GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 3: Differentiability and The Derivatives
(Concepts, Definitions, Methods & Applications of Differentiation)

 # The Derivatives

The definitions

Let $f: R \to R$ be a function and $y = f(x)$. Let $a \in R$.

The derivative of f at a is defined as the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. It is denoted as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

Note: This a may be arbitrary.

The derivative of f at x is defined as the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. It is denoted as $f'(x)$ or $\frac{dy}{dx}$.

Note: We may consider a function that maps x to $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. (If the limit exists as a real number, it must be unique.) This is called the derivative of f at x. We use the functional notation $f': R \to R$.

We write $\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Let $u = x + h$. Then, $u \to x \Leftrightarrow h \to 0$. We can re-write $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{u\to x} \frac{f(u)-f(x)}{u-x}$.

Let $u = a + h$. Then, $u \to a \Leftrightarrow h \to 0$. We can re-write $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{u\to a} \frac{f(u)-f(a)}{u-a}$.

We say **f is differentiable at a** if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists as a real number.

Let $\phi \neq S \subset R$ and S is open.

We say **f is differentiable on S** if f is differentiable at a for any $a \in S$.

Usually, we consider the open interval (a, b) for S. (where $a, b \in R$ with $a < b$)

Roughly speaking, "f is differentiable at a" means the curve $y = f(x)$ is **smooth/**has **no corners**/has **no sharp edges** so that we can find a line tangent to the curve.

Note: **The derivative of f at a** is the slope of the line tangent to the curve $y = f(x)$ at $\big(a, f(a)\big)$ if it exists as a real number.

Examples

(i) Let C be a fixed real number (constant) and $a \in R$.

Let $f: R \to R$ be defined by $f(x) = C$ for any $x \in R$.

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{C-C}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$.

This a is arbitrary.

Thus, f is differentiable on R.

Note: the slope of the tangent is zero. (the graph of $y = C$ is a horizontal line and the tangent line will overlap with the given line.)

(ii) Let $f: R \to R$ be defined by $f(x) = x$ for any $x \in R$.

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{a+h-a}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$.

This a is arbitrary.

Thus, f is differentiable on R.

Note: the slope of the tangent is one. (the graph of $y = x$ is an inclined line with slope $= 1$ and the tangent line will overlap with the given line.)

Examples

(iii) Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$.

$f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \dfrac{(a+h)^2 - a^2}{h}$

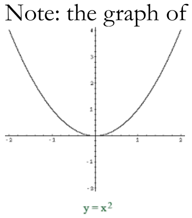
$= \lim_{h \to 0} \frac{a^2+2ah+h^2-a^2}{h} = \lim_{h \to 0} \frac{2ah+h^2}{h} = \lim_{h \to 0}(2a + h) = 2a.$

This a is arbitrary.

Thus, f is differentiable on R.

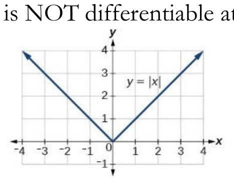
The slope of the line tangent to the curve at (a, a^2) is $2a$.

Note: the graph of $y = x^2$ is a parabola.



(iv) Let $f: R \to R$ be defined by $f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$.

Show that f is NOT differentiable at 0.



Proof:

$f(0+h) = f(h) = |h| = \begin{cases} h & if\ h \geq 0 \\ -h & if\ h < 0 \end{cases}$. $f(0) = 0$.

$\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{h-0}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$

$\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{-h-0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$

So, $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} \neq \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$.

Thus, $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ doesn't exist.

That is, f is NOT differentiable at 0.

Note: the graph of $y = |x|$ has no tangent lines at the origin. (not smooth/has corner/has sharp edge)

Example (v)

Let $f: R\backslash\{-3\} \to R$ be defined by $f(x) = \frac{x}{x+3}$.
Find an equation of the tangent at the origin $(0,0)$.

Solutions

Note: $f(0) = 0$.
Slope of required tangent is $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{\frac{h}{h+3}-0}{h} = \lim_{h\to 0}\frac{1}{h}\cdot\frac{h}{h+3} = \lim_{h\to 0}\frac{1}{h+3} = \frac{1}{0+3} = \frac{1}{3}$.

An equation of required tangent is

$\frac{y-0}{x-0} = \frac{1}{3}$

That is, $y = \frac{1}{3}x$.

Important notation:

Let $f: R \to R$ be a function and $y = f(x)$.
We write $\frac{d}{dx}y = \frac{dy}{dx}$. We say "differentiate y with respect to x" for $\frac{d}{dx}y$.

Note: We regard $\frac{d}{dx}$ maps the function f to the function f' provided that it is well defined.

Examples:

(i) Let C be a fixed real number (constant).
$\frac{d}{dx}C = 0$.

(ii) $\frac{d}{dx}x = 1$.

(iii) $\frac{d}{dx}x^2 = 2x$.

Theorem

For $n = 3,4,5,\cdots$, $\frac{d}{dx}x^n = nx^{n-1}$.

Proof:

Let $f: R \to R$ be a function defined by $f(x) = x^n$ for any $x \in R$, where $n = 3,4,5,\cdots$.

$\frac{d}{dx}x^n = f'(x)$

$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$ (Note: $(x+h)^n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}h^k$.)

$= \lim_{h\to 0}\frac{1}{h}\left[\left(\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}x^{n-k}h^k\right) - x^n\right]$

$= \lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{n}\frac{n!}{k!\,(n-k)!}x^{n-k}h^k$

$= \lim_{h\to 0}\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^{n-k}h^{k-1}$ (Note: $\frac{n!}{k!(n-k)!}x^{n-k}h^{k-1} \to 0$ as $h \to 0$ when $k = 2,3,\cdots,n$.)

$= \frac{n!}{1!\,(n-1)!}x^{n-1}$

$= nx^{n-1}$.

 # Rules for Differentiation (Sum, Difference and Scalar Multiple Rules)

Let $\lambda \in R$. Let $u: R \to R$ and $v: R \to R$ be functions.
Suppose u and v are differentiable at $x \in R$. Then,

1. $u + v$ is differentiable at $x \in R$ and $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$.
2. $u - v$ is differentiable at $x \in R$ and $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$.
3. λu is differentiable at $x \in R$ and $\frac{d}{dx}(\lambda u) = \lambda \frac{du}{dx}$.

Proof: Omitted

Corollary:

Let $p: R \to R$ be a polynomial function, say $p(x) = a_0 + \sum_{k=1}^{n} a_k x^k$ for any $x \in R$, where $a_0, a_1, \cdots, a_n \in R$ with $a_n \neq 0$.
Then, $p'(x) = \frac{d}{dx} p(x) = \sum_{k=1}^{n} k a_k x^{k-1}$.

An application:

Let $a, b, c, d, e \in R$ with $a \neq 0$. We have:

(i) $\frac{d}{dx}(ax + b) = a$

(ii) $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$

(iii) $\frac{d}{dx}(ax^3 + bx^2 + cx + d)$
$= 3ax^2 + 2bx + c$

(iv) $\frac{d}{dx}(ax^4 + bx^3 + cx^2 + dx + e)$
$= 4ax^3 + 3bx^2 + 2cx + d$

Examples:

(i) $\frac{d}{dx}(2x - 3) = 2$

(ii) $\frac{d}{dx}(5x^2 - 6x + 7) = 10x - 6$

(iii) $\frac{d}{dx}(9x^3 - 10x^2 + 11x + 12)$
$= 27x^2 - 20x + 11$

(iv) $\frac{d}{dx}(13x^4 - 14x^3 + 15x^2 + 16x - 17)$
$= 52x^3 - 42x^2 + 30x + 16$

Example

Find $\frac{d}{dx}\sqrt{1+x}$ **from first principles (that is, from the limit definitions)**.

Solutions

Let $f: \{x \in R: x > -1\} \to R$ be defined by $f(x) = \sqrt{1+x}$.

$$\frac{d}{dx}\sqrt{1+x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{1+x+h} - \sqrt{1+x}}{h} = \lim_{h \to 0} \frac{\sqrt{1+x+h} - \sqrt{1+x}}{h} \cdot \frac{\sqrt{1+x+h} + \sqrt{1+x}}{\sqrt{1+x+h} + \sqrt{1+x}}$$

$$= \lim_{h \to 0} \frac{(1+x+h) - (1+x)}{h\left(\sqrt{1+x+h} + \sqrt{1+x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{1+x+h} + \sqrt{1+x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+x+h} + \sqrt{1+x}} = \frac{1}{\sqrt{1+x+0} + \sqrt{1+x}}$$

$$= \frac{1}{2\sqrt{1+x}}$$

 # Rules for Differentiation (Product and Quotient Rules)

Let $\lambda \in R$. Let $u: R \to R$ and $v: R \to R$ be functions.
Suppose u and v are differentiable at $x \in R$. Then,

1. $u \cdot v$ is differentiable at $x \in R$ and $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$.
2. $\frac{u}{v}$ is differentiable at $x \in R$ and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
   (We assumed $v(x) \neq 0$ for concerned x.)

Idea of the proof:

1.
$$\lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0} u(x+h)\cdot\frac{v(x+h) - v(x)}{h} + v(x)\cdot\frac{u(x+h) - u(x)}{h}$$

2.
$$\lim_{h\to 0}\frac{1}{h}\left[\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}\right]$$
$$= \lim_{h\to 0}\frac{1}{v(x+h)v(x)}\cdot\frac{u(x+h)v(x) - u(x)v(x+h)}{h}$$

Note:
$$\frac{u(x+h)v(x) - u(x)v(x+h)}{h}$$
$$= v(x)\cdot\frac{u(x+h) - u(x)}{h} - u(x)\cdot\frac{v(x+h) - v(x)}{h}$$

Examples:

Find the derivatives of:

(i) $y = (1 - 4x^2)(3x^2 - 5x + 2)$

(ii) $y = (2x - 3x^2)^3$

(iii) $y = \dfrac{1}{x^2 + 1}$

(iv) $y = \dfrac{5x^4 - 6x + 7}{2x^2}$

Solutions (i)

Method 1 (Product Rule)

$$\frac{dy}{dx} = (1 - 4x^2) \cdot \frac{d}{dx}(3x^2 - 5x + 2) + (3x^2 - 5x + 2) \cdot \frac{d}{dx}(1 - 4x^2)$$

$$= (1 - 4x^2) \cdot (6x - 5) + (3x^2 - 5x + 2) \cdot (-8x)$$

$$= -5 + 6x + 20x^2 - 24x^3 - 16x + 40x^2 - 24x^3$$

$$= -5 - 10x + 60x^2 - 48x^3$$

Method 2 (Expansion before taking differentiation)

$$y = (1 - 4x^2)(3x^2 - 5x + 2)$$

$$= 2 - 5x + 3x^2 - 8x^2 + 20x^3 - 12x^4$$

$$= 2 - 5x - 5x^2 + 20x^3 - 12x^4$$

$$\frac{dy}{dx} = \frac{d}{dx}(2 - 5x - 5x^2 + 20x^3 - 12x^4) = -5 - 10x + 60x^2 - 48x^3$$

Solutions (ii)

Use $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we have

$$(2x - 3x^2)^3 = (2x)^3 + 3(2x)^2(-3x^2) + 3(2x)(-3x^2)^2 + (-3x^2)^3$$

$$= 8x^3 - 36x^4 + 54x^5 - 27x^6$$

$$\frac{d}{dx}(2x - 3x^2)^3 = \frac{d}{dx}(8x^3 - 36x^4 + 54x^5 - 27x^6)$$

$$= 24x^2 - 144x^3 + 270x^4 - 162x^5$$

Solutions (iii)

$$\frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{1}{(x^2+1)^2}\left[(x^2+1)\cdot\frac{d}{dx}1 - 1\cdot\frac{d}{dx}(x^2+1)\right]$$

$$= \frac{1}{(x^2+1)^2}[0-2x] = \frac{-2x}{(x^2+1)^2}$$

Solutions (iv)

$$\frac{d}{dx}\left(\frac{5x^4-6x+7}{2x^2}\right)$$

$$= \frac{1}{(2x^2)^2}\left[(2x^2)\cdot\frac{d}{dx}(5x^4-6x+7) - (5x^4-6x+7)\cdot\frac{d}{dx}(2x^2)\right]$$

$$= \frac{1}{4x^4}[(2x^2)\cdot(20x^3-6) - (5x^4-6x+7)\cdot(4x)]$$

$$= \frac{1}{4x^4}[40x^5-12x^2-20x^5+24x^2-28x]$$

$$= \frac{1}{4x^4}[20x^5+12x^2-28x]$$

$$= \frac{1}{x^3}[5x^4+3x-7]$$

Remark for Product Rule for differentiation

Let $\lambda \in R$. Let $u: R \to R$, $v: R \to R$ and $w: R \to R$ be functions. Suppose u, v and w are differentiable at $x \in R$.

Then, uvw is differentiable at $x \in R$ and $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$.

Proof:

$$\frac{d}{dx}(uvw) = u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u\left(v\frac{dw}{dx} + w\frac{dv}{dx}\right) + vw\frac{du}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Special Case: $u = v = w$, we have $\frac{d}{dx}u^3 = 3u^2\frac{du}{dx}$.

Solutions (ii)

$$\frac{d}{dx}(2x - 3x^2)^3 = 3(2x - 3x^2)^2 \cdot \frac{d}{dx}(2x - 3x^2) = 3(2x - 3x^2)^2 \cdot (2 - 6x)$$

$$= (4x^2 - 12x^3 + 9x^4) \cdot (6 - 18x)$$

$$= 24x^2 - 72x^3 + 54x^4 - 72x^3 + 216x^4 - 162x^5$$

$$= 24x^2 - 144x^3 + 270x^4 - 162x^5$$

Remark for Quotient Rule for differentiation

Show that $\frac{d}{dx}x^{-n} = -nx^{-n-1}$ for $n = 1,2,3,\cdots$.

Proof:

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{1}{(x^n)^2}\left[x^n \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}x^n\right]$$

$$= \frac{1}{(x^n)^2}[0 - nx^{n-1}] = \frac{-nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}} = -nx^{-n-1}$$

Summary

(i) $\frac{d}{dx}x^n = nx^{n-1}$ for $n = 1,2,3,\cdots$.

(ii) $\frac{d}{dx}1 = 0$

(iii) $\frac{d}{dx}x^{-n} = -nx^{-n-1}$ for $n = 1,2,3,\cdots$.

Note: We may regard $1 = x^0$ for $x \neq 0$. Then, the above rules can be written as:

$\frac{d}{dx}x^u = ux^{u-1}$ for $u = 0, \pm 1, \pm 2, \pm 3, \cdots$.

Solutions (iv)

$$\frac{d}{dx}\left(\frac{5x^4-6x+7}{2x^2}\right)$$

$$=\frac{d}{dx}\left(\frac{5}{2}x^2-3x^{-1}+\frac{7}{2}x^{-2}\right)=5x+3x^{-2}-7x^{-3}$$

$$=\frac{1}{x^3}[5x^4+3x-7]$$

Example:

Find $\frac{dz}{dt}$ if $z=\frac{1-t^3}{1+t^4}$.

Solutions

$$\frac{dz}{dt}=\frac{1}{(1+t^4)^2}\left[(1+t^4)\cdot\frac{d}{dt}(1-t^3)-(1-t^3)\cdot\frac{d}{dt}(1+t^4)\right]$$

$$=\frac{1}{(1+t^4)^2}[(1+t^4)\cdot(-3t^2)-(1-t^3)\cdot(4t^3)]$$

$$=\frac{1}{(1+t^4)^2}[-3t^2-3t^6-4t^3+4t^6]=\frac{1}{(1+t^4)^2}[-3t^2-4t^3+t^6]$$

 # Chain Rule (Rule for finding derivative of composite functions)

Let $g: R \to R$ be a function on x and $u=g(x)$. Let $f: R \to R$ be a function on u and $y=f(u)$. Then, we may regard $f: R \to R$ as a function on x.

As $y=f(u)$ and $u=g(x)$, so $y=f\big(g(x)\big)$. Then,

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ **(Chain Rule)**

$(f^\circ g)'(x)=f'\big(g(x)\big)\cdot g'(x)$ **(Chain Rule)**

Idea of Proof:

$$\frac{dy}{dx}=(f^\circ g)'(x)=\lim_{h\to 0}\frac{f\big(g(x+h)\big)-f\big(g(x)\big)}{h}$$

$$=\lim_{h\to 0}\frac{f(g(x)+k)-f\big(g(x)\big)}{k}\cdot\frac{g(x+h)-g(x)}{h}$$

(let $k=g\,(x+h)-g(x)$, assumed $k\to 0 \Leftrightarrow h\to 0$)

$$=\lim_{k\to 0}\frac{f(g(x)+k)-f\big(g(x)\big)}{k}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'\big(g(x)\big)\cdot g'(x)$$

$$=\frac{dy}{du}\cdot\frac{du}{dx}$$

Remark on Example (ii)

Find the derivative of $\frac{d}{dx}(2x-3x^2)^3$.

Solutions

Let $u = 2x - 3x^2$ and $y = u^3$.

$\frac{d}{dx}(2x-3x^2)^3 = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ (Chain Rule) $= 3u^2 \cdot (2-6x)$

$= 3(2x-3x^2)^2 \cdot (2-6x) = \cdots = 24x^2 - 144x^3 + 270x^4 - 162x^5$

Examples:

Find the derivatives of:

(i) $(3x^2+5)^{17}$

(ii) $\dfrac{1}{(2x^3-x+7)^2}$

(iii) $\left(\dfrac{x-1}{x+1}\right)^5$

Solutions (i)

$$\frac{d}{dx}(3x^2+5)^{17} = 17(3x^2+5)^{16}\cdot\frac{d}{dx}(3x^2+5) = 17(3x^2+5)^{16}\cdot 6x$$

$= 102x(3x^2+5)^{16}$.

Solutions (ii)

$$\frac{d}{dx}\left[\frac{1}{(2x^3-x+7)^2}\right] = \frac{d}{dx}(2x^3-x+7)^{-2}$$

$$= -2(2x^3-x+7)^{-3}\cdot\frac{d}{dx}(2x^3-x+7) = -2(2x^3-x+7)^{-3}\cdot(6x^2-1)$$

$$= \frac{-2(6x^2-1)}{(2x^3-x+7)^3}$$

<u>**Another Method**</u>

$$\frac{d}{dx}\left[\frac{1}{(2x^3-x+7)^2}\right]$$

$$= \frac{1}{(2x^3-x+7)^4}\left[(2x^3-x+7)^2\cdot\frac{d}{dx}1 - 1\cdot\frac{d}{dx}(2x^3-x+7)^2\right]$$

$$= \frac{1}{(2x^3-x+7)^4}\left[0 - 2(2x^3-x+7)\cdot\frac{d}{dx}(2x^3-x+7)\right] = \frac{-2(6x^2-1)}{(2x^3-x+7)^3}$$

Solutions (iii)

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\right)^5 = 5\left(\frac{x-1}{x+1}\right)^4 \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$= 5\left(\frac{x-1}{x+1}\right)^4 \cdot \frac{1}{(x+1)^2}\left[(x+1)\cdot\frac{d}{dx}(x-1) - (x-1)\cdot\frac{d}{dx}(x+1)\right]$$

$$= 5\left(\frac{x-1}{x+1}\right)^4 \cdot \frac{(x+1)-(x-1)}{(x+1)^2}$$

$$= 5\left(\frac{x-1}{x+1}\right)^4 \cdot \frac{2}{(x+1)^2} = \frac{10(x-1)^4}{(x+1)^6}$$

An application of Chain Rule:

$\frac{d}{dx}x^{\frac{m}{n}} = \frac{m}{n}x^{\frac{m}{n}-1}$ for $m \in \{0, \pm1, \pm2, \pm3, \cdots\}$ and $n \in \{1,2,3,\cdots\}$.

Proof:

Let $y = x^{\frac{m}{n}}$. Then, $y^n = x^m$.

$ny^{n-1}\frac{dy}{dx} = \frac{d}{dx}y^n = \frac{d}{dx}x^m = mx^{m-1}$.

$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}} = \frac{mx^{m-1}}{nx^m y^{-1}}$ (Note: $y^n = x^m$)

$$= \frac{mx^{-1}}{ny^{-1}} = \frac{m}{n}\cdot\frac{y}{x} = \frac{m}{n}\cdot\frac{x^{\frac{m}{n}}}{x} = \frac{m}{n}x^{\frac{m}{n}-1}$$

Summary:

Note: We may regard $1 = x^0$ for $x \neq 0$.

Then, we have $\frac{d}{dx}x^u = ux^{u-1}$ for any $u \in Q$.

Q is the set of all rational numbers (or the sets of all fractions).

That is, $Q = \left\{\frac{m}{n} : m \in \{0, \pm1, \pm2, \pm3, \cdots\} \text{ and } n \in \{1,2,3,\cdots\}\right\}$.

Claim:

We have $\frac{d}{dx}x^u = ux^{u-1}$ for any $u \in R$.

R is the set of all real numbers.

Proof: Omitted

Example

Find $\frac{dy}{dx}$ if $y = u^r$, u is a function on x, r is a non-zero real number.

Solutions

$$\frac{dy}{dx} = \frac{d}{dx}u^r = ru^{r-1}\frac{du}{dx}$$

Examples

Find the derivatives of:

(i) $\sqrt{x}$

(ii) $\sqrt{x^3}$

(iii) $\dfrac{1}{\sqrt[3]{x^2}}$

(iv) $5\sqrt{x^3} - \dfrac{2}{\sqrt[3]{x^2}}$

(v) $\sqrt{4 - x^2}$

(vi) $\sqrt{2x^2 - 3x + 5}$

Solutions

(i) $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$$

(ii) $$\frac{d}{dx}\sqrt{x^3} = \frac{d}{dx}x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

(iii) $$\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right) = \frac{d}{dx}x^{\frac{-2}{3}} = \frac{-2}{3}x^{\frac{-5}{3}} = \frac{-2}{3x^{5/3}}$$

(iv) $$\frac{d}{dx}\left(5\sqrt{x^3} - \frac{2}{\sqrt[3]{x^2}}\right) = 5 \cdot \frac{3}{2}\sqrt{x} - 2 \cdot \frac{-2}{3x^{5/3}} = \frac{15}{2}\sqrt{x} + \frac{4}{3x^{5/3}}$$

(v) $$\frac{d}{dx}\sqrt{4 - x^2} = \frac{1}{2\sqrt{4 - x^2}} \cdot \frac{d}{dx}(4 - x^2) = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

(vi) $$\frac{d}{dx}\sqrt{2x^2 - 3x + 5} = \frac{1}{2\sqrt{2x^2 - 3x + 5}} \cdot \frac{d}{dx}(2x^2 - 3x + 5)$$

$$= \frac{4x - 3}{2\sqrt{2x^2 - 3x + 5}}$$

An application of Chain Rule:

Let $f: D \to E$ be a function and let $y = f(x)$, $\phi \neq D \subset R$, $\phi \neq E \subset R$ and f is bijective.

Then, we can consider the inverse function of f, say $g: E \to D$ and $x = g(y)$.

Note: g is also bijective.

[That is, f and g are inverse functions to each other.]

We can consider both $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

Theorem: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ (assumed concerned items are meaningful.)

Proof:

$\frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{d}{dy} y$ (Chain Rule) $= 1$ So, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Example:

(i) Suppose we know $\frac{d}{dx} e^x = e^x$. Show that $\frac{d}{dx} lnx = \frac{1}{x}$.

(ii) Suppose we know $\frac{d}{dx} lnx = \frac{1}{x}$. Show that $\frac{d}{dx} e^x = e^x$.

Proof (i)

Let $y = e^x$. Then, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{e^x} = \frac{1}{y}$. So, $\frac{d}{dy} lny = \frac{1}{y}$. Hence, $\frac{d}{dx} lnx = \frac{1}{x}$.

Proof (ii)

Let $y = lnx$. Then, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{1}{x}} = x = e^y$. So, $\frac{d}{dy} e^y = e^y$. Hence, $\frac{d}{dx} e^x = e^x$.

An application of Chain Rule:

Note: We can use the Product rule and Chain Rule to prove the quotient rule.

Let $u: R \to R$ and $v: R \to R$ be functions. Suppose u and v are differentiable at $x \in R$.
Then, $\frac{u}{v}$ is differentiable at $x \in R$ and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
(We assumed $v(x) \neq 0$ for concerned x.)

Proof

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}(u \cdot v^{-1}) = u \cdot \frac{d}{dx}v^{-1} + v^{-1} \cdot \frac{d}{dx}u$$

$$= u \cdot \frac{d}{dv}v^{-1} \cdot \frac{dv}{dx} + \frac{1}{v} \cdot \frac{du}{dx} = u(-v^{-2})\frac{dv}{dx} + \frac{1}{v} \cdot \frac{du}{dx}$$

$$= \frac{1}{v^2}\left[-u\frac{dv}{dx} + v\frac{du}{dx}\right] = \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right].$$

 # Notation for differential:

Let $f: R \to R$ be a function on x. The derivative of f at x is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
Suppose $|h|$ is a very small positive number.
Let P be $(x, f(x))$ and Q be $(x+h, f(x+h))$.
We define $dx = \Delta x = h$ and $\Delta y = f(x+h) - f(x)$.
We say $\Delta x =$ change in x and $\Delta y =$ change in y.
We define $dy = \frac{dy}{dx} \cdot dx$. (Note: we can consider $\frac{dy}{dx}$ as $dy \div dx$.)
We say $dx =$ differential change in x and $dy =$ differential change in y.
Note:

(i) $\frac{\Delta y}{\Delta x}$ is the slope of the secant joining the points P and Q.

(ii) $\frac{dy}{dx}$ is the slope of the tangent at the point P if it exists as a real number.

$f'(x) \cdot h = \frac{dy}{dx}dx = dy \approx \Delta y = f(x+h) - f(x)$ when $h \approx 0$.
So, $f(x+h) \approx f(x) + f'(x) \cdot h$ when $h \approx 0$.

We may change the point for consideration:
So, $f(a+h) \approx f(a) + f'(a) \cdot h$ when $h \approx 0$ and f is differentiable at a.
Let $x = a + h$.
$f(x) \approx f(a) + f'(a) \cdot (x-a)$ when $x \approx a$ and f is differentiable at a.

Definition:
$f(x) \approx f(a) + f'(a) \cdot (x-a)$ when $x \approx a$ and f is differentiable at a.
We may let $y = L(x)$ and $L(x) = f(a) + f'(a)(x-a)$.
$y = L(x)$ is an equation of a line.
$y = L(x)$ is **the** good approximation of $y = f(x)$ when $x \approx a$.
This is called **the linear approximation** of $y = f(x)$.

Notes:

(i) $y = L(x)$ is an equation of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

(ii) $y = L(x)$ is **the** linear approximation of $y = f(x)$ near to the point $(a, f(a))$.

(iii) $y = L(x)$ is **the** linear approximation of $y = f(x)$ near a.
("near a" means "$x \approx a$")

Example:

(a) Find **the** linear approximation of the function $y = f(x) = \sqrt{1+x}$ near $a = 0$.

(b) Hence find an approximation of $\sqrt{101}$.

Solutions
(a)
As $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}}$.
$f(0) = \sqrt{1+0} = 1$ and $f'(0) = \frac{1}{2}(1+0)^{\frac{-1}{2}} = \frac{1}{2}$.
The linear approximation is $y = L(x)$ where $L(x) = f(0) + f'(0)x = 1 + \frac{1}{2}x$.
That is, $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ when $x \approx 0$.
(b)

$$\sqrt{101} = \sqrt{100} \times \sqrt{1.01} = 10\sqrt{1.01} \approx 10\left(1 + \frac{1}{2} \times 0.01\right) = 10 \times 1.005 = 10.05$$

Note: $0.01 \approx 0$.

 # Another important notation:

Let $f: R \to R$ be a function on x and $y = f(x)$. The derivative of f at x is $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Let $h = \Delta x$. Then, we may re-write $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Note: $\Delta y = f(x + \Delta x) - f(x)$. Thus, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

An application:
Chain Rule (Rule for finding derivative of composite functions)

Let $g: R \to R$ be a function on x and let $u = g(x)$. Let $f: R \to R$ be a function on u and let $y = f(u)$. Then, we may regard $f: R \to R$ as a function on x.

$y = f(u)$ and $u = g(x)$, so $y = f\big(g(x)\big)$. Then,

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ **(Chain Rule)**

$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$ **(Chain Rule)**

Idea of the proof:

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$ (Assumed $\Delta u \to 0 \Leftrightarrow \Delta x \to 0$)

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

 # Applications (Functions on time t)

Suppose $Q: R \to R$ be a function on time t.
The change in Q from time t to time $t + \Delta t$ is $\Delta Q = Q(t + \Delta t) - Q(t)$ (the increment).
The average rate of change of Q from time t to time $t + \Delta t$ is $\frac{\Delta Q}{\Delta t} = \frac{Q(t+\Delta t)-Q(t)}{\Delta t}$.
The instantaneous rate of change of Q at time t (or at the point $\big(t, Q(t)\big)$) is defined as
$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t)-Q(t)}{\Delta t}$.
(Note: it is the slope of the tangent to the graph $y = Q(t)$ at the point $\big(t, Q(t)\big)$.)

Example

The right cylindrical tank (with a vertical axis) is initially filled with 600 gal. of water. This tank takes 60 min. to empty after a drain in its bottom is opened. Suppose that the drain is opened at time $t = 0$, Suppose also that the volume $V(t)$ of water remaining in the tank at t minutes is $V(t) = \frac{1}{6}(60 - t)^2 = 600 - 20t + \frac{1}{6}t^2$ (in gal.). Find:

(i) the instantaneous rate at which water is flowing out of the tank at time $t = 15$ (in min.)
(ii) the instantaneous rate at which water is flowing out of the tank at time $t = 45$ (in min.)
(iii) the average rate at which water is flowing out of the tank during the half hour from time $t = 15$ to $t = 45$ (in min.)

Solutions

$V(t) = \dfrac{1}{6}(60 - t)^2 = 600 - 20t + \dfrac{1}{6}t^2$

$V(45) = \frac{1}{6}(60 - 45)^2 = \frac{1}{6} \times 15^2 = 37.5.$

$V(15) = \frac{1}{6}(60 - 15)^2 = \frac{1}{6} \times 45^2 = 337.5.$

$V'(t) = -20 + \dfrac{1}{3}t$

$V'(15) = -20 + \frac{1}{3} \times 15 = -15$ (negative sign means it is decreasing)

The instantaneous rate at which water is flowing out of the tank at time $t = 15$ is decreasing at 15 gal./min..

$V'(45) = -20 + \frac{1}{3} \times 45 = -5$ (negative sign means it is decreasing)

The instantaneous rate at which water is flowing out of the tank at time $t = 45$ is decreasing at 5 gal./min..

The average rate at which water is flowing out of the tank during the half hour from time $t = 15$ to time $t = 45$ is $\frac{\Delta V}{\Delta t} = \frac{V(45) - V(15)}{45 - 15} = \frac{37.5 - 337.5}{30} = -10$ (decreasing at 10 gal. /min.)

 # Another Applications:

Consider a linear motion (motion along a straight line), we may specify the direction in two ways (one is positive and the other one is negative). This denotes a position vector (displacement) which is a function $x: R \to R$.

The average velocity is defined as $\frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$.

The instantaneous velocity is defined as $v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$.

The (instantaneous) velocity is the instantaneous rate of change of position.

The (instantaneous) acceleration is the instantaneous rate of change of velocity.

Example

Find the maximum height attained by a ball thrown straight upward from the ground with initial velocity $u = +96$ ft./s.. Also, find the velocity with which it hits the ground upon its return.

Solutions

We know $a(t) = -g$ (downward acceleration due to gravity).

Use remark (ii), we get $v = u - gt$. Note: $u = 96$, $g = 32$. So, $y'(t) = v = 96 - 32t$.

Also, $y(0) = 0$. Use remark (iii), we get $y(t) = 96t - 16t^2$.

When the ball attained the maximum height, $v = 0$. Put $v = 0$, that is, $96 - 32t = 0$, so $t = 3$.

The maximum height is $y(3) = 96 \times 3 - 16 \times 3^2 = 144$ (in ft.).

When the ball returns to the ground, $y(t) = 0$. Put $y(t) = 0$, that is, $96t - 16t^2 = 0$, so $t = 0$ (at the start) or $t = 6$ (when it returns).

The velocity with which it strikes the ground is $y'(6) = 96 - 32 \times 6 = -96$ (in ft./s.) (negative sign means it is in downward direction)

Remarks:

(i) **Theorem**

Suppose $f: R \to R$ and $g: R \to R$ are functions on x.

Let $a, b \in R$ with $a < b$.

Suppose $f'(x) = g'(x)$ for any $x \in (a, b)$.

Then, we can find a fixed real number C (constant) such that $f(x) = g(x) + C$ for any $x \in (a, b)$.

Proof: Will be discussed later

(ii) Let $p(t) = u - gt$.

Also, we know $v'(t) = a(t) = -g = \frac{d}{dt}(u - gt) = p'(t)$ and $v(0) = u = p(0)$, use the above theorem to get $v(t) = p(t) = u - gt$.

(iii) Let $q(t) = 96t - 16t^2$.

As we know $y'(t) = 96 - 32t = \frac{d}{dt}(96t - 16t^2) = q'(t)$ and $y(0) = 0 = q(0)$, use the above theorem to get $y(t) = q(t) = 96t - 16t^2$.

Other Rates of Change

The area of a square with edge length x (in cm) is $A = x^2$ (in cm^2).

Then, $\frac{dA}{dx} = 2x$ is the rate of change of the area A with respect to x.

Suppose $x(t) = 5t$. Then, $A\big(x(t)\big) = \big(x(t)\big)^2 = (5t)^2 = 25t^2$.

$\frac{dA}{dt} = 50t$ is the rate of change of the area A with respect to t.

$\frac{dx}{dt} = 5$ is the rate of change of the edge length x with respect to t.

When $\boldsymbol{t = 10}$, $x = 5 \times 10 = 50$.

$\left.\frac{dA}{dt}\right|_{t=10} = 50 \times 10 = 500$. **$\boldsymbol{A}$ is increasing at the rate of $\boldsymbol{500\ cm^2/\ s}$**.

$\left.\frac{dA}{dx}\right|_{t=10} = 2 \times 50 = 100$.

$\left.\frac{dx}{dt}\right|_{t=10} = 5$. **$\boldsymbol{x}$ is increasing at the rate of $\boldsymbol{5\ cm/\ s}$**.

Note: $\left.\frac{dA}{dt}\right|_{t=10} = \left.\frac{dA}{dx}\right|_{t=10} \times \left.\frac{dx}{dt}\right|_{t=10}$ (**Chain Rule**)

Exercises:

(i) Write an equation for the straight line that is tangent to the graph of

$y = 2x^3 - 7x^2 + 3x + 4$ at the point $(1,2)$.

(ii) The volume V (in cm^3) of a given sample of water varies with changing temperature T (in ^{O}C).

For T between $0^{O}C$ and $30^{O}C$, the relation is given almost by the formula:

$V = V_0[1 - (6.427 \times 10^{-5})T + (8.505 \times 10^{-6})T^2 - (6.790 \times 10^{-8})T^3]$ where

V_0 is the volume of the water (not ice) sample at $0^{O}C$.

Suppose that $V_0 = 10^5$ (in cm^3).

Find both the volume and the rate of change of volume with respect to temperature when $T = 20$ (in ^{O}C).

Solutions (i)

As $y = 2x^3 - 7x^2 + 3x + 4$, we have $\frac{dy}{dx} = 6x^2 - 14x + 3$. So, $\left.\frac{dy}{dx}\right|_{x=1} = 6 - 14 + 3 = -5$.

An equation of required tangent is

$$\frac{y-2}{x-1} = -5$$

(that is, $y = -5x + 7$)

Solutions (ii)

As $V = V_0[1 - (6.427 \times 10^{-5})T + (8.505 \times 10^{-6})T^2 - (6.790 \times 10^{-8})T^3]$ and $T = 20$,

the volume is $10^5[1 - (6.427 \times 10^{-5}) \times 20 + (8.505 \times 10^{-6}) \times 20^2 - (6.790 \times 10^{-8}) \times 20^3]$

$= 100157.3$ (in cm^3).

Required volume is $100157.3\ cm^3$.

$$\frac{dV}{dT} = V_0[-(6.427 \times 10^{-5}) + 2(8.505 \times 10^{-6})T - 3(6.790 \times 10^{-8})T^2]$$

$$\left.\frac{dV}{dT}\right|_{T=20} = 10^5[-(6.427 \times 10^{-5}) + 2 \times (8.505 \times 10^{-6}) \times 20 - 3 \times (6.790 \times 10^{-8}) \times 20^2]$$

$= 19.445$ (in $cm^3/{}^{O}C$).

Required rate of change of volume is $19.445\ cm^3/{}^{O}C$.

As $V(21) \approx 100177.22$ and $V(20) = 100157.3$,

$\frac{V(21)-V(20)}{21-20} \approx 100177.2 - 100157.34 = 19.88 \approx 19.445 = \left.\frac{dV}{dT}\right|_{T=20}$.

Exercises

(i) A spherical balloon is being inflated. The radius r of the balloon is increasing at the rate of $0.2\ cm./\ s.$ when $r = 5\ cm.$ At what rate is the volume V of the balloon increasing at that instant?

[Hint: $V = \frac{4}{3}\pi r^3$.]

(ii) Imagine a spherical raindrop that is falling through water vapor in the air. Suppose that the vapor adheres to the surface of the raindrop in such a way that the time rate of increase of the mass M of the droplet is proportional to the surface area S of the droplet. If the initial radius of the droplet is, in effect, zero and the radius $r = 1\ mm$ after 20 $s.$, when is the radius $r = 3\ mm$?

Solutions (i):

$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dt}(r^3) = \frac{4}{3}\pi\left(3r^2 \cdot \frac{dr}{dt}\right) = 4\pi r^2 \cdot \frac{dr}{dt}$.

Put $r = 5$ and $\frac{dr}{dt} = 0.2$, $\frac{dV}{dt} = 4\pi \times 5^2 \times 0.2 \approx 62.83$.

Volume V of the balloon is increasing at $\approx 62.83\ cm^3./\ s$ at that instant.

Solutions (ii):

$\frac{dM}{dt} = kS$, where k is a fixed real number (constant).

Also, $M = \frac{4}{3}\pi\rho r^3$, $S = 4\pi r^2$. Note: ρ is a fixed real number (constant).

$$4k\pi r^2 = kS = \frac{dM}{dt} = \frac{4}{3}\pi\rho \cdot \frac{d}{dt}r^3 = \frac{4}{3}\pi\rho\left(3r^2 \cdot \frac{dr}{dt}\right) = 4\pi\rho r^2 \frac{dr}{dt}$$

So, $\frac{dr}{dt} = \frac{4k\pi r^2}{4\pi\rho r^2} = \frac{k}{\rho}$ which is a constant

The radius of the droplet **grows at a constant rate**.

It takes 20 seconds to grow 1 mm.

It will take 60 seconds (that is, 1 minute) to grow to 3 mm.

Differentiability and Vertical Tangent Lines

Example:

Let $f: R \to R$ be defined by $f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$.

Note that:

(i) f is continuous on R.

(ii) $f'(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$.

$f'(x)$ exists for any $x \neq 0$ and doesn't exist at $x = 0$.

(iii) f is NOT differentiable at 0 and f is differentiable on $R\backslash\{0\}$.

(iv) the curve $y = |x|$ has a corner at $x = 0$ and therefore it has NO tangents at such point.

Definition:

Let $f: R \to R$ be a function. Let $a \in R$. Suppose that:

(i) "$f(x) \to f(a)$ as $x \to a^+$ and $|f'(x)| \to +\infty$ as $x \to a^+$" OR

(ii) "$f(x) \to f(a)$ as $x \to a^-$ and $|f'(x)| \to +\infty$ as $x \to a^-$"

We say the curve $y = f(x)$ has **a vertical tangent** at $x = a$.

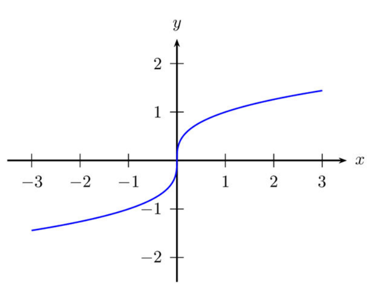
Example:

Let $f: R \to R$ be defined by $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Note: $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$.

The curve $y = f(x)$ has **a vertical tangent** at $x = 0$ (and f is NOT differentiable at $x = 0$).

(i) $f'(x) \to +\infty$ as $x \to 0^+$ and $f'(x) \to +\infty$ as $x \to 0^-$

(ii) $f'(x) \to +\infty$ as $x \to 0$



Exercises

(i) Find the points on the curve $y = f(x) = x \cdot \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$ at which the tangent line is either horizontal or vertical.

(ii) Show that the curve $y = f(x) = 1 - \sqrt[5]{x^2}$ for $-2 \leq x \leq 2$ has a vertical tangent line.

 # Theorem (Differentiability implies Continuity)

Let $f: R \to R$ be a function and let $a \in R$.

If f is differentiable at a, then f is continuous at a.

Proof:

Suppose f is differentiable at a.

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ exists as a real number.

$$\lim_{h \to 0} [f(a + h) - f(a)]$$

$$= \lim_{h \to 0} h \cdot \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= 0 \times f'(a)$$

$$= 0$$

So, f is continuous at a.

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 3: Differentiability and The Derivatives
(Concepts, Definitions, Methods & Applications of Differentiation)

Recall
 # Extreme Value Theorem
(Continuous Functions on a closed and bounded interval)

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x.
Suppose f is continuous on $[a, b]$.
Then, we can find $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$.

 # Definitions:

Let $a, b, m, M \in R$ with $a < b$ and $f: R \to R$ be a function on x. We say:

(i) f attains **the** global minimum value (or **the** absolute minimum value) m on $[a, b]$ if we can find $\alpha \in [a, b]$ such that $m = f(\alpha)$ **AND** $m \leq f(x)$ for any $x \in [a, b]$. In this case, $(\alpha, f(\alpha))$ is called **a** global minima (or **an** absolute minima).

(ii) f attains **the** global maximum value (or **the** absolute maximum value) M on $[a, b]$ if we can find $\beta \in [a, b]$ such that $M = f(\beta)$ **AND** $f(x) \leq M$ for any $x \in [a, b]$. In this case, $(\beta, f(\beta))$ is called **a** global maxima (or **an** absolute maxima).

Remark: "**the**" is used as if global minimum value / global maximum value exists, then it MUST be UNIQUE. However, α and β may not be unique.

Theorem (Corollary of Extreme Value Theorem):

If f is continuous on $[a, b]$, then f MUST attain BOTH the global minimum value AND the global maximum value on $[a, b]$.

Example 1:

Let $f: R \to R$ be a function defined by $f(x) = 2x$ for any $x \in R$.
Show that f attains the global minimum value 0 on $[0,1]$ and the global maximum value 2 on $[0,1]$.

Solutions

$0 \leq x \leq 1 \Longrightarrow 0 \leq 2x \leq 2$. So, $x \in [0,1] \Longrightarrow 0 \leq f(x) \leq 2$.

Also, $0 = f(0)$ and $2 = f(1)$.

Thus, f attains the global minimum value 0 on $[0,1]$ and the global maximum value 2 on $[0,1]$.

Example 2:

Let $g: R \to R$ be a function defined by $g(x) = \begin{cases} \frac{1}{x} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$.

Show that:

(i) g attains the global minimum value 1 on $[0,1]$ and does NOT attain the global maximum value on $[0,1]$.

(ii) g attains the global minimum value $\frac{1}{2}$ on $[1,2]$ and the global maximum value 1 on $[1,2]$.

Solutions (i):

$0 < x \leq 1 \Longrightarrow 1 \leq \frac{1}{x}$. So, $x \in (0,1] \Longrightarrow 1 \leq g(x)$. Also, $g(0) = 1$. Thus, $x \in [0,1] \Longrightarrow 1 \leq g(x)$.

Also, $1 = g(0)$. Thus, g attains the global minimum value 1 on $[0,1]$.

However, $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty$. So, g does NOT attain the global maximum value on $[0,1]$.

Solutions (ii):

$1 \leq x \leq 2 \Longrightarrow \frac{1}{2} \leq \frac{1}{x} \leq 1$. So, $x \in [1,2] \Longrightarrow \frac{1}{2} \leq g(x) \leq 1$. Also, $g(1) = 1$ and $g(2) = \frac{1}{2}$.

Thus, g attains the global minimum value $\frac{1}{2}$ on $[1,2]$ and the global maximum value 1 on $[1,2]$.

 # Definitions:

Let $a, b, c, d \in R$ with $a < c < b$, $a < d < b$ and $f: R \to R$ be a function on x. We say:

(i) f attains **a** local minimum value (or **a** relative minimum value) $f(c)$ on $[a, b]$ if we can find $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ AND $f(c) \leq f(x)$ for any $x \in (c - \delta, c + \delta)$.
In this case, $(c, f(c))$ is called **a** local minima (or **a** relative minima).

(ii) f attains **a** local maximum value (or **a** relative maximum value) $f(d)$ on $[a, b]$ if we can find $\delta > 0$ such that $(d - \delta, d + \delta) \subset [a, b]$ AND $f(x) \leq f(d)$ for any $x \in (d - \delta, d + \delta)$.
In this case, $(d, f(d))$ is called **a** local maxima (or **a** relative maxima).

Remark: "**a**" is used as f MAY attain more than **ONE** local minimum value or MAY attain more than **ONE** local maximum value.

Theorem 1: (The global minimum value at point other than the endpoints $\Longrightarrow$ A local minimum value)

Let $a, b, c \in R$ with $a < c < b$ and $f: R \to R$ be a function on x.

Suppose f attains **the** global minimum value (or **the** absolute minimum value) $f(c)$ on $[a, b]$.

Then, f attains **a** local minimum value (or **a** relative minimum value) $f(c)$ on $[a, b]$.

Theorem 2: The global maximum value at point other than the endpoints $\Longrightarrow$ A local maximum value)

Let $a, b, d \in R$ with $a < d < b$ and $f: R \to R$ be a function on x.

Suppose f attains **the** global maximum value (or **the** absolute maximum value) $f(d)$ on $[a, b]$.

Then, f attains **a** local maximum value (or **a** relative maximum value) $f(d)$ on $[a, b]$.

Definitions:

Let $a, b, E \in R$ with $a < b$ and $f: R \to R$ be a function on x. We say:
f attains **a** global extreme value (or **an** absolute extreme value) E on $[a, b]$ if f attains **the** global minimum value (or **the** absolute minimum value) E on $[a, b]$ **OR** f attains **the** global maximum value (or **the** absolute maximum value) E on $[a, b]$.

Remark: "**a**" is used as f MAY attain more than **ONE** global extreme value.

Suppose $E = f(e)$ for some $e \in [a, b]$. In this case, $(e, f(e))$ is called **a** global extrema (or **an** absolute extrema).

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 3: Differentiability and The Derivatives
(Concepts, Definitions, Methods & Applications of Differentiation)

Definitions:

Let $a, b, e \in R$ with $a < e < b$ and $f: R \to R$ be a function on x. We say:

f attains **a** local extreme value (or **a** relative extreme value) $f(e)$ on $[a, b]$ if f attains **a** local minimum value (or **a** relative minimum value) $f(e)$ on $[a, b]$ **OR** f attains **a** local maximum value (or **a** relative maximum value) $f(e)$ on $[a, b]$.

Remark: "**a**" is used as f MAY attain more than **ONE** local extreme value.

In this case, $(e, f(e))$ is called **a** local extrema (or **a** relative extrema).

Summary of Theorems 1 and 2:

Let $a, b, e \in R$ with $a < e < b$ and $f: R \to R$ be a function on x.

Then, we have $(e, f(e))$ is a global extrema $\Rightarrow$ $(e, f(e))$ is a local extrema

Definition

Let $a, b, c \in R$ with $a < c < b$ and f be a function on $x \in (a, b)$.

We say $(c, f(c))$ is **a** critical point of f on (a, b) if $f'(c) = 0$ OR $f'(c)$ doesn't exist as a real number.

In this case, c is called **a** critical value of f on (a, b).

Example 1:

Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$.

Show that $(0,0)$ is **the** critical point of f on R.

Proof

As $f'(x) = 2x$ for any $x \in R$, it always exists as a real number.

$f'(x) = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$ and $f(0) = 0$.

Thus, $(0,0)$ is **the** critical point of f on R.

Example 2:

Let $f: R \to R$ be defined by $f(x) = |x|$ for any $x \in R$.

Show that $(0,0)$ is **the** critical point of f on R.

Proof

$f'(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$ and $f'(0)$ doesn't exist as a real number.

$f'(x) \neq 0$ if $x \neq 0$.

Thus, $(0,0)$ is **the** critical point of f on R.

Fermat's Theorem

Let $c, \delta \in R$ with $\delta > 0$ and $f: R \to R$ be a function on x.

Suppose that:

(i) f is continuous on $[c - \delta, c + \delta]$ AND

(ii) f is differentiable on $(c - \delta, c + \delta)$ AND

(iii) $\left(c, f(c)\right)$ is a local extrema

Then, $f'(c) = 0$.

(In this case, $\left(c, f(c)\right)$ is a critical point.)

Proof for the case where $\left(c, f(c)\right)$ is a local minima:

Suppose we can find $\rho \in R$ with $\rho > 0$ such that $(c - \rho, c + \rho) \subset (c - \delta, c + \delta)$ AND $f(c) \leq f(x)$ for any $x \in (c - \rho, c + \rho)$.

For $h \in R$ with $0 < h < \rho$, we have $\frac{f(c+h)-f(c)}{h} \geq 0$ as $f(c + h) - f(c) \geq 0$. Hence, $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \geq 0$.

For $h \in R$ with $-\rho < h < 0$, we have $\frac{f(c+h)-f(c)}{h} \leq 0$ as $f(c + h) - f(c) \geq 0$. Hence, $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \leq 0$.

As f is differentiable at c, $f'(c)$ exists as a real number.

So, $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$.

From above, $f'(c) \geq 0$ and $f'(c) \leq 0$. Thus, $f'(c) = 0$.

Proof for the case where $\left(c, f(c)\right)$ is a local maxima: Similar Proof (Omitted)

Example (The converse of Fermat's Theorem is FALSE in general)

Let $f: R \to R$ be defined by $f(x) = x^3$ for any $x \in R$. We can check that:

(i) f is continuous on $[-1,1]$.

(ii) f is differentiable on $(-1,1)$.

(iii) $f'(x) = 3x^2$ for any $x \in (-1,1)$.

(iv) $(0,0)$ is **the** critical point of f on $(-1,1)$.

(v) $(0,0)$ is neither a local minima nor a local maxima

as $f(\alpha) = \alpha^3 < 0 < \beta^3 = f(\beta)$ for any $-1 < \alpha < 0 < \beta < 1$.

Important Corollary 1:

Let $a, b, c \in R$ with $a < c < b$ and $f: R \to R$ be a function on x.

Suppose that:

(i) f is continuous on $[a, b]$ AND

(ii) f is differentiable on (a, b) AND

(iii) $\big(c, f(c)\big)$ is a local extrema

Then, $f'(c) = 0$.

(In this case, $\big(c, f(c)\big)$ is a critical point.)

Important Corollary 2:

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x.

Suppose that:

(i) f is continuous on $[a, b]$ AND

(ii) f is differentiable on (a, b) AND

(iii) we can find some $e \in [a, b]$ such that $\big(e, f(e)\big)$ is a global extrema

Then, $e = a$ OR $e = b$ OR $f'(e) = 0$.

Important Observation:

Let $a, b. m, M \in R$ with $a < b$ and $f: R \to R$ be a function on x.

Suppose that:

(i) f is continuous on $[a, b]$ AND

(ii) f is differentiable on (a, b) AND

(iii) f attains its global minimum value m on $[a, b]$ and its global maximum value M on $[a, b]$.

We consider the set S of all critical values of f on (a, b), that is

$S = \{c \in (a, b): f'(c) = 0\}$.

Let $l = min\{f(c): c \in S\}$ AND $L = max\{f(c): c \in S\}$.

Then, we can show that $m = min\{f(a), f(b), l\}$ AND $M = max\{f(a), f(b), L\}$.

Example 1:

Let $f: R \to R$ be defined by $f(x) = \frac{3}{5}x(30 - x) = 18x - \frac{3}{5}x^2$ for any $x \in R$.

Find the global minimum value and the global maximum value of f on $[0,30]$.

Solutions

(i) f is continuous on $[0,30]$ and is differentiable on $(0,30)$.

(ii) $f(0) = 0 = f(30)$.

(iii) $f'(x) = 18 - \frac{6}{5}x$, $18 - \frac{6}{5}x = 0 \Leftrightarrow x = 15$.

The critical value of f on $(0,30)$ is 15.

$f(15) = \frac{3}{5} \times 15 \times (30 - 15) = 135$.

$(15,135)$ is the critical point of f on $(0,30)$.

(v) The global minimum value of f on $[0,30]$ is $min\{0,0,135\} = 0$.

$(0,0)$ and $(30,0)$ are global minima of f on $[0.30]$.

(vi) The global maximum value of f on $[0,30]$ is $max\{0,0,135\} = 135$.

$(15,135)$ is the global maxima of f on $[0,30]$.

Example 2A:

Find the global minimum value and the global maximum value of $f(x) = 2x^3 - 3x^2 - 12x + 15$ on $[0,3]$.

Solutions

(i) f is continuous on $[0,3]$ and is differentiable on $(0,3)$.

(ii) $f(0) = 15$, $f(3) = 54 - 27 - 36 + 15 = 6$.

(iii) $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$,

$f'(x) = 0 \Leftrightarrow x = 2 \ or - 1$. Note: $-1 \notin (0,3)$.

The critical value of f on $(0,3)$ is 2.

$f(2) = 16 - 12 - 24 + 15 = -5$.

$(2, -5)$ is the critical point of f on $(0,3)$.

(iv) The global minimum value of f on $[0,3]$ is $min\{15,6,-5\} = -5$.

$(2, -5)$ is the global minima of f on $[0,3]$.

(v) The global maximum value of f on $[0,3]$ is $max\{15,6,-5\} = 15$.

$(0,15)$ is the global maxima of f on $[0,3]$.

Example 2B:

Find the global minimum value and the global maximum value of $f(x) = 2x^3 - 3x^2 - 12x + 15$ on $[-2,3]$.

Solutions

(i) f is continuous on $[-2,3]$ and is differentiable on $(-2,3)$.

(ii) $f(-2) = -16 - 12 + 24 + 15 = 11, f(3) = 54 - 27 - 36 + 15 = 6$.

(iii) $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$,

$f'(x) = 0 \Leftrightarrow x = 2\ or - 1$.

The critical values of f on $(-2,3)$ are 2 and -1.

$f(2) = 16 - 12 - 24 + 15 = -5, f(-1) = -2 - 3 + 12 + 15 = 22$.

$(2, -5)$ and $(-1,22)$ are critical points of f on $(-2,3)$.

(iv) The global minimum value of f on $[-2,3]$ is $min\{11,6, -5,22\} = -5$.

$(2, -5)$ is the global minima of f on $[-2,3]$.

(v) The global maximum value of f on $[-2,3]$ is $max\{11,6, -5,22\} = 22$.

$(-1,22)$ is the global maxima of f on $[-2,3]$.

Example 3:

Find the global minimum value and the global maximum value of $f(x) = 3 - |x - 2|$ on $[1,4]$.

Solutions

$f(x) = 3 - |x - 2| = \begin{cases} -x + 5 & if\ x \geq 2 \\ x + 1 & if\ x < 2 \end{cases}$.

(Note: $3 - (x - 2) = -x + 5$ and $3 + (x - 2) = x + 1$)

Let $g(x) = -x + 5$ for any $x \in R$ and $h(x) = x + 1$ for any $x \in R$.

(i) g is continuous on $[2,4]$ and is differentiable on $(2,4)$.

(ii) $g(2) = -2 + 5 = 3$, $g(4) = -4 + 5 = 1$.

(iii) $g'(x) = -1$ for any $x \in (2,4)$. So, $g'(x) \neq 0$ for any $x \in (2,4)$.

g has NO critical values on $(2,4)$.

(iv) The global minimum value of g on $[2,4]$ is $min\{3,1\} = 1$.

(v) The global maximum value of g on $[2,4]$ is $max\{3,1\} = 3$.

(vi) h is continuous on $[1,2]$ and is differentiable on $(1,2)$.

(vii) $h(1) = 1 + 1 = 2$, $h(2) = 2 + 1 = 3$.

(viii) $h'(x) = 1$ for any $x \in (1,2)$. So, $h'(x) \neq 0$ for any $x \in (1,2)$.

h has NO critical values on $(1,2)$.

(ix) The global minimum value of h on $[1,2]$ is $min\{2,3\} = 2$.

(x) The global maximum value of h on $[1,2]$ is $max\{2,3\} = 3$.

The global minimum value of $f(x) = 3 - |x - 2|$ on $[1,4]$ is

$min\{\text{The global minimum value of } h \text{ on } [1,2], \text{The global minimum value of } g \text{ on } [2,4]\} = min\{2,1\} = 1$.

The global maximum value of $f(x) = 3 - |x - 2|$ on $[1,4]$ is

$max\{\text{The global maximum value of } h \text{ on } [1,2], \text{The global maximum value of } g \text{ on } [2,4]\} = max\{3,3\} = 3$.

Note: $f(x) = \begin{cases} g(x) & if\ x \geq 2 \\ h(x) & if\ x < 2 \end{cases}$ and f is continuous on $[1,4]$. However, f is NOT differentiable at 2.

Note:

Sometimes, we write "the maximum value" for "the global maximum value" and "the minimum value" for "the global minimum value" if there are no confusions.

Example 4:

Find the minimum and maximum values of $f(x) = 5x^{2/3} - x^{5/3}$ on $[-1,4]$.

Solution

Let $f(x) = 5x^{2/3} - x^{5/3}$ for any $x \in R$.

f is continuous on R and is differentiable on $R\backslash\{0\}$.

$f'(x) = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3x^{1/3}}(2-x)$ for $x \in R\backslash\{0\}$.

$f'(x) = 0 \Leftrightarrow x = 2$ for $x \in R\backslash\{0\}$ AND $f'(0)$ doesn't exist as a real number.

0 and 2 are critical values of f on $[-1,4]$.

$f(0) = 0 - 0 = 0, f(2) = 5 \times 2^{2/3} - 2^{\frac{5}{3}} = 5(\sqrt[3]{4}) - 2(\sqrt[3]{4}) = 3(\sqrt[3]{4}) \approx 4.762,$

$f(-1) = 5 + 1 = 6, f(4) = 5 \times \sqrt[3]{16} - 4 \times \sqrt[3]{16} = \sqrt[3]{16} \approx 2.520.$

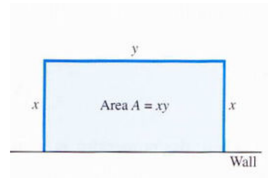
The minimum value of $f(x)$ on $[-1,4]$ is $min\{f(0), f(2), f(-1), f(4)\} = 0$.

The maximum value of $f(x)$ on $[-1,4]$ is $max\{f(0), f(2), f(-1), f(4)\} = 6$.

Applied Optimization Problems

Example 1:

A farmer has 200 yd. of fence with which to construct three sides of a rectangular pen; an existing long, straight wall will form the fourth side. What dimensions will maximize the area of the pen?



Solutions

Let the length be y yd. and the width be x yd. The length of the fence is $y + 2x$ yd.. So, $y + 2x = 200$.

Hence, $y = 200 - 2x$.

The area is $xy = x(200 - 2x) = 200x - 2x^2$ (in $yd.^2$)

Let A be a real-valued function defined as $A(x) = 200x - 2x^2$ for $0 \leq x \leq 100$. (Note: A is the area function of the pen.)

A is continuous on $[0,100]$ and is differentiable on $(0,100)$.

$A(0) = 0 = A(100)$.

$A'(x) = 200 - 4x = 4(50 - x)$.

$A'(x) = 0 \Leftrightarrow x = 50$

$A(50) = 50 \times (200 - 2 \times 50) = 5000$.

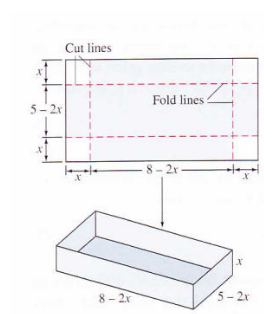
The critical point of A on $(0,100)$ is $(50,5000)$

The maximum value of A on $[0,100]$ is $max\{0,0,5000\} = 5000$ when $x = 50$ (and $y = 100$).

The maximum area is $5000\ yd.^2$ when the dimensions are $50\ yd. \times 100\ yd.$.

Example 2:

A piece of sheet metal is rectangular, 5 ft. wide and 8 ft. long. Congruent squares are to be cut from its four corners. The resulting piece of metal is to be folded and welded to form an open-topped box. How should this be done to get a box of largest possible volume?



Solutions

Let the length of the congruent squares that will be cut from the four corners be x ft.. Let V be a real-valued function defined by
$V(x) = x(8-2x)(5-2x) = 40x - 26x^2 + 4x^3$ for $0 \le x \le \frac{5}{2}$,

that is the volume function of the open-topped box (in $ft.^3$).

(Note: $x \ge 0$ and $2x \le min\{8,5\}$.)

V is continuous on $\left[0, \frac{5}{2}\right]$ and is differentiable on $\left(0, \frac{5}{2}\right)$.

$V'(x) = 40 - 52x + 12x^2 = 4(3x^2 - 13x + 10) = 4(3x-10)(x-1)$

$V'(x) = 0 \Leftrightarrow x = 1 \; or \; \frac{10}{3}$ ($x = \frac{10}{3}$ must be rejected as $0 \le x \le \frac{5}{2}$)

$V(1) = 40 - 26 + 4 = 18$.

The critical point of V on $\left(0, \frac{5}{2}\right)$ is $(1,18)$. Note: $V(0) = 0 = V\left(\frac{5}{2}\right)$.

The maximum value of V on $\left[0, \frac{5}{2}\right]$ is $max\{0,0,18\} = 18$ when $x = 1$.

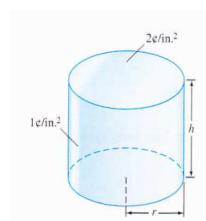
Exercise 1:

Suppose that the cost of publishing a small book is \$10,000 to set up the (annual) press run plus \$8 for each book printed. The publisher sold 7,000 copies last year at \$13 each, but sales dropped to 5,000 copies this year when the prices was raised to \$15 per copy. Assume that up to 10,000 copies can be printed in a single press run. How many copies should be printed, and what should be the selling price of each copy, to maximize the year's profit on this book?

Hint: Assume the unit price \$ p is related to the quantity supplied x as $p = A - Bx$, where A and B are constants.

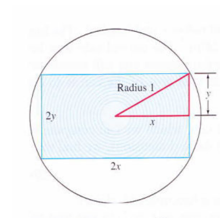
Exercise 2:

We need to design a cylindrical can with radius r in. and height h in.. The top and bottom must be made of copper, which will cost 2 cents/ $in.^2$. The curved side is to be made of aluminum, which will cost 1 cent/ $in.^2$. We seek the dimensions that will maximize the volume of the can. The only constraint is that the total cost of the can is to be 300π cents.



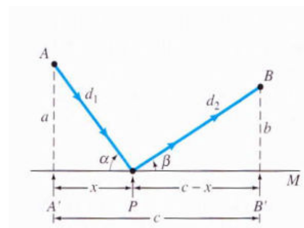
Exercise 3:

Suppose that you need to cut a beam with maximal rectangular cross section from a circular log of radius 1 ft.. (This is the geometric problem of finding the rectangle of greatest area that can be inscribed in a circle of radius 1.) What are the shape and cross-sectional area of such a beam?



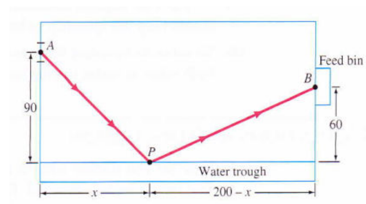
Exercise 4:

We consider the reflection of a ray of light by a mirror M, which shows a ray travelling from point A to point B via reflection of M at a point P. We assume that the location of the point of reflection is such that the total distance travelled by the light ray will be minimized. This is an application of Fermat's principle of least time for the propagation of light. The problem is to find P.



Exercise 5:

Consider a feedlot 200 ft. long with a water trough along one edge and a feed bin located on an adjacent edge, a cow enters the gate at the point A, 90 ft. from the water trough. She walks straight to point P, gets a drink from the trough, and then walks straight to the feed bin at point B, 60 ft. from the trough. If the cow knew Calculus, what point P along the water trough would she select to minimize the total distance she walks?



Derivatives of Trigonometric Functions

Theorem:

(i) $\frac{d}{dx}sinx = cosx$

(ii) $\frac{d}{dx}cosx = -sinx$

(iii) $\frac{d}{dx}tanx = sec^2x$

(iv) $\frac{d}{dx}cotx = -csc^2x$

(v) $\frac{d}{dx}cscx = -cotx \cdot cscx$

(vi) $\frac{d}{dx}secx = tanx \cdot secx$

Note: $tanx = \frac{sinx}{cosx}$, $cscx = \frac{1}{sinx}$, $secx = \frac{1}{cosx}$, $cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$.

Proof for (i):

$$\frac{d}{dx}sinx = \lim_{h\to 0}\frac{sin(x+h) - sinx}{h} = \lim_{h\to 0}\frac{1}{h}\cdot 2cos\frac{2x+h}{2}sin\frac{h}{2}$$

(Use the identity $sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$)

$$= \lim_{h\to 0}\frac{sin\frac{h}{2}}{\frac{h}{2}}\cdot cos\left(x+\frac{h}{2}\right) = 1\cdot cosx = cosx$$

(Note 1: $\lim_{h\to 0}\frac{sin\frac{h}{2}}{\frac{h}{2}} = \lim_{\theta\to 0}\frac{sin\theta}{\theta} = 1$ by letting $\theta = \frac{h}{2}$, $\theta \to 0 \Leftrightarrow h \to 0$)

(Note 2: $\lim_{h\to 0}cos\left(x+\frac{h}{2}\right) = cos(x+0) = cosx$)

Proof for (ii):

$\frac{d}{dx}cosx = \frac{d}{dx}sin\left(\frac{\pi}{2}-x\right)$ (Use the identity $cosx = sin\left(\frac{\pi}{2}-x\right)$.)

$= cos\left(\frac{\pi}{2}-x\right)\cdot\frac{d}{dx}\left(\frac{\pi}{2}-x\right)$ (Use Chain Rule)

$= sinx\cdot(-1)$ (Use the identity $sinx = cos\left(\frac{\pi}{2}-x\right)$.)

$= -sinx$

Proof for (iii):

$$\frac{d}{dx}tanx = \frac{d}{dx}\left(\frac{sinx}{cosx}\right)$$

$= \frac{cosx \cdot \frac{d}{dx}sinx - sinx \cdot \frac{d}{dx}cosx}{cos^2x}$ (Use Quotient Rule)

$$= \frac{cosx \cdot cosx - sinx \cdot (-sinx)}{cos^2x}$$

$$= \frac{cos^2x + sin^2x}{cos^2x}$$

$= \frac{1}{cos^2x}$ (Use the identity $cos^2x + sin^2x = 1$)

$$= sec^2x$$

Proof for (iv):

$$\frac{d}{dx}cotx = \frac{d}{dx}\left(\frac{1}{tanx}\right) = \frac{d}{dx}(tanx)^{-1}$$

$= -(tanx)^{-2} \cdot \frac{d}{dx}tanx$ (Use Chain Rule)

$$= -\frac{1}{tan^2x} \cdot sec^2x$$

$$= -\frac{cos^2x}{sin^2x} \cdot \frac{1}{cos^2x}$$

$$= -\frac{1}{sin^2x}$$

$$= -csc^2x$$

Proof for (v):

$$\frac{d}{dx}cscx = \frac{d}{dx}\left(\frac{1}{sinx}\right) = \frac{d}{dx}(sinx)^{-1}$$

$= -(sinx)^{-2} \cdot \frac{d}{dx}sinx$ (Use Chain Rule)

$$= -\frac{1}{sin^2x} \cdot cosx$$

$$= -\frac{cosx}{sinx} \cdot \frac{1}{sinx}$$

$$= -cotx \cdot cscx$$

Proof for (vi):

$$\frac{d}{dx}secx = \frac{d}{dx}\left(\frac{1}{cosx}\right) = \frac{d}{dx}(cosx)^{-1}$$

$= -(cosx)^{-2} \cdot \frac{d}{dx}cosx$ (Use Chain Rule)

$$= -\frac{1}{cos^2x} \cdot (-sinx)$$

$$= \frac{sinx}{cosx} \cdot \frac{1}{cosx}$$

$$= tanx \cdot secx$$

Examples:

Suppose k is a fixed real number (constant).

(i) $\frac{d}{dx}sinkx = kcoskx$

(ii) $\frac{d}{dx}coskx = -ksinkx$

[Hint: Use Chain Rule]

Examples:

Find:

(i) $\frac{d}{dx}(x^2 sinx)$

(ii) $\frac{d}{dt}cos^3 t$

(iii) $\frac{d}{dx}\left(\frac{cosx}{1-sinx}\right)$

(iv) $\frac{d}{dt}(2-3cost)^{\frac{3}{2}}$

(v) $\frac{d}{dx}(xtanx)$

(vi) $\frac{d}{dt}(cot^3 t)$

(vii) $\frac{d}{dz}\left(\frac{secz}{\sqrt{z}}\right)$

(viii) $\frac{d}{dt}(2sin10t + 3cos\pi t)$

(ix) $\frac{d}{dx}(sin^2 3xcos^4 5x)$

(x) $\frac{d}{dx}cos\sqrt{x}$

(xi) $\frac{d}{dx}sin^2\left[(2x-1)^{3/2}\right]$

(xii) $\frac{d}{dx}tan(2x^3)$

(xiii) $\frac{d}{dt}cot^3 2t$

(xiv) $\frac{d}{dy}sec\sqrt{y}$

(xv) $\frac{d}{dz}\sqrt{cscz}$

Solutions

(i)
$$\frac{d}{dx}(x^2 sinx)$$
$$= x^2 cosx + 2xsinx$$

(ii)
$$\frac{d}{dt}cos^3 t$$
$$= 3cos^2 t \cdot \frac{d}{dt}cost$$
$$= -3cos^2 tsint$$

(iii)
$$\frac{d}{dx}\left(\frac{cosx}{1-sinx}\right)$$
$$= \frac{(1-sinx)\cdot(-sinx) - cosx\cdot(-cosx)}{(1-sinx)^2}$$
$$= \frac{-sinx + sin^2 x + cos^2 x}{(1-sinx)^2}$$
$$= \frac{1-sinx}{(1-sinx)^2}$$
$$= \frac{1}{1-sinx}$$

(iv)
$$\frac{d}{dt}(2-3cost)^{\frac{3}{2}}$$
$$= \frac{3}{2}(2-3cost)^{\frac{1}{2}}\cdot\frac{d}{dt}(2-3cost)$$
$$= \frac{3}{2}\sqrt{2-3cost}\cdot(3sint)$$
$$= \frac{9sint\sqrt{2-3cost}}{2}$$

(v)
$$\frac{d}{dx}(xtanx)$$
$$= xsec^2 x + tanx$$

(vi)
$$\frac{d}{dt}(cot^3 t)$$
$$= 3cot^2 t\cdot\frac{d}{dt}(cott)$$
$$= -3cot^2 t\cdot csc^2 t$$

(vii)
$$\frac{d}{dz}\left(\frac{secz}{\sqrt{z}}\right)$$

(viii)
$$\frac{d}{dt}(2sin10t + 3cos\pi t)$$

$$= \frac{\sqrt{z} \cdot tanzsecz - secz \cdot \frac{1}{2\sqrt{z}}}{\left(\sqrt{z}\right)^2}$$

$$= \frac{secz}{2z^{3/2}}(2ztanz - 1)$$

$$= 20cos10t - 3\pi sin\pi t$$

(ix) $$\frac{d}{dx}(sin^2 3xcos^4 5x)$$

$$= sin^2 3x \cdot 4cos^3 5x \cdot (-sin5x) \cdot 5$$

$$+cos^4 5x \cdot 2sin3x \cdot (cos3x) \cdot 3$$

$$= 2sin3x \cdot cos^3 5x \cdot (-10sin3xsin5x + 3cos3xcos5x)$$

(x) $$\frac{d}{dx}cos\sqrt{x}$$

$$= -sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{-sin\sqrt{x}}{2\sqrt{x}}$$

(xi) $$\frac{d}{dx}sin^2(2x-1)^{\frac{3}{2}}$$

$$= 2sin(2x-1)^{\frac{3}{2}}cos(2x-1)^{\frac{3}{2}} \cdot \frac{3}{2}(2x-1)^{\frac{1}{2}} \cdot 2$$

$$= 6\sqrt{2x-1}sin(2x-1)^{\frac{3}{2}}cos(2x-1)^{\frac{3}{2}}$$

(xii) $$\frac{d}{dx}tan2x^3$$

$$= sec^2(2x^3) \cdot 6x^2$$

$$= 6x^2 sec^2(2x^3)$$

(xiii) $$\frac{d}{dt}cot^3 2t$$

$$= 3cot^2 2t \cdot (-csc^2 2t) \cdot 2$$

$$= -6cot^2 2t \cdot csc^2 2t$$

(xiv) $$\frac{d}{dy}sec\sqrt{y}$$

$$= tan\sqrt{y}sec\sqrt{y} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{tan\sqrt{y}sec\sqrt{y}}{2\sqrt{y}}$$

(xv) $$\frac{d}{dz}\sqrt{cscz}$$

$$= \frac{1}{2\sqrt{cscz}} \cdot (-cotzcscz)$$

$$= \frac{-cotz \cdot \sqrt{cscz}}{2}$$

Example 1:

Write an equation of the line tangent to the curve $y = cos^2 x$ at the point P on the graph where $x = 0.5$.

Solutions

As $y = cos^2 x, \frac{dy}{dx} = 2cosx \cdot (-sinx) = -sin2x$.

The slope of required tangent line is $\left.\frac{dy}{dx}\right|_{x=0.5} = -sin1$.

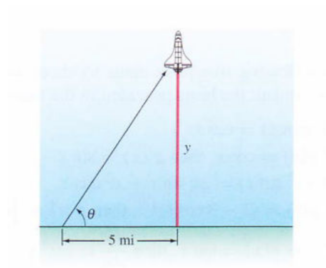
An equation of required tangent line is

$\frac{y - cos^2 0.5}{x - 0.5} = -sin1$

That is, $y = -(sin1)x + cos^2 0.5 + 0.5sin1$.

Example 2:

A rocket is launched vertically and is tracked by a radar station located on the ground 5 mi. from the launch pad. Suppose that the elevation angle θ of the line of sight to the rocket is increasing at 3^0 per second when $\theta = 60^0$. What is the velocity at this instant?



Solutions

As $y = 5tan\theta, \frac{dy}{dt} = 5sec^2\theta \cdot \frac{d\theta}{dt}$.

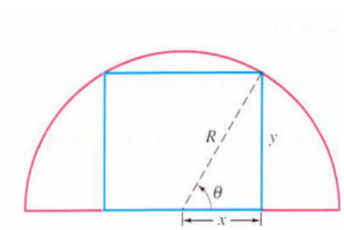
When, $\theta = 60^0 = \frac{\pi}{3}, \frac{d\theta}{dt} = \frac{3}{180}\pi = \frac{\pi}{60}$ $(rad./s.)$.

$\frac{dy}{dt} = 5sec^2\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{60} = 5 \cdot \frac{1}{\left(\frac{1}{2}\right)^2} \cdot \frac{\pi}{60} = \frac{\pi}{3}(mi./s.)$.

The velocity of the rocket at this instant is $\frac{\pi}{3}$ miles per second.

Exercise:

A rectangle is inscribed in a semicircle of radius R. What is the maximum possible area of such a rectangle?



Derivatives of Inverse Trigonometric Functions

Note: If we restrict the domains of the following trigonometric functions, each of them will be bijective.

$sin: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ $\quad$ $cos: [0, \pi] \to [-1,1]$ $\quad$ $tan: \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to R$

We can consider their inverse functions:

$sin^{-1}: [-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ $\quad$ $cos^{-1}: [-1,1] \to [0, \pi]$ $\quad$ $tan^{-1}: R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Theorem:

(i) $\frac{d}{dx} sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ for any $x \in (-1,1)$

(ii) $\frac{d}{dx} cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ for any $x \in (-1,1)$

(iii) $\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2}$ for any $x \in R$

Proof (i)

Let $y = sin^{-1}x$, so $siny = x$.

$cosy \cdot \frac{dy}{dx} = \frac{d}{dx} siny = \frac{d}{dx} x = 1.$

So, $\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

(Assumed $cosy > 0$)

Proof (ii)

Let $y = cos^{-1}x$, so $cosy = x$.

$-siny \cdot \frac{dy}{dx} = \frac{d}{dx} cosy = \frac{d}{dx} x = 1.$

So, $\frac{dy}{dx} = \frac{1}{-siny} = \frac{-1}{\sqrt{1-cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

(Assumed $siny > 0$)

Proof (iii)

Let $y = tan^{-1}x$, so $tany = x$.

$sec^2 y \cdot \frac{dy}{dx} = \frac{d}{dx} tany = \frac{d}{dx} x = 1.$

So, $\frac{dy}{dx} = \frac{1}{sec^2 y} = \frac{1}{1+tan^2 y} = \frac{1}{1+x^2}$

 # Derivatives of Exponential and Logarithmic Functions

Approach 1: (Will be discussed in the next course on Calculus)

Definition 1:

We can define $exp: R \to (0, \infty)$ so that $y(x) = exp(x)$ is the **unique** function that satisfies $\begin{cases} y(0) = 1 \\ \frac{d}{dx} y(x) = y(x) \end{cases}$.

Definition 2:

$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n$.

We can show that e is well defined.

Note:

$e^x = \left[\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n\right]^x = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{nx}$ (n is a positive integer)

$= \lim_{u \to +\infty} \left(1 + \frac{1}{u}\right)^{ux}$ (u is a real number) $= \lim_{v \to +\infty} \left(1 + \frac{x}{v}\right)^v$ (v is a real number)

$= \lim_{m \to +\infty} \left(1 + \frac{x}{m}\right)^m$ (m is a positive integer) $= \lim_{m \to +\infty} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left(\frac{x}{m}\right)^k$

$= 1 + x + \lim_{m \to +\infty} \sum_{k=2}^{m} \left(\prod_{l=1}^{k-1} \frac{m-l}{m}\right) \frac{x^k}{k!}$

$= 1 + x + \lim_{m \to +\infty} \sum_{k=2}^{m} \left(\prod_{l=1}^{k-1} \left(1 - \frac{l}{m}\right)\right) \frac{x^k}{k!}$

Thus, we may consider $1 + x + \lim_{m \to +\infty} \sum_{k=2}^{m} \frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$.

We can define $y: R \to (0, \infty)$ by $y(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$ for any $x \in R$.

Note: $\sum_{k=1}^{\infty} \frac{x^k}{k!} = \lim_{N \to +\infty} \sum_{k=1}^{N} \frac{x^k}{k!}$.

We can show that $y(x)$ is well defined and $\frac{d}{dx}\left(1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}\right) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$ for any $x \in R$. Thus, $y(x)$ satisfies $\begin{cases} y(0) = 1 \\ \frac{d}{dx} y(x) = y(x) \end{cases}$.

Hence, $exp(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$ for any $x \in R$. We write $e^x = exp(x)$.

Thus, $e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$ for any $x \in R$.

Approach 2: (Will be discussed in the next course on Calculus)

Definition:

We can define $ln: (0, \infty) \to R$ so that $y(x) = ln(x)$ is the **unique** function that satisfies $\begin{cases} y(1) = 0 \\ \frac{d}{dx} y(x) = \frac{1}{x} \end{cases}$.

We can also define $y: (0, \infty) \to R$ by $y(x) = \int_1^x \frac{1}{t} dt$. We can show that $y(x)$ satisfies $\begin{cases} y(1) = 0 \\ \frac{d}{dx} y(x) = \frac{1}{x} \end{cases}$

(Note: $\frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}$). Thus, $lnx = \int_1^x \frac{1}{t} dt$ for any $x \in (0, \infty)$.

Recall that $exp: R \to (0, \infty)$ and $ln: (0, \infty) \to R$ are inverse functions to each other. Use Chain Rule, we get $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Example:

(i) Suppose we know $\frac{d}{dx} e^x = e^x$. Show that $\frac{d}{dx} lnx = \frac{1}{x}$.

(ii) Suppose we know $\frac{d}{dx} lnx = \frac{1}{x}$. Show that $\frac{d}{dx} e^x = e^x$.

Therefore, we may use either approach 1 to define exponential function or approach 2 to define logarithmic function to get **BOTH** $\frac{d}{dx} e^x = e^x$ **AND** $\frac{d}{dx} lnx = \frac{1}{x}$.

Notations:

For $a > 0$, we define $a^x = exp(xlna) = e^{xlna}$ for any $x \in R$.

This defines a function from R to $(0, \infty)$.

When $a = e$, this function becomes exp.

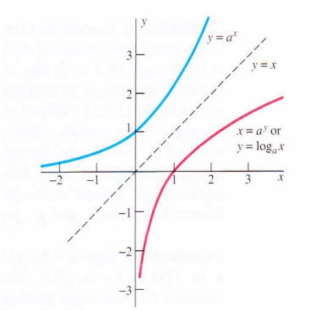
For $a > 0$ and $a \neq 1$, we define $log_a x = \frac{lnx}{lna}$ for any $x > 0$.

This defines a function from $(0, \infty)$ to R.

When $a = e$, this function becomes ln. It is called the **natural logarithmic function**.

When $a = 10$, this function becomes log_{10}. Sometimes, it is written as lg. It is called the **common logarithmic function**.

The above two functions are inverse functions to each other.



Summary:

$e^x = y \Leftrightarrow y = lnx$

$a^x = y \Leftrightarrow y = log_a x$

Important Results:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}lnx = \frac{1}{x}$$

Theorem:

$\frac{d}{dx}a^x = a^x \cdot lna$ (Assumes $a > 0$)

$\frac{d}{dx}log_a x = \frac{1}{xlna}$ (Assumes $a > 0$ and $a \neq 1$)

Proof: Use Chain Rule

Note:

We define $f: R\backslash\{0\} \to R$ by $f(x) = ln|x|$.

Show that $f'(x) = \frac{1}{x}$ for $x \neq 0$.

Proof:

For $x > 0$,

$$f'(x) = \frac{d}{dx}ln|x|$$

$$= \frac{d}{dx}lnx$$

$$= \frac{1}{x}$$

For $x < 0$,

$$f'(x) = \frac{d}{dx}ln|x| = \frac{d}{dx}ln(-x)$$

$= \frac{1}{-x} \cdot \frac{d}{dx}(-x)$ (Use Chain Rule)

$$= \frac{1}{-x} \cdot (-1)$$

$$= \frac{1}{x}$$

Thus, $f'(x) = \frac{1}{x}$ for $x \neq 0$.

Summary:

$\frac{d}{dx}lnx = \frac{1}{x}$ (Assumed $x > 0$)

$\frac{d}{dx}ln|x| = \frac{1}{x}$ (Assumed $x \neq 0$)

Note: the difference of these two domains.

Examples:

Find $\frac{dy}{dx}$ if:

(i) $y = e^{kx}$, where k is a fixed real number (constant) and $k \neq 0$.

(ii) $y = x^2 e^{-x}$

(iii) $y = \dfrac{lnx}{x}$

(iv) $y = ln(1 + x^2)$

(v) $y = \sqrt{1 + lnx}$

(vi) $y = ln\sqrt{\dfrac{2x+3}{4x+5}}$

(vii) $y = \dfrac{\sqrt{(x^2+1)^3}}{\sqrt[3]{(x^3+1)^4}}$

(viii) $y = x^{x+1}$ for $x > 0$

Solutions

(i) $\dfrac{dy}{dx} = ke^{kx}$

(ii) $\dfrac{dy}{dx}$

$= x^2(-e^{-x}) + e^{-x}(2x)$

$= e^{-x}(-x^2 + 2x)$

(iii) $\dfrac{dy}{dx} = \dfrac{x \cdot \frac{1}{x} - lnx \cdot 1}{x^2}$

$= \dfrac{1 - lnx}{x^2}$

(iv) $\dfrac{dy}{dx} = \dfrac{d}{dx} ln(1 + x^2)$

$= \dfrac{1}{1+x^2} \cdot \dfrac{d}{dx}(1 + x^2)$

$= \dfrac{1}{1+x^2} \cdot 2x$

$= \dfrac{2x}{1+x^2}$

(v) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1+lnx}} \cdot \dfrac{d}{dx}(1 + lnx)$

$= \dfrac{1}{2\sqrt{1+lnx}} \cdot \dfrac{1}{x}$

$= \dfrac{1}{2x \cdot \sqrt{1+lnx}}$

(vi) $y = ln\sqrt{\dfrac{2x+3}{4x+5}} = ln\left(\dfrac{2x+3}{4x+5}\right)^{\frac{1}{2}}$

$= \dfrac{1}{2} ln\left(\dfrac{2x+3}{4x+5}\right)$

$= \dfrac{1}{2}\big(ln(2x+3) - ln(4x+5)\big)$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{2}{2x+3} - \dfrac{4}{4x+5}\right)$

$= \dfrac{1}{2x+3} - \dfrac{2}{4x+5}$

(vii)

$$y = \frac{\sqrt{(x^2+1)^3}}{\sqrt[3]{(x^3+1)^4}} = \frac{(x^2+1)^{3/2}}{(x^3+1)^{4/3}}$$

$$lny = ln\frac{(x^2+1)^{\frac{3}{2}}}{(x^3+1)^{\frac{4}{3}}}$$

$$= ln(x^2+1)^{\frac{3}{2}} - ln(x^3+1)^{\frac{4}{3}}$$

$$= \frac{3}{2}ln(x^2+1) - \frac{4}{3}ln(x^3+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}lny$$

$$= \frac{3}{2}\cdot\frac{2x}{x^2+1} - \frac{4}{3}\cdot\frac{3x^2}{x^3+1}$$

$$= \frac{3x}{x^2+1} - \frac{4x^2}{x^3+1}$$

$$\frac{dy}{dx} = y\left(\frac{3x}{x^2+1} - \frac{4x^2}{x^3+1}\right)$$

$$= \frac{(x^2+1)^{3/2}}{(x^3+1)^{4/3}}\left(\frac{3x}{x^2+1} - \frac{4x^2}{x^3+1}\right)$$

(viii)

$$y = x^{x+1}$$

$$lny = ln(x^{x+1}) = (x+1)lnx$$

$$\frac{d}{dx}lny = \frac{d}{dx}(x+1)lnx$$

$$= (x+1)\cdot\frac{1}{x} + lnx\cdot 1$$

$$= 1 + \frac{1}{x} + lnx$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}lny = 1 + \frac{1}{x} + lnx$$

$$\frac{dy}{dx} = y\left(1 + \frac{1}{x} + lnx\right)$$

$$= x^{x+1}\left(1 + \frac{1}{x} + lnx\right)$$

Definition of Logarithmic Differentiation:

Finding $\frac{dy}{dx}$

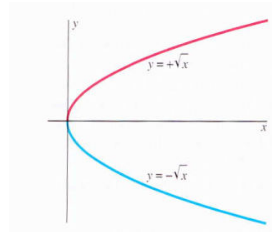
(i) Find lny and $\frac{d}{dx}lny$.

(ii) Use $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}lny$ to find $\frac{dy}{dx}$.

Examples (vii) and (viii)

Implicit Differentiation and Related Rates

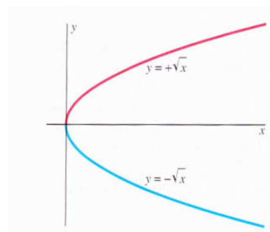
Consider the equation $x - y^2 = 0$, it doesn't define a functional relationship of y on x as both $(x, y) = (1,1)$ and $(x, y) = (1, -1)$ satisfy the equation. (that is, 1 is mapped to both 1 and -1.) However, we can get two explicit functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ from the original equation.



When (x, y) near to a particular point (a, b) (e.g. $(1,1)$), we may regard y as a function on x (e.g. $y = \sqrt{x}$). This is called the implicit function on x when (x, y) is near to a particular point. Then, we can find $\left.\frac{dy}{dx}\right|_{(x,y)=(a,b)}$.

Example 1:

Find $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}$ if $x - y^2 = 0$.



Solutions

Method 1 (Finding Explicit Function)

For the point $(1,1)$ on $x - y^2 = 0$, we consider the upper portion $y = \sqrt{x}$.

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$.

Method 2 (By Implicit Differentiation)

Assume $x - y^2 = 0$ defines y as a function on x when (x, y) is near to the point $(1,1)$.

Both $x - y^2$ and 0 are functions on x when (x, y) is near to the point $(1,1)$ AND they equal as functions, so they MUST have same derivative.

$\frac{d}{dx}(x - y^2) = \frac{d}{dx}0$.

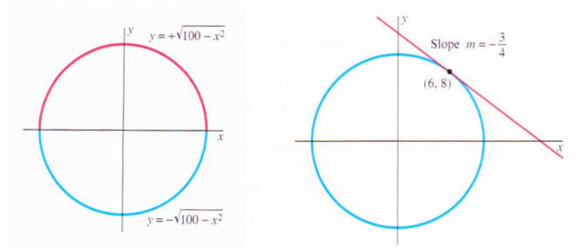
So, $1 - 2y\frac{dy}{dx} = 0$. (Note: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$.)

$\frac{dy}{dx} = \frac{1}{2y}$.

$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{1}{2\times 1} = \frac{1}{2}$.

Example 2:

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 100$.



Solutions

By implicit differentiation, $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}100$. So, $2x + 2y\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$.

Note 1: An equation of the tangent to the circle $x^2 + y^2 = 100$ at $(6,8)$ is

$\frac{y-8}{x-6} = \frac{-3}{4}$ (Note: $\left.\frac{dy}{dx}\right|_{(x,y)=(6,8)} = \frac{-6}{8} = \frac{-3}{4}$).

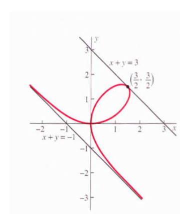
That is, $y = \frac{-3}{4}x + \frac{25}{2}$.

Note 2: $(6,8)$ lies on the upper portion of the circle $x^2 + y^2 = 100$, so it defines $y = \sqrt{100 - x^2}$. We can find the slope of required tangent as follows:

$\frac{dy}{dx} = \frac{1}{2\sqrt{100-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{100-x^2}}$, $\left.\frac{dy}{dx}\right|_{x=6} = \frac{-6}{\sqrt{100-6^2}} = \frac{-3}{4}$.

Example 3:

Find $\left.\frac{dy}{dx}\right|_{(x,y)=\left(\frac{3}{2},\frac{3}{2}\right)}$ if $x^3 + y^3 = 3xy$.



Solutions

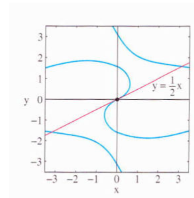
By implicit differentiation, $\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}3xy$.

So, $3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$. Thus, $\frac{dy}{dx} = \frac{3y-3x^2}{3y^2-3x} = \frac{y-x^2}{y^2-x}$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(\frac{3}{2},\frac{3}{2}\right)} = \frac{\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)} = -1$$

Example 4:

Find $\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)}$ if $sin(x+2y) = 2xcosy$.



Solutions

By implicit differentiation, $\frac{d}{dx} sin(x+2y) = \frac{d}{dx} 2xcosy$.

So, $cos(x+2y) \cdot \left(1 + 2\frac{dy}{dx}\right) = 2cosy - 2xsiny\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{2cosy - cos(x+2y)}{2cos(x+2y) + 2xsiny}$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)} = \frac{2-1}{2+0} = \frac{1}{2}$$

Related Rates

Example 1:

Suppose $x(t)$ and $y(t)$ are the $x-$ coordinate and $y-$ coordinate at time t of a particle moving around the circle with equation $x^2+y^2=25$. Given that $x=3$, $y=4$ and $\frac{dx}{dt}=12$ at a particular instant. Find $\frac{dy}{dt}$ at the same instant.

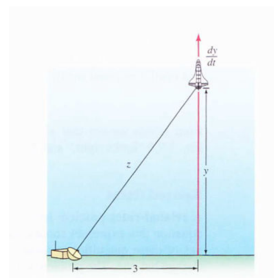
Solutions

As $x^2+y^2=25$, $\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}25$, $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$.

So, $\frac{dy}{dt}=\frac{-2x\frac{dx}{dt}}{2y}=\frac{-x}{y}\cdot\frac{dx}{dt}$. Thus, $\frac{dy}{dt}$ at that instant is $\frac{-3}{4}\cdot 12=-9$.

Example 2:

A rocket that is launched vertically is tracked by a radar station located on the ground 3 mi. from the launch site. What is the vertical speed of the rocket at the instant that its distance from the radar station is 5 mi. and this distance is increasing at the rate of 5000 mi./h.?



Solutions

Let the attitude of the rocket (in miles) be denoted by $y(t)$ and its distance (in miles) from the radar station by $z(t)$.

We have $y^2+3^2=z^2$.

To find $\frac{dy}{dt}$ when $z=5$ and $\frac{dz}{dt}=5000$.

When $z=5$, $y=4$ (as $y^2+3^2=z^2$, $y\geq 0$ and $z\geq 0$).

As $y^2+3^2=z^2$, $\frac{d}{dt}(y^2+3^2)=\frac{d}{dt}z^2$, $2y\frac{dy}{dt}+0=2z\frac{dz}{dt}$.

So, $2\times 4\times\frac{dy}{dt}=2\times 5\times 5000$, $\frac{dy}{dt}=\frac{2\times 5\times 5000}{2\times 4}=6250$ (in mi./h.)

The vertical speed of the rocket at the instant is 6250 miles per hour.

Successive Approximations and Newton's Method

A fundamental problem: Solve $f(x) = 0$.

For example, solve $f(x) = x^5 - 3x^3 + x^2 - 23x + 19 = 0$.

[Hint: Numerical methods will be useful for getting approximate solution(s) (or root(s).]

Example 1: (The Babylonian Iteration Method)

Find $\sqrt{A}$ for $A > 0$.

Solutions

We would like to get a convergent sequence of real numbers $a_1, a_2, a_3, \cdots$ such that $\lim_{n\to+\infty} a_n = \sqrt{A}$.

Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$ for $n = 1,2,3,\cdots$.

Suppose $\lim_{n\to+\infty} a_n$ exists as a real number, say $\lim_{n\to+\infty} a_n = B$.

Claim: $B = \sqrt{A}$.

Reason:

As $a_n > 0$ for $n = 1,2,3,\cdots$, we have $B = \lim_{n\to+\infty} a_n \geq 0$.

Obviously, $B \neq 0$. So $B > 0$.

$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$. So, $B = \frac{1}{2}\left(B + \frac{A}{B}\right)$. $2B = B + \frac{A}{B}$, $B = \frac{A}{B}$, $B^2 = A$.

Thus, $B = \sqrt{A}$ (by taking positive square root).

Remark: This is an algorithm to get an approximate value of a square root.

Find an approximate value of $\sqrt{2}$.

	Correct to 4 decimal places
$a_1 = 1$	1.0000
$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right)$	1.5000
$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) \approx \frac{1}{2}\left(1.5000 + \frac{2}{1.5000}\right)$	1.4167
$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) \approx \frac{1}{2}\left(1.4167 + \frac{2}{1.4167}\right)$	1.4142
$a_5 = \frac{1}{2}\left(a_4 + \frac{2}{a_4}\right) \approx \frac{1}{2}\left(1.4142 + \frac{2}{1.4142}\right)$	1.4142

Claim: $\sqrt{2} = 1.414$ (correct to 3 decimal places).

Questions:

1. How to get such algorithm?
2. Why it works? (For example, why $\lim_{n\to+\infty} a_n$ exists as a real number?)
3. What is the convergence rate? (Number of steps required for a given precision of accuracy)

Definition:

A sequence of real numbers $a_1, a_2, a_3, \cdots$ converges to a real number A if any $\varepsilon > 0$, we can find a positive integer N such that $n \geq N \Longrightarrow |a_n - A| < \varepsilon$. In this case, we write $\lim_{n\to+\infty} a_n = A$.

Example 2 (Newton's Method):

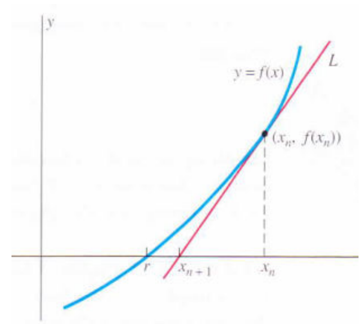
Consider $f(x) = 0$. Let **a** root of $f(x) = 0$ be r. That is, $f(r) = 0$.

Let x_1 be an initial guess of r.

Claim: $\lim_{n\to+\infty} x_n = r$ where $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 1,2,3,\cdots$.

Reason:

Consider the point $P_n(x_n, f(x_n))$ on the graph of the function, then the $x-$ intercept of the tangent at such point will be very close to r if the function is smooth.



An equation of the tangent at $P_n(x_n, f(x_n))$ is

$\frac{y-f(x_n)}{x-x_n} = f'(x_n)$. Its $x-$ intercept is x_{n+1}.

So, $\frac{0-f(x_n)}{x_{n+1}-x_n} = f'(x_n)$. Then, $\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$.

Thus, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

This is called the iterative formula of Newton's Method.

Example 2A:

Use Newton's method to find $\sqrt{2}$ correct to 9 decimal places.

Solutions

Let $f(x) = x^2 - 2$ for any $x \in R$. Then, $f'(x) = 2x$ for any $x \in R$.

Let $x_1 = 1$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 1,2,3,\cdots$.

So, $x_{n+1} = x_n - \frac{x_n{}^2-2}{2x_n} = \frac{1}{2x_n}(2x_n{}^2 - x_n{}^2 + 2) = \frac{1}{2x_n}(x_n{}^2 + 2) = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$.

(Same as the Babylonian Iteration Method)

Find an approximate value of $\sqrt{2}$.

	Correct to 9 decimal places
$x_1 = 1$	1.000000000
$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right)$	1.500000000
$x_3 = \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) \approx \frac{1}{2}\left(1.500000000 + \frac{2}{1.500000000}\right)$	1.416666667
$x_4 = \frac{1}{2}\left(x_3 + \frac{2}{x_3}\right) \approx \frac{1}{2}\left(1.416666667 + \frac{2}{1.416666667}\right)$	1.414215686
$x_5 = \frac{1}{2}\left(x_4 + \frac{2}{x_4}\right) \approx \frac{1}{2}\left(1.414215686 + \frac{2}{1.414215686}\right)$	1.414213562
$x_6 = \frac{1}{2}\left(x_5 + \frac{2}{x_5}\right) \approx \frac{1}{2}\left(1.414213562 + \frac{2}{1.414213562}\right)$	1.414213562

Claim: $\sqrt{2} = 1.414213562$ (correct to 9 decimal places).

Example 2B:

It is known that the equation $x = \frac{1}{2}cosx$ has a solution r near 0.5. Use Newton's Method to find a root correct to 5 decimal places.

Solutions

Let $f(x) = 2x - cosx$ for any $x \in R$. Then, $f'(x) = 2 + sinx$ for any $x \in R$.

Let $x_1 = 0.5$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 1,2,3,\cdots$.

So, $x_{n+1} = x_n - \frac{2x_n - cosx_n}{2+sinx_n}$.

	Correct to 5 decimal places
$x_1 = 0.5$	0.50000
$x_2 = x_1 - \frac{2x_1 - cosx_1}{2 + sinx_1} = 0.5 - \frac{2 \cdot 0.5 - cos0.5}{2 + sin0.5}$	0.45063
$x_3 = x_2 - \frac{2x_2 - cosx_2}{2 + sinx_2} \approx 0.45063 - \frac{2 \cdot 0.45063 - cos0.45063}{2 + sin0.45063}$	0.45018
$x_4 = x_3 - \frac{2x_3 - cosx_3}{2 + sinx_3} \approx 0.45018 - \frac{2 \cdot 0.45018 - cos0.45018}{2 + sin0.45018}$	0.45018

Guess: $r = 0.45018$ (Correct to 5 decimal places).

Example 2C:

It is known that the equation $3sinx = lnx$ has many positive solutions. We can approximate the solution r near 3 by using Newton's Method. Find such approximation correct to 5 decimal places.

Solutions

Let $f(x) = 3sinx - lnx$ for any $x > 0$. Then, $f'(x) = 3cosx - \frac{1}{x}$ for any $x > 0$.

Let $x_1 = 3$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 1,2,3, \cdots$.

So, $x_{n+1} = x_n - \frac{3sinx_n - lnx_n}{3cosx_n - \frac{1}{x_n}}$.

	Correct to 5 decimal places
$x_1 = 3$	3.00000
$x_2 = x_1 - \dfrac{3sinx_1 - lnx_1}{3cosx_1 - \frac{1}{x_1}} = 3 - \dfrac{3sin3 - ln3}{3cos3 - \frac{1}{3}}$	2.79558
$x_3 = x_2 - \dfrac{3sinx_2 - lnx_2}{3cosx_2 - \frac{1}{x_2}} \approx 2.79558 - \dfrac{3sin2.79558 - ln2.79558}{3cos2.79558 - \frac{1}{2.79558}}$	2.79225
$x_4 = x_3 - \dfrac{3sinx_3 - lnx_3}{3cosx_3 - \frac{1}{x_3}} \approx 2.79225 - \dfrac{3sin2.79225 - ln2.79225}{3cos2.79225 - \frac{1}{2.79225}}$	2.79225

Guess: $r = 2.79225$ (Correct to 5 decimal places).

Remark

Suppose $f: R \to R$ is differentiable on R.

Suppose we can find a sequence of real numbers $\{a_n\}$ with $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$ and $\lim_{n\to+\infty} a_n$ exists as a real number, say $\lim_{n\to+\infty} a_n = r$.

Claim: $f(r) = 0$

Reason:

$$r = \lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty}\left(a_n - \frac{f(a_n)}{f'(a_n)}\right) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty}\frac{f(a_n)}{f'(a_n)} = r - \lim_{n\to+\infty}\frac{f(a_n)}{f'(a_n)}$$

So, $\lim_{n\to+\infty}\frac{f(a_n)}{f'(a_n)} = 0$

$\Rightarrow \lim_{n\to+\infty} f(a_n) = 0$

$\Rightarrow f\left(\lim_{n\to+\infty} a_n\right) = 0$ (as f is continuous)

$\Rightarrow f(r) = 0$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 4: Applications of Differentiation

 # Approximations and Differential

Recall:
 # Notation for differential:
Let $f: R \to R$ be a function on x. The derivative of f at x is $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
Suppose $|h|$ is a very small positive number.
Let P be $(x, f(x))$ and Q be $(x+h, f(x+h))$.
We define $dx = \Delta x = h$ and $\Delta y = f(x+h) - f(x)$.
We say $\Delta x =$ change in x and $\Delta y =$ change in y.
We define $dy = \frac{dy}{dx} \cdot dx$. (Note: we can consider $\frac{dy}{dx}$ as $dy \div dx$.)
We say $dx =$ differential change in x and $dy =$ differential change in y.
Note:

(i) $\frac{\Delta y}{\Delta x}$ is the slope of the secant joining the points P and Q.
(ii) $\frac{dy}{dx}$ is the slope of the tangent at the point P if it exists as a real number.

$f'(x) \cdot h = \frac{dy}{dx} dx = dy \approx \Delta y = f(x+h) - f(x)$ when $h \approx 0$.
So, $f(x+h) \approx f(x) + f'(x) \cdot h$ when $h \approx 0$.

We may change the point for consideration:
So, $f(a+h) \approx f(a) + f'(a) \cdot h$ when $h \approx 0$ and f is differentiable at a.
Let $x = a + h$.
$f(x) \approx f(a) + f'(a) \cdot (x - a)$ when $x \approx a$ and f is differentiable at a.

 # Definition:
$f(x) \approx f(a) + f'(a) \cdot (x - a)$ when $x \approx a$ and f is differentiable at a.
We may let $y = L(x)$ and $L(x) = f(a) + f'(a)(x - a)$.
$y = L(x)$ is an equation of a line.
$y = L(x)$ is **the** good approximation of $y = f(x)$ when $x \approx a$.
This is called **the linear approximation** of $y = f(x)$.

Notes:

(i) $y = L(x)$ is an equation of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

(ii) $y = L(x)$ is **the** linear approximation of $y = f(x)$ near to the point $(a, f(a))$.

(iii) $y = L(x)$ is **the** linear approximation of $y = f(x)$ near a. ("near a" means "$x \approx a$")

Example 1:

(a) Find **the** linear approximation of the function $y = f(x) = \sqrt{1+x}$ near $a = 0$.

(b) Hence find an approximation of $\sqrt{101}$.

Solutions
(a)
As $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}}$.
$f(0) = \sqrt{1+0} = 1$ and $f'(0) = \frac{1}{2}(1+0)^{\frac{-1}{2}} = \frac{1}{2}$.
The linear approximation is $y = L(x)$ where $L(x) = f(0) + f'(0)x = 1 + \frac{1}{2}x$.
That is, $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ when $x \approx 0$.
(b)
$$\sqrt{101} = \sqrt{100} \times \sqrt{1.01} = 10\sqrt{1.01} \approx 10\left(1 + \frac{1}{2} \times 0.01\right) = 10 \times 1.005 = 10.05$$
Note: $0.01 \approx 0$.

Remark: $\sqrt[k]{1+x} \approx 1 + \frac{1}{k}x$ when $x \approx 0$ and $k = 2,3,4,\cdots$.
Reason
Let $f(x) = \sqrt[k]{1+x} = (1+x)^{\frac{1}{k}}$, $f'(x) = \frac{1}{k}(1+x)^{\frac{-k+1}{k}}$.
$f(0) = \sqrt[k]{1+0} = 1$ and $f'(0) = \frac{1}{k}(1+0)^{\frac{-k+1}{k}} = \frac{1}{k}$.
The linear approximation is $y = L(x)$ where $L(x) = f(0) + f'(0)x = 1 + \frac{1}{k}x$.
That is, $\sqrt[k]{1+x} \approx 1 + \frac{1}{k}x$ when $x \approx 0$.

Example 2:
Find an approximate value of $(122)^{2/3}$ by using the linear approximation formula.

Solutions
Note 1: $(125)^{2/3} = (5^3)^{2/3} = 5^2 = 25$.
Note 2: $(122)^{2/3} = 25\left(\frac{122}{125}\right)^{2/3} = 25\left(1 - \frac{3}{125}\right)^{2/3}$.
Let $f(x) = (1-x)^{2/3}$ for any $x \in R$.
The linear approximation formula to $f(x)$ when x is near to 0 is $f(0) + f'(0)x$.
$f(0) = 1$, $f'(x) = \frac{2}{3}(1-x)^{-1/3}(-1) = \frac{-2}{3}(1-x)^{-1/3}$, $f'(0) = \frac{-2}{3}$.
So, $\left(1 - \frac{3}{125}\right)^{2/3} \approx 1 + \frac{-2}{3} \times \frac{3}{125} = 1 - \frac{2}{125}$.
$$(122)^{2/3} = 25\left(1 - \frac{3}{125}\right)^{2/3} \approx 25\left(1 - \frac{2}{125}\right) = 25 - \frac{2}{5} = 24.6$$
Note 3: $\frac{3}{125} \approx 0$

Example 3:
A hemispherical bowl of radius 10 inches is filled with water to a depth of x inches. The volume of water in the bowl (in cubic inches) is given by the formula
$V(x) = \frac{\pi}{3}x^2(30 - x) = \pi\left(10x^2 - \frac{1}{3}x^3\right)$.

Suppose that you measure the depth of water in the bowl to be 5 inches with a maximum possible measured error of $\frac{1}{16}$ inch.

Estimate the maximum error in the calculated volume of water in the bowl.

Solutions
$V(x) = \pi\left(10x^2 - \frac{1}{3}x^3\right)$. $V'(x) = \pi(20x - x^2)$.
$dx = \Delta x = \frac{1}{16}$, $\Delta V \approx dV = V'(x)dx = V'(x)\Delta x$.
So, $\Delta V \approx V'(5) \times \frac{1}{16} = \pi(20 \times 5 - 5^2) \times \frac{1}{16} = \frac{75\pi}{16} \approx 14.73$
The maximum error in the calculated volume of water in the bowl is estimated as 14.73 $in.^3$.

Note 1:

$$V(5) = \pi\left(10 \times 5^2 - \frac{1}{3} \times 5^3\right) = \frac{625}{3}\pi$$

% error in the calculated volume of water in the bowl is

$\frac{\Delta V}{V} \times 100\% \approx \frac{\frac{75}{16}\pi}{\frac{625}{3}\pi} \times 100\% \approx 2.25\%$.

Note 2:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\pi(20x - x^2)\Delta x}{\pi\left(10x^2 - \frac{1}{3}x^3\right)} = \frac{20 - x}{10 - \frac{1}{3}x} \cdot \frac{\Delta x}{x}$$

When $x = 5$, $\frac{\Delta V}{V} \approx \frac{20-5}{10-\frac{1}{3}\times 5} \cdot \frac{\Delta x}{x} = 1.8 \cdot \frac{\Delta x}{x}$.

% error in the calculated volume $\approx 1.8 \times$ % error in the measured depth of water

Note 3:

$$\frac{\Delta x}{x} \cdot 100\% = \frac{\frac{1}{16}}{5} \times 100\% = 1.25\%$$

$$\frac{\Delta V}{V} \times 100\% \approx 2.25\% = 1.8 \times 1.25\%$$

Note 4:

We regard $x \approx 5 \pm \frac{1}{16}$ and $V \approx \frac{625}{3}\pi \pm \frac{75\pi}{16}$,

that is, $x \in \left[5 - \frac{1}{16}, 5 + \frac{1}{16}\right]$ and_$V \in \left[\frac{625}{3}\pi - \frac{75\pi}{16}, \frac{625}{3}\pi + \frac{75\pi}{16}\right]$.

Definitions

Absolute error = Actual Value – Approximated Value

Relative error $= \frac{Absolute\ error}{Actual\ Value} \times 100\% = \frac{\text{Actual Value – Approximated Value}}{Actual\ Value} \times 100\%$

Example 3:

$V\left(5+\frac{1}{16}\right) = \pi\left(10 \times 5.0625^2 - \frac{1}{3}5.0625^3\right) \approx 669.2858$

$V\left(5-\frac{1}{16}\right) = \pi\left(10 \times 4.9375^2 - \frac{1}{3}4.9375^3\right) \approx 639.8339$

$V(5) + V'(5) \times \frac{1}{16} = \frac{625}{3}\pi + \frac{75}{16}\pi \approx 669.2247$ which is close to 669.2858

$V(5) - V'(5) \times \frac{1}{16} = \frac{625}{3}\pi - \frac{75}{16}\pi \approx 639.7723$ which is close to 639.8339

$669.2858 - 639.8339 = 29.4519 \approx 29.4524 \approx 2 \times \frac{75}{16}\pi$

$|Absolute\ Error| \approx \frac{75}{16}\pi \approx 14.7262$ (in $in.^3$.)

(Assumed $|Absolute\ Error| \leq \frac{1}{2}(669.2858 - 639.8339)$ as we don't know the actual value)

Relative Error $\approx \frac{14.7262}{654.4985} \times 100\% \approx 2.25\%$

Note 5:

When $x = 5$, $\frac{\Delta V}{V} \approx 1.8 \cdot \frac{\Delta x}{x}$.

(i) Suppose we want % error in the calculated volume $\leq 1.8\%$.
Then, we may regard % error in the measured depth $\leq 1\%$.
That is $\Delta x \approx 0.05$.

(ii) Suppose we want % error in the calculated volume $\leq 0.5\%$.
Then, we may regard % error in the measured depth $\leq 0.2778\%$.
That is $\Delta x \approx 0.0139$.

Remarks on Linear Approximation:

Note:

$L(x) = f(a) + f'(a)(x-a)$ is the linear approximation of $y = f(x)$ near to a.

Consider absolute error $\varepsilon(x) = f(x) - L(x)$,

Note 1: $\lim_{x\to a} \varepsilon(x) = 0$

Note 2:

$$\lim_{x\to a}\frac{\varepsilon(x)}{x-a} = \lim_{x\to a}\frac{f(x)-L(x)}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$$
$$= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) - f'(a)$$
$$= f'(a) - f'(a)$$
$$= 0.$$

So, $f(x) = f(a) + f'(a)(x-a) + R_1(x)$, $\lim_{x\to a} R_1(x) = 0$ and $\lim_{x\to a}\frac{R_1(x)}{x-a} = 0$.

(Note: $R_1(x) = \varepsilon(x)$.)

We may repeat the argument for $\frac{R_1(x)}{x-a}$, it suggests:

$\frac{R_1(x)}{x-a} = 0 + a_2(x-a) + T_2(x)$, $\lim_{x\to a} T_2(x) = 0$ and $\lim_{x\to a}\frac{T_2(x)}{x-a} = 0$, a_2 is a fixed real number.

That is, $f(x) = f(a) + f'(a)(x-a) + a_2(x-a)^2 + R_2(x)$, $\lim_{x\to a} R_2(x) = 0$ and $\lim_{x\to a}\frac{R_2(x)}{(x-a)^2} = 0$.

(Note: $R_2(x) = T_2(x) \cdot (x-a)$.)

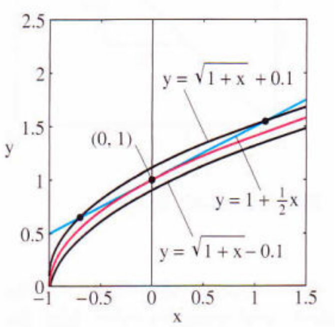
Observation for the general case:
$f(x) = f(a) + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + R_n(x)$
where $\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0$ and $a_1, a_2, \cdots, a_n$ are fixed real numbers.
This is a result of **Taylor's Theorem**.
(Will be discussed in the next course on Calculus)

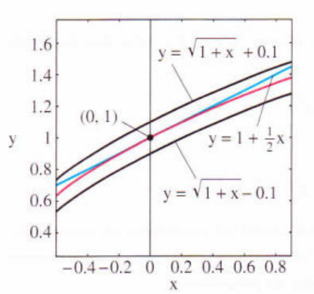
Example 4:
Let $y = f(x) = x^3$ for any $x \in R$.
$\Delta y = (x+\Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3$
$= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$
$dy = f'(x)dx = 3x^2\Delta x$
$\Delta y \approx dy$ as $3x(\Delta x)^2 + (\Delta x)^3 \approx 0$ when $\Delta x \approx 0$

When $x = a = 1$ and $\Delta x = 0.1$,
$dy = 3(0.1) = 0.3$, $\Delta y = 3(0.1) + 3(0.1)^2 + (0.1)^3 = 0.331$.

Example 5:
Find **an** interval on which the approximate $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ is accurate to within 0.1.

Solutions
We want $\left|\sqrt{1+x} - \left(1 + \frac{1}{2}x\right)\right| < 0.1$.
That is, $-0.1 < \sqrt{1+x} - \left(1 + \frac{1}{2}x\right) < 0.1$.
$\sqrt{1+x} - 0.1 < 1 + \frac{1}{2}x < \sqrt{1+x} + 0.1$
$\sqrt{1+x} - 1.1 < \frac{1}{2}x < \sqrt{1+x} - 0.9$
$2 \cdot \sqrt{1+x} - 2.2 < x < 2 \cdot \sqrt{1+x} - 1.8$
Trying $x = 0.1, 0.2, 0.3, \cdots$ and $x = -0.1, -0.2, -0.3, \cdots$, we can find out
$x \in (-0.6, 0.9) \Longrightarrow 2 \cdot \sqrt{1+x} - 2.2 < x < 2 \cdot \sqrt{1+x} - 1.8$
So, we may choose the interval $(-0.6, 0.9)$.





GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 4: Applications of Differentiation

 # Differential
Let $y = f(x)$ and f is differentiable for concerned x.
We call dx the differential of x and dy the differential of y where we define $dy = f'(x)dx$.

Suppose u, v are functions on x and u, v are differentiable for concerned x.

Product Rule:
$d(uv) = udv + vdu$

$$\frac{d(uv)}{uv} = \frac{udv + vdu}{uv} = \frac{dv}{v} + \frac{du}{u}$$

Let $A = uv$.

% error of $A = \frac{\Delta A}{A} \times 100\%$
$\approx \frac{dA}{A} \times 100\% = \frac{du}{u} \times 100\% + \frac{dv}{v} \times 100\%$
$\approx \frac{\Delta u}{u} \times 100\% + \frac{\Delta v}{v} \times 100\%$
$=$ % error of $u +$ % error of v

A generalization:
Suppose u, v, w are functions on x and u, v, w are differentiable for concerned x.
Let $V = uvw$.
Then, % error of $V \approx$ % error of $u +$ % error of $v +$ % error of w.

Example:
Find dy if:
(i) $y = 3x^2 - 2x^{3/2}$
(ii) $y = sin^2 t - cos2t$
(iii) $y = ze^z$

Solutions (i)
As $y = 3x^2 - 2x^{3/2}$, $\frac{dy}{dx} = 6x - 3x^{1/2}$, $dy = \left(6x - 3x^{1/2}\right)dx$.

Solutions (ii)
As $y = sin^2 t - cos2t$, $\frac{dy}{dt} = 2sintcost + 2sin2t = 3sin2t$, $dy = 3sin2tdt$.

Solutions (iii)
As $y = ze^z$, $\frac{dy}{dz} = ze^z + e^z = (z+1)e^z$, $dy = (z+1)e^z dz$.

Strictly Increasing and Strictly Decreasing Functions

Definitions:

Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$. We say:

(i) f is **strictly increasing** on $[a, b]$
if for any $\alpha, \beta \in [a, b]$ with $\alpha < \beta \Longrightarrow f(\alpha) < f(\beta)$.

(ii) f is **strictly decreasing** on $[a, b]$
if for any $\alpha, \beta \in [a, b]$ with $\alpha < \beta \Longrightarrow f(\alpha) > f(\beta)$.

(iii) f is **monotonic increasing** on $[a, b]$
if for any $\alpha, \beta \in [a, b]$ with $\alpha < \beta \Longrightarrow f(\alpha) \leq f(\beta)$.

(iv) f is **monotonic decreasing** on $[a, b]$
if for any $\alpha, \beta \in [a, b]$ with $\alpha < \beta \Longrightarrow f(\alpha) \geq f(\beta)$.

Corollary

(i) If f is **strictly increasing** on $[a, b]$, then f is **monotonic increasing** on $[a, b]$.

(ii) If f is **strictly decreasing** on $[a, b]$, then f is **monotonic decreasing** on $[a, b]$.

Definitions:

Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$. We say:

(i) f is **strictly increasing** on (a, b)
if for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta \Longrightarrow f(\alpha) < f(\beta)$.

(ii) f is **strictly decreasing** on (a, b)
if for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta \Longrightarrow f(\alpha) > f(\beta)$.

(iii) f is **monotonic increasing** on (a, b)
if for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta \Longrightarrow f(\alpha) \leq f(\beta)$.

(iv) f is **monotonic decreasing** on (a, b)
if for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta \Longrightarrow f(\alpha) \geq f(\beta)$.

Corollary

(i) If f is **strictly increasing** on (a, b), then f is **monotonic increasing** on (a, b).

(ii) If f is **strictly decreasing** on (a, b), then f is **monotonic decreasing** on (a, b).

Remark:

Let C be a fixed real number (constant).
Let $f: R \to R$ be defined by $f(x) = C$ for any $x \in R$.
Let $a, b \in R$ with $a < b$.
Then, f is BOTH **monotonic increasing** on $[a, b]$ AND **monotonic decreasing** on $[a, b]$.
Also, f is neither **strictly increasing** on $[a, b]$ nor **strictly decreasing** on $[a, b]$.

Example:

Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$.
Then,

(i) f is strictly increasing on $(0, \infty)$.
(ii) f is strictly decreasing on $(-\infty, 0)$.
(iii) f is neither strictly increasing on R nor strictly decreasing on R.

Proof (i)

For any $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$, we have $\alpha - \beta < 0$ and $\alpha + \beta > 0$.

So, $f(\alpha) - f(\beta) = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) < 0$.

$f(\alpha) < f(\beta)$.

Thus, f is strictly increasing on $(0, \infty)$.

Proof (ii)

For any $\alpha, \beta \in (-\infty, 0)$ with $\alpha < \beta$, we have $\alpha - \beta < 0$ and $\alpha + \beta < 0$.

So, $f(\alpha) - f(\beta) = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) > 0$.

$f(\alpha) > f(\beta)$.

Thus, f is strictly decreasing on $(0, \infty)$.

Proof (iii)

For any $\alpha \in (0, \infty)$ we have $-2\alpha < -\alpha < \alpha < 2\alpha$.

$f(-\alpha) - f(2\alpha) = (-\alpha)^2 - (2\alpha)^2 = \alpha^2 - 4\alpha^2 = -3\alpha^2 < 0$, so f is NOT strictly decreasing on R.

$f(-2\alpha) - f(\alpha) = (-2\alpha)^2 - \alpha^2 = 4\alpha^2 - \alpha^2 = 3\alpha^2 > 0$, so f is NOT strictly increasing on R.

Thus, f is neither strictly increasing on R nor strictly decreasing on R.

 # Rolle's Theorem

Let f be a real-valued function on x.

Let $a, b \in R$ with $a < b$.

Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).

If $f(a) = f(b)$, then we can find $c \in (a, b)$ such that $f'(c) = 0$.

Proof:

As f is continuous on $[a, b]$, we can find $\alpha, \beta \in [a, b]$ such that

$f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$.

Case 1: $f(\alpha) = f(\beta)$

Then, $f(x) = C$ for any $x \in [a, b]$ where $C = f(\alpha) = f(\beta)$ is a constant.

Let $c = \frac{a+b}{2} \in (a, b)$, we can easily check that $f'(c) = 0$.

Case 2: $f(\alpha) < f(\beta)$

Case 2.1: $f(a) = f(b) = f(\alpha)$

Then, $\beta \in (a, b)$ and $\big(\beta, f(\beta)\big)$ is a global maxima.

Thus, $\big(\beta, f(\beta)\big)$ is a local maxima.

By Fermat's Theorem, $f'(\beta) = 0$.

Choose $c = \beta$, we have $c \in (a, b)$ and $f'(c) = 0$.

Case 2.2: $f(a) = f(b) \neq f(\alpha)$

Then, $\alpha \in (a, b)$ and $\big(\alpha, f(\alpha)\big)$ is a global minima.

Thus, $\big(\alpha, f(\alpha)\big)$ is a local minima.

By Fermat's Theorem, $f'(\alpha) = 0$.

Choose $c = \alpha$, we have $c \in (a, b)$ and $f'(c) = 0$.

Example 1:

Let $f: \{x \in R: x \geq 0\} \to R$ be defined by $f(x) = x^{1/2} - x^{3/2}$. Note: $x \geq 0$.

We can check that:

(i) $f(0) = 0 = f(1)$

(ii) $f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}(1 - 3x)$

(iii) f is continuous on $[0,1]$ and is differentiable on $(0,1)$

Note: choose $c = \frac{1}{3}$, then $c \in (0,1)$ and $f'(c) = 0$.

Example 2:

Let $f: R \to R$ be defined by $f(x) = 1 - x^{2/3}$.

We can check that:

(i) $f(-1) = f(1)$

(ii) $f'(x) = -\frac{2}{3}x^{\frac{-1}{3}} = \frac{-2}{3 \cdot \sqrt[3]{x}}$

(iii) f is continuous on $[-1,1]$ and is differentiable on $(-1,1)\backslash\{0\}$

(iv) If $c \in (0,1)$, $f'(c) = \frac{-2}{3 \cdot \sqrt[3]{c}} < 0$

(v) If $c \in (-1,0)$, $f'(c) = \frac{-2}{3 \cdot \sqrt[3]{c}} > 0$

(vi) $f'(0)$ doesn't exist as a real number

Thus, we CANNOT find $c \in (-1,1)$ such that $f'(c) = 0$.

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Mean Value Theorem

Let f be a real-valued function on x.

Let $a, b \in R$ with $a < b$.

Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).

Then we can find $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$ or $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof:

Let g be a real-valued function on x and $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$.

As f is continuous on $[a, b]$ and is differentiable on (a, b), we have:

g is continuous on $[a, b]$ and is differentiable on (a, b).

$g(a) = 0 = g(b)$.

By Rolle's Theorem, we can find $c \in (a, b)$ such that $g'(c) = 0$.

As $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ and $g'(c) = 0$, we have $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$.

That is, $f(b) - f(a) = f'(c)(b - a)$ or $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary 1:

Let f be a real-valued function on x.

Let $a, b \in R$ with $a < b$.

Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).

If $f'(x) = 0$ for any $x \in (a, b)$, then we can find a fixed real number C (constant) such that $f(x) = C$ for any $x \in [a, b]$.

Corollary 2 (Important Result):

Let f, g be real-valued functions on x.

Let $a, b \in R$ with $a < b$.

Suppose **BOTH** f and g are continuous on $[a, b]$ **AND** are differentiable on (a, b).

If $f'(x) = g'(x)$ for any $x \in (a, b)$, then we can find a fixed real number C (constant) such that $f(x) = g(x) + C$ for any $x \in [a, b]$.

Proof for Corollary 1:

For any $\alpha, \beta \in (a, b)$ with $\alpha < \beta$.

We have: f is continuous on $[\alpha, \beta]$ and is differentiable on (α, β).

By Mean Value Theorem, we can find $\gamma \in (\alpha, \beta)$ such that $f(\beta) - f(\alpha) = f'(\gamma)(\beta - \alpha) = 0$ as $f'(\gamma) = 0$.

Note: $\gamma \in (\alpha, \beta) \subset (a, b)$.

Thus, $f(\beta) = f(\alpha)$.

So, choose $C = f\left(\frac{a+b}{2}\right)$. Then, $f(x) = C$ for any $x \in (a, b)$.

By continuity, $f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^+} C = C$ and $f(b) = \lim_{x \to b^-} f(x) = \lim_{x \to b^-} C = C$.

Thus, $f(x) = C$ for any $x \in [a, b]$.

Proof for Corollary 2:

Let $h(x) = f(x) - g(x)$ for any $x \in [a, b]$.

Then, use Corollary 1 to get the result.

Example 1:

Let $f: R \to R$ be a function on x.

Suppose $f'(x) = 6e^{2x}$ for any $x \in R$ and $f(0) = 7$.

Determine f.

Solutions

$f'(x) = 6e^{2x} = \frac{d}{dx}(3e^{2x})$ for any $x \in R$.

By Corollary 2, we can find a fixed real number C (constant) such that $f(x) = 3e^{2x} + C$ for any $x \in R$.

Put $x = 0$, $7 = f(0) = 3 + C$, so $C = 4$.

Thus, $f(x) = 3e^{2x} + 4$ for any $x \in R$.

Corollary 4 (Important Result):
Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$.
Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).
Then we have:

(i) If $f'(x) > 0$ for any $x \in (a, b)$, then f is **strictly increasing** on $[a, b]$.

(ii) If $f'(x) < 0$ for any $x \in (a, b)$, then f is **strictly decreasing** on $[a, b]$.

(iii) If $f'(x) \geq 0$ for any $x \in (a, b)$, then f is **monotonic increasing** on $[a, b]$.

(iv) If $f'(x) \leq 0$ for any $x \in (a, b)$, then f is **monotonic decreasing** on $[a, b]$.

Proof: Omitted (As Exercises)

Example 1:

Show that the equation $e^x + x - 2 = 0$ has exactly one real solution.

Solutions

Let $f: R \to R$ be the function defined by $f(x) = e^x + x - 2$ for any $x \in R$.
Then, f is differentiable on R and $f'(x) = e^x + 1 > 0$ for any $x \in R$.
Thus, f is strictly increasing on R. $f(x) = 0$ has **at most one** real solution.
As $f(0) = 1 + 0 - 2 = -1 < 0, f(1) = e + 1 - 2 = e - 1 > 0$ and f is continuous on R, by Intermediate Value Theorem, $f(x) = 0$ has **at least one** real solution.
Therefore, $f(x) = 0$ has **exactly one** real solution. That is, the equation $e^x + x - 2 = 0$ has **exactly one** real solution.

Example 2:

Determine the interval(s) on the $x-$ axis on which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is strictly increasing and those on which it is strictly decreasing.

Solutions

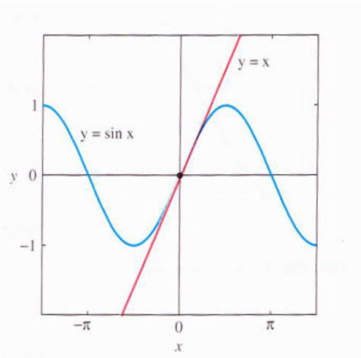
$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1).$
$f'(x) = 0 \Leftrightarrow x(x - 2)(x + 1) = 0 \Leftrightarrow x = 0 \text{ or } 2 \text{ or } -1$

	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$
x	$-ve$	$-ve$	$+ve$	$+ve$
$x - 2$	$-ve$	$-ve$	$-ve$	$+ve$
$x + 1$	$-ve$	$+ve$	$+ve$	$+ve$
$f'(x)$	$-ve$	$+ve$	$-ve$	$+ve$
$f(x)$	strictly decreasing	strictly increasing	strictly decreasing	strictly increasing

The intervals on which f is strictly increasing are $(-1,0)$ and $(2, \infty)$.
The intervals on which f is strictly decreasing are $(-\infty, -1)$ and $(0,2)$.

Example 3:

Show that $sinx < x$ for any $x > 0$.



Solutions

Let $f: R \to R$ be the function defined by $f(x) = sinx - x$ for any $x \in R$.
f is differentiable on R and $f'(x) = cosx - 1$ for any $x \in R$.
$f'(x) = cosx - 1 < 0$ for $x \in (0, \infty)\backslash\{2n\pi : n = 1,2,3, \cdots\}$.
f is strictly decreasing on $[2n\pi, 2(n+1)\pi]$ for $n = 0,1,2,3, \cdots$.
$f(0) = 0, f(2n\pi) = -2n\pi < 0$ for $n = 1,2,3, \cdots$.
Thus, $f(x) < 0$ for any $x > 0$.
That is, $sinx < x$ for any $x > 0$.

 # The First Derivative Test and Applications

Theorems:

Let f be a real-valued function on x. Let $a, b, c, \delta \in R$ with $a < b$ and $\delta > 0$.
Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).
Suppose $(c - \delta, c + \delta) \subset (a, b)$. Then,

(i) If $f'(x) < 0$ for any $x \in (c - \delta, c)$ AND $f'(x) > 0$ for any $x \in (c, c + \delta)$,
Then $\big(c, f(c)\big)$ is **a** local minima.

(ii) If $f'(x) > 0$ for any $x \in (c - \delta, c)$ AND $f'(x) < 0$ for any $x \in (c, c + \delta)$,
Then $\big(c, f(c)\big)$ is **a** local maxima.

(iii) If $f'(x) < 0$ for any $x \in (c - \delta, c)$ AND $f'(x) < 0$ for any $x \in (c, c + \delta)$,
Then $\big(c, f(c)\big)$ is neither a local minima nor a local maxima.

(iv) If $f'(x) > 0$ for any $x \in (c - \delta, c)$ AND $f'(x) > 0$ for any $x \in (c, c + \delta)$,
Then $\big(c, f(c)\big)$ is neither a local minima nor a local maxima.

Proof: Omitted (As exercises)

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 # Classification of Critical Points / Finding Global Extreme Values

Example 1:

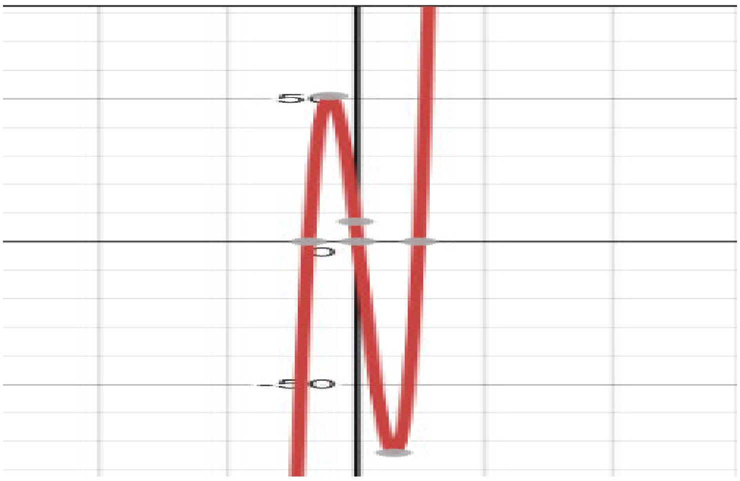
Find and classify the critical point(s) of the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ on R.

Solutions

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$
$f'(x) = 0 \Leftrightarrow (x-3)(x+2) = 0 \Leftrightarrow x = 3\ or -2$
$f(3) = 54 - 27 - 108 + 7 = -74$
$f(-2) = -16 - 12 + 72 + 7 = 51$
The critical points of f on R are $(3, -74)$ and $(-2, 51)$.

	$x < -2$	$-2 < x < 3$	$x > 3$
$x - 3$	$-ve$	$-ve$	$+ve$
$x + 2$	$-ve$	$+ve$	$+ve$
$f'(x)$	$+ve$	$-ve$	$+ve$
$f(x)$	strictly increasing	strictly decreasing	strictly increasing

The critical point $(3, -74)$ is a local minima.
The critical point $(-2, 51)$ is a local maxima.
The graph of $y = 2x^3 - 3x^2 - 36x + 7$ is:



Example 2:

Determine the global maximum value (or absolute maximum value) of $f(x) = \frac{2lnx}{x}$ for $x > 0$.

Solutions

$f'(x) = \frac{2}{x^2}\left(x \cdot \frac{1}{x} - lnx \cdot 1\right) = \frac{2(1-lnx)}{x^2}$ for $x > 0$.

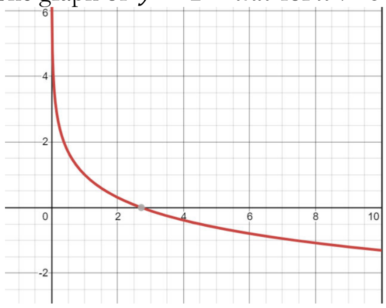
$f'(x) = 0 \Leftrightarrow 1 - lnx = 0 \Leftrightarrow lnx = 1 \Leftrightarrow x = e$

$f(e) = \frac{2lne}{e} = \frac{2}{e}$.

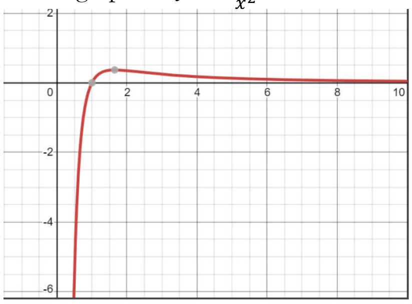
The critical point of f on $(0, \infty)$ is $\left(e, \frac{2}{e}\right)$.

	$0 < x < e$	$x > e$
$1 - lnx$	$+ve$	$-ve$
$f'(x)$	$+ve$	$-ve$
$f(x)$	strictly increasing	strictly decreasing

The graph of $y = 1 - lnx$ for $x > 0$ is:



The graph of $y = \frac{2lnx}{x^2}$ for $x > 0$ is:



Thus, the critical point $\left(e, \frac{2}{e}\right)$ is the global maxima of f on $(0, \infty)$.

The global maximum value of f on $(0, \infty)$ is $\frac{2}{e}$.

Example 3:

Determine the global minimum value (or absolute minimum value) of $f(x) = x + \frac{4}{x}$ for $x > 0$.

Solutions

$f(x) = x + 4x^{-1}$ for $x > 0$.

$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$ for $x > 0$.

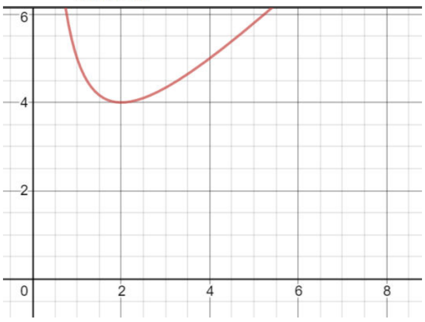
$f'(x) = 0 \Leftrightarrow (x-2)(x+2) = 0 \Leftrightarrow x = 2\ or - 2$.

$f(2) = 2 + \frac{4}{2} = 4$. Note: We don't need to consider $x = -2$ as $x > 0$.

The critical point of f on $(0, \infty)$ is $(2,4)$.

	$0 < x < 2$	$x > 2$
$x - 2$	$-ve$	$+ve$
$x + 2$	$+ve$	$+ve$
$f'(x)$	$-ve$	$+ve$
$f(x)$	strictly decreasing	strictly increasing

The graph of $y = x + \frac{4}{x}$ for $x > 0$ is:



Thus, the critical point $(2,4)$ is the global minima of f on $(0, \infty)$.

The global minimum value of f on $(0, \infty)$ is 4.

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Theorems:
Let f be a real-valued function on x. Let $a, c \in R$ with $a < c$.
Suppose f is differentiable on (a, ∞).
Then,
(i) If $f'(x) < 0$ for any $x \in (a, c)$ AND $f'(x) > 0$ for any $x \in (c, \infty)$,
Then $\big(c, f(c)\big)$ is **the** global minima.

(ii) If $f'(x) > 0$ for any $x \in (a, c)$ AND $f'(x) < 0$ for any $x \in (c, \infty)$,
Then $\big(c, f(c)\big)$ is **the** global maxima.

Theorems:
Let f be a real-valued function on x. Let $c, b \in R$ with $c < b$.
Suppose f is differentiable on $(-\infty, b)$.
Then,
(i) If $f'(x) < 0$ for any $x \in (-\infty, c)$ AND $f'(x) > 0$ for any $x \in (c, b)$,
Then $\big(c, f(c)\big)$ is **the** global minima.

(ii) If $f'(x) > 0$ for any $x \in (-\infty, c)$ AND $f'(x) < 0$ for any $x \in (c, b)$,
Then $\big(c, f(c)\big)$ is **the** global maxima.

Theorems:
Let f be a real-valued function on x. Let $a, c, b \in R$ with $a < c < b$.
Suppose f is differentiable on (a, b).
Then,
(i) If $f'(x) < 0$ for any $x \in (a, c)$ AND $f'(x) > 0$ for any $x \in (c, b)$,
Then $\big(c, f(c)\big)$ is **the** global minima.

(ii) If $f'(x) > 0$ for any $x \in (a, c)$ AND $f'(x) < 0$ for any $x \in (c, b)$,
Then $\big(c, f(c)\big)$ is **the** global maxima.

Proofs: Omitted (As Exercises)

 # Simple Curve Sketching (Version 1)
(i) Find and classify the critical point(s).
(ii) Find the interval(s) on which the function is strictly increasing or strictly decreasing.
(iii) Consider the behaviour of the function when $x \to +\infty$ and $x \to -\infty$
(iv) Find $x -$ intercept(s) and $y -$ intercept.

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Lecture Notes for Chapter 4: Applications of Differentiation

Example 1:

Sketch the graph of $f(x) = x^3 - 27x$ on R.

Solutions

Step 1: Finding the coordinates of the critical point(s)

$f'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x-3)(x+3)$
$f'(x) = 0 \Leftrightarrow (x-3)(x+3) = 0 \Leftrightarrow x = 3 \; or -3$
$f(3) = 27 - 81 = -54, f(-3) = -27 + 81 = 54$
The critical points of f on R are $(3, -54)$ and $(-3, 54)$.

Step 2: Finding the interval(s) on which the function is strictly increasing or strictly decreasing

	$x < -3$	$-3 < x < 3$	$x > 3$
$x - 3$	$-ve$	$-ve$	$+ve$
$x + 3$	$-ve$	$+ve$	$+ve$
$f'(x)$	$+ve$	$-ve$	$+ve$
$f(x)$	strictly increasing	strictly decreasing	strictly increasing

The intervals where f is strictly increasing are $(-\infty, -3)$ and $(3, \infty)$.
The interval where f is strictly decreasing is $(-3, 3)$.

Step 3: Classify Critical Point(s)

The critical point $(3, -54)$ is a local minima.
The critical point $(-3, 54)$ is a local maxima.

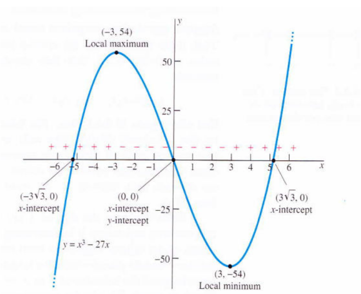
Step 4: Consider the behaviour of the function at infinity

$$\lim_{x \to +\infty} (x^3 - 27x) = +\infty$$
$$\lim_{x \to -\infty} (x^3 - 27x) = -\infty$$

Step 5: Find x – intercept(s) and y – intercept

$y = f(x) = x^3 - 27x$
Put $x = 0$, $y = 0$, $y -$ intercept is 0.
Put $y = 0$, $0 = x^3 - 27x = x(x - \sqrt{27})(x + \sqrt{27})$, $x -$ intercepts are 0, $-\sqrt{27}$ and $\sqrt{27}$.

Step 6: The sketch of the graph



Example 2:

Sketch the graph of $f(x) = 8x^5 - 5x^4 - 20x^3$ on R.

Solutions

Step 1: Finding the coordinates of the critical point(s)

$f'(x) = 40x^4 - 20x^3 - 60x^2 = 20x^2(2x^2 - x - 3) = 20x^2(2x - 3)(x + 1)$

$f'(x) = 0 \Leftrightarrow x^2(2x - 3)(x + 1) = 0 \Leftrightarrow x = 0 \; or \; \frac{3}{2} \; or - 1$

$f(0) = 0 - 0 - 0 = 0,$

$f\left(\frac{3}{2}\right) = \frac{243}{4} - \frac{405}{16} - \frac{135}{2} = \frac{-513}{16} = -32.0625,$

$f(-1) = -8 - 5 + 20 = 7.$

The critical points of f on R are $(0,0)$, $\left(\frac{3}{2}, \frac{-513}{16}\right)$ and $(-1,7)$.

Step 2: Finding the interval(s) on which the function is strictly increasing or strictly decreasing

	$x < -1$	$-1 < x < 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$2x - 3$	$-ve$	$-ve$	$-ve$	$+ve$
$x + 1$	$-ve$	$+ve$	$+ve$	$+ve$
$f'(x)$	$+ve$	$-ve$	$-ve$	$+ve$
$f(x)$	strictly increasing	strictly decreasing	strictly decreasing	strictly increasing

The intervals where f is strictly increasing are $(-\infty, -1)$ and $\left(\frac{3}{2}, \infty\right)$.

The intervals where f is strictly decreasing are $(-1,0)$ and $\left(0, \frac{3}{2}\right)$.

Step 3: Classify Critical Point(s)

The critical point $(-1,7)$ is a local maxima.

The critical point $\left(\frac{3}{2}, \frac{-513}{16}\right)$ is a local minima.

The critical point $(0,0)$ is neither a local maxima nor a local minima.

Step 4: Consider the behaviour of the function at infinity

$\lim_{x \to +\infty} (8x^5 - 5x^4 - 20x^3) = +\infty$

$\lim_{x \to -\infty} (8x^5 - 5x^4 - 20x^3) = -\infty$

Step 5: Find $x -$ intercept(s) and $y -$ intercept

$y = f(x) = 8x^5 - 5x^4 - 20x^3$

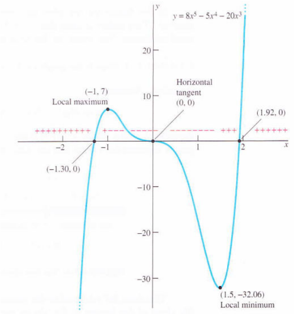
Put $x = 0$, $y = 0$, $y -$ intercept is 0.

Put $y = 0$, $0 = 8x^5 - 5x^4 - 20x^3 = x^3(8x^2 - 5x - 20)$,

$x = 0 \; or \; x = \frac{5 \pm \sqrt{665}}{16}$

$x -$ intercepts are $0, \frac{5+\sqrt{665}}{16}$ and $\frac{5-\sqrt{665}}{16}$.

Step 6: The sketch of the graph



Example 3:

Sketch the graph of $f(x) = x^{2/3}(x^2 - 2x - 6) = x^{8/3} - 2x^{\frac{5}{3}} - 6x^{2/3}$ on R.

Solutions

Step 1: Finding the coordinates of the critical point(s)

$f'(x) = \frac{8}{3}x^{\frac{5}{3}} - \frac{10}{3}x^{\frac{2}{3}} - 4x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}}(4x^2 - 5x - 6) = \frac{2}{3x^{1/3}}(x-2)(4x+3)$

$f'(x) = 0 \Leftrightarrow (x-2)(4x+3) = 0 \Leftrightarrow x = 2 \ or \ \frac{-3}{4}$

$f'(x)$ is undefined when $x = 0$.

$f(0) = 0$,

$f(2) = 2^{2/3}(4 - 4 - 6) = -6 \cdot \sqrt[3]{4} \approx -9.52$,

$f\left(\frac{-3}{4}\right) = \left(\frac{-3}{4}\right)^{2/3} \cdot \left(\frac{9}{16} + \frac{3}{2} - 6\right) \approx -3.25$.

The critical points of f on R are $(0,0)$, $(2, f(2))$ and $\left(\frac{-3}{4}, f\left(\frac{-3}{4}\right)\right)$.

Step 2: Finding the interval(s) on which the function is strictly increasing or strictly decreasing

	$x < \frac{-3}{4}$	$\frac{-3}{4} < x < 0$	$0 < x < 2$	$x > 2$
$x - 2$	$-ve$	$-ve$	$-ve$	$+ve$
$4x + 3$	$-ve$	$+ve$	$+ve$	$+ve$
$x^{1/3}$	$-ve$	$-ve$	$+ve$	$+ve$
$f'(x)$	$-ve$	$+ve$	$-ve$	$+ve$
$f(x)$	strictly decreasing	strictly increasing	strictly decreasing	strictly increasing

The intervals where f is strictly increasing are $\left(\frac{-3}{4}, 0\right)$ and $(2, \infty)$.

The intervals where f is strictly decreasing are $\left(-\infty, \frac{-3}{4}\right)$ and $(0,2)$.

Step 3: Classify Critical Point(s)

The critical point $(0,0)$ is a local maxima.

The critical points $\left(\frac{-3}{4}, f\left(\frac{-3}{4}\right)\right)$ and $(2, f(2))$ are local minima.

Step 4: Consider the behaviour of the function at infinity

$$\lim_{x \to +\infty} x^{2/3}(x^2 - 2x - 6) = +\infty$$
$$\lim_{x \to -\infty} x^{2/3}(x^2 - 2x - 6) = +\infty$$

Step 5: Find $x -$ intercept(s) and $y -$ intercept

$y = f(x) = x^{2/3}(x^2 - 2x - 6)$

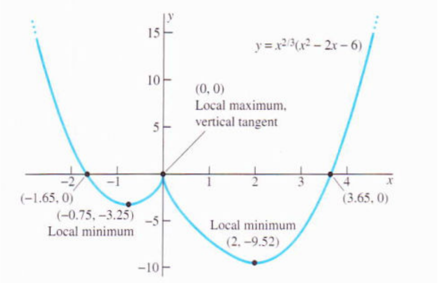
Put $x = 0$, $y = 0$, $y -$ intercept is 0.

Put $y = 0$, $0 = x^{2/3}(x^2 - 2x - 6)$,

$x = 0 \ or \ x = 1 \pm \sqrt{7}$

$x -$ intercepts are 0, $1 + \sqrt{7}$ and $1 - \sqrt{7}$.

Step 6: The sketch of the graph



 # Higher Derivatives and Concavity

Let f be a real-valued function on x and $y = f(x)$.
We write $\frac{dy}{dx} = f'(x)$.
We define the higher derivatives:
$\frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^3y}{dx^3} = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right), \frac{d^4y}{dx^4} = f^{(4)}(x) = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right),$
$\frac{d^{n+1}y}{dx^{n+1}} = f^{(n+1)}(x) = \frac{d}{dx}\left(\frac{d^ny}{dx^n}\right)$ for $n = 4,5,6, \cdots$.
We also write $\left.\frac{dy}{dx}\right|_{x=a} = f'(a)$.
We define the higher derivatives:
$\left.\frac{d^2y}{dx^2}\right|_{x=a} = f''(a), \left.\frac{d^3y}{dx^3}\right|_{x=a} = f'''(a), \left.\frac{d^4y}{dx^4}\right|_{x=a} = f^{(4)}(a),$
$\left.\frac{d^{n+1}y}{dx^{n+1}}\right|_{x=a} = f^{(n+1)}(a)$ for $n = 4,5,6, \cdots$.
Sometimes, we write $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$ and $y''' = \frac{d^3y}{dx^3}$.
Also, we write $y'(a) = \left.\frac{dy}{dx}\right|_{x=a}, y''(a) = \left.\frac{d^2y}{dx^2}\right|_{x=a}$ and $y'''(a) = \left.\frac{d^3y}{dx^3}\right|_{x=a}$.

Example 1:

Find the first four derivatives of $y = f(x) = 2x^3 + \frac{1}{x^2} + 16x^{7/2}$.

Solutions

$$y = f(x) = 2x^3 + x^{-2} + 16x^{7/2}$$
$$\frac{dy}{dx} = f'(x) = 6x^2 - 2x^{-3} + 56x^{5/2}$$
$$\frac{d^2y}{dx^2} = f''(x) = 12x + 6x^{-4} + 140x^{3/2}$$
$$\frac{d^3y}{dx^3} = f'''(x) = 12 - 24x^{-5} + 210x^{1/2}$$
$$\frac{d^4y}{dx^4} = f^{(4)}(x) = 120x^{-6} + 105x^{-1/2}$$

Example 2:

Find the second derivative $y''(x)$ of a function $y(x)$ that is defined implictly by the equation $x^2 - xy + y^2 = 9$.

Solutions

By implicit differentiation, $\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}9$.

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Method 1

$$y''(x) = \frac{d^2y}{dx^2} = \frac{1}{(2y-x)^2}\left[(2y - x)\left(\frac{dy}{dx} - 2\right) - (y - 2x)\left(2\frac{dy}{dx} - 1\right)\right]$$

$$= \frac{1}{(2y-x)^2}\left[3x\frac{dy}{dx} - 3y\right]$$

$$= \frac{3}{(2y-x)^2}\left[x\left(\frac{y-2x}{2y-x}\right) - y\right]$$

$$= \frac{3}{(2y-x)^3}[x(y - 2x) - y(2y - x)]$$

$$= \frac{3}{(2y-x)^3}[2xy - 2x^2 - 2y^2]$$

$$= \frac{-6}{(2y-x)^3}[x^2 - xy + y^2]$$

$= \frac{-6}{(2y-x)^3} \cdot 9$ (Note: $x^2 - xy + y^2 = 9$)

$$= \frac{-54}{(2y-x)^3}$$

$$= \frac{54}{(x-2y)^3}$$

Method 2

By Implicit Differentiation, $\frac{d}{dx}\left(2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx}\right) = \frac{d}{dx}0$.

$$2 - x\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) \cdot \left(\frac{dy}{dx}\right) = 0$$

$$(2y - x)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 2 = 0$$

$$(2y - x)\frac{d^2y}{dx^2} + 2\left(\frac{y-2x}{2y-x}\right)^2 - 2\left(\frac{y-2x}{2y-x}\right) + 2 = 0$$

$$(2y - x)\frac{d^2y}{dx^2} = \frac{-2}{(2y-x)^2}[(y - 2x)^2 - (y - 2x)(2y - x) + (2y - x)^2]$$

$$y''(x) = \frac{d^2y}{dx^2}$$

$$= \frac{-2}{(2y-x)^3}[y^2 - 4xy + 4x^2 - 2y^2 + 5xy - 2x^2 + 4y^2 - 4xy + x^2]$$

$$= \frac{-2}{(2y-x)^3}[3y^2 - 3xy + 3x^2]$$

$$= \frac{-6}{(2y-x)^3}[y^2 - xy + x^2]$$

$= \frac{-6}{(2y-x)^3} \cdot 9$ (Note: $x^2 - xy + y^2 = 9$)

$$= \frac{-54}{(2y-x)^3}$$

$$= \frac{54}{(x-2y)^3}$$

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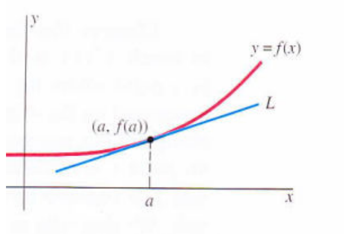
The sign of Second Derivative and Convexity

Definitions:

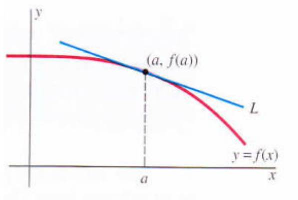
Let f be a real-valued function on x and $a \in R$.

Suppose f is differentiable at a and L is the line tangent to the graph $y = f(x)$ at the point $(a, f(a))$. We say:

f is **concave upward at a** if, on some open interval containing a, the graph of f lies above L



f is **concave downward at a** if, on some open interval containing a, the graph of f lies below L



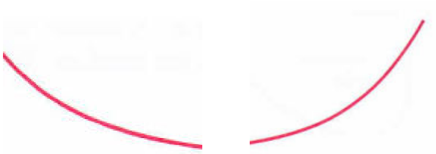
Observations:

Let f be a real-valued function on x and let $a, b \in R$ with $a < b$.

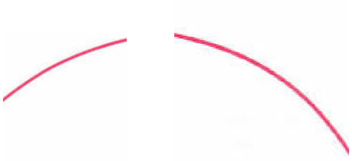
Suppose f is differentiable on (a, b).

We can check that:

(i) Suppose for any $\lambda \in R$ with $0 < \lambda < 1$ and for any $\alpha, \beta \in (a, b)$, we MUST have $f(\lambda\alpha + (1-\lambda)\beta) < \lambda f(\alpha) + (1-\lambda) f(\beta)$. Then, f is **concave upward** at any $x \in (a, b)$.



(ii) Suppose for any $\lambda \in R$ with $0 < \lambda < 1$ and for any $\alpha, \beta \in (a, b)$, we MUST have $f(\lambda\alpha + (1-\lambda)\beta) > \lambda f(\alpha) + (1-\lambda) f(\beta)$. Then, f is **concave downward** at any $x \in (a, b)$.



Recall an Important Result:

Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$.

Suppose f is continuous on $[a, b]$ and is differentiable on (a, b).

Then we have:

(i) If $f'(x) > 0$ for any $x \in (a, b)$, then f is **strictly increasing** on $[a, b]$.

(ii) If $f'(x) < 0$ for any $x \in (a, b)$, then f is **strictly decreasing** on $[a, b]$.

Corollary:

Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$.

Suppose f' is continuous on $[a, b]$ and is differentiable on (a, b).

Then we have:

(i) If $f''(x) > 0$ for any $x \in (a, b)$, then f' is **strictly increasing** on $[a, b]$.

(ii) If $f''(x) < 0$ for any $x \in (a, b)$, then f' is **strictly decreasing** on $[a, b]$.

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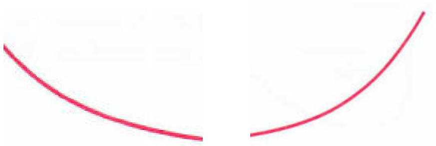
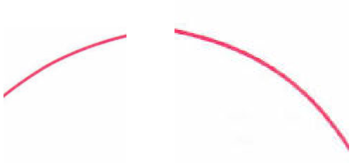
Theorems:

Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$.
Suppose f' is continuous on $[a, b]$ and is differentiable on (a, b). We have:

(i) If $f''(x) > 0$ for any $x \in (a, b)$, then f is **concave upward** at any $x \in (a, b)$.



(ii) If $f''(x) < 0$ for any $x \in (a, b)$, then f is **concave downward** at any $x \in (a, b)$.



Proof: Omitted (As Exercises)

Important Result:
Let $f: R \to R$ be a function and $a, b \in R$ with $a < b$.
Suppose f' is continuous on $[a, b]$ and is differentiable on (a, b).
Then the shape of f is:

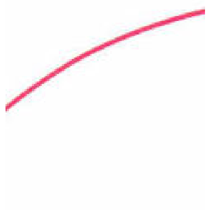
$f'(x) > 0$ for any $x \in (a, b)$ AND
$f''(x) > 0$ for any $x \in (a, b)$

Concave upward & Strictly Increasing

$f'(x) < 0$ for any $x \in (a, b)$ AND
$f''(x) > 0$ for any $x \in (a, b)$

Concave upward & Strictly Decreasing

$f'(x) > 0$ for any $x \in (a, b)$ AND
$f''(x) < 0$ for any $x \in (a, b)$



Concave downward & Strictly Increasing

$f'(x) < 0$ for any $x \in (a, b)$ AND
$f''(x) < 0$ for any $x \in (a, b)$

Concave downward & Strictly Decreasing

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An Application (Curve Sketching)

Example:

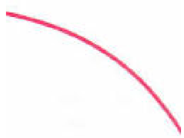
Sketch the graph of $y = f(x) = x^3 - 3x^2 + 3$ on R.

Solutions

Step 1: Finding the coordinates of the point $(c, f(c))$ where $(c, f(c))$ is a critical point or $f''(c) = 0$ or $f''(c)$ is undefined.

$f'(x) = 3x^2 - 6x = 3x(x-2)$
$f'(x) = 0 \Leftrightarrow x(x-2) = 0 \Leftrightarrow x = 0 \; or \; 2$
$f(0) = 3, f(2) = 8 - 12 + 3 = -1.$
$f''(x) = 6x - 6 = 6(x-1).$
$f''(x) = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1.$
$f(1) = 1 - 3 + 3 = 1.$
The critical points of f on R are $(0,3)$ and $(2,-1)$.
$f''(x) = 0$ at the point $(1,1)$.

Step 2: Finding the interval(s) on which the function is strictly increasing or strictly decreasing AND it is concave upward or concave downward.

	$x < 0$	$0 < x < 1$	$1 < x < 2$	$x > 2$
$x - 2$	$-ve$	$-ve$	$-ve$	$+ve$
x	$-ve$	$+ve$	$+ve$	$+ve$
$f'(x)$	$+ve$	$-ve$	$-ve$	$+ve$
$x - 1$	$-ve$	$-ve$	$+ve$	$+ve$
$f''(x)$	$-ve$	$-ve$	$+ve$	$+ve$
$f(x)$	concave downward & strictly increasing	concave downward & strictly decreasing	concave upward & strictly decreasing	concave upward & strictly increasing
Shape of f				

The intervals where f is strictly increasing are $(-\infty, 0)$ and $(2, \infty)$.
The interval where f is strictly decreasing is $(0,2)$.

The interval where f is concave upward is $(1, \infty)$.
The interval where f is concave downard is $(-\infty, 1)$.

Step 3: Classify Critical Point(s)
The critical point $(0,3)$ is a local maxima.
The critical point $(2,-1)$ is a local minima.

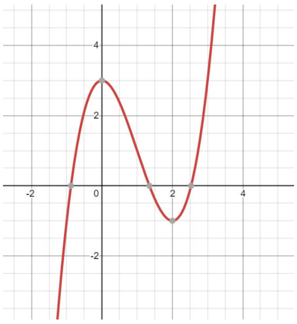
Step 4: Consider the behaviour of the function at infinity

$$\lim_{x \to +\infty} (x^3 - 3x^2 + 3) = +\infty$$
$$\lim_{x \to -\infty} (x^3 - 3x^2 + 3) = -\infty$$

Step 5: Find $x -$ intercept(s) and $y -$ intercept

$y = f(x) = x^3 - 3x^2 + 3$
Put $x = 0$, $y = 3$, $y -$ intercept is 3.
Put $y = 0$, $0 = x^3 - 3x^2 + 3$,
We can check that three $x -$ intercepts, one in $(-1,0)$, the other one in $(1,2)$ and the last one in $(2,3)$.
Note: $f(-1) < 0$, $f(0) > 0$, $f(1) > 0$, $f(2) < 0$, $f(3) > 0$.

Step 6: The sketch of the graph



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Definitions:
Let f be a real-valued function on x and let $a, b, c \in R$ with $a < c < b$.

$(c, f(c))$ is called **a** critical point of f on (a, b) if $f'(c) = 0$ or $f'(c)$ is undefined.

$(c, f(c))$ is called **a** point of inflection of f on (a, b) if we can find $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ **AND**
[f changes from concave upward on $(c - \delta, c)$ to concave downward on $(c, c + \delta)$] OR
[f changes from concave downward on $(c - \delta, c)$ to concave upward on $(c, c + \delta)$].

$(c, f(c))$ is called **a candidate** for a point of inflection of f on (a, b) if $f''(c) = 0$ or $f''(c)$ is undefined.

Remarks:
$(1,1)$ is the point of inflection of $f(x) = x^3 - 3x^2 + 3$ on R.

Step 1: Finding the coordinates of the critical point(s) or candidate(s) for a point of inflection.

 # Second Derivative Test

Let f be a real-valued function on x and let $c, \delta \in R$ with $\delta > 0$.
Suppose f'' is continuous on $(c-\delta, c+\delta)$ AND $f'(c) = 0$.
We have:

(i) If $f''(c) > 0$, then $(c, f(c))$ is **a** local minima.

(ii) If $f''(c) < 0$, then $(c, f(c))$ is **a** local maxima.

(iii) If $f''(c) = 0$, then we have NO conclusions on the nature of $(c, f(c))$.

Proof: Omitted (As Exercises)

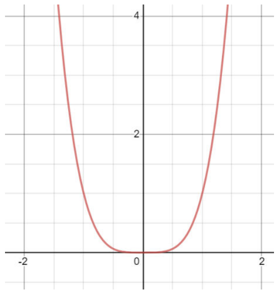
Remarks:

Example 1:

Let $f: R \to R$ be defined by $f(x) = x^4$.

We can show that $(0,0)$ is the critical point of f on R AND it is a local minima.

Note: $0 = f'(0) = f''(0)$.

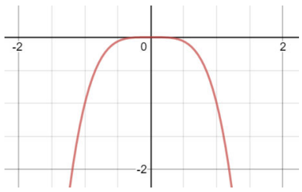


Example 2:

Let $f: R \to R$ be defined by $f(x) = -x^4$.

We can show that $(0,0)$ is the critical point of f on R AND it is a local maxima.

Note: $0 = f'(0) = f''(0)$.

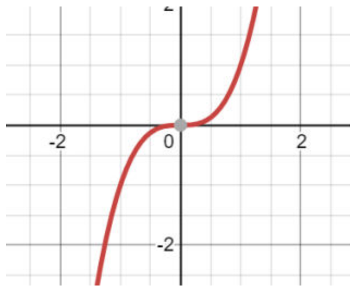


Example 3:

Let $f: R \to R$ be defined by $f(x) = x^3$.

We can show that $(0,0)$ is the critical point of f on R AND it is neither a local maxima nor a local minima.

$(0,0)$ is a point of inflection.
Note: $0 = f'(0) = f''(0)$.



An Application (Curve Sketching) (Version 2)

Step 1 Finding the coordinates of the point $(c, f(c))$ where $(c, f(c))$ is a critical point or a candidate for a point of inflection

Step 2 Finding the interval(s) on which the function is strictly increasing or strictly decreasing AND it is concave upward or concave downward

Step 3 Classify Critical Point(s) and find point(s) of inflection

Step 4 Consider the behaviour of the function at infinity

Step 5 Find $x -$ intercept(s) and $y -$ intercept

Step 6 The sketch of the graph

Example 1:

Sketch the graph of $y = f(x) = (2x^2 - 3x - 1)e^{-x}$ for $x \in R$.

Solutions

Step 1: Finding the coordinates of the point $(c, f(c))$ where $(c, f(c))$ is a critical point or a candidate for a point of inflection

$f'(x) = (2x^2 - 3x - 1)(-e^{-x}) + e^{-x}(4x - 3)$
$= (-2x^2 + 7x - 2)e^{-x} = -(2x^2 - 7x + 2)e^{-x}$.

$$f'(x) = 0 \Leftrightarrow 2x^2 - 7x - 2 = 0 \Leftrightarrow x = \frac{7 \pm \sqrt{33}}{4}$$

$f\left(\frac{7+\sqrt{33}}{4}\right) \approx 0.403$, $f\left(\frac{7-\sqrt{33}}{4}\right) \approx -1.275$

$f''(x) = -(2x^2 - 7x + 2)(-e^{-x}) - e^{-x}(4x - 7) = (2x^2 - 11x + 9)e^{-x}$
$= (2x - 9)(x - 1)e^{-x}$

$$f''(x) = 0 \Leftrightarrow (2x - 9)(x - 1) = 0 \Leftrightarrow x = \frac{9}{2} \text{ } or \text{ } 1$$

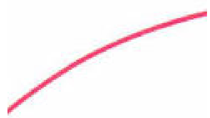
$f\left(\frac{9}{2}\right) \approx 0.289$, $f(1) \approx -0.736$

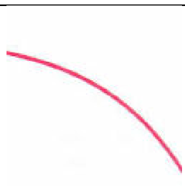
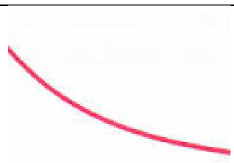
The critical points of f on R are $\left(\frac{7+\sqrt{33}}{4}, f\left(\frac{7+\sqrt{33}}{4}\right)\right)$ and $\left(\frac{7-\sqrt{33}}{4}, f\left(\frac{7-\sqrt{33}}{4}\right)\right)$.

The candidates for a point of inflection of f on R are $\left(\frac{9}{2}, f\left(\frac{9}{2}\right)\right)$ and $(1, f(1))$.

Note: $\frac{7-\sqrt{33}}{4} \approx 0.31$; $\frac{7+\sqrt{33}}{4} \approx 3.19$; $\frac{9}{2} = 4.5$

Step 2: Finding the interval(s) on which the function is strictly increasing or strictly decreasing AND it is concave upward or concave downward.

	$x < \frac{7-\sqrt{33}}{4}$	$\frac{7-\sqrt{33}}{4} < x < 1$	$1 < x < \frac{7+\sqrt{33}}{4}$
$2x^2 - 7x - 2$	$+ve$	$-ve$	$-ve$
$f'(x)$	$-ve$	$+ve$	$+ve$
$2x - 9$	$-ve$	$-ve$	$-ve$
$x - 1$	$-ve$	$-ve$	$+ve$
$f''(x)$	$+ve$	$+ve$	$-ve$
$f(x)$	concave upward & strictly decreasing	concave upward & strictly increasing	concave downward & strictly increasing
Shape of f			

	$\frac{7+\sqrt{33}}{4} < x < \frac{9}{2}$	$x > \frac{9}{2}$
$2x^2 - 7x - 2$	$+ve$	$+ve$
$f'(x)$	$-ve$	$-ve$
$2x - 9$	$-ve$	$+ve$
$x - 1$	$+ve$	$+ve$
$f''(x)$	$-ve$	$+ve$
$f(x)$	concave downward & strictly decreasing	concave upward & strictly decreasing
Shape of f		

The interval where f is strictly increasing is $\left(\frac{7-\sqrt{33}}{4}, \frac{7+\sqrt{33}}{4}\right)$.

The intervals where f is strictly decreasing are $\left(-\infty, \frac{7-\sqrt{33}}{4}\right)$ and $\left(\frac{7+\sqrt{33}}{4}, \infty\right)$.

The intervals where f is concave upward are $(-\infty, 1)$ and $\left(\frac{9}{2}, \infty\right)$.

The interval where f is concave downard is $\left(1, \frac{9}{2}\right)$.

Step 3: Classify Critical Point(s) and Find point(s) of inflection

The critical point $\left(\frac{7-\sqrt{33}}{4}, f\left(\frac{7-\sqrt{33}}{4}\right)\right)$ is a local minima.

The critical point $\left(\frac{7+\sqrt{33}}{4}, f\left(\frac{7+\sqrt{33}}{4}\right)\right)$ is a local maxima.

The points $(1, f(1))$ and $\left(\frac{9}{2}, f\left(\frac{9}{2}\right)\right)$ are points of inflection.

Step 4: Consider the behaviour of the function at infinity

$$\lim_{x\to+\infty} (2x^2 - 3x - 1)e^{-x} = 0$$
$$\lim_{x\to-\infty} (2x^2 - 3x - 1)e^{-x} = +\infty$$

Step 5: Find $x-$ intercept(s) and $y-$ intercept

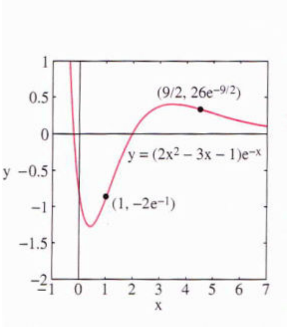
$y = f(x) = (2x^2 - 3x - 1)e^{-x}$

Put $x = 0, y = -1, y-$ intercept is -1.

Put $y = 0, 0 = (2x^2 - 3x - 1)e^{-x}$, We can check that two $x-$ intercepts, one in $(-1,0)$, the other one in $(1,2)$.

Note: $f(-1) > 0, f(0) < 0, f(1) < 0, f(2) > 0$.

Step 6: The sketch of the graph



Example 2:

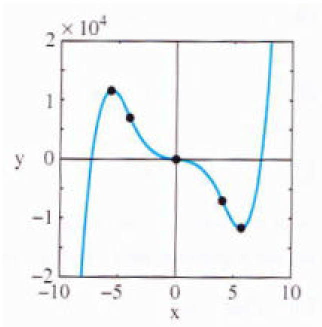
Find the critical point(s) and point(s) of inflection of f on R and classify the nature of each critical point where $f(x) = x^3 - 3x^2 + 3$ for $x \in R$.

Solutions

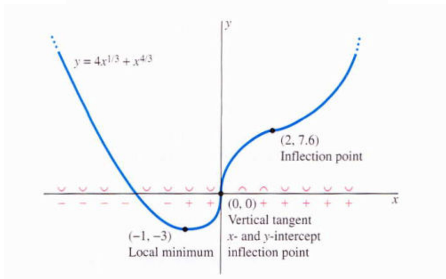
$f'(x) = 3x^2 - 6x = 3x(x-2)$
$f'(x) = 0 \Leftrightarrow x(x-2) = 0 \Leftrightarrow x = 0 \text{ } or \text{ } 2$
$f(0) = 3$ and $f(2) = 8 - 12 + 3 = -1$
The critical points of f on R are $(0,3)$ and $(2,-1)$.
$f''(x) = 6x - 6 = 6(x-1)$
$f''(x) = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$
$f(1) = 1 - 3 + 3 = 1$
A candidate for a point of inflection is $(1,1)$.
f'' is continuous on R.
$f''(x) < 0$ for any $x < 1$ AND $f''(x) > 0$ for any $x > 1$, so $(1,1)$ is the point of inflection of f on R.
$f''(0) = -6 < 0$, $(0,3)$ is a local maxima.
$f''(2) = 6 > 0$, $(2,-1)$ is a local minima.

Exercises

(i) Sketch the graph of $y = f(x) = 3x^5 - 160x^3$ on R.



(ii) Sketch the graph of $y = f(x) = 4x^{1/3} + x^{4/3}$ on R.



Curve Sketching and Asymptotes

Vertical Asymptote(s)

Let f be a real-valued function on x and let $a \in R$.

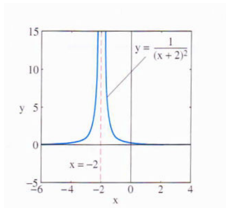
Suppose $\lim_{x\to a^+} f(x) = +\infty$ or $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = +\infty$ or $\lim_{x\to a^-} f(x) = -\infty$.

Then, the vertical line $x = a$ will be very close to the graph $y = f(x)$ when $x \to a^+$ or $x \to a^-$ or $x \to a$.

The vertical line $x = a$ is called **a** vertical asymptote of $y = f(x)$.

Example 1:

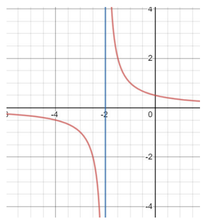
$x = -2$ is **a** vertical asymptote of $y = \frac{1}{(x+2)^2}$.



Note: $\lim_{x\to(-2)^+} \frac{1}{(x+2)^2} = +\infty$, $\lim_{x\to(-2)^-} \frac{1}{(x+2)^2} = +\infty$, $\lim_{x\to-2} \frac{1}{(x+2)^2} = +\infty$.

Example 2:

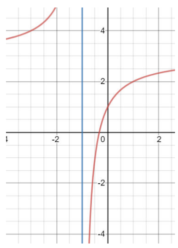
$x = -2$ is **a** vertical asymptote of $y = \frac{1}{x+2}$.



Note: $\lim_{x\to(-2)^+} \frac{1}{x+2} = +\infty$, $\lim_{x\to(-2)^-} \frac{1}{x+2} = -\infty$, $\lim_{x\to-2} \frac{1}{x+2}$ doesn't exist.

Example 3:

Find an equation of a vertical asymptote of the curve $y = \frac{x-1}{x+1} + 2$.



Solutions

$\lim_{x\to(-1)^+} \left(\frac{x-1}{x+1} + 2\right) = -\infty$, $\lim_{x\to(-1)^-} \left(\frac{x-1}{x+1} + 2\right) = +\infty$, $\lim_{x\to-1} \left(\frac{x-1}{x+1} + 2\right)$ doesn't exist.

$x = -1$ is **a** vertical asymptote of $y = \frac{x-1}{x+1} + 2$.

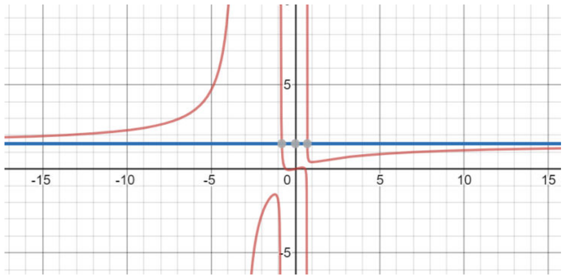
Horizontal Asymptote(s)

Let f be a real-valued function on x and let $l \in R$.
Suppose $\lim_{x\to+\infty} f(x) = l$ or $\lim_{x\to-\infty} f(x) = l$. Then, the horizontal line $y = l$ will be very close to the graph $y = f(x)$ when $x \to +\infty$ or $x \to -\infty$ or $x \to \infty$. The horizontal line $y = l$ is called **a** horizontal asymptote of $y = f(x)$.

Example 1:

Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if

$$y = f(x) = \frac{3x^3 - x}{2x^3 + 7x^2 - 4}$$



Solutions

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{3x^3-x}{2x^3+7x^2-4} = \lim_{x\to+\infty} \frac{3-\frac{1}{x^2}}{2+\frac{7}{x}-\frac{4}{x^3}} = \frac{3}{2}.$$

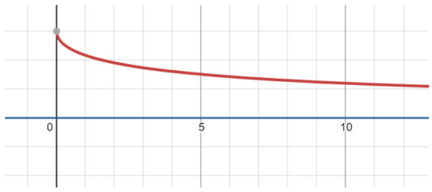
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^3-x}{2x^3+7x^2-4} = \lim_{x\to-\infty} \frac{3-\frac{1}{x^2}}{2+\frac{7}{x}-\frac{4}{x^3}} = \frac{3}{2}.$$

Note 1: $\lim_{x\to\infty} f(x) = \frac{3}{2}$

Note 2: $y = \frac{3}{2}$ is a horizontal asymptote of $y = f(x)$.

Example 2:

Find $\lim_{x\to+\infty} f(x)$ if $y = f(x) = \sqrt{x+a} - \sqrt{x}$ for $a > 0$.



(when $a = 4$)

Solutions

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(\sqrt{x+a} - \sqrt{x}\right) = \lim_{x\to+\infty} \frac{\left(\sqrt{x+a}-\sqrt{x}\right)\left(\sqrt{x+a}+\sqrt{x}\right)}{\left(\sqrt{x+a}+\sqrt{x}\right)}$$

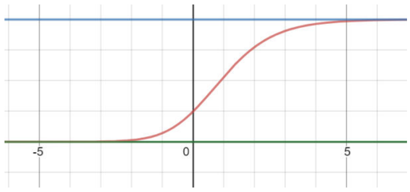
$$= \lim_{x\to+\infty} \frac{(x+a)-x}{\sqrt{x+a}+\sqrt{x}} = \lim_{x\to+\infty} \frac{a}{\sqrt{x+a}+\sqrt{x}} = \lim_{x\to+\infty} \frac{a}{\sqrt{1+\frac{a}{x}}+1} \cdot \frac{1}{\sqrt{x}}$$

$$= \frac{a}{\sqrt{1}+1} \cdot 0 = 0$$

Note: $y = 0$ is a horizontal asymptote of $y = f(x)$.

Example 3:

Find $\lim_{x\to+\infty} f(x)$ if $y = f(x) = \frac{4e^{2x}}{(1+e^x)^2}$.



Solutions

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4e^{2x}}{(1+e^x)^2} = \lim_{x\to+\infty} \frac{4}{(e^{-x}+1)^2} = \frac{4}{(0+1)^2} = 4$$

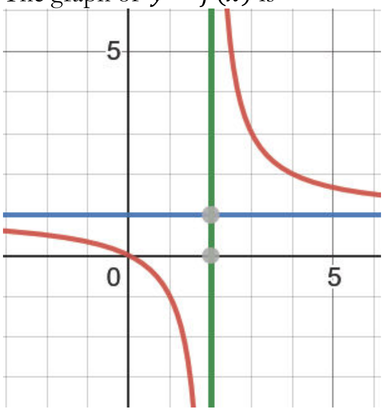
Note 1: $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4e^{2x}}{(1+e^x)^2} = \frac{4\times 0}{(1+0)^2} = 0$
Note 2: $\lim_{x\to\infty} f(x)$ doesn't exist.
Note 3: BOTH $y = 4$ and $y = 0$ are horizontal asymptotes of $y = f(x)$.

Example 4:

Sketch the graph of $y = f(x) = \frac{x}{x-2}$. Indicate any horizontal or vertical asymptotes.

Solutions

Note: $y = f(x) = \frac{x}{x-2} = \frac{2+x-2}{x-2} = \frac{2}{x-2} + 1$.
Horizontal asymptote is $y = 1$.
Vertical asymptote is $x = 2$.
The graph of $y = f(x)$ is



Example 5:

Find horizontal and vertical asymptotes of $y = f(x) = \frac{x}{(x+2)^2}$.

Solutions

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{(x+2)^2} = \lim_{x\to+\infty} \frac{1}{x} \cdot \frac{1}{\left(1+\frac{2}{x}\right)^2} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{(x+2)^2} = \lim_{x\to-\infty} \frac{1}{x} \cdot \frac{1}{\left(1+\frac{2}{x}\right)^2} = 0$$

So, $\lim_{x\to\infty} f(x) = 0$

$y = 0$ is a horizontal asymptote of $y = f(x)$.

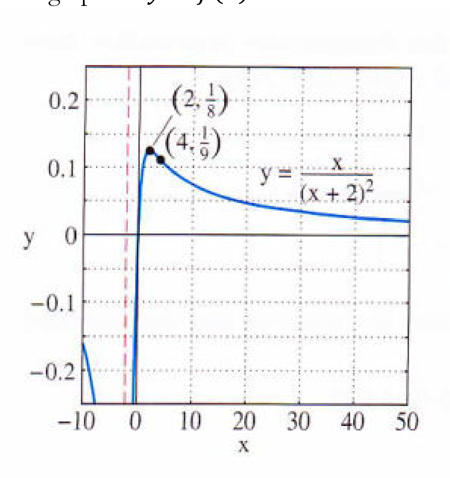
$$\lim_{x\to(-2)^+} f(x) = \lim_{x\to(-2)^+} \frac{x}{(x+2)^2} = -\infty$$

$$\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^-} \frac{x}{(x+2)^2} = -\infty$$

So, $\lim_{x\to-2} f(x) = -\infty$

$x = -2$ is a vertical asymptote of $y = f(x)$.

The graph of $y = f(x)$ is:



Example 6:

Find horizontal and vertical asymptotes of $y = f(x) = \frac{2+x-x^2}{(x-1)^2}$.

Solutions

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{2+x-x^2}{(x-1)^2} = \lim_{x\to+\infty} \frac{\frac{2}{x^2}+\frac{1}{x}-1}{\left(1-\frac{1}{x}\right)^2} = \frac{0+0-1}{(1-0)^2} = -1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2+x-x^2}{(x-1)^2} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}+\frac{1}{x}-1}{\left(1-\frac{1}{x}\right)^2} = \frac{0+0-1}{(1-0)^2} = -1$$

So, $\lim_{x\to\infty} f(x) = -1$

$y = -1$ is a horizontal asymptote of $y = f(x)$.

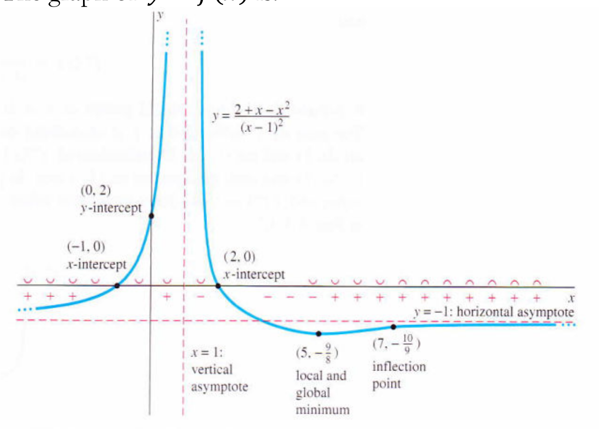
$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{2+x-x^2}{(x-1)^2} = +\infty$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{2+x-x^2}{(x-1)^2} = +\infty$$

So, $\lim_{x\to1} f(x) = +\infty$

$x = 1$ is a vertical asymptote of $y = f(x)$.

The graph of $y = f(x)$ is:



Exercises:

(i) Sketch the graph of $y = f(x) = \frac{x}{(x+2)^2}$.

(ii) Sketch the graph of $y = f(x) = \frac{2+x-x^2}{(x-1)^2}$.

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 4: Applications of Differentiation

Slant Asymptote(s)

The line with equation $y = mx + b$ is called **a** slant asymptote of $y = f(x)$ if

(i) $\lim\limits_{x \to +\infty} [f(x) - mx - b] = 0$ OR

(ii) $\lim\limits_{x \to -\infty} [f(x) - mx - b] = 0$ OR

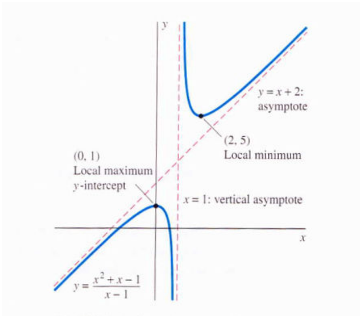
(iii) $\lim\limits_{x \to \infty} [f(x) - mx - b] = 0$

Note: m and b are fixed real numbers (constants) and $m \neq 0$.

Example

Find an equation of a slant asymptote of

$y = f(x) = \frac{x^2+x-1}{x-1}$.



Solutions

By long division, $x^2 + x - 1 = (x-1)(x+2) + 1$.

So, $\frac{x^2+x-1}{x-1} = \frac{(x-1)(x+2)+1}{x-1} = x + 2 + \frac{1}{x-1}$.

An equation of a slant asymptote of $y = f(x)$ is $y = x + 2$.

Note 1:

$$\lim_{x \to +\infty} [f(x) - x - 2] = \lim_{x \to +\infty} \left[\frac{x^2+x-1}{x-1} - x - 2\right] = \lim_{x \to +\infty} \frac{1}{x-1} = 0$$

$$\lim_{x \to -\infty} [f(x) - x - 2] = \lim_{x \to -\infty} \left[\frac{x^2+x-1}{x-1} - x - 2\right] = \lim_{x \to -\infty} \frac{1}{x-1} = 0$$

So, $\lim\limits_{x \to \infty} [f(x) - x - 2] = 0$.

Note 2:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2+x-1}{x-1} = \lim_{x \to +\infty} x \cdot \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x}} = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2+x-1}{x-1} = \lim_{x \to -\infty} x \cdot \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x}} = -\infty$$

Thus, $y = f(x)$ has NO horizontal asymptotes.

Note 3:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2+x-1}{x-1} = +\infty$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2+x-1}{x-1} = -\infty$$

$\lim\limits_{x \to 1} f(x)$ doesn't exist as a real number

Thus, $y = f(x)$ has a vertical asymptote $x = 1$.

Theorems:

Suppose the line with equation $y = mx + b$ is **a** slant asymptote of $y = f(x)$.
Then, we have:

(i) $m = \lim\limits_{x \to +\infty} \frac{f(x)}{x}$ OR

(ii) $m = \lim\limits_{x \to -\infty} \frac{f(x)}{x}$ OR

(iii) $$m = \lim_{x \to \infty} \frac{f(x)}{x}$$

Also, we have:

(i) $b = \lim\limits_{x \to +\infty} [f(x) - mx]$ OR

(ii) $b = \lim\limits_{x \to -\infty} [f(x) - mx]$ OR

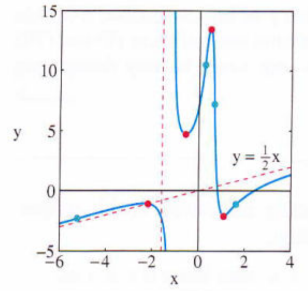
(iii) $b = \lim\limits_{x \to \infty} [f(x) - mx]$

Proof: Omitted (As Exercises)

Hint: $\lim\limits_{x \to +\infty} [f(x) - mx - b] = 0 \Longrightarrow 0 = \lim\limits_{x \to +\infty} \frac{f(x) - mx - b}{x} = \left(\lim\limits_{x \to +\infty} \frac{f(x)}{x}\right) - m$

Example

Find an equation of **the** slant asymptote of
$y = f(x) = \frac{x^6-4x^3+5x}{2x^5-5x^3+5}$.



Solutions

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{1}{x}\cdot\frac{x^6-4x^3+5x}{2x^5-5x^3+5} = \lim_{x\to+\infty}\frac{1-\frac{4}{x^3}+\frac{5}{x^5}}{2-\frac{5}{x^2}+\frac{5}{x^5}} = \frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{1}{x}\cdot\frac{x^6-4x^3+5x}{2x^5-5x^3+5} = \lim_{x\to-\infty}\frac{1-\frac{4}{x^3}+\frac{5}{x^5}}{2-\frac{5}{x^2}+\frac{5}{x^5}} = \frac{1}{2}$$

So, $\lim_{x\to\infty}\frac{f(x)}{x} = \frac{1}{2}$

Let $m = \frac{1}{2}$.

$$\lim_{x\to+\infty}[f(x)-mx] = \lim_{x\to+\infty}\left[\frac{x^6-4x^3+5x}{2x^5-5x^3+5}-\frac{1}{2}x\right]$$

$$= \lim_{x\to+\infty}\frac{2(x^6-4x^3+5x)-x(2x^5-5x^3+5)}{2(2x^5-5x^3+5)}$$

$$= \lim_{x\to+\infty}\frac{2x^4+5x}{2(2x^5-5x^3+5)}$$

$$= \lim_{x\to+\infty}\frac{1}{2x}\cdot\frac{2+\frac{5}{x^3}}{2-\frac{5}{x^2}+\frac{5}{x^5}} = 0$$

$$\lim_{x\to-\infty}[f(x)-mx] = \lim_{x\to-\infty}\left[\frac{x^6-4x^3+5x}{2x^5-5x^3+5}-\frac{1}{2}x\right] = \lim_{x\to-\infty}\frac{1}{2x}\cdot\frac{2+\frac{5}{x^3}}{2-\frac{5}{x^2}+\frac{5}{x^5}} = 0$$

So, $\lim_{x\to\infty}[f(x)-mx] = 0$.

Let $b = 0$.

Thus, $y = \frac{1}{2}x$ is **the** slant asymptote of $y = f(x)$.

An Application (Curve Sketching) (Final Version)

Step 1 Finding the coordinates of the point $(c, f(c))$ where $(c, f(c))$ is a critical point or a candidate for a point of inflection

Step 2 Finding the interval(s) on which the function is strictly increasing or strictly decreasing AND it is concave upward or concave downward

Step 3 Classify Critical Point(s) and find point(s) of inflection

Step 4 Consider the behaviour of the function at infinity

Step 5 Find $x-$ intercept(s) and $y-$ intercept

Step 6 Find horizontal, vertical and slant asymptote(s) of $y = f(x)$ if any.

Step 7 The sketch of the graph

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 4: Applications of Differentiation

Indeterminate Forms and L'Hopital's Rule

Definitions:

Let f and g be real-valued functions on x and let $a \in R$.

(i) If $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$, then we call $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ an indeterminate form.

(ii) If $\lim_{x\to a^-} f(x) = 0 = \lim_{x\to a^-} g(x)$, then we call $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ an indeterminate form.

(iii) If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then we call $\lim_{x\to a} \frac{f(x)}{g(x)}$ an indeterminate form.

These are called $\frac{0}{0}$ forms.

Examples:

(i) $$\lim_{x\to 0} \frac{sinx}{x} = 1$$

(ii) $$\lim_{x\to 0^+} \frac{sinx}{x^2} = +\infty,\ \lim_{x\to 0^-} \frac{sinx}{x^2} = -\infty,\ \lim_{x\to 0} \frac{sinx}{x^2} \text{ doesn't exist}$$

(iii) $$\lim_{x\to 0} \frac{sin(x^2)}{x} = 0$$

Proofs: Omitted (As Exercises)

Note: All are $\frac{0}{0}$ forms.

L'Hopital's Rule for $\frac{0}{0}$ form:

Theorem 1:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

(i) f and g are continuous on $[a, a+\delta]$ **AND**
(ii) f and g are differentiable on $(a, a+\delta)$ **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a, a+\delta)$ **AND**
(iv) $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$ **AND**
(v) $f(a) = 0 = g(a)$.

Then, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$.

Theorem 2:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

(i) f and g are continuous on $[a-\delta, a]$ **AND**
(ii) f and g are differentiable on $(a-\delta, a)$ **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a-\delta, a)$ **AND**
(iv) $\lim_{x\to a^-} f(x) = 0 = \lim_{x\to a^-} g(x)$ **AND**
(v) $f(a) = 0 = g(a)$.

Then, $\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}$.

Theorem 3:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

(i) f and g are continuous on $[a-\delta, a+\delta]$ **AND**
(ii) f and g are differentiable on $(a-\delta, a+\delta)\backslash\{a\}$**AND**
(iii) $g'(x) \neq 0$ for any $x \in (a-\delta, a+\delta)\backslash\{a\}$ **AND**
(iv) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ **AND**
(v) $f(a) = 0 = g(a)$.

Then, $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Cauchy's Mean Value Theorem

Let f and g be real-valued functions on x and let $a, b \in R$ with $a < b$.
Suppose that:

(i) both f and g are continuous on $[a, b]$ **AND**
(ii) both f and g are differentiable on (a, b) **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a, b)$.

Then, we can find $c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Remark:

$g(b) - g(a) \neq 0$, so the denominators of the fractions in both sides of the equation will not be zero. This equation is meaningful.

Reason:

By Rolle's Theorem, $g(b) - g(a) = 0$ will contradict $g'(x) \neq 0$ for any $x \in (a, b)$.

Proof:

Let λ be a fixed real number (constant).
Let F be a real-valued function defined by $F(x) = f(x) + \lambda g(x)$.
We can choose λ such that $F(a) = F(b)$.
Reason:
$F(a) = F(b) \Leftrightarrow f(a) + \lambda g(a) = f(b) + \lambda g(b) \Leftrightarrow \lambda = -\frac{f(b)-f(a)}{g(b)-g(a)}$.
So, $F(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(x)$.
Then, F is continuous on $[a, b]$ and is differentiable on (a, b).
By Rolle's Theorem, we can find $c \in (a, b)$ such that $F'(c) = 0$.
That is, $f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(c) = 0$.
Hence, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof for Theorem 1: (L'Hopital's Rule for $\frac{0}{0}$ form)
(Similarly for Theorems 2 & 3)

By Cauchy's Mean Value Theorem, we can find $c \in (a, a + \delta)$ such that
$\frac{f(a+\delta)-f(a)}{g(a+\delta)-g(a)} = \frac{f'(c)}{g'(c)}$.

$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f(x)-f(a)}{g(x)-g(a)}$ (Note: $f(a) = 0 = g(a)$)
$= \lim_{\delta \to 0^+} \frac{f(a+\delta)-f(a)}{g(a+\delta)-g(a)}$ (Note: letting $x = a + \delta, \delta \to 0^+ \Leftrightarrow x \to a^+$)
$= \lim_{c \to a^+} \frac{f'(c)}{g'(c)}$ (Note: $\delta \to 0^+ \Leftrightarrow c \to a^+$ as $c \in (a, a + \delta)$)
$= \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$

Example 1:

Find $\lim_{x \to 0} \frac{e^x-1}{sin2x}$.

Solutions

$\lim_{x \to 0} \frac{e^x - 1}{sin2x}$
$= \lim_{x \to 0} \frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}sin2x}$

Let $f(x) = e^x - 1$ and $g(x) = sin2x$ for any $x \in R$.
Both f and g are differentiable on R.
$f(0) = 0 = g(0)$
$\frac{0}{0}$ form (Use L'Hopital's Rule)

$= \lim_{x \to 0} \frac{e^x}{2cos2x}$
$= \frac{1}{2}$

Sometimes, we write the solution as:

$\lim_{x \to 0} \frac{e^x - 1}{sin2x}$ — $\frac{0}{0}$ form (Use L'Hopital's Rule)
$= \lim_{x \to 0} \frac{e^x}{2cos2x}$
$= \frac{1}{2}$

Example 2:

Find $\lim_{x\to 1}\frac{1-x+lnx}{1+cos\pi x}$.

Solutions

$$\lim_{x\to 1}\frac{1-x+lnx}{1+cos\pi x}$$

$$=\lim_{x\to 1}\frac{\frac{d}{dx}(1-x+lnx)}{\frac{d}{dx}(1+cos\pi x)}$$

Let $f(x)=1-x+lnx$ and $g(x)=1+cos\pi x$ for any $x>0$.
Both f and g are differentiable on $(0,\infty)$.
$f(1)=0=g(1)$
$\frac{0}{0}$ form (Use L'Hopital's Rule)

$$=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi sin\pi x}$$

Let $p(x)=-1+\frac{1}{x}$ and $q(x)=-\pi sin\pi x$ for any $x>0$.
Both p and q are differentiable on $(0,\infty)$.
$p(1)=0=q(1)$
$\frac{0}{0}$ form (Use L'Hopital's Rule)

$$=\lim_{x\to 1}\frac{\frac{d}{dx}\left(-1+\frac{1}{x}\right)}{\frac{d}{dx}(-\pi sin\pi x)}$$

$$=\lim_{x\to 1}\frac{\frac{-1}{x^2}}{-\pi^2 cos\pi x}$$

$$=\frac{-1}{-\pi^2\cdot(-1)}=\frac{-1}{\pi^2}$$

Sometimes, we write the solution as:

$$\lim_{x\to 1}\frac{1-x+lnx}{1+cos\pi x}$$ $\frac{0}{0}$ form (Use L'Hopital's Rule)

$$=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi sin\pi x}$$ $\frac{0}{0}$ form (Use L'Hopital's Rule)

$$=\lim_{x\to 1}\frac{\frac{-1}{x^2}}{-\pi^2 cos\pi x}$$

$$=\frac{-1}{-\pi^2\cdot(-1)}=\frac{-1}{\pi^2}$$

Example 3:

Find $\lim_{x\to 0}\frac{sinx}{x+x^2}$.

Solutions

$$\lim_{x\to 0}\frac{sinx}{x+x^2}$$ $\frac{0}{0}$ form (Use L'Hopital's Rule)

$$=\lim_{x\to 0}\frac{cosx}{1+2x}$$

$$=\frac{1}{1}=1$$

Note 1: $\lim_{x\to 0}\frac{cosx}{1+2x}$ is NOT of $\frac{0}{0}$ form.

Note 2: $\lim_{x\to 0}\frac{cosx}{1+2x}=1\neq 0=\lim_{x\to 0}\frac{-sinx}{2}$.

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 4: Applications of Differentiation

L'Hopital's Rule for $\frac{\infty}{\infty}$ form:

Theorem 1:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

(i) f and g are continuous on $[a, a+\delta]$ **AND**
(ii) f and g are differentiable on $(a, a+\delta)$ **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a, a+\delta)$ **AND**
(iv) $\lim_{x \to a^+} |f(x)| = +\infty = \lim_{x \to a^+} |g(x)|$

Then, $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$.

Theorem 2:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

(i) f and g are continuous on $[a-\delta, a]$ **AND**
(ii) f and g are differentiable on $(a-\delta, a)$ **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a-\delta, a)$ **AND**
(iv) $\lim_{x \to a^-} |f(x)| = +\infty = \lim_{x \to a^-} |g(x)|$

Then, $\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}$.

Theorem 3:

Let f and g be real-valued functions on x and let $a, \delta \in R$ with $\delta > 0$.
Suppose that:

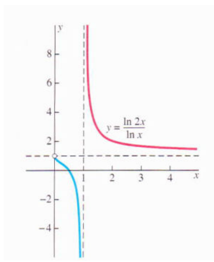
(i) f and g are continuous on $[a-\delta, a+\delta]$ **AND**
(ii) f and g are differentiable on $(a-\delta, a+\delta)\backslash\{a\}$**AND**
(iii) $g'(x) \neq 0$ for any $x \in (a-\delta, a+\delta)\backslash\{a\}$ **AND**
(iv) $\lim_{x \to a} |f(x)| = +\infty = \lim_{x \to a} |g(x)|$

Then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Proofs: Omitted (As Exercises)

Example:

Find $\lim_{x \to 0^+} \frac{ln2x}{lnx}$.



Solutions

$\lim_{x \to 0^+} \frac{ln2x}{lnx}$ $\qquad \frac{\infty}{\infty}$ form (Use L'Hopital's Rule)

$= \lim_{x \to 0^+} \frac{\frac{1}{2x} \cdot 2}{\frac{1}{x}}$

$= \lim_{x \to 0^+} 1 = 1$

L'Hopital's Rule for $\frac{\infty}{\infty}$ form:

Theorem 4:

Let f and g be real-valued functions on x and $a \in R$.
Suppose that:

(i) f and g are continuous on $[a, \infty)$ **AND**
(ii) f and g are differentiable on (a, ∞) **AND**
(iii) $g'(x) \neq 0$ for any $x \in (a, \infty)$ **AND**
(iv) $\lim_{x \to +\infty} |f(x)| = +\infty = \lim_{x \to +\infty} |g(x)|$

Then, $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$.

Theorem 5:

Let f and g be real-valued functions on x and $a \in R$.
Suppose that:

(i) f and g are continuous on $(-\infty, a]$ **AND**
(ii) f and g are differentiable on $(-\infty, a)$ **AND**
(iii) $g'(x) \neq 0$ for any $x \in (-\infty, a)$ **AND**
(iv) $\lim_{x \to -\infty} |f(x)| = +\infty = \lim_{x \to -\infty} |g(x)|$

Then, $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)}$.

Proofs: Omitted (As Exercises)

Example 1:

Find $\lim_{x \to +\infty} \frac{e^x}{lnx}$.

Solutions

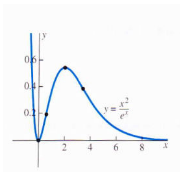
$\lim_{x \to +\infty} \frac{e^x}{lnx}$ $\frac{\infty}{\infty}$ form (Use L'Hopital's Rule)

$= \lim_{x \to +\infty} \frac{e^x}{\frac{1}{x}}$

$= \lim_{x \to +\infty} xe^x = +\infty$

Example 2:

Find $\lim_{x \to +\infty} \frac{x^2}{e^x}$ and $\lim_{x \to -\infty} \frac{x^2}{e^x}$.



Solutions

$\lim_{x \to +\infty} \frac{x^2}{e^x}$ $\frac{\infty}{\infty}$ form (Use L'Hopital's Rule)

$= \lim_{x \to +\infty} \frac{2x}{e^x}$ $\frac{\infty}{\infty}$ form (Use L'Hopital's Rule)

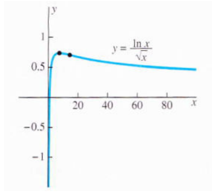
$= \lim_{x \to +\infty} \frac{2}{e^x} = 0$

$\lim_{x \to -\infty} \frac{x^2}{e^x} = +\infty$. So, $\lim_{x \to \infty} \frac{x^2}{e^x}$ doesn't exist.

Remark: $\lim_{x \to +\infty} \frac{x^k}{e^x} = 0$, $\lim_{x \to -\infty} \frac{x^{2k}}{e^x} = +\infty$ and $\lim_{x \to -\infty} \frac{x^{2k-1}}{e^x} = -\infty$ for $k = 1,2,3,\cdots$.

Example 3:

Find $\lim\limits_{x\to+\infty} \frac{lnx}{\sqrt{x}}$.



Solutions

$$\lim_{x\to+\infty} \frac{lnx}{\sqrt{x}} \qquad \frac{\infty}{\infty} \text{ form (Use L'Hopital's Rule)}$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x\to+\infty} \frac{2\sqrt{x}}{x}$$

$$= \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0$$

Exercises:

Find:

(i) $\lim\limits_{x\to10} \dfrac{ln(x-9)}{x-10}$

(ii) $\lim\limits_{x\to0} \dfrac{e^x+e^{-x}-2}{xsinx}$

(iii) $\lim\limits_{x\to+\infty} \dfrac{ln(lnx)}{xlnx}$

(iv) $\lim\limits_{x\to2} \dfrac{x^5-5x^2-12}{x^{10}-500x-24}$

(v) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{secx}{tanx}$

More Indeterminate Forms

(i) $0 \times \infty$ form

(ii) $\infty - \infty$ form

(iii) 0^0 form

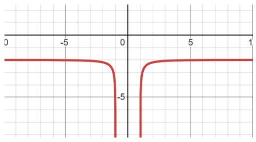
(iv) ∞^0 form

(v) 1^∞ form

Usual Technique: Change the above form into $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

Example 1:

Find $\lim_{x\to\infty} x ln\left(\frac{x-1}{x+1}\right)$.



Solutions

$$\lim_{x\to+\infty} x ln\left(\frac{x-1}{x+1}\right) \ (0\times\infty \text{ form})$$
$$= \lim_{x\to+\infty} \frac{ln\left(1-\frac{2}{x+1}\right)}{\frac{1}{x}} \ (\tfrac{0}{0} \text{ form})$$
$$= \lim_{x\to+\infty} \frac{\left(\frac{x+1}{x-1}\right)\cdot\frac{2}{(x+1)^2}}{\frac{-1}{x^2}}$$
$$= \lim_{x\to+\infty} \frac{-2x^2}{(x-1)(x+1)}$$
$$= \lim_{x\to+\infty} \frac{-2}{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}$$
$$= -2$$

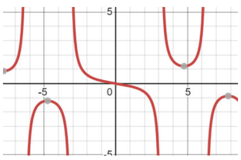
Thus, $\lim_{x\to\infty} x ln\left(\frac{x-1}{x+1}\right) = -2$

Solutions

$$\lim_{x\to-\infty} x ln\left(\frac{x-1}{x+1}\right)$$
$$= \lim_{u\to+\infty} -u ln\left(\frac{-u-1}{-u+1}\right) \ (\text{let } u = -x)$$
$$= -\lim_{u\to+\infty} u ln\left(\frac{u+1}{u-1}\right) \ (0\times\infty \text{ form})$$
$$= -\lim_{u\to+\infty} \frac{ln\left(1+\frac{2}{u-1}\right)}{\frac{1}{u}} \ (\tfrac{0}{0} \text{ form})$$
$$= -\lim_{u\to+\infty} \frac{\left(\frac{u-1}{u+1}\right)\cdot\frac{-2}{(u-1)^2}}{\frac{-1}{u^2}}$$
$$= -\lim_{u\to+\infty} \frac{2}{\left(1+\frac{1}{u}\right)\left(1-\frac{1}{u}\right)}$$
$$= -2$$

Example 2:

Find $\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{sinx}\right)$.

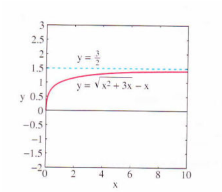


Solutions

$$\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{sinx}\right) \ (\infty-\infty \text{ form})$$
$$= \lim_{x\to0}\frac{sinx-x}{xsinx} \ (\tfrac{0}{0} \text{ form})$$
$$= \lim_{x\to0}\frac{cosx-1}{xcosx+sinx} \ (\tfrac{0}{0} \text{ form})$$
$$= \lim_{x\to0}\frac{-sinx}{-xsinx+2cosx}$$
$$= 0$$

Example 3:

Find $\lim_{x\to+\infty}\left(\sqrt{x^2+3x}-x\right)$.

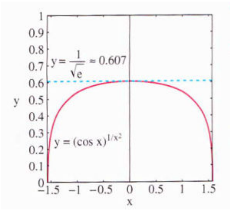


Solutions

$$\lim_{x\to+\infty}\left(\sqrt{x^2+3x}-x\right) \ (\infty-\infty \text{ form})$$
$$= \lim_{x\to+\infty} x\left(\sqrt{1+\frac{3}{x}}-1\right) \ (0\times\infty \text{ form})$$
$$= \lim_{x\to+\infty}\frac{\sqrt{1+\frac{3}{x}}-1}{\frac{1}{x}} \ (\tfrac{0}{0} \text{ form})$$
$$= \lim_{x\to+\infty}\frac{\frac{1}{2\sqrt{1+\frac{3}{x}}}\cdot\frac{-3}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to+\infty}\frac{3}{2\sqrt{1+\frac{3}{x}}} = \frac{3}{2}$$

Example 4:

Find $\lim\limits_{x\to 0}(cosx)^{1/x^2}$. ($1^\infty$ form)



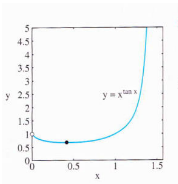
Solutions

$$ln\left(\lim_{x\to 0}(cosx)^{1/x^2}\right)$$
$$= \lim_{x\to 0} ln(cosx)^{\frac{1}{x^2}}$$
$$= \lim_{x\to 0}\frac{1}{x^2}ln(cosx)$$
$$= \lim_{x\to 0}\frac{ln(cosx)}{x^2} \quad (\tfrac{0}{0}\text{ form})$$
$$= \lim_{x\to 0}\frac{\frac{-sinx}{cosx}}{2x}$$
$$= \lim_{x\to 0}\frac{sinx}{x}\cdot\frac{-1}{2}\cdot\frac{1}{cosx}$$
$$= \frac{-1}{2}$$

So, $\lim\limits_{x\to 0}(cosx)^{1/x^2} = e^{-1/2} = \frac{1}{\sqrt{e}}$

Example 5:

Find $\lim\limits_{x\to 0^+} x^{tanx}$. ($0^0$ form)



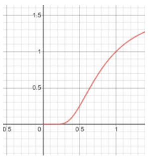
Solutions

$$ln\left(\lim_{x\to 0^+} x^{tanx}\right)$$
$$= \lim_{x\to 0^+} ln(x^{tanx})$$
$$= \lim_{x\to 0^+} tanx\cdot lnx$$
$$= \lim_{x\to 0^+}\frac{lnx}{cotx} \quad (\tfrac{\infty}{\infty}\text{ form})$$
$$= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-csc^2x}$$
$$= \lim_{x\to 0^+}\frac{sinx}{x}\cdot sinx\cdot(-1)$$
$$= 0$$

So, $\lim\limits_{x\to 0^+} x^{tanx} = e^0 = 1$

Example 6:

Find $\lim\limits_{x\to 0^+} x^{1/x}$.



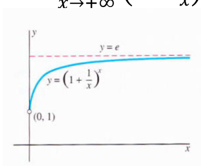
Solutions

$$ln\left(\lim_{x\to 0^+} x^{1/x}\right)$$
$$= \lim_{x\to 0^+} ln(x^{1/x})$$
$$= \lim_{x\to 0^+}\frac{1}{x}lnx$$
$$= \lim_{x\to 0^+}\frac{lnx}{x}$$
$$= -\infty$$

So, $\lim\limits_{x\to 0^+} x^{1/x} = 0$

Example 7:

Find $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$.



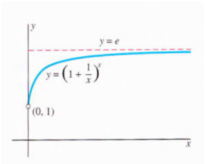
Solutions

$$ln\left(\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right)$$
$$= \lim_{x\to+\infty} ln\left(1+\frac{1}{x}\right)^x$$
$$= \lim_{x\to+\infty} xln\left(1+\frac{1}{x}\right)$$
$$= \lim_{x\to+\infty}\frac{ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad (\tfrac{0}{0}\text{ form})$$
$$= \lim_{x\to+\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to+\infty}\frac{1}{1+\frac{1}{x}} = 1$$

So, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$

Example 8:

Find $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$.



Solutions

$$ln\left(\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x\right)$$
$$=\lim_{x\to 0^+} ln\left(1+\frac{1}{x}\right)^x$$
$$=\lim_{x\to 0^+} xln\left(1+\frac{1}{x}\right)$$
$$=\lim_{x\to 0^+}\frac{ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\ (\tfrac{\infty}{\infty}\text{ form})$$
$$=\lim_{x\to 0^+}\frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}}$$
$$=\lim_{x\to 0^+}\frac{1}{1+\frac{1}{x}}=\lim_{x\to 0^+}\frac{x}{x+1}=0$$

So, $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=e^0=1$

Example 9:

Find $\lim_{x\to +\infty}\left(1+\frac{u}{x}\right)^x$.

Solutions

$$ln\left(\lim_{x\to +\infty}\left(1+\frac{u}{x}\right)^x\right)$$
$$=\lim_{x\to +\infty} ln\left(1+\frac{u}{x}\right)^x$$
$$=\lim_{x\to +\infty} xln\left(1+\frac{u}{x}\right)$$
$$=\lim_{x\to +\infty}\frac{ln\left(1+\frac{u}{x}\right)}{\frac{1}{x}}\ (\tfrac{0}{0}\text{ form})$$
$$=\lim_{x\to +\infty}\frac{\frac{1}{1+\frac{u}{x}}\cdot\frac{-u}{x^2}}{\frac{-1}{x^2}}$$
$$=\lim_{x\to +\infty}\frac{u}{1+\frac{u}{x}}=u$$

So, $\lim_{x\to +\infty}\left(1+\frac{u}{x}\right)^x=e^u$.

Exercises:

Find:

(i) $\lim_{x\to +\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

(ii) $\lim_{x\to 0^+} x^x$

(iii) $\lim_{x\to +\infty}\left(\frac{2x-1}{2x+1}\right)^x$

(iv) $\lim_{x\to 2^+}\left(\frac{1}{x-2}-\frac{1}{ln(x-1)}\right)$