GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 5: Indefinite and Definite Integrals

 # Antiderivatives and Initial Value Problems

Definition:

Let f and F be real-valued functions on x.
Suppose $\frac{d}{dx}F(x) = f(x)$ for all concerned x.
Then, we say F is **a** primitive function of f on concerned domain.
Sometimes, we also say F is **an** antiderivative of f on concerned domain.

Note:

If f has a primitive function on concerned domain, then it will has infinitely many primitive functions on concerned domain.

Reason:

Suppose F is a primitive function of f on concerned domain and α is a fixed real number (constant). We can define a new function G by $G(x) = F(x) + \alpha$ for all concerned x. Then, $\frac{d}{dx}G(x) = \frac{d}{dx}F(x) = f(x)$ for all concerned x. That is, G is also a primitive function of f on concerned domain. This α can be any real number. Thus, f has infinitely many primitive functions on concerned domain.

Example:

Let $f: R \to R$ be defined by $f(x) = 2x$ for any $x \in R$.
Let $F: R \to R$ be defined by $F(x) = x^2$ for any $x \in R$.
Let $G: R \to R$ be defined by $G(x) = x^2 + 3$ for any $x \in R$.
We can check that BOTH F and G are primitive functions of f on R.

Theorem 1:

Let $a, b \in R$ with $a < b$.
Let $f: (a, b) \to R$, $F: (a, b) \to R$ and $G: (a, b) \to R$ be functions on x.
Suppose $\frac{d}{dx}F(x) = f(x) = \frac{d}{dx}G(x)$ for any $x \in (a, b)$.
Then, we can find a fixed real number α (constant) such that $F(x) = G(x) + \alpha$ for any $x \in (a, b)$.

Theorem 2:

Let $a \in R$.
Let $f: (a, \infty) \to R$, $F: (a, \infty) \to R$ and $G: (a, \infty) \to R$ be functions on x.
Suppose $\frac{d}{dx}F(x) = f(x) = \frac{d}{dx}G(x)$ for any $x \in (a, \infty)$.
Then, we can find a fixed real number α (constant) such that $F(x) = G(x) + \alpha$ for any $x \in (a, \infty)$.

Theorem 3:

Let $b \in R$.
Let $f: (-\infty, b) \to R$, $F: (-\infty, b) \to R$ and $G: (-\infty, b) \to R$ be functions on x.
Suppose $\frac{d}{dx}F(x) = f(x) = \frac{d}{dx}G(x)$ for any $x \in (-\infty, b)$.
Then, we can find a fixed real number α (constant) such that $F(x) = G(x) + \alpha$ for any $x \in (-\infty, b)$.

Proofs: Omitted (Use Mean Value Theorem)

Remark 1:

Suppose F is **a** primitive function of f on concerned domain.
We can define a new function F_α by $F_\alpha(x) = F(x) + \alpha$ on concerned domain for any $\alpha \in R$.
We can show that the set $\{F_\alpha : \alpha \in R\}$ is **the** set of all primitive functions of f on concerned domain.

Proof: Use above theorems.

In this case, F is called an representative function of **the** set of all primitive functions of f on concerned domain and we say **the** set of all primitive functions of f on concerned domain is generated by F.

Remark 2:

Suppose F and G are primitive functions of f on concerned domain.
We can show that $\{F_\alpha : \alpha \in R\} = \{G_\beta : \beta \in R\}$.

Proof: Use above theorems.

Remark 3:

If f has a primitive function on concerned domain, then the set of all primitive functions of f on concerned domain is **UNIQUE**.

Remark 4:

If f has a primitive function F on concerned domain, we use C to denote an arbitrary constant and $F(x) + C$ to denote $\{F_C: C \in R\}$.

Remark 5:

We write $\int f(x)dx = F(x) + C$ to denote **the** set of all primitive functions of f on concerned domain is generated by F. That is, F is a primitive function of f on concerned domain.

Remark 6:

$\int f(x)dx = F(x) + C$ just means $\frac{d}{dx}F(x) = f(x)$ for any $x \in$ concerned domain.

Remark 7:

$\int f(x)dx$ is called **the** indefinite integral of f with respect to x on concerned domain.
We say: indefinite integrate f with respect to x.
$\int \square$ is called an integral sign.
$f(x)$ is called the integrand.
dx is called the differential.
x is called the variable of integration.

Remark 8:

Finding **the** indefinite integral of f with respect to x on concerned domain is a reverse process of differentiation.
We call it anti-differentiation or indefinite integration.

Remark 9:

For a concerned domain, let Ω be the set containing all functions f defined on concerned domain and f has a primitive function F on the same concerned domain.
The assignment $f \in \Omega \mapsto \{F_C: C \in R\}$ is a function.

Remark 9:

Taking Differentiation

Suppose α is a fixed real number (constant).

$\frac{d}{dx}\alpha = 0$

$\frac{d}{dx}x = 1$

$\frac{d}{dx}x^n = nx^{n-1}$ for $n = 2,3,4,\cdots$

$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r$ for $r \in R\backslash\{-1\}$.

$\frac{d}{dx}sinx = cosx$

$\frac{d}{dx}cosx = -sinx$

$\frac{d}{dx}tanx = sec^2x$

$\frac{d}{dx}cscx = -cotx \cdot cscx$

$\frac{d}{dx}secx = tanx \cdot secx$

$\frac{d}{dx}cotx = -csc^2x$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}lnx = \frac{1}{x}$ (Assumed $x > 0$)

$\frac{d}{dx}ln|x| = \frac{1}{x}$ (Assumed $x \neq 0$)

Taking Indefinite Integration

C denotes an arbitrary constant

$\int 0dx = C$

$\int 1dx = x + C$

$\int nx^{n-1}dx = x^n + C$ for $n = 2,3,4,\cdots$

$\int x^r dx = \frac{x^{r+1}}{r+1} + C$ for $r \in R\backslash\{-1\}$.

$\int cosxdx = sinx + C$

$\int -sinxdx = cosx + C$

$\int sec^2xdx = tanx + C$

$\int -cotx \cdot cscxdx = cscx + C$

$\int tanx \cdot secxdx = secx + C$

$\int -csc^2xdx = cotx + C$

$\int e^xdx = e^x + C$

$\int \frac{1}{x}dx = ln|x| + C$

(Choose the one with "largest" domain)

Exercises:

Suppose k is a fixed real number (constant) and $k \neq 0$.

(i) $$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$$

(ii) $$\int sinkxdx = \frac{-1}{k}coskx + C$$

(iii) $$\int coskxdx = \frac{1}{k}sinkx + C$$

Theorems (Rules for Indefinite Integration):

Let f and g be real-valued functions on x and are defined on an open interval of R.
Let λ be a fixed real number (constant).
Then,

(i) $$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

(ii) $$\int \big(f(x) - g(x)\big)\,dx = \int f(x)\,dx - \int g(x)\,dx$$

(iii) $$\int \lambda f(x)dx = \lambda \int f(x)dx$$

Proof: Omitted (As Exercises)

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Example 1:

Evaluate the indefinite integral $\int(6x^3+5x^2-4x-2020)\,dx$.

Solutions

$$\int(6x^3+5x^2-4x-2020)\,dx$$
$$=6\int x^3\,dx+5\int x^2\,dx-4\int x dx-2020\int 1\,dx$$
$$=6\left(\frac{x^4}{4}\right)+5\left(\frac{x^3}{3}\right)-4\left(\frac{x^2}{2}\right)-2020x+C$$
$$=\frac{3}{2}x^4+\frac{5}{3}x^3-2x^2-2020x+C$$

Note: We don't need to write four arbitrary constants C_1, C_2, C_3 and C_4 for the four indefinite integrals on the right hand side. We just write one arbitrary constant C as we only consider ONE representative primitive function for the integrand.

Example 2:

Evaluate $\int(2x-1)^3\,dx$.

Solutions

Method 1

$$\int(2x-1)^3\,dx$$
$$=\int(8x^3-12x^2+6x-1)dx$$
$$=2x^4-4x^3+3x^2-x+C$$

Method 2

As $\frac{d}{dx}(2x-1)^4$
$=4(2x-1)^3\cdot 2$
$=8(2x-1)^3$,
so $\frac{d}{dx}\frac{(2x-1)^4}{8}=(2x-1)^3$

$$\int(2x-1)^3\,dx$$
$$=\frac{(2x-1)^4}{8}+C$$

Remark:

$2x^4-4x^3+3x^2-x$ and $\frac{(2x-1)^4}{8}$ must differ by a fixed real number (constant).

$$\frac{(2x-1)^4}{8}=\frac{1}{8}(16x^4-32x^3+24x^2-8x+1)=2x^4-4x^3+3x^2-x+\frac{1}{8}$$

Example 3:

Evaluate the following indefinite integrals:

(i) $\displaystyle\int(x+5)^{10}\,dx$

Solutions

$\frac{d}{dx}(x+5)^{11}=11(x+5)^{10}$.
So, $\int(x+5)^{10}\,dx$
$=\dfrac{1}{11}(x+5)^{11}+C$

(ii) $\displaystyle\int\frac{20}{(4-5x)^3}\,dx$

Solutions

$\dfrac{d}{dx}(4-5x)^{-2}$
$=-2(4-5x)^{-3}\cdot(-5)$
$=10(4-5x)^{-3}$.
So, $\int\frac{20}{(4-5x)^3}\,dx$
$=2(4-5x)^{-2}+C$
$=\dfrac{2}{(4-5x)^2}+C$

Example 4:
Evaluate the following indefinite integrals:

(i) $$\int \left(x^3 + 3x \cdot \sqrt{x} - \frac{4}{x^2}\right) dx$$

Solutions

$$\int \left(x^3 + 3x \cdot \sqrt{x} - \frac{4}{x^2}\right) dx$$
$$= \int x^3\, dx + 3\int x^{\frac{3}{2}}\, dx - 4\int x^{-2}\, dx$$
$$= \frac{1}{4}x^4 + \frac{6}{5}x^{\frac{5}{2}} + \frac{4}{x} + C$$

(ii) $$\int (2cos3t + 5sin4t + 3e^{7t})\, dt$$

Solutions

$$\int (2cos3t + 5sin4t + 3e^{7t})\, dt$$
$$= 2\int cos3tdt + 5\int sin4t\, dt + 3\int e^{7t}\, dt$$
$$= \frac{2}{3}sin3t - \frac{5}{4}cos4t + \frac{3}{7}e^{7t} + C$$

Very Simple Differential Equation (An application):

Solve $\frac{dy}{dx} = f(x)$.

Solutions

$$y(x) = \int f(x)dx$$

Remark 1:

Suppose F is a primitive function of f on concerned domain.
Then, we can find a fixed real number C (constant) such that $y(x) = F(x) + C$ for any concerned x.

Remark 2:
Let $x_0 \in$ concerned domain and $y_0 = y(x_0)$.
Then, $C = y_0 - F(x_0)$. That is, it is fixed.

Hence, if there is a solution for the system $\begin{cases} \frac{d}{dx}y(x) = f(x) \\ y(x_0) = y_0 \end{cases}$, then it MUST be unique.

Remark 3:

$y_0 = y(x_0)$ is called the initial value and $\begin{cases} \frac{d}{dx}y(x) = f(x) \\ y(x_0) = y_0 \end{cases}$ is called differential equation with initial value (Initial Value Problem).

Example 1:
Solve $\frac{dy}{dx} = 3x^2$.

Solutions

$$y(x) = \int 3x^2\, dx = x^3 + C$$

where C is any arbitrary constant

Example 2:
Solve $\frac{dy}{dx} = 2x + 3$ and $y(1) = 2$.

Solutions

$$y(x) = \int (2x + 3)\, dx = x^2 + 3x + C$$

where C is a constant.
$2 = y(1) = 1 + 3 + C$. So, $C = -2$.
Thus, $y(x) = x^2 + 3x - 2$.

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Rectilinear Motion (An application)
For Acceleration function $a(t)$, Velocity function $v(t)$, Displacement function $s(t)$, we have:

$$a(t) = v'(t) = \frac{d}{dt}v(t)$$

$$\int a(t)\,dt = v(t) + C_1$$

C_1 is a constant

$$v(t) = s'(t) = \frac{d}{dt}s(t)$$

$$\int v(t)\,dt = s(t) + C_2$$

C_2 is a constant

If there is a solution for the system $\begin{cases} \frac{d^2}{dt^2}s(t) = a(t) \\ v(t_0) = v_0 \\ s(t_0) = s_0 \end{cases}$, then it MUST be unique.

Note 1: two initial values $v(t_0) = v_0$ and $s(t_0) = s_0$ are needed to fix the constants C_1 and C_2.

Note 2: For action under a constant force, we have constant acceleration $a(t) = \boldsymbol{a}$ for all concerned t, where $\boldsymbol{a}$ is a fixed real number (constant).

Example 1 (constant acceleration):

$\begin{cases} \frac{d^2}{dt^2}s(t) = \boldsymbol{a} \text{ for } t \geq 0 \\ s'(0) = u \\ s(0) = 0 \end{cases}$, where $\boldsymbol{a}$ is a fixed real number.

Then, $v(t) = u + \boldsymbol{a}t$, $s(t) = ut + \frac{1}{2}\boldsymbol{a}t^2$ and $v^2 - u^2 = 2\boldsymbol{a}s$
(Note: we write $v = v(t)$ and $s = s(t)$)

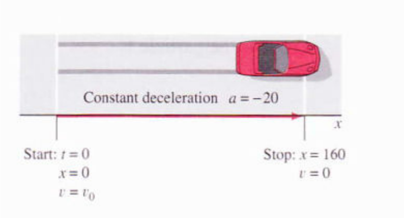
Proof
$v(t) = \int \boldsymbol{a}\,dt = \boldsymbol{a}t + C_1$, where C_1 is a constant.
$u = v(0) = 0 + C_1 \Longrightarrow C_1 = u \Longrightarrow v(t) = u + \boldsymbol{a}t$ for $t \geq 0$.
$s(t) = \int v(t)\,dt = \int (u + \boldsymbol{a}t)dt = ut + \frac{1}{2}\boldsymbol{a}t^2 + C_2$, where C_2 is a constant.
$0 = s(0) = 0 + 0 + C_2 \Longrightarrow C_2 = 0 \Longrightarrow s(t) = ut + \frac{1}{2}\boldsymbol{a}t^2$ for $t \geq 0$.
$v^2 - u^2 = (v - u)(v + u) = \boldsymbol{a}t(2u + \boldsymbol{a}t) = 2\boldsymbol{a}\left(ut + \frac{1}{2}\boldsymbol{a}t^2\right) = 2\boldsymbol{a}s$.

Example 2 (non-constant acceleration)
A particle starts from the rest (that is, with initial velocity 0) at the point $x = 10$ and moves along the $x-$ axis with acceleration function $a(t) = 12t$. Find the resulting position function $x(t)$.

Solutions
Let $v(t) = x'(t)$ for $t \geq 0$.
$v(t) = \int a(t)\,dt = \int 12t\,dt = 6t^2 + C_1$, C_1 is a constant.
$0 = v(0) = 0 + C_1 \Longrightarrow C_1 = 0 \Longrightarrow v(t) = 6t^2$.
$x(t) = \int v(t)\,dt = \int 6t^2\,dt = 2t^3 + C_2$, C_2 is a constant.
$10 = x(0) = 0 + C_2 \Longrightarrow C_2 = 10 \Longrightarrow x(t) = 2t^3 + 10$.
Answer: $x(t) = 2t^3 + 10$ for $t \geq 0$.

Example 3

The skid marks made by an automobile indicates that it brakes were fully applied for s distance of 160 ft. before it came to a stop. Suppose that the car in question has a constant deceleration of 20 ft. $/s^2$ under the conditions of the skid. How fast was the car travelling when its brakes were first applied?

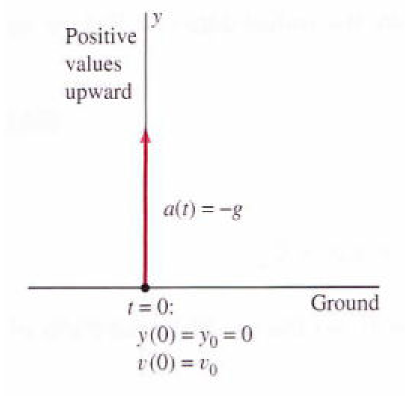


Solutions

$a(t) = -20$ for $t \geq 0$. Let $v(0) = v_0 = u$.
$v(t) = \int a(t)dt = \int -20\,dt = -20t + C_1$, C_1 is a constant.
$u = v(0) = 0 + C_1 \Rightarrow C_1 = u \Rightarrow v(t) = u - 20t$
$s(t) = \int v(t)dt = \int (u - 20t)\,dt = ut - 10t^2 + C_2$, C_2 is a constant.
$0 = s(0) = 0 - 0 + C_2 \Rightarrow C_2 = 0 \Rightarrow s(t) = ut - 10t^2$.
When $v(t) = 0$, $t = \frac{u}{20}$, $160 = s\left(\frac{u}{20}\right) = u\left(\frac{u}{20}\right) - 10\left(\frac{u}{20}\right)^2$.
So $u^2\left(\frac{1}{20} - \frac{1}{40}\right) = 160 \Rightarrow u^2 = 6400 \Rightarrow u = 80$ (note: $u \geq 0$).
The car was travelling at 80 ft. /s. when its brakes were first applied.

Exercise 1:

Suppose that a bolt was fired vertically upward from a crossbow at ground level and that it struck the ground 20 s. later. If air resistance may be neglected, find the initial velocity of the bolt and the maximum altitude that it reached.



Exercise 2:

Evaluate:

(i) $\displaystyle\int \left(x^{5/2} - \frac{5}{x^{10}} - \sqrt{x}\right) dx$

(ii) $\displaystyle\int (2t+1)^{10}\,dt$

(iii) $\displaystyle\int (e^x + e^{-x})^2\,dx$

Exercise 3:

Solve for $y = y(x)$ if $\frac{dy}{dx} = \frac{1}{x^2}$ and $y(1) = 5$.

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 # Some useful identities

(i) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1,2,3,\cdots$.

(ii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n = 1,2,3,\cdots$.

Proof for (i) by Mathematical Induction:

Let $S(n)$ be the statement that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1,2,3,\cdots$.
When $n = 1$,
LHS (Left Hand Side) is $\sum_{i=1}^{1} i = 1$.
RHS (Right Hand Side) is $\frac{1\times 2}{2} = 1$.
So, LHS = RHS. $S(1)$ is true.
Assume $S(k)$ is true, that is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ where k is a positive integer.
When $n = k + 1$,
LHS is $\sum_{i=1}^{k+1} i$

$$= \left(\sum_{i=1}^{k} i\right) + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{k+1}{2}(k+2) = \frac{(k+1)(k+1+1)}{2}$$

= RHS. So, $S(k+1)$ is also true.
By Mathematical Induction, $S(n)$ is true for $n = 1,2,3,\cdots$.

Proof for (ii) by Mathematical Induction:

Let $S(n)$ be the statement that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n = 1,2,3,\cdots$.
When $n = 1$,
LHS (Left Hand Side) is $\sum_{i=1}^{1} i^2 = 1^2 = 1$.
RHS (Right Hand Side) is $\frac{1\times 2\times 3}{6} = 1$.
So, LHS = RHS. $S(1)$ is true.
Assume $S(k)$ is true, that is $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ where k is a positive integer.
When $n = k + 1$,
LHS is $\sum_{i=1}^{k+1} i^2$

$$= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k+1}{6}[k(2k+1) + 6(k+1)]$$

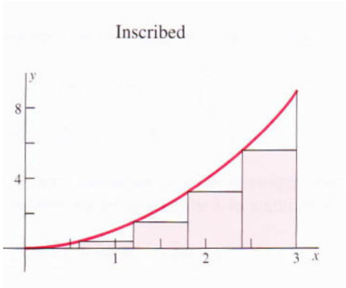
$$= \frac{k+1}{6}[2k^2 + 7k + 6]$$

$$= \frac{k+1}{6}(k+2)(2k+3)$$

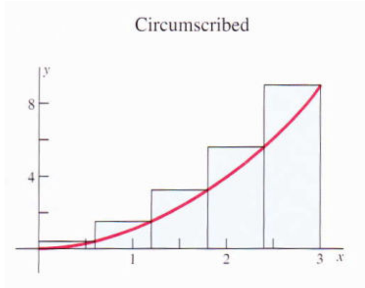
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

= RHS. So, $S(k+1)$ is also true.
By Mathematical Induction, $S(n)$ is true for $n = 1,2,3,\cdots$.

Area under a curve



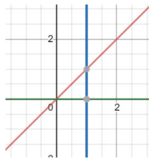
Rectangles inscribed under the curve $y = f(x)$



Rectangles circumscribed above the curve $y = f(x)$

Example 1:

Find the area bounded by the lines
$y = x, y = 0$ and $x = 1$.



Solutions

Method 1:

The region bounded by the lines $y = x, y = 0$ and $x = 1$ is a right-angled triangle with sides of lengths 1, 1 and $\sqrt{2}$.
Its area is $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

Method 2:

Let $f: R \to R$ be defined by $f(x) = x$ for any $x \in R$.
We sub-divide the interval $[0,1]$ into sub-intervals $[x_{j-1}, x_j]$ where $j = 1,2,3,\cdots,n$.
That is, $x_0 < x_1 < \cdots < x_n$, $x_0 = 0$ and $x_n = 1$.
We choose $x_j = \frac{j}{n}$ for $j = 0,1,2,3,\cdots,n$. Note: $x_j - x_{j-1} = \frac{1}{n}$ for $j = 1,2,3,\cdots,n$.
We call $\{x_0, x_1, \cdots, x_n\}$ a partition of $[0,1]$, say it is denoted as P_n.
Let $L(f, P_n)$ be the sum of the areas of the rectangles inscribed under the curve
$y = f(x) = x$ by considering the sub-intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ as the bases of such rectangles.
Let $U(f, P_n)$ be the sum of the areas of the rectangles circumscribed above the curve
$y = f(x) = x$ by considering the sub-intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ as the bases of such rectangles.
Let $A(f)$ be the area of the region bounded by $y = f(x) = x, y = 0$ and $x = 1$.

Observation: $L(f, P_n) \leq A(f) \leq U(f, P_n)$.

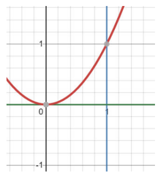
$$L(f, P_n) = \sum_{j=1}^{n} \frac{j-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{j=1}^{n} (j-1) = \frac{1}{n^2} \sum_{j=1}^{n-1} j$$
$= \frac{1}{n^2} \times \frac{n \times (n-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) \to \frac{1}{2}$ as $n \to +\infty$.

$$U(f, P_n) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{j=1}^{n} j$$
$= \frac{1}{n^2} \times \frac{n \times (n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}$ as $n \to +\infty$.

Thus, $A(f) = \frac{1}{2}$.
The area of the region bounded by the lines $y = f(x) = x, y = 0$ and $x = 1$ is $\frac{1}{2}$.

Example 2:

Find the area bounded by the lines
$y = x^2$, $y = 0$ and $x = 1$.



Solutions

Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$.
We sub-divide the interval $[0,1]$ into sub-intervals $[x_{j-1}, x_j]$ where $j = 1,2,3,\cdots,n$.
That is, $x_0 < x_1 < \cdots < x_n$, $x_0 = 0$ and $x_n = 1$.
We choose $x_j = \frac{j}{n}$ for $j = 0,1,2,3,\cdots,n$. Note: $x_j - x_{j-1} = \frac{1}{n}$ for $j = 1,2,3,\cdots,n$.
We call $\{x_0, x_1, \cdots, x_n\}$ a partition of $[0,1]$, say it is denoted as P_n.
Let $L(f, P_n)$ be the sum of the areas of the rectangles inscribed under the curve
$y = f(x) = x^2$ by considering the sub-intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ as the bases of such rectangles.
Let $U(f, P_n)$ be the sum of the areas of the rectangles circumscribed above the curve
$y = f(x) = x^2$ by considering the sub-intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ as the bases of such rectangles.
Let $A(f)$ be the area of the region bounded by $y = f(x) = x^2$, $y = 0$ and $x = 1$.

Observation: $L(f, P_n) \leq A(f) \leq U(f, P_n)$.

$$L(f, P_n) = \sum_{j=1}^{n} \left(\frac{j-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{j=1}^{n} (j-1)^2 = \frac{1}{n^3}\sum_{j=1}^{n-1} j^2$$
$$= \frac{1}{n^3} \times \frac{(n-1)\times n\times[2(n-1)+1]}{6} = \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \to \frac{1}{3} \text{ as } n \to +\infty.$$

$$U(f, P_n) = \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{j=1}^{n} j^2$$
$$= \frac{1}{n^3} \times \frac{n\times(n+1)\times(2n+1)}{6} = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \to \frac{1}{3} \text{ as } n \to +\infty.$$

Thus, $A(f) = \frac{1}{3}$.
The area of the region bounded by the curve $y = f(x) = x^2$ and the lines $y = 0$ and $x = 1$ is $\frac{1}{3}$.

Generalization 1:
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose $f(x) \geq 0$ for any $x \in [a, b]$.

Let $\{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$. That is, $x_0 < x_1 < \cdots < x_n$, $x_0 = a$ and $x_n = b$.
We use P_n to denote this partition.

Suppose that we can find the global maximum and the global minimum values of f on $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \cdots, n$.
(This is possible, for example, if f is continuous on $[a, b]$.)

Let $M_j = max\{f(x): x \in [x_{j-1}, x_j]\}$ and $m_j = min\{f(x): x \in [x_{j-1}, x_j]\}$ for $j = 1,2,3, \cdots, n$.

Let $L(f, P_n)$ be the sum of the areas of the rectangles inscribed under the curve
$y = f(x)$ by considering the sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$ as the bases of such rectangles.

Let $U(f, P_n)$ be the sum of the areas of the rectangles circumscribed above the curve
$y = f(x)$ by considering the sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$ as the bases of such rectangles.

Let $A(f)$ be the area of the region bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$.

Observation: $L(f, P_n) \leq A(f) \leq U(f, P_n)$.

$$L(f, P_n) = \sum_{j=1}^{n} m_j(x_j - x_{j-1})$$
$$U(f, P_n) = \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

Let $\|P_n\| = max\{x_j - x_{j-1}: j = 1,2,3, \cdots n\}$. This is called the norm of the partition P_n.

Suppose we can find a real number l such that $L(f, P_n) \to l$ and $U(f, P_n) \to l$ whenever $\|P_n\| \to 0$.

In this case, $A(f) = l$.

That is, the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is l.

Generalization 2:
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.

Let $\{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$. That is, $x_0 < x_1 < \cdots < x_n$, $x_0 = a$ and $x_n = b$.
We use P_n to denote this partition.

Suppose that we can find the global maximum and the global minimum values of f on $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \cdots, n$.
(This is possible, for example, if f is continuous on $[a, b]$.)

Let $M_j = max\{f(x): x \in [x_{j-1}, x_j]\}$ and $m_j = min\{f(x): x \in [x_{j-1}, x_j]\}$ for $j = 1,2,3, \cdots, n$.

Let $L(f, P_n) = \sum_{j=1}^{n} m_j(x_j - x_{j-1})$ and $U(f, P_n) = \sum_{j=1}^{n} M_j(x_j - x_{j-1})$.

$L(f, P_n)$ is called the Lower Riemann Sum of f for P_n.
$U(f, P_n)$ is called the Upper Riemann Sum of f for P_n.

Suppose we choose $\{c_1, \cdots, c_n\}$ so that $c_j \in [x_{j-1}, x_j]$ for $j = 1,2,3, \cdots, n$.
We denote P_n with this choice of $\{c_1, \cdots, c_n\}$ as $\dot{P_n}$.
Let $A(f, \dot{P_n}) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$.

Observation: $L(f, P_n) \le A(f, \dot{P_n}) \le U(f, P_n)$.

Let $\|P_n\| = max\{x_j - x_{j-1}: j = 1,2,3, \cdots n\}$. This is called the norm of the partition P_n.

Suppose we can find a real number l such that $L(f, P_n) \to l$ and $U(f, P_n) \to l$ whenever $\|P_n\| \to 0$.

In this case, $\lim\limits_{\|P_n\| \to 0} A(f, \dot{P_n}) = l$.

We say f is Riemann Integrable on $[a, b]$ and we define the definite integral of f on $[a, b]$ as such number l.
We denote it as $\int_a^b f(x)dx$.

From above arguments,
$\int_a^b f(x)dx = \lim\limits_{\|P_n\| \to 0} \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$ where $x_0 < x_1 < \cdots < x_n$, $x_0 = a$, $x_n = b$,
$c_j \in [x_{j-1}, x_j]$ for $j = 1,2,3, \cdots, n$ and $\|P_n\| = max\{x_j - x_{j-1}: j = 1,2,3, \cdots n\}$.

For $\int_a^b f(x)dx$, we say: integrate f with respect to x from $x = a$ to $x = b$.

$x = a$ is called the lower limit of the definite integral

$x = b$ is called the upper limit of the definite integral

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Lecture Notes for Chapter 5: Indefinite and Definite Integrals

Theorem:
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is Riemann Integrable on $[a, b]$.
Suppose $\boldsymbol{f(x) \geq 0}$ for any $\boldsymbol{x \in [a, b]}$.
Then, the area of the region bounded by the curve $y = f(x)$, the lines $y = 0$, $x = a$ and $x = b$ is $\int_a^b f(x)dx$.

Theorem:
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is continuous on $[a, b]$.
Then, f is Riemann Integrable on $[a, b]$.
Proof: Omitted (Will be included in the course on "Real Analysis", such as "Mathematical Analysis I".)

Theorem (Fundamental Theorem of Calculus)
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is continuous on $[a, b]$.
Suppose F is a primitive function of f on $[a, b]$.
Then, $\int_a^b f(x)dx = F(b) - F(a)$.
Sometimes, we write $\int_a^b f(x)dx = F(x)|_{x=a}^{b} = F(b) - F(a)$.
Note: $\frac{d}{dx}F(x) = f(x)$ for any $x \in (a, b)$.

Remark 1:
Suppose both F and G are primitive functions of f on $[a, b]$.
We can easily show that $F(b) - F(a) = G(b) - G(a)$.
Thus, the choice of an representative function F of the set of all primitive functions of f on $[a, b]$ will get the same value $\int_a^b f(x)dx$ by the equation $\int_a^b f(x)dx = F(b) - F(a)$.

Remark 2:
$\int_a^b f(x)dx$ is called the definite integral of f with respect to x from $x = a$ to $x = b$.
$\int f(x)\, dx$ is called the indefinite integral of f with respect to x.

Remark 3:
As F is a primitive function of f on $[a, b]$, we have $\int f(x)\, dx = F(x) + C$ where C is an arbitrary constant.

Proof
As f is continuous on $[a,b]$, f is Riemann Integrable on $[a,b]$.
Suppose we can find a real number l such that $\int_a^b f(x)dx = l$.
Let us consider a partition $P_n = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$, that is, $x_0 < x_1 < \cdots < x_n$, $x_0 = a$, $x_n = b$ and $\|P_n\| = max\{x_j - x_{j-1}: j = 1,2,3,\cdots n\}$.

By Mean Value Theorem, we can find $c_j \in [x_{j-1}, x_j]$ such that
$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1})$ for $j = 1,2,3,\cdots,n$.
That is, $F(x_j) - F(x_{j-1}) = f(c_j)(x_j - x_{j-1})$ for $j = 1,2,3,\cdots,n$.

Then, we have:
$F(b) - F(a) = F(x_n) - F(x_0)$
$= \sum_{j=1}^{n} \left(F(x_j) - F(x_{j-1})\right)$
$= \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$
So, $L(f,P_n) \le F(b) - F(a) \le U(f,P_n)$ as $L(f,P_n) \le \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \le U(f,P_n)$.
As $L(f,P_n) \to \int_a^b f(x)dx$ and $U(f,P_n) \to \int_a^b f(x)dx$ as $\|P_n\| \to 0$.
Thus, $\int_a^b f(x)dx \le F(b) - F(a) \le \int_a^b f(x)dx$.
Hence, $\int_a^b f(x)dx = F(b) - F(a)$.

Example 1:

Find $\int_0^1 xdx$ and $\int_0^1 x^2dx$.

Solutions

$$\int_0^1 xdx = \frac{x^2}{2}\bigg|_{x=0}^{1} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\int_0^1 x^2dx = \frac{x^3}{3}\bigg|_{x=0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}$$

Example 2:

(i) $$\int_0^\pi sinxdx$$

Solutions

$$\int_0^\pi sinxdx$$
$$= -cosx|_{x=0}^{\pi}$$
$$= (-cos\pi) - (-cos0)$$
$$= 1 - (-1)$$
$$= 2$$

(ii) $$\int_0^1 (2x+1)^3 dx$$

Solutions

$$\int_0^1 (2x+1)^3 dx$$
$$= \frac{1}{8}(2x+1)^4\Big|_{x=0}^{1}$$
$$= \frac{1}{8}(3^4 - 1)$$
$$= 10$$

(iii) $$\int_0^1 e^{2x} dx$$

Solutions

$$\int_0^1 e^{2x} dx$$
$$= \frac{1}{2}e^{2x}\Big|_{x=0}^{1}$$
$$= \frac{1}{2}(e^2 - 1)$$

(iv) $$\int_1^5 \sqrt{3x+1}dx$$

Solutions

$$\int_1^5 \sqrt{3x+1}dx$$
$$= \frac{2}{9}(3x+1)^{\frac{3}{2}}\Big|_{x=1}^{5}$$
$$= \frac{2}{9}\left(16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$
$$= \frac{2}{9}(64 - 8)$$
$$= \frac{112}{9}$$

Example 3:

Use definite integrals to evaluate the following limits:

(i) $$\lim_{n\to+\infty} \frac{1+2+3+\cdots+n}{n^2}$$

Solutions

$$\lim_{n\to+\infty} \frac{1+2+3+\cdots+n}{n^2}$$
$$= \lim_{n\to+\infty}\left(\frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \frac{3}{n}\cdot\frac{1}{n} + \cdots + \frac{n}{n}\cdot\frac{1}{n}\right)$$
$$= \int_0^1 xdx$$

(Consider the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\right\}$ of $[0,1]$ for the function $f(x) = x$.)

$$= \frac{1}{2}x^2\Big|_{x=0}^{1}$$
$$= \frac{1}{2}(1-0)$$
$$= \frac{1}{2}$$

(ii) $$\lim_{n\to+\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

Solutions

$$\lim_{n\to+\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$
$$= \lim_{n\to+\infty}\left(\left(\frac{1}{n}\right)^2\cdot\frac{1}{n} + \left(\frac{2}{n}\right)^2\cdot\frac{1}{n} + \left(\frac{3}{n}\right)^2\cdot\frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2\cdot\frac{1}{n}\right)$$
$$= \int_0^1 x^2dx$$

(Consider the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\right\}$ of $[0,1]$ for the function $f(x) = x^2$.)

$$= \frac{1}{3}x^3\Big|_{x=0}^{1}$$
$$= \frac{1}{3}(1-0)$$
$$= \frac{1}{3}$$

Exercises:

Use definite integrals to evaluate the following limits:

(i) $$\lim_{n\to+\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n\cdot\sqrt{n}}$$

(ii) $$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}-1\right)$$

Solutions

(i)
$$\lim_{n\to+\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n\cdot\sqrt{n}}$$
$$= \lim_{n\to+\infty}\left(\sqrt{\frac{1}{n}}\cdot\frac{1}{n}+\sqrt{\frac{2}{n}}\cdot\frac{1}{n}+\sqrt{\frac{3}{n}}\cdot\frac{1}{n}+\cdots+\sqrt{\frac{n}{n}}\cdot\frac{1}{n}\right)$$
(Consider the partition $\left\{0,\frac{1}{n},\frac{2}{n},\cdots,1\right\}$ of $[0,1]$ for the function $f(x)=\sqrt{x}$.)
$$= \int_0^1 \sqrt{x}\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_{x=0}^{1}$$
$$= \frac{2}{3}(1-0)$$
$$= \frac{2}{3}$$

(ii)
$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}-1\right)$$
$$= \lim_{n\to+\infty}\left(\left(\frac{2}{n}-1\right)\cdot\frac{1}{n}+\left(\frac{4}{n}-1\right)\cdot\frac{1}{n}+\left(\frac{6}{n}-1\right)\cdot\frac{1}{n}+\cdots+\left(\frac{2n}{n}-1\right)\cdot\frac{1}{n}\right)$$
(Consider the partition $\left\{0,\frac{1}{n},\frac{2}{n},\cdots,1\right\}$ of $[0,1]$ for the function $f(x)=2x-1$.)
$$= \int_0^1 (2x-1)\,dx$$
$$= x^2 - x\big|_{x=0}^{1}$$
$$= (1-1)-(0-0)$$
$$= 0$$

Example 4:

Use Riemann Sums to evaluate $\int_a^b x dx$ where $a, b \in R$ with $a < b$.

Solutions

Let $f: R \to R$ be defined by $f(x) = x$ for any $x \in R$.
Consider the partition P_n: $\left\{a, a + \frac{b-a}{n}, a + 2 \times \frac{b-a}{n}, \cdots, b\right\}$ of $[a, b]$ for the function $f(x) = x$, (that is, let $x_j = a + j \times \frac{b-a}{n}$ for $j = 0,1,2, \cdots, n$), we have

$$L(f, P_n) = \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$$
$$= \sum_{j=1}^{n} x_{j-1} \cdot \left(\frac{b-a}{n}\right)$$
$$= \sum_{j=1}^{n} \left(a + (j-1) \cdot \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$
$$= a\left(\frac{b-a}{n}\right)\left(\sum_{j=1}^{n} 1\right) + \left(\frac{b-a}{n}\right)^2 \left(\sum_{j=1}^{n} (j-1)\right)$$
$$= a\left(\frac{b-a}{n}\right) \cdot n + \left(\frac{b-a}{n}\right)^2 \cdot \frac{n(n-1)}{2}$$
$$= a(b-a) + \frac{1}{2}(b-a)^2\left(1 - \frac{1}{n}\right)$$
$$\to a(b-a) + \tfrac{1}{2}(b-a)^2 \text{ as } n \to +\infty$$

$$U(f, P_n) = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$
$$= \sum_{j=1}^{n} x_j \cdot \left(\frac{b-a}{n}\right)$$
$$= \sum_{j=1}^{n} \left(a + j \cdot \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$
$$= a\left(\frac{b-a}{n}\right)\left(\sum_{j=1}^{n} 1\right) + \left(\frac{b-a}{n}\right)^2 \left(\sum_{j=1}^{n} j\right)$$
$$= a\left(\frac{b-a}{n}\right) \cdot n + \left(\frac{b-a}{n}\right)^2 \cdot \frac{n(n+1)}{2}$$
$$= a(b-a) + \frac{1}{2}(b-a)^2\left(1 + \frac{1}{n}\right)$$
$$\to a(b-a) + \tfrac{1}{2}(b-a)^2 \text{ as } n \to +\infty$$

Note 1: $f'(x) = 1 > 0$ for any $x \in R$, so f is strictly increasing on $[x_{j-1}, x_j]$.
The global maximum value of f on $[x_{j-1}, x_j]$ is $f(x_j)$.
The global minimum value of f on $[x_{j-1}, x_j]$ is $f(x_{j-1})$.

Note 2:
$a(b-a) + \frac{1}{2}(b-a)^2 = ab - a^2 + \frac{1}{2}(b^2 - 2ab + a^2) = \frac{1}{2}(b^2 - a^2)$.

Thus, $\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$.

Example 5:

Use Riemann Sums to evaluate $\int_0^2 e^x dx$.

Solutions

Let $f: R \to R$ be defined by $f(x) = e^x$ for any $x \in R$.
Consider the partition P_n: $\left\{0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \cdots, 2\right\}$ of $[0,2]$ for the function $f(x) = e^x$, and let $x_0 = 0$, $x_j = \frac{2j}{n}$ and $c_j = x_j$ for $j = 1,2,\cdots,n$, we have

$$A(f, P_n) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$
$$= \sum_{j=1}^{n} e^{\frac{2j}{n}} \times \frac{2}{n}$$
$= \frac{2}{n}(e^u + e^{2u} + e^{3u} + \cdots + e^{nu})$ where $u = \frac{2}{n}$
$= \frac{2}{n}(a + ar + ar^2 + \cdots + ar^{n-1})$ where $a = r = e^u$
$$= \frac{2}{n} \cdot \frac{a(1 - r^n)}{1 - r}$$
$$= \frac{2}{n} \cdot \frac{e^{2/n}(1 - e^2)}{1 - e^{2/n}}$$
$$= 2(1 - e^2) \cdot \frac{1}{n\left(1 - e^{\frac{2}{n}}\right)} \cdot e^{\frac{2}{n}}$$
$\to 2(1 - e^2) \cdot \frac{-1}{2} \cdot 1 = e^2 - 1$ as $n \to +\infty$

Note the claim: $n\left(1 - e^{\frac{2}{n}}\right) \to -2$ as $n \to +\infty$

Proof of the claim:

$\lim_{n \to +\infty} n\left(1 - e^{\frac{2}{n}}\right)$ ($0 \times \infty$ form) (n is a positive integer)
$= \lim_{x \to +\infty} x\left(1 - e^{\frac{2}{x}}\right)$ (x is a real number)
$= \lim_{x \to +\infty} \frac{1 - e^{\frac{2}{x}}}{\frac{1}{x}}$ ($\frac{0}{0}$ form, L'Hopital's Rule)
$$= \lim_{x \to +\infty} \frac{-e^{\frac{2}{x}} \cdot \frac{-2}{x^2}}{\frac{-1}{x^2}}$$
$= \lim_{x \to +\infty} -2e^{\frac{2}{x}}$
$= -2$

$\int_0^2 e^x dx = \lim_{\|P_n\| \to 0} \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \lim_{n \to +\infty} A(f, P_n) = e^2 - 1.$

Theorems (Rules for Definite Integration):
Let $a, b \in R$ with $a < b$. Let f and g be real-valued functions on x and are defined on $[a, b]$.
Suppose both f and g are Riemann Integrable on $[a, b]$. Let λ be a fixed real number (constant).
Then,

(i) $f + g$ is Riemann Integrable on $[a, b]$ and
$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

(ii) $f - g$ is Riemann Integrable on $[a, b]$ and
$$\int_a^b \big(f(x) - g(x)\big)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

(iii) λf is Riemann Integrable on $[a, b]$ and
$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$$

Proof: Omitted (As Exercises)

Definition
Let $a, b \in R$ with $a < b$. Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is Riemann Integrable on $[a, b]$.
We define $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

Theorem:
Let $a, b \in R$ with $a < b$. Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is Riemann Integrable on $[a, b]$. For any $\alpha, \beta, \gamma \in [a, b]$, we have
$\int_\alpha^\gamma f(x)\, dx = \int_\alpha^\beta f(x)\, dx + \int_\beta^\gamma f(x)\, dx$.
Proof: Omitted (As Exercises)

Remark: $\int_\alpha^\alpha f(x)\, dx = 0$.

Example 1:
Evaluate:

(i) $$\int_0^{\frac{\pi}{2}} 2sinx\, dx$$

Solutions
$$\int_0^{\frac{\pi}{2}} 2sinx\, dx$$
$$= 2\int_0^{\frac{\pi}{2}} sinx\, dx$$
$$= 2\left(-cosx\Big|_0^{\frac{\pi}{2}}\right)$$
$$= 2\big(0 - (-1)\big)$$
$$= 2$$

(ii) $$\int_0^{\pi} \left(3\sqrt{x} + cos\frac{x}{2}\right) dx$$

Solutions
$$\int_0^{\pi} \left(3\sqrt{x} + cos\frac{x}{2}\right) dx$$
$$= 3\int_0^{\pi} \sqrt{x}\, dx + \int_0^{\pi} cos\frac{x}{2} dx$$
$$= 3\left(\frac{2}{3}x^{\frac{3}{2}}\Big|_0^{\pi}\right) + \left(2sin\frac{x}{2}\Big|_0^{\pi}\right)$$
$$= 3\left(\frac{2}{3}\pi^{\frac{3}{2}} - 0\right) + (2 - 0)$$
$$= 2\pi\sqrt{\pi} + 2$$

Example 2:

Let $f: R \to R$ be defined by $f(x) = 2|x|$ for any $x \in R$. Evaluate $\int_{-1}^{3} f(x)\,dx$.

Solutions

$$\int_{-1}^{3} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx$$
$$= \int_{-1}^{0} -2x\,dx + \int_{0}^{3} 2x\,dx$$
$$= (-x^2|_{-1}^{0}) + (x^2|_{0}^{3})$$
$$= \big(0-(-1)\big) + (9-0) = 10$$

Example 3:

Evaluate $\int_{0}^{2\pi} |cosx - sinx|\,dx$.

Solutions

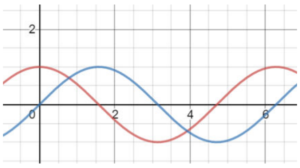
Consider $cosx - sinx = 0$ for $x \in [0, 2\pi]$,

$tanx = 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

Note 1: $cosx > sinx$ for $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$.

Note 2: $cosx < sinx$ for $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.



$$\int_{0}^{2\pi} |cosx - sinx|\,dx$$
$$= \int_{0}^{\frac{\pi}{4}} (cosx - sinx)\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} -(cosx - sinx)\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (cosx - sinx)\,dx$$
$$= \left(sinx + cosx|_{0}^{\frac{\pi}{4}}\right) - \left(sinx + cosx|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right) + \left(sinx + cosx|_{\frac{5\pi}{4}}^{2\pi}\right)$$
$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] - \left[\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right] + \left[(0+1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$
$$= [\sqrt{2} - 1] - [-2\sqrt{2}] + [1 + \sqrt{2}]$$
$$= 4\sqrt{2}.$$

Theorem 1 (Comparison Principle):
Let $a, b \in R$ with $a < b$.
Let f and g be real-valued functions on x and are defined on $[a, b]$.
Suppose both f and g are Riemann Integrable on $[a, b]$.
Suppose $f(x) \leq g(x)$ for any $x \in [a, b]$.
Then, $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
Proof: Omitted (As Exercise)

Theorem 2 (Comparison Principle):
Let $a, b, m, M \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is Riemann Integrable on $[a, b]$.
Suppose $m \leq f(x) \leq M$ for any $x \in [a, b]$.
Then, $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$.
Proof: Omitted (As Exercise)

Example:
Show that $1 \leq \int_0^1 \sqrt{1 + x^2}\,dx \leq \int_0^1 \sqrt{1 + x}\,dx$.

Proof
$0 \leq x \leq 1 \Longrightarrow 0 \leq x^2 \leq x \Longrightarrow 1 \leq 1 + x^2 \leq 1 + x \Longrightarrow 1 \leq \sqrt{1 + x^2} \leq \sqrt{1 + x}$.
Thus, $\int_0^1 1\,dx \leq \int_0^1 \sqrt{1 + x^2}\,dx \leq \int_0^1 \sqrt{1 + x}\,dx$. (Use Comparison Principle)
So, $1 \leq \int_0^1 \sqrt{1 + x^2}\,dx \leq \int_0^1 \sqrt{1 + x}\,dx$.

Exercises:

(i) Suppose that an animal population $P(t)$ initially numbers $P(0) = 100$ and that its rate of growth after t months is given by $P'(t) = 10 + t + (0.06)t^2$. What is the population after 10 months?

(ii) Find $\int_0^5 sin\frac{\pi x}{10}\,dx$.

(iii) Find $\int_{-1}^2 |x^3 - x|\,dx$.

(iv) Find $\int_0^6 \big|5 - 2|x|\big|\,dx$.

(v) Show that $\int_1^2 \sqrt{1 + x}\,dx \leq \int_1^2 \sqrt{1 + x^3}\,dx \leq 3$.

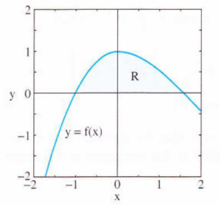
Finding area of the region bounded by curves (An application)

Example 1:

The figure shows the graph of the function f defined by

$f(x) = \begin{cases} cosx & if\ x \geq 0 \\ 1 - x^2 & if\ x \leq 0 \end{cases}$.

Find the area A of the region $\boldsymbol{R}$ bounded above by the graph $y = f(x)$ and below by the $x-$ axis.



Solutions

Put $1 - x^2 = 0$, we have $x = 1$ or -1. $x = 1$ is rejected for our consideration.

Put $cosx = 0$, we have $x = \frac{\pi}{2}$ (from the diagram).

$$A = \int_{-1}^{0} (1 - x^2)dx + \int_{0}^{\frac{\pi}{2}} cosxdx$$

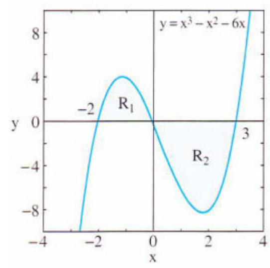
$$= \left(x - \frac{1}{3}x^3 \Big|_{x=-1}^{0} \right) + \left(sinx \Big|_{x=0}^{\frac{\pi}{2}} \right)$$

$$= \left[(0 - 0) - \left(-1 + \frac{1}{3} \right) \right] + (1 - 0)$$

$$= \frac{5}{3}$$

Example 2:

The figure shows the graph of $f(x) = x^3 - x^2 - 6x$.
Find the area A of the entire region $\boldsymbol{R}$ bounded above/below by the graph f and the $x-$ axis.
The region $\boldsymbol{R}$ consists of the two regions R_1 and R_2 as shown in the figure.



Solutions

As $f(x) = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x-3)(x+2)$,
$f(x) = 0 \Leftrightarrow x = -2 \text{ or } 0 \text{ or } 3$.

Area of $R_1 = \int_{-2}^{0}(x^3 - x^2 - 6x)\,dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\Big|_{x=-2}^{0}$

$= \left[(0-0-0) - \left(4 + \frac{8}{3} - 12\right)\right] = \frac{16}{3}$

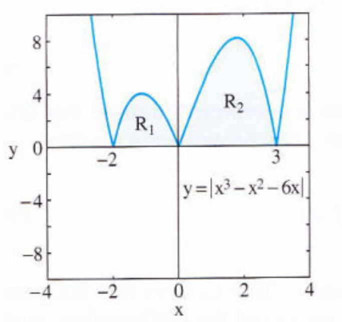
Area of $R_2 = \int_{0}^{3} -(x^3 - x^2 - 6x)\,dx = \frac{-1}{4}x^4 + \frac{1}{3}x^3 + 3x^2\Big|_{x=0}^{3}$

$= \left[\left(\frac{-81}{4} + 9 + 27\right) - (0+0+0)\right] = \frac{63}{4}$

$A = \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$

Remark

$$A = \int_{-2}^{3} |x^3 - x^2 - 6x|\,dx$$



Average Value of A Function:

Definition:

Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is Riemann Integrable on $[a, b]$.
For $y = f(x)$, we define the average value of y on $[a, b]$ as:
$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

Example:

Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$. For $y = f(x)$, find the average value of y on $[0,2]$.

Solutions

$\int_0^2 f(x)\, dx = \int_0^2 x^2\, dx = \frac{1}{3}x^3\Big|_{x=0}^{2} = \frac{1}{3}(8-0) = \frac{8}{3}$.
So, the average value of y on $[0,2]$ is $\bar{y} = \frac{1}{2-0}\int_0^2 f(x)\, dx = \frac{1}{2} \times \frac{8}{3} = \frac{4}{3}$.

Average Value Theorem:

Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is continuous on $[a, b]$.
Then, we can find $c \in [a, b]$ such that $\int_a^b f(x)\, dx = f(c) \cdot (b - a)$.
That is, $\bar{y} = f(c)$.
Sometimes, we write $c = \bar{x}$. So, $\bar{y} = f(\bar{x})$.
Proof: Omitted (As Exercise)

 # Fundamental Theorem of Calculus
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on x and is defined on $[a, b]$.
Suppose f is continuous on $[a, b]$.
We can define a function $F: [a, b] \to R$ by $F(x) = \int_a^x f(t)dt$.
(Note: it is well defined as f is continuous on $[a, b] \Longrightarrow f$ is Riemann Integrable on $[a, b]$.)

Show that $F'(x) = f(x)$ for any $x \in (a, b)$.
That is,
(i) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ for any $x \in (a, b)$.
(ii) $\int_a^x f(t)dt$ is a primitive function of f on $[a, b]$.

Proof:
$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt$$

Case 1: $h > 0$
As f is continuous on $[x, x+h]$, by Extreme Value Theorem, we can find
$m_h, M_h \in R$ such that $m_h \le f(t) \le M_h$ for any $t \in [x, x+h]$
and $m_h = f(\alpha)$, $M_h = f(\beta)$ where $\alpha, \beta \in [x, x+h]$.
By Comparison Principle, we have
$m_h \cdot h \le \int_x^{x+h} f(t)dt \le M_h \cdot h$ as $\int_x^{x+h} 1dt = h$.
So, $m_h \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M_h$.
$\lim_{h\to 0^+} m_h = f(x) = \lim_{h\to 0^+} M_h$.
Thus, $\lim_{h\to 0^+}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$.
Case 2: $h < 0$
Proof: Omitted (As Exercise)
Thus, $F'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$.

Remark:
Let G be a real-valued function on x and is defined on $[a, b]$.
Suppose G is a primitive function of f on $[a, b]$.
Then, $\int_a^b f(t)dt = G(b) - G(a) = F(b) - F(a) = F(b)$ as $F(a) = 0$.
So, $G(b) = G(a) + F(b) = G(a) + \int_a^b f(t)dt$.

Initial Value Problem (An application):

Solve for $y = y(x)$ if $\frac{dy}{dx} = f(x)$ and $y(a) = b$.

Solutions

$y(x) = y(a) + \int_a^x f(t)dt$.
That is, $y(x) = b + \int_a^x f(t)dt$.

Example:

Solve for $y = y(x)$ if $\frac{dy}{dx} = secx$ and $y(2) = 3$.

Solutions

$y(x) = y(2) + \int_2^x sectdt = 3 + \int_2^x sectdt.$

Example:

(i) Find $h'(x)$ if $h(x) = \int_0^{x^2} t^3 sintdt$.

Solutions

$h'(x) = (x^2)^3 sin(x^2) \cdot \frac{d}{dx} x^2$
$= x^6 sin(x^2) \cdot 2x$
$= 2x^7 sin(x^2)$

(ii) Find $h'(x)$ if $h(x) = \int_{cosx}^{sinx} t^2 dt$.

Solutions

$h(x) = \int_0^{sinx} t^2 dt - \int_0^{cosx} t^2 dt$
$h'(x)$
$= (sinx)^2 \cdot cosx - (cosx)^2 \cdot (-sinx)$
$= sinxcosx(sinx + cosx)$

Another Method:

$h(x) = \int_{cosx}^{sinx} t^2 dt$
$= \frac{1}{3} t^3 \Big|_{t=cosx}^{sinx}$
$= \frac{1}{3}(sin^3x - cos^3x)$
$h'(x)$
$= (sinx)^2 \cdot cosx - (cosx)^2 \cdot (-sinx)$
$= sinxcosx(sinx + cosx)$

 # Integration by Substitution for Indefinite Integral

Let $a, b \in R$ with $a < b$.
Let $f: R \to R$ and $g: R \to R$ be functions.
Suppose both f and g are differentiable on R.
Let $y = f(u)$ and $u = g(x)$.
Then, we may regard y as a function on x, that is $y = f\big(g(x)\big)$.
By Chain Rule, we have
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ OR $(f{\circ}g)'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Notes:

(i) $\int f'\big(g(x)\big) \cdot g'(x)\, dx = \int (f{\circ}g)'(x)\, dx = (f{\circ}g)(x) + C = f\big(g(x)\big) + C$

(ii) $\int f'(u)\, du = f(u) + C = f\big(g(x)\big) + C$ (as $\boldsymbol{u = g(x)}$)

(iii) We may regard $\int f'\big(g(x)\big) \cdot g'(x)\, dx = \int f'(u)\, du$
as $\boldsymbol{u = g(x)}$ and $\boldsymbol{du = g'(x)dx}$.

(iv) This suggests a way to evaluate the indefinite integral by letting $\boldsymbol{u = g(x)}$.

Example 1:
Evaluate:

(i) $\int (2x+1)^5 \cdot 2\,dx$

Solutions

$\int (2x+1)^5 \cdot 2\,dx$
$= \int u^5\,du$
(let $u = 2x+1$, $du = 2dx$)
$= \frac{1}{6}u^6 + C$
$= \frac{1}{6}(2x+1)^6 + C$

(ii) $\int 2x \cdot \sqrt{1+x^2}\,dx$

Solutions

$\int 2x \cdot \sqrt{1+x^2}\,dx$
$= \int \sqrt{1+x^2} \cdot 2xdx$
$= \int \sqrt{u}\,du$
(let $u = 1+x^2$, $du = 2xdx$)
$= \frac{2}{3}u^{3/2} + C$
$= \frac{2}{3}(1+x^2)^{3/2} + C$

Example 2:
Evaluate:

(i) $\int x^2 \cdot \sqrt{x^3+9}\,dx$

Solutions

$\int x^2 \cdot \sqrt{x^3+9}\,dx$
$= \frac{1}{3}\int \sqrt{x^3+9} \cdot 3x^2dx$
$= \frac{1}{3}\int \sqrt{u}\,du$
(let $u = x^3+9$, $du = 3x^2dx$)
$= \frac{2}{9}u^{3/2} + C$
$= \frac{2}{9}(x^3+9)^{3/2} + C$

(ii) $\int 3 \cdot \sqrt{x} \cdot e^{1+\sqrt{x^3}}\,dx$

Solutions

$\int 3 \cdot \sqrt{x} \cdot e^{1+\sqrt{x^3}}\,dx$
$= 2e \int e^{\sqrt{x^3}} \cdot \frac{3}{2}\sqrt{x}dx$

$= 2e \int e^u du$
(let $u = \sqrt{x^3}$, $du = \frac{3}{2}\sqrt{x}dx$)
$= 2e \cdot e^u + C$
$= 2e^{1+\sqrt{x^3}} + C$

Example 3:
Evaluate:

(i) $\int \frac{3 \cdot \sqrt{x}}{1+\sqrt{x^3}}dx$

Solutions

$\int \frac{3 \cdot \sqrt{x}}{1+\sqrt{x^3}}dx$
$= 2\int \frac{1}{1+\sqrt{x^3}} \cdot \frac{3}{2}\sqrt{x}dx$
$= 2\int \frac{1}{u}du$
(let $u = 1+\sqrt{x^3}$, $du = \frac{3}{2}\sqrt{x}dx$)
$= 2ln|u| + C$
$= 2ln\left|1+\sqrt{x^3}\right| + C$
$= 2ln\left(1+\sqrt{x^3}\right) + C$
Note: We assumed $x \geq 0$,
so $1+\sqrt{x^3} > 0$ for any $x \geq 0$

(ii) $\int \frac{e^x}{1+e^x}dx$

Solutions

$\int \frac{e^x}{1+e^x}dx$
$= \int \frac{1}{1+e^x} \cdot e^x dx$
$= \int \frac{1}{u}du$
(let $u = 1+e^x$, $du = e^x dx$)
$= ln|u| + C$
$= ln|1+e^x| + C$
$= ln(1+e^x) + C$
Note:
$1+e^x > 0$ for any $x \in R$

Example 4:

Evaluate:

(i) $\int sin(3x+4)\,dx$

Solutions

$\int sin(3x+4)\,dx$

$=\frac{1}{3}\int sin(3x+4)\cdot 3dx$

$=\frac{1}{3}\int sinu\,du$

(let $u=3x+4, du=3dx$)

$=\frac{-1}{3}cosu+C$

$=\frac{-1}{3}cos(3x+4)+C$

(ii) $\int 3x\cdot cos(x^2)\,dx$

Solutions

$\int 3x\cdot cos(x^2)\,dx$

$=\frac{3}{2}\int cos(x^2)\cdot 2xdx$

$=\frac{3}{2}\int cosudu$

(let $u=x^2, du=2xdx$)

$=\frac{3}{2}sinu+C$

$=\frac{3}{2}sin(x^2)+C$

Example 5:

Evaluate:

(i) $\int sec^2 3x\,dx$

Solutions

$\int sec^2 3x\,dx$

$=\frac{1}{3}\int sec^2 3x\cdot 3dx$

$=\frac{1}{3}\int sec^2 u\,du$

(let $u=3x, du=3dx$)

$=\frac{1}{3}tanu+C$

$=\frac{1}{3}tan3x+C$

(ii) $\int 2sin^3 x\cdot cosx\,dx$

Solutions

$\int 2sin^3 x\cdot cosx\,dx$

$=\int 2u^3\,du$

(let $u=sinx, du=cosxdx$)

$=\frac{2}{4}u^4+C$

$=\frac{1}{2}sin^4 x+C$

Example 6:

Evaluate:

(i) $\int \frac{x}{(30-x^2)^2}dx$

Solutions

$\int \frac{x}{(30-x^2)^2}dx$

$=\frac{-1}{2}\int (30-x^2)^{-2}\cdot(-2x)dx$

$=\frac{-1}{2}\int u^{-2}\,du$

(let $u=30-x^2, du=-2xdx$)

$=\frac{1}{2}u^{-1}+C$

$=\frac{1}{2(30-x^2)}+C$

(ii) $\int \frac{cos2t}{1+sin2t}dt$

Solutions

$\int \frac{cos2t}{1+sin2t}dt$

$=\frac{1}{2}\int \frac{1}{1+sin2t}\cdot 2cos2tdt$

$=\frac{1}{2}\int \frac{1}{u}du$

(let $u=1+sin2t, du=2cos2tdt$)

$=\frac{1}{2}ln|u|+C$

$=\frac{1}{2}ln|1+sin2t|+C$

Example 7:

Evaluate $\int \left(g(x)\right)^n \cdot g'(x)dx$ where $n = 1,2,3,\cdots$.

Solutions

$$\int \left(g(x)\right)^n \cdot g'(x)dx$$
$$= \int u^n\,du$$
(let $u = g(x)$, $du = g'(x)dx$)
$$= \frac{1}{n+1}u^{n+1} + C$$
$$= \frac{1}{n+1}\left(g(x)\right)^{n+1} + C$$

 # Integration by Substitution for Definite Integral

Let $a, b, \alpha, \beta \in R$ with $a < \alpha < \beta < b$.
Let f and g be real-valued functions.
Suppose g is continuous $[a, b]$ and is differentiable on (a, b).
Suppose g' is continuous on $[\alpha, \beta]$.
Suppose f is continuous on $g([\alpha, \beta])$.
Let $y = f(u)$ and $u = g(x)$.
Then,
$\int_\alpha^\beta f\left(g(x)\right) \cdot g'(x)\,dx = \int_{g(\alpha)}^{g(\beta)} f(u)\,du$.

Proof:

Suppose F is a real-valued function such that $F'(x) = f(x)$ for concerned x.
That is, F is a primitive function of f for concerned domain.
This is possible as we may let $F(x) = \int_\alpha^x f(t)\,dt$. (This is well defined as f is continuous on concerned domain.)

Note:

$$\int_\alpha^\beta f\left(g(x)\right) \cdot g'(x)\,dx$$
$$= \int_\alpha^\beta F'\left(g(x)\right) \cdot g'(x)\,dx$$
$$= F\left(g(x)\right)\Big|_{x=\alpha}^{\beta}$$
$$= F\left(g(\beta)\right) - F\left(g(\alpha)\right)$$

$$\int_{g(\alpha)}^{g(\beta)} f(u)\,du$$
$$= F(u)\big|_{u=g(\alpha)}^{g(\beta)}$$
$$= F\left(g(\beta)\right) - F\left(g(\alpha)\right)$$

Thus, $\int_\alpha^\beta f\left(g(x)\right) \cdot g'(x)\,dx = \int_{g(\alpha)}^{g(\beta)} f(u)\,du$.

Example:
Evaluate:

(i) $\int_3^5 \frac{x}{(30-x^2)^2}dx$

Solutions

$\int_3^5 \frac{x}{(30-x^2)^2}dx$
$= \frac{-1}{2}\int_3^5 (30-x^2)^{-2}\cdot(-2x)dx$
$= \frac{-1}{2}\int_{21}^5 u^{-2}du$
(let $u = 30 - x^2, du = -2xdx$,
when $x = 3, u = 21$,
when $x = 5, u = 5$)
$= \frac{-1}{2}(-u^{-1}|_{u=21}^5)$
$= \frac{-1}{2}\left(\frac{-1}{5} - \frac{-1}{21}\right)$
$= \frac{8}{105}$

Another Method

Use $\int \frac{x}{(30-x^2)^2}dx$
$= \frac{1}{2(30-x^2)} + C$
$\int_3^5 \frac{x}{(30-x^2)^2}dx$
$= \frac{1}{2(30-x^2)}\Big|_{x=3}^5$
$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{21}\right)$
$= \frac{8}{105}$

(ii) $\int_0^{\frac{\pi}{4}} \frac{cos2t}{1+sin2t}dt$

Solutions

$\int_0^{\frac{\pi}{4}} \frac{cos2t}{1+sin2t}dt$
$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{1+sin2t}\cdot 2cos2tdt$
$= \frac{1}{2}\int_1^2 \frac{1}{u}du$
(let $u = 1 + sin2t, du = 2cos2tdt$,
when $t = 0, u = 1$,
when $t = \frac{\pi}{4}, u = 2$)
$= \frac{1}{2}ln|u|\Big|_{u=1}^2$
$= \frac{1}{2}(ln2 - ln1) = \frac{1}{2}ln2$

Another Method

Use $\int \frac{cos2t}{1+sin2t}dt$
$= \frac{1}{2}ln|1+sin2t| + C$
$\int_0^{\frac{\pi}{4}} \frac{cos2t}{1+sin2t}dt$
$= \frac{1}{2}ln|1+sin2t|\Big|_{t=0}^{\frac{\pi}{4}}$
$= \frac{1}{2}(ln2 - ln1) = \frac{1}{2}ln2$

Exercises:
Evaluate:

(i) $\int \frac{x}{1+x^2}dx$

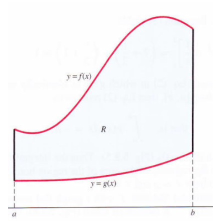
(ii) $\int sin^5 3z\cdot cos3zdz$

(iii) $\int_0^{\frac{\pi}{2}} e^{sinx}\cdot cosxdx$

(iv) $\int_0^{3\pi} cos^2xdx$

Areas of Plane Regions (Integrate w.r.t. x)

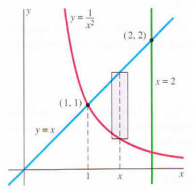
Let $a, b \in R$ with $a < b$.
Let f and g be real-valued functions on x and is defined on $[a, b]$.
Suppose $f(x) \geq g(x)$ for any $x \in [a, b]$.
Then, the area of the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is given by
$\int_a^b \big(f(x) - g(x)\big)\, dx$.



Proof: Omitted (As Exercise)

Example 1:

Find the area of the region bounded by the lines $y = x$ and $x = 2$ and the curve $y = \frac{1}{x^2}$.



Solutions

Put $x = \frac{1}{x^2}$, we have $x^3 = 1$, the real solution is $x = 1$. So, the point of intersection of $y = \frac{1}{x^2}$ and $y = x$ is $(1,1)$.

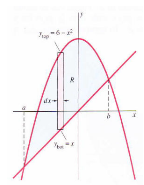
Required Area is $\int_1^2 \left(x - \frac{1}{x^2}\right) dx$

$$= \frac{1}{2}x^2 + \frac{1}{x}\bigg|_{x=1}^{2}$$

$$= \left(2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{2} + 1\right) = 1.$$

Example 2:

Find the area A of the region R bounded by the line $y = x$ and the parabola $y = 6 - x^2$.



Solutions

Put $x = 6 - x^2$, we have $x^2 + x - 6 = 0$, $(x + 3)(x - 2) = 0$, $x = -3$ or 2.
The points of intersection of $y = x$ and $y = 6 - x^2$ are $(-3, -3)$ and $(2,2)$.

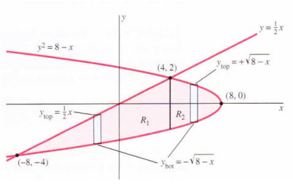
$$A = \int_{-3}^{2} (6 - x^2 - x)\, dx$$

$$= 6x - \frac{1}{3}x^3 - \frac{1}{2}x^2\bigg|_{x=-3}^{2}$$

$$= \left(12 - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right) = \frac{125}{6}$$

Example 3:

Find the area A of the region R bounded by the line $y = \frac{1}{2}x$ and the parabola $y^2 = 8 - x$.



Solutions

Put $y = \frac{1}{2}x$ into $y^2 = 8 - x$, we have $\frac{1}{4}x^2 = 8 - x$, $x^2 + 4x - 32 = 0$, $(x+8)(x-4) = 0$, so $x = -8$ or 4.

The points of intersection of $y = \frac{1}{2}x$ and $y^2 = 8 - x$ are $(-8, -4)$ and $(4,2)$.

Note: the upper portion and the lower portion of the parabola $y^2 = 8 - x$ are given by $y = \sqrt{8-x}$ and $y = -\sqrt{8-x}$ respectively.

The area of region R_1 is $\int_{-8}^{4} \left(\frac{1}{2}x + \sqrt{8-x}\right) dx$

$$= \frac{1}{4}x^2 - \frac{2}{3}(8-x)^{\frac{3}{2}}\Big|_{x=-8}^{4}$$

$$= \left(4 - \frac{16}{3}\right) - \left(16 - \frac{128}{3}\right) = \frac{76}{3}$$

The area of region R_2 is $\int_{4}^{8} 2 \cdot \sqrt{8-x}\, dx$

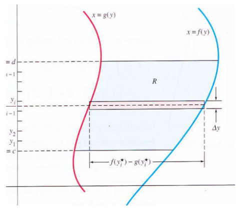
$$= \frac{-4}{3}(8-x)^{\frac{3}{2}}\Big|_{x=4}^{8}$$

$$= 0 - \frac{-32}{3} = \frac{32}{3}$$

Thus, $A = \frac{76}{3} + \frac{32}{3} = \frac{108}{3} = 36$.

Areas of Plane Regions (Integrate w.r.t. y)

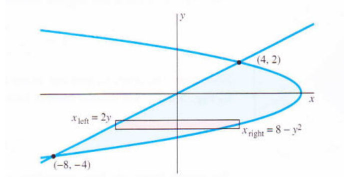
Let $c, d \in R$ with $c < d$.
Let f and g be real-valued functions on y and is defined on $[c, d]$.
Suppose $f(y) \geq g(y)$ for any $y \in [c, d]$.
Then, the area of the region bounded by $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$ is given by
$\int_c^d \big(f(y) - g(y)\big)\, dy$.



Proof: Omitted (As Exercise)

Example 3:

Find the area A of the region R bounded by the line $y = \frac{1}{2}x$ and the parabola $y^2 = 8 - x$.



Solutions

Put $y = \frac{1}{2}x$ into $y^2 = 8 - x$, we have $\frac{1}{4}x^2 = 8 - x$, $x^2 + 4x - 32 = 0$,
$(x + 8)(x - 4) = 0$, so $x = -8$ or 4.
The points of intersection of $y = \frac{1}{2}x$ and $y^2 = 8 - x$ are $(-8, -4)$ and $(4,2)$.
Note: $y = \frac{1}{2}x \Leftrightarrow x = 2y$ and $y^2 = 8 - x \Leftrightarrow x = 8 - y^2$.

$$A = \int_{-4}^{2} (8 - y^2 - 2y)\, dy$$

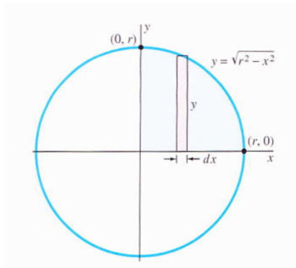
$$= 8y - \frac{1}{3}y^3 - y^2 \Big|_{y=-4}^{2}$$

$$= \left(16 - \frac{8}{3} - 4\right) - \left(-32 + \frac{64}{3} - 16\right) = 36.$$

Example 4:

Show that the area of a circle of radius r is $A = \pi r^2$.

Proof:



Consider $y = f(x) = \sqrt{r^2 - x^2}$ for $0 \le x \le r$ and let A be the area of whole circle, then $\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2}\, dx$.

$$\int_0^r \sqrt{r^2 - x^2}\, dx$$

$$= \int_{\frac{\pi}{2}}^{0} rsin\theta \cdot (-rsin\theta)\, d\theta$$

(let $x = rcos\theta$, $dx = -rsin\theta d\theta$,
$r^2 - x^2 = r^2(1 - cos^2\theta) = r^2 sin^2\theta$,
$\sqrt{r^2 - x^2} = rsin\theta$ (Note: $cos\theta \ge 0$ and $sin\theta \ge 0$).
When $x = 0$, $\theta = \frac{\pi}{2}$.
When $x = r$, $\theta = 0$.)

$$= -r^2 \int_{\frac{\pi}{2}}^{0} sin^2\theta\, d\theta = r^2 \int_0^{\frac{\pi}{2}} sin^2\theta\, d\theta$$

$$= r^2 \int_0^{\frac{\pi}{2}} \frac{1 - cos2\theta}{2}\, d\theta$$

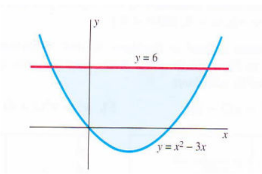
$$= \frac{r^2}{2}\left(\theta - \frac{1}{2}sin2\theta\Big|_{\theta=0}^{\frac{\pi}{2}}\right)$$

$$= \frac{r^2}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \frac{1}{4}\pi r^2$$

Thus, $\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2}\, dx = \frac{1}{4}\pi r^2$.
So, $A = \pi r^2$.

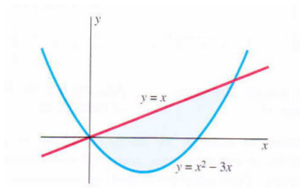
Exercise 1:

Find the area A of the region R bounded by the line $y = 6$ and the parabola
$y = x^2 - 3x$.



Exercise 2:

Find the area A of the region R bounded by the line $y = x$ and the parabola
$y = x^2 - 3x$.



 # Numerical Integration
Let $a, b \in R$ with $a < b$.
Let f be a real-valued function on $\boldsymbol{x}$.
Suppose f is Riemann Integrable on $[a, b]$.
Let $\{x_0, x_1, x_2, \cdots, x_n\}$ be a partition P_n of $[a, b]$, that is, $x_0 = a$, $x_n = b$, $x_0 < x_1 < x_2 < \cdots < x_n$.
$\|P_n\| = max\{x_j - x_{j-1}: j = 1,2,3, \cdots, n\}$
We choose $c_j \in [x_{j-1}, x_j]$ for $j = 1,2, \cdots, n$ and consider $A(f, P_n) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$.
We know when $\|\boldsymbol{P_n}\| \approx \mathbf{0}$, $\sum_{j=1}^{n} \boldsymbol{f(c_j)(x_j - x_{j-1})} \approx \int_a^b \boldsymbol{f(x)dx}$.

#1 Left End Approximation

$$\boldsymbol{L_n = \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})}$$

#2 Right End Approximation

$$\boldsymbol{R_n = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})}$$

#3 Trapezoidal Approximation

$$\boldsymbol{T_n = \frac{1}{2}(L_n + R_n)}$$

#4 Mid-Point Approximation

$$\boldsymbol{M_n = \sum_{j=1}^{n} f\left(\frac{x_{j-1} + x_j}{2}\right)(x_j - x_{j-1})}$$

#5 Simpson's Approximation

$$\boldsymbol{S_{2n} = \frac{2M_n + T_n}{3}}$$

Usual Case: Evenly spaced sub-intervals for the partition $x_j = a + j \times \frac{b-a}{n}$ for $j = 0,1,2, \cdots, n$.

Example 1

Find $\boldsymbol{L_n}$, $\boldsymbol{R_n}$, $\boldsymbol{T_n}$, $\boldsymbol{M_n}$ and $\boldsymbol{S_{2n}}$ for $\int_0^3 \boldsymbol{x^3 dx}$ where $x_j = a + j \times \frac{b-a}{n}$, $a = 0, b = 3, n = 5$ or 10.

For $n = 5$:

x	0.0	0.6	1.2	1.8	2.4	3.0
$f(x)$	0.000	0.216	1.728	5.832	13.824	27.000

$\boldsymbol{L_5} = \frac{3}{5}(0.000 + 0.216 + 1.728 + 5.832 + 13.824) = \frac{3}{5} \times 21.6 = 12.96$

$\boldsymbol{R_5} = \frac{3}{5}(0.216 + 1.728 + 5.832 + 13.824 + 27.000) = \frac{3}{5} \times 48.6 = 29.16$

$\boldsymbol{T_5} = 0.5 \times (12.96 + 29.16) = 21.06$

x	0.3	0.9	1.5	2.1	2.7
$f(x)$	0.027	0.729	3.375	9.261	19.683

$\boldsymbol{M_5} = \frac{3}{5}(0.027 + 0.729 + 3.375 + 9.261 + 19.683) = \frac{3}{5} \times 33.075 \approx 19.85$

$\boldsymbol{S_{10} \approx \frac{1}{3}(2 \times 19.85 + 21.06) \approx 20.25}$

For $n = 10$:

x	0.0	0.3	0.6	0.9	1.2	1.5
$f(x)$	0.000	0.027	0.216	0.729	1.728	3.375

x	1.8	2.1	2.4	2.7	3.0
$f(x)$	5.832	9.261	13.824	19.683	27.000

$\boldsymbol{L_{10}} = \frac{3}{10}(0.000 + 0.027 + 0.216 + 0.729 + 1.728 + 3.375 + 5.832 + 9.261 + 13.824 + 19.683)$

$= \frac{3}{10} \times 54.675 \approx 16.40$

$\boldsymbol{R_{10}} = \frac{3}{10}(0.027 + 0.216 + 0.729 + 1.728 + 3.375 + 5.832 + 9.261 + 13.824 + 19.683 + 27.000)$

$= \frac{3}{10} \times 81.675 \approx 24.50$

$\boldsymbol{T_{10}} \approx 0.5 \times (16.403 + 24.503) = 20.45$

x	0.15	0.45	0.75	1.05	1.35	1.65
$f(x)$	0.003375	0.091125	0.421875	1.157625	2.460375	4.492125

x	1.95	2.25	2.55	2.85
$f(x)$	7.414875	11.39063	16.58138	23.14913

$0.003375 + 0.091125 + 0.421875 + 1.157625 + 2.460375 = 4.134375$

$4.492125 + 7.414875 + 11.39063 + 16.58138 + 23.14913 = 63.02813$

$\boldsymbol{M_{10}} = \frac{3}{10}(4.134375 + 63.02813) = \frac{3}{10} \times 67.1625 \approx 20.15$

$\boldsymbol{S_{20} \approx \frac{1}{3}(2 \times 20.15 + 20.45) = 20.25}$

$$\int_0^3 \boldsymbol{x^3 dx} = \boldsymbol{\frac{1}{4}x^4}\Big|_{x=0}^{3} = \boldsymbol{\frac{81}{4} = 20.25}$$

Observations:

The case when $\boldsymbol{n = 10}$ will give a better approximation than the case when $\boldsymbol{n = 5}$.
In this example, $\boldsymbol{T_n}$, $\boldsymbol{M_n}$ **or** $\boldsymbol{S_{2n}}$ will give a better approximation than $\boldsymbol{L_n}$ or $\boldsymbol{R_n}$, $\boldsymbol{S_{2n}}$ is the best approximation.

Example 2

Find $\boldsymbol{L_n}$, $\boldsymbol{R_n}$, $\boldsymbol{T_n}$, $\boldsymbol{M_n}$ and $\boldsymbol{S_{2n}}$ for $\int_0^3 \boldsymbol{x^2 dx}$ where $x_j = a + j \times \frac{b-a}{n}$, $a = 0$, $b = 3$, $n = 3$ or 6.

x	0.0	1.0	2.0	3.0
$f(x)$	0.00	1.00	4.00	9.00

$\boldsymbol{L_3} = 1 \times (0.00 + 1.00 + 4.00) = 5.00$
$\boldsymbol{R_3} = 1 \times (1.00 + 4.00 + 9.00) = 14.00$
$\boldsymbol{T_3} = 0.5 \times (\boldsymbol{L_3} + \boldsymbol{R_3}) = 0.5 \times (5.00 + 14.00) = 9.50$

x	0.5	1.5	2.5
$f(x)$	0.25	2.25	6.25

$\boldsymbol{M_3} = 1 \times (0.25 + 2.25 + 6.25) = 8.75$

$\boldsymbol{S_6} = \frac{1}{3}(2 \times 8.75 + 9.50) = 9.00$

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	0.00	0.25	1.00	2.25	4.00	6.25	9.00

$\boldsymbol{L_6} = 0.5 \times (0.00 + 0.25 + 1.00 + 2.25 + 4.00 + 6.25) = 6.8750$
$\boldsymbol{R_6} = 0.5 \times (0.25 + 1.00 + 2.25 + 4.00 + 6.25 + 9.00) = 11.3750$
$\boldsymbol{T_6} = 0.5 \times (\boldsymbol{L_6} + \boldsymbol{R_6}) = 0.5 \times (6.8750 + 11.3750) = 9.1250$

x	0.25	0.75	1.25	1.75	2.25	2.75
$f(x)$	0.0625	0.5625	1.5625	3.0625	5.0625	7.5625

$\boldsymbol{M_6} = 0.5 \times (0.0625 + 0.5625 + 1.5625 + 3.0625 + 5.0625 + 7.5625) = 8.9375$

$\boldsymbol{S_{12}} = \frac{1}{3}(2 \times 8.9375 + 9.1250) = 9.0000$

$$\int_0^3 \boldsymbol{x^2 dx} = \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{x^3}\Big|_{x=0}^{3} = \mathbf{9}$$

Observations:

The case when $\boldsymbol{n = 6}$ will give a better approximation than the case when $\boldsymbol{n = 3}$.
In this example, $\boldsymbol{T_n}$, $\boldsymbol{M_n}$ **or** $\boldsymbol{S_{2n}}$ will give a better approximation than $\boldsymbol{L_n}$ or $\boldsymbol{R_n}$, $\boldsymbol{S_{2n}}$ is the best approximation.

 # Parabolic Approximations & Simpson's Approximations

Omitted (Left as Reading Assignment)

 # Error Estimates for $\boldsymbol{T_n}$, $\boldsymbol{M_n}$ and $\boldsymbol{S_{2n}}$

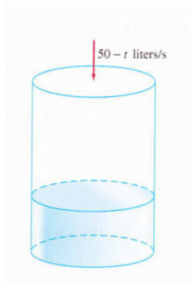
Omitted (Left as Reading Assignment)

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 6: Applications of Integration

Riemann Sums Approximation

Example 1:

Suppose that water is pumped into the initially empty tank.
The rate of water into the tank at time t (in seconds) is $50 - t$ liters (L) per second.
How much water flows into the tank during the first $30\ s.$?



Solutions

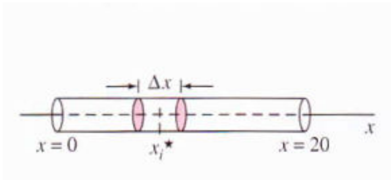
$$\frac{dV}{dt} = 50 - t$$

Required volume is

$$\int_0^{30} \frac{dV}{dt} dt = \int_0^{30} (50 - t)\, dt = 50t - \frac{1}{2}t^2 \Big|_{t=0}^{30} = 1500 - 450 = 1050 \text{ (liters).}$$

Example 2:

For a thin rod $20\ cm$ long, its (linear) density at the point x is $15 + 2x$ grams of mass per centimeter of the rod's length (g/cm).
The rod's density thus varies from 15 g/cm at the left end $x = 0$ to 55 g/cm at the right end $x = 20$.
Find the total mass of the rod.



Solutions

The density is $\delta(x) = 15 + 2x$ (in g/cm).
The mass can be found as $dM = (15 + 2x)dx$ (in g).

$$M(x) = \int_0^{20} (15 + 2x)dx = 15x + x^2|_{x=0}^{20} = 300 + 400 = 700 \text{ (grams).}$$

Example 3:

Calculate Q if $Q = \lim_{n\to+\infty} \sum_{i=1}^{n} 2x_i \cdot e^{-x_i^2} \cdot \Delta x$, where $x_0, x_1, x_2, \cdots, x_n$ are the endpoints of a partition of the interval $[1,2]$ into n subintervals, all with the same length $\Delta x = \frac{1}{n}$.

Solutions

$$Q = \int_1^2 2x \cdot e^{-x^2} dx = -e^{-x^2}\Big|_{x=1}^{2} = (-e^{-4}) - (-e^{-1}) = \frac{1}{e} - \frac{1}{e^4} \approx 0.3496$$

Note: $Q = \lim_{n\to+\infty} \sum_{i=1}^{n} 2\left(1 + \frac{i}{n}\right) \cdot e^{-x_i^2} \cdot \frac{1}{n}$

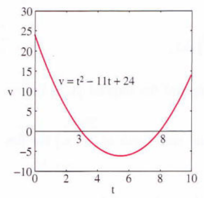
Displacement and Distance Travelled

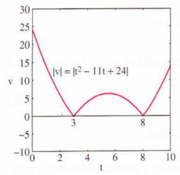
Displacement function $s(t) = \int_0^t v(\tau)d\tau$.

Distance travelled $\boldsymbol{S} = \int_0^t |v(\tau)|d\tau$.

Example 4:

Suppose that the velocity of a moving particle is $v(t) = t^2 - 11t + 24$ (ft./s.).
Find both the net distance (displacement) s and the total distance $\boldsymbol{S}$ it travels between time $t = 0$ and $t = 10$.





Solutions

The net distance (displacement) is $s(10) = \int_0^{10} (\tau^2 - 11\tau + 24)d\tau$

$= \frac{1}{3}\tau^3 - \frac{11}{2}\tau^2 + 24\tau\Big|_{\tau=0}^{10} = \frac{1000}{3} - 550 + 2400 = \frac{70}{3}$ (in ft.).

The total distance is $S = \int_0^{10} |\tau^2 - 11\tau + 24| d\tau$

$$= \int_0^3 (\tau^2 - 11\tau + 24)d\tau + \int_3^8 -(\tau^2 - 11\tau + 24)d\tau + \int_8^{10} (\tau^2 - 11\tau + 24)d\tau$$

$$= \left(\frac{1}{3}\tau^3 - \frac{11}{2}\tau^2 + 24\tau\Big|_{\tau=0}^{3}\right) - \left(\frac{1}{3}\tau^3 - \frac{11}{2}\tau^2 + 24\tau\Big|_{\tau=3}^{8}\right) + \left(\frac{1}{3}\tau^3 - \frac{11}{2}\tau^2 + 24\tau\Big|_{\tau=8}^{10}\right)$$

$= 65$ (in ft.).

Note: $\tau^2 - 11\tau + 24 = 0 \Leftrightarrow (\tau - 3)(\tau - 8) = 0 \Leftrightarrow \tau = 3 \; or \; 8$,
$\tau^2 - 11\tau + 24 < 0 \Leftrightarrow 3 < \tau < 8$ and $\tau^2 - 11\tau + 24 > 0 \Leftrightarrow \tau < 3 \; or \; \tau > 8$

Fluid Flow in Circular Pipes

(Reading Assignment)

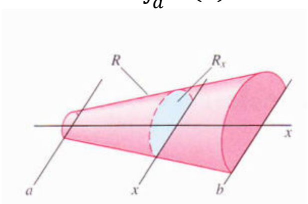
Flow Rates and Cardiac Output

(Reading Assignment)

Volumes by the Method of Cross Sections

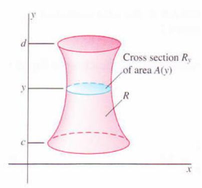
Cavalier's Principle (Perpendicular to $x -$ axis)

If the solid R lies alongside the interval $[a, b]$ on the $x -$ axis and has continuous cross-sectional area function $A(x)$, then its volume $V = \int_a^b A(x)dx$.



Cavalier's Principle (Perpendicular to $y -$ axis)

If the solid R lies alongside the interval $[c, d]$ on the $y -$ axis and has continuous cross-sectional area function $A(y)$, then its volume $V = \int_c^d A(y)dy$.

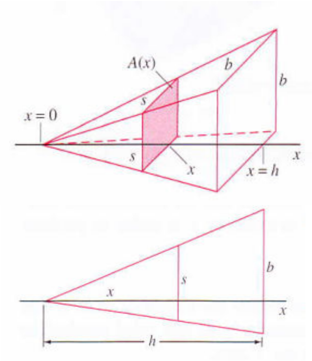


Example:

The figure shows a square-based pyramid oriented so that its height h corresponds to the interval $[0, h]$on the $x-$ axis. Its base is a $b \times b$ square, and each cross-section perpendicular to the $x-$ axis is also a square.

Let $A(x)$ be the area of the $s \times s$ cross section at x.

Show that the volume of the pyramid is given by $V = \frac{1}{3}b^2h$.



Solutions

Using similar triangles, $\frac{s}{x} = \frac{b}{h}$, so $s = \frac{b}{h}x$.

Therefore, $A(x) = s^2 = \frac{b^2}{h^2}x^2$

$V = \int_0^h A(x)dx = \int_0^h \frac{b^2}{h^2}x^2 dx = \frac{b^2}{h^2} \cdot \frac{x^3}{3}\Big|_{x=0}^{h} = \frac{b^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}b^2h.$

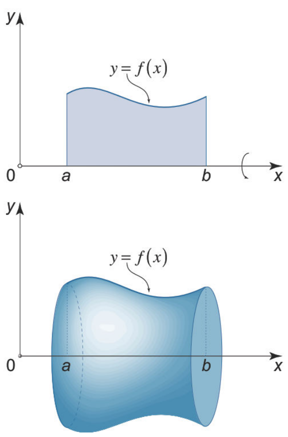
Note: $V = \frac{1}{3}Ah.$

 # Solids of Revolution (Disc/Washer Method)

Let f be a real-valued function on x.

Suppose $f(x) \geq 0$ for any $x \in [a, b]$.

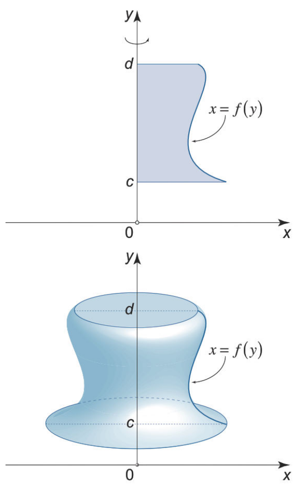
For the solid obtained by revolving around the $x-$ axis the region under the graph of $y = f(x)$ over the interval $[a, b]$, its volume is given by:
$V = \int_a^b \pi y^2 dx = \int_a^b \pi[f(x)]^2 dx.$



Let f be a real-valued function on y.

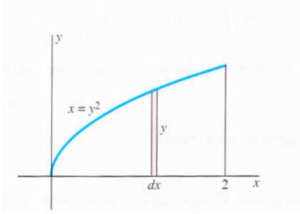
Suppose $f(y) \geq 0$ for any $y \in [c, d]$.

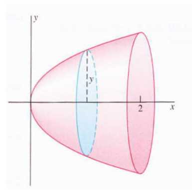
For the solid obtained by revolving around the $y-$ axis the region under the graph of $x = f(y)$ over the interval $[c, d]$, its volume is given by:
$V = \int_c^d \pi x^2 dy = \int_c^d \pi[f(y)]^2 dy.$



Example 1:

The figure shows the region that lies below the parabola $y^2 = x$ and above the $x-$ axis over the interval $[0,2]$. Find the volume V of the solid paraboloid obtained by revolving the region around the $x-$ axis.



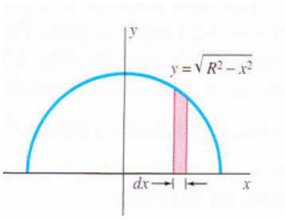


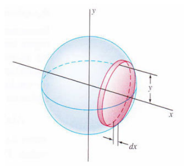
Solutions

$V = \pi\int_0^2 y^2 dx = \pi\int_0^2 x dx = \frac{1}{2}\pi x^2\Big|_{x=0}^{2} = 2\pi.$

Example 2:

Show that the volume of the solid sphere with radius $\boldsymbol{R}$ is $\frac{4}{3}\pi\boldsymbol{R}^3$.





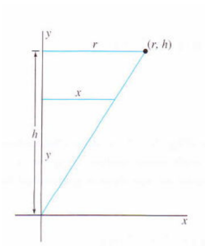
Solutions

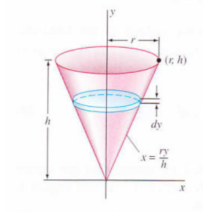
$$V = \pi\int_{-\boldsymbol{R}}^{\boldsymbol{R}} y^2 dx = \pi\int_{-\boldsymbol{R}}^{\boldsymbol{R}} (R^2 - x^2)dx = \pi\left(\boldsymbol{R}^2 x - \frac{1}{3}x^3\right)\Big|_{x=-\boldsymbol{R}}^{\boldsymbol{R}}$$

$= 2\pi\left(\boldsymbol{R}^3 - \frac{1}{3}\boldsymbol{R}^3\right) = \frac{4}{3}\pi\boldsymbol{R}^3.$

Example 3:

Show that the volume V of the solid right circular cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.





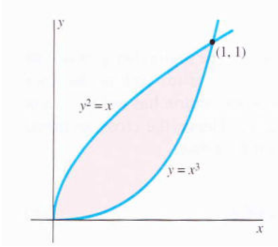
Solutions

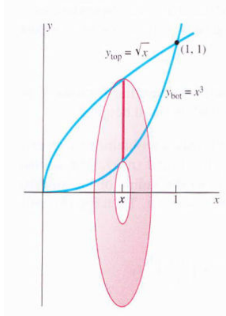
As $\frac{x}{y} = \frac{r}{h}$, so $x = \frac{r}{h}y$.

$V = \pi\int_0^h x^2 dy = \pi\int_0^h \frac{r^2}{h^2}y^2 dy = \frac{\pi r^2}{h^2}\cdot\frac{y^3}{3}\Big|_{y=0}^{h} = \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$

Example 4A:

Consider the plane region bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points $(0,0)$ and $(1,1)$, find the volume V of the solid of revolution when the region is revolved around the $x-$ axis.





Solutions

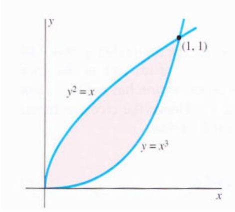
$y_{top} = \sqrt{x}, y_{bottom} = x^3$

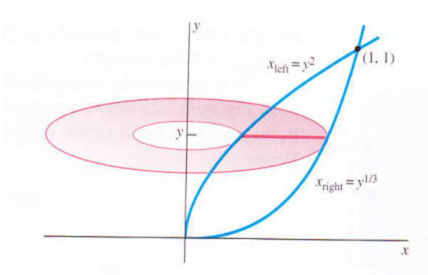
$$V = \pi \int_0^1 {y_{top}}^2 \, dx - \pi \int_0^1 {y_{bottom}}^2 \, dx = \pi \int_0^1 (x - x^6) \, dx$$

$$= \pi \left(\frac{1}{2} x^2 - \frac{1}{7} x^7 \right) \Big|_{x=0}^{1} = \pi \left(\frac{1}{2} - \frac{1}{7} \right) = \frac{5\pi}{14}.$$

Example 4B:

Consider the plane region bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points $(0,0)$ and $(1,1)$, find the volume V of the solid of revolution when the region is revolved around the $y-$ axis.





Solutions

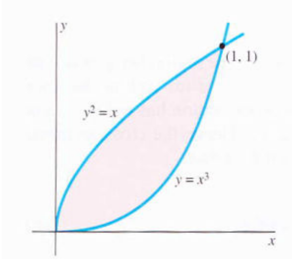
$x_{left} = y^2, x_{right} = y^{1/3}$

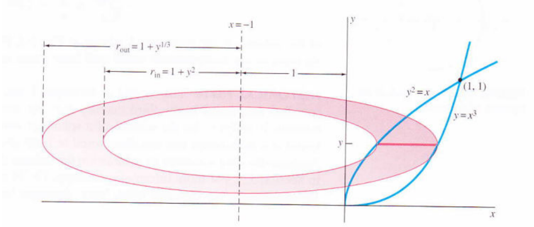
$$V = \pi \int_0^1 {x_{right}}^2 \, dy - \pi \int_0^1 {x_{left}}^2 \, dy = \pi \int_0^1 \left(y^{2/3} - y^4 \right) dy$$

$$= \pi \left(\frac{3}{5} y^{5/3} - \frac{1}{5} y^5 \right) \Big|_{y=0}^{1} = \pi \left(\frac{3}{5} - \frac{1}{5} \right) = \frac{2\pi}{5}.$$

Example 4C:

Consider the plane region bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points $(0,0)$ and $(1,1)$, find the volume V of the solid of revolution when the region is revolved around the vertical line $x = -1$.





Solutions

$r_{out} = 1 + x_{right} = 1 + y^{1/3}$, $r_{in} = 1 + x_{left} = 1 + y^2$,

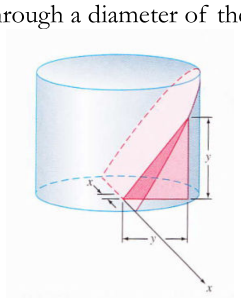
$$V = \pi \int_0^1 {r_{out}}^2\, dy - \pi \int_0^1 {r_{in}}^2\, dy = \pi \int_0^1 \left[\left(1 + y^{1/3}\right)^2 - (1 + y^2)^2\right] dy$$

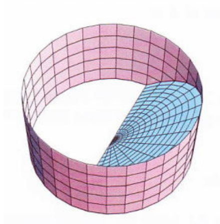
$$= \pi \int_0^1 \left[2y^{1/3} + y^{2/3} - 2y^2 - y^4\right] dy$$

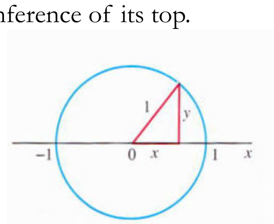
$$= \pi \left(\frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} - \frac{2}{3}y^3 - \frac{1}{5}y^5\right)\Big|_{y=0}^{1} = \pi\left(\frac{3}{2} + \frac{3}{5} - \frac{2}{3} - \frac{1}{5}\right) = \frac{37\pi}{30}.$$

Example 5:

Find the volume V_{wedge} of the wedge that is cut from a circular cylinder with unit radius and unit height by a plane that passes through a diameter of the base of the cylinder and through a point on the circumference of its top.







Solutions

Note 1: the cross-sectional area function $A(x) = \frac{1}{2}y^2$ and $y = \sqrt{1 - x^2}$.

Thus, $A(x) = \frac{1}{2}(1 - x^2)$.

The volume of the wedge is $V_{wedge} = \int_{-1}^{1} A(x)dx = 2\int_0^1 A(x)dx$

(By Symmetry) $= 2\int_0^1 \frac{1}{2}(1 - x^2)dx = x - \frac{1}{3}x^3\Big|_{x=0}^{1} = 1 - \frac{1}{3} = \frac{2}{3}$.

Note 2: The volume of the cylinder is $V_{cylinder} = \pi \times 1^2 \times 1 = \pi$.

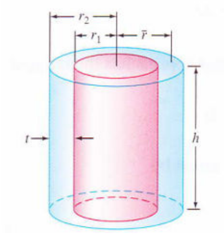
Note 3: $\frac{V_{wedge}}{V_{cylinder}} = \frac{\frac{2}{3}}{\pi} \approx 21\%$

Remark: Archimedes used a method of exhaustion for volume similar to that discussed for area to find out the volume of the wedge.

 # Volumes by the Method of Cylindrical Shells

 # Volume of a Cylindrical Shell

A cylinrical shell is a region bounded by two concentric circular cylinders of the same height h.
Suppose the inner cylinder has radius r_1 and the outer one has radius r_2.
Let $\bar{r} = \frac{r_1+r_2}{2}$ and $t = r_2 - r_1$.
Find the volume V_{shell} of the cylindical shell.

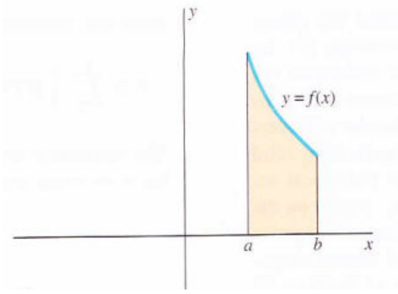


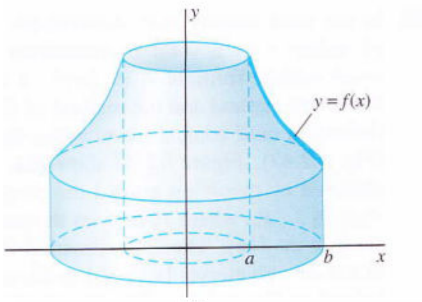
Solutions

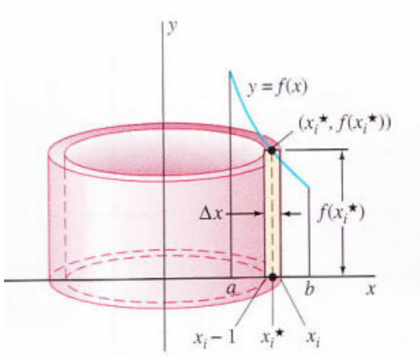
$$V_{shell} = \pi {r_2}^2 h - \pi {r_1}^2 h = 2\pi \cdot \frac{r_1 + r_2}{2} \cdot (r_2 - r_1)h = 2\pi \cdot \bar{r} \cdot t \cdot h$$
$$= \boldsymbol{2\pi} \times \boldsymbol{average\ radius} \times \boldsymbol{thickness} \times \boldsymbol{height}$$

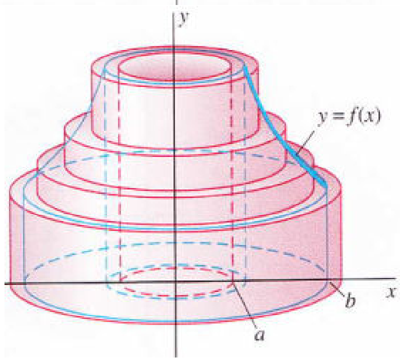
 # A solid of revolution (Shell Method)

Let $a, b \in R$ with $a < b$ and f be a real-valued function on x.
Suppose $f(x) \geq 0$ for any $x \in [a, b]$.
The region bounded by $y = f(x)$, $x = a$, $x = b$ and $y = 0$ is revolved along the $y-$ axis, find its volume V.

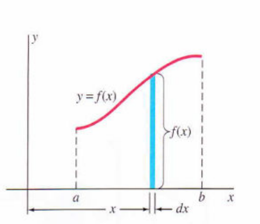


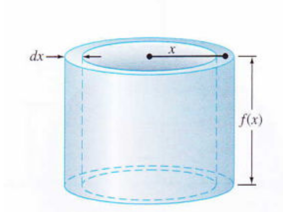


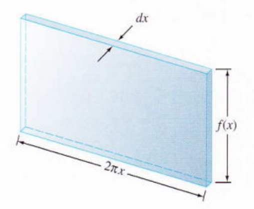




Solutions (Cylindrical Shell Method)
Idea:

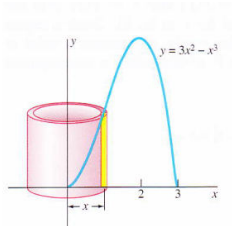






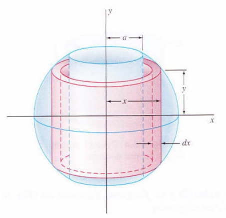
$dV = 2\pi \cdot x \cdot f(x) \cdot dx$
$V = \int_a^b 2\pi x f(x)\, dx.$

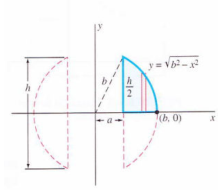
Example 1:
Find the volume V of the solid generated by revolving around the $y-$ axis the region under $y = 3x^2 - x^3$ from $x = 0$ to $x = 3$.



Solutions
$V = \int_0^3 2\pi x(3x^2 - x^3)\, dx = 2\pi \left(\frac{3}{4}x^4 - \frac{1}{5}x^5\right)\Big|_{x=0}^{3} = 2\pi\left(\frac{243}{4} - \frac{243}{5}\right) = \frac{243\pi}{10}.$

Example 2:
Find the volume V of the solid that remains after you bore a circular hole of radius a through the center of a solid sphere of radius $b > a$.





Solutions
By symmetry, $\frac{V}{2} = 2\pi \int_a^b x \cdot \sqrt{b^2 - x^2}\, dx.$
So, $V = 4\pi \int_a^b x \cdot \sqrt{b^2 - x^2}\, dx$

$$= -2\pi \int_a^b \sqrt{b^2 - x^2} \cdot (-2x)dx$$

(let $u = b^2 - x^2$, $du = -2xdx$.
When $x = a$, $u = b^2 - a^2$. When $x = b$, $u = 0$.)

$$= -2\pi \int_{b^2-a^2}^{0} \sqrt{u}\, du = 2\pi \int_0^{b^2-a^2} \sqrt{u}\, du$$

$$= \frac{4\pi}{3}\left(u^{\frac{3}{2}}\Big|_{u=0}^{b^2-a^2}\right) = \frac{4\pi}{3}(b^2 - a^2)^{3/2}$$

Remark (Special Case):
When $a \to 0^+$, $V \to \frac{4\pi}{3}b^3$.
When $a \to b^-$, $V \to 0$.

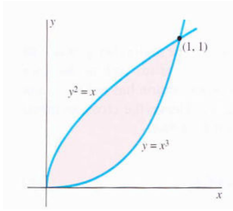
Examples 4A and 4B (re-visisted)

Consider the plane region bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points $(0,0)$ and $(1,1)$, find the volume V of the solid of revolution when the region is revolved:

(i) along the $x-$ axis

(ii) along the $y-$ axis

by Shell Method.



Solutions (i)

$$V = 2\pi \int_0^1 (x_{right} - x_{left}) \cdot y\, dy$$

$$= 2\pi \int_0^1 (y^{1/3} - y^2) \cdot y\, dy$$

$$= 2\pi \left(\frac{3}{7} x^{\frac{7}{3}} - \frac{1}{4} y^4 \Big|_{y=0}^{1} \right)$$

$$= 2\pi \left(\frac{3}{7} - \frac{1}{4} \right) = \frac{2\pi}{28}(12 - 7) = \frac{5\pi}{14}$$

Solutions (ii)

$$V = 2\pi \int_0^1 (y_{upper} - y_{lower}) \cdot x\, dx$$

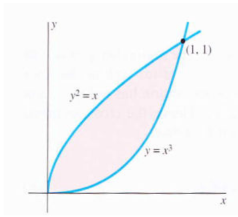
$$= 2\pi \int_0^1 \left(x^{\frac{1}{2}} - x^3 \right) \cdot x\, dx$$

$$= 2\pi \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{1}{5} x^5 \Big|_{x=0}^{1} \right)$$

$$= 2\pi \left(\frac{2}{5} - \frac{1}{5} \right) = \frac{2\pi}{5}$$

Example 4C (re-visited)

Consider the plane region bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points $(0,0)$ and $(1,1)$, find the volume V of the solid of revolution when the region is revolved around the vertical line $x = -1$ by Shell Method.



Solutions

$$V = 2\pi \int_0^1 (y_{upper} - y_{lower}) \cdot (1 + x)\, dx$$

$$= 2\pi \int_0^1 (x^{1/2} - x^3) \cdot (1 + x)\, dx$$

$$= 2\pi \int_0^1 \left(x^{\frac{1}{2}} - x^3 + x^{\frac{3}{2}} - x^4 \right) dx$$

$$= 2\pi \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4} x^4 + \frac{2}{5} x^{\frac{5}{2}} - \frac{1}{5} x^5 \Big|_{x=0}^{1} \right)$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} \right)$$

$$= \frac{37\pi}{30}$$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 6: Applications of Integration

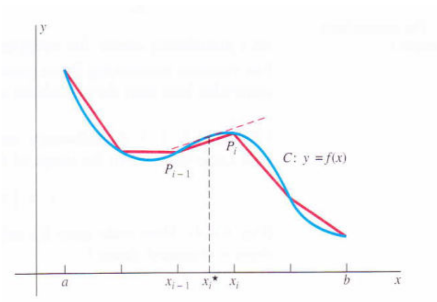
Arc Length and Surface Area of Revolution

Definition

Let $a, b \in R$ with $a < b$ and f be a real-valued function on x. Suppose f' is continuous on $[a, b]$.
Then, f is called a **smooth function** on $[a, b]$. In this case, the graph of f is called a **smooth arc**. Sometimes, we denote it as C.

Definition

Suppose the graph of a smooth function f on $[a, b]$ is C. Let $y = f(x)$. We define the **arc length** of C is
$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.



Idea behind the definition:

Let P_n be the partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, that is $x_0 < x_1 < \cdots < x_n$, $x_0 = a$, $x_n = b$,
$\|P_n\| = max\{x_j - x_{j-1} \colon j = 1,2, \cdots, n\}$.
Consider the points $Q_j\left(x_j, f(x_j)\right)$ on C for $j = 0,1,2, \cdots, n$, the length of the line segment $Q_{j-1}Q_j$ is denoted as $|Q_{j-1}Q_j|$. Then,
$|Q_{j-1}Q_j| = \sqrt{\left(x_j - x_{j-1}\right)^2 + \left[f(x_j) - f(x_{j-1})\right]^2}$.
We can observe that $s = \lim_{\|P_n\| \to 0} \sum_{j=1}^{n} |Q_{j-1}Q_j|$.
By Mean Value Theorem, we can find $c_j \in [x_{j-1}, x_j]$ such that
$f(x_j) - f(x_{j-1}) = f'(c_j)(x_j - x_{j-1})$.

Thus, $|Q_{j-1}Q_j| = \sqrt{\left(x_j - x_{j-1}\right)^2 + \left[f(x_j) - f(x_{j-1})\right]^2}$
$= \sqrt{\left(x_j - x_{j-1}\right)^2 + \left(f'(c_j)\right)^2 \left(x_j - x_{j-1}\right)^2}$
$= \sqrt{1 + \left(f'(c_j)\right)^2} \cdot \left(x_j - x_{j-1}\right)$

$s = \lim_{\|P_n\| \to 0} \sum_{j=1}^{n} |Q_{j-1}Q_j| = \lim_{\|P_n\| \to 0} \sum_{j=1}^{n} \sqrt{1 + \left(f'(c_j)\right)^2} \cdot \left(x_j - x_{j-1}\right)$
$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Definitions:

Let $a, b \in R$ with $a < b$ and f be a real-valued function on x. Suppose f' is continuous on $[a, b]$. Suppose the graph of f on $[a, b]$ is C.
Let $y = f(x)$.
The **arc length** of C is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

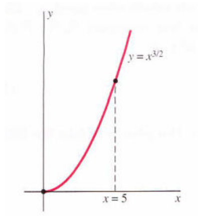
Let $c, d \in R$ with $c < d$ and f be a real-valued function on y. Suppose f' is continuous on $[c, d]$. Suppose the graph of f on $[c, d]$ is C.
Let $x = f(y)$.
The **arc length** of C is

$$s = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy$$

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example 1:

Find the length of the so-called semicubical parabola (it is not really a parabola) $y = x^{3/2}$ on $[0,5]$.



Solutions

$y = x^{3/2}, \frac{dy}{dx} = \frac{3}{2}x^{1/2}, 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{3}{2}x^{1/2}\right)^2 = 1 + \frac{9}{4}x$

Required Length is

$$\int_0^5 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9}\int_1^{\frac{49}{4}} \sqrt{u}\, du$$

(let $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$. When $x = 0, u = 1$. When $x = 5, u = \frac{49}{4}$.)

$$= \frac{4}{9} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big]_{u=1}^{\frac{49}{4}}$$

$$= \frac{8}{27}\left(\frac{343}{8} - 1\right)$$

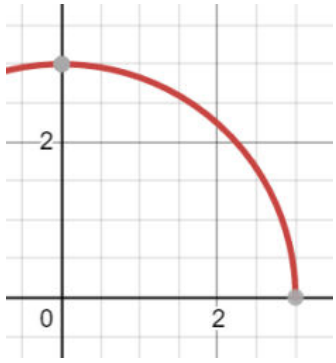
$$= \frac{335}{27}$$

Example 2:

Show that the circumference of a circle with radius r is $2\pi r$.

Proof

Let s be the circumference of a circle with radius r.



$y = \sqrt{r^2 - x^2}$ when $r = 3$

Let $y = \sqrt{r^2 - x^2}$ for $0 \le x \le r$.

Then, $\frac{1}{4}s = \int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2 = 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}}$$

Thus, $s = 4\int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}} dx$$

$$= 4\int_{\frac{\pi}{2}}^0 \frac{r}{rsin\theta} \cdot (-rsin\theta) d\theta = -4r\int_{\frac{\pi}{2}}^0 1 d\theta$$

(let $x = rcos\theta, dx = -rsin\theta d\theta$.

$r^2 - x^2 = r^2 - (rcos\theta)^2 = r^2(1 - cos^2\theta)$

$= r^2 sin^2\theta$.

Then, $\sqrt{r^2 - x^2} = rsin\theta$.

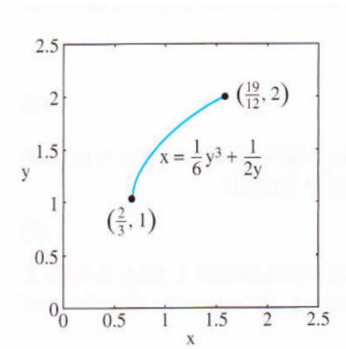
Note: $sin\theta \ge 0$ and $cos\theta \ge 0$.

When $x = 0, \theta = \frac{\pi}{2}$. When $x = r, \theta = 0$.)

$= 4r\int_0^{\frac{\pi}{2}} 1 d\theta = 4r \times \frac{\pi}{2} = 2\pi r$.

Example 3:

Find the length s of the curve $x = \frac{1}{6}y^3 + \frac{1}{2y}$, $1 \le y \le 2$.



Solutions

$$x = \frac{1}{6}y^3 + \frac{1}{2y} = \frac{1}{6}y^3 + \frac{1}{2}y^{-1}$$

$$\frac{dx}{dy} = \frac{1}{2}y^2 - \frac{1}{2}y^{-2} = \frac{1}{2}(y^2 - y^{-2})$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left[\frac{1}{2}(y^2 - y^{-2})\right]^2 = 1 + \frac{1}{4}(y^2 - y^{-2})^2$$

$$= 1 + \frac{1}{4}(y^4 - 2 + y^{-4}) = \frac{1}{4}(y^4 + 2 + y^{-4}) = \frac{1}{4}(y^2 + y^{-2})^2$$

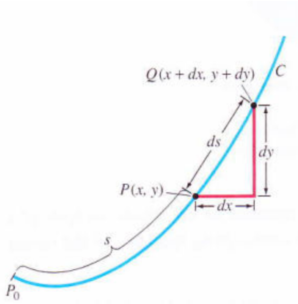
$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{1}{2}(y^2 + y^{-2})$$

$$s = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 \frac{1}{2}(y^2 + y^{-2})\, dy = \frac{1}{2}\left(\frac{y^3}{3} - \frac{1}{y}\right)\Bigg|_{y=1}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)\right] = \frac{17}{12}$$

Differential Forms and Arc Length

Consider two points $P(x,y)$ and $Q(x+dx, y+dy)$ on a smooth curve C,
$ds = \sqrt{(dx)^2 + (dy)^2}$.



Note 1:

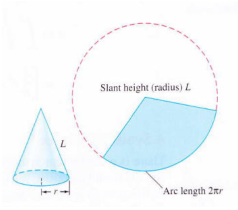
$$\sqrt{(dx)^2 + (dy)^2}$$
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Note 2:

$$s = \int_{s_0}^{s_1} ds = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{y_0}^{y_1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Surface Area of a Cone

Show that the surface area of the curved surface of the cone is $A = \pi r L$.
(where the radius of the base circle is r and slant height is L)

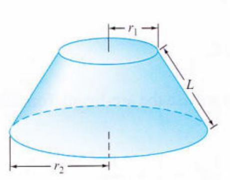


Proof

$$A = \pi L^2 \times \frac{2\pi r}{2\pi L} = \pi r L$$

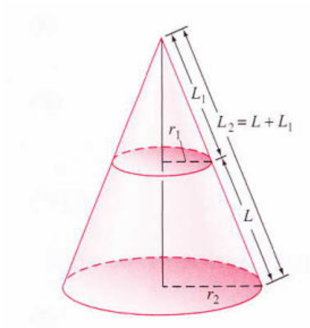
Surface Area of a Frustum

Show that the surface area of the curved surface of the frustum is $A = 2\pi \bar{r} L$.
(where the radii of the top circle and base circle are r_1 and r_2, slant height L, $\bar{r} = \frac{1}{2}(r_1 + r_2)$)



Proof

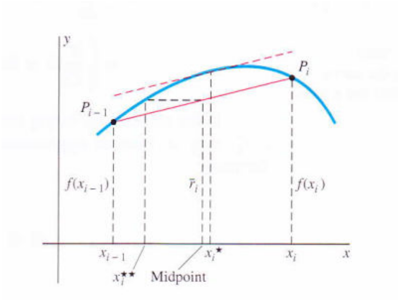
Let L_1 and L_2 be the slant heights of the cones with base radii r_1 and r_2 so that the larger cone minus the smaller cone will give the frustum.



Let $L_2 = L + L_1$.
By Similar Triangles, $\frac{L_2}{L_1} = \frac{r_2}{r_1}$.
So, $L_2 r_1 = L_1 r_2$
$(L + L_1) r_1 = L_1 r_2$
$L r_1 = L_1 (r_2 - r_1)$

$A = \pi r_2 L_2 - \pi r_1 L_1 = \pi r_2 (L + L_1) - \pi r_1 L_1$
$= \pi r_2 L + \pi L_1 (r_2 - r_1)$
$= \pi r_2 L + \pi L r_1$
$= \pi (r_2 + r_1) L$
$= 2\pi \bar{r} L$

Surface Area of Revolution



Consider two points $P_{i-1}(x_{i-1}, y_{i-1})$ and $P_i(x_i, y_i)$ on a smooth curve C, let the conical frustum obtained by revolving the line segment $P_{i-1}P_i$ around the $x-$ axis has slant height L_i.

$r_{i-1} = f(x_{i-1}), r_i = f(x_i)$,

$\overline{r_i} = \frac{1}{2}(r_{i-1} + r_i) = \frac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)$

As $\overline{r_i}$ is lying between $f(x_{i-1})$ and $f(x_i)$, by Intermediate Value Theorem, we can find

$d_i \in [x_{i-1}, x_i]$ such that $f(d_i) = \overline{r_i}$.

By Mean Value Theorem, we can find $c_i \in [x_{i-1}, x_i]$ such that

$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$.

$$L_i = \overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

$$= \sqrt{(x_i - x_{i-1})^2 + \big(f'(c_i)\big)^2 (x_i - x_{i-1})^2}$$

$$= \sqrt{1 + \big(f'(c_i)\big)^2}(x_i - x_{i-1})$$

Area of the curved surface of the frustum is

$$A_i = 2\pi\overline{r_i}L_i = 2\pi f(d_i)\sqrt{1 + \big(f'(c_i)\big)^2}(x_i - x_{i-1}).$$

Sum of the area of the curved surfaces of all frustums is $\sum_{i=1}^{n} A_i$

$$= \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + \big(f'(c_i)\big)^2}(x_i - x_{i-1})$$

We define Surface Area of Revolution

$$= \lim_{\|P_n\|\to 0} \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + \big(f'(c_i)\big)^2}(x_i - x_{i-1})$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

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Lecture Notes for Chapter 6: Applications of Integration

Definitions:

Let $a, b \in R$ with $a < b$ and f be a real-valued function on x. Suppose f' is continuous on $[a, b]$. Suppose the graph of f on $[a, b]$ is C.
Let $y = f(x)$.
The curved surface area of revolution of C along the $x-$ axis is

$$A = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

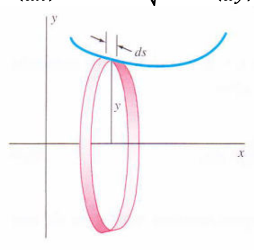
$$= 2\pi \int_a^b y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Let $c, d \in R$ with $c < d$ and f be a real-valued function on y. Suppose f' is continuous on $[c, d]$. Suppose the graph of f on $[c, d]$ is C.
Let $x = f(y)$.
The curved surface area of revolution of C along the $y-$ axis is

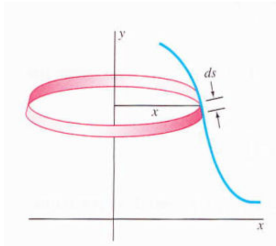
$$A = 2\pi \int_c^d f(y) \cdot \sqrt{1 + [f'(y)]^2}\, dy$$

$$= 2\pi \int_c^d x \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Note: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$



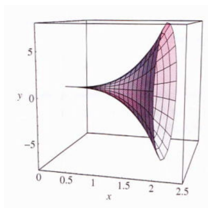
$$A = 2\pi \int_a^b y ds$$



$$A = 2\pi \int_c^d x ds$$

Example 1:

The figure shows the horn-shaped surface generated by revolving the curve $y = x^3, 0 \le x \le 2$ around the $x-$ axis. Find its surface area of revolution.



Solutions

$y = x^3; \frac{dy}{dx} = 3x^2; 1 + \left(\frac{dy}{dx}\right)^2 = 1 + (3x^2)^2 = 1 + 9x^4$

Its Surface Area of Revolution is

$$2\pi \int_0^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^2 x^3 \cdot \sqrt{1 + 9x^4} dx$$

$$= \frac{\pi}{18} \int_0^2 \sqrt{1 + 9x^4} \cdot 36x^3 dx$$

$$= \frac{\pi}{18} \int_1^{145} \sqrt{u} du$$

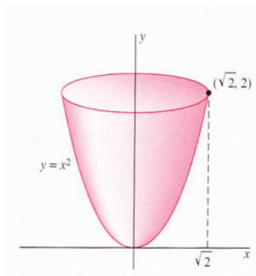
(let $u = 1 + 9x^4$, $du = 36x^3 dx$. When $x = 0, u = 1$. When $x = 2, u = 145$.)

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{145}$$

$$= \frac{\pi}{18} \cdot \frac{2}{3}\left(145^{\frac{3}{2}} - 1\right) = \frac{\pi}{27}\left(145^{\frac{3}{2}} - 1\right) \approx 203.04$$

Example 2:

Find the area of the paraboloid shown in the figure, which is obtained by revolving the parabolic arc
$y = x^2, 0 \leq x \leq \sqrt{2}$ around the $y -$ axis.



Solutions

As $y = x^2$, when $x = 0, y = 0$; when $x = \sqrt{2}, y = 2$.

$x = \sqrt{y}; \frac{dx}{dy} = \frac{1}{2\sqrt{y}}; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{2\sqrt{y}}\right)^2 = 1 + \frac{1}{4y}$

Its Surface Area of Revolution is

$$2\pi \int_0^2 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^2 \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}} dy$$

$$= 2\pi \int_0^2 \sqrt{y + \frac{1}{4}} dy$$

$$= 2\pi \int_{\frac{1}{4}}^{\frac{9}{4}} \sqrt{u} du$$

(let $u = y + \frac{1}{4}, du = dy$. When $y = 0, u = \frac{1}{4}$. When $y = 2, u = \frac{9}{4}$.)

$$= 2\pi \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=\frac{1}{4}}^{\frac{9}{4}}$$

$$= \frac{4\pi}{3}\left(\frac{27}{8} - \frac{1}{8}\right) = \frac{13\pi}{3}$$

Example 3:

Show that the surface area of a sphere with radius r is $4\pi r^2$.

Proof

Let A be the surface area of a sphere with radius r.

Let $y = \sqrt{r^2 - x^2}$ for $0 \le x \le r$.

Then, $\frac{1}{2}A = 2\pi \int_0^r y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2 = 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}}$$

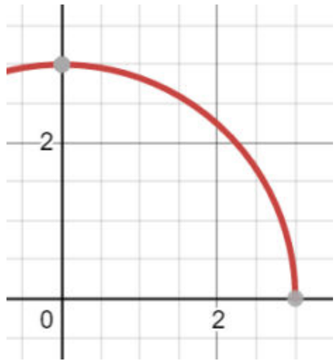
Thus, $A = 4\pi \int_0^r y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= 4\pi \int_0^r \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}\,dx$$

$$= 4\pi \int_0^r r\,dx = 4\pi r x|_{x=0}^{r}$$

$$= 4\pi r(r - 0)$$

$$= 4\pi r^2$$



$y = \sqrt{r^2 - x^2}$ when $r = 9$

Use of Parametric Substitutions for x and y in Arc Length and Surface Area of Revolution

Recall:

Let $a, b \in R$ with $a < b$ and f be a real-valued function on x.
Suppose f' is continuous on $[a, b]$. Suppose the graph of f on $[a, b]$ is C.
Let $y = f(x)$.

The **arc length** of C is $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

The **curved surface area of revolution** of C along the $x-$ axis is

$A = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_a^b y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example:

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$,

(i) show that the arc length s is $4\int_0^{\frac{\pi}{2}} \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$

(ii) it is revolved along the $x-$axis, show that the surface area of the revolution A is

$$4\pi b \int_0^{\frac{\pi}{2}} sint \cdot \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$$

(iii) it is revolved along the $x-$axis, show that the volume of the revolution V is $\frac{4\pi}{3} ab^2$.

Proof:

Let $x = acost$ and $y = bsint$ for $t \in \left[0, \frac{\pi}{2}\right]$.

$x'(t) = -asint, y'(t) = bcost$

$[x'(t)]^2 + [y'(t)]^2 = (-asint)^2 + (bcost)^2 = a^2 sin^2 t + b^2 cos^2 t$

$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2 sin^2 t + b^2 cos^2 t}$

$\frac{1}{4} s = \int_0^{\frac{\pi}{2}} \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$.

So, $s = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$.

$\frac{1}{2} A = 2\pi \int_0^{\frac{\pi}{2}} (bsint) \cdot \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$.

So, $A = 4\pi b \int_0^{\frac{\pi}{2}} sint \cdot \sqrt{a^2 sin^2 t + b^2 cos^2 t}\, dt$.

$$\frac{1}{2} V = \pi \int_{\frac{\pi}{2}}^{0} (bsint)^2 \cdot (-asint)\, dt = -\pi ab^2 \int_{\frac{\pi}{2}}^{0} sin^3 t\, dt = \pi ab^2 \int_0^{\frac{\pi}{2}} sin^3 t\, dt$$

So, $V = 2\pi ab^2 \int_0^{\frac{\pi}{2}} sin^3 t\, dt = 2\pi ab^2 \left(-cost + \frac{1}{3} cos^3 t \Big|_{t=0}^{\frac{\pi}{2}} \right)$

$$= 2\pi ab^2 \left[(0 + 0) - \left(-1 + \frac{1}{3} \right) \right] = \frac{4\pi}{3} ab^2$$

$$\int sin^3 t dt = -\int (1 - cos^2 t)(-sint) dt = -\int (1 - cos^2 t) d(cost) = -cost + \frac{1}{3} cos^3 t + C$$

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Lecture Notes for Chapter 6: Applications of Integration

Force and Work
Reading Assignment

Centroids of Plane Regions and Curves
Reading Assignment

Natural Logarithm as an Integral
Recall:
Definition:

We can define $ln: (0,\infty) \to R$ so that $y(x) = ln(x)$ is the **unique** function that satisfies $\begin{cases} y(1) = 0 \\ \frac{d}{dx}y(x) = \frac{1}{x} \end{cases}$.

We can also define $y: (0,\infty) \to R$ by $y(x) = \int_1^x \frac{1}{t}dt$. We can show that $y(x)$ satisfies $\begin{cases} y(1) = 0 \\ \frac{d}{dx}y(x) = \frac{1}{x} \end{cases}$

(Note: $\frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$). Thus, $lnx = \int_1^x \frac{1}{t}dt$ for any $x \in (0,\infty)$.

Some Properties:
For any $\alpha, \beta, \gamma \in (0,\infty)$, we can show that:

(i) $ln(\alpha\beta) = ln\alpha + ln\beta$

(ii) $ln\left(\frac{1}{\alpha}\right) = -ln\alpha$

(iii) $ln\left(\frac{\alpha}{\beta}\right) = ln\alpha - ln\beta$

(iv) $ln(\alpha^\gamma) = \gamma ln\alpha$

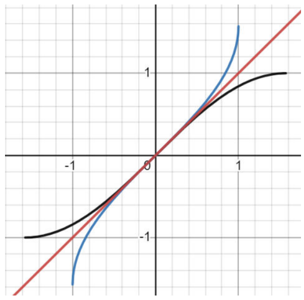
Logarithms and Experimental Data
Reading Assignments

Inverse Trigonometric Functions

Sine and Inverse Sine Function

$sin: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ is bijective. Its inverse is $sin^{-1}: [-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We say sin and sin^{-1} are inverse to each other.

Note: The graphs of $y = sinx$ and $y = sin^{-1}x$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx} sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ for any $x \in (-1,1)$.

Proof

Let $y = sin^{-1}x$ for any $x \in (-1,1)$. Note: $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Thus, $cosy > 0$.

Then, $siny = x$

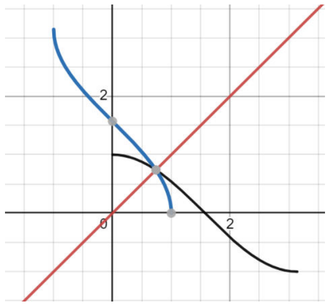
$$1 = \frac{d}{dx}x = \frac{d}{dx}siny = \left(\frac{d}{dy}siny\right) \cdot \frac{dy}{dx} = cosy \cdot \frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{cos^2y}} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$

Cosine and Inverse Cosine Function

$cos: [0,\pi] \to [-1,1]$ is bijective. Its inverse is $cos^{-1}: [-1,1] \to [0,\pi]$. We say cos and cos^{-1} are inverse to each other.

Note: The graphs of $y = cosx$ and $y = cos^{-1}x$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx} cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ for any $x \in (-1,1)$.

Proof

Let $y = cos^{-1}x$ for any $x \in (-1,1)$. Note: $y \in (0,\pi)$. Thus, $siny > 0$.

Then, $cosy = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}cosy = \left(\frac{d}{dy}cosy\right) \cdot \frac{dy}{dx} = -siny \cdot \frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\sqrt{sin^2y}} = \frac{-1}{\sqrt{1-cos^2y}} = \frac{-1}{\sqrt{1-x^2}}$

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Lecture Notes for Chapter 6: Applications of Integration

Remark 1

As $\frac{d}{dx}sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ for any $x \in (-1,1)$, we have
$\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + C$ and $\int \frac{1}{\sqrt{1-x^2}}dx = -cos^{-1}x + C$

Remark 2

Show that $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$ for any $x \in [-1,1]$

Proof

Let $sin^{-1}x = \theta$. Note: $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.
Then, $x = sin\theta = cos\left(\frac{\pi}{2} - \theta\right)$.
Note: $cos^{-1}x \in (0,\pi)$ and $\frac{\pi}{2} - \theta \in (0,\pi)$.
So, $cos^{-1}x = \frac{\pi}{2} - \theta$
That is, $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

Remark 3

Let $a > 0$ be a fixed real number (a constant).
For Integrand involving $a^2 - u^2$ or $\sqrt{a^2 - u^2}$, we shall use the substitution $u = asin\theta$.

Example 1

Evaluate $\int \frac{1}{\sqrt{16-x^2}}dx$

Solutions

Let $x = 4sin\theta$. Then, $dx = 4cos\theta d\theta$.
$16 - x^2 = 16 - (4sin\theta)^2 = 16(1 - sin^2\theta) = 16cos^2\theta$
$\sqrt{16 - x^2} = 4cos\theta$ (Assumed $cos\theta \geq 0$)

$$\int \frac{1}{\sqrt{16-x^2}}dx = \int \frac{1}{4cos\theta}4cos\theta d\theta = \int 1\, d\theta = \theta + C = sin^{-1}\left(\frac{x}{4}\right) + C$$

Example 2

Evaluate $\int \frac{x^2}{\sqrt{9-4x^2}}dx$

Solutions

Let $2x = 3sin\theta$. Then, $dx = \frac{3}{2}cos\theta d\theta$.
$9 - 4x^2 = 9 - (2x)^2 = 9 - (3sin\theta)^2 = 9(1 - sin^2\theta) = 9cos^2\theta$
$\sqrt{9 - 4x^2} = 3cos\theta$ (Assumed $cos\theta \geq 0$)

$$\int \frac{x^2}{\sqrt{9-4x^2}}dx = \int \frac{\left(\frac{3}{2}sin\theta\right)^2}{3cos\theta}\frac{3}{2}cos\theta d\theta = \frac{9}{8}\int sin^2\theta\, d\theta = \frac{9}{8}\int \frac{1 - cos2\theta}{2}d\theta$$

$$= \frac{9}{16}\left(\theta - \frac{sin2\theta}{2}\right) + C$$

$$= \frac{9}{16}(\theta - sin\theta cos\theta) + C$$

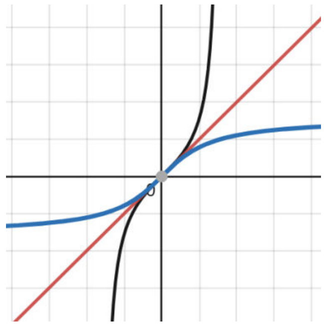
$$= \frac{9}{16}\left(sin^{-1}\left(\frac{2x}{3}\right) - \frac{2x}{3}\cdot\sqrt{1 - \frac{4x^2}{9}}\right) + C$$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 6: Applications of Integration

Tangent and Inverse Tangent Function

$tan: \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to R$ is bijective. Its inverse is $tan^{-1}: R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. We say tan and tan^{-1} are inverse to each other.

Note: The graphs of $y = tanx$ and $y = tan^{-1}x$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2}$ for any $x \in R$.

Proof

Let $y = tan^{-1}x$ for any $x \in R$.

Then, $tany = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}tany = \left(\frac{d}{dy}tany\right) \cdot \frac{dy}{dx} = sec^2y \cdot \frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$

Remark 1

Let $a > 0$ be a fixed real number (a constant).

For Integrand involving $a^2 + u^2$ or $\sqrt{a^2+u^2}$, we shall use the substitution $u = atan\theta$.

Example 1

Evaluate $\int \frac{x}{\sqrt{16+x^2}} dx$

Solutions

Let $x = 4tan\theta$. Then, $dx = 4sec^2\theta d\theta$.

$16 + x^2 = 16 + (4tan\theta)^2 = 16(1 + tan^2\theta) = 16sec^2\theta$

$\sqrt{16+x^2} = 4sec\theta$ (Assumed $sec\theta \geq 0$)

$$\int \frac{x}{\sqrt{16+x^2}} dx = \int \frac{4tan\theta}{4sec\theta} 4sec^2\theta d\theta$$

$$= 4\int tan\theta sec\theta \, d\theta$$

$$= 4sec\theta + C$$

$$= \sqrt{16+x^2} + C$$

Note: $tan\theta = \frac{x}{4}$. Then, $\sqrt{4^2+x^2} = \sqrt{16+x^2}$. So, $sin\theta = \frac{x}{\sqrt{16+x^2}}$ and $cos\theta = \frac{4}{\sqrt{16+x^2}}$

Thus, $sec\theta = \frac{\sqrt{16+x^2}}{4}$

Example 2

Evaluate $\int \frac{1}{(1+4x^2)^2} dx$

Solutions

Let $2x = tan\theta$. Then, $dx = \frac{1}{2} sec^2\theta d\theta$.

$1 + 4x^2 = 1 + (2x)^2 = 1 + (tan\theta)^2 = sec^2\theta$

$$\int \frac{1}{(1+4x^2)^2} dx = \int \frac{1}{(sec^2\theta)^2} \frac{1}{2} sec^2\theta d\theta = \frac{1}{2} \int cos^2\theta \, d\theta = \frac{1}{2} \int \frac{1 + cos2\theta}{2} d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{sin2\theta}{2}\right) + C$$

$$= \frac{1}{4}(\theta + sin\theta cos\theta) + C$$

$$= \frac{1}{4}\left(tan^{-1}(2x) + \frac{2x}{1+4x^2}\right) + C$$

Note: $tan\theta = \frac{2x}{1}$. Then, $\sqrt{(2x)^2 + 1} = \sqrt{1 + 4x^2}$. So, $sin\theta = \frac{2x}{\sqrt{1+4x^2}}$ and $cos\theta = \frac{1}{\sqrt{1+4x^2}}$

Thus, $sin\theta cos\theta = \frac{2x}{\sqrt{1+4x^2}} \cdot \frac{1}{\sqrt{1+4x^2}} = \frac{2x}{1+4x^2}$

Remark 2

Let $a > 0$ be a fixed real number (a constant).

For Integrand involving $u^2 - a^2$ or $\sqrt{u^2 - a^2}$, we shall use the substitution $u = asec\theta$.

Exercise

Evaluate $\int \frac{\sqrt{x^2-25}}{x} dx$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 6: Applications of Integration

Hyperbolic Functions

We define:

(i) Hyperbolic Sine Function

$sinh: R \to R$ by $sinh(x) = \frac{1}{2}(e^x - e^{-x})$

(ii) Hyperbolic Cosine Function

$cosh: R \to R$ by $cosh(x) = \frac{1}{2}(e^x + e^{-x})$

(iii) Hyperbolic Tangent Function

$tanh: R \to R$ by $tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) Hyperbolic Cosecant Function

$$csch(x) = \frac{1}{sinh(x)}$$

(v) Hyperbolic Secant Function

$$sech(x) = \frac{1}{cosh(x)}$$

(vi) Hyperbolic Cotangent Function

$$coth(x) = \frac{1}{tanh(x)}$$

Properties of Hyperbolic Trigonometric Functions

Example 1

Show that $cosh^2 x - sinh^2 x = 1$

Proof

$LHS = cosh^2 x - sinh^2 x$

$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$= 1$

$= RHS$

Theorems

(a) $\frac{d}{dx} sinh(x) = cosh\,(x)$

(b) $\frac{d}{dx} cosh(x) = sinh\,(x)$

(c) $\frac{d}{dx} tanh(x) = sech^2\,(x)$

(d) $\frac{d}{dx} coth(x) = -csch^2\,(x)$

(e) $\frac{d}{dx} sech(x) = -tanh\,(x) \cdot sech\,(x)$

(f) $\frac{d}{dx} csch(x) = -coth\,(x) \cdot csch\,(x)$

Proof

(a)
$$2\frac{d}{dx}sinh(x) = \frac{d}{dx}2\,sinh(x) = \frac{d}{dx}(e^x - e^{-x}) = e^x + e^{-x} = 2cosh\,(x)$$
So, $\frac{d}{dx}sinh(x) = cosh\,(x)$

(b)
$$2\frac{d}{dx}cosh(x) = \frac{d}{dx}2\,cosh(x) = \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x} = 2sinh\,(x)$$
So, $\frac{d}{dx}cosh(x) = sinh\,(x)$

(c)
$$\frac{d}{dx}tanh(x) = \frac{d}{dx}\left(\frac{sinh(x)}{cosh(x)}\right) = \frac{cosh(x)\cdot\frac{d}{dx}sinh(x) - sinh\,(x)\cdot\frac{d}{dx}cosh\,(x)}{cosh^2(x)}$$
$$= \frac{cosh(x)\cdot cosh(x) - \sinh(x)\cdot\sinh(x)}{cosh^2(x)}$$
$$= \frac{cosh^2x - sinh^2x}{cosh^2(x)}$$
$$= \frac{1}{cosh^2(x)}$$
$$= sech^2(x)$$

(d)
$$\frac{d}{dx}coth(x) = \frac{d}{dx}(tanh(x))^{-1} = -(tanh(x))^{-2}\cdot\frac{d}{dx}tanh(x)$$
$$= \frac{-\,cosh^2(x)}{sinh^2(x)}\cdot\frac{1}{cosh^2(x)}$$
$$= \frac{-1}{sinh^2(x)}$$
$$= -\,csch^2(x)$$

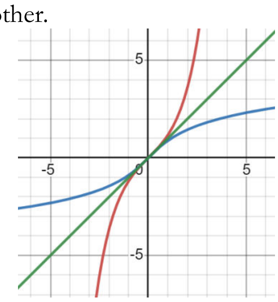
(e)
$$\frac{d}{dx}sech(x) = \frac{d}{dx}(cosh(x))^{-1} = -(cosh(x))^{-2}\cdot\frac{d}{dx}cosh(x)$$
$$= \frac{-1}{cosh^2(x)}\cdot sinh(x)$$
$$= -\,tanh(x)\,sech(x)$$

(f)
$$\frac{d}{dx}csch(x) = \frac{d}{dx}(sinh(x))^{-1} = -(sinh(x))^{-2}\cdot\frac{d}{dx}sinh(x)$$
$$= \frac{-1}{sinh^2(x)}\cdot cosh(x)$$
$$= -\,coth(x)\,csch(x)$$

Hyperbolic Sine and Inverse Hyperbolic Sine Function

$sinh: R \to R$ is bijective. Its inverse is $sinh^{-1}: R \to R$. We say $sinh$ and $sinh^{-1}$ are inverse to each other.

Note: The graphs of $y = sinh\,(x)$ and $y = sinh^{-1}(x)$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx}sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$ for any $x \in R$.

Proof

Let $y = sinh^{-1}x$ for any $x \in R$. Note: $y \in R$. Thus, $coshy > 0$.

Then, $sinh\ (y) = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}sinh\ (y) = \left(\frac{d}{dy}sinh\ (y)\right) \cdot \frac{dy}{dx} = cosh\ (y) \cdot \frac{dy}{dx}$$

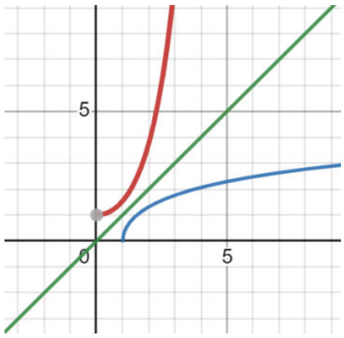
Thus, $\frac{dy}{dx} = \frac{1}{cosh\ (y)} = \frac{1}{\sqrt{cosh^2 y}} = \frac{1}{\sqrt{1+sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$

Hyperbolic Cosine and Inverse Hyperbolic Cosine Function

$cosh: \{x \in R: x \geq 1\} \to \{x \in R: x \geq 1\}$ is bijective. Its inverse is $cosh^{-1}: \{x \in R: x \geq 1\} \to \{x \in R: x \geq 1\}$.

We say $cosh$ and $cosh^{-1}$ are inverse to each other.

Note: The graphs of $y = cosh\ (x)$ and $y = cosh^{-1}(x)$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx}cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$ for any $x \in \{t \in R: t \geq 1\}$.

Proof

Let $y = cosh^{-1}x$ for any $x \in \{t \in R: t \geq 1\}$. Note: $y \in \{t \in R: t \geq 1\}$. Thus, $sinhy > 0$.

Then, $cosh\ (y) = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}cosh\ (y) = \left(\frac{d}{dy}cosh\ (y)\right) \cdot \frac{dy}{dx} = sinh\ (y) \cdot \frac{dy}{dx}$$

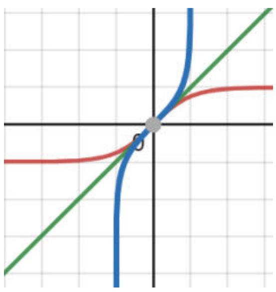
Thus, $\frac{dy}{dx} = \frac{1}{sinh\ (y)} = \frac{1}{\sqrt{sinh^2 y}} = \frac{1}{\sqrt{cosh^2 y-1}} = \frac{1}{\sqrt{x^2-1}}$

Hyperbolic Tangent and Inverse Hyperbolic Tangent Function

$tanh: R \to \{t \in R: -1 < t < 1\}$ is bijective. Its inverse is $tanh^{-1}: \{t \in R: -1 < t < 1\} \to R$.

We say $tanh$ and $tanh^{-1}$ are inverse to each other.

Note: The graphs of $y = tanh\ (x)$ and $y = tanh^{-1}(x)$ are symmetric about the line $y = x$.



Theorem

Show that $\frac{d}{dx} tanh^{-1}x = \frac{1}{1-x^2}$ for any $x \in \{t \in R: -1 < t < 1\}$.

Proof

Let $y = tanh^{-1}x$ for any $x \in \{t \in R: -1 < t < 1\}$.

Then, $tanh\ (y) = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}tan\,h(y) = \left(\frac{d}{dy}tanh(y)\right) \cdot \frac{dy}{dx} = sech^2\ (y) \cdot \frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = \frac{1}{sech^2\ (y)} = \frac{1}{1-tanh^2(y)} = \frac{1}{1-x^2}$

Note: $cosh^2(y) - sinh^2(y) = 1 \Rightarrow 1 - tanh^2(y) = sech^2\ (y)$

 # Inverse Hyperbolic Functions and Its Properties

Reading Assignment

Exercises

(a) Show that $sinh^{-1}x = ln\left(x + \sqrt{x^2+1}\right)$ for any $x \in R$.

(b) Show that $cosh^{-1}x = ln\left(x + \sqrt{x^2-1}\right)$ for any $x \in \{t \in R: t \geq 1\}$.

(c) Show that $tanh^{-1}x = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$ for any $x \in \{t \in R: -1 < t < 1\}$.