Entrega

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Una aclaración previa importante

Salvo que se indique lo contrario en la letra del ejercicio, o en la solución aquí propuesta, siempre se está asumiendo que en las distintas situaciones planteadas se está trabajando bajo los supuestos de la regresión lineal. En particular, el supuesto de que la matriz $\mathbf{X}$ es de rango completo por columnas fue utilizado ampliamente, sin aclaración explícita en las soluciones.

Práctico 1

Ejercicio 1

$$Z \sim \chi^2_d \Rightarrow Z \sim \text{Gamma}\left(\frac{d}{2}; \frac{1}{2}\right) \Rightarrow f_Z(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{I}_{[z \geq 0]}$$

- Esperanza de Z

$$E(Z) = \int_0^{+\infty} z \, \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{d}z = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \underbrace{\int_0^{+\infty} \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \, z^{(\alpha+1)-1} \, e^{-\beta z} \, \mathrm{d}z}_{=1} =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \frac{\alpha\Gamma(\alpha)}{\beta^\alpha \, \beta} = \frac{\alpha}{\beta} \Rightarrow E(Z) = \frac{{}^d/_2}{{}^1/_2} \Rightarrow \boxed{E(Z) = d}$$

- Varianza de Z: $V(Z) = E(Z^2) - E^2(Z)$
  - Esperanza de Z^2

$$E(Z^2) = \int_0^{+\infty} z^2 \, \frac{\beta^\alpha}{\Gamma(\alpha)} \, z^{\alpha-1} \, e^{-\beta z} \, \mathrm{d}z = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} \underbrace{\int_0^{+\infty} \frac{\beta^{\alpha+2}}{\Gamma(\alpha+2)} \, z^{(\alpha+2)-1} \, e^{-\beta z} \, \mathrm{d}z}_{=1} =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \frac{\alpha(\alpha+1)\Gamma(\alpha)}{\beta^\alpha \, \beta^2} = \frac{\alpha(\alpha+1)}{\beta^2}$$

  - Varianza de Z:

$$V(Z) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2} \, (\alpha + 1 - \alpha) = \frac{\alpha}{\beta} \Rightarrow V(Z) = \frac{{}^d/_2}{{}^1/_4} = \frac{4d}{2} \Rightarrow \boxed{V(Z) = 2d}$$

Ejercicio 2

1. $\sum\limits_{i=1}^n e_i = \sum\limits_{i=1}^n \left(y_i - \hat{y}_i\right) = \sum\limits_{i=1}^n y_i - \sum\limits_{i=1}^n \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) = \sum\limits_{i=1}^n y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum\limits_{i=1}^n x_i =$

   $= \bar{y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{x} = \bar{y} - \left(\bar{y} - \hat{\beta}_1 \bar{x}\right) - \hat{\beta}_1 \bar{x} = 0 \Rightarrow \boxed{\sum\limits_{i=1}^n e_i = 0}$

2. Si $\sum_{i=1}^{n} e_i = 0 \Rightarrow \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right) = 0 \Rightarrow \boxed{\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i}$

3. $\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i \left(y_i - \hat{y}_i\right) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \hat{y}_i = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) =$

$$= \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x}\right) \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 =$$

$$= \sum_{i=1}^{n} x_i y_i - \bar{y} \underbrace{\sum_{i=1}^{n} x_i}_{n\bar{x}} + \hat{\beta}_1 n \bar{x}^2 - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \hat{\beta}_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) - \hat{\beta}_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) \Rightarrow$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} x_i e_i = 0}$$

4. $\sum_{i=1}^{n} \hat{y}_i e_i = \sum_{i=1}^{n} \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right) e_i = \hat{\beta}_0 \underbrace{\sum_{i=1}^{n} e_i}_{=0} + \hat{\beta}_1 \underbrace{\sum_{i=1}^{n} x_i e_i}_{=0} \Rightarrow \boxed{\sum_{i=1}^{n} \hat{y}_i e_i = 0}$

5. $\sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n} \left(y_i - \hat{y}_i + \hat{y}_i - \bar{y}\right)^2 =$

$$= \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + 2 \underbrace{\sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)\left(\hat{y}_i - \bar{y}\right)}_{\sum_{i=1}^{n} e_i \left(\hat{y}_i - \bar{y}\right) = \sum_{i=1}^{n} e_i \hat{y}_i - \bar{y} \sum_{i=1}^{n} e_i} + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 =$$

$$= \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + 2 \underbrace{\sum_{i=1}^{n} e_i \hat{y}_i}_{=0} - \bar{y} \underbrace{\sum_{i=1}^{n} e_i}_{=0} + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 \Rightarrow$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 + \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2}$$

6. Interpretación geométrica:

   a) $\mathbb{1}'\mathbf{e} = \mathbf{0}$, implica que el vector de residuos es ortogonal a un vector de unos.

   b) $\mathbb{1}'\mathbf{y} = \mathbb{1}'\hat{\mathbf{y}}$

   c) $\mathbf{x}'\mathbf{e} = \mathbf{0}$, implica ortogonalidad entre los regresores y los residuos.

   d) $\mathbf{y}'\mathbf{e} = \mathbf{0}$, implica ortogonalidad entre el regresando y los residuos.

   e) Implica que $\|\mathbf{y} - \bar{\mathbf{y}}\|^2 = \|\hat{\mathbf{y}} - \bar{\mathbf{y}}\|^2 + \|\mathbf{y} - \hat{\mathbf{y}}\|^2$

Ejercicio 3

- Estimador MCO

$$\hat{\theta}_{MCO} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \left(\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) \Rightarrow \boxed{\hat{\theta}_{MCO} = \frac{1}{5}\Big(2y_1 + y_2\Big)}$$

- Suma de Cuadrados de Residuos (SSR):

$$SSR = \mathbf{y}'\mathbf{y} - \hat{\theta}(\mathbf{X}'\mathbf{y}) = \Big(y_1^2 + y_2^2\Big) - \frac{1}{5}\Big(2y_1 + y_2\Big)^2 =$$

$$= y_1^2 + y_2^2 - \frac{4}{5}\,y_1^2 - \frac{4}{5}\,y_1\,y_2 - \frac{1}{5}\,y_2^2 = \frac{y_1^2}{5} + \frac{4y_2^2}{5} - \frac{4}{5}\,y_1\,y_2 \Rightarrow \boxed{SSR = \frac{1}{5}\Big(y_1 - 2y_2\Big)^2}$$

Ejercicio 4

Caso general:

$$\min_{\hat{\beta}}\Big\{\hat{\mathbf{u}}'\hat{\mathbf{u}}\Big\} = \min_{\hat{\beta}}\Big\{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})\Big\} = \min_{\hat{\beta}}\Big\{\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\hat{\beta} - \hat{\beta}'\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}\Big\} =$$

$$= \min_{\hat{\beta}}\Big\{\mathbf{y}'\mathbf{y} - 2\hat{\beta}'\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}\Big\}$$

$$\frac{\partial(\hat{\mathbf{u}}'\hat{\mathbf{u}})}{\partial\hat{\beta}'} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\beta} = \mathbf{0}_{k+1} \Rightarrow (\mathbf{X}'\mathbf{X})\hat{\beta} = (\mathbf{X}'\mathbf{y}) \Rightarrow \boxed{\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}$$

$$\frac{\partial^2(\hat{\mathbf{u}}'\hat{\mathbf{u}})}{\partial^2\hat{\beta}} = \frac{\partial}{\partial\hat{\beta}}\Big(\mathbf{X}'\mathbf{X}\hat{\beta} - \mathbf{X}'\mathbf{y}\Big) = \mathbf{X}'\mathbf{X} > 0 \Rightarrow \hat{\beta} \text{ es mínimo}$$

Para el modelo $y_i = \beta_0 + \beta_1 x_i + u_i \ \forall i = 1; \ldots; n$

$$\mathbf{X} = \begin{pmatrix}1 & x_1\\1 & x_2\\ \vdots & \vdots\\ 1 & x_n\end{pmatrix} \quad \mathbf{y} = \begin{pmatrix}y_1\\y_2\\ \vdots\\ y_n\end{pmatrix} \quad \Rightarrow \quad (\mathbf{X}'\mathbf{X}) = \begin{pmatrix}n & \sum\limits_{i=1}^n x_i\\ \sum\limits_{i=1}^n x_i & \sum\limits_{i=1}^n x_i^2\end{pmatrix} \quad (\mathbf{X}'\mathbf{y}) = \begin{pmatrix}\sum\limits_{i=1}^n y_i\\ \sum\limits_{i=1}^n x_i y_i\end{pmatrix}$$

Por lo tanto,

$$\hat{\beta} = \begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\end{pmatrix} = \begin{pmatrix}\bar{y} - \hat{\beta}_1\bar{x}\\ \frac{S_{XY}}{S_X^2}\end{pmatrix}$$

Ejercicio 5

1.
   - $\bar{u} = \frac{1}{n}\sum\limits_{i=1}^n \frac{x_i - \bar{x}}{S_X} = \frac{1}{nS_X}\left(\sum\limits_{i=1}^n x_i - n\bar{x}\right) = \frac{1}{nS_X}\,(n\bar{x} - n\bar{x}) = 0$
   - $\bar{v} = \frac{1}{n}\sum\limits_{i=1}^n \frac{y_i - \bar{y}}{S_Y} = \frac{1}{nS_Y}\left(\sum\limits_{i=1}^n y_i - n\bar{y}\right) = \frac{1}{nS_Y}\,(n\bar{y} - n\bar{y}) = 0$

- $S_u^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(u_i-\bar{u})^2 = \frac{1}{n}\sum\limits_{i=1}^{n}u_i^2 - \bar{u}^2 = \frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{x_i-\bar{x}}{S_X}\right)^2 = \frac{1}{S_X^2}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}_{S_X^2}\right) =$

  $= \frac{1}{S_X^2}\, S_X^2 = 1$

- $S_v^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(v_i-\bar{v})^2 = \frac{1}{n}\sum\limits_{i=1}^{n}v_i^2 - \bar{v}^2 = \frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{y_i-\bar{y}}{S_Y}\right)^2 = \frac{1}{S_Y^2}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2}_{S_Y^2}\right) =$

  $= \frac{1}{S_Y^2}\, S_Y^2 = 1$

- $\hat{b}_1 = (\mathbf{u}'\mathbf{u})^{-1}(\mathbf{u}'\mathbf{v}) = \left(\sum\limits_{i=1}^{n}u_i^2\right)^{-1}\left(\sum\limits_{i=1}^{n}u_iv_i\right) = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}u_iv_i}{\frac{1}{n}\sum\limits_{i=1}^{n}u_i^2} = \frac{S_{uv}}{\underbrace{S_u^2}_{=1}} =$

  $= \frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{x_i-\bar{x}}{S_X}\right)\left(\frac{y_i-\bar{y}}{S_Y}\right) = \frac{1}{S_XS_Y}\left(\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}\right) = \frac{S_{XY}}{S_XS_Y}$

  Por lo tanto: $\hat{b}_1\,\frac{S_Y}{S_X} = \frac{S_{XY}}{S_XS_Y}\,\frac{S_Y}{S_X} = \frac{S_{XY}}{S_X^2}\,\frac{S_Y}{S_Y} = \frac{S_{XY}}{S_X^2} = \hat{\beta}_1$

- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ se desprende de la derivación de $\hat{\beta}_{MCO}$

2. Usando las demostraciones del punto 1, tenemos que:

$$\left.\begin{aligned} r_{uv} &= \tfrac{S_{uv}}{S_uS_v} \\ \hat{b}_1 &= \tfrac{S_{uv}}{S_u^2} \\ S_u &= S_v = 1 \end{aligned}\right\} \Rightarrow \hat{b}_1 = S_{uv} = r_{uv}$$

Ejercicio 6

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \frac{1}{n\sum\limits_{i=1}^{n}x_i^2 - \left(\sum\limits_{i=1}^{n}x_i\right)^2}\begin{pmatrix} \sum\limits_{i=1}^{n}x_i^2 & -\sum\limits_{i=1}^{n}x_i \\ -\sum\limits_{i=1}^{n}x_i & n \end{pmatrix}\begin{pmatrix} \sum\limits_{i=1}^{n}y_i \\ \sum\limits_{i=1}^{n}x_iy_i \end{pmatrix}$$

Por lo tanto:

$$\hat{\beta}_1 = \frac{-\left(\sum\limits_{i=1}^{n}x_i\right)\left(\sum\limits_{i=1}^{n}y_i\right) + n\sum\limits_{i=1}^{n}x_iy_i}{n\sum\limits_{i=1}^{n}x_i^2 - \left(\sum\limits_{i=1}^{n}x_i\right)^2} = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}x_iy_i - \bar{x}\bar{y}}{\frac{1}{n}\sum\limits_{i=1}^{n}x_i^2 - \bar{x}^2} = \frac{S_{XY}}{S_X^2} =$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \Rightarrow \boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

$$\hat{\beta}_0 = \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\frac{1}{n^2}\left[\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)\right]}{\frac{1}{n^2}\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]} =$$

$$= \frac{\frac{1}{n^2}\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i}{S_X^2} - \frac{\frac{1}{n^2}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{S_X^2} = \bar{y}\,\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2}{S_X^2} - \bar{x}\,\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i}{S_X^2} =$$

$$= \bar{y}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 + \bar{x}^2}{S_X^2}\right) - \bar{x}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y} + \bar{x}\bar{y}}{S_X^2}\right) =$$

$$= \bar{y}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}{S_X^2}\right) + \frac{\bar{y}\bar{x}^2}{S_X^2} - \bar{x}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{S_X^2}\right) - \frac{\bar{x}^2\bar{y}}{S_X^2} =$$

$$= \bar{y}\,\frac{S_X^2}{S_X^2} - \bar{x}\,\frac{S_{XY}}{S_X^2} \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

Ejercicio 7

Partimos de la distribución de Y: $Y \sim N_n\left(\mathbf{X}\beta;\ \sigma_\varepsilon^2\mathbf{I}_n\right)$

- Estimadores máximo verosímiles

$$\mathcal{L}(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = \prod_{i=1}^{n} f_{y_i}(x_i;y_i|\beta_0;\beta_1;\sigma_\varepsilon^2) =$$

$$= \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma_\varepsilon^2)^{-1/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(y_i - \beta_0 - \beta_1 x_i\right)^2\right\} =$$

$$= (2\pi)^{-n/2}(\sigma_\varepsilon^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i - \beta_0 - \beta_1 x_i\right)^2\right\}$$

$$l(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i - \beta_0 - \beta_1 x_i\right)^2$$

Luego entonces:

$$\frac{\partial l(\cdot)}{\partial \beta_0} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^n \Big(y_i - \beta_0 - \beta_1 x_i\Big)(-1) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^n y_i - n\beta_0 - \beta_1 \sum_{i=1}^n x_i = 0 \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$$

$$\frac{\partial l(\cdot)}{\partial \beta_1} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^n \Big(y_i - \beta_0 - \beta_1 x_i\Big)(-x_i) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^n \Big(y_i x_i - \beta_0 x_i - \beta_1 x_i^2\Big) = \sum_{i=1}^n x_i y_i - \beta_0 \sum_{i=1}^n x_i - \beta_1 \sum_{i=1}^n x_i^2 = 0 \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i y_i - \hat{\beta}_0 \sum_{i=1}^n x_i \Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} - \Big(\bar{y} - \hat{\beta}_1 \bar{x}\Big)\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \left[1 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n\sum_{i=1}^n x_i^2}\right] = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} - \frac{\sum_{i=1}^n y_i \sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2}$$

$$\Rightarrow \hat{\beta}_1 = \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}\right)\left(\frac{n\sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2}\right) - \left(\frac{\sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2}\right)\left(\frac{n\sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2}\right) \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 = \frac{\frac{1}{n^2}\left[n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i\right]}{\frac{1}{n^2}\left[n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2\right]} \Rightarrow \boxed{\hat{\beta}_1 = \frac{S_{XY}}{S_X^2}}$$

$$\frac{\partial l(\cdot)}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} - \frac{1}{2}\left(\frac{-1}{2(\sigma_\varepsilon^2)^2}\right)\underbrace{\sum_{i=1}^n \Big(y_i - \beta_0 - \beta_1 x_i\Big)^2}_{\hat{\varepsilon}'\hat{\varepsilon}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{2(\sigma_\varepsilon^2)^2} - \frac{n}{2\sigma_\varepsilon^2} = 0 \Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{(\sigma_\varepsilon^2)^2} = \frac{n}{\sigma_\varepsilon^2} \Rightarrow \boxed{\hat{\sigma}_\varepsilon^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}}$$

- Distribución de $\hat{\beta}$

$$\left.\begin{array}{l}\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) \\ \mathbf{y} \sim N_n\Big(\mathbf{X}\beta;\, \sigma_\varepsilon^2 \mathbf{I}_n\Big)\end{array}\right\} \Rightarrow \hat{\beta} \sim N_{k+1} \text{ por ser CL de normales}$$

- Esperanza de $\hat{\beta}$

$$E(\hat{\beta}) = E\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{y}) = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\beta = \beta$$

- Varianza de $\hat{\beta}$

$$V(\hat{\beta}) = V\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V(\mathbf{y})\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right]' =$$

$$= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma_\varepsilon^2\mathbf{I}_n\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}$$

Por lo tanto: $\boxed{\hat{\beta} \sim N_{k+1}\Big(\beta;\ \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}\Big)}$

Práctico 2

Ejercicio 4

Partimos de la distribución de Y

$$Y \sim N_n\Big(\mathbf{X}\beta;\ \sigma_\varepsilon^2\mathbf{I}_n\Big)$$

- Estimadores máximo verosímiles

$$\mathcal{L}(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = \prod_{i=1}^{n} f_{y_i}(x_i;y_i|\beta_0;\beta_1;\sigma_\varepsilon^2) =$$

$$= \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma_\varepsilon^2)^{-1/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2\right\} =$$

$$= (2\pi)^{-n/2}(\sigma_\varepsilon^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2\right\}$$

$$l(\beta_0;\beta_1;\sigma_\varepsilon^2|\mathbf{X};\mathbf{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2$$

Luego entonces:

$$\frac{\partial l(\cdot)}{\partial \beta_0} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)(-1) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} y_i - n\beta_0 - \beta_1\sum_{i=1}^{n} x_i = 0 \Rightarrow$$

$$\Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

$$\frac{\partial l(\cdot)}{\partial \beta_1} = -\frac{1}{2\sigma_\varepsilon^2}(2)\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)(-x_i) = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n}\Big(y_i x_i - \beta_0 x_i - \beta_1 x_i^2\Big) = \sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0 \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i \Rightarrow \hat{\beta}_1 = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \Big(\bar{y} - \hat{\beta}_1 \bar{x}\Big) \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} x_i^2} \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 \left[1 - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n \sum\limits_{i=1}^{n} x_i^2}\right] = \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2} - \frac{\sum\limits_{i=1}^{n} y_i \sum\limits_{i=1}^{n} x_i}{n \sum\limits_{i=1}^{n} x_i^2}$$

$$\Rightarrow \hat{\beta}_1 = \left(\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2}\right) \left(\frac{n \sum\limits_{i=1}^{n} x_i^2}{n \sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) - \left(\frac{\sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i}{n \sum\limits_{i=1}^{n} x_i^2}\right) \left(\frac{n \sum\limits_{i=1}^{n} x_i^2}{n \sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2}\right) \Rightarrow$$

$$\Rightarrow \hat{\beta}_1 = \frac{\frac{1}{n^2}\left[n \sum\limits_{i=1}^{n} x_i y_i - \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i\right]}{\frac{1}{n^2}\left[n \sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right]} \Rightarrow \boxed{\hat{\beta}_1 = \frac{S_{XY}}{S_X^2}}$$

$$\frac{\partial l(\cdot)}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} - \frac{1}{2}\left(\frac{-1}{2(\sigma_\varepsilon^2)^2}\right) \underbrace{\sum_{i=1}^{n}\Big(y_i - \beta_0 - \beta_1 x_i\Big)^2}_{\hat{\varepsilon}'\hat{\varepsilon}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{2(\sigma_\varepsilon^2)^2} - \frac{n}{2\sigma_\varepsilon^2} = 0 \Rightarrow \frac{\hat{\varepsilon}'\hat{\varepsilon}}{(\sigma_\varepsilon^2)^2} = \frac{n}{\sigma_\varepsilon^2} \Rightarrow \boxed{\hat{\sigma}_\varepsilon^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}}$$

- Distribución de $\hat{\beta}$

$$\left.\begin{array}{l} \hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) \\ \mathbf{y} \sim N_n\Big(\mathbf{X}\beta;\ \sigma_\varepsilon^2 \mathbf{I}_n\Big) \end{array}\right\} \Rightarrow \hat{\beta} \sim N_{k+1} \text{ por ser CL de normales}$$

- Esperanza de $\hat{\beta}$

$$E(\hat{\beta}) = E\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{y}) = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\beta = \beta$$

- Varianza de $\hat{\beta}$

$$V(\hat{\beta}) = V\Big[(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})\Big] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V(\mathbf{y})\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right]' =$$

$$= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma_\varepsilon^2 \mathbf{I}_n \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}$$

Por lo tanto: $\boxed{\hat{\beta} \sim N_{k+1}\Big(\beta;\ \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}\Big)}$

Ejercicio 5

Partimos del modelo:

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad \forall i = 1; \ldots ; 6$$

De la estimación MCO obtenemos:

$$\hat{\beta}_1 = \frac{S_{XY}}{S_X^2} = \frac{400{,}7}{466{,}7} = 0{,}8586$$

$$\hat{sd}(\hat{\beta}_1) = 0{,}0398$$

Con estos datos construimos el intervalo de confianza:

$$IC_{\beta_1}^{95\,\%} = \left[\hat{\beta}_1 \pm t_4(1 - {}^{\alpha}/_{2})\ \hat{sd}(\hat{\beta}_1)\right] \Rightarrow \boxed{IC_{\beta_1}^{95\,\%} = [0{,}748;\ 0{,}969]}$$

Ejercicio 6

Partimos del modelo:

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad \forall i = 1; \ldots ; 12$$

De la estimación MCO obtenemos:

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \begin{pmatrix} -120{,}21 \\ 1{,}11 \end{pmatrix}$$

$$\hat{\sigma}_{\varepsilon}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-k-1} = 1{,}32$$

Luego realizamos la prueba:

$$H_0)\ \ \beta_1 = 0 \quad Vs. \quad H_1)\ \ \beta_1 \neq 0$$

Con región crítica:

$$RC = \left\{(\mathbf{X}\mathbf{y})\ /\ |e| > t_{n-k-1}(1 - {}^{\alpha}/_{2})\right\}$$

Y estadístico de prueba:

$$e = \frac{\hat{\beta}_1}{\hat{sd}(\hat{\beta}_1)} \overset{\mathrm{H_0}}{\sim} t_{n-k-1}$$

Dado que $e = 11{,}58661 > 2{,}23 = t_10(0{,}975) \Rightarrow$ rechazo H_0, lo cual implica que, la altura es significativa para explicar el peso, al 95 %.

Ejercicio 7

1. Sea el modelo: $y_i = \beta_0 + \beta_1 t_i + u_i \;\; \forall i = 1; \ldots ; 11$

   Para el cual se obtienen las siguientes estimaciones MCO:

$$\hat{\beta} = \begin{pmatrix} 9{,}27 \\ 1{,}44 \end{pmatrix} \quad \hat{V}(\hat{\beta}) = \begin{pmatrix} 0{,}2145 & 0 \\ 0 & 0{,}02145 \end{pmatrix} \quad \hat{\sigma}_u^2 = 2{,}36$$

2. Se realiza la prueba

$$H_0) \;\; \beta_1 = 0 \quad Vs. \quad H_1) \;\; \beta_1 \neq 0$$

   Con región crítica:

$$RC = \left\{ (\mathbf{Xy}) \;/\; |e| > t_{n-k-1}(1 - {}^{\alpha}/_{2}) \right\}$$

   Y estadístico de prueba:

$$e = \frac{\hat{\beta}_1}{\hat{sd}(\hat{\beta}_1)} \overset{\mathrm{H}_0}{\sim} t_{n-k-1}$$

   Dado que $e = 9{,}87 > 2{,}26 = t_9(0{,}975) \Rightarrow$ rechazo H_0, lo cual implica que, la temperatura es significativa para explicar la cantidad producida, al 95 %.

3. Predicción puntual: $\hat{E}(y|t = 3) = \hat{\beta}_0 + \hat{\beta}_1(3) = 13{,}58$ Intervalo de confianza:

$$IC^{95\,\%}_{E(y|t=3)} = \left[\hat{E}(y|t = 3) \pm t_9(0{,}975) \sqrt{\hat{\sigma}_u^2 \begin{pmatrix} 1 & 3 \end{pmatrix} (\mathbf{X}'\mathbf{X})^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}} \right] = [12{,}44; \; 15{,}03]$$

4. Predicción puntual: $\hat{y}|t = 3 = \hat{\beta}_0 + \hat{\beta}_1(3) = 13{,}58$ Intervalo de confianza:

$$IC^{95\,\%}_{y|t=3} = \left[\hat{y}|t = 3 \pm t_9(0{,}975) \sqrt{\hat{\sigma}_u^2 \left[1 + \begin{pmatrix} 1 & 3 \end{pmatrix} (\mathbf{X}'\mathbf{X})^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]} \right] = [9{,}82; \; 17{,}34]$$

Ejercicio 8

1. Recta de regresión: $velocidad_i = 61{,}3 - 0{,}63 \; densidad_i \;\; \forall i = 1; \ldots ; 24$
2. $\sigma_\varepsilon^2 = 16{,}31$; $\hat{se}(\hat{\beta}_0) = 1{,}96$; $\hat{se}(\hat{\beta}_1) = 0{,}03$
3. $IC^{95\,\%}_{\beta_0} = \Big[57{,}25; \; 65{,}39 \Big]$; $IC^{95\,\%}_{\beta_1} = \Big[-0{,}70; \; -0{,}56 \Big]$
4. Tabla de significación del modelo:

	df	**SC**	**SCMe**	**F**
Densidad	1	6002.4	6002.4	368.13
Residuos	22	358.7	16.3	

   Contraste de significación global:

$$H_0) \;\; \beta_1 = 0 \quad Vs. \quad H_1) \;\; \beta_1 \neq 0$$

   Con región crítica:

$$RC = \left\{ (\mathbf{Xy}) \;/\; F_0 > F_{k-1;\, n-k}(1 - \alpha) \right\}$$

   Y estadístico de prueba:

$$F_0 = \frac{R^2/(k-1)}{(1 - R^2)/(n-k)} \overset{\mathrm{H}_0}{\sim} F_{k-1;\, n-k}$$

   Dado que $F_0 = 368{,}13 > 4{,}3 = F_{1;\,22}(0{,}95) \Rightarrow$ rechazo H_0, lo cual implica que, el modelos es significativo, al 95 %.

5. Estimación puntual: $\hat{E}(velocidad|densidad = 50) = 29{,}96$

   Intervalo de confianza: $IC^{90\,\%}_{E(velocidad|densidad=50)} = \Big[28{,}31;\ 31{,}70\Big]$

6. Con la raíz cuadrada de la velocidad:

   - Recta de regresión: $\sqrt{velocidad_i} = 8{,}09 - 0{,}06\ densidad_i \quad \forall i = 1; \ldots ; 24$
   - $\sigma^2_\varepsilon = 0{,}07$; $\hat{se}(\hat{\beta}_0) = 0{,}13$; $\hat{se}(\hat{\beta}_1) = 0{,}002$
   - $IC^{95\,\%}_{\beta_0} = \Big[7{,}82;\ 8{,}36\Big]$; $IC^{95\,\%}_{\beta_1} = \Big[-0{,}06;\ -0{,}05\Big]$
   - Tabla de significación del modelo:

	df	**SC**	**SCMe**	**F**
$\sqrt{\textbf{Velocidad}}$	1	48.92	48.92	676.39
Residuos	22	1.59	0.072	

   Contraste de significación global:

$$H_0)\ \ \beta_1 = 0 \quad Vs. \quad H_1)\ \ \beta_1 \neq 0$$

   Con región crítica:

$$RC = \Big\{(\mathbf{Xy})\ /\ F_0 > F_{k-1;\, n-k}(1-\alpha)\Big\}$$

   Y estadístico de prueba:

$$F_0 = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \overset{\mathrm{H_0}}{\sim} F_{k-1;\, n-k}$$

   Dado que $F_0 = 676{,}39 > 4{,}3 = F_{1;\,22}(0{,}95) \Rightarrow$ rechazo H_0, lo cual implica que, el modelos es significativo, al 95 %.

   - Estimación puntual: $\hat{E}(velocidad|densidad = 50) = 7{,}689$

     Intervalo de confianza: $IC^{90\,\%}_{E(velocidad|densidad=50)} = \Big[7{,}576;\ 7{,}803\Big]$

Ejercicio 9

1. Maximización del R^2:

$$\text{máx}\{R^2\} = \text{máx}\left\{1 - \frac{SCR}{SCT}\right\} = \text{máx}\left\{1 - \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\} = \text{máx}\left\{1 - \frac{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\} =$$

$$= \text{máx}\left\{1 - \frac{\mathbf{y}'\mathbf{y} - 2\hat{\beta}\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\right\}$$

$$\frac{\partial R^2}{\partial \hat{\beta}'} = 0 - \frac{1}{\mathbf{y}'\mathbf{y} - n\bar{y}^2}\left(-2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\beta}\right) = 0 \Rightarrow \boxed{\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}$$

2. - Partimos de la regresión particionada.

     Sea el siguiente modelo de regresión:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon = \mathbf{X}_1\beta + \mathbf{X}_2\alpha + \varepsilon$$

Con ecuaciones normales:

$$\begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

De estas ecuaciones se desprende que:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'(\mathbf{y} - \mathbf{X}_2\hat{\alpha})$$

Por el teorema de Frisch-Waugh-Lovel los estimadores MCO serán:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$$

$$\hat{\alpha} = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}(\mathbf{X}_2'\mathbf{M}_1\mathbf{y})$$

donde:

$$\mathbf{M}_1 = \mathbf{I}_n - \mathbf{P}_1 = \mathbf{I}_n - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$$
$$\mathbf{M}_2 = \mathbf{I}_n - \mathbf{P}_2 = \mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'$$

Dado que $\mathbf{M}$ es simétrica e idempotente, podemos escribir:

$$\mathbf{X}_2^* = \mathbf{M}_1\mathbf{X}_2$$

$$\mathbf{y}_* = \mathbf{M}_1\mathbf{y}$$

Por lo que entonces:

$$\hat{\beta} = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$$
$$\hat{\alpha} = (\mathbf{X}_2^{*'}\mathbf{X}_2^*)^{-1}(\mathbf{X}_2^{*'}\mathbf{y}_*)$$

- El caso particular en que $\mathbf{X}_2 = \mathbf{z}_{n\times 1}$

  Este es el caso en que $\mathbf{z}$ contiene una sola variable. Para simplificar notación llamaremos $\mathbf{X}_1 = \mathbf{X}$, dado que no hay riesgo de confusión.

  De la regresión: $\mathbf{y} = \mathbf{X}\beta + \mathbf{z}\alpha + \varepsilon$ podemos obtener la estimación del coeficiente asociado a $\mathbf{z}$ siguiendo lo visto anteriormente:

$$\hat{\alpha} = (\mathbf{z}'\mathbf{M}\mathbf{z})^{-1}(\mathbf{z}'\mathbf{M}\mathbf{y}) = (\mathbf{z}_*'\mathbf{z}_*)^{-1}(\mathbf{z}_*'\mathbf{y}_*)$$

- Cambio en la SCR al agregar un regresor.

  Sean $\hat{\mathbf{e}}'\hat{\mathbf{e}}$ la SCR de la regresión de $\mathbf{y}$ sobre $\mathbf{X}$, y $\hat{\mathbf{u}}'\hat{\mathbf{u}}$ la SCR de la regresión de $\mathbf{y}$ sobre $\mathbf{X}$ y un regresor adicional $\mathbf{z}$. El vector de residuos de la segunda regresión será entonces $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\check{\beta} - \mathbf{z}\hat{\alpha}$. Por otra parte, de la regresión particionada sabemos que:

$$\check{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big(\mathbf{y} - \mathbf{z}\hat{\alpha}\Big) = \underbrace{(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y})}_{\hat{\beta}} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha} = \hat{\beta} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha}$$

  Luego entonces,

$$\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\Big(\hat{\beta} - (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{z})\hat{\alpha}\Big) - \mathbf{z}\hat{\alpha} = \underbrace{\mathbf{y} - \mathbf{X}\hat{\beta}}_{\hat{\mathbf{e}}} + \underbrace{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}_{\mathbf{P}}\mathbf{z}\hat{\alpha} - \mathbf{z}\hat{\alpha} =$$

$$= \hat{\mathbf{e}} + \mathbf{Pz}\hat{\alpha} - \mathbf{z}\hat{\alpha} = \hat{\mathbf{e}} + (\mathbf{P} - \mathbf{I}_n)\mathbf{z}\hat{\alpha} = \hat{\mathbf{e}} - \mathbf{Mz}\hat{\alpha} = \hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}$$

Por lo tanto,

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \Big(\hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}\Big)'\Big(\hat{\mathbf{e}} - \mathbf{z}_*\hat{\alpha}\Big) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \underbrace{\hat{\mathbf{e}}'\mathbf{z}_*}_{1\times 1}\hat{\alpha} - \hat{\alpha}'\underbrace{\mathbf{z}_*{}'\hat{\mathbf{e}}}_{1\times 1} - \hat{\alpha}'\mathbf{z}_*{}'\mathbf{z}_*\hat{\alpha} =$$

$$= \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}(\underbrace{\mathbf{z}_*{}'\hat{\mathbf{e}}}_{=\mathbf{z}_*{}'\mathbf{y}_*}) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}(\underbrace{\mathbf{z}_*{}'\mathbf{y}_*}_{=\hat{\alpha}(\mathbf{z}_*{}'\mathbf{z}_*)}) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) =$$

$$= \hat{\mathbf{e}}'\hat{\mathbf{e}} - 2\hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) + \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*)$$

Por lo tanto:

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \hat{\mathbf{e}}'\hat{\mathbf{e}} - \hat{\alpha}^2(\mathbf{z}_*{}'\mathbf{z}_*) \leq \hat{\mathbf{e}}'\hat{\mathbf{e}}$$

Con lo que queda demostrado que, al agregar un regresor, $\downarrow$ SCR $\Rightarrow\uparrow$ R^2

- Algunas aclaraciones:
  - En la conclusión se sostuvo que el SCR cae, y por tanto el R^2 aumenta. Existe la posibilidad de que el SCR se mantenga incambiado (y por tanto tampoco cambie el R^2). Esto se da únicamente si $\mathbf{X}$ y $\mathbf{z}$ son ortogonales (es decir, están incorrelacionadas).
  - A la igualdad $\mathbf{z}_*{}'\hat{\mathbf{e}} = \mathbf{z}_*{}'\mathbf{y}_*$ se llega recordando que $\hat{\mathbf{e}} = \mathbf{My}$ (en el modelo sin la variable $\mathbf{z}$), pero a su vez, $\mathbf{My}$ se definió como $\mathbf{y}_*$. Por lo tanto, $\hat{\mathbf{e}} = \mathbf{y}_*$.
  - La igualdad $\mathbf{z}_*{}'\mathbf{y} = \hat{\alpha}(\mathbf{z}_*{}'\mathbf{z}_*)$ se desprende de las ecuaciones normales del modelo que incluye la variable $\mathbf{z}$ (ver segundo item, Regresión Particionada con $\mathbf{X}_2 = \mathbf{z}$). La posición de $\hat{\alpha}$ es intercambiable, dado que $\hat{\alpha}$ es un real.

Ejercicio 10

1. Sean $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ y $\mathbf{M} = \mathbf{I}_n - \mathbf{P}$
   - $\mathbf{P}' = \Big[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big]' = \Big(\mathbf{X}'\Big)'\Big((\mathbf{X}'\mathbf{X})^{-1}\Big)'\Big(\mathbf{X}\Big)' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{P}$
   - $\mathbf{PP} = \Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big)\Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big) = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\underbrace{(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}}_{\mathbf{I}_{k+1}}\mathbf{X}' =$

     $= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}_n\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{P}$
   - $\mathbf{MM} = \Big(\mathbf{I}_n - \mathbf{P}\Big)\Big(\mathbf{I}_n - \mathbf{P}\Big) = \mathbf{I}_n\mathbf{I}_n - \mathbf{I}_n\mathbf{P} - \mathbf{P}\mathbf{I}_n + \mathbf{PP} = \mathbf{I}_n - \mathbf{P} - \mathbf{P} + \mathbf{P} =$

     $= \mathbf{I}_n - \mathbf{P} = \mathbf{M}$
   - $\mathbf{M}' = \Big(\mathbf{I}_n - \mathbf{P}\Big)' = \mathbf{I}_n' - \mathbf{P}' = \mathbf{I}_n - \mathbf{P} = \mathbf{M}$
2. $E(\mathbf{e}) = E(\mathbf{y} - \hat{\mathbf{y}}) = E(\mathbf{y}) - E(\hat{\mathbf{y}}) = E(\mathbf{y}) - E(\mathbf{Py}) = E(\mathbf{y}) - \mathbf{P}E(\mathbf{y})$

   $= \Big(\mathbf{I}_n - \mathbf{P}\Big)E(\mathbf{y}) = \Big(\mathbf{I}_n - \mathbf{P}\Big)\mathbf{X}\beta = \Big(\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big)\mathbf{X}\beta =$

   $= \Big(\mathbf{X} - \mathbf{X}\underbrace{(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})}_{\mathbf{I}_{k+1}}\Big)\beta = \Big(\mathbf{X} - \mathbf{X}\Big)\beta = \mathbf{0}_{n\times(k+1)}\,\beta = \mathbf{0}_n$
3. - $V(\mathbf{e}) = V(\mathbf{y} - \hat{\mathbf{y}}) = V(\mathbf{y} - \mathbf{Py}) = V(\mathbf{My}) = \mathbf{M}V(\mathbf{y})\mathbf{M}' = \mathbf{M}\Big(\sigma_\varepsilon^2\mathbf{I}_n\Big)\mathbf{M} =$

     $= \sigma_\varepsilon^2\mathbf{MM} = \sigma_\varepsilon^2\mathbf{M}$

- $COV(\mathbf{e};\mathbf{y}) = COV(\mathbf{y}-\hat{\mathbf{y}};\mathbf{y}) = COV(\mathbf{y};\mathbf{y}) - COV(\hat{\mathbf{y}};\mathbf{y}) =$
  $= V(\mathbf{y}) - COV(\mathbf{P}\mathbf{y};\mathbf{y}) = V(\mathbf{y}) - \mathbf{P}COV(\mathbf{y};\mathbf{y}) = V(\mathbf{y}) - \mathbf{P}V(\mathbf{y}) =$
  $= \Big(\mathbf{I}_n - \mathbf{P}\Big)V(\mathbf{y}) = \Big(\mathbf{I}_n - \mathbf{P}\Big)\sigma^2_\varepsilon \mathbf{I}_n = \sigma^2_\varepsilon\Big(\mathbf{I}_n - \mathbf{P}\Big)$
- $COV(\mathbf{e};\hat{\mathbf{y}}) = COV(\mathbf{y}-\hat{\mathbf{y}};\hat{\mathbf{y}}) = COV(\mathbf{y};\hat{\mathbf{y}}) - V(\hat{\mathbf{y}}) = COV(\mathbf{y};\mathbf{P}\mathbf{y}) - V(\hat{\mathbf{y}}) =$
  $= V(\mathbf{y})\mathbf{P}' - V(\mathbf{P}\mathbf{y}) = V(\mathbf{y})\mathbf{P} - \mathbf{P}V(\mathbf{y})\mathbf{P}' = V(\mathbf{y})\mathbf{P} - \mathbf{P}V(\mathbf{y})\mathbf{P} =$
  $= \sigma^2_\varepsilon \mathbf{I}_n\mathbf{P} - \mathbf{P}\sigma^2_\varepsilon\mathbf{I}_n\mathbf{P} = \sigma^2_\varepsilon\mathbf{P} - \sigma^2_\varepsilon\mathbf{P}\mathbf{P} = \sigma^2_\varepsilon\Big(\mathbf{P} - \mathbf{P}\Big) = \sigma^2_\varepsilon\mathbf{0}_n = \mathbf{0}_n$

4. De las CPO del problema de minimización sabemos que:

$$\mathbf{X}'\mathbf{X}\hat{\beta} - \mathbf{X}'\mathbf{y} = \mathbf{0}_k \Rightarrow \mathbf{X}'\Big(\mathbf{y} - \mathbf{X}\hat{\beta}\Big) = \mathbf{0}_k \Rightarrow \mathbf{X}'\mathbf{e} = \mathbf{0}_k \Rightarrow \sum_{i=1}^{n} e_i x_{ij} = 0$$

   En particular, esto debe cumplirse para $X_1 = (1\dots1)'$, por lo que:

$$\sum_{i=1}^{n} e_i x_{i1} = \sum_{i=1}^{n} e_i 1 = \sum_{i=1}^{n} e_i = 0$$

   Por lo tanto: $\bar{e} = \frac{1}{n}\sum\limits_{i=1}^{n} e_i = \frac{1}{n}0 = 0$

5.
   - $\mathbf{e}'\mathbf{y} = (\mathbf{y} - \hat{\mathbf{y}})'\mathbf{y} = (\mathbf{y} - \mathbf{P}\mathbf{y})'\mathbf{y} = \Big((\mathbf{I}_n - \mathbf{P})\mathbf{y}\Big)'\mathbf{y} = \mathbf{y}'\mathbf{M}'\mathbf{y} = \mathbf{y}'\mathbf{M}\mathbf{y}$
   - $\mathbf{e}'\hat{\mathbf{y}} = \Big(\mathbf{y} - \hat{\mathbf{y}}\Big)'\Big(\mathbf{P}\mathbf{y}\Big) = \Big(\mathbf{y} - \mathbf{P}\mathbf{y}\Big)'\Big(\mathbf{P}\mathbf{y}\Big) = \Big(\mathbf{y}' - \mathbf{y}'\mathbf{P}'\Big)\Big(\mathbf{P}\mathbf{y}\Big) =$
     $= \mathbf{y}'\mathbf{P}\mathbf{y} - \mathbf{y}'\mathbf{P}'\mathbf{P}\mathbf{y} = \mathbf{y}'\mathbf{P}\mathbf{y} - \mathbf{y}'\mathbf{P}\mathbf{y} = 0$
   - $\mathbf{e}'\mathbf{X} = \Big(\mathbf{y} - \hat{\mathbf{y}}\Big)'\mathbf{X} = \Big(\mathbf{y}' - \mathbf{y}'\mathbf{P}'\Big)\mathbf{X} = \mathbf{y}'\mathbf{X} - \mathbf{y}'\mathbf{P}'\mathbf{X} = \mathbf{y}'\mathbf{X} - \mathbf{y}'\mathbf{X} = \mathbf{0}'_{k+1}$

6. $tr(\mathbf{P}) = tr\Big(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big) = tr\Big((\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\Big) = tr(\mathbf{I}_{k+1}) = k+1$

Práctico 3

Ejercicio 3

1.
   - Por idempotencia de $\mathbf{H}$:

$$0 \leq h_{ii} = h_{i1}^2 + h_{i2}^2 + \ldots + h_{ii}^2 + \ldots + h_{in}^2 = h_{ii}^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \Rightarrow$$

$$\Rightarrow \frac{h_{ii}}{h_{ii}} = \frac{h_{ii}^2}{h_{ii}} + \frac{1}{h_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \Rightarrow 1 = h_{ii} + \underbrace{\frac{1}{h_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2}_{\geq 0} \Rightarrow h_{ii} \leq 1$$

     Para probar que $\frac{1}{n} < h_{ii}$ debemos primero centrar $\mathbf{H}$. Sean

$$\tilde{\mathbf{X}} = \begin{bmatrix} 1 & x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & \cdots & x_{1k}-\bar{x}_k \\ 1 & x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 & \cdots & x_{2k}-\bar{x}_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1}-\bar{x}_1 & x_{n2}-\bar{x}_2 & \cdots & x_{nk}-\bar{x}_k \end{bmatrix}_{n\times(k+1)}$$

$$\tilde{\mathbf{H}} = \tilde{\mathbf{X}}(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}' = \begin{bmatrix} \tilde{h}_{11} & \cdots & \tilde{h}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1} & \cdots & \tilde{h}_{nn} \end{bmatrix}_{n\times n}$$

y el modelo

$$\mathbf{y} = \alpha \nVdash_n + \tilde{\mathbf{X}}\beta + \varepsilon \Rightarrow \hat{\mathbf{y}} = \hat{\alpha} \nVdash_n + \tilde{\mathbf{X}}\hat{\beta} \Rightarrow \hat{\mathbf{y}} = \bar{y} \nVdash_n + \tilde{\mathbf{X}}\hat{\beta} = \bar{y} \nVdash_n + \tilde{\mathbf{X}}(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}(\tilde{\mathbf{X}}'\mathbf{y}) \Rightarrow$$

$$\Rightarrow \hat{\mathbf{y}} = \left[\frac{1}{n}\Big(\nVdash'_n \mathbf{y}\Big)\right] \nVdash_n + \tilde{\mathbf{H}}\mathbf{y} = \left(\frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n\times n)} + \tilde{\mathbf{H}}\right)\mathbf{y} = \mathbf{H}\mathbf{y}$$

Por lo tanto

$$\mathbf{H} = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n\times n)} + \tilde{\mathbf{H}} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(n\times n)} + \begin{bmatrix} \tilde{h}_{11} & \cdots & \tilde{h}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1} & \cdots & \tilde{h}_{nn} \end{bmatrix}_{n\times n} =$$

$$= \begin{bmatrix} \tilde{h}_{11}+\frac{1}{n} & \cdots & \tilde{h}_{1n}+\frac{1}{n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{n1}+\frac{1}{n} & \cdots & \tilde{h}_{nn}+\frac{1}{n} \end{bmatrix}_{n\times n}$$

Luego, dado que

$$\left.\begin{array}{l} h_{ii} = \tilde{h}_{ii} + \frac{1}{n} \\ h_{ii} \geq 0 \text{ por idempotencia de } \mathbf{H} \\ \tilde{h}_{ii} \geq 0 \text{ por idempotencia de } \tilde{\mathbf{H}} \end{array}\right\} \Rightarrow h_{ii} \geq \frac{1}{n}$$

Podemos concluir entonces que:

$$\boxed{\frac{1}{n} \leq h_{ii} \leq 1}$$

- Por idempotencia de $\mathbf{H}$:

$$h_{ii} = h_{ii}^2 + h_{ij}^2 + \sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n} h_{ik}^2 \Rightarrow \underbrace{h_{ii}(1-h_{ii})}_{\leq 1/4} = h_{ij}^2 + \sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n} h_{ik}^2$$

$$\text{Si } h_{ii}(1-h_{ii}) = \frac{1}{4} \Rightarrow h_{ii} = \frac{1}{2} \Rightarrow h_{ij}^2 < \frac{1}{4} \Rightarrow \boxed{-\frac{1}{2} < h_{ij} < \frac{1}{2} \ \ \forall i \neq j}$$

2. En la regresión lineal simple:

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} n & \sum\limits_{i=1}^{n} x_i \\ \sum\limits_{i=1}^{n} x_i & \sum\limits_{i=1}^{n} x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2}{n^2\, S_X^2} & -\frac{\sum_{i=1}^{n} x_i}{n^2\, S_X^2} \\ -\frac{\sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{n}{n^2\, S_X^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2 - x_1 \sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{-\sum_{i=1}^{n} x_i + n x_1}{n^2\, S_X^2} \\ \vdots & \vdots \\ \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i}{n^2\, S_X^2} & \frac{-\sum_{i=1}^{n} x_i + n x_n}{n^2\, S_X^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2 - x_1 \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1^2}{n^2\, S_X^2} & & & \\ \frac{\sum_{i=1}^{n} x_i^2 - x_2 \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1 x_2}{n^2\, S_X^2} & & & \\ \vdots & & & \\ \vdots & & & \\ \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i - x_1 \sum_{i=1}^{n} x_i + n x_1 x_n}{n^2\, S_X^2} & \cdots & \cdots & \frac{\sum_{i=1}^{n} x_i^2 - x_n \sum_{i=1}^{n} x_i - x_n \sum_{i=1}^{n} x_i + n x_n^2}{n^2\, S_X^2} \end{bmatrix}_{n \times n}$$

Operando con el i-ésimo elemento de la diagonal, obtenemos que:

$$h_{ii} = \frac{\sum\limits_{i=1}^{n} x_i^2 - x_i \sum\limits_{i=1}^{n} x_i - x_i \sum\limits_{i=1}^{n} x_i + n x_i^2}{n^2\, S_X^2} = \frac{\sum\limits_{i=1}^{n} x_i^2 - 2x_i \sum\limits_{i=1}^{n} x_i + n x_i^2}{n^2\, S_X^2} =$$

$$= \frac{\sum\limits_{i=1}^{n} x_i^2 - n\bar{x}^2 - 2x_i \sum\limits_{i=1}^{n} x_i + n x_i^2 + n\bar{x}^2}{n^2\, S_X^2} = \frac{n S_X^2}{n^2\, S_X^2} + \frac{-2x_i n\bar{x} + n x_i^2 + n\bar{x}^2}{n^2\, S_X^2} =$$

$$= \frac{1}{n} + \frac{n\left(x_i - \bar{x}\right)^2}{n^2\, S_X^2} \Rightarrow \boxed{h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{n\, S_X^2}}$$

Operando con el ij-ésimo elemento fuera de la diagonal, obtenemos que:

$$h_{ij} = \frac{\sum_{i=1}^{n} x_i^2 - x_i \sum_{i=1}^{n} x_i - x_j \sum_{i=1}^{n} x_i + nx_i^2}{n^2\, S_X^2} = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 - x_i \sum_{i=1}^{n} x_i - x_j \sum_{i=1}^{n} x_i + nx_ix_j + n\bar{x}^2}{n^2\, S_X^2} =$$

$$= \frac{nS_X^2}{n^2\, S_X^2} + \frac{n(x_i-\bar{x})(x_j-\bar{x})}{n^2\, S_X^2} \Rightarrow \boxed{h_{ij} = \frac{1}{n} + \frac{(x_i-\bar{x})(x_j-\bar{x})}{n\, S_X^2}}$$

3. Esta relación es más sencilla de visualizar si vemos la matriz $\tilde{\mathbf{X}}$ por filas y a la matriz $(\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}$ por columnas:

$$\tilde{\mathbf{X}} = \begin{bmatrix} \text{——} & \tilde{\mathbf{x}}_\mathbf{1} & \text{——} \\ \text{——} & \tilde{\mathbf{x}}_\mathbf{2} & \text{——} \\ & \vdots & \\ \text{——} & \tilde{\mathbf{x}}_\mathbf{n} & \text{——} \end{bmatrix}_{n\times k} \qquad (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1} = \begin{bmatrix} | & | & & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_k} \\ | & | & & | \end{bmatrix}_{k\times k}$$

Luego el ii-ésimo elemento de $\tilde{\mathbf{H}}$ está dado por $h_{ii} = \frac{1}{n} + \tilde{h}_{ii}$ (tal como se demostró en la parte 1), donde:

$$\tilde{h}_{ii} = \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_1}\, \tilde{\mathbf{x}}_\mathbf{i} + \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_2}\, \tilde{\mathbf{x}}_\mathbf{i} + \ldots + \tilde{\mathbf{x}}_\mathbf{i}'\, \mathbf{a_k}\, \tilde{\mathbf{x}}_\mathbf{i}$$

Por lo tanto:

$$\tilde{h}_{ii} = \tilde{\mathbf{x}}_\mathbf{i}'\, (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{x}}_\mathbf{i}$$

Luego entonces:

$$h_{ii} = \frac{1}{n} + \tilde{\mathbf{x}}_\mathbf{i}'\, (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{x}}_\mathbf{i} = \frac{1}{n}\left(1 + \tilde{\mathbf{x}}_\mathbf{i}'\, \hat{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{x}}_\mathbf{i}\right) \Rightarrow \boxed{\tilde{h}_{ii} = \frac{1}{n}\left(1 + \hat{D}_i^2\right)}$$

Ejercicio 4

$$Dffits_i = \frac{|\hat{y}_i - \hat{y}_{i(i)}|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{|y_i - e_i - y_i + e_{(i)}|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{|e_{(i)} - e_i|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\left|\frac{e_i}{1-h_{ii}} - e_i\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\left|\frac{e_i - e_i(1-h_{ii})}{1-h_{ii}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} =$$

$$= \frac{\left|\frac{e_i h_{ii}}{1-h_{ii}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\frac{h_{ii}}{\sqrt{1-h_{ii}}}\left|\frac{e_i}{\sqrt{1-h_{ii}}}\right|}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{\sqrt{h_{ii}}}{\sqrt{1-h_{ii}}}\left|\frac{e_i}{\sqrt{S_{(i)}^2(1-h_{ii})}}\right| = |t_i|\sqrt{\frac{h_{ii}}{1-h_{ii}}}$$

Práctico 4

Ejercicio 4

1. Expresión de F_0 para la prueba $H_0 : \beta_2 = 0$

$$F_0 = \frac{\left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)' \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\right)^{-1} \left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

   Teniendo en cuenta que:

   - $q = 1$
   - $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
   - $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$
   - $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix}'$
   - $(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{pmatrix}$
   - $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \frac{1}{4}$

   Llegamos a que:

$$F_0 = \frac{4\hat{\beta}_2^2}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

2. Expresión de F_0 para la prueba $H_0 : \beta_1 = \beta_2$

$$F_0 = \frac{\left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)' \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\right)^{-1} \left(\mathbf{R}\hat{\beta} - \mathbf{r}\right)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

   Teniendo en cuenta que:

   - $q = 1$
   - $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
   - $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$
   - $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix}'$
   - $(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{pmatrix}$
   - $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \frac{46}{5}$

   Llegamos a que:

$$F_0 = \frac{5}{46} \frac{\left(\hat{\beta}_1 - \hat{\beta}_2\right)^2}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Ejercicio 5

Las matrices de datos:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 6{,}6 \\ 7{,}8 \\ 2{,}1 \\ 0{,}4 \end{bmatrix} \quad \Rightarrow \quad (\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 0{,}1 & 0 \\ 0 & 0{,}25 \end{bmatrix} \quad (\mathbf{X}'\mathbf{y}) = \begin{bmatrix} 20{,}9 \\ 16{,}1 \end{bmatrix}$$

Por lo tanto, $\hat{\beta} = \begin{pmatrix} 2{,}090 \\ 4{,}025 \end{pmatrix}$ y $\hat{\sigma}_u = 0{,}4932$

Para hallar el estadístico de prueba, partimos de:

$$F_0 = \frac{\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)'\Big(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\Big)^{-1}\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

Teniendo en cuenta que:

- $q = 1$
- $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{1\times 1} = r = 0$
- $\mathbf{R}_{q\times(k+1)} = \begin{pmatrix} -2 & 1 \end{pmatrix}$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = 0{,}65$

Llegamos a que:

$$F_0 = \frac{1}{0{,}65} \frac{\Big(-2\hat{\beta}_1 + \hat{\beta}_2\Big)^2}{0{,}4932^2} = 0{,}1520$$

Luego: $F_{1;2}(1-\alpha) = 38{,}5063 > 0{,}1520 = F_0 \Rightarrow$ no rechazo H_0

Ejercicio 6

- Bajo $H_0 : \mathbf{X}\beta = \mathbf{0}_n \Rightarrow \mathbf{y} = \varepsilon \Rightarrow SSR_{H_0} = \mathbf{y}'\mathbf{y}$

  Por otra parte, $SCT_{SR} = SCE_{SR} + SCR_{SR} \Rightarrow \mathbf{y}'\mathbf{y} - n\bar{y}^2 = SCR_{SR} + \hat{\mathbf{y}}'\hat{\mathbf{y}} - n\bar{y}^2 \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SCR_{SR} + \hat{\mathbf{y}}'\hat{\mathbf{y}} \Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + (\mathbf{X}\hat{\beta})'\mathbf{X}\hat{\beta} \Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta} \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'(\mathbf{X}'\mathbf{X})\Big((\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}\Big) = SRC_{SR} + \hat{\beta}'\underbrace{(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}}_{\mathbf{I}_k}(\mathbf{X}'\mathbf{y}) \Rightarrow$

  $\Rightarrow \mathbf{y}'\mathbf{y} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{y}$

  Luego entonces, $SCR_{H_0} = SRC_{SR} + \hat{\beta}'\mathbf{X}'\mathbf{y} \Rightarrow \boxed{\hat{\beta}'\mathbf{X}'\mathbf{y} = SCR_{H_0} - SRC_{SR}}$

- Estadístico F_0

$$F_0 = \frac{\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)'\Big(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}\Big)^{-1}\Big(\mathbf{R}\hat{\beta} - \mathbf{r}\Big)/q}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}$$

  Teniendo en cuenta que:

  - $q = k + 1$
  - $\mathbf{r}_{q\times 1} \Rightarrow \mathbf{r}_{k+1\times 1} = \mathbf{0}_{k+1}$
  - $\mathbf{R}_{q\times(k+1)} = \mathbf{I}_{k+1}$

- $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \cdots & \hat{\beta}_k \end{pmatrix}'$
- $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}' = \mathbf{I}_{k+1}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}'_{k+1} = (\mathbf{X}'\mathbf{X})^{-1}$

Llegamos a que:

$$F_0 = \frac{\hat{\beta}'\Big((\mathbf{X}'\mathbf{X})^{-1}\Big)^{-1}\hat{\beta}/(k+1)}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)} \Rightarrow \boxed{F_0 = \frac{\hat{\beta}'(\mathbf{X}'\mathbf{X})\hat{\beta}/(k+1)}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-k-1)}}$$