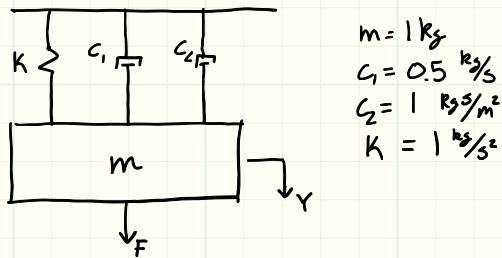


1)



$$\begin{aligned}m &= 1 \text{ kg} \\c_1 &= 0.5 \text{ kg/s} \\c_2 &= 1 \text{ kg s/m}^2 \\K &= 1 \text{ kg/s}^2\end{aligned}$$

a) $m\ddot{y} + c_1\dot{y} + c_2\dot{y}^3 + Ky = F$

b) No,

$$\begin{aligned}x_1 &= y \\x_2 &= \dot{y}\end{aligned} \quad \dot{x} = \left(\frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{K}{m}x_1 \right)$$

$$u = F \quad y = x_1$$

c)

$$f_1(x, u) = x_2 : \text{Linear? Yes.}$$

Linearity wrt x : True

$$f_1(x^{(1)} + x^{(2)}, 0) = x_2^{(1)} + x_2^{(2)} \quad f_1(x^{(1)}, 0) + f_1(x^{(2)}, 0) = x_2^{(1)} + x_2^{(2)}$$

$$f_1(\alpha x^{(1)}, 0) = \alpha x_2^{(1)} \quad \alpha f_1(x^{(1)}, 0) = \alpha x_2^{(1)}$$

Linearity wrt u : True

$$f_1(0, u^{(1)} + u^{(2)}) = 0 \quad f_1(0, u^{(1)}) + f_1(0, u^{(2)}) = 0$$

$$f_1(0, \alpha u^{(1)}) = 0 \quad \alpha f_1(0, u^{(1)}) = 0$$

$$f_2(x, u) = \frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{K}{m}x_1$$

Linearity wrt x : False

$$f_2(x^{(1)} + x^{(2)}, 0) = -\frac{c_1}{m}(x_2^{(1)} + x_2^{(2)}) - \frac{c_2}{m}(x_2^{(1)} + x_2^{(2)})^3 - \frac{K}{m}(x_1^{(1)} + x_1^{(2)})$$

$$f_2(x^{(1)}, 0) + f_2(x^{(2)}, 0) = -\frac{c_1}{m}(x_2^{(1)}) - \frac{c_2}{m}(x_2^{(1)})^3 - \frac{K}{m}(x_1^{(1)})$$

$$f_2(\alpha x^{(1)}, 0) = -\frac{c_1}{m}(\alpha x_2^{(1)}) - \frac{c_2}{m}\alpha^3(x_2^{(1)})^3 - \frac{K}{m}\alpha x_1^{(1)}$$

$$\alpha f_2(x^{(1)}, 0) = -\frac{c_1}{m}\alpha x_2^{(1)} - \frac{c_2}{m}\alpha x_2^{(1)} - \alpha \frac{K}{m}x_1^{(1)}$$

linearity wrt u : True

$$f_2(0, u^{(1)} + u^{(2)}) = \frac{u^{(1)}}{m} + \frac{u^{(2)}}{m} \quad f_2(0, \alpha u^{(1)}) = \alpha \frac{u^{(1)}}{m}$$

$$f_2(0, u^{(1)}) + f_2(0, u^{(2)}) = \frac{u^{(1)}}{m} + \frac{u^{(2)}}{m} \quad \alpha f_2(0, u^{(1)}) = \alpha \frac{u^{(1)}}{m}$$

Is the system linear? No.

d) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{u}{m} - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^3 - \frac{k}{m}x_1 \end{pmatrix}$ Linearized about $y(t) = \dot{y}(t) = F(t) = 0 \quad \forall t \geq 0$

$$(y) = (x_1)$$

$$\begin{aligned} \dot{\delta}_x &= A\delta_x + B\delta_u \\ \dot{\delta}_y &= C\delta_x + D\delta_u \end{aligned} \quad A_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{x^*, u^*}, \quad B_{ij} = \left. \frac{\partial f_i}{\partial u_j} \right|_{x^*, u^*}, \quad C_{ij} = \left. \frac{\partial h_i}{\partial x_j} \right|_{x^*, u^*}, \quad D_{ij} = \left. \frac{\partial h_i}{\partial u_j} \right|_{x^*, u^*}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c_1}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0]$$

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c_1}{m} \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \delta_u$$

$$\delta_y = [1 \quad 0] \delta_x$$

- e) The linear system overestimates the displacement of the mass because the nonlinear damper is not contributing to the linearized dynamics. This can be observed as the speed of the mass increases past 0.5 m/s and is amplified as the speed increases. The graphs are attached at the end of this PDF.

f) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{u}{m} - \frac{k}{m}x_1 - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2^2 \end{pmatrix}$ Linearized about $y(t) = \dot{y}(t) = F(t) = 0 \quad \forall t \geq 0$

$$\dot{\delta}_x = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c_1}{m} \end{pmatrix} \delta_x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \delta_u$$

$$\delta_y = [1 \quad 0] \delta_x$$

- i) The linearized system of part e & part f behave the same because they are the same system
- ii) The biggest difference between the two nonlinear systems is that for some values of $x_2(0)$, the square damper system moves more than the cubed damper and for others it displaces less.
- iii) I learned how small the linearized region of the dynamics can be and that although linearized systems can be useful; they are only useful in their regions.

2)

- a) Is the system $\dot{x}(t) = -\alpha x(t) + u(t)$ linear?
 $y(t) = x(t)$

Assuming $x(0) = 0$:

$$X(s) = \left(\frac{1}{s+\alpha}\right) U(s) \rightarrow y(t) = x(t) = \int_0^t e^{-\alpha(t-\tau)} u(\tau) d\tau$$

$$y = S(u) = \int_0^t e^{-\alpha(t-\tau)} u(\tau) d\tau$$

$$y^{(1)} = S(u^{(1)}) = \int_0^t e^{-\alpha(t-\tau)} u^{(1)}(\tau) d\tau \quad \text{Let } u^{(3)} \triangleq u^{(1)} + u^{(2)}$$

$$y^{(2)} = S(u^{(2)}) = \int_0^t e^{-\alpha(t-\tau)} u^{(2)}(\tau) d\tau \quad y^{(3)} = \int_0^t e^{-\alpha(t-\tau)} (u^{(1)}(\tau) + u^{(2)}(\tau)) d\tau$$

$$y^{(3)} = S(u^{(3)}) = \int_0^t e^{-\alpha(t-\tau)} u^{(3)}(\tau) d\tau \quad y^{(3)} = \int_0^t e^{-\alpha(t-\tau)} u^{(1)}(\tau) d\tau + \int_0^t e^{-\alpha(t-\tau)} u^{(2)}(\tau) d\tau$$

$$y^{(3)} = y^{(1)} + y^{(2)} \therefore S(u^{(1)} + u^{(2)}) = S(u^{(1)}) + S(u^{(2)})$$

$$y^{(1)} = S(\alpha u^{(1)}) = \int_0^t e^{-\alpha(t-\tau)} (\alpha u^{(1)}(\tau)) d\tau \quad y^{(2)} = \alpha S(u^{(2)}) = \alpha \int_0^t e^{-\alpha(t-\tau)} u^{(2)}(\tau) d\tau$$

$$u^{(1)} = u^{(2)} \Leftrightarrow y^{(1)} = y^{(2)} \therefore S(\alpha u^{(1)}) = \alpha S(u^{(1)})$$

Since: $S(u^{(1)} + u^{(2)}) = S(u^{(1)}) + S(u^{(2)})$ & $S(\alpha u^{(1)}) = \alpha S(u^{(1)})$ the system is linear.

b) Is the system $\dot{x}(t) = -\alpha x(t) + u(t)$ linear?
 $y(t) = x^2(t)$

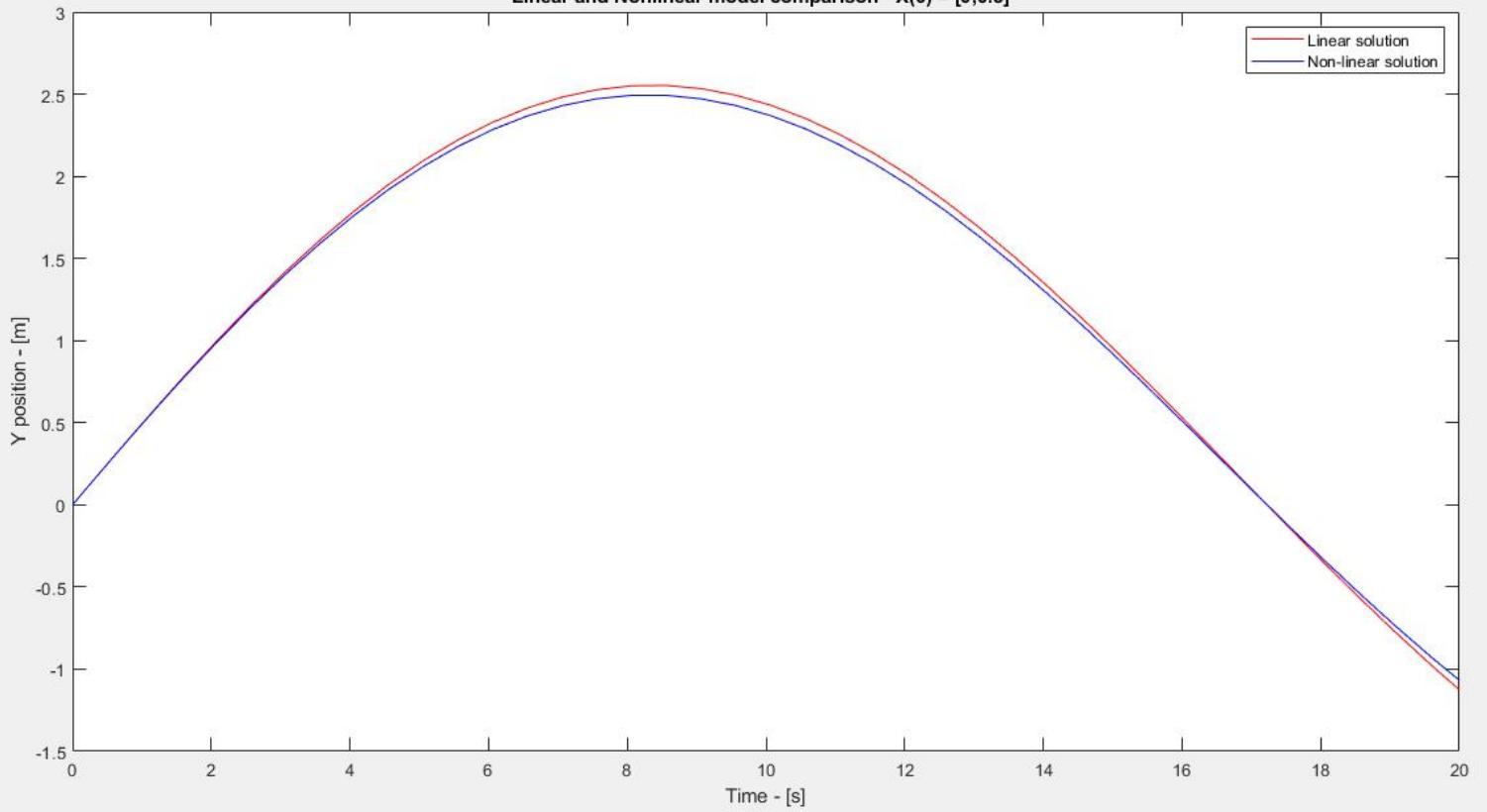
from previous problem it was found that $x(t) = \int_0^t e^{-\alpha(t-\tau)} u(\tau) d\tau$.

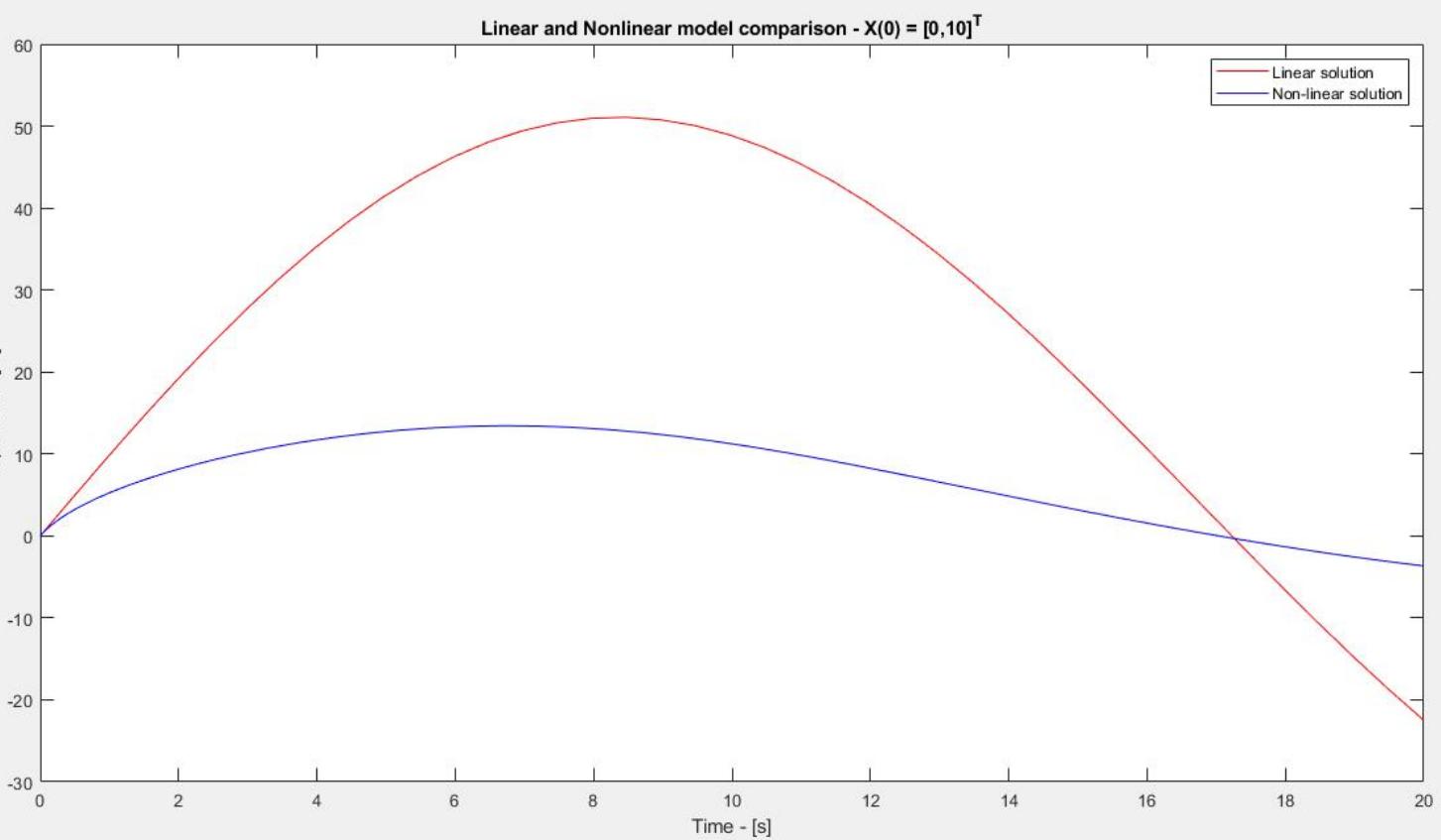
$$y = S(u) = \left(\int_0^t e^{-\alpha(t-\tau)} u(\tau) d\tau \right)^2$$

$$S(\alpha u^{(1)}) = \left(\int_0^t e^{-\alpha(t-\tau)} \alpha u(\tau) d\tau \right)^2 = \alpha^2 \left(\int_0^t e^{-\alpha(t-\tau)} u(\tau) d\tau \right)^2 = \alpha^2 S(u^{(1)})$$

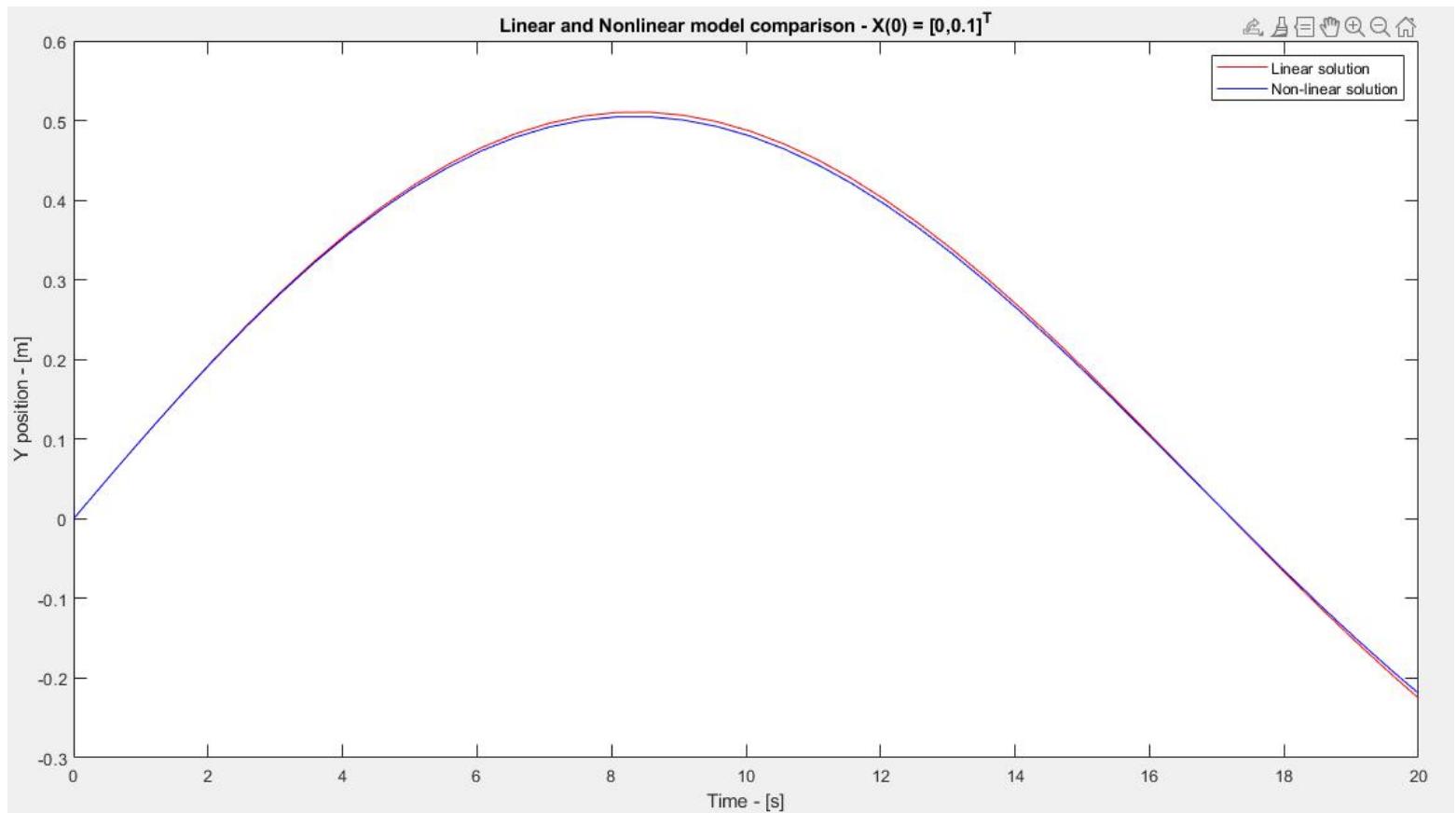
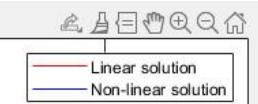
$\alpha^2 S(u^{(1)}) \neq \alpha S(u^{(1)})$ \therefore system is nonlinear.

Linear and Nonlinear model comparison - $X(0) = [0, 0.5]^T$





Linear and Nonlinear model comparison - $X(0) = [0, 0.1]^T$



Linear and Nonlinear model comparison - $X(0) = [0, 10]^T$

