Práctico 1 - Solución

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Ejercicio 1

1. Hallar la distribución de $Y = 1/X$.

$$X \sim \text{Gamma}(\alpha;\, \beta) \Rightarrow f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta\, x} \mathrm{I}_{[x \geq 0]}$$

$$Y = 1/X \Rightarrow X = 1/Y \Rightarrow \frac{\partial X}{\partial Y} = -\frac{1}{Y^2}$$

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} \exp\left\{-\frac{\beta}{y}\right\} \mathrm{I}_{[{}^{1}/_{y} \geq 0]} \left|-\frac{1}{y^2}\right| = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{-(\alpha+1)} \exp\left\{-\frac{\beta}{y}\right\} \mathrm{I}_{[y \geq 0]}$$

$$\therefore Y \sim \text{Inv-Gamma}(\alpha;\, \beta)$$

2. Calcular $E(Y)$ y $V(Y)$.

$$E(Y) = \int_0^{+\infty} y\, f_Y(y)\, \mathrm{d}y = \int_0^{+\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{-\alpha} \exp\left\{-\frac{\beta}{y}\right\} \mathrm{d}y =$$

Sea $Z = 1/Y \Rightarrow Y = 1/Z \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}y} = -\frac{1}{z^2}$, entonces:

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\int_0^{+\infty} z^{\alpha-2} e^{-\beta\, z} \mathrm{d}z}_{\text{Gamma}(\alpha-2;\, \beta)} = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} = \frac{\beta}{\alpha-1} \quad \forall \alpha > 1$$

Siguiendo el mismo razonamiento:

$$E(Y^2) = \frac{\beta^2}{(\alpha-2)(\alpha-1) \ \ \forall \alpha > 2}$$

Por lo tanto:

$$V(Y) = E(Y^2) - E^2(Y) = \frac{\beta^2}{(\alpha-2)(\alpha-1)} - \left(\frac{\beta}{\alpha-1}\right)^2 = \frac{\beta^2}{(\alpha-2)(\alpha-1)^2} \quad \forall \alpha > 2$$

3.

```
a <- b <- 2
n <- 1000
set.seed(123456789)
x <- rgamma(n, shape=a, rate=b)
y <- 1/x
mean(y)
```

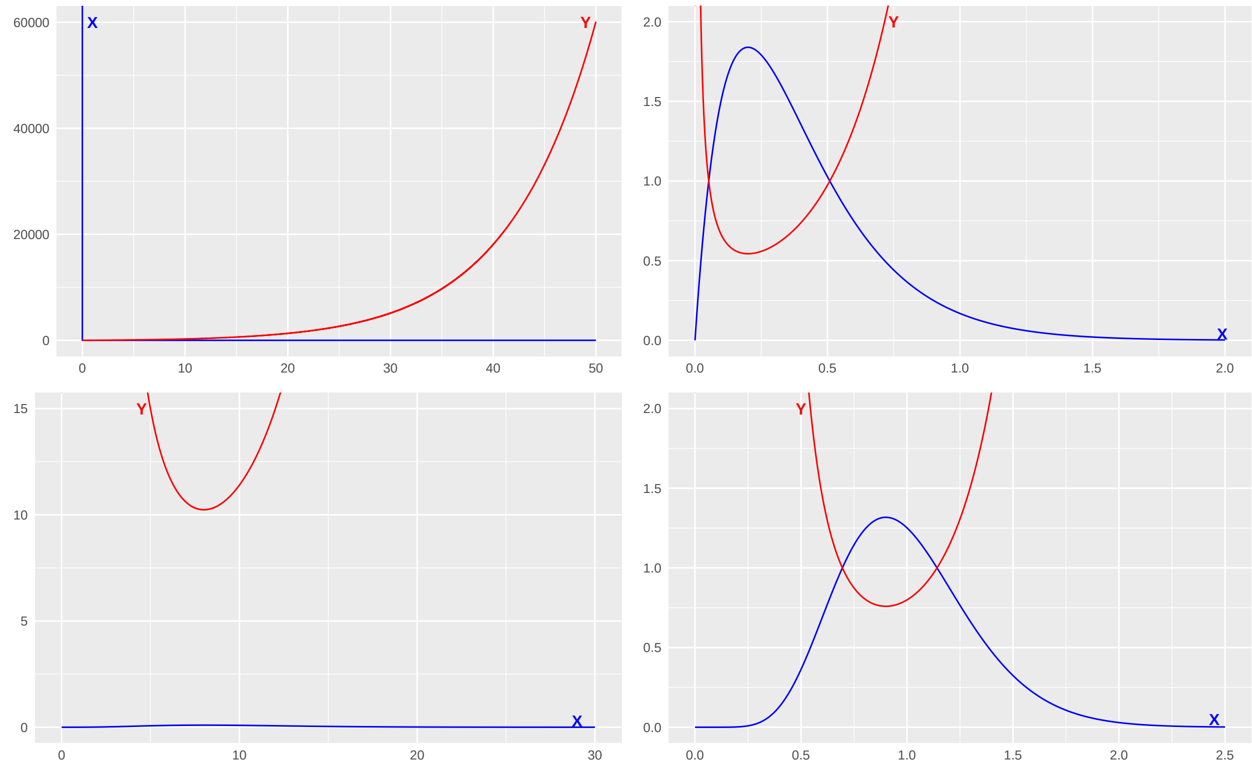
```
## [1] 2.059289
```

```r
var(y)
```

```
## [1] 29.41411
```

4.

```
## Warning: `data_frame()` is deprecated, use `tibble()`.
## This warning is displayed once per session.
```



Ejercicio 2

1. Sea $Y = \sum_{i=1}^{n} X_i$

$$M_Y(t) = E\left(e^{Y\,t}\right) = E\left(\exp\left\{t\sum_{i=1}^{n} X_i\right\}\right) = E\left(\prod_{i=1}^{n} e^{t\,X_i}\right) = \prod_{i=1}^{n} E\left(e^{t\,X_i}\right) = \prod_{i=1}^{n} M_{X_i}(t) = [M_X(t)]^n$$

2. Si $X \sim \mathrm{Exp}(\lambda) \Rightarrow M_Y(t) = \left[\frac{\lambda}{\lambda - t}\right]^n \Rightarrow Y \sim \mathrm{Gamma}(n;\, \lambda)$

Ejercicio 3

1. Ley de esperanzas iteradas

$$E_Y\left[E_{X|Y}(X|Y)\right] = \int_Y E_{X|Y}(X|Y)\, f_Y(y)\,\mathrm{d}y = \int_Y \left[\int_X x\, f_{X|Y}(x|y)\,\mathrm{d}x\right] f_Y(y)\,\mathrm{d}y =$$

$$= \int_Y \int_X x\, f_{X|Y}(x|y)\, f_Y(y)\,\mathrm{d}x\,\mathrm{d}y = \int_Y \int_X x\, f_{X;Y}(x;y)\,\mathrm{d}x\,\mathrm{d}y = \int_X x\left[\int_Y f_{X;Y}(x;y)\mathrm{d}y\right]\mathrm{d}x =$$

$$= \int_X x\, f_X(x) \mathrm{d}x = E_X(X)$$

2. Ley de varianzas iteradas

$$V_X(X) = E_X\left[\Big(X - E_X(X)\Big)^2\right] = E_X\left[\Big(X - E_{X|Y}(X|Y) + E_{X|Y}(X|Y) - E_X(X)\Big)^2\right] =$$

$$= E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)^2 + \Big(E_{X|Y}(X|Y) - E_X(X)\Big)^2 + 2\Big(X - E_{X|Y}(X|Y)\Big)\Big(E_{X|Y}(X|Y) - E_X(X)\Big)\right] =$$

$$= \underbrace{E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)^2\right]}_{(1)} + \underbrace{E_X\left[\Big(E_{X|Y}(X|Y) - E_X(X)\Big)^2\right]}_{(2)} + 2\, \underbrace{E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)\Big(E_{X|Y}(X|Y) - E_X(X)\Big)\right]}_{(3)}$$

$$(1)\;\; E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)^2\right] = E_Y\left[E_X\left(\Big(X - E_{X|Y}(X|Y)\Big)^2\right)\Bigg| Y\right] = E_Y\Big(V_{X|Y}(X|Y)\Big)$$

$$(2)\;\; E_X\left[\Big(E_{X|Y}(X|Y) - E_X(X)\Big)^2\right] = V_Y\Big(E_{X|Y}(X|Y)\Big)$$

$$(3)\;\; E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)\Big(E_{X|Y}(X|Y) - E_X(X)\Big)\right] =$$

$$= E_Y\left[E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)\Big(E_{X|Y}(X|Y) - E_X(X)\Big)\right]\Bigg| Y\right] =$$

$$= E_Y\left[\Big(E_{X|Y}(X|Y) - E_X(X)\Big) E_X\left[\Big(X - E_{X|Y}(X|Y)\Big)\right]\Bigg| Y\right] =$$

$$= E_Y\left[\Big(E_{X|Y}(X|Y) - E_X(X)\Big)\Big(E_{X|Y}(X|Y) - E_{X|Y}(X|Y)\Big)\right] = 0$$

Por lo tanto: $V_X(X) = E_Y\Big(V_{X|Y}(X|Y)\Big) + V_Y\Big(E_{X|Y}(X|Y)\Big)$

Ejercicio 4

1.

$$E(Y) = \sum_{Rec(Y)} y \binom{n}{y} \theta^y (1-\theta)^{n-y} = \sum_{y=0}^{n} y \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

Pero si $y = 0$ el primer t?rmino de la sumatoria es cero, por lo que podemos descartarlo, obteniendo:

$$E(Y) = \sum_{y=1}^{n} y \binom{n}{y} \theta^y (1-\theta)^{n-y} = \sum_{y=1}^{n} y \frac{n!}{y!(n-y)!} \theta^y (1-\theta)^{n-y} =$$

$$= \sum_{y=1}^{n} \frac{n(n-1)!}{(y-1)!(n-y)!} \theta\, \theta^{y-1} (1-\theta)^{n-y} = n\,\theta \sum_{y=1}^{n} \frac{(n-1)!}{(y-1)!(n-y)!} \theta^{y-1} (1-\theta)^{n-y}$$

Se definen $a = y - 1 \Rightarrow y = a + 1$, y $b = n - 1 \Rightarrow n = b + 1$ de donde obtenemos los siguientes límites para la sumatoria:

$$\begin{array}{rcl} y & = & 1 \\ y & = & n \end{array} \Rightarrow \begin{array}{rcl} a+1 & = & 1 \\ a+1 & = & b+1 \end{array} \Rightarrow \begin{array}{rcl} a & = & 0 \\ a & = & b \end{array}$$

Luego entonces:

$$E(Y) = n\,\theta \sum_{a=0}^{b} \frac{b!}{a!(b-a)!}\,\theta^{a}\,(1-\theta)^{b-a} = n\,\theta \underbrace{\sum_{Rec(A)} \binom{b}{a}\,\theta^{a}\,(1-\theta)^{b-a}}_{\text{Bin}(b;\,\theta)} = n\,\theta$$

2. Partimos de que $V(Y) = E(Y^2) - E^2(Y)$, y procedemos a calcular $E(Y^2)$ de forma análoga a la parte anterior:

$$E\big[Y(Y-1)\big] = \sum_{Rec(Y)} y(y-1)\binom{n}{y}\theta^{y}(1-\theta)^{n-y} = \sum_{y=1}^{n} y(y-1)\frac{n(n-1)(n-2)!}{y(y-1)(y-2)!(n-y)!}\theta^{2}\,\theta^{y-2}(1-\theta)^{n-y} =$$

$$= n(n-1)\,\theta^{2}\sum_{y=1}^{n}\frac{(n-2)!}{(y-2)!(n-y)!}\theta^{y-2}(1-\theta)^{n-y} = n(n-1)\,\theta^{2}\sum_{a=0}^{n}\binom{b}{a}\theta^{a}(1-\theta)^{b-a} = n(n-1)\,\theta^{2}$$

Luego entonces:

$$E\big[Y(Y-1)\big] = E\big[Y^2 - Y\big] = E(Y^2) - E(Y) \Rightarrow n(n-1)\,\theta^2 = E(Y^2) - n\,\theta \Rightarrow E(Y^2) = n(n-1)\,\theta^2 + n\,\theta$$

Por lo tanto:

$$V(Y) = n(n-1)\,\theta^2 + n\,\theta - n^2\,\theta^2 = n\,\theta\big[(n-1)\,\theta + 1 - n\,\theta\big] = n\,\theta\big[n\,\theta - \theta + 1 - n\,\theta\big] = n\,\theta\,(1-\theta)$$