

Práctico 0 - Solución

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Ejercicio 1

Partido	P(partido)	P(votó partido)	P(no votó partido)
Conservador	0.30	0.65	0.35
Liberal	0.50	0.82	0.18
Independiente	0.20	0.50	0.50

Parte a

$$\begin{aligned} P(\text{liberal} \mid \text{votó}) &= \frac{P(\text{liberal}; \text{votó})}{P(\text{votó})} = \frac{P(\text{votó} \mid \text{liberal}) P(\text{liberal})}{P(\text{votó})} = \\ &= \frac{(0.82)(0.5)}{(0.3)(0.65) + (0.5)(0.82) + (0.2)(0.5)} = \frac{82}{141} \approx 0.581 \end{aligned}$$

Parte b

$$\begin{aligned} P(\text{liberal} \mid \text{no votó}) &= \frac{P(\text{liberal}; \text{no votó})}{P(\text{no votó})} = \frac{P(\text{no votó} \mid \text{liberal}) P(\text{liberal})}{P(\text{no votó})} = \\ &= \frac{(0.18)(0.5)}{(0.3)(0.35) + (0.5)(0.18) + (0.2)(0.5)} = \frac{18}{59} \approx 0.305 \end{aligned}$$

Ejercicio 2

9 monedas:

- 3 monedas de dos caras
- 4 monedas de dos cruces
- 2 monedas comunes

Parte a

$$P(\text{cara}) = \sum_{\text{tipo de moneda}} P(\text{cara} \mid \text{tipo de moneda}) P(\text{tipo de moneda}) = \frac{3}{9} (1) + \frac{4}{9} (0) + \frac{2}{9} (0.5) = \frac{4}{9} \approx 0.44$$

Parte b

$$P(\text{común} \mid \text{cara}) = \frac{P(\text{común}; \text{cara})}{P(\text{cara})} = \frac{P(\text{común} \mid \text{cara}) P(\text{común})}{P(\text{cara})} = \frac{(0.5) \frac{2}{9}}{\frac{4}{9}} = 0.25$$

Parte c

$$P(\text{común} \mid n \text{ caras}) = \frac{P(\text{común}; n \text{ caras})}{P(n \text{ caras})} = \frac{P(n \text{ caras} \mid \text{común}) P(\text{común})}{P(n \text{ caras})} = \frac{(1/2)^n (2/9)}{(4/9)^n} = \left(\frac{9}{8}\right)^n \left(\frac{2}{9}\right)$$

Ejercicio 3

Parte a y Parte b

Ocupación del Padre	Ocupación del Hijo					
	Agricultor	Operativo	Artesano	Ventas	Profesional	Mg. del padre (Y_1)
Agricultor	0.018	0.035	0.031	0.008	0.018	0.11
Operativo	0.002	0.112	0.064	0.032	0.069	0.279
Artesano	0.001	0.066	0.094	0.032	0.084	0.277
Ventas	0.001	0.018	0.019	0.010	0.051	0.099
Profesional	0.001	0.029	0.032	0.043	0.130	0.235
Mg. del hijo (Y_2)	0.023	0.26	0.24	0.125	0.352	1

Parte c

$$P(Y_2|Y_1 = \text{"Agricultor"}) = \frac{P(Y_2; Y_1 = \text{"Agricultor"})}{P(Y_1 = \text{"Agricultor"})}$$

	Agricultor	Operativo	Artesano	Ventas	Profesional
Agricultor	0.1636364	0.3181818	0.2818182	0.0727273	0.1636364

Parte d

$$P(Y_1|Y_2 = \text{"Agricultor"}) = \frac{P(Y_2; Y_1 = \text{"Agricultor"})}{P(Y_2 = \text{"Agricultor"})}$$

	Agricultor
Agricultor	0.7826087
Operativo	0.0869565
Artesano	0.0434783
Ventas	0.0434783
Profesional	0.0434783

Ejercicio 4

Parte a

$$E(a_1 Y_1 + a_2 Y_2) = E(a_1 Y_1) + E(a_2 Y_2) = a_1 E(Y_1) + a_2 E(Y_2) = a_1 \mu_1 + a_2 \mu_2$$

$$\begin{aligned} V(a_1 Y_1 + a_2 Y_2) &= V(a_1 Y_1) + V(a_2 Y_2) + 2 \text{COV}(a_1 Y_1; a_2 Y_2) = \\ &= a_1^2 V(Y_1) + a_2^2 V(Y_2) + 2 a_1 a_2 \underbrace{\text{COV}(Y_1; Y_2)}_{=0 \text{ por independencia}} = a_1^2 V(Y_1) + a_2^2 V(Y_2) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 \end{aligned}$$

Parte b

$$E(a_1 Y_1 - a_2 Y_2) = E(a_1 Y_1) - E(a_2 Y_2) = a_1 E(Y_1) - a_2 E(Y_2) = a_1 \mu_1 - a_2 \mu_2$$

$$\begin{aligned} V(a_1 Y_1 - a_2 Y_2) &= V(a_1 Y_1) + V(a_2 Y_2) - 2 \text{COV}(a_1 Y_1; a_2 Y_2) = \\ &= a_1^2 V(Y_1) + a_2^2 V(Y_2) - 2 a_1 a_2 \underbrace{\text{COV}(Y_1; Y_2)}_{=0 \text{ por independencia}} = a_1^2 V(Y_1) + a_2^2 V(Y_2) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 \end{aligned}$$

Ejercicio 5

Parte a

Por teorema de la multiplicidad:

$$\begin{aligned} p(x; y; z) &= p(z) p(y|z) p(x|y; z) \Rightarrow \\ \Rightarrow p(x|y; z) &= \frac{p(x; y; z)}{p(z) p(z|y)} \propto \frac{f(x; z) g(y; z) h(z)}{p(z) \frac{p(z; y)}{p(z)}} = \frac{f(x; z) g(y; z) h(z)}{p(z; y)} \propto \frac{f(x; z) g(y; z) h(z)}{g(z; y)} = \\ &= f(x; z) h(z) \propto f(x; z) \Rightarrow \boxed{p(x|y; z) \propto f(x; z)} \end{aligned}$$

Otra solución:

$$\begin{aligned}
p(x; y; z) &\propto f(x; z) g(y; z) h(z) \\
\int_Y p(x; y; z) dy &\propto \int_Y f(x; z) g(y; z) h(z) dy \\
\int_Y p(x; y; z) dy &\propto f(x; z) h(z) \int_Y g(y; z) dy \\
p(x; z) &\propto f(x; z) h(z) p(z) \\
\frac{p(x; z)}{p(z)} &\propto f(x; z) h(z) \\
p(x|z) &\propto f(x; z) h(z) \\
p(x|y; z) &\propto f(x; z) h(z) \\
p(x|y; z) &\propto f(x; z)
\end{aligned}$$

Parte b

Por teorema de la multiplicidad:

$$\begin{aligned}
p(x; y; z) &= p(x) p(z|x) p(y|x; z) \Rightarrow \\
\Rightarrow p(y|x; z) &= \frac{p(x; y; z)}{p(x) p(z|x)} \propto \frac{f(x; z) g(y; z) h(z)}{p(x) \frac{p(z; x)}{p(x)}} = \frac{f(x; z) g(y; z) h(z)}{p(z; x)} \propto \frac{f(x; z) g(y; z) h(z)}{f(z; x)} = \\
&= g(y; z) h(z) \propto g(y; z) \Rightarrow \boxed{p(y|x; z) \propto g(y; z)}
\end{aligned}$$

Parte c

$$\begin{aligned}
p(x; y|z) &= \frac{p(x; y; z)}{p(z)} \propto \frac{f(x; z) g(y; z) h(z)}{p(z)} = \frac{\overbrace{f(x; z)}^{\propto p(x; z)}}{p(z)} \underbrace{g(y; z)}_{\propto p(y; z)} \overbrace{h(z)}^{\propto p(z)} \propto \frac{p(x; z)}{p(z)} p(y; z) p(z) = \\
&= p(x|z) p(y|z) p(z) \propto p(x|z) p(y|z)
\end{aligned}$$

Ejercicio 6

Parte a

$$E(y) = E_n[E_Y(y|n)] = E_n[n p] = \lambda p$$

Parte b

$$\begin{aligned} V(y) &= V_n[E_Y(y|n)] + E_n[V_Y(y|n)] = V_n[n p] + E_n[n p(1-p)] = p^2 V_n[n] + p(1-p)E_n[n] = \\ &= p^2 \lambda + p(1-p)\lambda = p^2 \lambda + \lambda p - p^2 \lambda = p^2 \lambda \end{aligned}$$

Parte c

$$\begin{aligned} p(y; n) &= p(y|n) p(n) = \binom{n}{y} p^y (1-p)^{n-y} \mathbf{I}_{[y \in \mathbb{N}_0]} \frac{e^{-\lambda} \lambda^n}{n!} \mathbf{I}_{[n \in \mathbb{N}_0]} \\ p(y) &= \sum_{n=0}^{+\infty} p(y; n) = \sum_{n=0}^{+\infty} p(y|n) p(n) = \sum_{n=0}^{+\infty} \binom{n}{y} p^y (1-p)^{n-y} \mathbf{I}_{[y \in \mathbb{N}_0]} \frac{e^{-\lambda} \lambda^n}{n!} = \\ &= \frac{e^{-\lambda} p^y \lambda^y}{y!} \sum_{n=0}^{+\infty} \frac{(\lambda(1-p))^{n-y}}{(n-y)!} \end{aligned}$$

Sean $x = n - y$ y $A = \lambda(1-p)$, entonces:

$$= \frac{e^{-\lambda} p^y \lambda^y}{y!} \sum_{x=-y}^{+\infty} \frac{A^x}{x!} = \frac{e^{-\lambda} p^y \lambda^y}{y!} e^A = \frac{e^{-\lambda} p^y \lambda^y}{y!} e^{\lambda(1-p)} = \frac{e^{-\lambda p} (\lambda p)^y}{y!}$$

Por lo tanto, $y \sim \text{Poisson}(\lambda p)$ (asumiendo que no pasa nada al sumar desde $x = -y \dots$ consultar!)

Ejercicio 7

Probar la regla de Bayes

$$p(H_j|E) = \frac{p(E|H_j) p(H_j)}{\sum_{k=1}^K p(E|H_k) p(H_k)}$$

Sabiendo que se cumple $P(A|B) = \frac{p(A \cap B)}{p(B)}$ para cualesquiera sucesos A y B , donde E es un evento cualquiera y H_1, \dots, H_k forman una partición de Ω .

Demostración:

$$p(H_j|E) = \frac{p(H_j; E)}{p(E)} = \frac{p(E|H_j) p(H_j)}{\sum_{k=1}^K p(E; H_k)} = \frac{p(E|H_j) p(H_j)}{\sum_{k=1}^K p(E|H_k) p(H_k)}$$

Ejercicio 7

Parte a

X Y

	1	0
1	$(0.5)(0.4) = 0.2$	$(0.5)(0.6) = 0.3$
0	$(0.5)(0.6) = 0.3$	$(0.5)(0.4) = 0.2$

Parte b

$$\begin{aligned}
 E(Y) &= \sum_Y y p(y) = (1) p(y) + (0) p(y) = p(y) = \sum_X p(x; y) = P(\text{verde} \mid \text{cara}) P(\text{cara}) + P(\text{verde} \mid \text{cruz}) P(\text{cruz}) = \\
 &= (0.5)(0.4) + (0.5)(0.6) = 0.5
 \end{aligned}$$