Derivation of Lorentz Transform

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Derivation

Consider two inertia frames x and x', we could write the relation in following form:

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

The norm of matrix must equals 1, we have

$$AD - BC = 1 \tag{1}$$

Case 1, if a beam of light is moving at positive x direction and at $t = 0$, $x = 0$. we have

$$\begin{pmatrix} ct \\ ct \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} ct' \\ ct' \end{pmatrix}$$

So we derive

$$A + B = C + D \tag{2}$$

Case 2, if point is at rest and at the origin of x' frame

$$\begin{pmatrix} vt \\ ct \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 \\ ct' \end{pmatrix}$$

So we derive

$$\frac{B}{D} = \frac{v}{c} \tag{3}$$

Case 3, if point is at rest and at the origin of x frame

$$\begin{pmatrix} 0 \\ ct \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} -vt' \\ ct' \end{pmatrix}$$

So we derive

$$\frac{B}{A} = \frac{v}{c} \tag{4}$$

From (2), (3) and (4) we know $A = D$ and $B = C$, and let the ratio $\frac{v}{c} = \beta$

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} A & \beta A \\ \beta A & A \end{pmatrix} \cdot \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Also we have (1)

$$A^2(1-\beta^2) = 1 \tag{5}$$

Then we have

$$\gamma = A = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6}$$