GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 2: Limits and Continuity

Tangents Lines and Slope Predictors

Example 1
Determine the slope of the line L tangent to the parabola $y = x^2$ at the point $P(a, a^2)$.

Solutions
Consider a point $Q(a + h, (a + h)^2)$ on the given parabola $y = x^2$.
The slope of the secant PQ is $\frac{(a+h)^2-a^2}{(a+h)-a} = \frac{a^2+2ah+h^2-a^2}{h} = \frac{2ah+h^2}{h} = 2a + h$.
As $Q(a + h, (a + h)^2) \to P(a, a^2) \Leftrightarrow h \to 0$ (the secant will overlap with the tangent), the slope of the tangent is $\lim_{h\to 0}(2a + h) = 2a$.

Observation for General Slope Predictor
Determine the slope of the line L tangent to the graph $y = f(x)$ at the point $P\big(a, f(a)\big)$.

Solutions
Consider a point $Q\big(a + h, f(a + h)\big)$ on the given curve $y = f(x)$.
The slope of the secant PQ is $\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$.
As $Q\big(a + h, f(a + h)\big) \to P\big(a, f(a)\big) \Leftrightarrow h \to 0$ (the secant will overlap with the tangent), the slope of the tangent is $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Example 2
Determine the slope of the line L tangent to the curve $y = ax^2 + bx + c$ at the point $P(x_0, a{x_0}^2 + bx_0 + c)$.

Solutions
Let $y = f(x) = ax^2 + bx + c$.
The slope of the tangent is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

$$= \lim_{h\to 0}\frac{[a(x_0 + h)^2 + b(x_0 + h) + c] - [a{x_0}^2 + bx_0 + c]}{h}$$
$$= \lim_{h\to 0}\frac{[a{x_0}^2 + 2ax_0h + ah^2 + bx_0 + bh + c] - [a{x_0}^2 + bx_0 + c]}{h}$$
$$= \lim_{h\to 0}\frac{2ax_0h + ah^2 + bh}{h}$$
$$= \lim_{h\to 0}(2ax_0 + ah + b)$$
$$= 2ax_0 + b$$

Example 3A
Determine an equation of the line L_1 tangent to the curve $y = 2x^2 - 3x + 5$ when $x = -1$.

Solutions
When $x = -1$, $y = 2(-1)^2 - 3(-1) + 5 = 10$.
$a = 2, b = -3, c = 5$.
Slope of required tangent L_1 is $2 \times 2 \times (-1) + (-3) = -7$
An equation of required tangent L_1 is

$$\frac{y - 10}{x - (-1)} = -7$$
$$y - 10 = -7(x + 1)$$

$y = -7x + 3$ (slope intercept form) or $7x + y - 3 = 0$ (general form).

Example 3B

Determine an equation of the line L_2 normal to the curve $y = 2x^2 - 3x + 5$ when $x = -1$.

Solutions

Slope of required normal L_2 is $\frac{-1}{-7} = \frac{1}{7}$.

An equation of required normal L_2 is

$$\frac{y - 10}{x - (-1)} = \frac{1}{7}$$

$7(y - 10) = x + 1$

$7y = x + 71$

$y = \frac{1}{7}x + \frac{71}{7}$ (slope intercept form) or $x - 7y + 71 = 0$ (general form)

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Lecture Notes for Chapter 2: Limits and Continuity

The Limit Concept

#1 $\varepsilon - \delta$ **Definition of a right hand limit for** $\lim_{x \to a^+} f(x) = L$

Suppose $f: R \to R$ is a function and $a, L \in R$.

For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that $0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon$

Remarks:

1. Sometimes, we write $f(x) \to L$ as $x \to a^+$
2. Sometimes, we say as $x \to a^+$, $f(x) \to L$
3. L is called **the limit** of f as $x \to a^+$.
   We say $\lim_{x \to a^+} f(x)$ exists as a real number L. (if the limit exists, it MUST be UNIQUE.)
4. $x \to a^+ \Leftrightarrow x \to a$ and $x > a$
5. Roughly speaking, when $x > a$ and x is very close to a, $f(x)$ will be very close to L.

Example

Show that $\lim_{x \to 2^+} (-2 + x) = 0$.

Proof:
Idea:
For any $\varepsilon > 0$, we need to find $\delta > 0$ such that $0 < x - 2 < \delta \Longrightarrow |(-2 + x) - 0| < \varepsilon$

Observe that $|(-2 + x) - 0| = |-2 + x| = x - 2$

Note: $0 < x - 2 \Longrightarrow -2 + x > 0 \Longrightarrow |-2 + x| = x - 2$

Thus, we may choose $\delta = \varepsilon$.

Formal way of writing:
For any $\varepsilon > 0$, we choose $\delta = \varepsilon > 0$ such that $0 < x - 2 < \delta$
$\Longrightarrow |(-2 + x) - 0| = |-2 + x| = x - 2 < \delta = \varepsilon$

Note: $0 < x - 2 \Longrightarrow -2 + x > 0 \Longrightarrow |-2 + x| = x - 2$

Observation:

x	$-2 + x$
2.1	0.1
2.01	0.01
2.001	0.001
2.0001	0.0001
2.00001	0.00001

when $x > 2$ and x is very close to 2, $-2 + x$ will be very close to 0.

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Lecture Notes for Chapter 2: Limits and Continuity

Theorem (Uniqueness of Limit)
Suppose $f: R \to R$ is a function and $a, L_1, L_2 \in R$.
If $\lim_{x \to a^+} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$, then $L_1 = L_2$.

Proof:
Case 1: $L_1 > L_2$
For $\varepsilon_0 = \frac{1}{3}(L_1 - L_2) > 0$,

as $\lim_{x \to a^+} f(x) = L_1$, we can find $\delta_1 > 0$ such that $0 < x - a < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_0$
that is, $0 < x - a < \delta_1 \Rightarrow L_1 - \varepsilon_0 < f(x) < L_1 + \varepsilon_0$

Note: $|f(x) - L_1| < \varepsilon_0$
$\Leftrightarrow -\varepsilon_0 < f(x) - L_1 < \varepsilon_0$
$\Leftrightarrow L_1 - \varepsilon_0 < f(x) < L_1 + \varepsilon_0$

as $\lim_{x \to a^+} f(x) = L_2$, we can find $\delta_2 > 0$ such that $0 < x - a < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon_0$
that is, $0 < x - a < \delta_2 \Rightarrow L_2 - \varepsilon_0 < f(x) < L_2 + \varepsilon_0$

We choose $\delta = \min(\delta_1, \delta_2) > 0$, then
$0 < x - a < \delta$
$\Rightarrow$ both $L_1 - \varepsilon_0 < f(x) < L_1 + \varepsilon_0$ and $L_2 - \varepsilon_0 < f(x) < L_2 + \varepsilon_0$
$(L_1 - \varepsilon_0, L_1 + \varepsilon_0) = \left(\frac{2L_1 + L_2}{3}, \frac{4L_1 - L_2}{3}\right)$
$(L_2 - \varepsilon_0, L_2 + \varepsilon_0) = \left(\frac{4L_2 - L_1}{3}, \frac{2L_2 + L_1}{3}\right)$

$L_1 > L_2 \Rightarrow L_1 + L_1 + L_2 > L_2 + L_1 + L_2 \Rightarrow \frac{2L_1 + L_2}{3} > \frac{2L_2 + L_1}{3}$
(that is, $2L_1 + L_2 > 2L_2 + L_1$)

Thus, $(L_1 - \varepsilon_0, L_1 + \varepsilon_0) \cap (L_2 - \varepsilon_0, L_2 + \varepsilon_0) = \phi$.

We get a contraction.
[we can't find $f(x) \in (L_1 - \varepsilon_0, L_1 + \varepsilon_0) \cap (L_2 - \varepsilon_0, L_2 + \varepsilon_0)$]

Case 2: $L_2 > L_1$ (Similar proof as Case 1)

Cases 1 & 2: We get a contraction.

Thus, $L_1 = L_2$.

#1 **$\varepsilon - \delta$ Definition of a left hand limit for $\lim_{x \to a^-} f(x) = L$**

Suppose $f: R \to R$ is a function and $a, L \in R$.

For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that $0 < a - x < \delta \Longrightarrow |f(x) - L| < \varepsilon$

Remarks:

1. Sometimes, we write $f(x) \to L$ as $x \to a^-$
2. Sometimes, we say as $x \to a^-$, $f(x) \to L$
3. L is called **the limit** of f as $x \to a^-$.
   We say $\lim_{x \to a^-} f(x)$ exists as a real number L. (if the limit exists, it MUST be UNIQUE.)
4. $x \to a^- \Leftrightarrow x \to a$ and $x < a$
5. Roughly speaking, when $x < a$ and x is very close to a, $f(x)$ will be very close to L.

Example

Show that $\lim_{x \to 2^-} (x + 2) = 4$.

Proof:
Idea:
For any $\varepsilon > 0$, we need to find $\delta > 0$ such that $0 < 2 - x < \delta \Longrightarrow |x + 2 - 4| < \varepsilon$

Observe that $|x + 2 - 4| = |x - 2| = 2 - x$

Note: $0 < 2 - x \Longrightarrow x - 2 < 0 \Longrightarrow |x - 2| = 2 - x$

Thus, we may choose $\delta = \varepsilon$.

Formal way of writing:
For any $\varepsilon > 0$, we choose $\delta = \varepsilon > 0$ such that $0 < 2 - x < \delta \Longrightarrow |x + 2 - 4| = |x - 2| = 2 - x < \delta = \varepsilon$

Note: $0 < 2 - x \Longrightarrow x - 2 < 0 \Longrightarrow |x - 2| = 2 - x$

Observation:

x	$x + 2$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999

when $x < 2$ and x is very close to 2, $x + 2$ will be very close to 4.

Theorem (Uniqueness of Limit)
Suppose $f: R \to R$ is a function and $a, L_1, L_2 \in R$.
If $\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^-} f(x) = L_2$, then $L_1 = L_2$.

Proof: Omitted

#1 **$\varepsilon - \delta$ Definition of a two-sided limit for $\lim_{x \to a} f(x) = L$**

Suppose $f: R \to R$ is a function and $a, L \in R$.

For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that $0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$

Remarks:

1. Sometimes, we write $f(x) \to L$ as $x \to a$
2. Sometimes, we say as $x \to a$, $f(x) \to L$
3. L is called **the limit** of f as $x \to a$.
   We say $\lim_{x \to a} f(x)$ exists as a real number L. (if the limit exists, it MUST be UNIQUE.)
4. $x \to a \Leftrightarrow x \to a^+$ or $x \to a^-$
5. Roughly speaking, when x is very close to a, $f(x)$ will be very close to L.

Example

$\lim_{x \to 3} \frac{x-1}{x+2} = \frac{2}{5} = 0.4.$

Observation:

x	$\frac{x-1}{x+2}$ (to 4 decimal places)	x	$\frac{x-1}{x+2}$ (to 4 decimal places)
3.1	0.4118	2.9	0.3878
3.01	0.4012	2.99	0.3988
3.001	0.4001	2.999	0.3999
3.0001	0.4000	2.9999	0.4000

Theorem (Uniqueness of Limit)
Suppose $f: R \to R$ is a function and $a, L_1, L_2 \in R$.
If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$, then $L_1 = L_2$.

Proof: Omitted

Theorem (Relationship between Two Sided Limit and Right Hand & Left Hand Limits)
Suppose $f: R \to R$ is a function and $a, L \in R$.
If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Proof:
Suppose $\lim_{x \to a} f(x) = L$.
For any $\varepsilon > 0$, we can find $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
Thus, $0 < x - a < \delta \Rightarrow 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
So, $\lim_{x \to a^+} f(x) = L$.

Also, $0 < a - x < \delta \Rightarrow 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
So, $\lim_{x \to a^-} f(x) = L$.

Collorary (Relationship between Two Sided Limit and Right Hand & Left Hand Limits)
Suppose $f: R \to R$ is a function and $a, L_1, L_2 \in R$.
If $\lim_{x \to a^+} f(x) = L_1$, $\lim_{x \to a^-} f(x) = L_2$ and $L_1 \neq L_2$, then $\lim_{x \to a} f(x)$ doesn't exist in R.

Proof: Obvious

Example 1
Let $f(x) = \begin{cases} -x + 2 & if\ x \geq 2 \\ x + 2 & if\ x < 2 \end{cases}$.
Show that $\lim_{x \to 2} f(x)$ doesn't exist in R.

Proof:
$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x + 2) = 0$
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 2) = 4$
$\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$
So, $\lim_{x \to 2} f(x)$ doesn't exist in R.
Note: $f(2)$ is defined.

Example 2
Let $f(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$.
Show that $\lim_{x \to 0} f(x)$ doesn't exist in R.

Proof:
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$
$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1$
$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$
So, $\lim_{x \to 0} f(x)$ doesn't exist in R.
Note: $f(0)$ isn't defined.

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Theorem (Relationship between Two Sided Limit and Right Hand & Left Hand Limits)
Suppose $f: R \to R$ is a function and $a, L \in R$.
If $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

Proof:
Suppose $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.
For any $\varepsilon > 0$,
as $\lim_{x\to a^+} f(x) = L$, we can find $\delta_1 > 0$ such that $0 < x - a < \delta_1 \Longrightarrow |f(x) - L| < \varepsilon$;
as $\lim_{x\to a^-} f(x) = L$, we can find $\delta_2 > 0$ such that $0 < a - x < \delta_2 \Longrightarrow |f(x) - L| < \varepsilon$.

We choose $\delta = min(\delta_1, \delta_2) > 0$,
$0 < x - a < \delta \Longrightarrow 0 < x - a < \delta_1 \Longrightarrow |f(x) - L| < \varepsilon$;
Also, $0 < a - x < \delta \Longrightarrow 0 < a - x < \delta_2 \Longrightarrow |f(x) - L| < \varepsilon$.
Thus, $0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.
(Note: $0 < |x - a| < \delta \Leftrightarrow 0 < x - a < \delta$ or $0 < a - x < \delta$.)

Example

Find $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} 1 & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$.

Solutions
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$.
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1$.
So, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1$. Thus, $\lim_{x\to 0} f(x) = 1$.

Note 1: $\lim_{x\to 0} f(x)$ exists as a real number.
Note 2: $f(0)$ is defined but $\lim_{x\to 0} f(x) = 1 \neq 0 = f(0)$.

 # The Limit Rules
Constant Rule

Suppose C is a fixed real number (constant) and let $a \in R$.
Suppose $f: R \to R$ is a function defined by $f(x) = C$ for any $x \in R$.
Then, $\lim_{x\to a} f(x) = C$.

Proof:
For any $\varepsilon > 0$, we choose $\delta = \varepsilon > 0$, then $0 < |x - a| < \delta$
$\Longrightarrow |f(x) - C| < \varepsilon$ as $|f(x) - C| = |C - C| = 0 < \varepsilon$ for any $x \in R$.

Remark: We write the above result as $\lim_{x\to a} C = C$.

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Rules for Limits (Sum, Difference, Product, Quotient and Scalar Multiple Rules)

Let $a, L, M, \lambda \in R$. Let $f: R \to R$ and $g: R \to R$ be functions.
Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then,

1. $\lim_{x\to a}(f+g)(x) = L + M$
2. $\lim_{x\to a}(f-g)(x) = L - M$
3. $\lim_{x\to a}(f \cdot g)(x) = LM$
4. $\lim_{x\to a}\frac{f}{g}(x) = \frac{L}{M}$
   (Assumed $M \neq 0$ and $g(x) \neq 0$ when x is near to a)
5. $\lim_{x\to a}(\lambda f)(x) = \lambda L$

Proof: Omitted

Remark: We have similar rules for Right Hand and Left Hand Limits.

Example 1

Find $\lim_{x\to 3}(x^2 + 2x + 4)$.

Solutions

$\lim_{x\to 3}(x^2+2x+4) = \lim_{x\to 3} x^2 + \lim_{x\to 3} 2x + \lim_{x\to 3} 4$

$= \left(\lim_{x\to 3} x\right)^2 + 2\lim_{x\to 3} x + \lim_{x\to 3} 4 = 3^2 + 2\times 3 + 4 = 19$

Example 2

Find $\lim_{x\to 3}\frac{2x+5}{x^2+2x+4}$.

Solutions

$$\lim_{x\to 3}\frac{2x+5}{x^2+2x+4} = \frac{2\times 3+5}{3^2+2\times 3+4} = \frac{11}{19}$$

Example 3

Find $\lim_{x\to 3}\frac{x-1}{x+2}$.

Solutions

$\lim_{x\to 3}\frac{x-1}{x+2} = \frac{3-1}{3+2} = \frac{2}{5} = 0.4$.

Example 4

Find $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$.

Solutions

$$\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$$
$$= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2}\frac{x+2}{x+3}$$
$$= \frac{2+2}{2+3} = \frac{4}{5}$$

Remarks:

(i) Let $f: R\backslash\{2\} \to R$ is a function defined by $f(x) = \frac{x^2-4}{x^2+x-6}$ if $x \neq 2$.
$\lim_{x\to 2} f(x)$ exists as a real number but $f(2)$ isn't defined.

(ii) Let $g: R \to R$ is a function defined by
$$g(x) = \begin{cases} \frac{x^2-4}{x^2+x-6} & if\ x \neq 2 \\ 0 & if\ x = 2 \end{cases}.$$
Note: $f \neq g$.

(iii) $\lim_{x\to 2} g(x) = \lim_{x\to 2} f(x)$ exists as a real number and $g(2)$ is defined.

Example 5

Find $\lim_{t\to 0}\frac{\sqrt{t+25}-5}{t}$.

Solutions

$$\lim_{t\to 0}\frac{\sqrt{t+25}-5}{t}$$
$$=\lim_{t\to 0}\frac{\sqrt{t+25}-5}{t}\cdot\frac{\sqrt{t+25}+5}{\sqrt{t+25}+5}$$
$$=\lim_{t\to 0}\frac{\left(\sqrt{t+25}\right)^2-5^2}{t\left(\sqrt{t+25}+5\right)}$$
$$=\lim_{t\to 0}\frac{t+25-25}{t\left(\sqrt{t+25}+5\right)}$$
$$=\lim_{t\to 0}\frac{t}{t\left(\sqrt{t+25}+5\right)}=\lim_{t\to 0}\frac{1}{\sqrt{t+25}+5}$$
$$=\frac{1}{\sqrt{0+25}+5}=\frac{1}{\sqrt{25}+5}=\frac{1}{5+5}=\frac{1}{10}$$

Substitution Rule

Let $f\colon R\to R$ and $g\colon R\to R$ be functions. Let $a, L\in R$.
Suppose $\lim_{x\to a} g(x)=L$ and $\lim_{x\to L} f(x)=f(L)$.
Then, $\lim_{x\to a} f\big(g(x)\big)=f(L)$. That is, $\lim_{x\to a} f\big(g(x)\big)=f\left(\lim_{x\to a} g(x)\right)$.

Proof:
For any $\varepsilon>0$,

as $\lim_{x\to L} f(x)=f(L)$, we can find $\theta>0$ such that
$0<|x-L|<\theta \Longrightarrow |f(x)-f(L)|<\varepsilon$ **AND**

as $\lim_{x\to a} g(x)=L$, we can find $\delta>0$ such that
$0<|x-a|<\delta \Longrightarrow |g(x)-L|<\theta$.

So, for this $\delta>0$, we have
$0<|x-a|<\delta$
$\Longrightarrow |g(x)-L|<\theta$
$\Longrightarrow \left|f\big(g(x)\big)-f(L)\right|<\varepsilon$.

Thus, $\lim_{x\to a} f\big(g(x)\big)=f(L)$.

Root Rule

Let $a\in R$ and $a>0$. Let $n=2,4,6,8,\cdots$.
Then, $\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$.

Proof: Use Substitution Rule

Index Rule

Let $a\in R$ and $a>0$. Let $n=2,4,6,8,\cdots$.
Then, $\lim_{x\to a} x^{m/n}=a^{m/n}$.

Proof: Use Root and Product Rules

Example

Find $\lim\limits_{x\to 4} \sqrt[3]{3\cdot\sqrt{x^3}+20\cdot\sqrt{x}}$

Solutions

$\mathrm{Lim}_{x\to 4} \sqrt[3]{3\cdot\sqrt{x^3}+20\cdot\sqrt{x}} = \sqrt[3]{3\cdot\sqrt{4^3}+20\cdot\sqrt{4}} = \sqrt[3]{3\times 8+20\times 2} = \sqrt[3]{64} = 4.$

Observation

Investigate $\lim\limits_{x\to 1}\frac{1}{(x-1)^2}$.

Solutions

x	$\frac{1}{(x-1)^2}$	x	$\frac{1}{(x-1)^2}$
1.1	100	0.9	100
1.01	10000	0.99	10000
1.001	1000000	0.999	1000000

As $x\to 1$, $\frac{1}{(x-1)^2}\to +\infty$ (it can be a very large real number).

We say $\lim\limits_{x\to 1}\frac{1}{(x-1)^2}$ doesn't exist as a real number.

We shall explain the definition $\lim\limits_{x\to 1}\frac{1}{(x-1)^2} = +\infty$ later.

Squeeze Rule

Let $f: R \to R$, $g: R \to R$ and $h: R \to R$ be functions.
Let $a, \theta, L \in R$ and $\theta > 0$.
Suppose $f(x) \leq g(x) \leq h(x)$ for any x with $0 < |x - a| < \theta$.
Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$.
Then, $\lim_{x \to a} g(x) = L$.

Proof:
For any $\varepsilon > 0$,

as $\lim_{x \to a} f(x) = L$, we can find $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Longrightarrow |f(x) - L| < \varepsilon$
(that is, $L - \varepsilon < f(x) < L + \varepsilon$)

AND
as $\lim_{x \to a} h(x) = L$, we can find $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Longrightarrow |h(x) - L| < \varepsilon$
(that is, $L - \varepsilon < h(x) < L + \varepsilon$)

We choose $\delta = min(\delta_1, \delta_2, \theta) > 0$.

$0 < |x - a| < \delta \Longrightarrow 0 < |x - a| < \delta_1 \Longrightarrow L - \varepsilon < f(x)$

$0 < |x - a| < \delta \Longrightarrow 0 < |x - a| < \theta \Longrightarrow f(x) \leq g(x) \leq h(x)$

$0 < |x - a| < \delta \Longrightarrow 0 < |x - a| < \delta_2 \Longrightarrow h(x) < L + \varepsilon$

Combining them, $0 < |x - a| < \delta \Longrightarrow L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$
(that is, $|g(x) - L| < \varepsilon$)

Thus, $\lim_{x \to a} g(x) = L$.

 # Basic Trigonometric Limits

1. $\lim\limits_{\theta\to 0} sin\theta = 0$
2. $\lim\limits_{\theta\to 0} cos\theta = 1$
3. $\lim\limits_{\theta\to 0} \frac{sin\theta}{\theta} = 1$

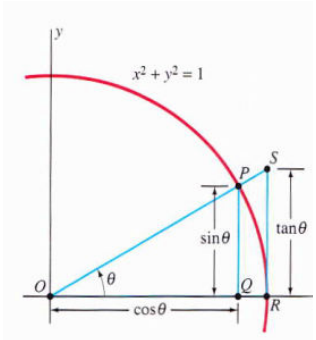
Note: θ is in radian measure.

Proof for (3):

Suppose $0 < \theta < \frac{\pi}{2}$.

Consider two right angled triangles ΔOPQ and ΔOSR, where $\angle OQP = \angle ORS = 90^o$.

Let $\angle POQ = \angle SOR = \theta$, P and R are the points on the circle $x^2 + y^2 = 1$.



Consider ΔOPQ, $OQ = cos\theta$ and $PQ = sin\theta$.

Area of ΔOPQ is $\frac{1}{2} \times OQ \times PQ = \frac{1}{2} sin\theta cos\theta$.

Consider the sector OPR, area of the sector is $\frac{\theta}{2\pi} \times \pi \times 1^2 = \frac{1}{2}\theta$.

Consider ΔOSR, $OR = 1$ and $SR = tan\theta$.

Area of ΔOSR is $\frac{1}{2} \times OR \times SR = \frac{1}{2} \times 1 \times tan\theta = \frac{1}{2} tan\theta$.

As Area of $\Delta OPQ <$ Area of sector $OPR <$ Area of ΔOSR,

$\frac{1}{2} sin\theta cos\theta < \frac{1}{2}\theta < \frac{1}{2} tan\theta$.

So, $sin\theta cos\theta < \theta$ and $\theta < tan\theta$.

So, $\frac{sin\theta}{\theta} < \frac{1}{cos\theta}$ and $cos\theta < \frac{sin\theta}{\theta}$.

That is, $cos\theta < \frac{sin\theta}{\theta} < \frac{1}{cos\theta}$.

$\lim\limits_{\theta\to 0^+} cos\theta = 1$ and $\lim\limits_{\theta\to 0^+} \frac{1}{cos\theta} = \frac{1}{\lim\limits_{\theta\to 0^+} cos\theta} = \frac{1}{1} = 1$.

By Squeeze Rule, $\lim\limits_{\theta\to 0^+} \frac{sin\theta}{\theta} = 1$.

Case 1: $\lim\limits_{\theta\to 0^+} \frac{sin\theta}{\theta} = 1$ (Proof as above)

Case 2: $\lim\limits_{\theta\to 0^-} \frac{sin\theta}{\theta} = 1$

Proof:

$\lim\limits_{\theta\to 0^-} \frac{sin\theta}{\theta}$

$= \lim\limits_{\alpha\to 0^+} \frac{sin(-\alpha)}{-\alpha}$ (let $\theta = -\alpha, \theta \to 0^- \Leftrightarrow \alpha \to 0^+$)

$= \lim\limits_{\alpha\to 0^+} \frac{-sin\alpha}{-\alpha}$ ($sin(-\alpha) = -sin\alpha$)

$= \lim\limits_{\alpha\to 0^+} \frac{sin\alpha}{\alpha}$

$= 1$ (Use the result of Case 1)

Thus, $\lim\limits_{\theta\to 0} \frac{sin\theta}{\theta} = 1$.

Example 1

Find $\lim_{\theta\to 0}\frac{1-cos\theta}{\theta}$.

Solutions

$$\lim_{\theta\to 0}\frac{1-cos\theta}{\theta}$$
$$=\lim_{\theta\to 0}\frac{1-cos\theta}{\theta}\cdot\frac{1+cos\theta}{1+cos\theta}$$
$$=\lim_{\theta\to 0}\frac{1-cos^2\theta}{\theta(1+cos\theta)}$$
$$=\lim_{\theta\to 0}\frac{sin^2\theta}{\theta(1+cos\theta)}$$
$$=\lim_{\theta\to 0}\frac{sin\theta}{\theta}\cdot sin\theta\cdot\frac{1}{1+cos\theta}$$
$$=1\times 0\times\frac{1}{1+1}=0$$

Example 2

Find $\lim_{x\to 0}\frac{tan3x}{x}$.

Solutions

$$\lim_{x\to 0}\frac{tan3x}{x}$$
$$=\lim_{x\to 0}\frac{sin3x}{3x}\cdot 3\cdot\frac{1}{cos3x}$$
$$=1\times 3\times\frac{1}{1}=3$$

Notes:

$\lim_{x\to 0}\frac{sin3x}{3x}=\lim_{\theta\to 0}\frac{sin\theta}{\theta}=1$ (let $\theta=3x$, $x\to 0\Leftrightarrow\theta\to 0$)

$\lim_{x\to 0}cos3x=\lim_{\theta\to 0}cos\theta=1$ (let $\theta=3x$, $x\to 0\Leftrightarrow\theta\to 0$)

Example 3

Find $\lim_{x\to 0}xsin\frac{1}{x}$.

Solutions

$-1\le sin\frac{1}{x}\le 1$ for any $x\ne 0$.

For $x>0$, we have $-x\le xsin\frac{1}{x}\le x$.
$\lim_{x\to 0^+}-x=-\lim_{x\to 0^+}x=-0=0$ and
$\lim_{x\to 0^+}x=0$.
By Squeeze Rule, $\lim_{x\to 0^+}xsin\frac{1}{x}=0$.

For $x<0$, we have $-x\ge xsin\frac{1}{x}\ge x$.
$\lim_{x\to 0^-}x=0$ and
$\lim_{x\to 0^-}-x=-\lim_{x\to 0^-}x=-0=0$.
By Squeeze Rule, $\lim_{x\to 0^-}xsin\frac{1}{x}=0$.

So, $\lim_{x\to 0^+}xsin\frac{1}{x}=\lim_{x\to 0^-}xsin\frac{1}{x}=0$.
Thus, $\lim_{x\to 0}xsin\frac{1}{x}=0$.

Remark: $\lim_{x\to 0}xsin\frac{1}{x}=0$ but $\lim_{x\to 0}sin\frac{1}{x}$ doesn't exist.

Example 4

Show that $\lim_{x\to 0}sin\frac{\pi}{x}$ doesn't exist.
(Hence, $\lim_{x\to 0}sin\frac{1}{x}$ doesn't exist.)

Proof:
Let $a_n=\frac{1}{2n+1}$ for $n=1,2,3,\cdots$.
Then, $a_n\to 0$ as $n\to+\infty$ AND
$sin\frac{\pi}{a_n}=sin(2n+1)\pi=sin\pi=0$.
$\lim_{n\to+\infty}sin\frac{\pi}{a_n}=0$.
Let $b_n=\frac{1}{2n+\frac{1}{2}}$ for $n=1,2,3,\cdots$.
Then, $b_n\to 0$ as $n\to+\infty$ AND
$sin\frac{\pi}{b_n}=sin\left(2n+\frac{1}{2}\right)\pi=sin\frac{\pi}{2}=1$.
$\lim_{n\to+\infty}sin\frac{\pi}{b_n}=1$.
$$\lim_{n\to+\infty}sin\frac{\pi}{a_n}=0\ne 1=\lim_{n\to+\infty}sin\frac{\pi}{b_n}$$
Thus, $\lim_{x\to 0}sin\frac{\pi}{x}$ doesn't exist.

Hence part: Obvious

Exercises

(i) Show that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Remark:
$\lim_{x \to 0^-} \sqrt{x}$ is not meaningful as $\sqrt{x}$ is not a real number when $x < 0$.

(ii) Let $f(x) = \begin{cases} x^2 & if\ x \leq 0 \\ x\sin\frac{1}{x} & if\ x > 0 \end{cases}$.

Show that $\lim_{x \to 0} f(x) = 0$.

(iii) Show that $\lim_{x \to 3^-} \left(\frac{x^2}{x^2+1} + \sqrt{9 - x^2} \right) = \frac{9}{10}$.

Remark:
$\lim_{x \to 3^+} \left(\frac{x^2}{x^2+1} + \sqrt{9 - x^2} \right)$ is not meaningful as $\sqrt{9 - x^2}$ is not a real number when $x > 3$.

 # Existence of Tangent Line

The slope of the line L tangent to the graph $y = f(x)$ at the point $P\big(a, f(a)\big)$ is $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if it exists as a real number.

In this case, an equation of the tangent line is given by:

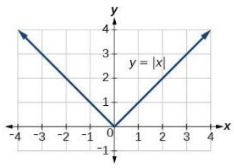
$$\frac{y - f(a)}{x - a} = m$$

$y = mx + c$ where $c = f(a) - ma$.

Remark: if $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ doesn't exist, we say there are no tangent lines to the graph of $y = f(x)$ at the point $P\big(a, f(a)\big)$.

Example

Show that the graph of $y = |x|$ has no tangent lines at the origin.



Solution:

Let $y = f(x) = |x|$ for any $x \in R$.

Note: $f(0 + h) = f(h) = |h|$, $f(0) = 0$. $f(0 + h) - f(0) = |h|$.

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$

So, $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1 \neq -1 = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$.

Thus, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ doesn't exist as a real number.

The Limit Concept

#2 **$M-\delta$ Definition of a right hand limit for $\lim_{x\to a^+} f(x) = +\infty$**

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < x - a < \delta \Rightarrow f(x) > M$

Remarks:
1. Sometimes, we write $f(x) \to +\infty$ as $x \to a^+$
2. Sometimes, we say as $x \to a^+$, $f(x) \to +\infty$
3. We say $\lim_{x\to a^+} f(x)$ don't exist as a real number and the limit is $+\infty$.
4. $x \to a^+ \Leftrightarrow x \to a$ and $x > a$
5. Roughly speaking, when $x > a$ and x is very close to a, $f(x)$ will be a very large real number.

Example

Show that $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

Proof:
Idea:

For any $M > 0$, we need to find $\delta > 0$ such that $0 < x < \delta \Rightarrow \frac{1}{x} > M$.
Observe that $\frac{1}{x} > M \Leftrightarrow x < \frac{1}{M}$
So, we choose $\delta = \frac{1}{M}$.

Formal way of writing:
For any $M > 0$, we choose $\delta = \frac{1}{M} > 0$ such that $0 < x < \frac{1}{M} \Rightarrow \frac{1}{x} > M$.
Thus, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

Observation:

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10000

when $x > 0$ and x is very close to 0, $\frac{1}{x}$ will be a very large real number.

#2 $M-\delta$ **Definition of a left hand limit for** $\lim_{x\to a^-} f(x) = +\infty$

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < a - x < \delta \Longrightarrow f(x) > M$

Remarks:
1. Sometimes, we write $f(x) \to +\infty$ as $x \to a^-$
2. Sometimes, we say as $x \to a^-$, $f(x) \to +\infty$
3. We say $\lim_{x\to a^-} f(x)$ don't exist as a real number and the limit is $+\infty$.
4. $x \to a^- \Leftrightarrow x \to a$ and $x < a$
5. Roughly speaking, when $x < a$ and x is very close to a, $f(x)$ will be a very large real number.

Example:

$$\lim_{x\to 0^-} \frac{-1}{x} = +\infty.$$

#2 $M-\delta$ **Definition of a two sided limit for** $\lim_{x\to a} f(x) = +\infty$

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < |x - a| < \delta \Longrightarrow f(x) > M$

Example:

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

Theorem

Suppose $f: R \to R$ is a function and $a \in R$.

$\lim_{x\to a} f(x) = +\infty \Leftrightarrow \lim_{x\to a^+} f(x) = +\infty$ and $\lim_{x\to a^-} f(x) = +\infty$.

Proof: Omitted

The Limit Concept

#3 $M-\delta$ **Definition of a right hand limit for** $\lim_{x\to a^+} f(x) = -\infty$

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < x - a < \delta \Longrightarrow f(x) < -M$

Example:

$$\lim_{x\to 0^+} \frac{-1}{x} = -\infty.$$

#3 $M-\delta$ **Definition of a left hand limit for** $\lim_{x\to a^-} f(x) = -\infty$

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < a - x < \delta \Longrightarrow f(x) < -M$

Example:

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

#3 $M-\delta$ **Definition of a two sided limit for** $\lim_{x\to a} f(x) = -\infty$

Suppose $f: R \to R$ is a function and $a \in R$.

For any $M > 0$, we can find $\delta > 0$ (δ may depend on M) such that $0 < |x - a| < \delta \Longrightarrow f(x) < -M$

Example:

$$\lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

Theorem

Suppose $f: R \to R$ is a function and $a \in R$.
$\lim_{x\to a} f(x) = -\infty \Leftrightarrow \lim_{x\to a^+} f(x) = -\infty$ and $\lim_{x\to a^-} f(x) = -\infty$.

Proof: Omitted

Example 1

Show that $\lim_{x\to 0} \frac{1}{x}$ doesn't exist.

Proof:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \neq -\infty = \lim_{x\to 0^-} \frac{1}{x}.$$

So, $\lim_{x\to 0} \frac{1}{x}$ doesn't exist.

Example 2

Show that $\lim_{x\to 1} \frac{2x+1}{x-1}$ doesn't exist.

Proof:

$$\lim_{x\to 1^+} \frac{2x+1}{x-1} = +\infty \neq -\infty = \lim_{x\to 1^-} \frac{2x+1}{x-1}.$$

So, $\lim_{x\to 1} \frac{2x+1}{x-1}$ doesn't exist.

Remark: $\lim_{x\to 1} \left|\frac{2x+1}{x-1}\right| = +\infty.$

The Limit Concept

#4 $\varepsilon - M$ **Definition of a left hand limit for** $\lim_{x \to +\infty} f(x) = L$

Suppose $f: R \to R$ is a function and $L \in R$.

For any $\varepsilon > 0$, we can find $M > 0$ (M may depend on ε) such that $x > M \Longrightarrow |f(x) - L| < \varepsilon$

Remarks:

1. Sometimes, we write $f(x) \to L$ as $x \to +\infty$
2. Sometimes, we say as $x \to +\infty$, $f(x) \to L$
3. L is called **the limit** of f as $x \to +\infty$.
   We say $\lim_{x \to +\infty} f(x)$ exists as a real number L. (if the limit exists, it MUST be UNIQUE.)
4. Roughly speaking, when x is a very large real number, $f(x)$ will be very close to L.

Exercise

Show that $\lim_{x \to +\infty} \frac{1}{x} = 0$.

Proof:
Idea:

For any $\varepsilon > 0$, we need to find $M > 0$ such that $x > M \Longrightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$.
Observe that $\frac{1}{x} < \varepsilon \Longleftrightarrow x > \frac{1}{\varepsilon}$
So, we choose $M = \frac{1}{\varepsilon}$.

Formal way of writing:
For any $\varepsilon > 0$, we choose $M = \frac{1}{\varepsilon} > 0$ such that $x > M$
$\Longrightarrow \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$. Thus, $\lim_{x \to +\infty} \frac{1}{x} = 0$.

Observation:

x	$\frac{1}{x}$
10	0.1
100	0.01
1000	0.001

when x is very large real number, $\frac{1}{x}$ will be very close to 0.

#4 $\varepsilon - M$ **Definition of a right hand limit for** $\lim_{x \to -\infty} f(x) = L$

Suppose $f: R \to R$ is a function and $L \in R$.

For any $\varepsilon > 0$, we can find $M > 0$ (M may depend on ε) such that $x < -M \Longrightarrow |f(x) - L| < \varepsilon$

Remarks:

1. Sometimes, we write $f(x) \to L$ as $x \to -\infty$

2. Sometimes, we say as $x \to -\infty$, $f(x) \to L$

3. L is called **the limit** of f as $x \to -\infty$.
We say $\lim_{x \to -\infty} f(x)$ exists as a real number L. (if the limit exists, it MUST be UNIQUE.)

4. Roughly speaking, when $-x$ is a very large real number, $f(x)$ will be very close to L.

Example:

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

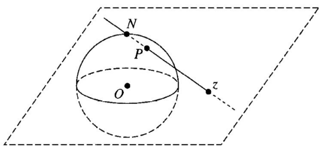
#4 **Definition of a two sided limit for** $\lim_{x \to \infty} f(x) = L$

We say $\lim_{x \to \infty} f(x) = L \Leftrightarrow \lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$

Question

What do we mean "$x \to \infty \Leftrightarrow x \to +\infty$ *and* $x \to -\infty$"?

Stereographic projection



Example:

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

The Limit Concept

We can define $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to +\infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = +\infty$ and $\lim_{x \to \infty} f(x) = -\infty$.

Examples:

$\lim_{x \to +\infty} x^2 = +\infty$ $\quad$ $\lim_{x \to +\infty} -x^2 = -\infty$ $\quad$ $\lim_{x \to -\infty} x^2 = +\infty$

$\lim_{x \to -\infty} -x^2 = -\infty$ $\quad$ $\lim_{x \to \infty} x^2 = +\infty$ $\quad$ $\lim_{x \to \infty} -x^2 = -\infty$

GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 2: Limits and Continuity

The Concept of Continuity

Definitions:

Let f be a function on $x \in R$ and let $a \in R$.

Suppose:

(i) $(a-\delta, a+\delta) \subset$ the domain of f for some $\delta > 0$
(that is, f is defined at all the points in a neighborhood of a.) **AND**

(ii) $\lim_{x \to a} f(x)$ exists as a real number **AND**

(iii) $\lim_{x \to a} f(x) = f(a)$.

Then, we say ***f* is continuous at *a***. Otherwise, we say f is NOT continuous at a or f is dis-continuous at a.

Roughly speaking, "f is continuous at a" means the graph of $y = f(x)$ is **connected**/is **not broken**/has **no holes**/has **no jumps** near to the point $(a, f(a))$.

Open Set

Let $\phi \neq S \subset R$. We say S is open if for any $s \in S$, we can find $\delta > 0$ such that $(s-\delta, s+\delta) \subset S$.

Closed Set

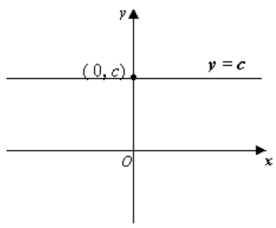
Let $\phi \neq T \subset R$. We say T is closed if $R \backslash T$ is open.

Definitions:

1. Let $\phi \neq S \subset R$ and S is open.
   We say ***f* is continuous on *S*** if f is continuous at a for any $a \in S$.
2. Let $a, b \in R$ with $a < b$.
   We say ***f* is continuous on [*a*, *b*]** if f is continuous on (a, b) **and** $\lim_{x \to a^+} f(x) = f(a)$ **and** $\lim_{x \to b^-} f(x) = f(b)$

Examples

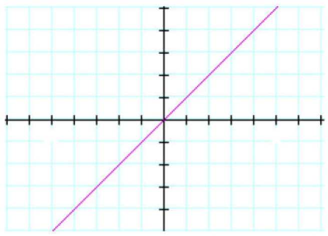
1. Suppose C is a fixed real number (constant) and let $a \in R$.
   Suppose $f: R \to R$ is a function defined by $f(x) = C$ for any $x \in R$.
   Then, $\lim_{x \to a} f(x) = C = f(a)$.
   So, f is continuous at a.
   This a is arbitrary.
   Thus, f is continuous on R.

   Note: the graph of $y = f(x)$ is a **connected** horizontal line.
   e.g. when $C > 0$,

   Constant Valued Functions are continuous on R.

2. Let $a \in R$.
   Suppose $f: R \to R$ is a function defined by $f(x) = x$ for any $x \in R$.
   Then, $\lim_{x \to a} f(x) = a = f(a)$.
   So, f is continuous at a.
   This a is arbitrary.
   Thus, f is continuous on R.

   $y = x$
   The identity function

   Note: the graph of $y = f(x) = x$ is a **connected** line.

   The Identity Function $y = f(x) = x$ is continuous on R.

 # Rules for Continuous Functions

Let f and g be functions on $x \in R$ and let $a, \lambda \in R$.
Suppose BOTH f and g are continuous at a.
Then,

(i) $f + g$ is continuous at a.
(ii) $f - g$ is continuous at a.
(iii) $f \cdot g$ is continuous at a.
(iv) $\frac{f}{g}$ is continuous at a.
(We assumed $g(x) \neq 0$ in a neighborhood of a.)
(v) λf is continuous at a.

Proof: Omitted (from Sum, Difference, Product, Quotient and Scalar Multiple Laws for Limits)

Example 3:

All polynomial functions are continuous on R.

Proof: Use the results of examples 1 and 2 and apply the above rules for continuous functions.

Example 4:

Let $f(x) = \frac{1}{x-2}$ for $x \neq 2$.
f is NOT continuous at 2 as it is undefined at 2.

Example 5:

Let $g(x) = sgn(x) = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$.
g is NOT continuous at 0.

Proof:

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} 1 = 1$.
$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} -1 = -1$.
$\lim_{x \to 0^+} g(x) \neq \lim_{x \to 0^-} g(x)$.
So, $\lim_{x \to 0} g(x)$ doesn't exist.

Note: g is defined at all the points in a neighborhood of 0.

Example 6:

Let $h(x) = \begin{cases} \frac{sinx}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$.
h is NOT continuous at 0.

Proof:

$\lim_{x \to 0} h(x) = 1$.
$h(0) = 0$.
$\lim_{x \to 0} h(x) \neq h(0)$.

Note: h is defined at all the points in a neighborhood of 0 AND $\lim_{x \to 0} h(x)$ exists as a real number.

Definition
Let f be a function on $x \in R$ and let $a \in R$.

Suppose:

(i) $(a-\delta, a+\delta) \subset$ the domain of f for some $\delta > 0$
(that is, f is defined at all the points in a neighborhood of a.) **AND**

(ii) $\lim_{x \to a} f(x)$ exists as a real number **AND**

(iii) $\lim_{x \to a} f(x) \neq f(a)$.

Then, we say ***f* has a removable discontinuity at *a***.
(We can re-define $f(a)$ as $\lim_{x \to a} f(x)$ so that f is continuous at a.)

Example 6:

Let $h(x) = \begin{cases} \frac{sinx}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$.
h has a removable discontinuity at 0.

Proof:

$\lim_{x \to 0} h(x) = 1$.
$h(0) = 0$.
$\lim_{x \to 0} h(x) \neq h(0)$.

Note: h is defined at all the points in a neighborhood of 0 AND $\lim_{x \to 0} h(x)$ exists as a real number.

Example 7:

Let $f(x) = \begin{cases} \frac{x-2}{x^2-3x+2} & if\ x \neq 2\ and\ x \neq 1 \\ 0 & if\ \ x = 2\ or\ x = 1 \end{cases}$.
Show that:

(i) f has a removable discontinuity at 2.
(ii) f has a non-removable discontinuity at 1.

Proof:
f is defined on R.

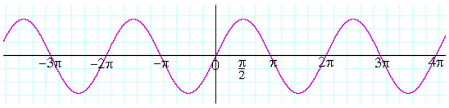
(i)
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x-2}{x^2-3x+2} = \lim_{x \to 2} \frac{x-2}{(x-2)(x-1)}$$
$$= \lim_{x \to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1 \neq 0 = f(2)$$

(ii)
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x-1} = +\infty$.
$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x-1} = -\infty$.
$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$.
So, $\lim_{x \to 1} f(x)$ doesn't exist.

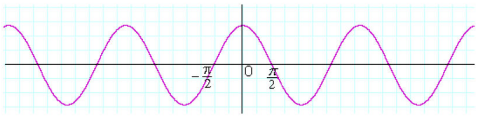
Examples 8 (Trigonometric Functions)

(i) Let $f: R \to R$ and $g: R \to R$ be defined as:
$f(x) = sinx$ for any $x \in R$ and $g(x) = cosx$ for any $x \in R$.
Then, BOTH f and g are continuous on R.

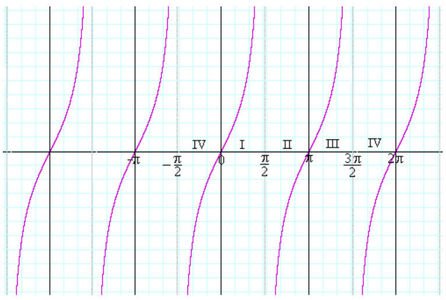
The graph of $y = f(x) = sinx$:



The graph of $y = g(x) = cosx$:



(ii) Let $h: R \to R$ be defined as
$h(x) = tanx$ for any $x \in R \backslash \left\{\frac{n\pi}{2} : n = \pm1, \pm2, \pm3, \cdots\right\}$
Then, h is continuous on $R \backslash \left\{\frac{n\pi}{2} : n = \pm1, \pm2, \pm3, \cdots\right\}$.
The graph of $y = h(x) = tanx$:



Rule on Composition of Continuous Functions

Let f be a function on $u \in R$ and say $y = f(u)$.
Let g be a function on $x \in R$ and say $u = g(x)$.
Then, we may regard $y = f(g(x))$ as a function on x.
Usually, we write $f(g(x)) = (f^\circ g)(x)$.
Let $a \in R$. Note: $g(a) \in R$.

Theorem

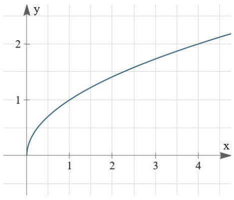
Suppose g is continuous at a AND f is continuous at $g(a)$. Then, $f^\circ g$ is continuous at a.

Proof:

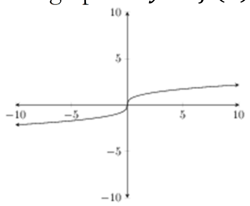
We can show that $f(g(x))$ is well defined in a neighborhood of a as f is continuous at $g(a)$.
As f is continuous at $g(a)$, $\lim_{u \to g(a)} f(u) = f(g(a))$.
For any $\varepsilon > 0$, we can find $\theta > 0$ such that $|u - g(a)| < \theta \Longrightarrow |f(u) - f(g(a))| < \varepsilon$.
As g is continuous at a, $\lim_{x \to a} g(x) = g(a)$.
For the above $\theta > 0$, we can find $\delta > 0$ such that $|x - a| < \delta \Longrightarrow |g(x) - g(a)| < \theta$.
Thus, we have $|x - a| < \delta \Longrightarrow |g(x) - g(a)| < \theta \Longrightarrow |f(g(x)) - f(g(a))| < \varepsilon$.
Therefore, $\lim_{x \to a} f(g(x)) = f(g(a))$.

Example 9 (n-th Root Function)

(a) Let $n = 2,4,6,8,\cdots$, let $f: \{x \in R: x \geq 0\} \to R$ be defined by $f(x) = \sqrt[n]{x}$.
f is continuous on $\{x \in R: x \geq 0\}$.
The graph of $y = f(x) = \sqrt{x}$ is:



(b) Let $n = 3,5,7,9,\cdots$, let $f: R \to R$ be defined by $f(x) = \sqrt[n]{x}$.
f is continuous on R.
The graph of $y = f(x) = \sqrt[3]{x}$ is:



Question:

Why we are interested at **continuous functions**?

Answer

Continuous Functions have many nice properties.

Some considerations:

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x and is defined on $[a, b]$.

Note 1:

Let $m_1, m_2 \in R$ such that

(i) $m_1 \leq f(x)$ for any $x \in [a, b]$ AND
(ii) $m_1 = f(\alpha_1)$ for some $\alpha_1 \in [a, b]$.

AND

(iii) $m_2 \leq f(x)$ for any $x \in [a, b]$ AND
(iv) $m_2 = f(\alpha_2)$ for some $\alpha_2 \in [a, b]$.

Show that $m_1 = m_2$.

Proof:
$m_1 \leq f(\alpha_2) = m_2$;
$m_2 \leq f(\alpha_1) = m_1$;
so, $m_1 = m_2$.

Note 2:

Let $M_1, M_2 \in R$ such that

(i) $f(x) \leq M_1$ for any $x \in [a, b]$ AND
(ii) $M_1 = f(\beta_1)$ for some $\beta_1 \in [a, b]$.

AND

(iii) $f(x) \leq M_2$ for any $x \in [a, b]$ AND
(iv) $M_2 = f(\beta_2)$ for some $\beta_2 \in [a, b]$.

Show that $M_1 = M_2$.

Proof:
$M_2 = f(\beta_2) \leq M_1$;
$M_1 = f(\beta_1) \leq M_2$;
so, $M_1 = M_2$.

Thus, we can call m is **the** global minimum value or **the** absolute minimum value of f on $[a, b]$ if:

(i) $m \leq f(x)$ for any $x \in [a, b]$ AND
(ii) $m = f(\alpha)$ for some $\alpha \in [a, b]$.

Also, we can call M is **the** global maximum value or **the** absolute maximum value of f on $[a, b]$ if:

(i) $f(x) \leq M$ for any $x \in [a, b]$ AND
(ii) $M = f(\beta)$ for some $\beta \in [a, b]$.

Extreme Value Theorem
(Continuous Functions on a closed and bounded interval)

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x.
Suppose f is continuous on $[a, b]$.
Then, we can find $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$.

Remarks:
(i) α may not be unique.
(ii) β may not be unique.
(iii) $f(\alpha)$ and $f(\beta)$ are unique.
(iv) $f(\alpha)$ is called **the global minimum value** or **the absolute minimum value** of f on $[a, b]$.
(v) $f(\beta)$ is called **the global maximum value** or **the absolute maximum value** of f on $[a, b]$.

Proof: Omitted

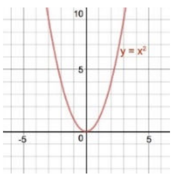
[Roughly speaking: The graph of $y = f(x)$ for $a \leq x \leq b$ is **a connected curve with two ends fixed**.
We can find the "Highest point(s)" and "Lowest point(s)".]

Examples

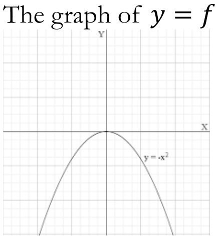
(i) Let $f: R \to R$ be defined by $f(x) = 3$ for any $x \in [1,2]$.
3 is the global minimum value of f on $[1,2]$.
3 is the global maximum value of f on $[1,2]$.
Note: $3 = f(1) = f(2) = f(c)$ for any $c \in [1,2]$.

(ii) Let $g: R \to R$ be defined by $g(x) = x$ for any $x \in [1,2]$.
1 is the global minimum value of g on $[1,2]$.
2 is the global maximum value of g on $[1,2]$.
Note:
$1 = g(1) \neq g(c)$ for any $c \in (1,2]$.
$2 = g(2) \neq g(d)$ for any $d \in [1,2)$.

(iii) Let $f: R \to R$ be defined by $f(x) = x^2$ for any $x \in R$.
0 is the global minimum value of f on $[-1,2]$.
4 is the global maximum value of f on $[-1,2]$.
0 is the global minimum value of f on $[-3,1]$.
9 is the global maximum value of f on $[-3,1]$.
The graph of $y = f(x) = x^2$ is



(iv) Let $f: R \to R$ be defined by $f(x) = -x^2$ for any $x \in R$.
-4 is the global minimum value of f on $[-1,2]$.
0 is the global maximum value of f on $[-1,2]$.
-9 is the global minimum value of f on $[-3,1]$.
0 is the global maximum value of f on $[-3,1]$.
The graph of $y = f(x) = -x^2$ is



GEST 1004 Quantitative Reasoning for Science and Technology
Lecture Notes for Chapter 2: Limits and Continuity

Intermediate Value Property
(Continuous Functions on a closed and bounded interval)

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x.
Suppose f is continuous on $[a, b]$.
Then, we can find $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$.
Let $m = f(\alpha)$ and $M = f(\beta)$.
For any value $L \in [m, M]$, we can find $\gamma \in [a, b]$ such that $L = f(\gamma)$.

Proof: Omitted

[Roughly speaking: The graph of $y = f(x)$ for $a \leq x \leq b$ is **a connected curve with two ends fixed**. We can find point(s) to connect the "Highest point(s)" and "Lowest point(s)".]

 # Intermediate Value Theorem (An application)

Let $a, b \in R$ with $a < b$ and $f: R \to R$ be a function on x.
Suppose f is continuous on $[a, b]$.
If $f(a)f(b) < 0$, then we can find $c \in [a, b]$ such that $f(c) = 0$.

Note: at least one (may be one or more than one)

Example 1

Let $f: R \to R$ be defined by $f(x) = x^2 - x - 12$ for any $x \in R$.
Show that there is at least one root of $f(x) = 0$ on $[3,5]$.

Proof:

f is a polynomial function, so f is continuous on R.
$f(3) = 3^2 - 3 - 12 = 9 - 3 - 12 = -6 < 0$.
$f(5) = 5^2 - 5 - 12 = 25 - 5 - 12 = 8 > 0$.
By Intermediate Value Theorem, there is at least one root of $f(x) = 0$ on $[3,5]$.

Example 2
(# Method of Bisection: an application of Intermediate Value Theorem)

Use method of bisection to find a root of $x^3 - 4x - 9 = 0$ (correct the answer to 2 decimal places).

Solutions

Let $f: R \to R$ be defined by $f(x) = x^3 - 4x - 9$ for any $x \in R$.
f is continuous on R.
$f(3) = 3^3 - 4 \times 3 - 9 = 27 - 12 - 9 = 6 > 0$.
$f(2) = 2^3 - 4 \times 2 - 9 = 8 - 8 - 9 = -9 < 0$.
Thus, we know there is at least one root of $x^3 - 4x - 9 = 0$ on $[2,3]$.

Correct to 3 decimal places

a	$c = \frac{1}{2}(a+b)$	b	**Sign of $f(a)$**	**Sign of $f(c)$**	**Sign of $f(b)$**
2.000	2.500	3.000	$-ve$	$-ve$	$+ve$
2.500	2.750	3.000	$-ve$	$+ve$	$+ve$
2.500	2.625	2.750	$-ve$	$-ve$	$+ve$
2.625	2.688	2.750	$-ve$	$-ve$	$+ve$
2.688	2.719	2.750	$-ve$	$+ve$	$+ve$
2.688	2.704	2.719	$-ve$	$-ve$	$+ve$
2.704	2.712	2.719	$-ve$	$+ve$	$+ve$
2.704	2.708	2.712	$-ve$	$+ve$	$+ve$
2.704	2.706	2.708	$-ve$	$-ve$	$+ve$
2.706		2.708	$-ve$		$+ve$

$2.706 = 2.71$ (correct to 2 decimal places)
$2.708 = 2.71$ (correct to 2 decimal places)

A root of $x^3 - 4x - 9 = 0$ is 2.71 (correct to 2 decimal places).

Exercise

Use method of bisection to find a root of $x^3 - 7x^2 + 14x - 6 = 0$ (correct the answer to 2 decimal places).