Entrega 1 - Inferencia II

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Ejercicio 1

Sea $\phi \sim \text{Gamma}(\alpha;\, \beta)$.

1. Obtener la distribución de $\sigma^2 = 1/\phi$.

$$\sigma^2 = \frac{1}{\phi} \Rightarrow \phi = \frac{1}{\sigma^2} \Rightarrow \frac{\partial \phi}{\partial \sigma^2} = -\frac{1}{(\sigma^2)^2}$$

$$f_\phi(\phi) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \phi^{\alpha-1} \, \exp\big\{ -\beta\, \phi \big\} \, \mathrm{I}_{[\phi \geq 0]}$$

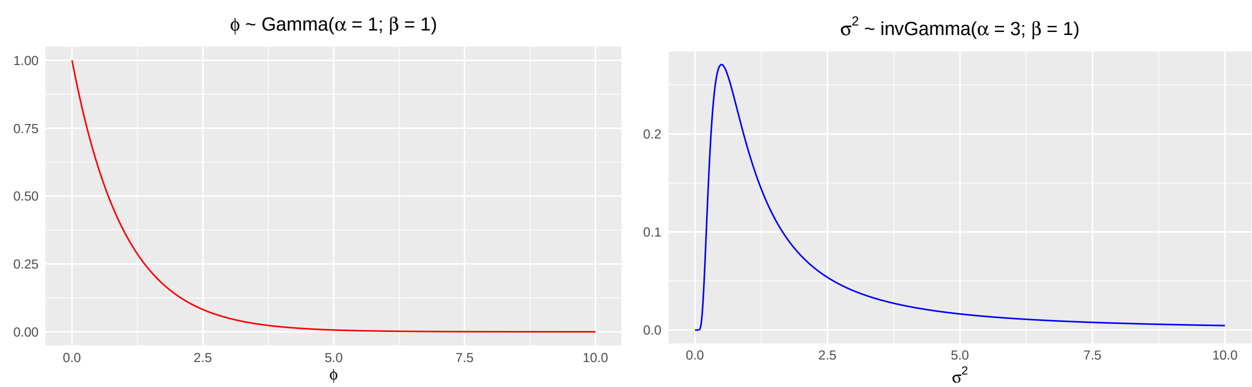
$$f_{\sigma^2}(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2} \right)^{\alpha-1} \exp\left\{ -\frac{\beta}{\sigma^2} \right\} \mathrm{I}_{[\frac{1}{\sigma^2} \geq 0]} \left| -\frac{1}{(\sigma^2)^2} \right| = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2} \right)^{(\alpha+2)-1} \exp\left\{ -\frac{\beta}{\sigma^2} \right\} \mathrm{I}_{[\sigma^2 > 0]}$$

Por lo tanto:

$$\boxed{\sigma^2 \sim \text{invGamma}(\alpha;\, \beta)}$$

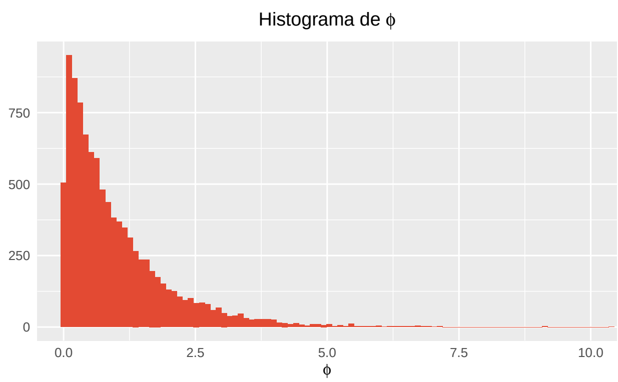
2. Para $\alpha = \beta = 1$, dibujar la densidad de ϕ y σ^2 en figuras distintas.

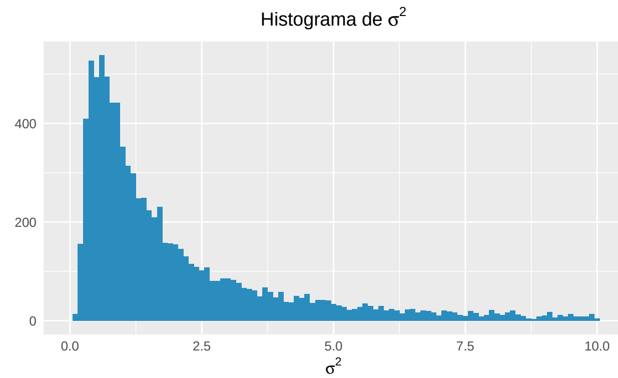
```r
alpha <- beta <- 1
x <- seq(0, 10, 0.001)
phi <- dgamma(x, shape=alpha, scale=1/beta)
sigma2 <- dgamma(1/x, shape=alpha+2, scale=1/beta)
```



3. Simular 10,000 realizaciones de ϕ y usarlas para obtener valores simulados de σ^2. Luego dibujar histogramas de cada conjunto de simulaciones por separado.

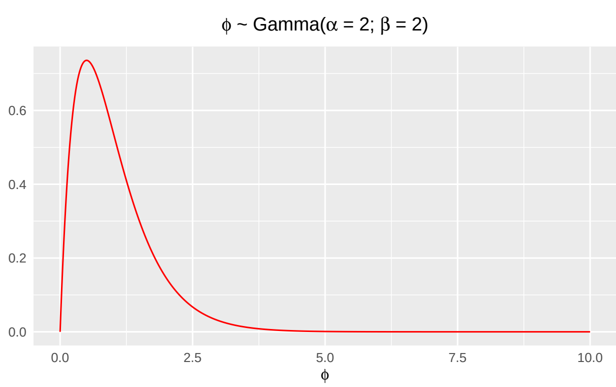
```r
set.seed(123456789)
phi <- rgamma(10000, shape=alpha, scale=1/beta)
sigma2 <- 1/phi
```

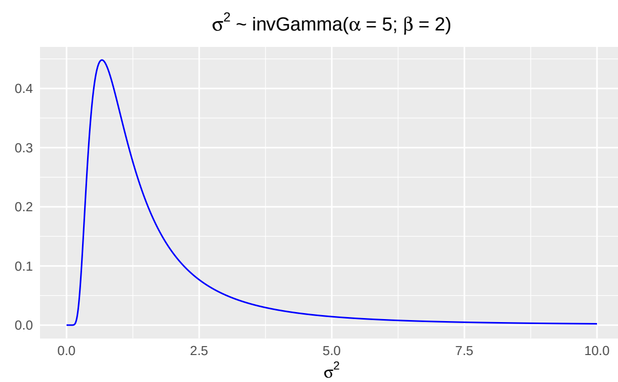




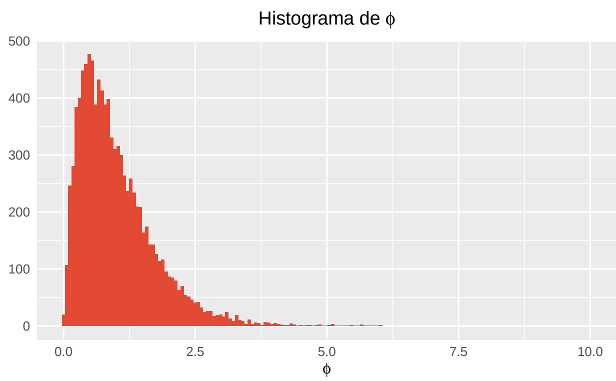
4. Repetir el procedimiento para $\alpha = \beta = 2$ y $\alpha = \beta = 0.5$.

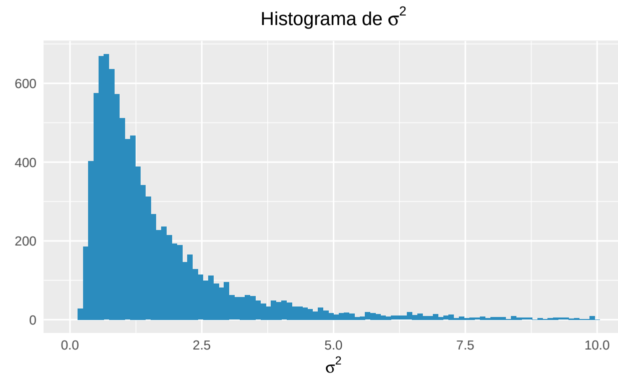
```r
alpha <- beta <- 2
x <- seq(0, 10, 0.001)
phi <- dgamma(x, shape=alpha, scale=1/beta)
sigma2 <- dgamma(1/x, shape=alpha+2, scale=1/beta)
```



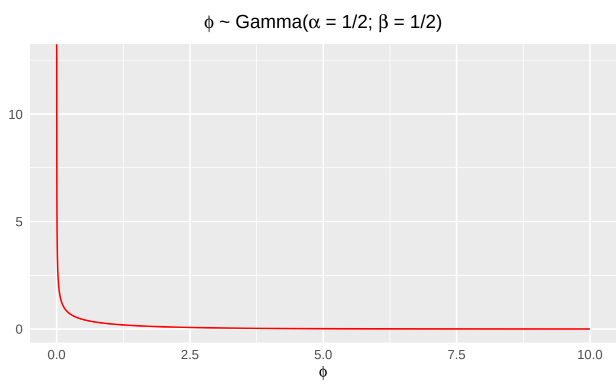


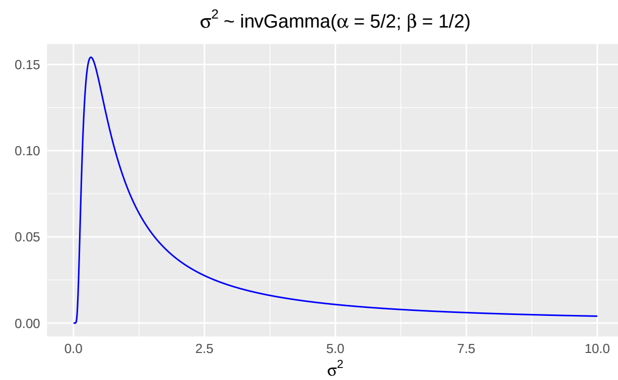
```r
set.seed(123456789)
phi <- rgamma(10000, shape=alpha, scale=1/beta)
sigma2 <- 1/phi
```



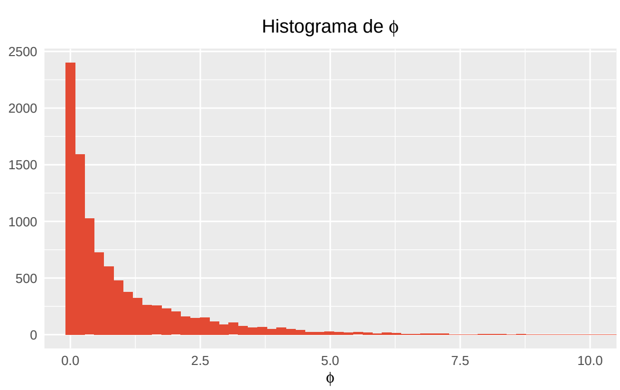


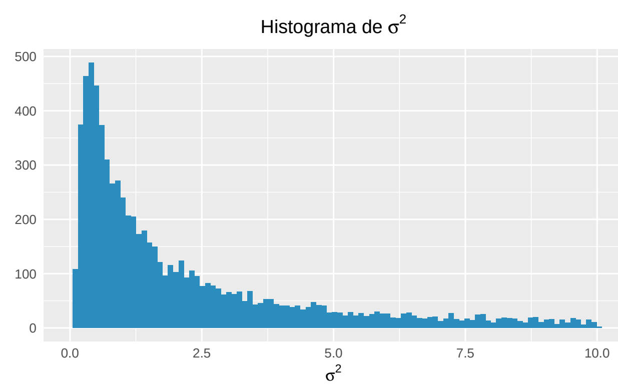

```r
alpha <- beta <- 1/2
x <- seq(0, 10, 0.001)
phi <- dgamma(x, shape=alpha, scale=1/beta)
sigma2 <- dgamma(1/x, shape=alpha+2, scale=1/beta)
```






```r
set.seed(123456789)
phi <- rgamma(10000, shape=alpha, scale=1/beta)
sigma2 <- 1/phi
```





Ejercicio 2

Supongamos que $Y_i \overset{iid}{\sim}$ Poisson(λ) con $\lambda \sim$ Gamma(a, b), esto es:

$$p(y) = \frac{e^{-\lambda}\,\lambda^{y}}{y!}\,\mathrm{I}_{[y\in\mathbb{N}_0]}$$

$$p(\lambda) = \frac{b^a}{\Gamma(a)}\,\lambda^{a-1}\,e^{-b\,\lambda}\,\mathrm{I}_{[\lambda\geq 0]}$$

Por otro lado, sea $\tilde{y} \sim$ Poisson(λ), es decir una observación futura, que suponemos condicionalmente independiente respecto a $y|\lambda$.

1. Encuentre la posterior $p(\lambda|y)$ donde $y = (y_1; \ldots; y_n)$.

$$p(\lambda|y) = \frac{p(\lambda;\, y)}{p(y)} = \frac{p(y|\lambda)\,p(\lambda)}{p(y)}$$

Luego entonces, la distribución en el muestreo es:

$$p(y|\lambda) = \mathscr{L}(y|\lambda) = \prod_{i=1}^{n} p(y_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\,\lambda^{y_i}}{y_i!}\mathrm{I}_{[y_i\in\mathbb{N}_0]} = \frac{e^{-n\lambda}\,\lambda^{\sum\limits_{i=1}^{n} y_i}}{\prod\limits_{i=1}^{n} y_i!}\prod_{i=1}^{n}\mathrm{I}_{[y_i\in\mathbb{N}_0]}$$

La probabilidad de y se consigue integrando:

$$p(y) = \int\limits_{Rec(\lambda)} p(y;\,\lambda)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} p(y|\lambda)\,p(\lambda)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} \frac{e^{-n\lambda}\,\lambda^{\sum\limits_{i=1}^{n} y_i}}{\prod\limits_{i=1}^{n} y_i!}\prod_{i=1}^{n}\mathrm{I}_{[y_i\in\mathbb{N}_0]}\,\frac{b^a}{\Gamma(a)}\,\lambda^{a-1}\,e^{-b\,\lambda}\,\mathrm{I}_{[\lambda\geq 0]}\,\mathrm{d}\lambda =$$

$$= \frac{b^a}{\Gamma(a)}\left(\prod_{i=1}^{n}\frac{\mathrm{I}_{[y_i\in\mathbb{N}_0]}}{y_i!}\right)\underbrace{\int_0^{+\infty} e^{-(b+n)\lambda}\,\lambda^{\sum\limits_{i=1}^{n} y_i+a-1}\,\mathrm{d}\lambda}_{=\left(\frac{(n+b)^{\sum_{i=1}^{n} y_i+a}}{\Gamma\left(\sum_{i=1}^{n} y_i+a\right)}\right)^{-1}} = \frac{b^a}{\Gamma(a)}\left(\prod_{i=1}^{n}\frac{\mathrm{I}_{[y_i\in\mathbb{N}_0]}}{y_i!}\right)\frac{\Gamma\left(\sum\limits_{i=1}^{n} y_i+a\right)}{(n+b)^{\sum\limits_{i=1}^{n} y_i+a}}$$

Por lo tanto, la probabilidad posterior $p(\lambda|y)$ es[1]:

$$p(\lambda|y) = \frac{e^{-n\lambda}\,\lambda^{\sum\limits_{i=1}^{n} y_i}\left(\prod\limits_{i=1}^{n}\frac{\mathrm{I}_{[y_i\in\mathbb{N}_0]}}{y_i!}\right)\frac{b^a}{\Gamma(a)}\,\lambda^{a-1}\,e^{-b\,\lambda}\,\mathrm{I}_{[\lambda\geq 0]}}{\frac{b^a}{\Gamma(a)}\left(\prod\limits_{i=1}^{n}\frac{\mathrm{I}_{[y_i\in\mathbb{N}_0]}}{y_i!}\right)\frac{\Gamma\left(\sum_{i=1}^{n} y_i+a\right)}{(n+b)^{\sum_{i=1}^{n} y_i+a}}} = \frac{(n+b)^{\sum\limits_{i=1}^{n} y_i+a)}}{\Gamma\left(\sum\limits_{i=1}^{n} y_i+a\right)}\,\lambda^{\sum\limits_{i=1}^{n} y_i+a-1}\,e^{-(n+b)\lambda}\,\mathrm{I}_{[\lambda\geq 0]}$$

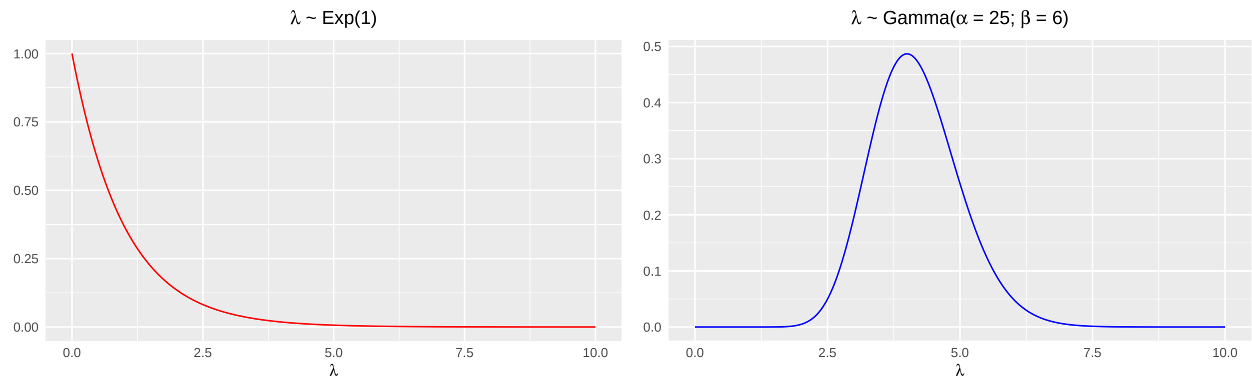
Con lo cual:

$$\boxed{\lambda|y \sim \text{Gamma}\left(\sum_{i=1}^{n} y_i + a;\, n+b\right)}$$

[1] A esta misma conclusión podría llegarse utilizando la proporcionalidad, dado que ya identificamos el kernel correspondiente.

2. Para $y = (4;4;5;8;3)'$ y $a = b = 1$, dibujar la previa y la posterior de λ.

```r
y <- c(4,4,5,8,3)
n <- length(y)
a <- b <- 1
x <- seq(0, 10, by=0.001)
previa <- dexp(x, rate=b)
posterior <-dgamma(x, shape=(sum(y)+a), scale=1/(n+b))
```



3. Derivar $p(\tilde{y})$ y $p(\tilde{y}|y)$, la distribución predictiva previa y posterior respectivamente. Dibujar ambas distribuciones en el mismo gráfico.

$$\star\ p(\tilde{y}) = \int\limits_{Rec(\lambda)} p(\tilde{y};\,\lambda)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} p(\tilde{y}\,|\,\lambda)\,p(\lambda)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} \frac{e^{-\lambda}\,\lambda^{\tilde{y}}}{\tilde{y}!}\,\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}\,\frac{b^a}{\Gamma(a)}\,\lambda^{a-1}\,e^{-b\,\lambda}\mathrm{d}\lambda =$$

$$= \frac{b^a}{\Gamma(a)}\,\frac{\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}}{\tilde{y}!}\int\limits_{Rec(\lambda)} \lambda^{\tilde{y}+a-1}e^{-(b+1)\lambda}\,\mathrm{d}\lambda = \frac{b^a}{\Gamma(a)}\,\frac{\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}}{\tilde{y}!}\,\frac{\Gamma(\tilde{y}+a)}{(b+1)^{\tilde{y}+a}} =$$

$$= \frac{\Gamma(\tilde{y}+a)}{\tilde{y}!\,\Gamma(a)}\,\frac{b^a}{(b+1)^{\tilde{y}+a}}\,\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]} = \frac{(\tilde{y}+a-1)!}{\tilde{y}!\,(a-1)!}\left[\frac{b}{b+1}\right]^a\left[\frac{1}{b+1}\right]^{\tilde{y}}\,\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}$$

Por lo tanto:

$$\boxed{\tilde{y} \sim \mathrm{BN}\left(a;\,\frac{b}{b+1}\right)}$$

$$\star\ p(\tilde{y}|y) = \int\limits_{Rec(\lambda)} p(\tilde{y},\,\lambda\,|\,y)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} p(\tilde{y}\,|\,\lambda,\,y)\,p(\lambda\,|\,y)\,\mathrm{d}\lambda = \int\limits_{Rec(\lambda)} p(\tilde{y}\,|\,\lambda)\,p(\lambda\,|\,y)\,\mathrm{d}\lambda =$$

$$= \int\limits_{Rec(\lambda)} \frac{e^{-\lambda}\,\lambda^{\tilde{y}}}{\tilde{y}!}\,\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}\,\frac{(n+b)^{\sum\limits_{i=1}^{n} y_i+a}}{\Gamma\left(\sum\limits_{i=1}^{n} y_i+a\right)}\,\lambda^{\sum\limits_{i=1}^{n} y_i+a-1}\,\exp\Big\{-(n+b)\lambda\Big\}\,\mathrm{I}_{[\lambda\geq 0]}\,\mathrm{d}\lambda =$$

$$= \left[\frac{\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}}{\tilde{y}!}\right]\left[\frac{(n+b)^{\sum\limits_{i=1}^{n} y_i+a}}{\Gamma\left(\sum\limits_{i=1}^{n} y_i+a\right)}\right]\underbrace{\int\limits_{0}^{+\infty}\lambda^{\sum\limits_{i=1}^{n} y_i+a+\tilde{y}-1}\,\exp\Big\{-(n+b+1)\lambda\Big\}\,\mathrm{d}\lambda}_{\text{kernel de una Gamma}\left(\sum\limits_{i=1}^{n} y_i+a+\tilde{y};\,n+b+1\right)} =$$

$$= \left[\frac{\mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}}{\tilde{y}!}\right]\left[\frac{(n+b)^{\sum\limits_{i=1}^{n} y_i + a}}{\Gamma\left(\sum\limits_{i=1}^{n} y_i + a\right)}\right]\left[\frac{\Gamma\left(\sum\limits_{i=1}^{n} y_i + a + \tilde{y}\right)}{(n+b+1)^{\sum\limits_{i=1}^{n} y_i + a + \tilde{y}}}\right] =$$

Asumiendo que $a \in \mathbb{N}$ y $b \in \mathbb{N}$:

$$= \mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}\left[\frac{\left(\sum\limits_{i=1}^{n} y_i + a + \tilde{y} - 1\right)!}{\tilde{y}!\left(\sum\limits_{i=1}^{n} y_i + a - 1\right)!}\right]\left[\frac{n+b}{n+b+1}\right]^{\sum\limits_{i=1}^{n} y_i + a}\left[\frac{1}{n+b+1}\right]^{\tilde{y}} =$$

$$= \mathrm{I}_{[\tilde{y}\in\mathbb{N}_0]}\begin{pmatrix} \sum\limits_{i=1}^{n} y_i + a + \tilde{y} - 1 \\ \sum\limits_{i=1}^{n} y_i + a - 1 \end{pmatrix}\left[\frac{n+b}{n+b+1}\right]^{\sum\limits_{i=1}^{n} y_i + a}\left[\frac{1}{n+b+1}\right]^{\tilde{y}}$$

Por lo tanto:

$$\boxed{\tilde{y}|y \sim \mathrm{BN}\left(\sum_{i=1}^{n} y_i + a; \frac{n+b}{n+b+1}\right)}$$

```r
y <- c(4,4,5,8,3)
n <- length(y)
a <- b <- 1
x <- seq(0, 100, by=1)
previa <- dnbinom(x, size=a, prob=b/(b+1))
posterior <- dnbinom(x, size=sum(y)+a, prob=(n+b)/(n+b+1))
```