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A class of tests for exponentiality against increasing failure rate average alternatives

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Summary

A class of tests is proposed for testing exponentiality against the increasing failure rate average class of nonexponential probability distributions. Some of these tests are consistent as well for the larger class, new better than used. Compared to its competitors these statistics are remarkably simple. They are U-statistics and hence asymptotically normally distributed. The asymptotic relative efficiency with respect to the Hollander–Proschan statistic and the cumulative total time on test statistic are reasonably high.

Some key words: Asymptotic relative efficiency; Cumulative total time on test statistic; Hollander–Proschan statistic; New better than used; Reliability theory; U-statistic.

1. Introduction

The class of increasing failure rate average distributions plays a central role in the statistical theory of reliability. It is the smallest class of probability distributions which contains the exponential distribution and is closed under formation of coherent systems. Also these distributions arise as life distributions from various useful shock models.

Let F and G be two probability distributions such that $F(0) = G(0) = 0$. It is well known that F is an increasing failure rate average distribution if and only if, for $x > 0$, $0 < b < 1$, $-\log \bar{F}(bx) < -b \log \bar{F}(x)$ or equivalently

$$\bar{F}(bx) \geqslant \{\bar{F}(x)\}^b, \tag{1·1}$$

where $\bar{F}(x) = 1 - F(x)$. The equality in (1·1) above holds if and only if F is an exponential distribution. For detailed discussion of these concepts, see Barlow & Proschan (1975, Chapter 4) or Doksum (1969).

Tests for exponentiality designed to detect the alternative hypotheses relevant in reliability theory include those of Proschan & Pyke (1967), Barlow (1968), Bickel & Doksum (1969), Bickel (1969), Ahmed (1975), Hollander & Proschan (1972, 1975) and Koul (1977, 1978). However, no test seems to have been developed specifically for the increasing failure rate average class. In this paper we develop tests for exponentiality which seem intuitively appropriate to detect this alternative. We also obtain tests which are useful for detecting as well the larger class of alternatives, new better than used.

In § 2 we define the new class of statistics which are seen to be U-statistics (Hoeffding, 1948). In § 3 the unbiasedness of these tests against increasing failure rate average alternatives is shown. A class of distributions is specified for each test such that the power of the test is one for any alternative belonging to this class. Also Monte Carlo estimates of the critical points as well as of the power of the tests for some specific alternatives are obtained for sample sizes up to 15. In § 4 we prove the asymptotic normality of these test statistics. Consistency of these tests for the increasing failure rate average and new better than used classes is also proved. Section 5 is devoted to Pitman asymptotic relative efficiency. Some tests of this class are compared with Hollander & Proschan's test (1972) and the test based on the 'cumulative total time on test' statistic.

2. The proposed class of statistics

Let $X_1, \ldots, X_n$ be a random sample from a continuous probability distribution with distribution function F such that $F(0) = 0$. We wish to test the null hypothesis

$$H_0\colon F(x) = 1 - e^{-\theta x} \quad (x > 0,\ \theta > 0),$$

with θ unspecified, against the alternative H_1 that F belongs to the increasing failure rate average class, but is not exponential.

The above H_0 and H_1 may also be stated as

$$H_0\colon \bar{F}(bx) = \{\bar{F}(x)\}^b \quad (x > 0,\ 0 \leqslant b \leqslant 1),$$

$$H_1\colon \bar{F}(bx) \geqslant \{\bar{F}(x)\}^b \quad (x > 0,\ 0 < b < 1),$$

with strict inequality for some x.

Define a parameter

$$M(F) = \int_0^\infty \bar{F}(bx)\, dF(x).$$

If F belongs to H_0, then $M(F) = (b+1)^{-1}$, whereas for all F belonging to H_1

$$M(F) = \int_0^\infty \bar{F}(bx)\, dF(x) > \int_0^\infty \{\bar{F}(x)\}^b\, dF(x) = (b+1)^{-1}.$$

Hence $M(F) - (b+1)^{-1}$ may be taken as a measure of deviation of F from the null hypothesis of exponentiality.

Define a function

$$h_b(X_1, X_2) = \begin{cases} 1 & (X_1 > bX_2), \\ 0 & \text{otherwise}, \end{cases}$$

where b is a fixed number belonging to $(0, 1)$. Define J_b as the U-statistic based on the kernel h_b, that is

$$J_b = n(n-1)^{-1} \Sigma^* h_b(X_i, X_j),$$

where Σ^* denotes summation over $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ such that $i \neq j$.

It is seen that

$$E(J_b) = E(h_b) = \mathrm{pr}\,(X_1 > bX_2) = M(F)$$

and intuitively large values of the statistic J_b indicate the alternative hypothesis. This leads to the rejection of H_0 if $J_b \geqslant c_{\alpha,n}$, where $c_{\alpha,n}$ is the appropriate critical point such that the test has the required size α.

The value of the statistic ranges from $\frac{1}{2}$ to 1. It is equal to $\frac{1}{2}$ if $X_{(i)} < bX_{(i+1)}$ for $i = 1, \ldots, n-1$, where $X_{(1)}, \ldots, X_{(n)}$ are the order statistics of $X_1, \ldots, X_n$. At the other extreme it equals 1 if $X_{(1)} > bX_{(n)}$.

The statistic J_b is easy to calculate. Multiply each observation by b. Arrange $X_1, \ldots, X_n$ and $bX_1, \ldots, bX_n$ together in increasing order of magnitude. Let R_i be the rank of X_i in the combined order. Then

$$S = \sum_{i=1}^{n} R_i - \tfrac{1}{2}n(n+1) - n$$

is the number of pairs of (X_i, bX_j) for $i \neq j$, such that X_i is larger than bX_j. Then obviously $J_b = \{n(n-1)\}^{-1} S$. This is just the Wilcoxon statistic for the data of X's and bX's.

3. Unbiasedness and estimates of power

To show the unbiasedness of the test based on J_b we have to prove that the probability of 'rejection' is not less than α whenever the alternative hypothesis is true. Let G be the exponential distribution with $\theta = 1$. Let F be a distribution in the alternative hypothesis, i.e. for all b in $(0, 1)$

$$-\log \bar{F}(bx) \leqslant -b \log \bar{F}(x).$$

Let $X_1, \ldots, X_n$ be a random sample from F. Let $Y'_i = G^{-1} F(X_i) = -\log \bar{F}(X_i)$. Then $Y'_1, \ldots, Y'_n$ have the same probability distribution as a random sample $Y_1, \ldots, Y_n$ from G.

Now $X_1 \leqslant bX_2$ implies $G^{-1} F(X_1) \leqslant G^{-1} F(bX_2) \leqslant bG^{-1} F(X_2)$, so that $X_1 \leqslant bX_2$ implies $Y'_1 \leqslant bY'_2$. Therefore $h_b(Y'_1, Y'_2) \leqslant h_b(X_1, X_2)$. But $h_b(Y_1, Y_2)$ has the same probability distribution as $h_b(Y'_1, Y'_2)$. Hence $J_b(Y_1, \ldots, Y_n)$ and $J_b(Y'_1, \ldots, Y'_n)$ have the same probability distribution, and $J_b(X_1, \ldots, X_n)$ is stochastically larger than these. Hence

$$\mathrm{pr}_F (J_b \geqslant c_{\alpha,n}) \geqslant \mathrm{pr}_G (J_b \geqslant c_{\alpha,n}). \tag{3·1}$$

The left-hand side of (3·1) represents the power of the test at a fixed alternative F of H_1 and the right-hand side is equal to α, which implies unbiasedness of the J_b test.

Let $\mathscr{G}_{(a_1,a_2)}$ be the class of all distributions with support $[a_1, a_2]$, where $a_1 > ba_2$. Then, easily, we have that 'rejection' has power one for any alternative belonging to the class $\mathscr{G}_{(a_1,a_2)}$, whenever the level of significance $\alpha \leqslant \mathrm{pr}_{H_0} (J_b = 1)$.

Not all members of $\mathscr{G}_{(a_1,a_2)}$ are in the increasing failure rate average class.

A Monte Carlo study was carried out to estimate critical points, the exact significance level and the power for two specific alternatives corresponding to significance levels near to $\alpha = 0·01$ and $\alpha = 0·05$ of two tests belonging to this class based on the statistics $J_{\frac{1}{2}}$ and $J_{0·9}$. The study was done for $n = 4, 5, \ldots, 15$ each value being based on 10,000 samples of the required size. The two alternatives to the null hypothesis H_0 of exponentiality with $\theta = 1$ for which the power has been estimated are H_1, the Weibull distribution of index 2, with $F(x) = 1 - \exp(-x^2)$ and H_2, a linear failure rate distribution with $\theta = 1$, with $F(x) = 1 - \exp(-x - \frac{1}{2}x^2)$. Both these distributions are increasing failure rate distributions, and hence also increasing failure rate average. Table 1 gives the critical region for $n(n-1) J_b$ for $\alpha = 0·05$. The probabilities which are estimated are

$$p(b, \alpha, n, i) = \mathrm{pr}_{H_i} \{n(n-1) J_b \geqslant c_\alpha\} \quad (i = 0, 1, 2;\ b = 0·5, 0·9)$$

Table 1. *Monte Carlo estimates of critical values, exact levels of significance and power of the $J_{\frac{1}{2}}$ and $J_{0·9}$ tests; $\alpha = 0·05$.*

	$b = 0·5$				$b = 0·9$			
n	Exact α $p(b,\alpha,n,0)$	c_α	$p(b,\alpha,n,1)$	$p(b,\alpha,n,2)$	Exact α $p(b,\alpha,n,0)$	c_α	$p(b,\alpha,n,1)$	$p(b,\alpha,n,2)$
5	0·130	16	0·578	0·193	0·073	12	0·233	0·096
	0·043	17	0·340	0·074	0·024	13	0·092	0·032
7	0·057	32	0·506	0·102	0·075	24	0·281	0·102
	0·035	33	0·405	0·068	0·021	25	0·117	0·030
9	0·052	51	0·572	0·100	0·056	40	0·278	0·082
	0·033	52	0·481	0·067	0·021	41	0·139	0·034
11	0·070	73	0·703	0·142	0·106	59	0·452	0·152
	0·048	74	0·631	0·099	0·043	60	0·279	0·066
13	0·059	100	0·724	0·130	0·090	83	0·436	0·124
	0·040	101	0·661	0·094	0·033	84	0·281	0·056
15	0·050	132	0·742	0·119	0·057	111	0·407	0·094
	0·037	133	0·690	0·091	0·025	112	0·266	0·046

and c_α is the estimated critical point for α near to 0·05. Values for $\alpha = 0{\cdot}01$ are available from the author.

4. Asymptotic normality and consistency

The statistic J_b is the U-statistic corresponding to the kernel h_b. Using the results of Hoeffding (1948), we have that the asymptotic distribution of $n^{\frac{1}{2}}\{J_b - M(F)\}$ is normal with mean zero and variance $4\zeta_1$, where

$$\zeta_1 = E\{\psi_1^2(X_1)\} - \{M(F)\}^2$$

and $\psi_1(x_1) = E\{h_b^*(x_1, X_2)\}$, provided $\zeta_1 > 0$. Here h^* is the symmetric version of h. Under the null hypothesis, $M(F) = (b+1)^{-1}$ and

$$\zeta_1 = \frac{1}{4}\left\{1 + \frac{b}{b+2} + \frac{1}{2b+1} + \frac{2(1-b)}{b+1} - \frac{2b}{b^2+b+1} - \frac{4}{(b+1)^2}\right\}.$$

This ensures the consistency of the J_b test whenever $E(J_b) > (b+1)^{-1}$ and the alternatives are continuous increasing failure rate average distributions.

Let $b = 1/k$, where $k > 2$ is an integer. Now F is a new better than used distribution if and only if, for every $x > 0$, $y > 0$,

$$\bar{F}(y)\,\bar{F}(x) \geqslant \bar{F}(x+y);$$

see, e.g. Hollander & Proschan (1972). Now F is new better than used implies that $\bar{F}(z/k) > \bar{F}(z)^{1/k}$ for every $z > 0$ and $k = 2, 3, \ldots$. The equality obtains only for the exponential distribution. Hence the J_b test for $b = k^{-1}$ $(k = 2, 3, \ldots)$ is consistent against continuous new better than used distributions.

5. Asymptotic relative efficiency

We calculate the Pitman asymptotic relative efficiency of two of the J_b tests, namely $b = 0{\cdot}5$, $b = 0{\cdot}9$, for three parametric families of distributions. These depend upon a real parameter θ in such a way that $\theta = \theta_0$ yields a distribution belonging to the null hypothesis whereas $\theta > \theta_0$ yields distributions from the alternative. These are:

(i) the Weibull distribution,

$$\bar{F}_\theta(x) = \exp(-x^\theta) \quad (\theta \geqslant 1,\ x > 0,\ \theta_0 = 1);$$

(ii) the Makeham distribution,

$$\bar{F}_\theta(x) = \exp[-\{x + \theta(x + e^{-x} - 1)\}] \quad (\theta \geqslant 0, x > 0,\ \theta_0 = 0);$$

(iii) the linear failure rate distribution,

$$\bar{F}(x) = \exp\{-(x + \tfrac{1}{2}\theta x^2)\} \quad (\theta \geqslant 0,\ x > 0,\ \theta_0 = 0).$$

The values of the Pitman asymptotic relative efficiency of these two tests with respect to Hollander & Proschan's (1972) test and to the 'cumulative total time' (Bickel & Doksum, 1969) test for these three parametric families are given in Table 2. The table indicates that these tests have high efficiency when compared with competitors.

Table 2. *Asymptotic relative efficiency of the J_b tests for $b = 0{\cdot}5$, 0·9*

Distribution	Weibull	Makeham	Linear failure rate
ARE ($J_{\frac{1}{2}}$, HP)	1·006	0·946	0·931
ARE ($J_{0\cdot9}$, HP)	1·022	1·020	1·020
ARE ($J_{\frac{1}{2}}$, BD)	0·937	0·787	0·418
ARE ($J_{0\cdot9}$, BD)	0·956	0·815	0·459

HP, Hollander–Proschan test; BD, cumulative total time (Bickel–Doksum) test.

6. Remarks

Compared to many other tests which are consistent for the increasing failure rate average class, the J_b tests have fairly high efficiency, as shown in Table 2 for some standard distributions. Also the statistic is very simply calculated. The tests with $b = 1/k$ for $k = 2, 3, \ldots$ are consistent for the bigger new better than used class.

Section 3 indicates that the $J_{\frac{1}{2}}$ test is most powerful, as it has power one, for any alternative belonging to the class $\mathscr{G}_{(a_1, a_2)}$ with $a_1 > \frac{1}{2} a_2$. All the distributions of this class are new better than used; see Hollander & Proschan (1972).

Thus one may recommend the test $J_{\frac{1}{2}}$ whenever the alternative is suspected to lie in the larger new better than used class. The test $J_{0\cdot 9}$ may be recommended whenever the alternative is the restricted increasing failure rate average class.

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