Entrega

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Una aclaración previa importante

Salvo que se indique lo contrario en la letra del ejercicio, o en la solución aquí propuesta, siempre se está asumiendo que en las distintas situaciones planteadas se está trabajando bajo los supuestos de la regresión lineal. En particular, el supuesto de que la matriz $\mathbf{X}$ es de rango completo por columnas fue utilizado ampliamente, sin aclaración explícita en las soluciones.

Práctico 1

Ejercicio 1

$$Z \sim \chi^2_d \Rightarrow Z \sim \text{Gamma}\left(\frac{d}{2}; \frac{1}{2}\right) \Rightarrow f_Z(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{I}_{[z \geq 0]}$$

- Esperanza de Z

$$E(Z) = \int_0^{+\infty} z \, \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{d}z = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \underbrace{\int_0^{+\infty} \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \, z^{(\alpha+1)-1} \, e^{-\beta z} \, \mathrm{d}z}_{=1} =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \frac{\alpha\Gamma(\alpha)}{\beta^\alpha \, \beta} = \frac{\alpha}{\beta} \Rightarrow E(Z) = \frac{{}^d/_2}{{}^1/_2} \Rightarrow \boxed{E(Z) = d}$$

- Varianza de Z: $V(Z) = E(Z^2) - E^2(Z)$
  - Esperanza de Z^2

$$E(Z^2) = \int_0^{+\infty} z^2 \, \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{d}z = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} \underbrace{\int_0^{+\infty} \frac{\beta^{\alpha+2}}{\Gamma(\alpha+2)} \, z^{(\alpha+2)-1} \, e^{-\beta z} \, \mathrm{d}z}_{=1} =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \frac{\alpha(\alpha+1)\Gamma(\alpha)}{\beta^\alpha \, \beta^2} = \frac{\alpha(\alpha+1)}{\beta^2}$$

  - Varianza de Z:

$$V(Z) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2} \, (\alpha + 1 - \alpha) = \frac{\alpha}{\beta} \Rightarrow V(Z) = \frac{{}^d/_2}{{}^1/_4} = \frac{4d}{2} \Rightarrow \boxed{V(Z) = 2d}$$

Ejercicio 2

1. $\sum\limits_{i=1}^n e_i = \sum\limits_{i=1}^n \left(y_i - \hat{y}_i\right) = \sum\limits_{i=1}^n y_i - \sum\limits_{i=1}^n \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) = \sum\limits_{i=1}^n y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum\limits_{i=1}^n x_i =$

$$= \bar{y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{x} = \bar{y} - \left(\bar{y} - \hat{\beta}_1 \bar{x}\right) - \hat{\beta}_1 \bar{x} = 0 \Rightarrow \boxed{\sum_{i=1}^n e_i = 0}$$

2. Si $\sum\limits_{i=1}^{n} e_i = 0 \Rightarrow \sum\limits_{i=1}^{n} \left(y_i - \hat{y}_i\right) = 0 \Rightarrow \boxed{\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i}$

3. $$\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i \left(y_i - \hat{y}_i\right) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \hat{y}_i = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) =$$

   $$= \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x}\right) \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 =$$

   $$= \sum_{i=1}^{n} x_i y_i - \bar{y} \underbrace{\sum_{i=1}^{n} x_i}_{n\bar{x}} + \hat{\beta}_1 n \bar{x}^2 - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \hat{\beta}_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) - \hat{\beta}_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) \Rightarrow$$

   $$\Rightarrow \boxed{\sum_{i=1}^{n} x_i e_i = 0}$$

4. $$\sum_{i=1}^{n} \hat{y}_i e_i = \sum_{i=1}^{n} \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) e_i = \hat{\beta}_0 \underbrace{\sum_{i=1}^{n} e_i}_{=0} + \hat{\beta}_1 \underbrace{\sum_{i=1}^{n} x_i e_i}_{=0} \Rightarrow \boxed{\sum_{i=1}^{n} \hat{y}_i e_i = 0}$$

5. $$\sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n} \left(y_i - \hat{y}_i + \hat{y}_i - \bar{y}\right)^2 =$$

   $$= \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + 2 \underbrace{\sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)\left(\hat{y}_i - \bar{y}\right)}_{\sum_{i=1}^{n} e_i \left(\hat{y}_i - \bar{y}\right) = \sum_{i=1}^{n} e_i \hat{y}_i - \bar{y} \sum_{i=1}^{n} e_i} + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 =$$

   $$= \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + 2 \underbrace{\sum_{i=1}^{n} e_i \hat{y}_i}_{=0} - \bar{y} \underbrace{\sum_{i=1}^{n} e_i}_{=0} + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 \Rightarrow$$

   $$\Rightarrow \boxed{\sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2}$$

6. Interpretación geométrica:

   a) $\mathbb{1}'\mathbf{e} = \mathbf{0}$, implica que el vector de residuos es ortogonal a un vector de unos.

   b) $\mathbb{1}'\mathbf{y} = \mathbb{1}'\hat{\mathbf{y}}$

   c) $\mathbf{x}'\mathbf{e} = \mathbf{0}$, implica ortogonalidad entre los regresores y los residuos.

   d) $\mathbf{y}'\mathbf{e} = \mathbf{0}$, implica ortogonalidad entre el regresando y los residuos.

   e) Implica que $\|\mathbf{y} - \bar{\mathbf{y}}\|^2 = \|\hat{\mathbf{y}} - \bar{\mathbf{y}}\|^2 + \|\mathbf{y} - \hat{\mathbf{y}}\|^2$

Ejercicio 3

- Estimador MCO

$$\hat{\theta}_{MCO} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \left(\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) \Rightarrow \boxed{\hat{\theta}_{MCO} = \frac{1}{5}\Big(2y_1 + y_2\Big)}$$

- Suma de Cuadrados de Residuos (SSR):

$$SSR = \mathbf{y}'\mathbf{y} - \hat{\theta}(\mathbf{X}'\mathbf{y}) = \Big(y_1^2 + y_2^2\Big) - \frac{1}{5}\Big(2y_1 + y_2\Big)^2 =$$

$$= y_1^2 + y_2^2 - \frac{4}{5}\,y_1^2 - \frac{4}{5}\,y_1\,y_2 - \frac{1}{5}\,y_2^2 = \frac{y_1^2}{5} + \frac{4y_2^2}{5} - \frac{4}{5}\,y_1\,y_2 \Rightarrow \boxed{SSR = \frac{1}{5}\Big(y_1 - 2y_2\Big)^2}$$

Ejercicio 4

Caso general:

$$\min_{\hat{\beta}}\Big\{\hat{\mathbf{u}}'\hat{\mathbf{u}}\Big\} = \min_{\hat{\beta}}\Big\{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})\Big\} = \min_{\hat{\beta}}\Big\{\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\hat{\beta} - \hat{\beta}'\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}\Big\} =$$

$$= \min_{\hat{\beta}}\Big\{\mathbf{y}'\mathbf{y} - 2\hat{\beta}'\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}\Big\}$$

$$\frac{\partial(\hat{\mathbf{u}}'\hat{\mathbf{u}})}{\partial\hat{\beta}'} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\beta} = \mathbf{0}_{k+1} \Rightarrow (\mathbf{X}'\mathbf{X})\hat{\beta} = (\mathbf{X}'\mathbf{y}) \Rightarrow \boxed{\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}$$

$$\frac{\partial^2(\hat{\mathbf{u}}'\hat{\mathbf{u}})}{\partial^2\hat{\beta}} = \frac{\partial}{\partial\hat{\beta}}\Big(\mathbf{X}'\mathbf{X}\hat{\beta} - \mathbf{X}'\mathbf{y}\Big) = \mathbf{X}'\mathbf{X} > 0 \Rightarrow \hat{\beta} \text{ es mínimo}$$

Para el modelo $y_i = \beta_0 + \beta_1 x_i + u_i\ \forall i = 1;\ldots;n$

$$\mathbf{X} = \begin{pmatrix}1 & x_1\\ 1 & x_2\\ \vdots & \vdots\\ 1 & x_n\end{pmatrix} \quad \mathbf{y} = \begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix} \quad \Rightarrow \quad (\mathbf{X}'\mathbf{X}) = \begin{pmatrix} n & \sum\limits_{i=1}^{n} x_i\\ \sum\limits_{i=1}^{n} x_i & \sum\limits_{i=1}^{n} x_i^2\end{pmatrix} \quad (\mathbf{X}'\mathbf{y}) = \begin{pmatrix}\sum\limits_{i=1}^{n} y_i\\ \sum\limits_{i=1}^{n} x_i y_i\end{pmatrix}$$

Por lo tanto,

$$\hat{\beta} = \begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\end{pmatrix} = \begin{pmatrix}\bar{y} - \hat{\beta}_1\bar{x}\\ \frac{S_{XY}}{S_X^2}\end{pmatrix}$$

Ejercicio 5

1. - $\bar{u} = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{x_i - \bar{x}}{S_X} = \frac{1}{nS_X}\left(\sum\limits_{i=1}^{n} x_i - n\bar{x}\right) = \frac{1}{nS_X}\,(n\bar{x} - n\bar{x}) = 0$

   - $\bar{v} = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{y_i - \bar{y}}{S_Y} = \frac{1}{nS_Y}\left(\sum\limits_{i=1}^{n} y_i - n\bar{y}\right) = \frac{1}{nS_Y}\,(n\bar{y} - n\bar{y}) = 0$

- $S_u^2 = \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2 = \frac{1}{n}\sum_{i=1}^{n}u_i^2 - \bar{u}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{S_X}\right)^2 = \frac{1}{S_X^2}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}_{S_X^2}\right) =$

  $= \frac{1}{S_X^2}\,S_X^2 = 1$

- $S_v^2 = \frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v})^2 = \frac{1}{n}\sum_{i=1}^{n}v_i^2 - \bar{v}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i-\bar{y}}{S_Y}\right)^2 = \frac{1}{S_Y^2}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2}_{S_Y^2}\right) =$

  $= \frac{1}{S_Y^2}\,S_Y^2 = 1$

- $\hat{b}_1 = (\mathbf{u}'\mathbf{u})^{-1}(\mathbf{u}'\mathbf{v}) = \left(\sum_{i=1}^{n}u_i^2\right)^{-1}\left(\sum_{i=1}^{n}u_iv_i\right) = \frac{\frac{1}{n}\sum_{i=1}^{n}u_iv_i}{\frac{1}{n}\sum_{i=1}^{n}u_i^2} = \frac{S_{uv}}{\underbrace{S_u^2}_{=1}} =$

  $= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{S_X}\right)\left(\frac{y_i-\bar{y}}{S_Y}\right) = \frac{1}{S_XS_Y}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}\right) = \frac{S_{XY}}{S_XS_Y}$

  Por lo tanto: $\hat{b}_1\,\frac{S_Y}{S_X} = \frac{S_{XY}}{S_XS_Y}\,\frac{S_Y}{S_X} = \frac{S_{XY}}{S_X^2}\,\frac{S_Y}{S_Y} = \frac{S_{XY}}{S_X^2} = \hat{\beta}_1$

- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ se desprende de la derivación de $\hat{\beta}_{MCO}$

2. Usando las demostraciones del punto 1, tenemos que:

$$\left.\begin{array}{r} r_{uv} = \frac{S_{uv}}{S_uS_v} \\[2ex] \hat{b}_1 = \frac{S_{uv}}{S_u^2} \\[2ex] S_u = S_v = 1 \end{array}\right\} \Rightarrow \hat{b}_1 = S_{uv} = r_{uv}$$

Ejercicio 6

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \frac{1}{n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2}\begin{pmatrix} \sum_{i=1}^{n}x_i^2 & -\sum_{i=1}^{n}x_i \\ -\sum_{i=1}^{n}x_i & n \end{pmatrix}\begin{pmatrix} \sum_{i=1}^{n}y_i \\ \sum_{i=1}^{n}x_iy_i \end{pmatrix}$$

Por lo tanto:

$$\hat{\beta}_1 = \frac{-\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right) + n\sum_{i=1}^{n}x_iy_i}{n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2} = \frac{\frac{1}{n}\sum_{i=1}^{n}x_iy_i - \bar{x}\bar{y}}{\frac{1}{n}\sum_{i=1}^{n}x_i^2 - \bar{x}^2} = \frac{S_{XY}}{S_X^2} =$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \Rightarrow \boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$\hat{\beta}_0 = \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\frac{1}{n^2}\left[\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)\right]}{\frac{1}{n^2}\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]} =$$

$$= \frac{\frac{1}{n^2}\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i}{S_X^2} - \frac{\frac{1}{n^2}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{S_X^2} = \bar{y}\,\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2}{S_X^2} - \bar{x}\,\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i}{S_X^2} =$$

$$= \bar{y}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 + \bar{x}^2}{S_X^2}\right) - \bar{x}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y} + \bar{x}\bar{y}}{S_X^2}\right) =$$

$$= \bar{y}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}{S_X^2}\right) + \frac{\bar{y}\bar{x}^2}{S_X^2} - \bar{x}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{S_X^2}\right) - \frac{\bar{x}^2\bar{y}}{S_X^2} =$$

$$= \bar{y}\,\frac{S_X^2}{S_X^2} - \bar{x}\,\frac{S_{XY}}{S_X^2} \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

Ejercicio 7

Partimos de la distribución de Y: $Y \sim N_n\left(\mathbf{X}\beta;\ \sigma_\varepsilon^2\mathbf{I}_n\right)$

- Estimadores máximo verosímiles

$$\mathcal{L}(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = \prod_{i=1}^{n} f_{y_i}(x_i;y_i|\beta_0;\beta_1;\sigma_\varepsilon^2) =$$

$$= \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma_\varepsilon^2)^{-1/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(y_i-\beta_0-\beta_1x_i\right)^2\right\} =$$

$$= (2\pi)^{-n/2}(\sigma_\varepsilon^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)^2\right\}$$

$$l(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)^2$$

Luego entonces:

$$\frac{\partial l(\cdot)}{\partial \beta_0} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)(-1) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0 \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$$

$$\frac{\partial l(\cdot)}{\partial \beta_1} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)(-x_i) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n}\Big(y_i x_i - \beta_0 x_i - \beta_1 x_i^2\Big) = \sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0 \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i \Rightarrow \hat{\beta}_1 = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \Big(\bar{y} - \hat{\beta}_1 \bar{x}\Big)\frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} x_i^2} \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \left[1 - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n\sum\limits_{i=1}^{n} x_i^2}\right] = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \frac{\sum\limits_{i=1}^{n} y_i \sum\limits_{i=1}^{n} x_i}{n\sum\limits_{i=1}^{n} x_i^2}$$

$$\Rightarrow \hat{\beta}_1 = \left(\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2}\right)\left(\frac{n\sum\limits_{i=1}^{n} x_i^2}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) - \left(\frac{\sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i}{n\sum\limits_{i=1}^{n} x_i^2}\right)\left(\frac{n\sum\limits_{i=1}^{n} x_i^2}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 = \frac{\frac{1}{n^2}\left[n\sum\limits_{i=1}^{n} x_i y_i - \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i\right]}{\frac{1}{n^2}\left[n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right]} \Rightarrow \boxed{\hat{\beta}_1 = \frac{S_{XY}}{S_X^2}}$$

$$\frac{\partial l(\cdot)}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} - \frac{1}{2}\left(\frac{-1}{2(\sigma_\varepsilon^2)^2}\right)\underbrace{\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2}_{\hat{\varepsilon}'\hat{\varepsilon}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{2(\sigma_\varepsilon^2)^2} - \frac{n}{2\sigma_\varepsilon^2} = 0 \Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{(\sigma_\varepsilon^2)^2} = \frac{n}{\sigma_\varepsilon^2} \Rightarrow \boxed{\hat{\sigma}_\varepsilon^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}}$$

- Distribución de $\hat{\beta}$

$$\left.\begin{array}{l} \hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) \\ \mathbf{y} \sim N_n\Big(\mathbf{X}\beta;\, \sigma_\varepsilon^2 \mathbf{I}_n\Big) \end{array}\right\} \Rightarrow \hat{\beta} \sim N_{k+1} \text{ por ser CL de normales}$$

- Esperanza de $\hat{\beta}$

$$E(\hat{\beta}) = E\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{y}) = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\beta = \beta$$

- Varianza de $\hat{\beta}$

$$V(\hat{\beta}) = V\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V(\mathbf{y})\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right]' =$$

$$= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma_\varepsilon^2\mathbf{I}_n\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}$$

Por lo tanto: $\boxed{\hat{\beta} \sim N_{k+1}\Big(\beta;\ \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}\Big)}$

Práctico 2

Ejercicio 4

Partimos de la distribución de Y

$$Y \sim N_n\Big(\mathbf{X}\beta;\ \sigma_\varepsilon^2\mathbf{I}_n\Big)$$

- Estimadores máximo verosímiles

$$\mathcal{L}(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = \prod_{i=1}^{n} f_{y_i}(x_i;y_i|\beta_0;\beta_1;\sigma_\varepsilon^2) =$$

$$= \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma_\varepsilon^2)^{-1/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\Big(y_i-\beta_0-\beta_1x_i\Big)^2\right\} =$$

$$= (2\pi)^{-n/2}(\sigma_\varepsilon^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\Big(y_i-\beta_0-\beta_1x_i\Big)^2\right\}$$

$$l(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\Big(y_i-\beta_0-\beta_1x_i\Big)^2$$

Luego entonces:

$$\frac{\partial l(\cdot)}{\partial\beta_0} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i-\beta_0-\beta_1x_i\Big)(-1) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n}y_i - n\beta_0 - \beta_1\sum_{i=1}^{n}x_i = 0 \Rightarrow$$

$$\Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

$$\frac{\partial l(\cdot)}{\partial\beta_1} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i-\beta_0-\beta_1x_i\Big)(-x_i) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n}\Big(y_i x_i - \beta_0 x_i - \beta_1 x_i^2\Big) = \sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0 \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i \Rightarrow \hat{\beta}_1 = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \Big(\bar{y} - \hat{\beta}_1 \bar{x}\Big)\frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} x_i^2} \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \left[1 - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n\sum\limits_{i=1}^{n} x_i^2}\right] = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \frac{\sum\limits_{i=1}^{n} y_i \sum\limits_{i=1}^{n} x_i}{n\sum\limits_{i=1}^{n} x_i^2}$$

$$\Rightarrow \hat{\beta}_1 = \left(\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2}\right)\left(\frac{n\sum\limits_{i=1}^{n} x_i^2}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) - \left(\frac{\sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i}{n\sum\limits_{i=1}^{n} x_i^2}\right)\left(\frac{n\sum\limits_{i=1}^{n} x_i^2}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 = \frac{\frac{1}{n^2}\left[n\sum\limits_{i=1}^{n} x_i y_i - \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i\right]}{\frac{1}{n^2}\left[n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right]} \Rightarrow \boxed{\hat{\beta}_1 = \frac{S_{XY}}{S_X^2}}$$

$$\frac{\partial l(\cdot)}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} - \frac{1}{2}\left(\frac{-1}{2(\sigma_\varepsilon^2)^2}\right)\underbrace{\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2}_{\hat{\varepsilon}'\hat{\varepsilon}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{2(\sigma_\varepsilon^2)^2} - \frac{n}{2\sigma_\varepsilon^2} = 0 \Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{(\sigma_\varepsilon^2)^2} = \frac{n}{\sigma_\varepsilon^2} \Rightarrow \boxed{\hat{\sigma}_\varepsilon^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}}$$

- Distribución de $\hat{\beta}$

$$\left.\begin{array}{r}\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) \\ \mathbf{y} \sim N_n\Big(\mathbf{X}\beta;\, \sigma_\varepsilon^2 \mathbf{I}_n\Big)\end{array}\right\} \Rightarrow \hat{\beta} \sim N_{k+1} \text{ por ser CL de normales}$$

- Esperanza de $\hat{\beta}$

$$E(\hat{\beta}) = E\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{y}) = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\beta = \beta$$

- Varianza de $\hat{\beta}$

$$V(\hat{\beta}) = V\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V(\mathbf{y})\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right]' =$$

$$= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma_\varepsilon^2\mathbf{I}_n\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}$$

Por lo tanto: $\boxed{\hat{\beta} \sim N_{k+1}\Big(\beta;\, \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}\Big)}$

Ejercicio 5

Partimos del modelo:

$$y_i = \beta_0 + \beta_1 x_i + u_i \ \ \forall i = 1; \ldots ; 6$$

De la estimación MCO obtenemos:

$$\hat{\beta}_1 = \frac{S_{XY}}{S_X^2} = \frac{400{,}7}{466{,}7} = 0{,}8586$$

$$\hat{sd}(\hat{\beta}_1) = 0{,}0398$$

Con estos datos construimos el intervalo de confianza:

$$IC_{\beta_1}^{95\,\%} = \left[\hat{\beta}_1 \pm t_4(1 - {}^{\alpha}/_{2})\ \hat{sd}(\hat{\beta}_1)\right] \Rightarrow \boxed{IC_{\beta_1}^{95\,\%} = [0{,}748;\ 0{,}969]}$$

Ejercicio 6

Partimos del modelo:

$$y_i = \beta_0 + \beta_1 x_i + u_i \ \ \forall i = 1; \ldots ; 12$$

De la estimación MCO obtenemos:

$$\hat{\beta} = (\mathbf{X'X})^{-1}(\mathbf{X'y}) = \begin{pmatrix} -120{,}21 \\ 1{,}11 \end{pmatrix}$$

$$\hat{\sigma}_{\varepsilon}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-k-1} = 1{,}32$$

Luego realizamos la prueba:

$$H_0)\ \ \beta_1 = 0 \quad Vs. \quad H_1)\ \ \beta_1 \neq 0$$

Con región crítica:

$$RC = \left\{(\mathbf{Xy})\ /\ |e| > t_{n-k-1}(1 - {}^{\alpha}/_{2})\right\}$$

Y estadístico de prueba:

$$e = \frac{\hat{\beta}_1}{\hat{sd}(\hat{\beta}_1)} \overset{\mathrm{H_0}}{\sim} t_{n-k-1}$$

Dado que $e = 11{,}58661 > 2{,}23 = t_10(0{,}975) \Rightarrow$ rechazo H_0, lo cual implica que, la altura es significativa para explicar el peso, al 95 %.

Ejercicio 7

1. Sea el modelo: $y_i = \beta_0 + \beta_1 t_i + u_i \ \ \forall i = 1;\dots;11$

   Para el cual se obtienen las siguientes estimaciones MCO:

$$\hat{\beta} = \begin{pmatrix} 9{,}27 \\ 1{,}44 \end{pmatrix} \quad \hat{V}(\hat{\beta}) = \begin{pmatrix} 0{,}2145 & 0 \\ 0 & 0{,}02145 \end{pmatrix} \quad \hat{\sigma}_u^2 = 2{,}36$$

2. Se realiza la prueba

$$H_0)\ \beta_1 = 0 \quad Vs. \quad H_1)\ \beta_1 \neq 0$$

   Con región crítica:

$$RC = \Big\{(\mathbf{Xy}) \ / \ |e| > t_{n-k-1}(1 - {}^{\alpha}/_{2})\Big\}$$

   Y estadístico de prueba:

$$e = \frac{\hat{\beta}_1}{\hat{sd}(\hat{\beta}_1)} \overset{\mathrm{H}_0}{\sim} t_{n-k-1}$$

   Dado que $e = 9{,}87 > 2{,}26 = t_9(0{,}975) \Rightarrow$ rechazo H_0, lo cual implica que, la temperatura es significativa para explicar la cantidad producida, al 95 %.

3. Predicción puntual: $\hat{E}(y|t=3) = \hat{\beta}_0 + \hat{\beta}_1(3) = 13{,}58$ Intervalo de confianza:

$$IC_{E(y|t=3)}^{95\,\%} = \left[\hat{E}(y|t=3) \pm t_9(0{,}975)\sqrt{\hat{\sigma}_u^2 \begin{pmatrix} 1 & 3 \end{pmatrix} (\mathbf{X}'\mathbf{X})^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}\ \right] = [12{,}44;\ 15{,}03]$$

4. Predicción puntual: $\hat{y}|t=3 = \hat{\beta}_0 + \hat{\beta}_1(3) = 13{,}58$ Intervalo de confianza:

$$IC_{y|t=3}^{95\,\%} = \left[\hat{y}|t=3 \pm t_9(0{,}975)\sqrt{\hat{\sigma}_u^2 \left[1 + \begin{pmatrix} 1 & 3 \end{pmatrix} (\mathbf{X}'\mathbf{X})^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]}\ \right] = [9{,}82;\ 17{,}34]$$

Ejercicio 8

1. Recta de regresión: $velocidad_i = 61{,}3 - 0{,}63\ densidad_i \ \ \forall i = 1;\dots;24$
2. $\sigma_\varepsilon^2 = 16{,}31$; $\hat{se}(\hat{\beta}_0) = 1{,}96$; $\hat{se}(\hat{\beta}_1) = 0{,}03$
3. $IC_{\beta_0}^{95\,\%} = \Big[57{,}25;\ 65{,}39\Big]$; $IC_{\beta_1}^{95\,\%} = \Big[-0{,}70;\ -0{,}56\Big]$
4. Tabla de significación del modelo:

	df	**SC**	**SCMe**	**F**
Densidad	1	6002.4	6002.4	368.13
Residuos	22	358.7	16.3	

   Contraste de significación global:

$$H_0)\ \beta_1 = 0 \quad Vs. \quad H_1)\ \beta_1 \neq 0$$

   Con región crítica:

$$RC = \Big\{(\mathbf{Xy}) \ / \ F_0 > F_{k-1;\,n-k}(1-\alpha)\Big\}$$

   Y estadístico de prueba:

$$F_0 = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \overset{\mathrm{H}_0}{\sim} F_{k-1;\,n-k}$$

   Dado que $F_0 = 368{,}13 > 4{,}3 = F_{1;\,22}(0{,}95) \Rightarrow$ rechazo H_0, lo cual implica que, el modelos es significativo, al 95 %.

5. Estimación puntual: $\hat{E}(velocidad|densidad = 50) = 29{,}96$

   Intervalo de confianza: $IC^{90\,\%}_{E(velocidad|densidad=50)} = \Big[28{,}31;\ 31{,}70\Big]$

6. Con la raíz cuadrada de la velocidad:
   - Recta de regresión: $\sqrt{velocidad_i} = 8{,}09 - 0{,}06\ densidad_i \quad \forall i = 1;\ldots;24$
   - $\sigma^2_\varepsilon = 0{,}07$; $\hat{se}(\hat\beta_0) = 0{,}13$; $\hat{se}(\hat\beta_1) = 0{,}002$
   - $IC^{95\,\%}_{\beta_0} = \Big[7{,}82;\ 8{,}36\Big]$; $IC^{95\,\%}_{\beta_1} = \Big[-0{,}06;\ -0{,}05\Big]$
   - Tabla de significación del modelo:

	df	**SC**	**SCMe**	**F**
$\sqrt{\textbf{Velocidad}}$	1	48.92	48.92	676.39
Residuos	22	1.59	0.072	

   Contraste de significación global:

$$H_0)\ \ \beta_1 = 0 \quad Vs. \quad H_1)\ \ \beta_1 \neq 0$$

   Con región crítica:

$$RC = \Big\{(\mathbf{X}\mathbf{y})\ /\ F_0 > F_{k-1;\,n-k}(1-\alpha)\Big\}$$

   Y estadístico de prueba:

$$F_0 = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \overset{\mathrm{H_0}}{\sim} F_{k-1;\,n-k}$$

   Dado que $F_0 = 676{,}39 > 4{,}3 = F_{1;\,22}(0{,}95) \Rightarrow$ rechazo H_0, lo cual implica que, el modelos es significativo, al 95 %.

   - Estimación puntual: $\hat{E}(velocidad|densidad = 50) = 7{,}689$

     Intervalo de confianza: $IC^{90\,\%}_{E(velocidad|densidad=50)} = \Big[7{,}576;\ 7{,}803\Big]$

Ejercicio 9

1. Maximización del R^2:

$$\text{máx}\{R^2\} = \text{máx}\left\{1 - \frac{SCR}{SCT}\right\} = \text{máx}\left\{1 - \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\} = \text{máx}\left\{1 - \frac{(\mathbf{y} - \mathbf{X}\hat\beta)'(\mathbf{y} - \mathbf{X}\hat\beta)}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\} =$$

$$= \text{máx}\left\{1 - \frac{\mathbf{y}'\mathbf{y} - 2\hat\beta\mathbf{X}'\mathbf{y} + \hat\beta'\mathbf{X}'\mathbf{X}\hat\beta}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\}$$

$$\frac{\partial R^2}{\partial \hat\beta'} = 0 - \frac{1}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\left(-2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat\beta\right) = 0 \Rightarrow \boxed{\hat\beta = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}$$

2. - Partimos de la regresión particionada.

     Sea el siguiente modelo de regresión:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon = \mathbf{X}_1\beta + \mathbf{X}_2\alpha + \varepsilon$$

Con ecuaciones normales:

$$\begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

De estas ecuaciones se desprende que:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'(\mathbf{y} - \mathbf{X}_2\hat{\alpha})$$

Por el teorema de Frisch-Waugh-Lovel los estimadores MCO serán:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$$

$$\hat{\alpha} = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}(\mathbf{X}_2'\mathbf{M}_1\mathbf{y})$$

donde:

$$\mathbf{M}_1 = \mathbf{I}_n - \mathbf{P}_1 = \mathbf{I}_n - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$$
$$\mathbf{M}_2 = \mathbf{I}_n - \mathbf{P}_2 = \mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'$$

Dado que $\mathbf{M}$ es simétrica e idempotente, podemos escribir:

$$\mathbf{X}_2^* = \mathbf{M}_1\mathbf{X}_2$$

$$\mathbf{y}_* = \mathbf{M}_1\mathbf{y}$$

Por lo que entonces:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$$
$$\hat{\alpha} = (\mathbf{X}_2^{*'}\mathbf{X}_2^*)^{-1}(\mathbf{X}_2^{*'}\mathbf{y}_*)$$

- El caso particular en que $\mathbf{X}_2 = \mathbf{z}_{n\times 1}$

  Este es el caso en que $\mathbf{z}$ contiene una sola variable. Para simplificar notación llamaremos $\mathbf{X}_1 = \mathbf{X}$, dado que no hay riesgo de confusión.

  De la regresión: $\mathbf{y} = \mathbf{X}\beta + \mathbf{z}\alpha + \varepsilon$ podemos obtener la estimación del coeficiente asociado a $\mathbf{z}$ siguiendo lo visto anteriormente:

$$\hat{\alpha} = (\mathbf{z}'\mathbf{M}\mathbf{z})^{-1}(\mathbf{z}'\mathbf{M}\mathbf{y}) = (\mathbf{z}_*'\mathbf{z}_*)^{-1}(\mathbf{z}_*'\mathbf{y}_*)$$

- Cambio en la SCR al agregar un regresor.

  Sean $\hat{\mathbf{e}}'\hat{\mathbf{e}}$ la SCR de la regresión de $\mathbf{y}$ sobre $\mathbf{X}$, y $\hat{\mathbf{u}}'\hat{\mathbf{u}}$ la SCR de la regresión de $\mathbf{y}$ sobre $\mathbf{X}$ y un regresor adicional $\mathbf{z}$. El vector de residuos de la segunda regresión será entonces $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\breve{\beta} - \mathbf{z}\hat{\alpha}$. Por otra parte, de la regresión particionada sabemos que:

$$\breve{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big(\mathbf{y} - \mathbf{z}\hat{\alpha}\Big) = \underbrace{(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}_{\hat{\beta}} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha} = \hat{\beta} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha}$$

  Luego entonces,

$$\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\Big(\hat{\beta} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha}\Big) - \mathbf{z}\hat{\alpha} = \underbrace{\mathbf{y} - \mathbf{X}\hat{\beta}}_{\hat{\mathbf{e}}} + \underbrace{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}_{\mathbf{P}}\mathbf{z}\hat{\alpha} - \mathbf{z}\hat{\alpha} =$$

$$= \hat{\mathbf{e}} + \mathbf{Pz}\hat{\alpha} - \mathbf{z}\hat{\alpha} = \hat{\mathbf{e}} + (\mathbf{P} - \mathbf{I}_n)\mathbf{z}\hat{\alpha} = \hat{\mathbf{e}} - \mathbf{Mz}\hat{\alpha} = \hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}$$

Por lo tanto,

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \Big(\hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}\Big)'\Big(\hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}\Big) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \underbrace{\hat{\mathbf{e}}'\mathbf{z}_*}_{1\times 1}\hat{\alpha} - \hat{\alpha}'\underbrace{\mathbf{z}_*{}'\hat{\mathbf{e}}}_{1\times 1} - \hat{\alpha}'\mathbf{z}_*{}'\mathbf{z}_*\hat{\alpha} =$$

$$= \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}(\underbrace{\mathbf{z}_*{}'\hat{\mathbf{e}}}_{=\mathbf{z}_*{}'\mathbf{y}_*}) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}(\underbrace{\mathbf{z}_*{}'\mathbf{y}_*}_{=\hat{\alpha}(\mathbf{z}_*{}'\mathbf{z}_*)}) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) =$$

$$= \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*)$$

Por lo tanto:

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) \leq \hat{\mathbf{e}}'\hat{\mathbf{e}}$$

Con lo que queda demostrado que, al agregar un regresor, $\downarrow$ SCR $\Rightarrow\uparrow R^2$

- Algunas aclaraciones:
  - En la conclusión se sostuvo que el SCR cae, y por tanto el R^2 aumenta. Existe la posibilidad de que el SCR se mantenga incambiado (y por tanto tampoco cambie el R^2). Esto se da únicamente si $\mathbf{X}$ y $\mathbf{z}$ son ortogonales (es decir, están incorrelacionadas).
  - A la igualdad $\mathbf{z}_*{}'\hat{\mathbf{e}} = \mathbf{z}_*{}'\mathbf{y}_*$ se llega recordando que $\hat{\mathbf{e}} = \mathbf{My}$ (en el modelo sin la variable $\mathbf{z}$), pero a su vez, $\mathbf{My}$ se definió como $\mathbf{y}_*$. Por lo tanto, $\hat{\mathbf{e}} = \mathbf{y}_*$.
  - La igualdad $\mathbf{z}_*{}'\mathbf{y} = \hat{\alpha}(\mathbf{z}_*{}'\mathbf{z}_*)$ se desprende de las ecuaciones normales del modelo que incluye la variable $\mathbf{z}$ (ver segundo item, Regresión Particionada con $\mathbf{X}_2 = \mathbf{z}$). La posición de $\hat{\alpha}$ es intercambiable, dado que $\hat{\alpha}$ es un real.

Ejercicio 10

1. Sean $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ y $\mathbf{M} = \mathbf{I}_n - \mathbf{P}$
   - $\mathbf{P}' = \Big[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big]' = \Big(\mathbf{X}'\Big)'\Big((\mathbf{X}'\mathbf{X})^{-1}\Big)'\Big(\mathbf{X}\Big)' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{P}$
   - $\mathbf{PP} = \Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big)\Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big) = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\underbrace{(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}}_{\mathbf{I}_{k+1}}\mathbf{X}' =$

     $= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}_n\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{P}$
   - $\mathbf{MM} = \Big(\mathbf{I}_n - \mathbf{P}\Big)\Big(\mathbf{I}_n - \mathbf{P}\Big) = \mathbf{I}_n\mathbf{I}_n - \mathbf{I}_n\mathbf{P} - \mathbf{P}\mathbf{I}_n + \mathbf{PP} = \mathbf{I}_n - \mathbf{P} - \mathbf{P} + \mathbf{P} =$

     $= \mathbf{I}_n - \mathbf{P} = \mathbf{M}$
   - $\mathbf{M}' = \Big(\mathbf{I}_n - \mathbf{P}\Big)' = \mathbf{I}_n' - \mathbf{P}' = \mathbf{I}_n - \mathbf{P} = \mathbf{M}$
2. $E(\mathbf{e}) = E(\mathbf{y} - \hat{\mathbf{y}}) = E(\mathbf{y}) - E(\hat{\mathbf{y}}) = E(\mathbf{y}) - E(\mathbf{Py}) = E(\mathbf{y}) - \mathbf{P}E(\mathbf{y})$

   $= \Big(\mathbf{I}_n - \mathbf{P}\Big)E(\mathbf{y}) = \Big(\mathbf{I}_n - \mathbf{P}\Big)\mathbf{X}\beta = \Big(\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big)\mathbf{X}\beta =$

   $= \Big(\mathbf{X} - \mathbf{X}\underbrace{(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})}_{\mathbf{I}_{k+1}}\Big)\beta = \Big(\mathbf{X} - \mathbf{X}\Big)\beta = \mathbf{0}_{n\times(k+1)}\,\beta = \mathbf{0}_n$
3. - $V(\mathbf{e}) = V(\mathbf{y} - \hat{\mathbf{y}}) = V(\mathbf{y} - \mathbf{Py}) = V(\mathbf{My}) = \mathbf{M}V(\mathbf{y})\mathbf{M}' = \mathbf{M}\Big(\sigma_\varepsilon^2\mathbf{I}_n\Big)\mathbf{M} =$

     $= \sigma_\varepsilon^2\mathbf{MM} = \sigma_\varepsilon^2\mathbf{M}$

- $COV(\mathbf{e};\mathbf{y}) = COV(\mathbf{y}-\hat{\mathbf{y}};\mathbf{y}) = COV(\mathbf{y};\mathbf{y}) - COV(\hat{\mathbf{y}};\mathbf{y}) =$
  $= V(\mathbf{y}) - COV(\mathbf{P}\mathbf{y};\mathbf{y}) = V(\mathbf{y}) - \mathbf{P}COV(\mathbf{y};\mathbf{y}) = V(\mathbf{y}) - \mathbf{P}V(\mathbf{y}) =$
  $= \Big(\mathbf{I}_n - \mathbf{P}\Big)V(\mathbf{y}) = \Big(\mathbf{I}_n - \mathbf{P}\Big)\sigma_\varepsilon^2\mathbf{I}_n = \sigma_\varepsilon^2\Big(\mathbf{I}_n - \mathbf{P}\Big)$
- $COV(\mathbf{e};\hat{\mathbf{y}}) = COV(\mathbf{y}-\hat{\mathbf{y}};\hat{\mathbf{y}}) = COV(\mathbf{y};\hat{\mathbf{y}}) - V(\hat{\mathbf{y}}) = COV(\mathbf{y};\mathbf{P}\mathbf{y}) - V(\hat{\mathbf{y}}) =$
  $= V(\mathbf{y})\mathbf{P}' - V(\mathbf{P}\mathbf{y}) = V(\mathbf{y})\mathbf{P} - \mathbf{P}V(\mathbf{y})\mathbf{P}' = V(\mathbf{y})\mathbf{P} - \mathbf{P}V(\mathbf{y})\mathbf{P} =$
  $= \sigma_\varepsilon^2\mathbf{I}_n\mathbf{P} - \mathbf{P}\sigma_\varepsilon^2\mathbf{I}_n\mathbf{P} = \sigma_\varepsilon^2\mathbf{P} - \sigma_\varepsilon^2\mathbf{P}\mathbf{P} = \sigma_\varepsilon^2\Big(\mathbf{P}-\mathbf{P}\Big) = \sigma_\varepsilon^2\mathbf{0}_n = \mathbf{0}_n$

4. De las CPO del problema de minimización sabemos que:

$$\mathbf{X}'\mathbf{X}\hat{\beta} - \mathbf{X}'\mathbf{y} = \mathbf{0}_k \Rightarrow \mathbf{X}'\Big(\mathbf{y} - \mathbf{X}\hat{\beta}\Big) = \mathbf{0}_k \Rightarrow \mathbf{X}'\mathbf{e} = \mathbf{0}_k \Rightarrow \sum_{i=1}^{n} e_i x_{ij} = 0$$

   En particular, esto debe cumplirse para $X_1 = (1\ldots1)'$, por lo que:

$$\sum_{i=1}^{n} e_i x_{i1} = \sum_{i=1}^{n} e_i 1 = \sum_{i=1}^{n} e_i = 0$$

   Por lo tanto: $\bar{e} = \frac{1}{n}\sum\limits_{i=1}^{n} e_i = \frac{1}{n}0 = 0$

5. - $\mathbf{e}'\mathbf{y} = (\mathbf{y}-\hat{\mathbf{y}})'\mathbf{y} = (\mathbf{y}-\mathbf{P}\mathbf{y})'\mathbf{y} = \Big((\mathbf{I}_n - \mathbf{P})\mathbf{y}\Big)'\mathbf{y} = \mathbf{y}'\mathbf{M}'\mathbf{y} = \mathbf{y}'\mathbf{M}\mathbf{y}$
   - $\mathbf{e}'\hat{\mathbf{y}} = \Big(\mathbf{y}-\hat{\mathbf{y}}\Big)'\Big(\mathbf{P}\mathbf{y}\Big) = \Big(\mathbf{y}-\mathbf{P}\mathbf{y}\Big)'\Big(\mathbf{P}\mathbf{y}\Big) = \Big(\mathbf{y}' - \mathbf{y}'\mathbf{P}'\Big)\Big(\mathbf{P}\mathbf{y}\Big) =$
     $= \mathbf{y}'\mathbf{P}\mathbf{y} - \mathbf{y}'\mathbf{P}'\mathbf{P}\mathbf{y} = \mathbf{y}'\mathbf{P}\mathbf{y} - \mathbf{y}'\mathbf{P}\mathbf{y} = 0$
   - $\mathbf{e}'\mathbf{X} = \Big(\mathbf{y}-\hat{\mathbf{y}}\Big)'\mathbf{X} = \Big(\mathbf{y}' - \mathbf{y}'\mathbf{P}'\Big)\mathbf{X} = \mathbf{y}'\mathbf{X} - \mathbf{y}'\mathbf{P}'\mathbf{X} = \mathbf{y}'\mathbf{X} - \mathbf{y}'\mathbf{X} = \mathbf{0}'_{k+1}$

6. $tr(\mathbf{P}) = tr\Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big) = tr\Big((\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\Big) = tr(\mathbf{I}_{k+1}) = k+1$

Práctico 3

Ejercicio 3

1. - Por idempotencia de $\mathbf{H}$:

$$0 \le h_{ii} = h_{i1}^2 + h_{i2}^2 + \ldots + h_{ii}^2 + \ldots + h_{in}^2 = h_{ii}^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \Rightarrow$$

$$\Rightarrow \frac{h_{ii}}{h_{ii}} = \frac{h_{ii}^2}{h_{ii}} + \frac{1}{h_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \Rightarrow 1 = h_{ii} + \underbrace{\frac{1}{h_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2}_{\ge 0} \Rightarrow h_{ii} \le 1$$

   Para probar que $\frac{1}{n} < h_{ii}$ debemos primero centrar $\mathbf{H}$. Sean

$$\tilde{\mathbf{X}} = \begin{bmatrix} 1 & x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 & \cdots & x_{1k} - \bar{x}_k \\ 1 & x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 & \cdots & x_{2k} - \bar{x}_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 & \cdots & x_{nk} - \bar{x}_k \end{bmatrix}_{n \times (k+1)}$$

$$\tilde{\mathbf{H}} = \tilde{\mathbf{X}}(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}' = \begin{bmatrix} \tilde{h}_{11} & \cdots & \tilde{h}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1} & \cdots & \tilde{h}_{nn} \end{bmatrix}_{n \times n}$$

y el modelo

$$\mathbf{y} = \alpha \nVdash_n + \tilde{\mathbf{X}}\beta + \varepsilon \Rightarrow \hat{\mathbf{y}} = \hat{\alpha} \nVdash_n + \tilde{\mathbf{X}}\hat{\beta} \Rightarrow \hat{\mathbf{y}} = \bar{y} \nVdash_n + \tilde{\mathbf{X}}\hat{\beta} = \bar{y} \nVdash_n + \tilde{\mathbf{X}}(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}(\tilde{\mathbf{X}}'\mathbf{y}) \Rightarrow$$

$$\Rightarrow \hat{\mathbf{y}} = \left[\frac{1}{n}\left(\nVdash'_n \mathbf{y}\right)\right] \nVdash_n + \tilde{\mathbf{H}}\mathbf{y} = \left(\frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n \times n)} + \tilde{\mathbf{H}} \right) \mathbf{y} = \mathbf{H}\mathbf{y}$$

Por lo tanto

$$\mathbf{H} = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n \times n)} + \tilde{\mathbf{H}} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n \times n)} + \begin{bmatrix} \tilde{h}_{11} & \cdots & \tilde{h}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1} & \cdots & \tilde{h}_{nn} \end{bmatrix}_{n \times n} =$$

$$= \begin{bmatrix} \tilde{h}_{11} + \frac{1}{n} & \cdots & \tilde{h}_{1n} + \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1} + \frac{1}{n} & \cdots & \tilde{h}_{nn} + \frac{1}{n} \end{bmatrix}_{n \times n}$$

Luego, dado que

$$\left.\begin{array}{l} h_{ii} = \tilde{h}_{ii} + \frac{1}{n} \\ h_{ii} \geq 0 \text{ por idempotencia de } \mathbf{H} \\ \tilde{h}_{ii} \geq 0 \text{ por idempotencia de } \tilde{\mathbf{H}} \end{array}\right\} \Rightarrow h_{ii} \geq \frac{1}{n}$$

Podemos concluir entonces que:

$$\boxed{\frac{1}{n} \leq h_{ii} \leq 1}$$

- Por idempotencia de **H**:

$$h_{ii} = h_{ii}^2 + h_{ij}^2 + \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{n} h_{ik}^2 \Rightarrow \underbrace{h_{ii}(1 - h_{ii})}_{\leq 1/4} = h_{ij}^2 + \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{n} h_{ik}^2$$

$$\text{Si } h_{ii}(1 - h_{ii}) = \frac{1}{4} \Rightarrow h_{ii} = \frac{1}{2} \Rightarrow h_{ij}^2 < \frac{1}{4} \Rightarrow \boxed{-\frac{1}{2} < h_{ij} < \frac{1}{2} \quad \forall i \neq j}$$

2. En la regresión lineal simple:

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2}{n^2\, S_X^2} & -\frac{\sum_{i=1}^{n} x_i}{n^2\, S_X^2} \\ -\frac{\sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{n}{n^2\, S_X^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2 - x_1 \sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{-\sum_{i=1}^{n} x_i + n x_1}{n^2\, S_X^2} \\ \vdots & \vdots \\ \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{-\sum_{i=1}^{n} x_i + n x_n}{n^2\, S_X^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2 - x_1 \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1^2}{n^2\, S_X^2} & & & \\ \frac{\sum_{i=1}^{n} x_i^2 - x_2 \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1 x_2}{n^2\, S_X^2} & & & \\ \vdots & & & \\ \vdots & & & \\ \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1 x_n}{n^2\, S_X^2} & \cdots & \cdots & \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i - x_n \sum_{i=1}^{n} x_i + n x_n^2}{n^2\, S_X^2} \end{bmatrix}_{n\times n}$$

Operando con el i-ésimo elemento de la diagonal, obtenemos que:

$$h_{ii} = \frac{\sum_{i=1}^{n} x_i^2 - x_i \sum_{i=1}^{n} x_i - x_i \sum_{i=1}^{n} x_i + n x_i^2}{n^2\, S_X^2} = \frac{\sum_{i=1}^{n} x_i^2 - 2x_i \sum_{i=1}^{n} x_i + n x_i^2}{n^2\, S_X^2} =$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 - 2x_i \sum_{i=1}^{n} x_i + n x_i^2 + n\bar{x}^2}{n^2\, S_X^2} = \frac{n S_X^2}{n^2\, S_X^2} + \frac{-2x_i n\bar{x} + n x_i^2 + n\bar{x}^2}{n^2\, S_X^2} =$$

$$= \frac{1}{n} + \frac{n\left(x_i - \bar{x}\right)^2}{n^2\, S_X^2} \Rightarrow \boxed{h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{n\, S_X^2}}$$

Operando con el ij-ésimo elemento fuera de la diagonal, obtenemos que:

$$h_{ij} = \frac{\sum_{i=1}^{n} x_i^2 - x_i \sum_{i=1}^{n} x_i - x_j \sum_{i=1}^{n} x_i + nx_i^2}{n^2\, S_X^2} = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 - x_i \sum_{i=1}^{n} x_i - x_j \sum_{i=1}^{n} x_i + nx_i x_j + n\bar{x}^2}{n^2\, S_X^2} =$$

$$= \frac{nS_X^2}{n^2\, S_X^2} + \frac{n(x_i - \bar{x})(x_j - \bar{x})}{n^2\, S_X^2} \Rightarrow \boxed{h_{ij} = \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{n\, S_X^2}}$$

3. Esta relación es más sencilla de visualizar si vemos la matriz $\tilde{\mathbf{X}}$ por filas y a la matriz $(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}$ por columnas:

$$\tilde{\mathbf{X}} = \begin{bmatrix} \text{—} & \tilde{\mathbf{x}}_\mathbf{1} & \text{—} \\ \text{—} & \tilde{\mathbf{x}}_\mathbf{2} & \text{—} \\ & \vdots & \\ \text{—} & \tilde{\mathbf{x}}_\mathbf{n} & \text{—} \end{bmatrix}_{n \times k} \qquad (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1} = \begin{bmatrix} | & | & & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_k} \\ | & | & & | \end{bmatrix}_{k \times k}$$

Luego el ii-ésimo elemento de $\tilde{\mathbf{H}}$ está dado por $h_{ii} = \frac{1}{n} + \tilde{h}_{ii}$ (tal como se demostró en la parte 1), donde:

$$\tilde{h}_{ii} = \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_1}\, \tilde{\mathbf{x}}_\mathbf{i} + \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_2}\, \tilde{\mathbf{x}}_\mathbf{i} + \ldots + \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_k}\, \tilde{\mathbf{x}}_\mathbf{i}$$

Por lo tanto:

$$\tilde{h}_{ii} = \tilde{\mathbf{x}}_\mathbf{i}'\, (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1} \tilde{\mathbf{x}}_\mathbf{i}$$

Luego entonces:

$$h_{ii} = \frac{1}{n} + \tilde{\mathbf{x}}_\mathbf{i}'\, (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1} \tilde{\mathbf{x}}_\mathbf{i} = \frac{1}{n}\Big(1 + \tilde{\mathbf{x}}_\mathbf{i}'\, \hat{\mathbf{\Sigma}}^{-1} \tilde{\mathbf{x}}_\mathbf{i}\Big) \Rightarrow \boxed{\tilde{h}_{ii} = \frac{1}{n}\Big(1 + \hat{D}_i^2\Big)}$$

Ejercicio 4

$$Dffits_i = \frac{|\hat{y}_i - \hat{y}_{i(i)}|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{|y_i - e_i - y_i + e_{(i)}|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{|e_{(i)} - e_i|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\left|\frac{e_i}{1-h_{ii}} - e_i\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\left|\frac{e_i - e_i(1-h_{ii})}{1-h_{ii}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} =$$

$$= \frac{\left|\frac{e_i h_{ii}}{1-h_{ii}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\frac{h_{ii}}{\sqrt{1-h_{ii}}}\left|\frac{e_i}{\sqrt{1-h_{ii}}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\sqrt{h_{ii}}}{\sqrt{1-h_{ii}}}\left|\frac{e_i}{\sqrt{S_{(i)}^2 (1-h_{ii})}}\right| = |t_i|\sqrt{\frac{h_{ii}}{1-h_{ii}}}$$

Práctico 4

Ejercicio 4

1. Expresión de F_0 para la prueba $H_0 : \beta_2 = 0$

$$F_0 = \frac{\left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)' \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\right)^{-1} \left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Teniendo en cuenta que:

- $q = 1$
- $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
- $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$
- $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix}'$
- $(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{pmatrix}$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \frac{1}{4}$

Llegamos a que:

$$F_0 = \frac{4\hat{\beta}_2^2}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

2. Expresión de F_0 para la prueba $H_0 : \beta_1 = \beta_2$

$$F_0 = \frac{\left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)' \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\right)^{-1} \left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Teniendo en cuenta que:

- $q = 1$
- $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
- $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$
- $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix}'$
- $(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{pmatrix}$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \frac{46}{5}$

Llegamos a que:

$$F_0 = \frac{5}{46}\,\frac{\left(\hat{\beta}_1 - \hat{\beta}_2\right)^2}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Ejercicio 5

Las matrices de datos:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 6{,}6 \\ 7{,}8 \\ 2{,}1 \\ 0{,}4 \end{bmatrix} \quad \Rightarrow \quad (\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 0{,}1 & 0 \\ 0 & 0{,}25 \end{bmatrix} \quad (\mathbf{X}'\mathbf{y}) = \begin{bmatrix} 20{,}9 \\ 16{,}1 \end{bmatrix}$$

Por lo tanto, $\hat{\beta} = \begin{pmatrix} 2{,}090 \\ 4{,}025 \end{pmatrix}$ y $\hat{\sigma}_u = 0{,}4932$

Para hallar el estadístico de prueba, partimos de:

$$F_0 = \frac{\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)' \Big(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\Big)^{-1} \Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Teniendo en cuenta que:

- $q = 1$
- $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
- $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} -2 & 1 \end{pmatrix}$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = 0{,}65$

Llegamos a que:

$$F_0 = \frac{1}{0{,}65} \frac{\Big(-2\hat{\beta}_1 + \hat{\beta}_2\Big)^2}{0{,}4932^2} = 0{,}1520$$

Luego: $F_{1;2}(1-\alpha) = 38{,}5063 > 0{,}1520 = F_0 \Rightarrow$ no rechazo H_0

Ejercicio 6

- Bajo $H_0 : \mathbf{X}\beta = \mathbf{0}_n \Rightarrow \mathbf{y} = \varepsilon \Rightarrow SSR_{H_0} = \mathbf{y}'\mathbf{y}$

  Por otra parte, $SCT_{SR} = SCE_{SR} + SCR_{SR} \Rightarrow \mathbf{y}'\mathbf{y} - n\bar{y}^2 = SCR_{SR} + \hat{\mathbf{y}}'\hat{\mathbf{y}} - n\bar{y}^2 \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SCR_{SR} + \hat{\mathbf{y}}'\hat{\mathbf{y}} \Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + (\mathbf{X}\hat{\beta})'\mathbf{X}\hat{\beta} \Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta} \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'(\mathbf{X}'\mathbf{X})\Big((\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}\Big) = SRC_{SR} + \hat{\beta}' \underbrace{(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}}_{\mathbf{I}_k}(\mathbf{X}'\mathbf{y}) \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{y}$

  Luego entonces, $SCR_{H_0} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{y} \Rightarrow \boxed{\hat{\beta}'\mathbf{X}'\mathbf{y} = SCR_{H_0} - SRC_{SR}}$

- Estadístico F_0

  $$F_0 = \frac{\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)' \Big(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\Big)^{-1} \Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

  Teniendo en cuenta que:

  - $q = k + 1$
  - $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{k+1\times 1} = \mathbf{0}_{k+1}$
  - $\mathbf{R}_{q\times(k+1)} = \mathbf{I}_{k+1}$

- $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \cdots & \hat{\beta}_k \end{pmatrix}'$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \mathbf{I}_{k+1}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}'_{k+1} = (\mathbf{X}'\mathbf{X})^{-1}$

Llegamos a que:

$$F_0 = \frac{\hat{\beta}'\left((\mathbf{X}'\mathbf{X})^{-1}\right)^{-1}\hat{\beta}/(k+1)}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)} \Rightarrow \boxed{F_0 = \frac{\hat{\beta}'(\mathbf{X}'\mathbf{X})\hat{\beta}/(k+1)}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}}$$