(4) $\alpha\to\beta,\ \forall x\alpha \vdash \beta$ $(\to -)$

(5) $\alpha\to\beta \vdash \forall x\alpha \to \beta$ $(\to +)$

(6) $\exists x(\alpha\to\beta) \vdash \forall x\alpha \to \beta$

(x不在$\forall x\alpha \to \beta$中自由出现) $(\exists -)$

例10(2)($\dashv$)的证明

2. $\exists x(\alpha\rightarrow\beta) \vdash\dashv \forall x\alpha\rightarrow\beta$ 若x不在β中自由出现.

证: ($\dashv$)

(1) $\forall x\neg(\alpha\rightarrow\beta) \vdash \forall x\neg(\alpha\rightarrow\beta)$ $(\in)$

(2) $\forall x\neg(\alpha\rightarrow\beta) \vdash \neg(\alpha\rightarrow\beta)$ $(\forall-)(1)$

(3) $\neg(\alpha\rightarrow\beta) \vdash \alpha\wedge\neg\beta$ (定理1)

(4) $\alpha\wedge\neg\beta \vdash \alpha,\ \neg\beta$ (定理1)

(5) $\forall x\neg(\alpha\rightarrow\beta) \vdash \alpha,\ \neg\beta$ $(Tr)(2,3,4)$

(6) $\forall x\neg(\alpha\rightarrow\beta) \vdash \forall x\alpha$ $(\forall+)(5)$

(7) $\forall x\alpha,\ \neg\beta \vdash \forall x\alpha\wedge\neg\beta$ (定理1)

例10(2)($\dashv$)的证明(续)

1.$\exists x(\alpha\to\beta) \vdash\dashv \alpha\to\exists x\beta$ 若x不在α中自由出现.

续证:

(8) $\forall x\alpha \wedge \neg\beta \vdash \neg(\forall x\alpha \to \beta)$ (定理1)

(9) $\forall x\neg(\alpha\to\beta) \vdash \neg(\forall x\alpha \to \beta)$ (Tr)(5,6,7,8)

(10) $\forall x\alpha \to \beta \vdash \neg\forall x\neg(\alpha\to\beta)$ (定理2)

(11) $\neg\forall x\neg(\alpha\to\beta) \vdash \exists x(\alpha\to\beta)$ (例9)

(12) $\forall x\alpha \to \beta \vdash \exists x(\alpha\to\beta)$ $(Tr)(10, 11)$

注：该方向也没用到条件"x不在α中自由出现"

例11

证明:

1. $\forall x(\alpha \leftrightarrow \beta) \vdash \forall x\alpha \leftrightarrow \forall x\beta$

2. $\forall x(\alpha \leftrightarrow \beta) \vdash \exists x\alpha \leftrightarrow \exists x\beta$

例11(1)的证明

1. $\forall x(\alpha \leftrightarrow \beta) \vdash \forall x\alpha \leftrightarrow \forall x\beta$

证:

(1) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \forall x(\alpha \leftrightarrow \beta)$ $(\in)$

(2) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \alpha \leftrightarrow \beta$ $(\forall -)(1)$

(3) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \forall x\alpha$ $(\in)$

(4) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \alpha$ $(\forall -)(3)$

(5) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \beta$ $(\leftrightarrow -)(2,4)$

(6) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\alpha \vdash \forall x\beta$ $(\forall +)(5)$

(7) $\forall x(\alpha \leftrightarrow \beta),\ \forall x\beta \vdash \forall x\alpha$ (同(6))

(8) $\forall x(\alpha \leftrightarrow \beta) \vdash \forall x\alpha \leftrightarrow \forall x\beta$ $(\leftrightarrow +)(6,7)$

例11(1)的证明

2. $\forall x(\alpha \leftrightarrow \beta) \vdash \exists x\alpha \leftrightarrow \exists x\beta$

证:

(1) $\forall x(\alpha \leftrightarrow \beta),\ \alpha \vdash \forall x(\alpha \leftrightarrow \beta)$ $(\in)$

(2) $\forall x(\alpha \leftrightarrow \beta),\ \alpha \vdash \alpha \leftrightarrow \beta$ $(\forall -)(1)$

(3) $\forall x(\alpha \leftrightarrow \beta),\ \alpha \vdash \alpha$ $(\in)$

(4) $\forall x(\alpha \leftrightarrow \beta),\ \alpha \vdash \beta$ $(\leftrightarrow -)(2, 3)$

(5) $\forall x(\alpha \leftrightarrow \beta),\ \alpha \vdash \exists x\beta$ $(\exists +)(4)$

(6) $\forall x(\alpha \leftrightarrow \beta),\ \exists x\alpha \vdash \exists x\beta$ $(\exists -)(5)$

(7) $\forall x(\alpha \leftrightarrow \beta),\ \exists x\beta \vdash \exists x\alpha$ (同(6))

(8) $\forall x(\alpha \leftrightarrow \beta) \vdash \exists x\alpha \leftrightarrow \exists x\beta$ $(\leftrightarrow +)(6, 7)$

例12

证明：若x不在α中自由出现，则

1. $\alpha \wedge \forall x\beta \vdash\dashv \forall x(\alpha \wedge \beta)$
2. $\alpha \wedge \exists x\beta \vdash\dashv \exists x(\alpha \wedge \beta)$
3. $\alpha \vee \forall x\beta \vdash\dashv \forall x(\alpha \vee \beta)$
4. $\alpha \vee \exists x\beta \vdash\dashv \exists x(\alpha \vee \beta)$

只证明1和3.

例12(1)($\vdash$)的证明

1. $\alpha \wedge \forall x\beta \dashv\vdash \forall x(\alpha \wedge \beta)$, 若$x$不在$\alpha$中自由出现.

证: ($\vdash$)

(1) $\alpha \wedge \forall x\beta \vdash \alpha \wedge \forall x\beta$ $\quad(\in)$

(2) $\alpha \wedge \forall x\beta \vdash \alpha$ $\quad(\wedge -)(1)$

(3) $\alpha \wedge \forall x\beta \vdash \forall x\beta$ $\quad(\wedge -)(1)$

(4) $\alpha \wedge \forall x\beta \vdash \beta$ $\quad(\forall -)(3)$

(5) $\alpha \wedge \forall x\beta \vdash \alpha \wedge \beta$ $\quad(\wedge +)(2,4)$

(6) $\alpha \wedge \forall x\beta \vdash \forall x(\alpha \wedge \beta)$

(x不在$\alpha \wedge \forall x\beta$中自由出现) $\quad(\forall +)(5)$

例12(1)($\dashv$)的证明

1. $\alpha \wedge \forall x\beta \vdash\dashv \forall x(\alpha \wedge \beta)$， 若$x$不在$\alpha$中自由出现.

证：($\dashv$)

(1) $\forall x(\alpha \wedge \beta) \vdash \forall x(\alpha \wedge \beta)$ $\quad(\in)$

(2) $\forall x(\alpha \wedge \beta) \vdash \alpha \wedge \beta$ $\quad(\forall -)(1)$

(3) $\forall x(\alpha \wedge \beta) \vdash \alpha$ $\quad(\wedge -)(2)$

(4) $\forall x(\alpha \wedge \beta) \vdash \beta$ $\quad(\wedge -)(2)$

(5) $\forall x(\alpha \wedge \beta) \vdash \forall x\beta$ $\quad(\forall +)(4)$

(6) $\forall x(\alpha \wedge \beta) \vdash \alpha \wedge \forall x\beta$ $\quad(\wedge +)(3,5)$

注：没有用到条件"x不在α中自由出现".

例12(3)($\vdash$)的证明

3. $\alpha \vee \forall x\beta \vdash\dashv \forall x(\alpha \vee \beta)$， 若$x$不在$\alpha$中自由出现.

证: ($\vdash$)

(1)	$\alpha \vdash \alpha$	$(\in)$
(2)	$\alpha \vdash \alpha \vee \beta$	$(\vee+)(1)$
(3)	$\alpha \vdash \forall x(\alpha \vee \beta)$ (x不在α中自由出现)	$(\forall+)(2)$
(4)	$\forall x\beta \vdash \forall x\beta$	$(\in)$
(5)	$\forall x\beta \vdash \beta$	$(\forall-)(4)$
(6)	$\forall x\beta \vdash \alpha \vee \beta$	$(\vee+)(5)$
(7)	$\forall x\beta \vdash \forall x(\alpha \vee \beta)$	$(\forall+)(6)$
(8)	$\alpha \vee \forall x\beta \vdash \forall x(\alpha \vee \beta)$	$(\vee-)(3,6)$

例12(3)($\dashv$)的证明

3. $\alpha \vee \forall x\beta \vdash\dashv \forall x(\alpha \vee \beta)$，若$x$不在$\alpha$中自由出现.

证：($\dashv$)

(1)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \forall x(\alpha \vee \beta)$	$(\in)$
(2)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \alpha \vee \beta$	$(\forall -)(1)$
(3)	$\alpha \vee \beta \vdash \neg\alpha \rightarrow \beta$	(命题内定理)
(4)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \neg\alpha \rightarrow \beta$	$(Tr)(2,3)$
(5)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \neg\alpha$	$(\in)$
(6)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \beta$	$(\rightarrow -)(4,5)$
(7)	$\forall x(\alpha \vee \beta), \neg\alpha \vdash \forall x\beta$	$(\forall +)(6)$
(8)	$\forall x(\alpha \vee \beta) \vdash \neg\alpha \rightarrow \forall x\beta$	$(\rightarrow +)(7)$
(9)	$\neg\alpha \rightarrow \forall x\beta \vdash \alpha \vee \forall x\beta$	(命题内定理)
(10)	$\forall x(\alpha \vee \beta) \vdash \alpha \vee \forall x\beta$	$(Tr)(8,9)$

$\boxed{\text{作业}}$

p.559(p.184)

6. 若y对x在α中自由，且y不在α中自由出现，则 $\forall x\alpha \vdash\dashv \forall y\alpha(x/y)$.
7.
9. 证明例12的(2),(4)
12. 证明：若$\Gamma, \alpha \vdash_{N_{\mathfrak{L}}} \beta$，且$x$不在$\Gamma$的任何公式中自由出现，则$\Gamma, \exists x\alpha \vdash_{N_{\mathfrak{L}}} \exists x\beta$.
13. 证明：若$\Gamma, \alpha \vdash_{N_{\mathfrak{L}}} \beta$，且$x$不在$\Gamma \cup \{\beta\}$的任何公式中自由出现，则$\Gamma, \exists x\alpha \vdash_{N_{\mathfrak{L}}} \forall x\beta$.
14. (2),(4),(5)

谢　谢

复习

$N_{\mathfrak{L}}$中的一些可证式子:

$$\begin{cases} \alpha \to \forall x\beta \vdash\dashv \forall x(\alpha \to \beta) \\ \alpha \to \exists x\beta \vdash\dashv \exists x(\alpha \to \beta) \end{cases}$$

若x不在α中自由出现

$$\begin{cases} \forall x\alpha \to \beta \vdash\dashv \exists x(\alpha \to \beta) \\ \exists x\alpha \to \beta \vdash\dashv \forall x(\alpha \to \beta) \end{cases}$$

若x不在β中自由出现

$$\begin{cases} \alpha \wedge \forall x\beta \vdash\dashv \forall x(\alpha \wedge \beta) \\ \alpha \wedge \exists x\beta \vdash\dashv \exists x(\alpha \wedge \beta) \\ \alpha \vee \forall x\beta \vdash\dashv \forall x(\alpha \vee \beta) \\ \alpha \vee \exists x\beta \vdash\dashv \exists x(\alpha \vee \beta) \end{cases}$$

若x不在α中自由出现

复习(续)

$N_{\mathfrak{L}}$中的一些可证式子:

$$\begin{cases} \neg\forall x\alpha \vdash\dashv & \exists x\neg\alpha \\ \neg\exists x\alpha \vdash\dashv & \forall x\neg\alpha \end{cases}$$

$$\begin{cases} \forall x\alpha \vdash\dashv & \forall y\alpha(x/y) \\ \exists x\alpha \vdash\dashv & \forall y\alpha(x/y) \end{cases} \quad \text{若}y\text{不在}\alpha\text{中出现}$$

思考题: $\forall x(\alpha \leftrightarrow \beta) \vdash \forall x\alpha \leftrightarrow \forall x\beta?$

$\forall x(\alpha \leftrightarrow \beta) \vdash \exists x\alpha \leftrightarrow \exists x\beta?$

$\exists x(\alpha \leftrightarrow \beta) \vdash \exists x\alpha \leftrightarrow \exists x\beta?$

$\exists x(\alpha \leftrightarrow \beta) \vdash \forall x\alpha \leftrightarrow \forall x\beta?$

替换定理

定理3 设α, β, γ是$\mathfrak{L}$的公式，满足$\beta \vdash\dashv \gamma$.

α'为将α中某些β换为γ得到的公式.

则$\alpha \vdash\dashv \alpha'$.

示意图:

$$
\begin{array}{llllllll}
\alpha: & \cdots & \beta & \cdots & \beta & \cdots & \beta & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\alpha': & \cdots & \gamma & \cdots & \beta & \cdots & \gamma & \cdots
\end{array}
$$

引理1

若$\mathfrak{L}$中的公式$\alpha, \alpha', \beta, \beta'$满足 $\alpha \vdash\dashv \alpha'$, $\beta \vdash\dashv \beta'$, 则:

1. $\neg\,\alpha \vdash\dashv\ \neg\,\alpha'$
2. $\alpha \vee \beta \vdash\dashv\ \alpha' \vee \beta'$
3. $\alpha \wedge \beta \vdash\dashv\ \alpha' \wedge \beta'$
4. $\alpha \rightarrow \beta \vdash\dashv\ \alpha' \rightarrow \beta'$
5. $\alpha \leftrightarrow \beta \vdash\dashv\ \alpha' \leftrightarrow \beta'$
6. $\forall x\alpha \vdash\dashv\ \forall x\alpha'$
7. $\exists x\alpha \vdash\dashv\ \exists x\alpha'$

只证(7).

引理1(7)的证明

若$\mathfrak{L}$中的公式$\alpha,\alpha',\beta,\beta'$满足 $\alpha \vdash\dashv \alpha'$, $\beta \vdash\dashv \beta'$, 则:

7. $\exists x\alpha \vdash\dashv \exists x\alpha'$

证: (1) $\alpha \vdash\dashv \alpha'$

(2) $\emptyset \vdash \alpha \leftrightarrow \alpha'$

(3) $\emptyset \vdash \forall x(\alpha \leftrightarrow \alpha')$

(4) $\forall x(\alpha \leftrightarrow \alpha') \vdash \exists x\alpha \leftrightarrow \exists x\alpha'$ (例11)

(5) $\emptyset \vdash \exists x\alpha \leftrightarrow \exists x\alpha'$ (Tr)

(6) $\exists x\alpha \vdash \exists x\alpha \leftrightarrow \exists x\alpha'$ (+)

(7) $\exists x\alpha \vdash \exists x\alpha$

(8) $\exists x\alpha \vdash \exists x\alpha'$

(9) $\exists x\alpha' \vdash \exists x\alpha$ (同理(8))

(10) $\exists x\alpha \vdash\dashv \exists x\alpha'$

引理1(7)的证明(续)

若$\mathfrak{L}$中的公式$\alpha, \alpha', \beta, \beta'$满足 $\alpha \vdash\dashv \alpha'$, $\beta \vdash\dashv \beta'$, 则:

7. $\exists x\alpha \vdash\dashv \exists x\alpha'$

另证:

(1)	$\alpha \vdash \alpha'$	(已知)
(2)	$\alpha \vdash \exists x\alpha'$	$(\exists+)(1)$
(3)	$\exists x\alpha \vdash \exists x\alpha'$	$(\exists-)(2)$
(4)	$\exists x\alpha' \vdash \exists x\alpha$	(同理(3))
(5)	$\exists x\alpha \vdash\dashv \exists x\alpha'$	

替换定理的证明

定理3 设α, β, γ是$\mathfrak{L}$的公式，满足$\beta \vdash\dashv \gamma$.

α'为将α中某些β换为γ得到的公式．则$\alpha \vdash\dashv \alpha'$.

证：对α中出现的量词与联结词的个数d归纳证明.

(1) 当$d = 0$时，α为原子公式，
则: $\alpha = \alpha'$，或$\alpha = \beta$且$\alpha' = \gamma$，从而$\alpha \vdash\dashv \alpha'$.

(2) 设$d \leq n$时命题成立．考察$d = n + 1$时情形.
α必为下列形式之一：$\neg\alpha_1$，$\alpha_1 \vee \alpha_2$，$\alpha_1 \wedge \alpha_2$，
$\alpha_1 \to \alpha_2$，$\alpha_1 \leftrightarrow \alpha_2$，$\exists x\alpha_1$，$\forall x\alpha_1$.
无论哪种形式,α_1及α_2中量词和联结词的个数都$\leq n$

替换定理的证明(续)

设对α_1与α_2中出现的某些β换为γ得到的公式分别为 α_1'与α_2'.

由归纳假设得: $\alpha_1 \vdash\dashv \alpha_2'$, $\alpha_2 \vdash\dashv \exists\alpha_2'$.

且α'分别为: $\neg\alpha_1'$, $\alpha_1' \vee \alpha_2'$, $\alpha_1' \wedge \alpha_2'$, $\alpha_1' \rightarrow \alpha_2'$, $\alpha_1' \leftrightarrow \alpha_2'$, $\exists x\alpha_1'$, $\forall x\alpha_1'$.

对它们分别应用引理1可得$\alpha \vdash\dashv \alpha'$.

归纳证毕.

替换定理的一个应用

例12(1) 证明：若x不在α中自由出现，则：

$$\forall x(\alpha \wedge \beta) \vdash\dashv \alpha \wedge \forall x\beta$$

证：(1) $\alpha \wedge \beta \vdash\dashv \neg(\alpha \to \neg\beta)$ (命题内定理)

(2) $\forall x(\alpha \wedge \beta) \vdash\dashv \forall x\neg(\alpha \to \neg\beta)$ 替换定理

(3) $\neg\forall x(\alpha \wedge \beta) \vdash\dashv \neg\forall x\neg(\alpha \to \neg\beta)$ (定理2)

(4) $\neg\forall x\neg(\alpha \to \neg\beta) \vdash\dashv \exists x(\alpha \to \neg\beta)$ (例9)

(5) $\exists x(\alpha \to \neg\beta) \vdash\dashv \alpha \to \exists x\neg\beta$ (例10)

(6) $\neg\forall x(\alpha \wedge \beta) \vdash\dashv \alpha \to \exists x\neg\beta$ (Tr)

(7) $\forall x(\alpha \wedge \beta) \vdash\dashv \neg(\alpha \to \exists x\neg\beta)$ (定理2)

(8) $\neg(\alpha \to \exists x\neg\beta) \vdash\dashv \alpha \wedge \neg\exists x\neg\beta$ (命题内定理)

(9) $\forall x(\alpha \wedge \beta) \vdash\dashv \alpha \wedge \neg\exists x\neg\beta$ (Tr)

(10) $\forall x(\alpha \wedge \beta) \vdash\dashv \alpha \wedge \forall x\beta$ (替换定理)

范式

定义10 $\mathfrak{L}$的一个公式α如要具有如下形状:

$$Q_1 v_1 Q_2 v_2 \cdots Q_n v_n\, \beta$$

其中:

Q_i为量词$\forall$或$\exists$ $(1 \leq i \leq n)$;

v_i为个体变元符号$(1 \leq i \leq n)$;

n为自然数(n可以$= 0$);

β中没有量词出现.

则称α为$\mathfrak{L}$的一个前束范式.

范式(存在)定理

定理4 对$\mathfrak{L}$任一个公式α，存在$\mathfrak{L}$的一个前束范式α'，使$\alpha \vdash\dashv \alpha'$.

也称此α'为α的一个前束范式.

下面先用例子来说明前束范式的求法.

例8

求下列各公式的前束范式.

1. $\neg(\forall x_2 \exists x_1 F_1^2(x_1, x_2))$

2. $\forall x_1 F^1(x_1) \rightarrow \forall x_2 F^1(x_2)$

3. $\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$

4. $(\forall x_1 F_1^2(x_1, x_2) \rightarrow \neg \exists x_2 F_2^1(x_2)) \rightarrow \forall x_1 \forall x_2 F_3^2(x_1, x_2)$

5. $\forall x_1 F_1^2(x_1, x_2) \leftrightarrow \forall x_2 F_1^2(x_1, x_2)$

例8(1)的解

1. $\neg(\forall x_2 \exists x_1 F_1^2(x_1, x_2))$

解:

$$\neg(\forall x_2 \exists x_1 F_1^2(x_1, x_2))$$

$$\vdash\dashv \exists x_2(\neg \exists x_1 F_1^2(x_1, x_2))$$

$$\vdash\dashv \exists x_2 \forall x_1(\neg F_1^2(x_1, x_2))$$

例8(2)的解

2. $\forall x_1 F^1(x_1) \to \forall x_2 F^1(x_2)$

解:

$\forall x_1 F^1(x_1) \to \forall x_2 F^1(x_2)$

$\vdash\dashv \forall x_2 \big(\forall x_1 F^1(x_1) \to F^1(x_2)\big)$

$\vdash\dashv \forall x_2 \exists x_1 \big(F^1(x_1) \to F^1(x_2)\big)$

例8(2)的解

2. $\forall x_1 F^1(x_1) \to \forall x_2 F^1(x_2)$

解:

$\forall x_1 F^1(x_1) \to \forall x_2 F^1(x_2)$

$\vdash\dashv \forall x_2 (\forall x_1 F^1(x_1) \to F^1(x_2))$

$\vdash\dashv \forall x_2 \exists x_1 (F^1(x_1) \to F^1(x_2))$

另解:

$\forall x_1 F^1(x_1) \to \forall x_2 F^1(x_2)$

$\vdash\dashv \exists x_1 (F^1(x_1) \to \forall x_2 F^1(x_2))$

$\vdash\dashv \exists x_1 \forall x_2 (F^1(x_1) \to F^1(x_2))$

注: 范式不一定唯一

例8(3)的解

3. $\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$

解:

$$\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$$

$$\vdash\dashv \forall x_2 \big(\forall x_1 F_1^2(x_1, x_2) \rightarrow F^1(x_2)\big)$$

$$\vdash\dashv \forall x_2 \exists x_1 \big(F_1^2(x_1, x_2) \rightarrow F^1(x_2)\big)$$

例8(3)的解

3. $\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$

解:

$\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$

$\vdash\!\dashv \forall x_2 \big(\forall x_1 F_1^2(x_1, x_2) \rightarrow F^1(x_2)\big)$ $\times$

$\vdash\!\dashv \forall x_2 \exists x_1 \big(F_1^2(x_1, x_2) \rightarrow F^1(x_2)\big)$

例8(3)的解

3. $\forall x_1 F_1^2(x_1,x_2)\rightarrow\forall x_2 F^1(x_2)$

解:

$$\forall x_1 F_1^2(x_1,x_2)\rightarrow\forall x_2 F^1(x_2)$$

$$\vdash\dashv \forall x_2\big(\forall x_1 F_1^2(x_1,x_2)\rightarrow F^1(x_2)\big)$$

$$\vdash\dashv \forall x_2\exists x_1\big(F_1^2(x_1,x_2)\rightarrow F^1(x_2)\big)$$

解:

$$\forall x_1 F_1^2(x_1,x_2)\rightarrow\forall x_2 F^1(x_2)$$

$$\vdash\dashv \forall x_1 F_1^2(x_1,x_3)\rightarrow\forall x_2 F^1(x_2)$$

$$\vdash\dashv \forall x_2\big(\forall x_1 F_1^2(x_1,x_3)\rightarrow F^1(x_2)\big)$$

$$\vdash\dashv \forall x_2\exists x_1\big(F_1^2(x_1,x_3)\rightarrow F^1(x_2)\big)$$

例8(3)的解

3. $\forall x_1 F_1^2(x_1, x_2) \to \forall x_2 F^1(x_2)$

解:

$\forall x_1 F_1^2(x_1, x_2) \to \forall x_2 F^1(x_2)$

$\vdash\!\dashv \forall x_2 \big(\forall x_1 F_1^2(x_1, x_2) \to F^1(x_2)\big)$

$\vdash\!\dashv \forall x_2 \exists x_1 \big(F_1^2(x_1, x_2) \to F^1(x_2)\big)$

解:

$\forall x_1 F_1^2(x_1, x_2) \to \forall x_2 F^1(x_2)$

$\vdash\!\dashv \forall x_1 F_1^2(x_1, x_3) \to \forall x_2 F^1(x_2)$

$\vdash\!\dashv \forall x_2 \big(\forall x_1 F_1^2(x_1, x_3) \to F^1(x_2)\big)$

$\vdash\!\dashv \forall x_2 \exists x_1 \big(F_1^2(x_1, x_3) \to F^1(x_2)\big)$

例8(3)的解

3. $\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$

解:

$$\forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_2 F^1(x_2)$$
$$\vdash\dashv \forall x_1 F_1^2(x_1, x_2) \rightarrow \forall x_3 F^1(x_3)$$
$$\vdash\dashv \forall x_3 \big(\forall x_1 F_1^2(x_1, x_2) \rightarrow F^1(x_3)\big)$$
$$\vdash\dashv \forall x_3 \exists x_1 \big(F_1^2(x_1, x_2) \rightarrow F^1(x_3)\big)$$

例8(3)的解

3. $\forall x_1 F_1^2(x_1,x_2) \to \forall x_2 F^1(x_2)$

解:

$\forall x_1 F_1^2(x_1,x_2) \to \forall x_2 F^1(x_2)$

$\vdash\!\dashv \forall x_1 F_1^2(x_1,x_2) \to \forall x_3 F^1(x_3)$

$\vdash\!\dashv \forall x_3 (\forall x_1 F_1^2(x_1,x_2) \to F^1(x_3))$

$\vdash\!\dashv \forall x_3 \exists x_1 (F_1^2(x_1,x_2) \to F^1(x_3))$

另解:

$\forall x_1 F_1^2(x_1,x_2) \to \forall x_2 F^1(x_2)$

$\vdash\!\dashv \exists x_1 (F_1^2(x_1,x_2) \to \forall x_2 F^1(x_2))$

$\vdash\!\dashv \exists x_1 (F_1^2(x_1,x_2) \to \forall x_3 F^1(x_3))$

$\vdash\!\dashv \exists x_1 \forall x_3 (F_1^2(x_1,x_2) \to F^1(x_3))$

例8(4)的解

4. $\big(\forall x_1 F_1^2(x_1,\ x_2) \to \neg\exists x_2 F_2^1(x_2)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2)$

解:

$$
\begin{aligned}
& \big(\forall x_1 F_1^2(x_1,\ x_2) \to \neg\exists x_2 F_2^1(x_2)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2) \\
\vdash\dashv\ & \big(\forall x_1 F_1^2(x_1,\ x_2) \to \forall x_2 \neg F_2^1(x_2)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2) \\
\vdash\dashv\ & \exists x_1 \big(F_1^2(x_1,\ x_2) \to \forall x_2 \neg F_2^1(x_2)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2) \\
\vdash\dashv\ & \exists x_1 \big(F_1^2(x_1,\ x_2) \to \forall x_3 \neg F_2^1(x_3)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2) \\
\vdash\dashv\ & \exists x_1 \forall x_3 \big(F_1^2(x_1,\ x_2) \to \neg F_2^1(x_3)\big) \to \forall x_1 \forall x_2 F_3^2(x_1,\ x_2) \\
\vdash\dashv\ & \exists x_1 \forall x_3 \big(F_1^2(x_1,\ x_2) \to \neg F_2^1(x_3)\big) \to \forall x_4 \forall x_5 F_3^2(x_4,\ x_5) \\
\vdash\dashv\ & \forall x_1 \exists x_3 \Big(\big(F_1^2(x_1,\ x_2) \to \neg F_2^1(x_3)\big) \to \forall x_4 \forall x_5 F_3^2(x_4,\ x_5)\Big) \\
\vdash\dashv\ & \forall x_1 \exists x_3 \forall x_4 \forall x_5 \Big(\big(F_1^2(x_1,\ x_2) \to \neg F_2^1(x_3)\big) \to F_3^2(x_4,\ x_5)\Big)
\end{aligned}
$$

例8(5)的解

5. $\forall x_1 F_1^2(x_1, x_2) \leftrightarrow \forall x_2 F_1^2(x_1, x_2)$

解: $\forall x_1 F_1^2(x_1, x_2) \leftrightarrow \forall x_2 F_1^2(x_1, x_2)$

$$\begin{aligned}
&\vdash\dashv \big(\forall x_1 F_1^2(x_1, x_2) \to \forall x_2 F_1^2(x_1, x_2)\big) \wedge \\
&\qquad \big(\forall x_2 F_1^2(x_1, x_2) \to \forall x_1 F_1^2(x_1, x_2)\big) \\
&\vdash\dashv \big(\forall x_3 F_1^2(x_3, x_2) \to \forall x_4 F_1^2(x_1, x_4)\big) \wedge \\
&\qquad \big(\forall x_5 F_1^2(x_1, x_5) \to \forall x_6 F_1^2(x_6, x_2)\big) \\
&\vdash\dashv \exists x_3 \big(F_1^2(x_3, x_2) \to \forall x_4 F_1^2(x_1, x_4)\big) \wedge \\
&\qquad \exists x_5 \big(F_1^2(x_1, x_5) \to \forall x_6 F_1^2(x_6, x_2)\big) \\
&\vdash\dashv \exists x_3 \forall x_4 \big(F_1^2(x_3, x_2) \to F_1^2(x_1, x_4)\big) \wedge \\
&\qquad \exists x_5 \forall x_6 \big(F_1^2(x_1, x_5) \to F_1^2(x_6, x_2)\big) \\
&\vdash\dashv \exists x_3 \forall x_4 \exists x_5 \forall x_6 \Big(\big(F_1^2(x_3, x_2) \to F_1^2(x_1, x_4)\big) \wedge \\
&\qquad \big(F_1^2(x_1, x_5) \to F_1^2(x_6, x_2)\big)\Big)
\end{aligned}$$

范式定理的证明

对$\mathfrak{L}$任一个公式α，存在$\mathfrak{L}$的前束范式α'，使$\alpha \vdash\dashv \alpha'$.

证：对α中所含的联结词与量词的个数d归纳证明.

(1) 当$d = 0$时，α为原子公式，从而α中没有量词，取$\alpha' = \alpha$，则$\alpha \vdash\dashv \alpha'$

(2) 设当$d \leq n$时命题成立，考察$d = n + 1$时情形. α为下列几种情形之一:

$\neg\alpha_1$， $\alpha_1 \vee \alpha_2$， $\alpha_1 \wedge \alpha_2$， $\alpha_1 \rightarrow \alpha_2$， $\alpha_1 \leftrightarrow \alpha_2$， $\exists x\alpha_1$， $\forall x\alpha_1$.

范式定理的证明(续1)

由归纳假设知：存在$\mathfrak{L}$的前束范式α_1', α_2',使得
$\alpha_1 \vdash\dashv \alpha_1'$， $\alpha_2 \vdash\dashv \alpha_2'$.

设α_1', α_2'分别为:

$$Q_1 v_1 Q_2 v_2 \cdots Q_m v_m \alpha_1''$$
$$Q_{m+1} v_{m+1} Q_{m+2} v_{m+2} \cdots Q_{m+n} v_{m+n} \alpha_2''$$

其中:

Q_i为$\forall$或$\exists$, v_i为个体变元符号$(1 \leq i \leq m+n)$.

α_1''与α_2''中没有量词出现.

范式定理的证明(续2)

(2.1) 当α为$\neg\alpha_1$时, $\alpha \vdash\dashv \neg Q_1v_1Q_2v_2\cdots Q_mv_m\alpha_1''$.

从而 $\alpha \vdash\dashv Q_1^*v_1Q_2^*v_2\cdots Q_m^*v_m\neg\alpha_1''$.

其中:

$$Q_i^* = \begin{cases} \forall & \text{若}Q_i\text{为}\exists \\ \exists & \text{若}Q_i\text{为}\forall \end{cases}$$

则$Q_1^*v_1Q_2^*v_2\cdots Q_m^*v_m\neg\alpha_1''$ 即为所求.

范式定理的证明(续3)

(2.2) 当α为$\alpha_1 \vee \alpha_2$时.

$$
\begin{aligned}
\alpha \vdash\dashv & (Q_1 v_1 \cdots Q_m v_m \alpha_1'') \vee \\
& (Q_{m+1} v_{m+1} \cdots Q_{m+n} v_{m+n} \alpha_2'') \\
\vdash\dashv & (Q_1 v_1' \cdots Q_m v_m' \alpha_1''(v_m/v_m') \cdots (v_1/v_1')) \vee \\
& (Q_{m+1} v_{m+1}' \cdots Q_{m+n} v_{m+n}' \\
& \quad \alpha_2''(v_{m+n}/v_{m+n}') \cdots (v_{m+1}/v_{m+1}'))
\end{aligned}
$$

其中:

$v_1', v_2', \cdots, v_{m+n}'$为$\mathfrak{L}$中互不相同的个体变元符号,
且不在α_1'及α_2'中出现.

范式定理的证明(续4)

从而由例12知:

$$\begin{aligned}\alpha \vdash\!\dashv\, & Q_1 v_1' \cdots Q_m v_m' \\ & Q_{m+1} v_{m+1}' \cdots Q_{m+n} v_{m+n}' \\ & \Big((\alpha_1''(v_m/v_m') \cdots (v_1/v_1')) \vee \\ & \ (\alpha_2''(v_{m+n}/v_{m+n}') \cdots (v_{m+1}/v_{m+1}')) \Big)\end{aligned}$$

此即为所求的前束范式.

范式定理的证明(续4)

(2.3) 当α为$\alpha_1 \wedge \alpha_2$时, 仿(2.2)可证.

(2.4) 当α为$\alpha_1 \rightarrow \alpha_2$时, 利用例6与例10 仿上可证.

(2.5) 当α为$\alpha_1 \leftrightarrow \alpha_2$时,

$$\alpha \vdash\dashv (\alpha_1 \rightarrow \alpha_2) \wedge (\alpha_2 \rightarrow \alpha_1).$$

由(2.3)及(2.4)知存在前束范式β使: $\alpha \vdash\dashv \beta$.

(2.6) 当α为$\forall x \alpha_1$或$\exists x \alpha_1$时, 由引理1之(6,7)易证.

归纳证完, 命题成立.

谓词公式按前束范式的分类

定义11　设n是一个非0的自然数.

(1) 若前束范式α的量词以全称量词开始，并且全称量词组与存在量词组有 $n-1$次交替，则称α为一个Π_n型前束范式，简称为Π_n型公式.

(2) 若前束范式α的量词以存在量词开始，并且全称量词组与存在量词组有 $n-1$次交替，则称α为一个Σ_n型前束范式，简称为Σ_n型公式.

例如: $\forall x_1 \forall x_2 (F^1(x_1) \to F^1(x_2))$为 Π_1型公式.

$\exists x_3 \exists x_1 \forall x_1 \forall x_2 (F^1(x_1) \to F^1(x_2))$为 Σ_2型公式.

$\mathbf{N}_{\mathfrak{L}}$的内定理

定义12　设α为$\mathbf{N}_{\mathfrak{L}}$的一个公式，

若$\emptyset \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha$，则称$\alpha$为$\mathbf{N}_{\mathfrak{L}}$的一个内定理，记为$\vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha$.

易证：$\alpha_1, \alpha_2, \cdots, \alpha_n \vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha$

当且仅当$\vdash_{\mathbf{N}_{\mathfrak{L}}} \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$

当且仅当$\vdash_{\mathbf{N}_{\mathfrak{L}}} (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \rightarrow \alpha$

作业

p.560(p.185)

17. (1), (2), (3)

18. (1), (3)

19. (2),(3)

谢　谢