Min-Graph Coloring

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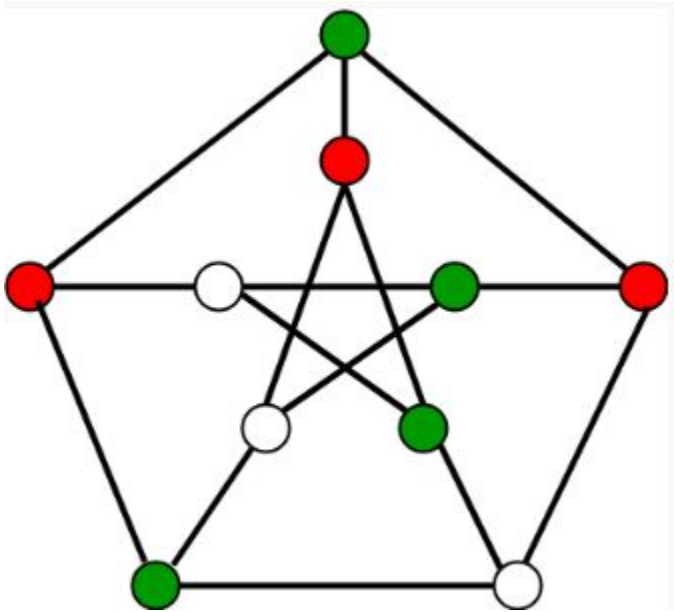
Problem Presentation



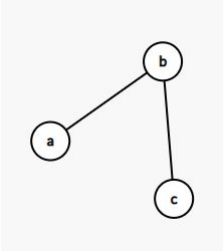
Exact Solution

Example Inputs and Outputs:

Incomplete Graphs:

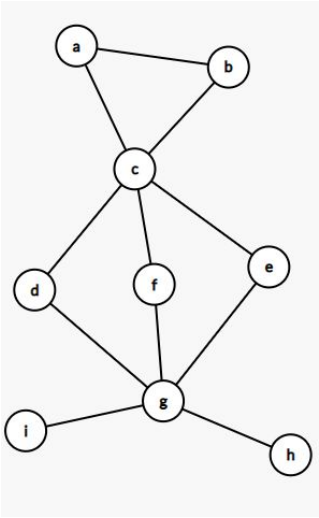
Input: 2

a b
b c



Input: 11

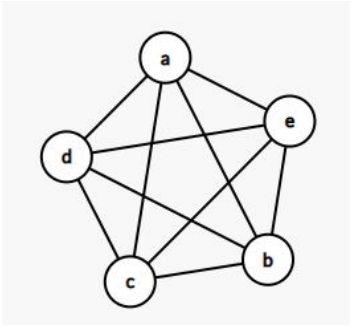
a b
a c
b c
c d
c e
c f
g d
g e
g f
g h
g i



Complete Graph:

Input: 10

a b
a c
a d
a e
b c
b d
b e
c d
c e
d e



Why is the problem important?

This problem has a wide range of applications in various fields such as:

1. Scheduling: This problem can be used to schedule tasks on a project network, where each vertex represents a task, and the edges represent the precedence constraints between the tasks.

2. Map coloring: This problem can be used to color regions on a map such that no two adjacent regions have the same color.

3. Sudoku: This game can be viewed as an instance of the problem, where the squares on the grid represent the vertices and the edges represent the constraints between the squares.

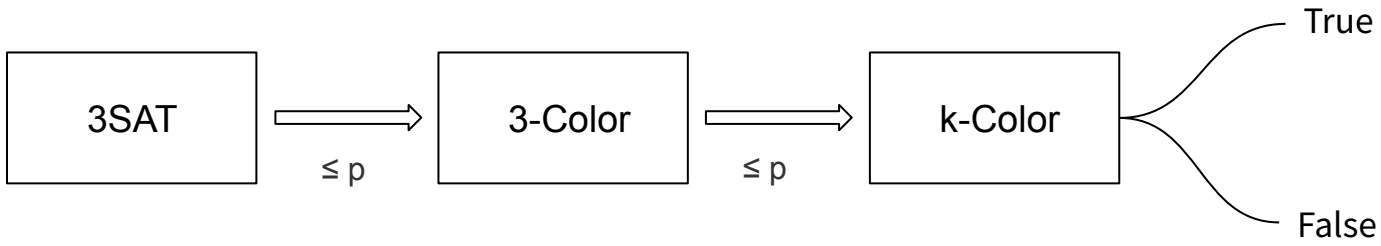
Explanation of the certifier process being polynomial

For each combination of colors, check if adjacent vertices have different colors. If any pair of adjacent vertices have the same color, the combination of colors is invalid.

The certifier process has a time complexity of O(n^2) where n is the number of vertices. Therefore, the certifier process is polynomial.

Reduction problem

3SAT → k-Coloring:



Explanation of the exact coded solution

Create a graph with the edges entered by the user.
Start at 1 color.
Use the itertool to create all possible combinations of colors for n vertices.
For each combination, check if it is valid.

If false, increment the number of colors, and do the same process.
If true, return that number of colors, and the color assigned to each vertex.

```
while not check:
    nums = []

    for i in range(num_colors):
        nums.append(i)

    lst = itertools.product(nums, repeat=len(graph))

    for val in lst:
        check, colors, nums = check_independent_set(val, graph)
        if check:
            break

    num_colors += 1
```

```
def check_independent_set(nums, graph):
    pair = zip(graph, nums)
    colors = {}

    for x in pair:
        colors[x[0]] = x[1]

    for u in graph:
        for v in graph[u]:
            if colors[u] == colors[v]:
                return False, colors, nums

    return True, colors, nums
```

Approximation

Degree of Saturation (DSatur) Approximation

The DSatur algorithm is a greedy approximation approach for the min-coloring problem

The algorithm iteratively colors each vertex in the graph by selecting the vertices with the highest degree of saturation (DS) value and coloring them with the first available color. The the DS values of the adjacent uncolored vertices after each color assignment, until all vertices are colored.

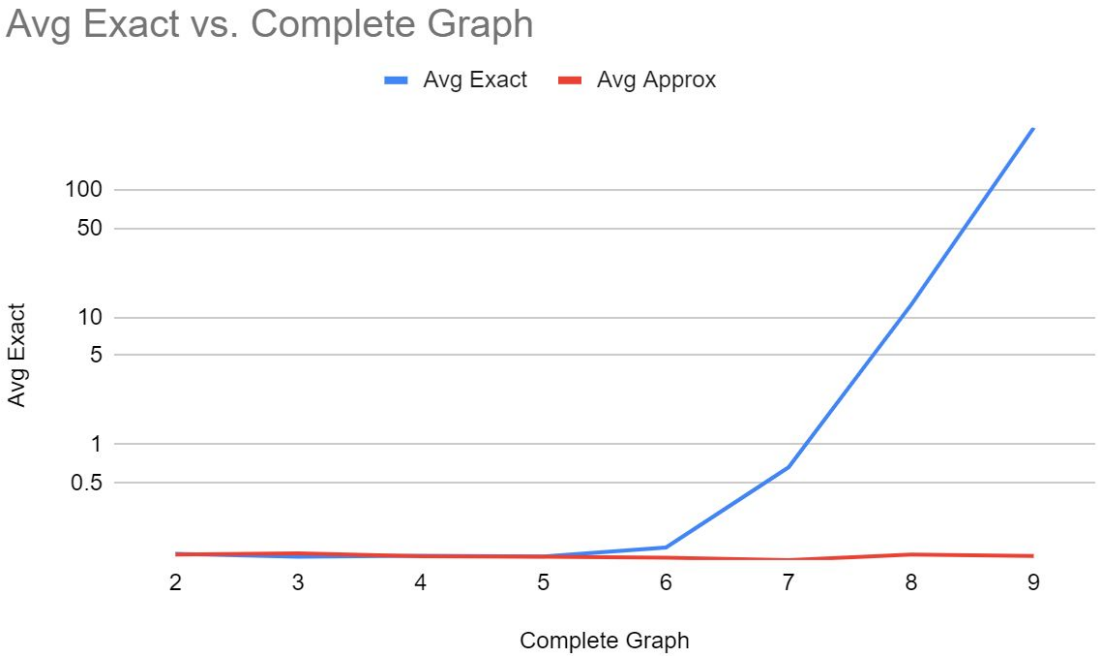
Pseudocode

```
1. Choose a vertex with maximum degree and color it with the first available
color.
2. While there are uncolored vertices:
    a. Choose the uncolored vertex with the highest DS value and color it
with the first available color.
    b. Update the DS values of all uncolored vertices adjacent to the newly
colored vertex.
    c. If there are no vertices with DS > 0, choose a vertex with maximum
degree that has not been colored and color it with the first available
color.
3. Return the final coloring.
```

This algorithm has a runtime of $O(n^2)$, what is interesting is DSatur is an exact algorithm if the input graph is a Bipartite, Cycle, or a Wheel graph.

Exact vs Approximation, side by side

The difference between O(e^n) and O(n^2) can be seen in this chart



Fail Cases

Lower Bound