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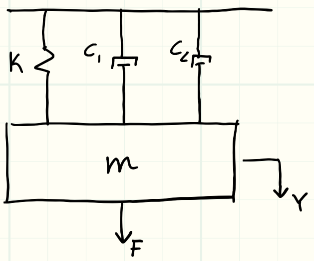
1)



$m = 1\,kg$
$c_1 = 0.5\ kg/s$
$c_2 = 1\ kg\,s/m^2$
$K = 1\ kg/s^2$

a) $m\ddot{y} + c_1\dot{y} + c_2\dot{y}^3 + Ky = F$

b) No,

$x_1 = y$
$x_2 = \dot{y}$

$$\dot{x} = \begin{pmatrix} x_2 \\ \frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{K}{m}x_1 \end{pmatrix}$$

$u = F$ $\qquad y = x_1$

c)

$f_1(x,u) = x_2$: Linear? Yes.

linearity wrt x : True

$f_1(x^{(1)}+x^{(2)},0) = x_2^{(1)} + x_2^{(2)}$ $\qquad f_1(x^{(1)},0) + f_1(x^{(2)},0) = x_2^{(1)} + x_2^{(2)}$

$f_1(\alpha x^{(1)},0) = \alpha x_2^{(1)}$ $\qquad \alpha f_1(x^{(1)},0) = \alpha x_2^{(1)}$

linearity wrt u: True

$f_1(0,u^{(1)}+u^{(2)}) = 0$ $\qquad f_1(0,u^{(1)}) + f_1(0,u^{(2)}) = 0$

$f_1(0,\alpha u^{(1)}) = 0$ $\qquad \alpha f_1(0,u^{(1)}) = 0$

$f_2(x,u) = \frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{K}{m}x_1$

linearity wrt x: False

$$f_2(x^{(1)}+x^{(2)},0) = -\frac{c_1}{m}(x_2^{(1)}+x_2^{(2)}) - \frac{c_2}{m}(x_2^{(1)}+x_2^{(2)})^3 - \frac{K}{m}(x_1^{(1)}+x_2^{(2)})$$

$$f_2(x^{(1)},0)+f(x^{(2)},0) = -\frac{c_1}{m}(x_2^{(1)}+x_2^{(2)}) - \frac{c_2}{m}\left((x_2^{(1)})^3+(x_2^{(2)})^3\right) - \frac{K}{m}(x_1^{(1)}+x_2^{(2)})$$

$$f_2(\alpha x^{(1)},0) = -\frac{c_1}{m}(\alpha x_2^{(1)}) - \frac{c_2}{m}\alpha^3(x_2^{(1)})^3 - \frac{K}{m}\alpha x_1^{(1)}$$

$$\alpha f_2(x^{(1)},0) = -\frac{c_1}{m}\alpha x_2^{(1)} - \frac{c_2}{m}\alpha x_2^{(1)} - \alpha\frac{K}{m}x_1^{(1)}$$

linearity wrt u: True

$f_2(0, u^{(1)}+u^{(2)}) = \frac{u^{(1)}}{m} + \frac{u^{(2)}}{m}$ $\qquad$ $f_2(0, \alpha u^{(1)}) = \alpha \frac{u^{(1)}}{m}$

$f_2(0, u^{(1)}) + f(0, u^{(2)}) = \frac{u^{(1)}}{m} + \frac{u^{(2)}}{m}$ $\qquad$ $\alpha f_2(0, u^{(1)}) = \alpha \frac{u^{(1)}}{m}$

Is the system linear? No.

d) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{K}{m}x_1 \end{pmatrix}$ Linearized about $y(t) = \dot{y}(t) = F(t) = 0 \quad \forall t \geq 0$

$(y) = (x_1)$

$\dot{\delta}_x = A\delta_x + B\delta_u$
$\delta_y = C\delta_x + D\delta_u$

$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x^*, u^*}$, $B_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{x^*, u^*}$, $C_{ij} = \left.\frac{\partial h_i}{\partial x_j}\right|_{x^*, u^*}$, $D_{ij} = \left.\frac{\partial h_i}{\partial u_j}\right|_{x^*, u^*}$

$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{c_1}{m} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \end{bmatrix}$

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{c_1}{m} \end{bmatrix}\delta_x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}\delta_u$$

$$\delta_y = \begin{bmatrix} 1 & 0 \end{bmatrix}\delta_x$$

e) The linear system overestimates the displacement of the mass because the nonlinear dampener is not contributing to the linearized dynamics. This can be observed as the speed of the mass increases past 0.5 m/s and is amplified as the speed increases. The graphs are attached at the end of this PDF.

f) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{u}{m} - \frac{K}{m}x_1 - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^2 \end{pmatrix}$ Linearized about $y(t) = \dot{y}(t) = F(t) = 0 \quad \forall t \geq 0$

$$\dot{\delta}_x = \begin{pmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{c_1}{m} \end{pmatrix}\delta_x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}\delta_u$$

$$\delta_y = \begin{pmatrix} 1 & 0 \end{pmatrix}\delta_x$$

i) The linearized system of part e & part f behave the same because they are the same system

ii) The biggest difference between the two nonlinear systems is that for some values of $x_2(0)$, the square dampener system move than the cubed dampener and for others it displaces less.

iii) I learned how small the linearized region of the dynamics can be and that although linearized systems can be useful; they are only useful in their regions.

2)

a) Is the system $\dot{x}(t) = -ax(t) + u(t)$ linear?
$y(t) = x(t)$

Assuming $x(0)=0$:

$$X(s) = \left(\frac{1}{s+a}\right) U(s) \quad \rightarrow \quad y(t) = x(t) = \int_0^t e^{-a(t-\tau)} u(\tau)\, d\tau$$

$$y = S(u) = \int_0^t e^{-a(t-\tau)} u(\tau)\, d\tau$$

$$y^{(1)} = S(u^{(1)}) = \int_0^t e^{-a(t-\tau)} u^{(1)}(\tau)\, d\tau \qquad \text{Let } u^{(3)} \triangleq u^{(1)} + u^{(2)}$$

$$y^{(2)} = S(u^{(2)}) = \int_0^t e^{-a(t-\tau)} u^{(2)}(\tau)\, d\tau \qquad y^{(3)} = \int_0^t e^{-a(t-\tau)} \left(u^{(1)}(\tau) + u^{(2)}(\tau)\right) d\tau$$

$$y^{(3)} = S(u^{(3)}) = \int_0^t e^{-a(t-\tau)} u^{(3)}(\tau)\, d\tau \qquad y^{(3)} = \int_0^t e^{-a(t-\tau)} u^{(1)}(\tau)\, d\tau + \int_0^t e^{-a(t-\tau)} u^{(2)}(\tau)\, d\tau$$

$$y^{(3)} = y^{(1)} + y^{(2)} \therefore S(u^{(1)} + u^{(2)}) = S(u^{(1)}) + S(u^{(2)})$$

$$y^{(1)} = S(\alpha u^{(1)}) = \int_0^t e^{-a(t-\tau)} (\alpha u^{(1)}(\tau))\, d\tau \qquad y^{(2)} = \alpha S(u^{(2)}) = \alpha \int_0^t e^{-a(t-\tau)} u^{(2)}(\tau)\, d\tau$$

$$u^{(1)} = u^{(2)} \iff y^{(1)} = y^{(2)} \quad \therefore \quad S(\alpha u^{(1)}) = \alpha S(u^{(1)})$$

Since: $S(u^{(1)} + u^{(2)}) = S(u^{(1)}) + S(u^{(2)})$ & $S(\alpha u^{(1)}) = \alpha S(u^{(1)})$ the system is linear.

b) Is the system $\dot{x}(t) = -a\,x(t) + u(t)$ linear?
$y(t) = x^2(t)$

from previous problem it was found that $x(t) = \int_0^t e^{-a(t-\tau)}\, u(\tau)\, d\tau$.

$$y = S(u) = \left(\int_0^t e^{-a(t-\tau)}\, u(\tau)\, d\tau\right)^2$$

$$S(\alpha u^{(1)}) = \left(\int_0^t e^{-a(t-\tau)}\, \alpha\, u(\tau)\, d\tau\right)^2 = \alpha^2\left(\int_0^t e^{-a(t-\tau)}\, u(\tau)\, d\tau\right)^2 = \alpha^2\, S(u^{(1)})$$

$\alpha^2 S(u^{(1)}) \neq \alpha S(u^{(1)})$ $\therefore$ system is nonlinear.