GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 1: Basic Knowledge on Functions

 # Definition of a function

A real valued function f defined on a set D (such that $\phi \neq D \subset R$) is a rule that assigns to each number $x \in D$ exactly one real number, denoted by $f(x)$.

Notes:

(a) R is the set of all real numbers.
(b) We say $f: D \to R$ is a function.
(c) The set D is called **the domain** of f.
(d) The set $\{y \in R: y = f(x)\ for\ some\ x \in D\}$ is called **the range** of f.

Examples

1. Let $D = R$ and f is defined by $f(x) = x^2$.
This f is a real valued function defined on D.

2. Let $D = R$ and g is defined by
$g(x) = \begin{cases} x^2 & if\ x \geq 0 \\ \sqrt{-x} & if\ x < 0 \end{cases}$.
This g is a real valued function defined on D.

3. Let $D = R$ and h is defined by $h(x) = \frac{1}{x}$.
This h is NOT a real valued function defined on D.

[Note: h is NOT defined at 0. (NO assignments for $0 \in D$)]

4. Let $D = R\backslash\{0\} = \{x \in R: x \neq 0\}$ and h is defined by $h(x) = \frac{1}{x}$.
This h is a real valued function defined on D.

5. Let $D = \{x \in R: x > 0\}$ and h is defined by $h(x) = \frac{1}{x}$.
This h is a real valued function defined on D.

Remark: Usually we choose the "largest set" D so that h is a real valued function defined on D. This set is called **the natural domain**.

Question: What do we mean "largest set"?

[Suppose $\phi \neq D_1 \subset R$, $\phi \neq D_2 \subset R$, $D_1 \subset D_2$ and $D_1 \neq D_2$, we may say D_1 is smaller than D_2 or D_2 is larger than D_1.]

6. Let $D = [-1,1] = \{x \in R: -1 \leq x \leq 1\}$ and f is defined by assigning x to y such that $x^2 + y^2 = 1$.

This f is NOT a real valued function defined on D.

[Reason: when $x = 0$, both $y = 1$ and $y = -1$ satisfy the equation $x^2 + y^2 = 1$. Such $x = 0$ will be assigned to more than one real number y.]

Suppose $f: D \to R$ is a function and $y = f(x)$. x is called **an independent variable** and y is called **a dependent variable**. Sometimes, we say y depends on x.

Examples

7. Let $f: R \to R$ be defined by $f(x) = x^2 + x - 3$.

   Is f is a real valued function defined on the entire real number line R?

   What are the values of $f(-2), f(0), f(3), f(x+1)$ and $f(2x)$?

   Answers: $-1, -3, 9, x^2 + 3x - 1, 4x^2 + 2x - 3$

8. Suppose $f: R \to R$ is a function and we know
   $f(x+1) = 2x^2 - x + 3$ for any $x \in R$.

   What is the formula for $f(x)$?

   Answer: $2x^2 - 5x + 6$

Examples for Intervals of Real Numbers

A closed interval	$[1,3] = \{x \in R: 1 \leq x \leq 3\}$
An open interval	$(-1,2) = \{x \in R: -1 < x < 2\}$
A half-open interval	$[0,1.5) = \{x \in R: 0 \leq x < 1.5\}$
A half-open interval	$(-7,3] = \{x \in R: -7 < x \leq 3\}$
A unbounded interval	$(-\infty, 2] = \{x \in R: x \leq 2\}$
A unbounded interval	$\left(\frac{1}{2}, \infty\right) = \left\{x \in R: \frac{1}{2} < x\right\}$
A unbounded interval	$(-\infty, \infty) = R =$ the set of all real numbers

Note:
An interval I (such that $\phi \neq I \subset R$) is **bounded** if there exists a real number $B > 0$ such that $|x| \leq B$ for any $x \in I$.

Such B is called an upper bound for I (this is NOT unique.) and $-B$ is called a lower bound for I (this is also NOT unique.).

We have $-B \leq x \leq B$ for any $x \in I$.

Two functions $f: D \to R$ and $g: E \to R$ are the **SAME** if:
(We say f equals to g. We write $f = g$.)

(a) $D = E$ (**SAME domain**) **AND**

(b) $f(x) = g(x)$ for any $x \in D$ (**SAME assignment**),

otherwise we say f and g are **DIFFERENT**.
(We say f doesn't equal to g. We write $f \neq g$.)

Examples

(a) Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = (x+3)^2$ and $g(x) = x^2 + 6x + 9$. Then, $f = g$.

(b) Let $f: R\backslash\{0\} \to R$ and $g: \{x \in R: x > 0\} \to R$ be defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x}$. Then, $f \neq g$. (**different domains**)

Exercises

1. Find the (natural) domain of f that is defined by $f(x) = \frac{1}{2x+4}$.

Note:
The only trouble point is when the denominator is zero
(that is when $2x + 4 = 0$).

The (natural) domain is $D = \{x \in R: 2x + 4 \neq 0\}$
$= (-\infty, -2) \cup (-2, \infty)$.

2. Find the (natural) domain of g that is defined by $g(x) = \frac{1}{\sqrt{2x+4}}$.

We need:
(i) $2x + 4 \geq 0$ (so that the square root is meaningful)
(ii) $2x + 4 \neq 0$ (so that the denominator is non-zero)

Thus $2x + 4 > 0$.

The (natural) domain is $D = \{x \in R: 2x + 4 > 0\}$
$= (-2, \infty)$.

Note:
$\sqrt{0} = 0$; $\sqrt{4} = 2$; $\sqrt{-4}$ is NOT a real number;
$\sqrt{x} = y$ means "$y \geq 0$ and $x = y^2$" when we consider real numbers only.
$\sqrt{x}$ is called the non-negative square root of x.
We don't write $\sqrt{4} = \pm 2$.

Mathematical Modeling (Reading assignments)

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 1: Basic Knowledge on Functions

Graphs of common curves given by equations (Revision)

Straight Line

(a) Write an equation of the straight line L passing through the point $P(3,5)$ and is parallel to the line with equation $y = 2x - 4$. Sketch the graph of L.

Answer: $y = 2x - 1$

(b) An equation of the straight line L is given by $Ax + By + C = 0$, where $A \neq 0$ and $B \neq 0$. Find the slope, $x -$ intercept and $y -$ intercept of L.

Answer: slope $= \frac{-A}{B}$, $x -$ intercept $= \frac{-C}{A}$, $y -$ intercept $= \frac{-C}{B}$

Circle

(c) Find the coordinates of the centre and radius of the circle C given by the equation $x^2 + y^2 - 6x - 8y - 75 = 0$.
Sketch the graph of C.

Answers: the center is $(3,4)$, radius $= 10$

(d) Assume $D^2 + E^2 - 4F \geq 0$, find the coordinates of the centre and radius of the circle given by the equation $x^2 + y^2 + Dx + Ey + F = 0$, where D, E and F are fixed real numbers (constants).

Answer: the center is $\left(\frac{-D}{2}, \frac{-E}{2}\right)$, radius $= \frac{1}{2}\sqrt{(D^2 + E^2 - 4F)}$

Parabola

(e) Sketch the following graphs on the same diagram:
- (i) $y = x^2$
- (ii) $y = 2x^2$
- (iii) $y = \frac{1}{2}x^2$

Any observations?

(f) Sketch the following graphs on the same diagram:
- (i) $y = x^2$
- (ii) $y = x^2 + 2$
- (iii) $y = x^2 - 2$

Any observations?

(g) Sketch the following graphs on the same diagram:
- (i) $y = x^2$
- (ii) $y = (x + 2)^2$
- (iii) $y = (x - 2)^2$

Any observations?

(h) Sketch the graph of $y = 2(x - 3)^2 + 5$

(i) Sketch the following graphs on the same diagram:
- (i) $y = -x^2$
- (ii) $y = -2x^2$
- (iii) $y = \frac{-1}{2}x^2$

Any observations?

(j) Sketch the following graphs on the same diagram:
- (i) $y = -x^2$

(ii) $y = -x^2 + 2$
(iii) $y = -x^2 - 2$

Any observations?

(k) Sketch the following graphs on the same diagram:
(i) $y = -x^2$
(ii) $y = -(x + 2)^2$
(iii) $y = -(x - 2)^2$

Any observations?

(l) Sketch the graph of $y = -2(x + 3)^2 - 5$

(m) Find the coordinates of the vertex point and an equation of the axis of symmetry of the parabola given by the equation:
(i) $y = a(x - h)^2 + k$
(ii) $y = ax^2 + bx + c$
(assumed $a \neq 0$, where a, b and c are fixed real numbers (constants).

Answer:
(i) the vertex is (h, k) and the axis of symmetry is given by $x = h$

(ii) the vertex is $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$ and the axis of symmetry is given by $x = \frac{-b}{2a}$
where $\Delta = b^2 - 4ac$

Note:
(i) When $a > 0$, the graph opens upward and the vertex is called the minimum point.

(ii) When $a < 0$, the graph opens downward and the vertex is called the maximum point.

(iii) The vertex is also called the turning point.

Absolute Value Function

Let $f: R \to R$ be defined by $f(x) = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$. This f is called the absolute value function and is denoted as $f(x) = |x|$.

Note: $|x| = \sqrt{x^2}$.

Exercise:
Sketch the graphs of:
(a) $y = 2|x - 3| - 5$
(b) $y = -2|x + 3| + 5$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 1: Basic Knowledge on Functions

Reciprocal Function

Let $f: R\setminus\{0\} \to R$ be defined by $f(x) = \frac{1}{x}$. This f is called the reciprocal function.

Exercise:
Sketch the graphs of:

(a) $$y = \frac{2}{x-3} - 5$$

(b) $$y = \frac{-2}{x+3} + 5$$

Square Root Function

Let $f: \{x \in R: x \geq 0\} \to R$ be defined by $f(x) = \sqrt{x}$. This f is called the square root function.

Exercise:
Sketch the graphs of:

(a) $y = 2\sqrt{x-3} - 5$ for $x \geq 3$

(b) $y = -2\sqrt{x+3} + 5$ for $x \geq -3$

Operations / Combinations of Functions

Let $f: R \to R$ and $g: R \to R$ be functions and c is a fixed real number (constant).
We can define the functions $f + g: R \to R, f - g: R \to R, f \cdot g: R \to R, \frac{f}{g}: R \to R$and $cf: R \to R$ as follows:

$$(f + g)(x) = f(x) + g(x),$$
$$(f - g)(x) = f(x) - g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x),$$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (Assumed $g(x) \neq 0$ for any $x \in R$)
$$(cf)(x) = c \cdot f(x)$$

Note the cases where Natural Domain of $f \neq R$ or Natural Domain of $g \neq R$

(i) We may use the natural domain D (such that $\phi \neq D \subset R$) of f instead of R for defining cf.

(ii) We may use the set D (such that $\phi \neq D \subset R$) instead of R for defining $f + g, f - g, f \cdot g, \frac{f}{g}$ if $D \subset$ Natural Domain of f and $D \subset$ Natural Domain of g.

Exercises

Let $f: R \to R$ and $g: R \to R$ be functions defined by $f(x) = x^2 + 1$ and $g(x) = x - 1$. Find:

(i) $(f + g)(5)$
(ii) $(f - g)(4)$
(iii) $(f \cdot g)(3)$
(iv) $\left(\frac{f}{g}\right)(2)$
(v) $(6f)(1)$

Answer: 30, 14, 20, 5, 12

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 1: Basic Knowledge on Functions

Polynomial Functions

A polynomial of degree 0	Suppose a is a fixed real number (constant). Let $P: R \to R$ be defined by $P(x) = a$ for any $x \in R$. P is called a **constant function**.
A polynomial of degree 1	Suppose a, b are fixed real numbers (constant) and $a \neq 0$. Let $P: R \to R$ be defined by $P(x) = ax + b$ for any $x \in R$. P is called a **linear function**.
A polynomial of degree 2	Suppose a, b, c are fixed real numbers (constant) and $a \neq 0$. Let $P: R \to R$ be defined by $P(x) = ax^2 + bx + c$ for any $x \in R$. P is called a **quadratic function**.
A polynomial of degree 3	Suppose a, b, c, d are fixed real numbers (constant) and $a \neq 0$. Let $P: R \to R$ be defined by $P(x) = ax^3 + bx^2 + cx + d$ for any $x \in R$. P is called a **cubic function**.
A polynomial of degree n	Suppose $a_n, a_{n-1}, \cdots, a_1, a_0$ are fixed real numbers (constant) and $a_n \neq 0$. Let $P: R \to R$ be defined by $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for any $x \in R$. P is called a **polynomial function of degree $\boldsymbol{n}$**.

Rational Functions

Suppose $P: R \to R$ and $Q: R \to R$ are polynomial functions and $D = R \backslash \{x \in R: Q(x) = 0\}$.
Then, we can define $\frac{P}{Q}: D \to R$ by $\frac{P}{Q}(x) = \frac{P(x)}{Q(x)}$.
$\frac{P}{Q}$ is called a rational function (that is, the fraction of two polynomials)

Power Functions

A power function of degree 0	Let $P_0: R \to R$ be defined by $P_0(x) = 1$ for any $x \in R$.
A power function of degree 1	Let $P_1: R \to R$ be defined by $P_1(x) = x$ for any $x \in R$.
A power function of degree 2	Let $P_2: R \to R$ be defined by $P_2(x) = x^2$ for any $x \in R$.
A power function of degree n, where $n = 3,4,5, \cdots$	Let $P_n: R \to R$ be defined by $P_n(x) = x^n$ for any $x \in R$.

Remark:

Suppose $a_n, a_{n-1}, \cdots, a_1, a_0$ are fixed real numbers (constant) and $a_n \neq 0$.
Let $P: R \to R$ be defined by $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for any $x \in R$.
Then, we write $P = \sum_{i=0}^{n} a_i P_i = a_0 P_0 + a_1 P_1 + \cdots + a_n P_n$.

Definition for a zero of a function

Suppose $f: R \to R$ is a function and a is a real number such that $f(a) = 0$. a is called a zero of f.
It may not be unique.
It is also called $x -$ intercept for $y = f(x)$.

Example:
Let $f: R \to R$ be defined by $f(x) = x^2 - 4$.
2 is a zero of f as $f(2) = 0$.
-2 is also a zero of f as $f(-2) = 0$.
2 and -2 are $x -$ intercepts for $y = x^2 - 4$.

Trigonometric Functions (Reading Assignments)

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Lecture Notes for Chapter 1: Basic Knowledge on Functions

Composition of function

Suppose that:

(i) $\phi \neq D \subset R$ and $\phi \neq E \subset R$ **AND**
(ii) $f: D \to R$ and $g: E \to R$ are functions **AND**
(iii) $f(x) \in E$ for any $x \in D$.

Sometimes, we write this as $f(D) \subset E$. Note: $f(D) = \{f(x): x \in D\}$.

We can define a new function $h: D \to R$ by $h(x) = g(y)$ where $y = f(x)$.
Sometimes, we write it as $h(x) = g\big(f(x)\big)$.
h is called **the composition of g and f**. Sometimes, we write as $h = g°f$.

Note:
We evaluate $h(x)$ in two steps: *1: $y = f(x)$; *2: $h(x) = g(y)$.
We write $g°f(x) = g\big(f(x)\big)$.

Examples

1. Let $D = \{x \in R: x \geq 0\}$ and $E = R$.
   Let $f: D \to R$ and $g: E \to R$ be defined by $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$.
   f and g are functions.
   Questions:
   (i) What is $g°f$?
   (ii) Is $f°g$ well defined on E?
   (iii) Is $f°g = g°f$ when both of them are well defined?

2. Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = x^2$ and $g(x) = cosx$.
   f and g are functions.
   Questions:
   (i) What is $f \cdot g$?
   (ii) What is $f°g$?
   (iii) What is $g°f$?

More considerations for the concept of a function

Suppose $\phi \neq D \subset R$ and $\phi \neq E \subset R$.

We say $f: D \to E$ is a function if f assigns to each number $x \in D$ exactly one real number in E.

[In other words, $f: D \to R$ is a function and $f(D) \subset E$]

We say f is **injective or one-to-one** if for any $a, b \in D$ with $f(a) = f(b) \Longrightarrow a = b$.

Examples

(i) Let $f: R \to R$ be defined by $f(x) = 2x + 1$.

f is **injective**.

Proof:

For any $a, b \in R$ with $f(a) = f(b)$,

we have $2a + 1 = 2b + 1 \Leftrightarrow 2a = 2b \Leftrightarrow a = b$

(ii) Let $g: R \to \{x \in R: x \geq 0\}$ be defined by $g(x) = x^2$.

g is **NOT injective**.

Proof:

$g(-1) = 1 = g(1)$, but $-1 \neq 1$

We say f is **surjective or onto** if for any $u \in E$, we can find $a \in D$ such that $f(a) = u$. Such a may not be unique.

Examples

(i) Let $f: R \to R$ be defined by $f(x) = 2x + 1$.

f is **surjective**.

Proof:

For any $u \in R$, let $a = \frac{u-1}{2}$, then $f(a) = 2\left(\frac{u-1}{2}\right) + 1 = u$.

(ii) Let $g: R \to \{x \in R: x \geq 0\}$ be defined by $g(x) = x^2$.

g is **surjective**.

Proof:

For any $u \geq 0$, let $a = \sqrt{u}$, then $g(a) = \left(\sqrt{u}\right)^2 = u$.

Also, $g(-a) = \left(-\sqrt{u}\right)^2 = u$.

(iii) Let $h: R \to R$ be defined by $h(x) = x^2$.

h is NOT **surjective**.

Proof:

For $u = -1$, we cannot find $a \in R$ such that $h(a) = u$.

That is, we cannot find $a \in R$ such that $a^2 = -1$.

We say f is **bijective** if f is both injective and surjective.

Examples

(i) Let $f: R \to R$ be defined by $f(x) = 2x + 1$.

f is **bijective**.

(ii) Let $g: R \to \{x \in R: x \geq 0\}$ be defined by $g(x) = x^2$.

g is NOT **bijective**.

Suppose $f: D \to E$ is a function and f is bijective.

For any $y \in E$, we have exactly one $x \in D$ such that $f(x) = y$.

We can define a function $g: E \to D$ by assigning $y \in E$ to $x \in D$.

Such g is also bijective.

Note 1:

Suppose $f: D \to E$ and $g: E \to D$ are functions.

For any $x \in D$, $g\big(f(x)\big) = x$ AND

For any $y \in E$, $f\big(g(y)\big) = y$.

We say f and g are **inverse functions to each other**.

Note 2:

Suppose f and g are **inverse functions to each other**.

Suppose (a, b) is a point on the graph of $y = f(x)$. That is, $b = f(a)$.

We have $a = g(b)$.

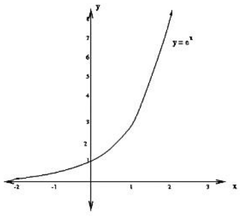
So, (b, a) is a point on the graph of $y = g(x)$.

Thus, the graphs of $y = f(x)$ and $y = g(x)$ are symmetric about the line $y = x$.

Exponential Function

Let $f: R \to R$ be defined by $f(x) = e^x$ for any $x \in R$ where $e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$.
This f is called **the exponential function**.
Sometimes, we write it as $f(x) = exp(x)$.
That is, $exp(x) = e^x$.

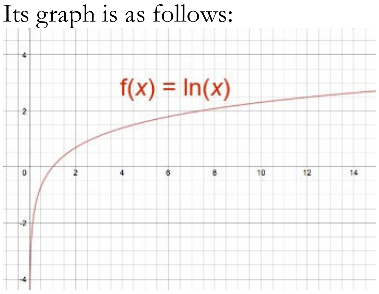
Its graph is as follows:



Logarithmic Function

Let $g: \{x \in R: x > 0\} \to R$ be defined by $g(x) = log_e x$ for any $x > 0$.
Sometimes, we write $log_e(x) = ln(x)$.
This g is called **the natural logarithmic function**.

Its graph is as follows:



Relationship between exponential and logarithmic functions

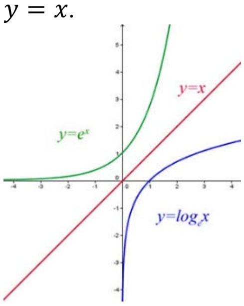
Notes:

(i) $exp: R \to \{x \in R: x > 0\}$ is bijective
(ii) $ln: \{x \in R: x > 0\} \to R$ is bijective
(iii) exp and ln are **inverse functions to each other**

That is,
$e^{lnx} = exp(lnx) = x$ for any $x > 0$ AND
$ln(e^x) = ln(exp(x)) = x$ for any $x \in R$

Notes:

(iv) The graphs of $exp: R \to \{x \in R: x > 0\}$ and $ln: \{x \in R: x > 0\} \to R$ are symmetric about the line $y = x$.



Exercises on graphs of functions inverse to each other:

Sketch the following graphs on the same diagram:

(a)

(i) $y = x^2$ for $x \geq 0$

(ii) $y = \sqrt{x}$ for $x \geq 0$

(b)

(i) $y = \frac{1}{x-1}$ for $x \neq 1$

(ii) $y = \frac{1}{x} + 1$ for $x \neq 0$

Transcendental Functions (Reading Assignments)