Práctico 0 - Solución

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September 9, 2017

Ejercicio 1

Partido	P(partido)	P(votó | partido)	P(no votó | partido)
Conservador	0.30	0.65	0.35
Liberal	0.50	0.82	0.18
Independiente	0.20	0.50	0.50

Parte a

$$P(\text{liberal} \mid \text{votó}) = \frac{P(\text{liberal; votó})}{P(\text{votó})} = \frac{P(\text{votó} \mid \text{liberal})\, P(\text{liberal})}{P(\text{votó})} =$$

$$= \frac{(0.82)(0.5)}{(0.3)(0.65) + (0.5)(0.82) + (0.2)(0.5)} = \frac{82}{141} \approx 0.581$$

Parte b

$$P(\text{liberal} \mid \text{no votó}) = \frac{P(\text{liberal; no votó})}{P(\text{no votó})} = \frac{P(\text{no votó} \mid \text{liberal})\, P(\text{liberal})}{P(\text{no votó})} =$$

$$= \frac{(0.18)(0.5)}{(0.3)(0.35) + (0.5)(0.18) + (0.2)(0.5)} = \frac{18}{59} \approx 0.305$$

Ejercicio 2

9 monedas:

- 3 monedas de dos caras
- 4 monedas de dos cruces
- 2 monedas comunes

Parte a

$$P(\text{cara}) = \sum_{\substack{\text{tipo de}\\ \text{moneda}}} P(\text{cara} \mid \text{tipo de moneda})\, P(\text{tipo de moneda}) = \frac{3}{9}\,(1) + \frac{4}{9}\,(0) + \frac{2}{9}\,(0.5) = \frac{4}{9} \approx 0.44$$

Parte b

$$P(\text{común} \mid \text{cara}) = \frac{P(\text{común; cara})}{P(\text{cara})} = \frac{P(\text{común} \mid \text{cara})\, P(\text{común})}{P(\text{cara})} = \frac{(0.5)\,\frac{2}{9}}{\frac{4}{9}} = 0.25$$

Parte c

$$P(\text{común} \mid n \text{ caras}) = \frac{P(\text{común}; n \text{ caras})}{P(n \text{ caras })} = \frac{P(n \text{ caras} \mid \text{común})\, P(\text{común})}{P(n \text{ caras})} = \frac{(^1\!/_2)^n\,(^2\!/_9)}{(^4\!/_9)^n} = \left(\frac{9}{8}\right)^n \left(\frac{2}{9}\right)$$

Ejercicio 3

Parte a y Parte b

`Ocupación del Padre` `Ocupación del Hijo`

	Agricultor	Operativo	Artesano	Ventas	Profesional	Mg. del padre (Y_1)
Agricultor	0.018	0.035	0.031	0.008	0.018	0.11
Operativo	0.002	0.112	0.064	0.032	0.069	0.279
Artesano	0.001	0.066	0.094	0.032	0.084	0.277
Ventas	0.001	0.018	0.019	0.010	0.051	0.099
Profesional	0.001	0.029	0.032	0.043	0.130	0.235
Mg. del hijo (Y_2)	0.023	0.26	0.24	0.125	0.352	1

Parte c

$$P(Y_2|Y_1 = \text{"Agricultor"}) = \frac{P(Y_2;\, Y_1 = \text{"Agricultor"})}{P(Y_1 = \text{"Agricultor"})}$$

	Agricultor	Operativo	Artesano	Ventas	Profesional
Agricultor	0.1636364	0.3181818	0.2818182	0.0727273	0.1636364

Parte d

$$P(Y_1|Y_2 = \text{"Agricultor"}) = \frac{P(Y_2;\, Y_1 = \text{"Agricultor"})}{P(Y_2 = \text{"Agricultor"})}$$

	Agricultor
Agricultor	0.7826087
Operativo	0.0869565
Artesano	0.0434783
Ventas	0.0434783
Profesional	0.0434783

Ejercicio 4

Parte a

$$E(a_1\,Y_1 + a_2\,Y_2) = E(a_1\,Y_1) + E(a_2\,Y_2) = a_1\,E(Y_1) + a_2\,E(Y_2) = a_1\,\mu_1 + a_2\,\mu_2$$

$$V(a_1\,Y_1 + a_2\,Y_2) = V(a_1\,Y_1) + V(a_2\,Y_2) + 2\,COV(a_1\,Y_1;\,a_2\,Y_2) =$$
$$= a_1^2\,V(Y_1) + a_2^2\,V(Y_2) + 2\,a_1\,a_2\,\underbrace{COV(Y_1;\,Y_2)}_{\substack{=0 \text{ por} \\ \text{independencia}}} = a_1^2\,V(Y_1) + a_2^2\,V(Y_2) = a_1^2\,\sigma_1^2 + a_2^2\,\sigma_2^2$$

Parte b

$$E(a_1\,Y_1 - a_2\,Y_2) = E(a_1\,Y_1) - E(a_2\,Y_2) = a_1\,E(Y_1) - a_2\,E(Y_2) = a_1\,\mu_1 - a_2\,\mu_2$$

$$V(a_1\,Y_1 - a_2\,Y_2) = V(a_1\,Y_1) + V(a_2\,Y_2) - 2\,COV(a_1\,Y_1;\,a_2\,Y_2) =$$
$$= a_1^2\,V(Y_1) + a_2^2\,V(Y_2) - 2\,a_1\,a_2\,\underbrace{COV(Y_1;\,Y_2)}_{\substack{=0 \text{ por} \\ \text{independencia}}} = a_1^2\,V(Y_1) + a_2^2\,V(Y_2) = a_1^2\,\sigma_1^2 + a_2^2\,\sigma_2^2$$

Ejercicio 5

Parte a

Por teorema de la multiplicidad:

$$p(x;y;z) = p(z)\,p(y|z)\,p(x|y;z) \Rightarrow$$

$$\Rightarrow p(x|y;z) = \frac{p(x;y;z)}{p(z)\,p(z|y)} \propto \frac{f(x;z)\,g(y;z)\,h(z)}{p(z)\,\frac{p(z;y)}{p(z)}} = \frac{f(x;z)\,g(y;z)\,h(z)}{p(z;y)} \propto \frac{f(x;z)\,g(y;z)\,h(z)}{g(z;y)} =$$

$$= f(x;z)\,h(z) \propto f(x;z) \Rightarrow \boxed{p(x|y;z) \propto f(x;z)}$$

Otra solución:

$$\begin{aligned}
p(x;y;z) &\propto f(x;z)\,g(y;z)\,h(z)\\
\textstyle\int_Y p(x;y;z)dy &\propto \textstyle\int_Y f(x;z)\,g(y;z)\,h(z)dy\\
\textstyle\int_Y p(x;y;z)dy &\propto f(x;z)\,h(z)\,\textstyle\int_Y g(y;z)dy\\
p(x;z) &\propto f(x;z)\,h(z)\,p(z)\\
\tfrac{p(x;z)}{p(z)} &\propto f(x;z)\,h(z)\\
p(x|z) &\propto f(x;z)\,h(z)\\
p(x|y;z) &\propto f(x;z)\,h(z)\\
p(x|y;z) &\propto f(x;z)
\end{aligned}$$

Parte b

Por teorema de la multiplicidad:

$$p(x;y;z) = p(x)\,p(z|x)\,p(y|x;z) \Rightarrow$$

$$\Rightarrow p(y|x;z) = \frac{p(x;y;z)}{p(x)\,p(z|x)} \propto \frac{f(x;z)\,g(y;z)\,h(z)}{p(x)\,\frac{p(z;x)}{p(x)}} = \frac{f(x;z)\,g(y;z)\,h(z)}{p(z;x)} \propto \frac{f(x;z)\,g(y;z)\,h(z)}{f(z;x)} =$$

$$= g(y;z)\,h(z) \propto g(y;z) \Rightarrow \boxed{p(y|x;z) \propto g(y;z)}$$

Parte c

$$p(x;y|z) = \frac{p(x;y;z)}{p(z)} \propto \frac{f(x;z)\,g(y;z)\,h(z)}{p(z)} = \frac{\overbrace{f(x;z)}^{\propto\, p(x;z)}}{p(z)}\,\underbrace{g(y;z)}_{\propto p(y;z)}\,\overbrace{h(z)}^{\propto\, p(z)} \propto \frac{p(x;z)}{p(z)}\,p(y;z)\,p(z) =$$

$$= p(x|z)\,p(y|z)\,p(z)\,p(z) \propto p(x|z)\,p(y|z)$$

Ejercicio 6

Parte a

$$E(y) = E_n\big[E_Y(y|n)\big] = E_n\big[n\,p\big] = \lambda\,p$$

Parte b

$$V(y) = V_n\big[E_Y(y|n)\big] + E_n\big[V_Y(y|n)\big] = V_n\big[n\,p\big] + E_n\big[n\,p\,(1-p)\big] = p^2\,V_n\big[n\big] + p\,(1-p)E_n\big[n\big] =$$
$$= p^2\,\lambda + p\,(1-p)\lambda = p^2\,\lambda + \lambda\,p - p^2\lambda = p^2\,\lambda$$

Parte c

$$p(y;n) = p(y|n)\,p(n) = \binom{n}{y}\,p^y\,(1-p)^{n-y}\,\mathrm{I}_{[y\in\mathbb{N}_0]}\,\frac{e^{-\lambda}\,\lambda^n}{n!}\mathrm{I}_{[n\in\mathbb{N}_0]}$$

$$p(y) = \sum_{n=0}^{+\infty} p(y;n) = \sum_{n=0}^{+\infty} p(y|n)\,p(n) = \sum_{n=0}^{+\infty}\binom{n}{y}\,p^y\,(1-p)^{n-y}\,\mathrm{I}_{[y\in\mathbb{N}_0]}\,\frac{e^{-\lambda}\,\lambda^n}{n!} =$$

$$= \frac{e^{-\lambda}\,p^y\,\lambda^y}{y!}\sum_{n=0}^{+\infty}\frac{\Big(\lambda(1-p)\Big)^{n-y}}{(n-y)!}$$

Sean $x = n - y$ y $A = \lambda(1-p)$, entonces:

$$= \frac{e^{-\lambda}\,p^y\,\lambda^y}{y!}\sum_{x=-y}^{+\infty}\frac{A^x}{x!} = \frac{e^{-\lambda}\,p^y\,\lambda^y}{y!}e^{A} = \frac{e^{-\lambda}\,p^y\,\lambda^y}{y!}e^{\lambda(1-p)} = \frac{e^{-\lambda p}\,(\lambda p)^y}{y!}$$

Por lo tanto, $y \sim \text{Poisson}(\lambda p)$ (asumiendo que no pasa nada al sumar desde $x = -y\ \ldots$ consultar!)

Ejercicio 7

Probar la regla de Bayes

$$p(H_j|E) = \frac{p(E|H_j)\,p(H_j)}{\sum\limits_{k=1}^{K} p(E|H_k)\,p(H_k)}$$

Sabiendo que se cumple $P(A|B) = \frac{p(A\cap B)}{p(B)}$ para cualesquiera sucesos A y B, donde E es un evento cualquiera y $H_1, \cdots, H_k$ forman una partición de Ω.

Demostración:

$$p(H_j|E) = \frac{p(H_j;E)}{p(E)} = \frac{p(E|H_j)\,p(H_j)}{\sum_{k=1}^{K} p(E;H_k)} = \frac{p(E|H_j)\,p(H_j)}{\sum\limits_{k=1}^{K} p(E|H_k)\,p(H_k)}$$

Ejercicio 7

Parte a

`X` `Y`

	1	0
1	$(0.5)(0.4) = 0.2$	$(0.5)(0.6) = 0.3$
0	$(0.5)(0.6) = 0.3$	$(0.5)(0.4) = 0.2$

Parte b

$$E(Y) = \sum_{Y} y\, p(y) = (1)\, p(y) + (0)\, p(y) = p(y) = \sum_{X} p(x; y) = P(\text{verde} \mid \text{cara})\, P(\text{cara}) + P(\text{verde} \mid \text{cruz})\, P(\text{cruz}) =$$

$$= (0.5)(0.4) + (0.5)(0.6) = 0.5$$