The Steenrod Algebra and Its Dual

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Abstract

These are notes for the seminar "Advanced Topics in Homotopy Theory" given by Prof. Stefan Schwede and Dr. Jack Davies in Bonn during the WS2023/24. Our goal is to present the main results of Milnor's paper "The Steenrod Algebra and its Dual" [Mil58]. The text in blue was not presented during the talk.

Contents

1 The Steenrod Algebra

Let p be a prime.

Definition 1 (Stable Cohomology operation) *A* ***stable*** $\bmod p$ ***cohomology operation*** θ *of type* $r \in \mathbb{Z}$ *is a family of natural transformations* $(\theta_n)_{n\in\mathbb{N}}$[1]

$$\theta_n \colon H^n(-, \mathbb{F}_p) \to H^{n+r}(-, \mathbb{F}_p)$$

[1] We view $\bmod p$ cohomology as a functor $\mathrm{Top}^2 \to \mathrm{Ab}$ where Top^2 denotes pairs of pointed topological spaces, we won't need the spaces to be Hausdorff or CW-complexes.

such that the following diagram commutes for every space X

$$\begin{array}{ccc} H^n(X, \mathbb{F}_p) & \xrightarrow{\theta_n} & H^{n+r}(X, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ H^{n+1}(\Sigma X, \mathbb{F}_p) & \xrightarrow[\theta_{n+1}]{} & H^{n+r+1}(\Sigma X, \mathbb{F}_p) \end{array}$$

We can trivially compose two cohomology operations θ, θ' of type r (resp. r') to obtain a cohomology operation of type $r + r'$, this motivates the following definition.

Definition 2 (Steenrod Algebra) *The* mod p ***Steenrod Algebra*** $\mathcal{A}_p$ *is the ring freely generated by the stable cohomology operations. This ring comes with a natural grading coming from the type of the cohomology operation.*

For those familiar with (maps of) spectra, the most natural way to define the Steenrod algebra is by the formula $\mathcal{A}_p = H\mathbb{F}_p^*(H\mathbb{F}) = \bigoplus_n H\mathbb{F}_p^n(H\mathbb{F}_p)$.

Remark 1 *Notice that if* θ *and* θ' *are two cohomology operations of different types, their sum* $\theta + \theta'$ *in* $\mathcal{A}_p$ *does* ***not*** *define a cohomology operation in any natural way.*
Despite this, $\mathcal{A}_p$ *still naturally acts on the* ***full*** *cohomology* $H^*(X)$ *of a space, when viewed as an abelian group.*

As we will establish, $\mathcal{A}_p$ carries a Hopf algebra structure which makes $H^*(X)$ into a (Hopf-)module. Before showing this, we present structural results about the Steenrod algebra.

1.1 Steenrod Powers

Definition 3 (Steenrod Powers) *Suppose* $p > 2$, *the* ***Steenrod powers*** *are the stable cohomology operations*

$$P^i \colon H^q(-, \mathbb{F}_p) \to H^{q+2i(p-1)}(-, \mathbb{F}_p)$$

uniquely determined by the following properties

1. $P^0 = \mathrm{Id}$
2. *if* $x \in H^{2n}(X, A, \mathbb{F}_p)$, *then* $P^n x = x^p$
3. *if* $x \in H^n(X, A)$, *then* $P^i x = 0$ *for all* $2i > n$
4. $\delta P^i = P^i \delta$ *where* δ *is the boundary homomorphism*
5. $P^i(xy) = \sum_{j+k=i} P^j x P^k y$

Definition 4 (Steenrod Squares) *The* ***Steenrod squares*** *are the unique stable* mod 2 *cohomology operations*

$$Sq^i \colon H^q(-, \mathbb{F}_2) \to H^{q+i}(-, \mathbb{F}_2)$$

uniquely determined by

1. $P^0 = \mathrm{Id}$
2. *if* $x \in H^n(X, A, \mathbb{F}_2)$, *then* $Sq^n(x) = x^2$
3. *if* $x \in H^n(X, A, \mathbb{F}_2)$, *then* $Sq^i x = 0$ *for all* $i > n$
4. $Sq^n(xy) = \sum_{i+j=n} Sq^i x Sq^j y$
5. $\delta Sq^i = Sq^i \delta$

For any prime, we can also define the Bockstein morphism:

Definition 5 (Bockstein) *The natural transformations*

$$\delta_n \colon H^n(-, \mathbb{F}_p) \to H^{n+1}(-, \mathbb{F}_p)$$

associated to the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ *define a stable cohomology operation* $\delta = (\delta_n)_{n \in \mathbb{N}}$ *called the* ***Bockstein morphism****.*

For $p = 2$, the Bockstein coincides with Sq^1. It is a famed result of Steenrod that these operations generate the Steenrod algebra.

Theorem 2 (Structure of the Steenrod Algebra) *[SE62, Ch. VI, Sec. 2] Let* p *be an odd prime. Call a sequence* $I = (\epsilon_0, s_1, \epsilon_1, s_2, \dots)$ ***admissible*** *if it is finite,* $s_i \geq 1, \epsilon = 0, 1$ *and* $s_i \geq p s_{i+1} + \epsilon_i$*. The set*

$$P^I := \beta^{\epsilon_0} P^{s_1} \beta^{\epsilon_1} P^{s_2}, \quad I \textit{ admissible}$$

is a basis for the Steenrod algebra.

There is a similar result for $p = 2$, which we do not make explicit.
The algebra structure of $\mathcal{A}_p$ is extremely complex, as is made apparent by the Adem relations, we will show that the dual of $\mathcal{A}_p$ (as a vector space) also inherits an algebra structure that is comparatively simple: it is a graded polynomial algebra.

2 Hopf Algebras

The first goal of this talk is to show that $\mathcal{A}_p$ is a Hopf algebra over $\mathbb{F}_p$, so we need to define what a Hopf algebra is.

2.1 Bi-Algebras

We start by studying Hopf algebras independently. Throughout, let k be a field.

Definition 6 (Algebra) *An **Algebra** is a triple* $(\mathcal{A}, \mu, \eta)$ *with* $\mathcal{A}$ *a* k*-vector space together with two maps* $\mu \colon A \otimes A \to A$ *(multiplication) ,* $\eta \colon k \to A$ *(unit) making the following diagrams commute*

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mathrm{Id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{Id}"'] & A \otimes \mathcal{A} \arrow[d, "\mu"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"'] & \mathcal{A} \end{tikzcd}$$

$$\begin{tikzcd} k \otimes \mathcal{A} \arrow[r, "i \otimes \eta"] \arrow[dr] & \mathcal{A} \otimes \mathcal{A} \arrow[d, dashed, "\mu"] & \mathcal{A} \otimes k \arrow[l, "\eta \otimes i"'] \arrow[dl] \\ & \mathcal{A} & \end{tikzcd}$$

Dualizing these definitions, we unsurprisingly obtain

Definition 7 (Coalgebra) *A **coalgebra** is a triple* (C, Δ, ϵ) *where* C *is a* k*-vector space togethere with two maps* $\Delta \colon C \to C \otimes C$ *(comultiplication) and* $\epsilon \colon C \to k$ *(augmentation) making the following diagrams commute*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{Id}"'] & C \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} & C \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\ k \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{Id}"] \arrow[r, "\mathrm{Id} \otimes \epsilon"'] & C \otimes k \end{tikzcd}$$

Since taking duals commutes with tensor products, notice that the dual $C^{\vee}$ naturally gets an algebra structure.
We define (co-)algebra morphisms in the obvious way.

Definition 8 (Bialgebra) *A **bialgebra** is a tuple* $(\mathcal{A}, \mu, \eta, \Delta, \epsilon)$ *where*

- $\mathcal{A}$ *is a* k*-vector space*
- $\mu \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
- $\eta \colon k \to \mathcal{A}$
- $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$
- $\epsilon \colon \mathcal{A} \to k$

such that $(\mathcal{A}, \mu, \eta)$ is an algebra, $(\mathcal{A}^*, \Delta^*, \epsilon^*)$ is an algebra and such that Δ and ϵ are algebra morphisms

Equivalently, one can also require μ and ϵ to be coalgebra morphisms.
If $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ is a graded algebra, we define the **dual algebra** by

$$\mathcal{A}^* \coloneqq A_n^*, \text{ with } A_n^* = \hom(A_n, k)$$

We call a graded algebra $\mathcal{A}$ **graded commutative** if for all homogeneous elements $\alpha, \beta \in \mathcal{A}$, we have $\alpha\beta = (-1)^{\dim \alpha \dim \beta} \beta\alpha$. (omitting μ for sanity reasons) The graded algebra $\mathcal{A}$ is **connected** if $\mathcal{A}_0$ is generated by 1, equivalently $\eta \colon k \to \mathcal{A}_0$ is an isomorphism.
We can similarly define the notion of a graded coalgebra and of a connected coalgebra.

2.2 Antipode maps

Though we will not talk about antipode maps again throughout this talk, we still define them to be able to properly define a Hopf algebra.
Let C be a bi-algebra as above and let $f, g \colon C \to C$ be linear maps, we define the convolution $f * g$ of f with g as the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} C \otimes C \xrightarrow{\mu} C.$$

Definition 9 (Antipode) *An antipode $S \colon C \to C$ is an endomorphism such that*

$$S * \mathrm{Id} = \mathrm{Id} * S = \eta \circ \epsilon.$$

Definition 10 (Hopf Algebra) *A Hopf Algebra is a bi-algebra with an antipode.*

For specific classes of bialgebras, there is a way of constructing an antipode map.

Theorem 3 ([MM65, prop 8.2]) *Let $\mathcal{A}$ be a connected graded bialgebra such that $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i a_i \otimes b_i$ with $\dim a_i, \dim b_i > 0$, then $\mathcal{A}$ admits an antipode map.*

Proof Let $x \in \mathcal{A}$, to define S, we proceed inductively on the degree of x. If $\dim x = 0$, we define $S(x) = x$.
Inductively, suppose we've defined S for all x of degree $< n$ and write $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i a_i \otimes b_i$ as above. Since Δ respects the grading, we may suppose that $\dim b_i < n$, we let

$$S(x) \coloneqq -x - \sum_i a_i S(b_i)$$

One now easily checks that S is an antipode. $\square$

3 The Diagonal Morphism

Our first goal is to prove that $\mathcal{A}_p$ has the structure of a Hopf algebra and to make its structure more explicit.
Throughout, let X be a space. We start by constructing the diagonal morphism $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of our (soon to be) Hopf algebra.

Proposition 4 *There is a unique diagonal morphism $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that*

1. *For all $\theta \in \mathcal{A}$, $\Delta(\theta) = \sum_i \theta_i' \otimes \theta_i$" and $\alpha, \beta \in H^*(X)$ we have*
$$\theta(\alpha \smile \beta) = \sum (-1)^{\dim \theta_i'' \dim \alpha} \theta_i'(\alpha) \smile \theta_i''(\beta)$$
2. *The morphism Δ is a ring morphism.*

Proof Let $\mathcal{A} \otimes \mathcal{A}$ act on $H^*(X) \otimes H^*(X)$ by

$$(\theta' \otimes \theta'')(\alpha \otimes \beta) = (-1)^{\dim \theta'' \dim \alpha} \theta'(\alpha) \otimes \theta''(\beta) \quad \text{where } \theta', \theta'' \in \mathcal{A}, \alpha, \beta \in H^*(X).$$

We let $c\colon H^*(X) \otimes H^*(X) \to H^*(X)$ be the cup product.
Δ exists
Let $R \subset \mathcal{A}$ be the set of all θ such that

$$\theta(\alpha \smile \beta) = c\rho(\alpha \otimes \beta)$$

for some $\rho \in \mathcal{A} \otimes \mathcal{A}$. We want to show that $R = \mathcal{A}$.
Notice that R is closed under multiplication and addition. If $\theta_1, \theta_2 \in R$, then

$$\theta_1\theta_2(\alpha \smile \beta) = c\rho_1\rho_2(\alpha \otimes \beta) \text{ and } (\theta_1 + \theta_2)(\alpha \smile \beta) = c((\rho_1 + \rho_2)(\alpha \otimes \beta))$$

Hence, it suffices to show that R contains the Bockstein and the Steenrod powers which follows from the formulas

$$\delta(\alpha \smile \beta) = \delta\alpha \smile \beta + (-1)^{\dim \alpha} \alpha \smile \delta(\beta)$$
$$P^n(\alpha \smile \beta) = \sum_{i+j=n} P^i(\alpha) \smile P^j(\beta)$$

Δ is unique
Let $K \coloneqq K(\mathbb{F}_p, n+1)$ and $\gamma \in H^{n+1}(K) \simeq [K, K]_*$ correspond to the identity map, the map

$$\begin{aligned} \mathrm{ev}_\gamma\colon \mathcal{A}_i &\to H^{n+1+i}(K) \\ \theta &\mapsto \theta\gamma \end{aligned}$$

is an isomorphism for all $i \leq n$, it follows that

$$\begin{aligned} j\colon (\mathcal{A} \otimes \mathcal{A})_i &\to H^{2n+2+i}(K \times K) \\ \theta \otimes \theta' &\mapsto (-1)^{\dim \theta' \dim \gamma} \theta(\gamma) \otimes \theta'(\gamma) \end{aligned}$$

is too.
Let $\theta \in \mathcal{A}_i$, suppose ρ, ρ' both satisfy the required equality, then

$$j(\rho) = c\rho\left((\gamma \otimes 1) + (1 \otimes \gamma)\right) = c\rho'\left((\gamma \otimes 1) + (1 \otimes \gamma)\right) = j(\rho')$$

The unicity of Δ implies that it is a ring morphism. $\square$

Remark 5 *From this proof, we can in particular single out the action of Δ on generators, namely, it follows that*

$$\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$$
$$\Delta(P^n) = \sum_{i+j=n} P^i \otimes P^j.$$

Theorem 6 (The Steenrod Algebra is a Hopf Algebra) *The maps*

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A},$$

where μ is composition, give $\mathcal{A}$ the structure of a Hopf algebra. Furthermore Δ is graded commutative.

Proof It suffices to show that Δ is associative and commutative.

Associativity

It suffices to check the identity

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

This identity clearly holds on generators, namely

$$\begin{aligned}(\Delta \otimes 1)(\delta \otimes 1 + 1 \otimes \delta) &= \delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta \\ &= (1 \otimes \Delta)(\delta \otimes 1 + 1 \otimes \delta)\end{aligned}$$

and

$$\begin{aligned}(\Delta \otimes 1)\left(\sum_{i+j=n} P^i \otimes P^j\right) &= \sum_{i+j=n}\left(\sum_{i'+j'=i} P^{i'} \otimes P^{j'}\right) \otimes P^j \\ &= \sum_{i+j+k=n} P^i \otimes P^j \otimes P^k \\ &= (1 \otimes \Delta)\left(\sum_{i+j=n} P^i \otimes P^j\right).\end{aligned}$$

(Graded) Commutativity

Let

$$\begin{aligned} T\colon \mathcal{A}\otimes\mathcal{A} &\to \mathcal{A}\otimes\mathcal{A} \\ \theta\otimes\theta' &\mapsto (-1)^{\dim\theta\dim\theta'}\theta'\otimes\theta. \end{aligned}$$

We have to check that $\Delta = T\Delta$, which one can check again on generators:

$$T(1\otimes\delta+\delta\otimes 1) = 1\otimes\delta+\delta\otimes 1$$

and

$$T(\sum_{i+j=n} P^i\otimes P^j) = \sum_{i+j=n}(-1)^{4ij(p-1)^2}P^j\otimes P^i \qquad \square$$

4 The dual Steenrod Algebra

There are a lot of excellent resources that describe the structure of the mod 2 Steenrod algebra, see Akhil Matthew's blog or [MT08, Chap. 6]. For the sake of originality, we present the complementary case.
From now on, p is a prime different from 2 and $\mathcal{A} := \mathcal{A}_p$, we follow [Mil58, Chap. 3].

For the rest of this talk, we focus on the dual Steenrod algebra $\mathcal{A}_* := \mathcal{A}^\vee$, whose multiplication is induced by Δ. Our goal is to fully determine the structure of $\mathcal{A}_*$, more precisely we will sketch the proof of the following theorem:

Theorem 7 *There is a graded isomorphism*

$$\mathcal{A}_* \simeq \Lambda[\tau_0,\tau_1,\ldots]\otimes\mathbb{F}_p[\xi_1,\xi_2,\ldots]$$

The grading will be determined later.

To single out an appropriate set of generators for $\mathcal{A}_*$, we analyze how $\mathcal{A}_*$ (co-)acts on the cohomology ring of a specific space. We start by describing this co-action formally and then introduce the relevant space.

4.1 The coaction of $\mathcal{A}_*$

Let $\langle\cdot,\cdot\rangle$ denote the evaluation pairing on $H_*\times H^*$.
Given that we are working over a field, cohomology and homology are dual. Hence, given $\theta\in\mathcal{A}_i$ and $\mu\in H_n$, we define $\theta\cdot\mu\in H_{n-i}$ by the rule

$$\langle\theta\cdot\mu,\alpha\rangle := \langle\mu,\theta\cdot\alpha\rangle \text{ for all } \alpha\in H^{n-i}.$$

This gives a well defined action

$$\lambda_*\colon \mathcal{A} \otimes H_* \to H_*.$$

We denote the dual of this action by $\lambda^*\colon H^* \to \mathcal{A}_* \otimes H^*$. The restriction of λ_*

$$\lambda_i\colon \mathcal{A} \otimes H^{n+i} \to H^n$$

also gives rise to dual morphisms $\lambda^i\colon H^n \to \mathcal{A}_* \otimes H^{n+i}$ which satisfy

$$\lambda^* = \lambda^1 + \lambda^2 + \dots .^2$$

We can also understand the action of $\mathcal{A}$ better in terms of λ^*: if we know $\lambda^*(\alpha)$, we know $\theta \cdot \alpha$ for any $\theta \in \mathcal{A}$.

Lemma 8 *Let $\lambda^*(\alpha) = \sum_i \alpha_i \otimes \omega_i$ and $\theta \in \mathcal{A}$, then*

$$\theta\alpha = \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \theta, \omega_i \rangle \alpha_i$$

Proof By definition of the action, we have

$$\begin{aligned} \langle \mu, \theta\alpha \rangle &= \langle \mu\theta, \alpha \rangle \\ &= \langle \mu \otimes \theta, \lambda^* \alpha \rangle \\ &= \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \mu, \alpha_i \rangle \langle \theta, \omega_i \rangle \end{aligned} \qquad \square$$

And the general equality follows.

4.2 Generators for $\mathcal{A}_*$

Fix some large integer N and let $X = S^{2N+1}/\mathbb{Z}_p = sk_{2N+1}K(\mathbb{F}_p, 1)$. The $(\bmod p)$ cohomology ring of X has the following properties

$$H^1(X) = \langle \alpha \rangle, H^2(X) = \langle \beta \rangle, H^{2i}(X) = \langle \beta^i \rangle, H^{2i+1}(X) = \langle \alpha\beta^i \rangle,$$

where $\beta = \delta\alpha$ and $i \leq N$.

Notation 9 *We define*

$$M^k := P^{p^{k-1}} \cdots P^p P^1$$

Lemma 10 *For all $\theta \in \mathcal{A}$*

$$\theta\beta = \begin{cases} \beta^{p^k} & \text{if } \theta = M_k \\ 0 & \text{else.} \end{cases}$$

[2] Elements in H^* are always finite sums, so this sum should be understood as $\bigoplus_i \lambda^i$

Proof Let $\mathcal{P} = 1 + P^1 + P^2 + \ldots$, from the properties of the Steenrod powers, we notice that

$$\mathcal{P}\beta = \beta + \beta^p \text{ thus } \mathcal{P}\left(\beta^{p^r}\right) = \beta^{p^r} + \beta^{p^{r+1}}.$$

Hence $P^{p^r}(\beta^{p^r}) = \beta^{p^{r+1}}$ and $P^j(\beta^{p^r})$ for $j \neq p^r$ and $j > 0$. From this, we deduce the statement. $\square$

We will now explicitly determine a basis for $\mathcal{A}_*$.

Lemma 11 *There exist elements $\tau_i, \in \mathcal{A}_*^{2p^k-1}$ such that*

$$\lambda^*\alpha = \alpha \otimes 1 + \beta \otimes \tau_0 + \ldots + \beta^{p^r} \otimes \tau_r.$$

Similarly, there exist elements $\xi_i \in \mathcal{A}_*^{2p^i-2}$ with $\xi_0 = 1$ such that

$$\lambda^*\beta = \beta \otimes \xi_0 + \beta^p \otimes \xi_1 + \ldots + \beta^{p^r} \otimes \xi_r$$

Proof From the above, it follows that

$$\lambda^*\beta = \lambda^0\beta + \lambda^{2p-2}\beta + \ldots + \lambda^{2p^k-2}\beta.$$

As the cohomology of X is one-dimensional in all degrees, we deduce that $\lambda^{2p^k-2}(\beta) = \beta^{p^k} \otimes \xi^k$. The exact same argument works for $\lambda^*\alpha$. $\square$

We now study the evaluation pairing $\mathcal{A}_* \times \mathcal{A} \to \mathbb{F}_p$. We easily establish the following lemma

Lemma 12 *We have $\langle \xi_k, M_k \rangle = 1$ but $\langle \xi_k, \theta \rangle = 0$ for any other monomial. Furthermore*

$$\langle M_k\delta, \tau_k \rangle = 1$$

and $\langle \theta, \tau_k \rangle$ for any other monomial.

Proof We know that

$$M_k\beta = \beta^{p^k} = \sum_i (-1)^{2p^i \dim \xi^i} \langle M_k, \xi_i \rangle \beta^{p^i}$$

Proving the equality. The second equality follows from the same argument applied to α and $M_k\delta$. $\square$

We are ready to prove the main structure theorem for the dual Hopf algebra.

Theorem 13 *There is a graded isomorphism*

$$\mathcal{A}_* \simeq \Lambda[\tau_0, \tau_1, \ldots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad \textit{where } \dim \tau_i = 2p^i - 1, \dim \xi_i = 2p^i - 2.$$

Here $\Lambda[\tau_0, \ldots]$ denotes the exterior algebra and $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$ is the polynomial algebra. This isomorphism is graded.

Proof (Sketch) Let $\mathcal{I}$ be the set of finite sequences $(\epsilon_0, r_1, \epsilon_1, \ldots)$ with $\epsilon_i = 0, 1$ and $r_i \in \mathbb{N}$. Given $I \in \mathcal{I}$, we define

$$\omega(I) := \tau_0^{\epsilon_0} \xi_1^{r_1} \tau_1^{\epsilon_1} \xi_2^{r_2} \cdots .$$

We claim it is sufficient to show that the set of $\omega(I)$ form a basis for $\mathcal{A}_*$. Indeed, the τ_i, ξ_j then don't observe any additional identities and the graded commutativity gives the desired isomorphism.
We may order the set $\mathcal{I}$ colexicographically, ie. $(a_1, \epsilon_1, a_2, \cdots) < (b_1, \epsilon_1', b_2, \cdots)$ if $a_i < b_i$ for the largest i such that a_i and b_i differ (remember that the sequences are finite).

We also associated to a $J = (\epsilon_0, r_1, \epsilon_1, \ldots) \in \mathcal{I}$ an element of $\mathcal{A}$.

$$\theta(J) = \delta^{\epsilon_0} P^{s_1} \delta^{\epsilon_1} P^{s_2} \cdots ,$$

where $s_j = \sum_{i=k}^{\infty} (\epsilon_i + r_i) p^{i-k}$.

One can check that the $\theta(J)$ are the basic monomials of the Cartan basis for $\mathcal{A}$.

To show the isomorphism, we show that the basic monomials in $\mathcal{A}$ form an "almost dual" basis to the set of $\omega(I)$.
For this, we use the following lemma.

$$\text{Let } I < J \in \mathcal{I}, \text{ then } \langle \theta(J), \omega(I) \rangle = 0 \text{ if } I < J, \text{ furthermore } \langle \theta(I), \omega(I) \rangle = \pm 1. \qquad (\star)$$

The proof of $(\star)$ is the main technical step in the proof and we skip it. Let $\mathcal{I}_n \subset \mathcal{I}$ be the set of sequences such that $\dim \omega(I) = \dim \theta(I) = n$. The matrix $(\langle \theta(J), \omega(I) \rangle)_{I,J \in \mathcal{I}_n}$ is upper-triangular with ± 1 on the diagonal, hence, the pairing is non-degenerate and the $\omega(I)$ generate the n-th graded part of $\mathcal{A}_*$. ☐

We also state the case for $p = 2$ without proof, the proof can be found in the original paper too and proceeds in very similar steps.

Theorem 14 (The mod 2 dual Steenrod Algebra) *Let $\mathcal{A}_2$ be the mod 2 Steenrod algebra and $\mathcal{A}_{2*}$ its dual Let $\xi_i \in \mathcal{A}_{2*}$ be the dual basis of the basis $\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 \in \mathcal{A}_2$, then there is a graded isomorphism*

$$\mathcal{A}_{2*} \simeq \mathbb{F}_2[\xi_1, \xi_2, \ldots].$$

4.3 The comultiplication in $\mathcal{A}_*$

If we want to fully describe $\mathcal{A}_*$ as a Hopf algebra, we also have to describe the comultiplication $\mu_* := (\mu)^\vee$, wher μ is the usual multiplication in $\mathcal{A}$.

Proposition 15 *We have*

$$\mu_*(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{i} \otimes \xi_i$$
$$\mu_*(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i + \tau_k \otimes 1$$

Proof We first notice that the commutativity of

$$\begin{array}{ccc} H_* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{1\otimes\phi^*} & H_* \otimes \mathcal{A} \\ {\scriptstyle \lambda_*\otimes 1}\downarrow & & \downarrow{\scriptstyle \lambda_*} \\ H_* \otimes \mathcal{A} & \xrightarrow[\lambda_*]{} & H_* \end{array}$$

implies the identity

$$(\lambda^* \otimes 1)\lambda^* = (1 \otimes \mu_*)\lambda^*.$$

Let $\alpha, \beta \in H^*(X)$ with X as before, then

$$\lambda^*(\beta) = \sum \beta^{p^j} \otimes \xi_j$$
$$\lambda^*(\beta^{p^i}) = \sum \beta^{p^{i+j}} \otimes \xi_j^{p^i}$$

Hence, from the identity above, we get

$$\begin{aligned}(\lambda^* \otimes 1)\lambda^*(\beta) &= \sum_{i,j} \beta^{p^{i+j}} \otimes \xi_j^{p^i} \otimes \xi_i \\ &= (1 \otimes \mu_*)\lambda^*(\beta) \\ &= \sum \beta^{p^k} \otimes \mu_*(\xi_k) \qquad \square\end{aligned}$$

And hence we deduce the identity for $\mu_*(\xi_k)$, the identity for $\mu_*(\tau_k)$ is deduce in the same way.

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